

Homogeneous almost complex structures in dimension 6 with semi-simple isotropy

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Abstract

We classify invariant almost complex structures on homogeneous manifolds of dimension 6 with semi-simple isotropy. Those with non-degenerate Nijenhuis tensor have the automorphism group of dimension either 14 or 9. An invariant almost complex structure with semi-simple isotropy is necessarily either of specified 6 homogeneous types or a left-invariant structure on a Lie group. For integrable invariant almost complex structures we classify all compatible invariant Hermitian structures on these homogeneous manifolds, indicate their integrability properties (Kähler, SNK, SKT) and mark the other interesting geometric properties (including the Gray-Hervella type).

1 Introduction and main results

Consider an almost complex manifold (M, J) , $J^2 = -\mathbf{1}$. If M is closed, the automorphisms of J form a Lie group [BKW]. In general the automorphism group can be infinite-dimensional, but finite-dimensionality can be guaranteed by additional local (non-degeneracy of the Nijenhuis tensor [K₁]) or global (Kobayashi partial hyperbolicity [Ko, KO]) conditions.

An almost complex structure is integrable if the Nijenhuis tensor N_J vanishes [NW]. In real dimension 6 (complex dimension 3) non-degeneracy of the Nijenhuis tensor means that $N_J : \Lambda_{\mathbb{C}}^2 TM \rightarrow TM$ is a (\mathbb{C} -antilinear) isomorphism. Such structures are important in applications to critical points of the Hitchin-type functionals and nearly Kähler geometry [Br, V].

As proven in [K₂] the local automorphism group G of the structure J on M^6 with non-degenerate N_J has dimension at most 14, and that this bound is achieved only for the G_2 -invariant almost complex structures¹: either G_2^c -invariant J on S^6 or G_2^* -invariant J on its non-compact version $S^{2,4}$. It

¹We denote the compact real form of G_2 by $G_2^c \subset SO(7)$ and the split real form with the trivial center by $G_2^* \subset SO(3, 4)$.

is natural to ask what is the next submaximal (= maximal among structures that are not locally G_2 -invariant) dimension of the automorphism group.

As one can expect this is still transitive, we confine to (locally) homogeneous structures. But their classification is cumbersome, and so we restrict further by requiring the isotropy group H to be semi-simple. This generalizes the assumption of [Wo]. In this reference the almost complex structures on G/H with irreducible isotropy H were classified. Here we extend this classification in dimension 6. The obtained structures J possess abundant symmetries ($\dim G \geq 9$) by construction.

Theorem 1. *The only homogeneous almost complex structures on the homogeneous space $M^6 = G/H$ with semi-simple isotropy group H are (up to a covering and a quotient by a discrete central subgroup):*

- (I) *the homogeneous almost complex structure on $S^6 = G_2^c/SU(3)$ or on $S^{2,4} = G_2^*/SU(1,2)$;*
- (II₁) *4-parametric family on $U(3)/SU(2)$, $U(2,1)/SU(2)$;*
- (II₂) *4-parametric family on $U(2,1)/SU(1,1)$,
and 2-parametric on $GL(3)/SU(1,1)$;*
- (III) *left-invariant almost complex structures on a 6D Lie group.*

The tables of the latter structures are given in the Appendix, see also Theorem 3. The structures of type II are described in Section 6.

Remark 1. *As written above, the possible M^6 are obtained from the universal covering group G by additional discrete quotient $M = \Gamma \backslash G/H$. These central subgroups $\Gamma \subset G$ can be completely described. For instance, instead of $U(3)/SU(2)$ we get $G = \mathbb{R}^1 \times SU(3)$, $H = SU(2)$ and Γ is one of the 4 obvious discrete subgroups of the center $Z(G) = \mathbb{R} \times \mathbb{Z}_3$. Similarly, we get $G = \mathbb{R} \times \widetilde{SU(2,1)}$ or $\mathbb{R} \times \widetilde{SL(3)}$ for other type II cases. However the invariant almost complex structure J on M depends only on 2 parameters in these cases (the torus is covered by a cylinder), see the details in Section 6.*

Remark 2. *In dimension 4 all complex representations with semi-simple $H \subset GL(2, \mathbb{C})$ lead to the flat structure $G = H \ltimes \mathbb{C}^2$ and so $M^4 = \mathbb{C}^2$. If we allow H to be reductive, then 4 new cases appear:*

- $SU(3)/U(2) = \mathbb{C}P^2$,
- $SU(2,1)/U(2) = B^4$,
- $SU(2,1)/U(1,1) = \mathbb{C}P^2 \setminus \{pt\} \simeq \mathbb{C}P^1 \times \mathbb{C}$
- $SL(3)/U(1,1) = \mathbb{R}P^2 \times \mathbb{R}P^{2\nu} \setminus \mathbb{P}\{(v,p) : v \cdot p = 0\} \simeq T\mathbb{R}P^2$.

In all these cases J is integrable (complex structure). Our method can be used to extend the classification to the reductive isotropy H in dimension 6 as well, including such complex manifolds as $\mathbb{C}P^3 = SU(4)/U(3)$.

We will also examine the invariant (pseudo-)Riemannian and almost symplectic structures on these homogeneous 6-manifolds G/H , specifying (in the case they are compatible with the almost complex structure) which of them are Hermitian, Kähler, strongly nearly Kähler (SNK), strongly Kähler with torsion (SKT) and discuss the Gray-Hervella classes of them.

The SNK condition is closely related with the condition of non-degeneracy of N_J (recall that this means $N_J(\Lambda^2 TM) = TM$). For the Calabi almost complex structure J on S^6 it is known that its automorphism group is the compact real form G_2^c . Similarly the split real form G_2^* is the symmetry group of the homogeneous structure J on the pseudo-sphere $S^{2,4}$ (in both cases $\dim \text{Aut}(M, J) = 14$). It turns out that for the other cases of Theorem 1 with non-degenerate tensor N_J the local symmetries of J (and hence the global ones) are only the obvious ones.

Theorem 2. *Let J be an invariant almost complex structure on the homogeneous space $M = G/H$ from Theorem 1 of types II or III (i.e. J is not G_2 -invariant). Assume that the Nijenhuis tensor N_J is non-degenerate. Then the (local and global) automorphisms of J in the connected component of unity are only those coming from G , whence $\dim \text{Aut}(M, J) = 9$.*

Some calculations from this work used symbolic packages of Maple; the corresponding worksheets are available from arXiv:1401.8187.

2 Classification result via representation theory

Consider a homogeneous manifold $M = G/H$, i.e. a connected manifold M on which a connected Lie group G acts transitively with the stabilizer H of a point $o \in M$. We will always assume that G acts effectively on M , i.e. no non-trivial subgroup of H is normal in G .

In this case the isotropy representation $j : H \rightarrow \text{GL}(T_o M)$ is almost faithful (has finite kernel) provided the stabilizer group H is reductive (in particular, semi-simple that is our running assumption). When M has a G -invariant almost complex structure J whose Nijenhuis tensor N_J is non-degenerate, then this is also the case by [K₂].

Let $\mathfrak{g}, \mathfrak{h}$ be the Lie algebras of the Lie groups G, H and $\mathfrak{m} = T_o M$ the model tangent space of G/H , $o = eH$. The isotropy representation makes the space \mathfrak{m} into \mathfrak{h} -module. The above data (\mathfrak{h} subalgebra, \mathfrak{m} representation)

can be summarized in the following exact 3-sequence of \mathfrak{h} -modules

$$0 \rightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{m} \rightarrow 0.$$

Our hypothesis that \mathfrak{h} is semi-simple yields the splitting of the sequence via an embedding $\mathfrak{m} \subset \mathfrak{g}$ as an H -invariant complement, and we get the reductive decomposition

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}, \quad [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}.$$

Our strategy for classification is to start with the pure algebraic data, and then reconstruct the Lie groups G, H and the manifold M with its geometry.

Reconstructing $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ from the representation $(\mathfrak{h}, \mathfrak{m})$ amounts to the following. The brackets $\mathcal{B}_{\mathfrak{h}} : \Lambda^2 \mathfrak{h} \rightarrow \mathfrak{h}$ and $\mathcal{B}_{\mathfrak{h}, \mathfrak{m}} : \mathfrak{h} \wedge \mathfrak{m} \rightarrow \mathfrak{m}$ are given by the Lie algebra structure of \mathfrak{h} and the \mathfrak{h} -module structure of \mathfrak{m} respectively. The missing ingredient is the map $\mathcal{B}_{\mathfrak{m}} : \Lambda^2 \mathfrak{m} \rightarrow \mathfrak{g}$, determining the full bracket

$$\mathcal{B} = \mathcal{B}_{\mathfrak{h}} + \mathcal{B}_{\mathfrak{h}, \mathfrak{m}} + \mathcal{B}_{\mathfrak{m}} : \Lambda^2 \mathfrak{g} = \Lambda^2 \mathfrak{h} \oplus (\mathfrak{h} \wedge \mathfrak{m}) \oplus \Lambda^2 \mathfrak{m} \rightarrow \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}.$$

Lemma 1. *The Jacobi identity of the resulting bracket $\mathcal{B} : \Lambda^2 \mathfrak{g} \rightarrow \mathfrak{g}$, involving an element from \mathfrak{h} , is equivalent to \mathfrak{h} -equivariancy of $\mathcal{B}_{\mathfrak{m}}$. If $\mathcal{B}_{\mathfrak{m}}$ is \mathfrak{h} -equivariant, then the bracket \mathcal{B} defines the Lie algebra structure on \mathfrak{g} iff the Jacobi map $\text{Jac}_{\mathfrak{m}} : \Lambda^3 \mathfrak{m} \rightarrow \mathfrak{g}$ vanishes, where*

$$\text{Jac}_{\mathfrak{m}}(x, y, z) = \mathcal{B}(x, \mathcal{B}_{\mathfrak{m}}(y, z)) + \mathcal{B}(y, \mathcal{B}_{\mathfrak{m}}(z, x)) + \mathcal{B}(z, \mathcal{B}_{\mathfrak{m}}(x, y)).$$

The Lie algebra \mathfrak{g} defined by such $\mathcal{B}_{\mathfrak{m}}$ is called *the Lie algebra extension* of the \mathfrak{h} -module \mathfrak{m} .

Proof. The Jacobi relation involving 3 elements from \mathfrak{h} holds as \mathfrak{h} is a Lie algebra. The Jacobi relation involving 2 elements from \mathfrak{h} and 1 from \mathfrak{m} holds as \mathfrak{m} is an \mathfrak{h} -representation. Finally the Jacobi relation involving 1 element from \mathfrak{h} and 2 from \mathfrak{m} is precisely the equivariancy of the map $\mathcal{B}_{\mathfrak{m}}$. \square

Since \mathfrak{h} is semi-simple, the construction of $\mathcal{B}_{\mathfrak{m}} \in \text{Hom}_{\mathfrak{h}}(\Lambda^2 \mathfrak{m}, \mathfrak{g})$ goes as follows. Decompose into irreducible \mathfrak{h} -modules (including the trivial): $\Lambda^2 \mathfrak{m} = \oplus r_i \cdot \mathfrak{u}_i = \oplus (\mathfrak{u}_i \otimes \mathbb{R}^{r_i})$, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} = \oplus s_i \cdot \mathfrak{u}_i = \oplus (\mathfrak{u}_i \otimes \mathbb{R}^{s_i})$. Then by Schur's lemma

$$\text{Hom}_{\mathfrak{h}}(\Lambda^2 \mathfrak{m}, \mathfrak{g}) = \bigoplus \mathfrak{gl}_{\mathfrak{h}}(\mathfrak{u}_i) \otimes \text{Hom}(\mathbb{R}^{r_i}, \mathbb{R}^{s_i}) = \oplus \text{Mat}_{s_i \times r_i},$$

where the space $\text{Mat}_{s \times r}$ consists of real, complex or quaternionic $s \times r$ matrices (the algebra of splitting operators $\mathfrak{gl}_{\mathfrak{h}}(\mathfrak{u}_i) = \mathbb{R}, \mathbb{C}$ or \mathbb{H}).

Thus \mathfrak{h} -equivariancy of $\mathcal{B}_{\mathfrak{m}}$ can be effectively checked via the representation theory. On the contrary, vanishing of $\text{Jac}_{\mathfrak{m}}(\Lambda^3 \mathfrak{m})$ is a set of linear and quadratic relations on $\mathcal{B}_{\mathfrak{m}}$ to be checked directly (representation theory helps here too through \mathfrak{h} -equivariancy of the map $\text{Jac}_{\mathfrak{m}}$).

Invariant almost complex structures J on M bijectively corresponds to \mathfrak{h} -invariant tensors (here endomorphisms²)

$$J \in (\mathfrak{m}^* \otimes \mathfrak{m})^{\mathfrak{h}} = \text{End}_{\mathfrak{h}}(\mathfrak{m}) \quad \text{with} \quad J^2 = -\mathbf{1}.$$

Similarly, invariant pseudo-Riemannian metrics and almost symplectic structures on M are in bijective correspondence with non-degenerate \mathfrak{h} -invariant tensors $g \in (S^2 \mathfrak{m}^*)^{\mathfrak{h}}$ and $\omega \in (\Lambda^2 \mathfrak{m}^*)^{\mathfrak{h}}$ respectively.

Our aim is to classify 6-dimensional homogeneous manifolds $M = G/H$ with semisimple H , admitting an invariant almost complex structure J . Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ be the associated reductive decomposition. By effectivity the isotropy representation $\text{ad} : \mathfrak{h} \rightarrow \mathfrak{gl}(\mathfrak{m})$ is exact (due to this all elements of \mathfrak{g} act as non-trivial symmetries) and it preserves the complex structure J on \mathfrak{m} . Therefore we identify $\mathfrak{h} \subset \mathfrak{gl}(\mathfrak{m}, J) \simeq \mathfrak{gl}_3(\mathbb{C})$.

2.1 Classification result

Our strategy is the following:

1. Enumerate all semi-simple subalgebras $\mathfrak{h} \subset \mathfrak{gl}_3(\mathbb{C})$, hence, all 6-dimensional \mathfrak{h} -modules \mathfrak{m} with an invariant complex structure J .
2. Describe all \mathfrak{h} -equivariant linear maps $\mathcal{B}_{\mathfrak{m}} : \Lambda^2 \mathfrak{m} \rightarrow \mathfrak{g}$ by decomposing the module $\Lambda^2 \mathfrak{m}$ into irreducible submodules.
3. Compute all Lie algebra extensions \mathfrak{g} of the \mathfrak{h} -module \mathfrak{m} by solving the equations $\text{Jac}_{\mathfrak{m}} = 0 \in \Lambda^3 \mathfrak{m}^* \otimes \mathfrak{g}$ on the parameters in $\mathcal{B}_{\mathfrak{m}}$.
4. Determine the homogeneous almost complex manifolds $M = G/H$ associated with the Lie algebra $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ and the complex structure J .

The trivial bracket $\mathcal{B}_{\mathfrak{m}} = 0$ defines the semidirect product Lie algebra $\mathfrak{g} = \mathfrak{h} \ltimes \mathbb{C}^3$ corresponding to the manifold $M = \mathbb{C}^3$ with the standard complex structure and the obvious action of the semi-direct Lie group $G = H \ltimes \mathbb{C}^3$. We call such structure *flat* and exclude them from consideration.

Below we use the following notations. For a classical simple Lie algebra \mathfrak{h} denote by V the standard (tautological) \mathfrak{h} -module. It has the natural complex structure if \mathfrak{h} is $\mathfrak{sl}_2(\mathbb{C})$, $\mathfrak{sl}_3(\mathbb{C})$, $\mathfrak{su}(3)$ or $\mathfrak{su}(2, 1)$. For $\mathfrak{h} = \mathfrak{su}(2)$ we

²Endomorphisms are \mathbb{R} -linear and \mathfrak{h} -equivariant transformations of the module.

identify the module V with the space \mathbb{H} of quaternions and the algebra \mathfrak{h} with imaginary quaternions $\text{Im}(\mathbb{H})$ acting from the left. Similarly for $\mathfrak{h} = \mathfrak{su}(1, 1)$ we identify the module V with the space \mathbb{H}_s of split quaternions and the algebra \mathfrak{h} with split imaginary quaternions $\text{Im}(\mathbb{H}_s)$ acting from the left. The space of invariant complex structures on V consists of right multiplications $R_q : x \mapsto xq$ by a quaternion q with $q^2 = -1$. Such complex structures (quaternions q) are parametrized by the unit sphere $\mathbb{S}^2 \subset \text{Im}(\mathbb{H})$ in the first case and the unit pseudosphere $\mathbb{S}^{1,1} \subset \text{Im}(\mathbb{H}_s)$ in the second case.

We identify the trivial 2-dimensional representation of \mathfrak{h} and a complex structure J with the standard pair (\mathbb{C}, i) . We denote by \mathfrak{ad} the adjoint representation of the Lie algebra \mathfrak{h} . If \mathfrak{h} is real, the invariant complex structures on the module $\mathfrak{ad}^{\mathbb{C}} = \mathfrak{ad} \oplus \mathfrak{ad}$ are parametrized by

$$J(\mathbf{v}, \mathbf{w}) = \left(r \mathbf{v} - \frac{1+r^2}{t} \mathbf{w}, t \mathbf{v} - r \mathbf{w} \right), \quad (1)$$

the same concerns the complexified tautological representation $V^{\mathbb{C}}$ of $\mathfrak{sl}_3(\mathbb{R})$.

Theorem 3. *There are 7 different real semi-simple subalgebras in the complex Lie algebra $\mathfrak{gl}_3(\mathbb{C})$ (up to conjugation):*

- $\mathfrak{h} = \mathfrak{su}(2)$ or $\mathfrak{su}(1, 1)$, representations $V + \mathbb{C}$, $\mathfrak{ad}^{\mathbb{C}}$;
- $\mathfrak{h} = \mathfrak{sl}_2(\mathbb{C})$, representations $V + \mathbb{C}$, \mathfrak{ad} ;
- $\mathfrak{h} = \mathfrak{sl}_3(\mathbb{R})$, representation $V^{\mathbb{C}}$;
- $\mathfrak{h} = \mathfrak{su}(3)$ or $\mathfrak{su}(2, 1)$ or $\mathfrak{sl}_3(\mathbb{C})$, representation V .

For the cases of $\mathfrak{su}(2)$, $\mathfrak{su}(1, 1)$ the possible Lie algebras \mathfrak{g} with the specified representations \mathfrak{m} are tabulated in the Appendix. For $\mathfrak{sl}_2(\mathbb{C})$ the adjoint representation gives only $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$ (so that $M = \text{SL}_2(\mathbb{C}) \oplus \text{SL}_2(\mathbb{C}) / \text{SL}_2^{\text{diag}}(\mathbb{C})$), while $V + \mathbb{C}$ leads to 2 cases for \mathfrak{g} from the Appendix. For $\mathfrak{su}(3)$ the corresponding \mathfrak{g} is the Lie algebra of the exceptional group G_2 . For $\mathfrak{su}(2, 1)$ the corresponding \mathfrak{g} is the Lie algebra of the exceptional group G_2^ . The other cases $\mathfrak{sl}_3(\mathbb{R})$ and $\mathfrak{sl}_3(\mathbb{C})$ give only the flat structures (in which case $M^6 = \mathbb{C}^3$ or its quotient).*

The proof of this theorem is a straightforward (but lengthy) calculation, we sketch it in the next section.

2.2 Proof of Theorem 1

On the level of Lie algebras Theorem 1 follows instantly from Theorem 3 and the tables from the Appendix.

The passage to the Lie groups is straightforward for types I and II, because in these cases we indicate the pair (G, H) , and it remains to treat only different discrete quotients. But for type III we need to establish existence of the Lie

group G such that \mathfrak{h} exponentiates into its Lie subgroup. In general a Lie groups H with a homomorphic embedding of Lie algebras $\iota : \mathfrak{h} = \text{Lie}(H) \subset \mathfrak{g}$ does not have a homomorphic embedding $j : H \hookrightarrow G$ into a Lie group G with $\mathfrak{g} = \text{Lie}(G)$ and $\iota = dj$, see counter-examples in [B, GOV].

In our case, however the Lie functor works nicely. The output of Theorem 3 yields the structure of Lie algebra on $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, and we should consider only the cases, when \mathfrak{m} is closed with respect to these brackets, i.e. $[\cdot, \cdot]_{\mathfrak{m}}^{\mathfrak{h}} = 0$ in terms of splitting of the bracket $\mathcal{B}_{\mathfrak{m}}$. Indeed, when $\mathfrak{m} \subset \mathfrak{g}$ is not a Lie subalgebra (types I and II in Theorem 1) we have an obvious Lie subgroup $H \subset G$ with $M = G/H$ the required homogeneous almost complex manifold.

Proposition 1. *The pairs $(\mathfrak{g}, \mathfrak{h})$ of Lie algebra/subalgebra from the Tables of the Appendix with \mathfrak{m} being a Lie algebra correspond to the pairs (G, H) of Lie group/subgroup of type III in Theorem 1.*

This statement concerns the cases A1-A6 from the Tables except for the cases A1.4 and A3.5, which correspond to type II (case A6 is rather simple and was already discussed in Theorem 3).

Proof. Let M be the simply connected Lie group corresponding to \mathfrak{m} (for the representation $V + \mathbb{C}$ the Lie algebra \mathfrak{m} is solvable and so $M \simeq \mathbb{R}^6$ as a manifold, for the representation $\mathfrak{ad}^{\mathbb{C}}$ the choice of M is obvious). Consider the representation $\rho : \mathfrak{h} \rightarrow \text{End}(\mathfrak{m})$. Then there exists a homomorphism $R : H \rightarrow GL(\mathfrak{m})$ such that $dR = \rho$. By virtue of Proposition 4.2 of [VO, Chapter 2] the semi-direct product $G = H \ltimes_R M$ is the desired simply-connected Lie group. The main idea of this approach follows Cartan's proof of the third Lie theorem [C].

Another proof is based on the Palais' criterion [P] for a transformation group to be a Lie group. Namely, M acts on itself by left translations and $H \subset \text{Diff}(M)$ is a closed subgroup (as it is the stabilizer of the unity and closed in $GL(T_e M)$). Both M and H generate a closed subgroup in the group of diffeomorphisms, which is a Lie group G with $\text{Lie}(G) = \mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{m}$. \square

Thus, when $\mathfrak{m} \subset \mathfrak{g}$ is a Lie subalgebra we also get representation $M = G/H$ and this finishes the proof of Theorem 1.

3 Proof of the classification result

Representation part of Theorem 3 (list of 7 cases) is obvious from the general theory of representations of semi-simple Lie algebras. The difficult part is the reconstruction of possible Lie brackets on $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$.

Let us consider the case, when $\mathfrak{h} = \mathfrak{su}(2)$ and the isotropy as a complex representation has type $\mathfrak{m} = V + \mathbb{C}$. Identify \mathfrak{h} with the Lie algebra of imaginary quaternions $\mathrm{Im}(\mathbb{H})$ and V with the left \mathfrak{h} -module \mathbb{H} . The endomorphism ring of V is the algebra \mathbb{H} acting from the right (we will use this freedom to change the basis in V). The following easy claim will be used repeatedly.

Lemma 2. *The \mathfrak{h} -invariant complex structure J on the module $\mathfrak{m} = V + \mathbb{C}$ is given via a unit imaginary quaternion $q \in \mathbb{S}^2 \subset \mathrm{Im}(\mathbb{H})$ by the formula*

$$J(x, \eta) = (xq, i\eta), \quad x \in V = \mathbb{H}, \eta \in \mathbb{C}.$$

It is possible to fix q to be equal to $i \in \mathbb{H}$ by an endomorphism, but as noticed above we use this freedom to simplify the brackets. On the other hand, though on the 2-dimensional trivial \mathfrak{h} -module \mathbb{R}^2 there is a 2-parametric family of complex structures, we fix one (equal to i , turning this submodule into \mathbb{C}) as this freedom does not help to simplify the brackets.

The first task in constructing the map $\mathcal{B}_{\mathfrak{m}}$ is to decompose the \mathfrak{h} -module $\Lambda^2 \mathfrak{m} = \Lambda^2(V + \mathbb{C}) = \Lambda^2 V + V \otimes \mathbb{C} + \Lambda^2 \mathbb{C}$ into irreducibles. From the representation viewpoint (without complex structure J) \mathbb{C} is the trivial 2-dimensional real module \mathbb{R}^2 , so $\Lambda^2 \mathbb{C} \simeq \mathbb{R}^1$ and $V \otimes \mathbb{C} = V + V$.

Lemma 3. *As an \mathfrak{h} -module $\Lambda^2 V = \mathfrak{ad} + \mathbb{R}^3$.*

Proof. Let us give two proofs, one via the representation theory of simple Lie algebras and the other straightforward.

The complexified Lie algebra $\mathfrak{h}^{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C})$ has the same standard representation V (it is not absolutely irreducible). Changing the real form to $\mathfrak{h}' = \mathfrak{su}(1, 1) \simeq \mathfrak{sl}(2, \mathbb{R})$ we obtain by the highest weight decomposition $V = W + W$, where W is the standard representation of $\mathfrak{sl}(2, \mathbb{R})$. Now $\Lambda^2(W + W) = \Lambda^2(W) \otimes \mathbb{R}^2 + S^2 W + \mathbb{R} = \mathfrak{ad} + \mathbb{R}^3$ yields the claim.

A more direct proof is as follows. Let us identify $V \simeq V^*$ using the \mathfrak{h} -invariant metric on $V = \mathbb{H}$: $g(x, y) = \mathrm{Re}(x\bar{y})$, $x, y \in V$. Consider the 2-forms $\omega_b, \omega^b \in \Lambda^2 V^*$ given by (check both are skew-symmetric!)

$$\omega_b(x, y) = \mathrm{Re}(xb\bar{y}), \quad \omega^b(x, y) = \mathrm{Re}(x\bar{y}b), \quad b \in \mathrm{Im}(\mathbb{H}).$$

The group $H = SU(2) \simeq \mathbb{S}^2 \subset \mathrm{Im}(\mathbb{H})$ acts on them as follows ($q \in H$)

$$\begin{aligned} \omega_b(qx, qy) &= \mathrm{Re}(qxb\bar{qy}) = \mathrm{Re}(qxb\bar{y}\bar{q}) = \mathrm{Re}(qxb\bar{y}\bar{q}) = \mathrm{Re}(xb\bar{y}|q|^2) = \omega_b(x, y), \\ \omega^b(qx, qy) &= \mathrm{Re}(qx\bar{qy}b) = \mathrm{Re}(qx\bar{y}\bar{q}b) = \mathrm{Re}(x\bar{y}q^{-1}bq) = \omega_{\mathrm{Ad}_q^{-1}b}(x, y). \end{aligned}$$

Consequently the 6-dimensional space $\Lambda^2 V^*$ has two 3-dimensional submodules $\{\omega^b\}$ and $\{\omega_b\}$. By the above the first of them has type \mathfrak{ad} and the second is trivial \mathbb{R}^3 . They do not intersect and so span the whole $\Lambda^2 V^*$. \square

Lemma 4. *Let $\mathfrak{g} = \mathfrak{ad} + V + \mathbb{C}$ be a Lie algebra extension of the \mathfrak{h} -module $V + \mathbb{C}$. Then the brackets on \mathfrak{m} are*

$$[V, V] \subset \mathfrak{ad} + \mathbb{C}, \quad [V, \mathbb{C}] \subset V, \quad [\mathbb{C}, \mathbb{C}] \subset \mathbb{C}.$$

Here from the naked representation theory viewpoint $\mathbb{C} = \mathbb{R}^2$, but we shall use the structure i on it. The lemma has the following implications.

- \mathbb{C} is a Lie subalgebra of \mathfrak{g} and it is solvable: $\Lambda^2 \mathbb{C} \mapsto \mathbb{R} \subset \mathbb{C}$. There exists a J -adapted basis e, ie of \mathbb{C} such that $[e, ie] = \varepsilon e$ for some $\varepsilon \in \{0, 1\}$ (for $\varepsilon = 1$ this determines e uniquely).
- Since \mathbb{C} is the trivial \mathfrak{h} -module, the bracket $V \otimes \mathbb{C} \rightarrow V$ is composed of endomorphisms, i.e. $[x, \eta] = A_\eta(x)$, where $\eta \mapsto -A_\eta$ is the homomorphism of Lie algebras $\mathbb{C} \rightarrow \text{Lie}(\text{End}_{\mathbb{R}}(V)) = \mathbb{R} \oplus \mathfrak{h} \simeq \mathfrak{u}(2)$. Since the latter has no non-trivial solvable subalgebras, the homomorphism is not injective when the subalgebra \mathbb{C} is not abelian ($\varepsilon = 1 \Rightarrow A_e = 0$).
- The V -bracket $\Lambda^2 V \rightarrow \mathfrak{g}$ for some $\lambda \in \mathbb{R}$, $b, b' \in \text{Im}(\mathbb{H})$ equals

$$[x, y] = \lambda \cdot \text{Im}(x\bar{y}) + (\omega_b(x, y)e + \omega_c(x, y)ie).$$

Elaborating upon Lemma 1 with these choices \mathfrak{g} is a Lie algebra iff the V -Jacobi identity holds – $\text{Jac}_{\mathfrak{m}} : \Lambda^3 V \rightarrow \mathfrak{g}$ is zero and in addition

$$\begin{aligned} \omega_b(A_{ie}x, y) + \omega_b(x, A_{ie}y) &= \varepsilon \omega_b(x, y), \\ \omega_b(A_e x, y) + \omega_b(x, A_e y) &= -\varepsilon \omega_c(x, y), \\ \omega_c(A_\eta x, y) + \omega_c(x, A_\eta y) &= 0, \quad \eta \in \mathbb{C}. \end{aligned} \tag{2}$$

It is also important to notice that since \mathbb{H} is a division algebra, then every nonzero operator $A_\eta \in \text{End}_{\mathbb{R}}(V) = \mathbb{H}$ is invertible. Now we consider the following possibilities (if $\lambda = 0$, then \mathfrak{m} is a Lie algebra).

(A1.1) $\lambda = 0$, $[V, V] \neq 0$ and the subalgebra $A_{\mathbb{C}} \subset \mathbb{H}$ is nonzero. We claim that the map A has a kernel. Indeed, for $\varepsilon = 1$ we have $A_e = 0$. For $\varepsilon = 0$ denoting $A_\eta(x) = xq$, $q \in \mathbb{H}$, then (2) implies $\text{Re}(x(qd + d\bar{q})\bar{y}) = 0$, where d is any linear combination of $b, c \in \text{Im}(\mathbb{H})$. As b, c are not simultaneously zero, this yields a kernel, which can be accommodated into e (using $GL(1, \mathbb{C})$ -freedom of change of basis in the \mathbb{C} -module for $\varepsilon = 0$).

Then the Jacobi identity $\text{Jac}_{\mathfrak{m}}(\Lambda^3 V) = 0$ and the remark about invertibility imply that $c = 0$. Using the endomorphism freedom in the choice of basis in V , we can assume $b = \alpha i$. From (2) we obtain $A_{ie}(x) = x(\frac{\varepsilon}{2} + ri)$ for some $r \in \mathbb{R}$. Thus in this case \mathfrak{m} is a Lie algebra with the structure

equations³

$$[x, y] = \alpha \operatorname{Re}(xi\bar{y})e, \quad [x, ie] = x\left(\frac{\varepsilon}{2} + ri\right), \quad [e, ie] = \varepsilon e.$$

In other words, \mathfrak{m} is obtained in two steps. First we construct the central extension of abelian algebra V by the 2-form ω_i - we get the Heisenberg Lie algebra $\mathfrak{heis}(V) = V + \mathbb{R}e$. Then we take its 1-dimensional extension of by the derivation⁴ ad_{ie} . In Table A1 this case is called A1.1.

(A1.2) $\lambda = 0$, $[V, V] \neq 0$ and the subalgebra $A_{\mathbb{C}} \subset \mathbb{H}$ is zero. In this case the subalgebra \mathbb{C} must be abelian, and then the Jacobi identity holds $\operatorname{Jac}_{\mathfrak{m}}(\Lambda^3 V) = 0$ identically. We can normalize (by rescaling e) $b = i$ and $c = p \in \operatorname{Im}(\mathbb{H})$ is arbitrary.

Thus \mathfrak{m} is the Lie algebra, which is a 2-dimensional central extension of abelian V by two 2-forms ω_i, ω_p . This is the case A1.2 of Table A1.

(A1.3) $\lambda = 0$, $[V, V] = 0$. Here the Jacobi identity is satisfied and we only need to normalize the map $\eta \mapsto A_{\eta}$. If $\varepsilon \neq 0$, then $A_e = 0$ and we choose a basis in V such that $A_{ie}(x) = x(\beta + ri)$, $\beta, r \in \mathbb{R}$.

On the other hand, if $\varepsilon = 0$, then $A_{\mathbb{C}} \subset \mathbb{H}$ is either 1-dimensional or 2-dimensional subalgebra, which is possible only for $A_e(x) = x\alpha$, $A_{ie}(x) = x(\beta + ri)$, $\alpha, \beta, r \in \mathbb{R}$. Clearly we can normalize $\alpha = 0$ or 1.

Thus \mathfrak{m} is a Lie algebra, which is either an extension of abelian V by two commuting derivations $\operatorname{ad}_e, \operatorname{ad}_{ie}$ or an extension of abelian $V + \mathbb{R}e$ by one derivation ad_{ie} . This is the case A1.3 of Table A1.

(A1.4) $\lambda \neq 0$. Here \mathfrak{m} is not a Lie subalgebra of \mathfrak{g} . In this case the Jacobi identity implies that \mathbb{C} has a central element (in particular \mathbb{C} is abelian). We choose it to be ie . Then $A_{ie} = 0$, $c = 0$. We can normalize (by endomorphisms) $\lambda = \pm 1$, $b = i$. The Jacobi identity yields $A_e(x) = 3\lambda xi$.

The obtained Lie algebra \mathfrak{g} is isomorphic to $\mathfrak{u}(3)$ for $\lambda = 1$ and to $\mathfrak{u}(1, 2)$ for $\lambda = -1$. The associated almost complex manifolds are $U(3)/SU(2)$ and $U(2, 1)/SU(2)$ respectively. In Table A1 this case is called A1.4.

Thus we obtained the complete classification of the homogeneous structures $M = G/H$ in the first case from the list of Theorem 3. Invariant almost

³Here and in what follows we adopt the convention that $x, y \in V$ are arbitrary elements, but $e \in \mathbb{C}$ is a fixed element, in particular e, ie is a real basis of \mathbb{C} .

⁴The central ("left") extension and extension by derivations ("right") $\tilde{\mathfrak{g}}$ of the Lie algebra \mathfrak{g} (via \mathfrak{f}) are given respectively by the exact sequences

$$0 \rightarrow \mathfrak{f} \rightarrow \tilde{\mathfrak{g}} \rightarrow \mathfrak{g} \rightarrow 0, \quad 0 \rightarrow \mathfrak{g} \rightarrow \tilde{\mathfrak{g}} \rightarrow \mathfrak{f} \rightarrow 0.$$

Then \mathfrak{g} is respectively the quotient/subalgebra of $\tilde{\mathfrak{g}}$ and its bracket can/cannot change upon the extension.

complex structure on M is obtained from the \mathfrak{h} -invariant structure J on \mathfrak{m} as in Lemma 2. Its Nijenhuis tensor is computed by the formula

$$N_J(X, Y) = \pi([JX, JY] - J[X, JY] - J[JX, Y] - [X, Y]), \quad X, Y \in \mathfrak{m},$$

where $[\cdot, \cdot] : \Lambda^2 \mathfrak{m} \rightarrow \mathfrak{g}$ are the brackets in the Lie algebra \mathfrak{g} and $\pi : \mathfrak{g} \rightarrow \mathfrak{m}$ is the projection along \mathfrak{h} . Similarly the differential of the almost symplectic structure ω on \mathfrak{m} is computed by the Cartan formula using only the brackets on \mathfrak{m} . This explains all entries in Table A1.

Consider the other representation $\mathfrak{m} = \mathfrak{ad}^{\mathbb{C}}$ of $\mathfrak{h} = \mathfrak{su}(2)$. Here it is more convenient to involve Levi decomposition: $\mathfrak{g} = \mathfrak{g}_{ss} \oplus \mathfrak{r}$, where the first summand is a semi-simple part and the second is the radical. The factor \mathfrak{g}_{ss} can be chosen to contain \mathfrak{h} , and so can be one of the following:

- $\mathfrak{g}_{ss} = \mathfrak{su}(2)^3$, $\mathfrak{h} = \mathfrak{su}(2)_{\text{diag}}$,
- $\mathfrak{g}_{ss} = \mathfrak{su}(2) \oplus \mathfrak{sl}(2, \mathbb{C})$, $\mathfrak{h} = \mathfrak{su}(2)_{\text{diag}}$,
- $\mathfrak{g}_{ss} = \mathfrak{su}(2)^2$, $\mathfrak{h} = \mathfrak{su}(2)_{\text{diag}}$,
- $\mathfrak{g}_{ss} = \mathfrak{sl}(2, \mathbb{C})$, $\mathfrak{h} \subset \mathfrak{sl}(2, \mathbb{C})$.

The last case is disqualified as \mathfrak{h} acts by zero on $\mathfrak{g}/\mathfrak{g}_{ss}$ (no 3-dimensional nontrivial representation for \mathfrak{g}_{ss}). The first three give the cases A2.1, A2.2, A2.3 of Table A2 respectively.

Finally, it is possible that $\mathfrak{g}_{ss} = \mathfrak{h}$, whence $\mathfrak{m} = \mathfrak{r}$. Since this is solvable of module type $\mathfrak{ad}^{\mathbb{C}}$, the only non-trivial brackets in terms of the splitting $\mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2$ (grading) are the following (case A2.4)

$$[x_1, y_1] = [x, y]_2, \quad x, y \in \mathfrak{ad} \simeq \mathfrak{h}.$$

The corresponding analysis for $\mathfrak{h} = \mathfrak{su}(1, 1)$ is similar (for instance, Lemma 2 holds true with \mathbb{H} changed to \mathbb{H}_s), but a special care should be taken as in this non-compact case there are null elements on the representation V with respect to its unique (up to scale) \mathfrak{h} -invariant metric (that's why Table A3 is bigger than A1). The cases $\mathfrak{h} = \mathfrak{sl}_2(\mathbb{C}), \mathfrak{sl}_3(\mathbb{R}), \mathfrak{sl}_3(\mathbb{C})$ are much simpler. The details of computations can be found in [Wi]. For the largest algebras $\mathfrak{su}(3)$ and $\mathfrak{su}(2, 1)$ arising in the symmetry analysis the computations are done in [K₂].

This finishes the proof of Theorem 3.

4 Automorphism groups of nondegenerate structures J

Here we prove Theorem 2 – find the symmetry algebra of the structures J with non-degenerate Nijenhuis tensor N_J . It follows from the tables that N_J can be non-degenerate only for isotropy algebras $\mathfrak{su}(2)$, $\mathfrak{su}(1, 1)$ or $\mathfrak{su}(3)$, $\mathfrak{su}(2, 1)$. In the latter two cases J is locally isomorphic to the G_2 -invariant almost complex structure on either S^6 or $S^{2,4}$ and if the automorphism group has dimension 14, it is one of the two forms of the group G_2 , see [K₂] for details. In what follows we consider the former two cases.

According to [K₂] the isotropy $\mathfrak{sym}(J)_o$ of the symmetry algebra $\mathfrak{sym}(J)$ at any point $o \in M$ is 1-jet determined: the proof of Theorem 1 loc.cit. implies

Theorem 4. *If the Nijenhuis tensor N_J on a connected almost complex manifold (M^6, J) is non-degenerate, then any vector field $X \in \mathfrak{sym}(J)$, is uniquely determined by its 1-jet $[X]_o^1$. Consequently, the isotropy algebra satisfies:*

$$\mathfrak{sym}(J)_o = \{X : L_X(J) = 0, X(o) = 0\} \subset \mathfrak{gl}(\mathfrak{m}, J).$$

4.1 Symmetries via derivations

Theorem 4 hints to the following statement concerning the symmetry algebra of the homogeneous models A1.1, A2, A3.1, A3.2 and A4 according to the numeration in Appendix.

Proposition 2. *Let $M = G/H$ be the homogeneous almost complex manifold associated with one of the Lie algebra extensions of the \mathfrak{h} -module \mathfrak{m} in the case when \mathfrak{m} is a Lie algebra (subalgebra in \mathfrak{g}) and \mathfrak{h} is either $\mathfrak{su}(2)$ or $\mathfrak{su}(1, 1)$. If the Nijenhuis tensor N_J of the almost complex structure J is non-degenerate, then the full symmetry algebra as a vector space is*

$$\mathfrak{sym}(J) = \mathfrak{m} + \mathfrak{sym}(J)_o$$

and the full isotropy algebra equals

$$\mathfrak{sym}(J)_o = \mathfrak{der}(\mathfrak{m}) \cap \mathfrak{gl}(\mathfrak{m}, J) = \{A \in \mathfrak{der}(\mathfrak{m}) : AJ = JA\}.$$

Otherwise said, we are given the pair $(\mathfrak{g}, \mathfrak{h})$ with $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$ and \mathfrak{h} -invariant J on \mathfrak{m} . The claim is that if we can find an extension $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}) \supset (\mathfrak{g}, \mathfrak{h})$ with the same property, then still $\tilde{\mathfrak{h}}$ acts on \mathfrak{m} by derivations.

Below we denote by $\pi : \tilde{\mathfrak{g}} \rightarrow \mathfrak{m}$ the projection along $\tilde{\mathfrak{h}}$ and use the labels for almost complex homogeneous spaces from the Appendix.

Proof. Let $\tilde{\mathfrak{g}} = \mathfrak{sym}(J)$ be the full symmetry algebra. By [K₁, K₂] the full isotropy algebra $\tilde{\mathfrak{h}} = \mathfrak{sym}(J)_o \subset \tilde{\mathfrak{g}}$ is at most 8-dimensional. If J is not locally isomorphic to the G_2^c -invariant almost complex structure on S^6 or G_2^* -invariant almost complex structure on $S^{2,4}$, then $\dim \tilde{\mathfrak{h}} \leq 5$. Indeed, this is so in any non-exceptional case of canonical forms NDG(1-4) of [K₁], exceptions are the two cases given by formulae (16) and (17) of [K₂, Section 7] when $\tilde{\mathfrak{h}}$ is equal to $\mathfrak{su}(2,1)$ or $\mathfrak{su}(3)$ respectively. The structure of \mathfrak{g} recovers uniquely and $M = G_2^*/SU(2,1)$ or $M = G_2^c/SU(3)$ respectively; the structure J in every case is unique. Any proper subalgebra of $\mathfrak{su}(2,1)$ is at most 5-dimensional, and that of $\mathfrak{su}(3)$ is at most 4-dimensional, whence the claim.

Since already J has 3-dimensional isotropy \mathfrak{h} by construction, the additional subspace $\mathfrak{r} \subset \tilde{\mathfrak{h}}$ is the radical of $\dim \mathfrak{r} \leq 2$.

We start with the $\mathfrak{ad}^{\mathbb{C}}$ representation of $\mathfrak{h} = \mathfrak{su}(2)$ on \mathfrak{m} . Then by \mathfrak{h} -equivariance of the brackets, \mathfrak{r} is in the radical $\tilde{\mathfrak{r}}$ of $\tilde{\mathfrak{g}}$.

If \mathfrak{m} is either $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ or $\mathfrak{sl}(2, \mathbb{C})$ (A2.1, A2.2), then \mathfrak{g} is semi-simple, so extension of \mathfrak{h} to $\tilde{\mathfrak{h}}$ is by radical \mathfrak{r} only, whence $[\mathfrak{r}, \mathfrak{m}] \subset \mathfrak{r}$ and hence the added summand \mathfrak{r} acts non-effectively, which is prohibited.

If $\mathfrak{m} = \mathfrak{su}(2) + \mathbb{R}^3$ (A2.3), the radical of $\tilde{\mathfrak{g}}$ is $\tilde{\mathfrak{r}} = \mathfrak{r} + \mathbb{R}^3$ and so we have: $[\mathfrak{r}, \mathfrak{su}(2)] = 0$ and $[\mathfrak{r}, \mathbb{R}^3] \subset \mathfrak{r} + \mathbb{R}^3$. By J -invariance of $\pi \circ \mathfrak{ad}_{\mathfrak{r}}(\mathfrak{m})$ we get $\pi([\mathfrak{r}, \mathfrak{m}]) = 0$, meaning the action of \mathfrak{r} on \mathfrak{m} is non-effective – contradiction.

If $\mathfrak{m} = \mathfrak{a}_1 \oplus \mathfrak{a}_2$ (A2.4), the Lie algebra structure is graded and the \mathfrak{h} -representation on $\mathfrak{a}_i \simeq \mathbb{R}^3$ is adjoint. In the radical $\tilde{\mathfrak{r}} = \mathfrak{m} + \mathfrak{r}$ the \mathfrak{h} -representation \mathfrak{r} is the trivial submodule, whence from \mathfrak{h} -equivariance we get: $[\mathfrak{r}, \mathfrak{m}] \subset \mathfrak{m}$. This means that \mathfrak{r} acts by derivations, as required.

The case $\mathfrak{h} = \mathfrak{su}(1,1)$ is similar except for the last type $\mathfrak{m} = \mathfrak{a}_1 \oplus \mathfrak{a}_2$. Then another possibility occurs that \mathfrak{r} is the standard representation $\mathcal{R}(\lambda_1)$ of $\mathfrak{h} \simeq \mathfrak{sl}_2(\mathbb{R})$, where $\mathcal{R}(w)$ is the representation of highest weight w . As $\mathfrak{a}_i = \mathcal{R}(2\lambda_1)$ and $\mathcal{R}(\lambda_1) \otimes \mathcal{R}(2\lambda_1) = \mathcal{R}(\lambda_1) + \mathcal{R}(3\lambda_1)$, \mathfrak{h} -equivariance implies $[\mathfrak{r}, \mathfrak{m}] \subset \mathfrak{r}$, so the action is non-effective.

Consider now the second possible representation of \mathfrak{h} : $\mathfrak{m} = V + \mathbb{C}$. Starting with $\mathfrak{h} = \mathfrak{su}(2)$ we note the additional subspace \mathfrak{r} is the trivial \mathfrak{h} -module. Again \mathfrak{h} -equivariance and Schur's lemma imply $[\mathfrak{r}, V] \subset V$. Since $0 \neq [V, V] \subset \mathbb{C}$ and $\pi \circ \mathfrak{ad}_{\mathfrak{r}}|_{\mathfrak{m}}$ commutes with J , the Jacobi identity implies $[\mathfrak{r}, \mathbb{C}] \subset \langle [V, V], J[V, V] \rangle = \mathbb{C}$. Thus $[\mathfrak{r}, \mathfrak{m}] \subset \mathfrak{m}$ and \mathfrak{r} acts by derivations, as claimed.

When $\mathfrak{h} = \mathfrak{su}(1,1)$ and \mathfrak{r} as \mathfrak{h} -representation is trivial, the argument is the same. So let $\mathfrak{r} \simeq \mathbb{R}^2$ be the standard representation U of $\mathfrak{h} \simeq \mathfrak{sl}_2(\mathbb{R})$. In this case $V = U^{\mathbb{C}}$ is $2\mathcal{R}(\lambda_1)$ as \mathfrak{h} -representation. Therefore as $\mathcal{R}(\lambda_1) \otimes \mathcal{R}(\lambda_1) = \mathcal{R}(0) + \mathcal{R}(2\lambda_1) = \mathbb{R} + S^2U$, we get by \mathfrak{h} -equivariance: $[\mathfrak{r}, V] \subset \mathfrak{h} + \mathbb{C}$. Since

$0 \neq [V, V] \subset \mathbb{C}$, $[V, \mathbb{C}] \subset V$, $[\mathfrak{h}, V] = V$ and $\pi \circ \mathfrak{ad}_{\mathfrak{r}}|_{\mathfrak{m}}$ commutes with J , the Jacobi identity yields $[\mathfrak{r}, \mathbb{C}] \subset [\mathfrak{h} + \mathbb{C}, V] = V$.

Let us show that $[\mathfrak{r}, V]$ does not have \mathfrak{h} component. Indeed, since $[\mathfrak{r}, \mathfrak{r}] \subset \mathfrak{h} + \mathbb{C}$ by \mathfrak{h} -equivariance, the Jacobi identity implies

$$V \supset [[\mathfrak{r}, \mathfrak{r}], V] \subset [\mathfrak{r}, [\mathfrak{r}, V]] \subset [\mathfrak{r}, \mathfrak{h} + \mathbb{C}] \subset \mathfrak{r} + V.$$

Since $[\mathfrak{r}, \mathfrak{h}] = \mathfrak{r}$, the presence of \mathfrak{h} component implies non-trivial \mathfrak{r} component in the last summand of the above display formula. To study it consider the bracket-maps $\psi : \mathfrak{r} \times \mathfrak{h} \rightarrow \mathfrak{r}$ and $\phi : \mathfrak{r} \times V \rightarrow \mathfrak{h}$ (for the latter we post-compose with the projection). The Jacobi identity and the above display formula imply $\psi(r_1, \phi(r_2, v)) = \psi(r_2, \phi(r_1, v))$ for all $r_1, r_2 \in \mathfrak{r}$, $v \in V$. Since the maps ψ, ϕ depend only on the \mathfrak{h} -module structure and as such $\mathfrak{r} = U$, $V = U \oplus U$, we change the maps to

$$\Psi : U \times \mathfrak{h} \rightarrow U, \quad \Phi : U \times U \rightarrow \mathfrak{h} \text{ with } \Psi(u_1, \Phi(u_2, u_3)) = \Psi(u_2, \Phi(u_1, u_3)).$$

Here Ψ is the standard representation, and Φ is proportional to the symmetric multiplication $(u_1, u_2) \mapsto \lambda u_1 u_2$, $\lambda \in \mathbb{R}$, because $S^2 U = \mathfrak{h}$. This isomorphism is given by a choice of \mathfrak{h} -invariant area form ω on U . Let $p, q \in U$ be the canonical basis, $\omega(p, q) = 1$. Then

$$\Psi(p, \Phi(q, p)) = \lambda \Psi(p, qp) = \lambda p \neq \Psi(q, \Phi(p, p)) = \lambda \Psi(q, p^2) = -2\lambda p$$

unless $\lambda = 0$. Thus the \mathfrak{h} component vanishes and $[\mathfrak{r}, V] \subset \mathbb{C}$.

Therefore $[\mathfrak{r}, \mathfrak{m}] \subset \mathfrak{m}$ and \mathfrak{r} acts by derivations.

This finishes the proof of Proposition 2. □

4.2 Proof of Theorem 2

To find the derivations we can use the exact sequence

$$0 \rightarrow Z(\mathfrak{m}) \rightarrow \mathfrak{m} \xrightarrow{\text{ad}} \mathfrak{der}(\mathfrak{m}) \rightarrow H^1(\mathfrak{m}, \mathfrak{m}) \rightarrow 0, \quad (3)$$

where $Z(\mathfrak{m})$ is the center of the Lie algebra \mathfrak{m} .

Consider at first the case $\mathfrak{h} = \mathfrak{su}(2)$, representation $\mathfrak{ad}^{\mathbb{C}}$. In all four cases here $\mathfrak{m} \subset \mathfrak{g}$ is a Lie subalgebra.

In the first two cases A2.1, A2.2 \mathfrak{m} is semi-simple: $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ or $\mathfrak{sl}_2(\mathbb{C})$. By Whitehead lemma $H^1(\mathfrak{m}, \mathfrak{m}) = 0$, so all derivations are inner. Thus $\mathfrak{der}(\mathfrak{m}) = \{\text{ad}_X : X \in \mathfrak{m}\} \simeq \mathfrak{m}$ from (3).

We claim that if ad_X commutes with J , then ad_{JX} does not. Elsewise

$$N_J(X, Y) = [\text{ad}_{JX}, J](Y) - J[\text{ad}_X, J](Y) = 0,$$

and the Nijenhuis tensor is degenerate (even DG_2 in terminology of [K₁]). Therefore $\mathfrak{snm}(J)_o$ is totally real in \mathfrak{m} and so cannot have dimension > 3 . But dimension 3 is guaranteed since $\mathfrak{snm}(J)_o \supset \mathfrak{h}$. Consequently $\mathfrak{snm}(J) = \mathfrak{g}$.

Next case A2.3 is $\mathfrak{m} = \mathfrak{h} \times \mathbb{R}^3$. Clearly $\mathfrak{snm}(J)_o \subset \mathfrak{der}(\mathfrak{m})$ must preserve the radical \mathbb{R}^3 . The operator $J_r = J - r\mathbf{1}$ is invariant, where $\mathbf{1}$ is the identity operator – see formula (1). Therefore the full isotropy also preserves the semi-simple part $J_r\mathbb{R}^3 = \mathfrak{h}$, and the action on \mathfrak{h} induces the action on \mathbb{R}^3 . Again any derivation on \mathfrak{h} is inner and so $\dim \mathfrak{snm}(J)_o \leq 3$ implying $\mathfrak{snm}(J) = \mathfrak{g}$.

In the last case A2.4 the graded nilpotent Lie algebra $\mathfrak{m} = \mathfrak{a}_1 \oplus \mathfrak{a}_2$ with $\mathfrak{a}_i \simeq \mathbb{R}^3$, $J_r : \mathfrak{a}_1 \rightarrow \mathfrak{a}_2$ and the bracket is $[\xi, \eta] = J_r(\xi \times \eta)$, $\xi, \eta \in \mathfrak{a}_1$, with the cross product \times being the Lie bracket on $\mathbb{R}^3 = \mathfrak{su}(2)$. This relation shows that $\mathfrak{a}_2 = [\mathfrak{m}, \mathfrak{m}]$ equipped with \times product (so the bracket is $\Lambda^2 \mathfrak{a}_2 \ni \xi \wedge \eta \mapsto [J_r^{-1}\xi, J_r^{-1}\eta] \in \mathfrak{a}_2$) must be preserved by the derivations. Since this algebra (\mathfrak{a}_2, \times) is isomorphic to $\mathfrak{su}(2)$, we obtain $\mathfrak{snm}(J) = \mathfrak{g}$ as before.

Secondly let representation \mathfrak{m} of \mathfrak{h} be $V \oplus \mathbb{C}$. Only one of the cases A1.1, with \mathfrak{m} being a Lie algebra, has non-degenerate N_J . In this case the space of derivations of \mathfrak{m} commuting with J is obtained by the straightforward computation with the case split according to parameters; these tedious computations are done in Maple. The result is the same as above.

The case $\mathfrak{h} = \mathfrak{su}(1, 1)$ is very similar to the considered $\mathfrak{su}(2)$. The only difference is that in $V + \mathbb{C}$ representation there is one more case.

Now to complete the proof of Theorem 2, we have to consider the homogeneous structures of type II in Theorem 1 (when \mathfrak{m} is not a Lie algebra: A1.4, A3.5). For such $M = G/H$ the Lie group G is reductive, $\mathfrak{g} = \mathfrak{g}_{ss} + \mathfrak{z}$ with 1-dimensional center \mathfrak{z} and 8-dimensional semi-simple part \mathfrak{g}_{ss} , and the Lie algebra $\tilde{\mathfrak{g}} = \mathfrak{snm}(J)$ contains \mathfrak{g} .

From the proof of Proposition 2 we know that $\tilde{\mathfrak{h}} = \mathfrak{h} + \mathfrak{r}$, where the semi-direct summand \mathfrak{r} is the radical in $\tilde{\mathfrak{h}}$ and $\dim \mathfrak{r} \leq 2$. This implies (by inspection of the Levi decomposition of $\tilde{\mathfrak{g}}$) that $\mathfrak{z} + \mathfrak{r} \subset \tilde{\mathfrak{g}}$ is a subalgebra, which is either semi-simple or the radical of $\tilde{\mathfrak{g}}$. In any case, because $\mathfrak{m} = \mathfrak{m}_0 + \mathfrak{z}$ for $\mathfrak{m}_0 = \mathfrak{m} \cap \mathfrak{g}_{ss} \subset \mathfrak{g}_{ss}$, we get $[\mathfrak{r}, \mathfrak{m}] \subset [\mathfrak{r}, \mathfrak{m}_0] + [\mathfrak{r}, \mathfrak{z}] \subset \mathfrak{r} + \mathfrak{z}$. Consequently $\pi \circ \mathfrak{ad}_{\mathfrak{r}}(\mathfrak{m}) \subset \mathfrak{z}$ for the projection $\pi : \tilde{\mathfrak{g}} \rightarrow \mathfrak{m}$ along $\tilde{\mathfrak{h}}$. Since $\pi \circ \mathfrak{ad}_{\mathfrak{r}} \subset \mathfrak{gl}(\mathfrak{m}, J)$ yields J -invariance of $\pi \circ \mathfrak{ad}_{\mathfrak{r}}(\mathfrak{m})$, we conclude that by dimensional reasons $\pi([\mathfrak{r}, \mathfrak{m}]) = 0$. Consequently the action of \mathfrak{r} on \mathfrak{m} is not effective, so $\mathfrak{r} = 0$.

This finishes the proof of Theorem 2.

5 Almost Hermitian structures and their integrability

The existence of a homogeneous almost Hermitian, almost symplectic or almost complex structure depends only on the isotropy representation, in contrast with the various integrability conditions (Kähler, etc.) for such structures which generally depend on the Lie algebra structure.

Pseudo-Riemannian metrics on the almost complex homogeneous manifold $M = G/H$ with the isotropy \mathfrak{h} -module \mathfrak{m} and \mathfrak{h} -invariant complex structure J on \mathfrak{m} correspond to non-degenerate \mathfrak{h} -invariant quadratic forms $g \in S^2\mathfrak{m}^*$. Invariant almost Hermitian structures are elements of the set

$$S_J^2\mathfrak{m}^* = \{g \in (S^2\mathfrak{m}^*)^{\mathfrak{h}} : g(J\xi, J\eta) = g(\xi, \eta), \det(g) \neq 0\}.$$

Likewise invariant compatible almost symplectic structures are elements of the set

$$\Lambda_J^2\mathfrak{m}^* = \{\omega \in (\Lambda^2\mathfrak{m}^*)^{\mathfrak{h}} : \omega(J\xi, J\eta) = \omega(\xi, \eta), \omega^3 \neq 0\}.$$

The Kähler form $\omega \in \Lambda_J^2\mathfrak{m}^*$ associated to $g \in S_J^2\mathfrak{m}^*$ is defined by $\omega(\xi, \eta) = g(J\xi, \eta)$. This formula makes a bijective correspondence $S_J^2\mathfrak{m}^* \simeq \Lambda_J^2\mathfrak{m}^*$.

Note that two invariant almost Hermitian metrics g, \tilde{g} define a symmetric (with respect to both g and \tilde{g}) invertible operator $A : \mathfrak{m} \rightarrow \mathfrak{m}$ by $\tilde{g}(\xi, \eta) = g(A\xi, \eta)$, which has to commute with both \mathfrak{h} and J . Thus the operator A belongs to the complex endomorphism ring $\text{End}_{\mathfrak{h}}(\mathfrak{m}, J)$.

5.1 Classification of almost Hermitian structures

Let us list all invariant almost Hermitian structures according to the types of \mathfrak{h} -modules as in Theorem 3 (we'll omit the word "almost" for the metric).

Case 1: $\mathfrak{h} = \mathfrak{su}(2)$, $\mathfrak{m} = V \oplus \mathbb{C}$, where $V \simeq \mathbb{H}$. There are Hermitian metrics g_1 on V , g_2 on \mathbb{C} . Since $\text{End}_{\mathfrak{h}}(\mathfrak{m}, J) = \mathbb{C} \oplus \mathbb{C}$ and the symmetric endomorphisms are $A \in \mathbb{R} \oplus \mathbb{R}$, the general invariant compatible metric on \mathfrak{m} is $g = a g_1 + b g_2$. Its signature is $(2k, 6 - 2k)$ depending on $a, b \neq 0$.

The almost-symplectic form is up to scaling $\omega(\xi, \eta) = g_1(\xi_1 q, \eta_1) + g_2(i\xi_2, \eta_2)$ for some $q \in \text{Im}(\mathbb{H}) \setminus 0$, where $\xi = \xi_1 + \xi_2, \eta = \eta_1 + \eta_2 \in V \oplus \mathbb{C}$.

Case 2: $\mathfrak{h} = \mathfrak{su}(2)$, $\mathfrak{m} = \mathfrak{ad}^{\mathbb{C}}$. The operator J induces the equivariant splitting $\mathfrak{m} = \mathfrak{ad} \oplus J\mathfrak{ad}$. The Riemannian metric g , which is the direct sum of the Killing forms on each summand, is compatible. Since $\text{End}_{\mathfrak{h}}(\mathfrak{m}, J) = \mathbb{C}$

and the symmetric endomorphisms are $A \in \mathbb{R}$, the invariant compatible metric is unique up to scaling (but g depends on J).

Decomposing into irreducible \mathfrak{h} -modules $\Lambda^2 \mathfrak{m} = 3\mathfrak{ad} \oplus W^5 \oplus \mathbb{R}^1$, we conclude that the only almost symplectic form ω is the Kähler form of the metric g .

Case 3: $\mathfrak{h} = \mathfrak{su}(1, 1)$, $\mathfrak{m} = V \oplus \mathbb{C}$. Here $V = U^{\mathbb{C}}$ for the standard \mathfrak{sl}_2 representation U . Similar to Case 1, the general invariant metric is $g = a g_1 + b g_2$, $a, b \neq 0$. The metric g_1 is of split signature, g_2 is Riemannian, so g has type $(4, 2)$ (or $(2, 4)$, but we will not distinguish the opposite signatures).

Since $\Lambda^2 V = \mathfrak{ad} \oplus \mathbb{R}^3$, the space of invariant 2-forms is 3-dimensional. Indeed, $\text{End}_{\mathfrak{h}}(V) = \mathfrak{gl}_2 \simeq \mathbb{H}_s$, and so up to scaling an almost symplectic form is $\omega(\xi, \eta) = g_1(\xi_1 q, \eta_1) + g_2(i\xi_2, \eta_2)$, $q \in \text{Im}(\mathbb{H}_s)$, $q^2 \neq 0$.

Case 4: $\mathfrak{h} = \mathfrak{su}(1, 1)$, $\mathfrak{m} = \mathfrak{ad}^{\mathbb{C}}$. This is similar to Case 2: the (invariant) almost symplectic structure ω is unique up to scale; it is J -independent and is J -compatible for every $J \in \text{End}_{\mathfrak{h}}(\mathfrak{m})$, $J^2 = -\mathbf{1}$. The Hermitian metric $g = -i_J \omega$ depends on J and has signature $(4, 2)$.

Case 5: $\mathfrak{h} = \mathfrak{sl}_2(\mathbb{C})$, $\mathfrak{m} = V + \mathbb{C}$. There are no $\mathfrak{sl}_2(\mathbb{C})$ -invariant metric on the V component. The almost symplectic form $\omega_1(x, y) = g_1(xq, y)$ on V from Case 1 is $\mathfrak{sl}_2(\mathbb{C})$ -invariant iff $q \in \text{Im}(\mathbb{H}) \cap i\text{Im}(\mathbb{H})$, i.e. $q \perp i$. Thus the space of invariant almost symplectic structures on \mathfrak{m} up to scaling is given by 2 parameters: $\omega = \omega_1 + \omega_2$.

Case 6: $\mathfrak{h} = \mathfrak{sl}_2(\mathbb{C})$, $\mathfrak{m} = \mathfrak{ad}$, $J = i$. The Killing form K provides an invariant metric on \mathfrak{ad} , but it is not Hermitian as $K(J\xi, J\eta) = -K(\xi, \eta)$. Since any other metric or 2-form must be related to K by an operator $A \in \text{End}_{\mathfrak{h}}(\mathfrak{m}) = \mathbb{C}$, no compatible metric and no almost-symplectic form exists. Instead we have two invariant anti-compatible metrics $K(\xi, \eta)$ and $K(J\xi, \eta)$. The homogeneous space $M = \text{SL}_2(\mathbb{C})^2 / \text{SL}_2(\mathbb{C})_{\text{diag}} \simeq \text{SL}_2(\mathbb{C})$ is complex.

Case 7: $\mathfrak{h} = \mathfrak{su}(3)$, $\mathfrak{m} = V$. \mathfrak{m} has an invariant Hermitian metric g of signature $(6, 0)$. Since $\text{End}_{\mathfrak{h}}(\mathfrak{m}, J) = \mathbb{C}$ and only $A \in \mathbb{R}$ are symmetric, g is unique up to scaling. The almost symplectic form is unique and compatible.

The corresponding homogeneous manifold is \mathbb{S}^6 , and J is the unique invariant almost complex structure. Known as the Calabi structure, it is well studied. In particular, the triple (g, J, ω) is strongly nearly Kähler (SNK) and the Hermitian metric g is 3-symmetric and Einstein.

Case 8: $\mathfrak{h} = \mathfrak{su}(2, 1)$, $\mathfrak{m} = V$. \mathfrak{m} has an invariant pseudo-Hermitian metric g of signature $(4, 2)$. As in Case 7 the pseudo-Riemannian metric g and

the almost symplectic form ω are both unique (up to scaling) and compatible.

The corresponding homogeneous manifold is $\mathbb{S}^{2,4} \simeq \mathbb{S}^2 \times \mathbb{R}^4$, and J is the unique invariant almost complex structure. It is non-degenerate, and is the split analog of the Calabi structure. The triple (g, J, ω) is strongly nearly pseudo-Kähler and the Hermitian metric g is Einstein.

5.2 Kähler and nearly Kähler structures

Examining the list of all our homogeneous structures we conclude that the only pseudo-Kähler metrics are the cases A1.1, A3.1 and A1.3, A3.4 from the Appendix. Even though the groups on which the structures live are solvable (the topology is rather simple), the metric properties are non-trivial. We summarize the results.

Theorem 5. *The only pseudo-Kähler homogeneous 6D manifolds with semi-simple (nontrivial) isotropy are quotients $M = G/H$ with $H = SU(2)$ or $H = SU(1,1)$ with reducible isotropy representation $\mathfrak{m} = V + \mathbb{C}$. As an H -module $V = \mathbb{H}$ or resp. $V = \mathbb{H}_s$.*

This M is a Lie group; its Lie algebra \mathfrak{m} given by the following relations (two cases). Below $0 \neq \alpha, r \in \mathbb{R}$, $\varepsilon \in \{0, 1\}$ are the parameters, and the vectors $x, y \in V$, $e, ie \in \mathbb{C}$. The almost complex structure $J(x, e) = (xi, ie)$.

$$1) \quad \mathfrak{m} : \quad [x, y] = \alpha \operatorname{Re}(xi\bar{y})e, \quad [x, ie] = x(\frac{1}{2} + ri), \quad [e, ie] = e.$$

This \mathfrak{m}^6 is a 1-dimensional "right⁵ extension" of the 5D Heisenberg algebra. The symplectic form is $\omega = \omega_V + c\omega_{\mathbb{C}}$, $\omega_V(x, y) = \operatorname{Re}(xi\bar{y})$, $\omega_{\mathbb{C}}(e, ie) = 1$. The pseudo-Hermitian metric is $g(\xi, \eta) = \omega(\xi, J\eta)$; its signature is $(6, 0)$ for $\mathfrak{h} = \mathfrak{su}(2)$, $c > 0$ and $(4, 2)$ else. Moreover g is Einstein with the cosmological constant -4 , and is not conformally flat.

$$2) \quad \mathfrak{m} : \quad [x, ie] = rxi, \quad [e, ie] = \varepsilon e.$$

This \mathfrak{m}^6 is a 1-dimensional "right extension" of the 5D Abelian algebra. The symplectic form is $\omega = \omega_V + c\omega_{\mathbb{C}}$. The pseudo-Hermitian metric is $g(\xi, \eta) = \omega(\xi, J\eta)$; its signature is $(6, 0)$ for $\mathfrak{h} = \mathfrak{su}(2)$, $c > 0$ and $(4, 2)$ else. The metric is not Einstein or conformally flat unless $\varepsilon = 0$, when g is flat.

It is important to study when (M, g, J, ω) is strongly nearly Kähler (SNK), meaning that for the Levi-Civita connection ∇ the tensor $\nabla\omega$ is (nonzero) totally skew symmetric: $\nabla\omega = \frac{1}{3}d\omega \neq 0$. This is a restrictive condition.

⁵This means extension by derivations; terminology comes from Fuks [F], and is opposed to left=central extensions. For a Lie algebra \mathfrak{g} its "right extensions" are enumerated by the cohomology group $H^1(\mathfrak{g}, \mathfrak{g})$ and "left extensions" by $H^2(\mathfrak{g})$.

For instance, the Nijenhuis tensor N_J is non-degenerate and the geometry is constrained by the 'splitting principle' of P.-A. Nagy [Na]. As classified by J.-B. Butruilles [Bu], homogeneous SNK structures in 6D up to quotient are:

- $\mathbb{S}^6 = G_2^c/SU(3)$,
- $\mathbb{S}^3 \times \mathbb{S}^3 = SU(2) \times SU(2) \times SU(2)/SU(2)_{\text{diag}}$
- $\mathbb{C}P^3 = SU(4)/(SU(3) \times U(1)) = Sp(4)/(SU(2) \times U(1))$,
- the flag variety $\mathbb{F}(1, 2) = SU(3)/(U(1) \times U(1))$.

The first two belong to our list (the invariant structure J on $\mathbb{S}^3 \times \mathbb{S}^3$ corresponds to the case A2.1 from the Tables with parameters $(r, t) = \pm(\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}})$, so it has more symmetry than observed in [Bu]), while the last two do not (as they have reductive and Abelian isotropy respectively).

Homogeneous pseudo-SNK structures of signature $(2, 4)$ (this is given by the same condition: $\nabla\omega$ nonzero totally skew symmetric) with semi-simple isotropy can be extracted from our classification⁶:

- $\mathbb{S}^{2,4} = G_2^*/SU(1, 2)$,
- $SL(2) \times SL(2) = SU(2, 1) \times SU(2, 1) \times SU(2, 1)/SU(2, 1)_{\text{diag}}$ (the invariant structure J on this M^6 corresponds to the case A4.1 of the Tables with parameters $(r, t) = \pm(\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}})$)
- the left-invariant structure on the solvable Lie group corresponding to the case A3.2 with the parameters (after rescaling ω) $r = -\frac{3}{2t}$, $\epsilon = +1$, $\alpha = 0$, $p = t(i + j) \in \mathbb{H}_s$, $u = -\frac{1}{2}k \in \mathbb{H}_s$, $q = i$ and $b = \frac{1}{2}ti \in \mathbb{H}_s$.

Remark 3. *Since the latter homogeneous pseudo-SNK structure on $M^6 = G^9/H^3$ has no SNK analog, we write it explicitly. Let e_i be a basis of \mathfrak{m} and θ_i be the dual basis of \mathfrak{m}^* . The structure equations are $(\theta_{ij} = \theta_i \wedge \theta_j, t \neq 0)$:*

$$\begin{aligned} d\theta_1 &= \frac{3}{2}(\theta_{35} - \theta_{25}) - \theta_{16} + \frac{1}{2}\theta_{46}, & d\theta_2 &= \frac{3}{2}(\theta_{15} + \theta_{45}) - \theta_{26} - \frac{1}{2}\theta_{36}, \\ d\theta_3 &= \frac{3}{2}(\theta_{15} + \theta_{45}) - \theta_{36} - \frac{1}{2}\theta_{26}, & d\theta_4 &= \frac{3}{2}(\theta_{25} - \theta_{35}) - \theta_{46} + \frac{1}{2}\theta_{16}, \\ d\theta_5 &= t(\theta_{13} - \theta_{12} + \theta_{24} - \theta_{34}) - \theta_{56}, & d\theta_6 &= 0 \end{aligned}$$

and the almost complex structure J and the metric g are given by the formulae

$$\begin{aligned} J &= (e_2 \otimes \theta_1 - e_1 \otimes \theta_2 + e_3 \otimes \theta_4 - e_4 \otimes \theta_3) + (e_6 \otimes \theta_5 - e_5 \otimes \theta_6), \\ g &= \frac{1}{2}t(\theta_1^2 + \theta_2^2 - \theta_3^2 - \theta_4^2) + \theta_5^2 + \theta_6^2. \end{aligned}$$

⁶There are obvious pseudo-SNK analogs of signature $(2, 4)$ of the last two entries in Butruilles' list, but we present here only the spaces G/H with semi-simple H .

5.3 SKT and Gray-Hervella classes

Strong Kähler with torsion (SKT) structures are defined by property $\partial\bar{\partial}\omega = 0$ (in addition to $N_J = 0$). They are important in generalized Kähler geometry and supersymmetric nonlinear sigma models, see e.g. [FPS]. The SKT property is equivalent to

$$d_J^2\omega = 0, \quad d_J = d \circ J$$

(where $J \circ \sigma = \sigma(J \cdot, J \cdot, \dots)$), and we shall study generalizations when $d_J^k\omega = 0$ for larger k (and J not necessarily integrable). For instance, the standard almost Hermitian structure (g, J, ω) on \mathbb{S}^6 is not SKT, $d_J^3\omega \neq 0$, but $d_J^4\omega = 0$.

There are many structures J of type III, which are not SKT, but satisfy the condition $d_J^3\omega = 0$. The only occasions of SKT are these:

Theorem 6. *The only homogeneous Hermitian manifolds M^6 with semi-simple isotropy, which satisfy the SKT property but do not belong to either Kähler or pseudo-Kähler class, are equivalent to the following.*

1) *The structure of case A1.2 with parameters $q = \cos\theta \cdot i + \sin\theta \cdot j$, $p = \pm\sqrt{3\sin^2\theta - 1} \cdot q + \sin\theta \cdot k$. The Lie algebra \mathfrak{m} is the central extension*

$$0 \rightarrow \mathbb{R}^2 \rightarrow \mathfrak{m} \rightarrow \mathbb{R}^4 \rightarrow 0,$$

whence the homogeneous space is $M = G/H$ is an \mathbb{R}^2 -bundle over \mathbb{R}^4 .

2) *The structure of case A1.3 with parameters $q = i$, $\alpha = 0$, $\beta = -\frac{1}{2}$, $\epsilon = 1$ or of case A3.4 with the same parameters and in addition $p = 0$, $u = \lambda i$. The Lie algebra \mathfrak{m} is the "right" extension ($\mathbb{R}^4 = V = \mathbb{H}$ or resp. \mathbb{H}_s)*

$$0 \rightarrow \mathbb{R}^4 \rightarrow \mathfrak{m} \rightarrow \mathfrak{s}_2 \rightarrow 0,$$

where $\mathfrak{s}_2 = \text{Lie}(S_2)$ is the solvable non-abelian 2D Lie algebra of the Lie group S_2 , represented via rank 1 homomorphism $S_2 \rightarrow \mathbb{C}^ \xrightarrow{\text{diag}} \text{GL}_2(\mathbb{C}) \subset \text{GL}_4(\mathbb{R})$. The homogeneous space is $M = G/H \simeq \mathbb{R}^4 \rtimes S_2$.*

3) *The structure of case A3.4 with parameters $q = i$, $\alpha = 0$, $\epsilon = 1$, $p = \frac{1}{2}(i + j)$, $u = -\frac{1}{2}k$ and $\beta = -1$ or $\beta = \frac{1}{2}$. The Lie algebra \mathfrak{m} is the "right" extension given by the same sequence as in 2), but now the homomorphism $\varphi : S_2 \rightarrow \text{GL}_2(\mathbb{C}) \subset \text{GL}_4(\mathbb{R})$ has rank 2: $\varphi(e) = R_{\frac{1}{2}(i+j)}$, $\varphi(ie) = R_{\beta-\frac{1}{2}k}$, where $R_h(x) = xh$, $h \in \mathbb{H}_s$. Again $M = G/H \simeq \mathbb{R}^4 \rtimes S_2$.*

It is interesting which Gray-Hervella (GH) classes of almost Hermitian manifolds are realizable within our class of homogeneous 6D manifolds with semi-simple isotropy. In the work [GH] 16 classes of such manifolds were

encoded by subsets $\sigma \subset \{1, 2, 3, 4\}$ and $\mathfrak{u}(3)$ -modules $W_\sigma = \cup_{i \in \sigma} W_i$. Denote the corresponding class of structures by \mathcal{W}_σ .

The class corresponding to the empty set \emptyset is the Kähler class \mathcal{K} . The class \mathcal{W} for $\sigma = \{1, 2, 3, 4\}$ consists of all almost Hermitian manifolds.

In order to represent the general class \mathcal{W} by a disjoint union, we modify the Gray-Hervella classes to the following (strict) GH-classes

$$\tilde{\mathcal{W}}_\sigma = \mathcal{W}_\sigma \setminus (\cup_{i \in \sigma} \mathcal{W}_{\sigma \setminus \{i\}}).$$

For instance, $\tilde{\mathcal{W}}_i = \mathcal{W}_i \setminus \mathcal{K}$, $\tilde{\mathcal{W}}_{i,j} = (\mathcal{W}_i \oplus \mathcal{W}_j) \setminus (\mathcal{W}_i \cup \mathcal{W}_j)$ for $i \neq j$ etc.

We already discussed and classified the Kähler \mathcal{K} and nearly Kähler $\mathcal{W}_1 = \mathcal{NK}$ classes, which yields the description of strictly nearly Kähler class $\tilde{\mathcal{W}}_1$. Inspection of the tables (we refer only to Tables A1 and A2 corresponding to almost Hermitian manifolds) rules out the class $\mathcal{W}_2 = \mathcal{AK}$ of almost Kähler manifolds with the exception of the Kähler structures (observe from the Tables A1-A2 that whenever $d\omega = 0$, then also $N_J = 0$, so the structure belongs to the Kähler class \mathcal{K}). The class $\mathcal{W}_3 = \mathcal{H} \cap \mathcal{SK}$ of special Hermitian manifolds is realized only (again with the exception of the Kähler class \mathcal{K}) by the Tables A1.2 (several parametric cases) and A2.2 ($r = 0, t = \pm 1$); the latter is $M = SL_2(\mathbb{C})$ with the standard complex structure. The class $\tilde{\mathcal{W}}_4$ (containing locally conformally Kähler but non-Kähler manifolds) is realized only by the Table A1.4 ($q = \pm i$), in particular for $\varepsilon = -1$ we obtain the Calabi-Eckmann structure on $S^1 \times S^5$. For the other classes we have:

Theorem 7. *The GH-classes, realized as homogeneous almost Hermitian manifolds M^6 with semi-simple isotropy, are precisely the following: \mathcal{K} , $\tilde{\mathcal{W}}_1$, $\tilde{\mathcal{W}}_3$, $\tilde{\mathcal{W}}_4$, $\tilde{\mathcal{W}}_{1,2}$, $\tilde{\mathcal{W}}_{1,3}$, $\tilde{\mathcal{W}}_{2,3}$, $\tilde{\mathcal{W}}_{3,4}$, $\tilde{\mathcal{W}}_{1,2,3}$, $\tilde{\mathcal{W}}_{1,2,4}$, $\tilde{\mathcal{W}}_{1,3,4}$, $\tilde{\mathcal{W}}_{2,3,4}$, $\tilde{\mathcal{W}} = \tilde{\mathcal{W}}_{1,2,3,4}$.*

Thus the non-realizable (via our models) GH-classes are $\tilde{\mathcal{W}}_2$, $\tilde{\mathcal{W}}_{1,4}$, $\tilde{\mathcal{W}}_{2,4}$.

Proof. The proof is the direct calculation (in this algebraic computation we used Maple). Let us indicate, which sub-classes in Tables A1 and A2 realize the GH-classes (omitting the precise values of the parameters):

A1.1: \mathcal{K} , $\tilde{\mathcal{W}}_{3,4}$, $\tilde{\mathcal{W}}_{1,2,3}$, $\tilde{\mathcal{W}}$.	A2.1: $\tilde{\mathcal{W}}_1$, $\tilde{\mathcal{W}}_{1,3}$.
A1.2: $\tilde{\mathcal{W}}_3$, $\mathcal{W}_{1,2}$, $\tilde{\mathcal{W}}_{3,4}$, $\tilde{\mathcal{W}}_{1,2,3}$, $\tilde{\mathcal{W}}$.	A2.2: $\tilde{\mathcal{W}}_3$, $\tilde{\mathcal{W}}_{1,3}$.
A1.3: \mathcal{K} , $\tilde{\mathcal{W}}_{3,4}$, $\tilde{\mathcal{W}}_{1,2,3}$, $\tilde{\mathcal{W}}$.	A2.3: $\tilde{\mathcal{W}}_{1,3}$.
A1.4: $\tilde{\mathcal{W}}_4$, $\mathcal{W}_{1,2}$, $\mathcal{W}_{1,3}$, $\tilde{\mathcal{W}}_{2,3}$, $\tilde{\mathcal{W}}_{i,j,k}$ ($i < j < k$), $\tilde{\mathcal{W}}$.	A2.4: $\tilde{\mathcal{W}}_{1,3}$.

Notice that for Table A2 the structures are never in the general $\tilde{\mathcal{W}}$ class because they always satisfy the condition $\delta\omega = 0$ of $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$ by the \mathfrak{h} -equivariance of the divergence and the module type, but in fact they all satisfy the stronger condition of $\mathcal{W}_1 \oplus \mathcal{W}_3$. \square

Remark 4. *The cocalibrated structures of Hervella-Vidal type $\mathcal{G}_1 = \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ are those admitting a Hermitian connection with totally skew-symmetric torsion studied in [AFS, S]. Our computation confirms the results of these papers about the GH-type of such structures with parallel torsion.*

6 Investigation of homogeneous models of type II

The G_2 -invariant almost complex structures on \mathbb{S}^6 and $\mathbb{S}^{2,4}$ (type I) are well-studied [E, G, Ka, K₂], type III structures are described in Appendix - the corresponding manifolds have simple topology. In this section we describe the almost complex models of type II (some examples of these appeared in [S]).

Notice that by Section 5 none of type II models possesses a Kähler structure with the symmetry group G of dimension 9. This does not mean that they do not have Kähler structures at all. We shall describe them topologically and see that in some cases such a structure exists, however it is not G -invariant.

6.1 Homogeneous models of type II₁

Structures of type II₁ have $\mathfrak{m} = V + \mathbb{C}$ as \mathfrak{h} -representation. Interpreting $\mathfrak{h} = \mathfrak{su}(2)$ as imaginary quaternions we can identify V with the space of quaternions \mathbb{H} (as the module, not algebra). In particular, the set of \mathfrak{h} -invariant complex structures is the standard unit sphere $\mathbb{S}^2 = \{q \in \text{Im}(\mathbb{H}) : q \cdot q = -1\}$.

Thus the set of \mathfrak{h} -invariant (almost) complex structures on (representation) \mathfrak{m} is $\mathcal{J}_e = \mathbb{S}^2 \times \{1\} \simeq \mathbb{S}^2$.

II₁^a. The Lie group $SU(3)$ acts transitively on \mathbb{S}^5 with the stabilizer $SU(2)$. Therefore we have the following diffeomorphism:

$$M = U(3)/SU(2) = U(1) \times SU(3)/SU(2) \simeq \mathbb{S}^1 \times \mathbb{S}^5.$$

The complex structure J on this manifold is obtained from Hopf fibration $\mathbb{S}^5 \rightarrow \mathbb{C}P^2$ with the fiber \mathbb{S}^1 and the standard connection $H \subset T\mathbb{S}^5$, so that M is the \mathbb{T}^2 fibration over $\mathbb{C}P^2$.

This is the well-known Calabi-Eckmann complex structure J_0 : at every point it is the sum of the standard complex structure on the connection (flat CR-structure) and a complex structure on $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ (given by 2

parameters, which disappear if we pass to the universal cover). Clearly, M is not symplectic and so is not Kähler.

There is however a 2-parameter family of deformations of this structure to almost complex structure on the universal cover $\tilde{M} = \mathbb{R}^1 \times \mathbb{S}^5$. Indeed, this is given by the construction of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{m}, J)$, see Tables in the Appendix. To describe J fix a point, say $a = (0, 0, 1) \in \mathbb{S}^5 \subset \mathbb{C}^3$. The CR-hyperplane $\mathbb{C}^2 \simeq H_a \subset T_a \mathbb{S}^5$ as \mathfrak{h} -representation is isomorphic to \mathbb{H} , and so the space of invariant structures is \mathbb{S}^2 . We translate this to any other point $b \in \mathbb{S}^5$ by an element of $SU(3)$ and obtain an invariant CR-structure J' on the connection H (in fact, we have the trivial \mathbb{S}^2 -bundle of almost complex structures on the connection H over \mathbb{S}^5). This is extended to $J = J' + J''$ by complementing J' with (any) complex shift invariant structure J'' on the cylinder $\mathbb{R}^1 \times \mathbb{S}^1 \simeq \mathbb{C}/\mathbb{Z}$. The space of obtained structures is $\mathcal{J}_e \simeq \mathbb{S}^2$.

Since normalization of the Lie algebra structure on \mathfrak{g} involves complex multiplication $J_0 = i$ on \mathfrak{m} , there are two preferred complex structures $\pm J_0$ on this sphere. Moving along the fiber of the Hopf fibration $\mathbb{S}^5 \rightarrow \mathbb{C}P^2$ rotates the sphere \mathbb{S}^2 along the axis through these antipodal points (the differential of this maps $\text{Im}(\mathbb{H}) \ni q \mapsto [q, i] \in T\mathbb{S}^2$). Only $\pm J_0$ are (integrable) complex structures; for all other choices of J , the Nijenhuis tensor N_J is non-degenerate.

When we compactify M to $\mathbb{S}^1 \times \mathbb{S}^5$, we get two more parameters coming from the torus $\mathbb{T}^2 \subset U(3)/SU(2)$ (quotient of the cylinder), as the space of complex structures on it is given by the fundamental domain $\Sigma^2 = \{|z| \geq 1, |\text{Re}(z)| \leq \frac{1}{2}, \text{Im}(z) > 0\} \subset \mathbb{C}$. Thus the space of parameters of $U(3)$ -invariant complex structures on this compact M^6 is $\mathcal{J}_+ = \mathcal{J}_e \times \Sigma^2$. The latter deformation does not change integrability: N_J depends only on the first factor, and so is as described above.

II₁^b. Similarly, the Lie group $SU(2, 1)$ acts transitively on the pseudo-sphere $N^5 = \{z \in \mathbb{C}^3 : |z_1|^2 + |z_2|^2 - |z_3|^2 = -1\}$ with the stabilizer $SU(2)$ (inducing the action on the unit ball $B^4 \simeq \mathbb{P}N^5$ with the stabilizer $U(2)$, important for the canonical/flat CR-structure on $\mathbb{S}^3 = \partial B^4$). Since $N^5 \simeq \mathbb{S}^1 \times B^4$, we obtain the following diffeomorphism:

$$M = U(2, 1)/SU(2) \simeq U(1) \times SU(2, 1)/SU(2) \simeq \mathbb{T}^2 \times B^4.$$

The invariant complex structure J on this M , as well as on the universal cover $\tilde{M} = \mathbb{C} \times B^4$, is however not the obvious one, since (M, J, ω) is not Kähler: the structure J' on the factor B^4 has to be $SU(2, 1)$ -invariant.

Again the space of moduli of all $SU(2)$ -invariant structures J on \tilde{M} is $\mathcal{J}_e \simeq \mathbb{S}^2$ and on M it is $\mathcal{J}_+ = \mathcal{J}_e \times \Sigma^2$. Indeed, at every point $a \in N^5$ the natural contact space $H_a \subset T_a N^5$ is isomorphic to \mathbb{H} as \mathfrak{h} -module, and the space of invariant complex structures on H is the trivial \mathbb{S}^2 -bundle over N^5 . This

gives the complex structure J' , and the shift invariant complex structure J'' on the factor \mathbb{T}^2 resp. \mathbb{C} yields the structure $J = J' + J''$ on M resp. \tilde{M} . The integrable structures among these J are $J_0 = \pm i$ only, for others the tensor N_J is non-degenerate.

Concerning the integrability properties for the structures from \mathcal{J}_e the following statement describes the generalized SKT property.

Proposition 3. *For the compatible invariant almost symplectic form ω and the operator $d_J = d \circ J$ we have: $d_J^2 \omega \neq 0$ always; $d_J^3 \omega \neq 0$ unless $J = J_0 = \pm i$ or $J \perp J_0 \Leftrightarrow J = \cos t \cdot j + \sin t \cdot k$, $t \in \mathbb{R}$ (in these cases $d_J^3 \omega$ vanishes); $d_J^4 \omega = 0$ for any other parameter value.*

Thus we obtain that not only the poles, but also the equator between them in $\mathcal{J}_e = \mathbb{S}^2$ consists of distinguished almost complex structures.

6.2 Homogeneous models of type II_2

Structures of type II_2 also have $\mathfrak{m} = V + \mathbb{C}$ as \mathfrak{h} -representation. For $\mathfrak{h} = \mathfrak{su}(1, 1)$, considered as imaginary split quaternions, we identify V with the module of split quaternions \mathbb{H}_s . The set of \mathfrak{h} -invariant complex structures $J_q(v) = v \cdot q$, $q \in \text{Im}(\mathbb{H}_s)$ is the two sheet hyperboloid $Z^2 = \{q \in \text{Im}(\mathbb{H}_s) : q \cdot q = -1\}$. Thus the set of \mathfrak{h} -invariant (almost) complex structures on $V + \mathbb{C}$ is $\mathcal{J}_h = Z^2 \times \{1\} \simeq \mathbb{D}^2 \times \mathbb{Z}_2$.

Notice also that on V there are (almost) product structures $I_q(v) = v \cdot q$ forming the set $Y^2 = \{q \in \text{Im}(\mathbb{H}_s) : q \cdot q = +1\}$, which is the one sheet hyperboloid homeomorphic to the cylinder $\mathbb{S}^1 \times \mathbb{R}^1$.

II_2^q . The Lie group $SU(2, 1)$ acts transitively on the unit pseudo-sphere in $\mathbb{C}^3 \simeq \mathbb{R}^{4+2}$ with the metric of the signature $(4, 2)$: $Q^5 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 + |z_2|^2 - |z_3|^2 = 1\} \simeq \mathbb{S}^3 \times \mathbb{C}$. Thus

$$M = U(2, 1)/SU(1, 1) = U(1) \times SU(2, 1)/SU(1, 1) \simeq \mathbb{S}^1 \times \mathbb{S}^3 \times \mathbb{C}.$$

To see the invariant complex structure, notice that \mathbb{S}^1 acts on Q^5 , $w \mapsto e^{it}w$, and the quotient is

$$Q^5/\mathbb{S}^1 = \mathbb{C}^2 \# \overline{\mathbb{C}P^2} \simeq \mathbb{C}P^2 \setminus \{pt\} \simeq \mathbb{C}P^1 \times \mathbb{C};$$

the structure J is obtained similarly to case 1 above.

Namely, the CR-connection $H \subset TQ$ carries the space \mathcal{J}_h of G -invariant almost complex structures J' , which form a trivial 2D bundle over Q . Complementing this with any shift invariant complex structure J'' on the torus \mathbb{T}^2 , which is the fiber of the discussed map $M^6 \rightarrow \mathbb{C}P^1 \times \mathbb{C}$, we get the almost complex structure $J = J' + J''$ on $M = U(2, 1)/SU(1, 1)$. The moduli space of these structures is $\mathcal{J}_+ = \mathcal{J}_h \times \Sigma^2$.

If we consider the universal cover $\hat{M} = \mathbb{R}^1 \times \mathbb{S}^3 \times \mathbb{C}$, then the above torus becomes the cylinder $\mathbb{R}^1 \times \mathbb{S}^1$ and (since on the cylinder all complex structures are equivalent) the moduli space becomes not 4- but 2-dimensional, namely we get \mathcal{J}_h .

Again there are two preferred structures, corresponding to $J' = \pm i$. Indeed, moving along the orbit \mathbb{S}^1 of $U(1)$ -action the space $\mathcal{J}_h \simeq \mathbb{D}^2 \times \mathbb{Z}_2$ rotates around its axis of symmetry through $\pm i$. Only these two corresponding structures J are integrable, the others have non-degenerate Nijenhuis tensor N_J .

II₂^b. Finally, let us discuss

$$M = GL(3)/SU(1, 1) = GL(3)/SL(2).$$

It has two connected components, each being simply-connected.

We can identify M with the space $\{(v, \Pi^2, \omega)\}$, where $v \in \mathbb{R}^3$ is a nonzero vector, $\Pi^2 \not\supset v$ a transversal 2-plane and $\omega \in \Lambda^2 \Pi^* \setminus 0$ an area form on it. Indeed, $GL(3)$ acts transitively on this space with the stabilizer $SL(2)$. Furthermore, we have

$$M^6 = \mathbb{R}_* \times N^5, \text{ where } N^5 = SL(3)/SL(2),$$

and $SL(2)$ is embedded into $SL(3)$ as the lower 2×2 block.

We identify $N^5 = \{(v, p) \in \mathbb{R}^3 \times (\mathbb{R}^3)^* : \langle v, p \rangle = 1\}$ by choosing covector p in the annihilator of Π . At $a = (v, p) \in N$ the stabilizer $SL(2)$ acts on $T_a N = \{(w, q) : \langle w, p \rangle + \langle v, q \rangle = 0\}$. This has the invariant subspace $H_a = H_a^1 \oplus H_a^2$, where $H_a^1 = \{(w, 0) : \langle w, p \rangle = 0\}$ and $H_a^2 = \{(0, q) : \langle v, q \rangle = 0\}$.

We have: $H_a^2 = (H_a^1)^*$. The $SL(2)$ -invariant area form ω on H_a^1 is obtained as $\iota_v \Omega$, where Ω is the volume form in \mathbb{R}^3 . The invariant almost complex structure is now given by $J_{rt}(\xi) = r\xi + t\hat{\xi}$, where $\xi \in H_a^1$ and $\hat{\xi} = \iota_\xi \omega \in H_a^2$, $r, t \in \mathbb{R}$. Requirement $J_{rt}^2 = -1$ implies $J_{rt}(\hat{\xi}) = -\frac{1+r^2}{t}\xi - r\hat{\xi}$. Complementing this by requirement that J_{rt} maps the complement $\{(\lambda v, -\lambda p)\} \subset T_a N^5$ to $T_a \mathbb{R}_*$ we obtain the almost complex structure on M parametrized by the pair (r, t) or equivalently by the space \mathcal{J}_h . Direct calculation shows that none of these structures J_{rt} is integrable.

The integrability properties generalizing SKT for the structures from \mathcal{J}_h are given by the following statement.

Proposition 4. *For the compatible invariant almost symplectic form ω and the operator $d_J = d \circ J$ we have: $d_J^2 \omega \neq 0$ always; $d_J^3 \omega \neq 0$ unless $J = J_0 = \pm i$ in the case $G = U(2, 1)$ ($p = i$ in the case A3.5) or $J = \cosh t \cdot i + \sinh t \cdot k$, $t \in \mathbb{R}$ in the case $G = GL(3)$ ($p = j$ in the case A3.5; in terms of the parameters of the Table this means $q = \alpha i + \beta k$, $\alpha^2 - \beta^2 = 1$) – only in these cases $d_J^3 \omega$ vanishes; and $d_J^4 \omega = 0$ for any other parameter value.*

Notice that for $H = SU(1, 1)$ there are less invariant structures (J, ω) with $d^3\omega = 0$ than for $H = SU(2)$. Actually, for $G = U(2, 1)$ the additional plane $\{q \in \text{Im}(\mathbb{H}_s) : q \perp i\} = \langle j, k \rangle$ consists of the product structures $q^2 = 1$; for $G = GL(3)$ and $p = j$ (we refer to the case A3.5 in Table A3) the plane $\{q \in \text{Im}(\mathbb{H}_s) : q \perp j\} = \langle i, k \rangle$ contains the indicated two-component curve of complex structures $q^2 = -1$.

7 Concluding remarks

In this paper we classified special homogeneous almost complex structures in dimension 6. Most of them turn out to be the left-invariant structures on Lie groups and, with some notable exceptions, most of these are solvable. Passing to quotient nil- or solv-manifolds destroys the isotropy, so our classification easily implies

Theorem 8. *The only compact homogeneous almost complex manifolds in dimension 6 with semi-simple (nontrivial) isotropy are \mathbb{S}^6 , $\mathbb{S}^1 \times \mathbb{S}^5$ and $\mathbb{S}^3 \times \mathbb{S}^3$ and their finite quotients, equipped with the almost complex structures as described in Theorem 1.*

Remark 5. *The possible quotients, that carry general parameter almost complex structures J from Theorem 1, are $\mathbb{S}^1 \times (\mathbb{S}^5/\mathbb{Z}_3)$, $\mathbb{S}^1 \tilde{\times} (\mathbb{S}^5/\mathbb{Z}_3) = (\mathbb{S}^1 \times \mathbb{S}^5)/\mathbb{Z}_3$, $\mathbb{RP}^3 \times \mathbb{S}^3$ and $\mathbb{RP}^3 \times \mathbb{RP}^3$. For some exceptional values of parameters there are further quotients, e.g. the Calabi-Eckmann structure descends onto the direct product $\mathbb{S}^1 \times (\mathbb{S}^5/\mathbb{Z}_n)$ for any n .*

Some items from our list easily generalize to higher dimensions, providing examples of non-integrable almost complex structures with abundant symmetries. Previously some invariant almost complex structure appeared in [Wo, WG]. Here are some new examples.

For every Lie algebra \mathfrak{h} and a complex vector space (V, i) consider the space $\mathfrak{m} = \mathfrak{h} \otimes V$ with the \mathfrak{h} -invariant almost complex structure given by $J(h \otimes v) = h \otimes (i \cdot v)$. Fixing a commutative associative bilinear multiplication $Q : S^2V \rightarrow V$ we define the Lie algebra structure on \mathfrak{m} by the formula

$$[h \otimes v, h' \otimes v'] = [h, h']_{\mathfrak{h}} \otimes Q(v, v').$$

Then $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{m}$ is a Lie algebra, and the corresponding Lie group $M = G/H$ carries G -invariant almost complex structure J .

All 4 structures in Table A2 for $\mathfrak{h} = \mathfrak{su}(2)$ resp. Table A4 for $\mathfrak{h} = \mathfrak{su}(1, 1)$ are of this type with $V = \mathbb{R}^2$. The above family of generalized gradings contains the 2-step graded nilpotent algebra $\mathfrak{m} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$, with the bracket $[h_1, h'_1] = [h, h']_2$, where $\mathfrak{h} \ni h \leftrightarrow h_i \in \mathfrak{h}_i \simeq \mathfrak{h}$ (and similarly for h'). The corresponding Lie group $M = \exp(\mathfrak{m})$ carries the left-invariant structure J with non-degenerate Nijenhuis tensor in the sense $N_J(\Lambda^2 TM) = TM$.

Another example is the "right extension" M^{4n+2} of the Heisenberg group over the quaternion space \mathbb{H}^n . We describe the corresponding Lie algebra. Let I, J, K be the standard $\mathrm{Sp}(n)$ -invariant complex structures on \mathbb{H}^n , $g(x, y) = x^t \cdot \bar{y}$ be the standard hyper-Hermitian structure and $\omega_q(x, y) = g(xq, y)$ be the symplectic structure parametrized by $q \in \mathbb{S}^2 \subset \mathbb{R}^3(I, J, K)$, $x, y \in \mathbb{H}^n$. Then with fixed $e \in \mathbb{C}$ we have the following brackets on \mathfrak{m} ($\varepsilon \in \{0, 1\}$):

$$[x, y] = \omega_I(x, y)e, \quad [x, ie] = x(\frac{1}{2}\varepsilon + \beta I), \quad [e, ie] = \varepsilon e.$$

Notice that $\mathbb{H}^n \oplus \langle e \rangle$ is the Heisenberg algebra and \mathfrak{m} its extension by derivations. The almost complex structure $J_q(x, e) = (xq, ie)$ on M is invariant with respect to the Lie group $G = \mathrm{Sp}(n) \ltimes \exp(\mathfrak{m})$. The "twistor space" of this M (= the bundle of moduli of the invariant almost complex structures) is the $4(n+1)$ -dimensional manifold $M^\dagger = M \times \mathbb{S}^2$ equipped with the canonical almost complex structure

$$J_\dagger(x, e, \xi) = (xq, ie, \xi q), \quad \xi \in T_q\mathbb{S}^2.$$

This structure J_\dagger is non-integrable (its Nijenhuis tensor is encoded by N_J), but has some natural pseudoholomorphic foliations.

Finally let us notice that for type II spaces from Theorem 1 the set of invariant almost complex structures \mathcal{J} is two-dimensional (if we consider simply connected model M): it is parametrized by \mathbb{S}^2 for $\mathfrak{h} = \mathfrak{su}(2)$ and by $\mathbb{Z}_2 \times \mathbb{D}^2$ for $\mathfrak{h} = \mathfrak{su}(1, 1)$. The space of all these structures — the 8-dimensional "twistor space" $M^6 \times \mathcal{J}$ — has a natural almost complex structure, but its Nijenhuis tensor is nonzero. For type II_1 spaces on $M = U(3)/SU(2)$ or $M = U(2, 1)/SU(2)$ (where the space of almost complex structures is given by 4 parameters) the corresponding "twistor space" $M^6 \times \mathcal{J} \times \Sigma^2$ is 10-dimensional and it also possesses the natural non-integrable almost complex structure.

A Tables of the structure of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{m}, J, \omega)$

Here we list the data for Theorems 1, 3 and indicate some integrability properties for the structures. These completely encode all non-flat homogeneous almost complex manifolds with semi-simple isotropy, only excluding the special cases $G_2^c/SU(3) = S^6$ and $G_2^*/SU(2, 1) = S^{2,4}$.

Imaginary quaternions are generated by anti-commuting $i, j, k = ij$, which satisfy $i^2 = -1, j^2 = -1, k^2 = -1$. In tables A1, A2 we identify $\mathfrak{h} = \mathrm{Im}(\mathbb{H})$.

Imaginary split quaternions are generated by anti-commuting $i, j, k = ij$, with $i^2 = -1, j^2 = 1, k^2 = 1$. In tables A3, A4 we identify $\mathfrak{h} = \mathrm{Im}(\mathbb{H}_s)$.

The brackets $[\mathfrak{h}, \mathfrak{h}]$ and $[\mathfrak{h}, \mathfrak{m}]$ are straightforward and are not included into the tables. We include only the non-trivial brackets $[\mathfrak{m}, \mathfrak{m}]$.

The almost complex structure J is indicated in terms of its 2 parameters for tables A1-A4 (beware that the parameter r has different meaning in different tables). J has no parameters in tables A5-A6. In the case of representation $\mathfrak{ad}^{\mathbb{C}}$ we use formula (1) for J .

We list only non-trivial values of the Nijenhuis tensor in the minimal amount due to the identity $N_J(Jx, y) = N_J(x, Jy) = -JN_J(x, y)$.

We write NDG to indicate that the tensor N_J is non-degenerate; DG_2 indicates that the image of $N_J : \Lambda_{\mathbb{C}}^2 TM \rightarrow TM$ is a real rank 2 subdistribution of TM , DG_1 means it is a real rank 4 subdistribution (in DG_k the number k is the complex codimension of $\text{Im}(N_J)$, see [K₁]).

The Hermitian metric is given by $g(\xi, \eta) = \omega(\xi, J\eta)$ as in Section 5. Thus we describe only the almost symplectic form ω and its differential.

Recall that V stands for the standard representation, \mathbb{C} for the trivial complex and \mathfrak{ad} for the adjoint representation.

Table A1: isotropy $\mathfrak{h} = \mathfrak{su}(2)$, representation $V + \mathbb{C}$:

m as \mathfrak{h} -representation	$\mathfrak{m} = V \oplus \mathbb{C} = \mathbb{H} \oplus \mathbb{C}$. \mathfrak{h} acts from the left, $\mathfrak{ad}_{\mathbb{Im}\mathbb{H}} \subset \text{End}_{\mathbb{R}}(\mathbb{H})$
vectors & parameters	$e \in \mathbb{C}$ is fixed. $x, y \in \mathbb{H}$ are arbitrary. $q, p, b \in \text{Im}(\mathbb{H})$, $\alpha, \beta, r, \varepsilon \in \mathbb{R}$
structure J on \mathfrak{m}	$J = (q, i)$, $q^2 = -1$. q acts from the right: $J(x, e) = (xq, ie)$
structure ω on \mathfrak{m}	$\omega = \omega_{\mathbb{H}} + \omega_{\mathbb{C}}$, $\omega_{\mathbb{H}}(x, y) = \text{Re}(xb\bar{y})$, $\omega_{\mathbb{C}}(e, ie) = 1$ ($b \neq 0$)
compatibility of J and ω	$b \in \mathbb{R}q$

	Lie algebra structure of \mathfrak{g}	tensor N_J & condition for $d\omega = 0$	notes
A1.1	$[x, y] = \alpha \text{Re}(xi\bar{y})e$ $[x, ie] = x(\frac{\varepsilon}{2} + ri)$ $[e, ie] = \varepsilon e$	$N_J(x, y) = \alpha(\text{Re}(xq[q, i]\bar{y}) - i\text{Re}(x[q, i]\bar{y}))e$ $N_J(x, e) = rx[q, i]$ $d\omega = 0$ iff $\varepsilon = 1, b = \alpha i$	$\varepsilon \in \{0, 1\}$, $\alpha \neq 0$ NDG unless $r = 0$ or $q = \pm i$
A1.2	$[x, y] =$ $= (\text{Re}(xi\bar{y}) + i\text{Re}(xp\bar{y}))e$	$N_J(x, y) = \text{Re}(x(q[q, i] + [q, p])\bar{y})e +$ $+ \text{Re}(x(q[q, p] - [q, i])\bar{y})ie$, $d\omega \neq 0$	DG ₂ unless $[p, q] = q[q, i]$
A1.3	$[x, e] = \alpha x$ $[x, ie] = x(\beta + ri)$ $[e, ie] = \varepsilon e$	$N_J(x, e) = rx[q, i]$ $d\omega = 0$ iff $\alpha = \beta = 0, b \in \mathbb{R}i$	$(\varepsilon, \alpha) \in \{(0, 0),$ $(0, 1), (1, 0)\}$ DG ₁ or NDG
A1.4	$[x, y] = \text{Re}(xi\bar{y})e + \varepsilon \cdot \mathfrak{ad}_{\text{Im}(x\bar{y})}$ $[x, e] = 3\varepsilon xi$	$N_J(x, y) = (\text{Re}(xq[q, i]\bar{y}) - i\text{Re}(x[q, i]\bar{y}))e$ $N_J(x, e) = 3\varepsilon xq[q, i]$, $d\omega \neq 0$	$\varepsilon = \pm 1$ $\varepsilon = -1 \Rightarrow \mathfrak{g} = \mathfrak{u}(3)$ $\varepsilon = 1 \Rightarrow \mathfrak{g} = \mathfrak{u}(2, 1)$ NDG unless $q = \pm i$

Table A2: isotropy $\mathfrak{h} = \mathfrak{su}(2)$, representation $\mathfrak{ad}^{\mathbb{C}}$:

m as \mathfrak{h} -representation	$\mathfrak{m} = \mathfrak{ad} \oplus \mathfrak{ad} = \mathfrak{ad}^{\mathbb{C}}$, the complex adjoint representation of \mathfrak{h}
vectors & parameters	$x, y \in \mathfrak{m}$. If $\mathfrak{m} = \mathfrak{m}_a \oplus \mathfrak{m}_b$, we decompose $x = x_a + x_b$. $r, t \in \mathbb{R}$
structure J on \mathfrak{m}	$J(u, 0) = (ru, tu)$, $u \in \mathfrak{ad}$ and $J^2 = -1$
structure ω on \mathfrak{m}	Let K_1, K_2 be the Killing forms of $\mathfrak{su}(2)$ on each copy of \mathfrak{ad} and $J_0(u, v) = (-v, u)$, $u, v \in \mathfrak{ad}$. Then $\omega(x, y) = (K_1 + K_2)(J_0x, y)$
compatibility of J and ω	ω is always compatible, never closed

	Lie algebra structure of \mathfrak{g}	tensor N_J & condition for $d\omega = 0$	notes
A2.1	$\mathfrak{m} = \mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_2$ semi-simple	$N_J(x_1, y_1) = -(r^2 + 1)[x, y]_1 + t(t - 2r)[x, y]_2$	1, 2 are not gradings, NDG always
A2.2	$\mathfrak{m} = \mathfrak{sl}_2(\mathbb{C})$ - simple, $\mathfrak{su}(2) \oplus i\mathfrak{su}(2)$	$N_J(x, y) = (1 + r^2 - t^2 + 2rti)[x, y]$	$N_J = 0$ for $J = \pm i$, NDG else
A2.3	$\mathfrak{m} = \mathfrak{su}(2)_0 \oplus \mathfrak{su}(2)_1$ $\mathfrak{su}(2)_1$ abelian.	$N_J(x_0, y_0) = -(r^2 + 1)[x, y]_0 - 2rt[x, y]_1$	Graded NDG always
A2.4	$\mathfrak{m} = \mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_2$ 2-step nilpotent	$N_J(x_1, y_1) = \frac{2(r^3+r)}{t}[x, y]_1 + (3r^2 - 1)[x, y]_2$	Graded NDG always

Table A3: isotropy $\mathfrak{h} = \mathfrak{su}(1, 1)$, representation $V + \mathbb{C}$:

m as \mathfrak{h} -representation	$\mathfrak{m} = V \oplus \mathbb{C} = \mathbb{H}_s \oplus \mathbb{C}$. \mathfrak{h} acts from the left, $\mathfrak{ad}_{\mathfrak{Im}\mathbb{H}_s} \subset \text{End}_{\mathbb{R}}(\mathbb{H}_s)$
vectors & parameters	$e \in \mathbb{C}$ - fixed. $x, y \in \mathbb{H}_s$ - arbitrary. $q, p, u, b \in \text{Im}(\mathbb{H}_s)$, $\alpha, \beta, r, \varepsilon \in \mathbb{R}$
structure J on \mathfrak{m}	$J = (q, i)$, $q^2 = -1$. q acts from the right: $J(x, e) = (xq, ie)$
structure ω on \mathfrak{m}	$\omega = \omega_{\mathbb{H}_s} + \omega_{\mathbb{C}}$, $\omega_{\mathbb{H}_s}(x, y) = \text{Re}(xb\bar{y})$, $\omega_{\mathbb{C}}(e, ie) = 1$ ($b \neq 0$)
compatibility of J and ω	$b \in \mathbb{R}q$

	Lie algebra structure of \mathfrak{g}	tensor N_J & condition for $d\omega = 0$	notes
A3.1	$[x, y] = \text{Re}(xp\bar{y})e$ $[x, ie] = x(\frac{\varepsilon}{2} + rp)$ $[e, ie] = \varepsilon e$	$N_J(x, y) = (\text{Re}(xq[q, p]\bar{y}) - i\text{Re}(x[q, p]\bar{y}))e$ $N_J(x, e) = rx[q, p]$ $d\omega = 0$ iff $\varepsilon = 1$, $[p, b] = p - b$	$\varepsilon \in \{1, 0\}$, $p^2 \neq 0$ NDG unless $r = 0$ or $q \in \mathbb{R}p$
A3.2	$[x, y] = \text{Re}(xp\bar{y})e$ $[x, e] = rxp$ $[x, ie] = x(\varepsilon + \alpha\delta_r^0 + u)$ $[e, ie] = \varepsilon e$	$N_J(x, y) = (\text{Re}(xq[q, p]\bar{y}) - i\text{Re}(x[q, p]\bar{y}))e$ $N_J(x, e) = x(rq[q, p] + [q, u])$ $d\omega = 0$ iff $[u, b] = p - 2(\varepsilon + s\delta_r^0)b$	$\varepsilon \in \{1, 0\}$, $p^2 = 0$ $[p, u] = (\varepsilon + 2\alpha\delta_r^0)p$ $p \neq 0$; NDG unless $[u, q] = rq[q, p]$
A3.3	$[x, y] = (\text{Re}(xp\bar{y}) + i\text{Re}(xu\bar{y}))e$	$N_J(x, y) = \text{Re}(x(q[q, p] + [q, u])\bar{y})e +$ $+ \text{Re}(x(q[q, u] - [q, p])\bar{y})ie$, $d\omega \neq 0$	DG ₂ if $[u, q] \neq q[q, p]$ $N_J = 0$ else
A3.4	$[x, e] = x(\alpha + p)$ $[x, ie] = x(\beta + u)$ $[e, ie] = \varepsilon e$	$N_J(x, e) = x(q[q, p] + [q, u])$ $d\omega = 0$ iff $[b, p] = [b, u] = 0$, $\alpha = \beta = 0$	$\varepsilon = p = 0$; or $\varepsilon = 1$, $[p, u] = p$, $p^2 = 0$, $\alpha = 0$ DG ₁ or $N_J = 0$
A3.5	$[x, y] = \text{Re}(xp\bar{y})e + \mathfrak{ad}_{\text{Im}(x\bar{y})}$ $[x, e] = 3\varepsilon xp$ $p^2 = -\varepsilon = \pm 1$	$N_J(x, y) = (\text{Re}(xq[q, p]\bar{y}) - i\text{Re}(x[q, p]\bar{y}))e$ $N_J(x, e) = 3\varepsilon xq[q, p]$ $d\omega \neq 0$	$p \in \{i, j\}$ $p = i \Rightarrow \mathfrak{g} = \mathfrak{u}(2, 1)$ $p = j \Rightarrow \mathfrak{g} = \mathfrak{gl}_3$ NDG unless $p = \pm q$

Table A4: isotropy $\mathfrak{h} = \mathfrak{su}(1, 1)$, representation $\mathfrak{ad}^{\mathbb{C}}$:

m as \mathfrak{h} -representation	$\mathfrak{m} = \mathfrak{ad} \oplus \mathfrak{ad} = \mathfrak{ad}^{\mathbb{C}}$, the complex adjoint representation of \mathfrak{h}
structure J on \mathfrak{m}	$J(u, 0) = (ru, tu)$, $u \in \mathfrak{ad}$ and $J^2 = -1$
structure ω on \mathfrak{m}	Let K_1, K_2 be the Killing forms of $\mathfrak{su}(1, 1)$ on each copy of \mathfrak{ad} and $J_0(u, v) = (-v, u)$, $u, v \in \mathfrak{ad}$. Then $\omega(x, y) = (K_1 + K_2)(J_0x, y)$
compatibility of J and ω	ω is always compatible, never closed

	Lie algebra structure of \mathfrak{g}	tensor N_J & condition for $d\omega = 0$	notes
A4.1	$\mathfrak{m} = \mathfrak{su}(1, 1)_1 \oplus \mathfrak{su}(1, 1)_2$ semi-simple	$N_J(x_1, y_1) = -(r^2 + 1)[x, y]_1 + t(t - 2r)[x, y]_2$	1, 2 are not gradings, NDG always
A4.2	$\mathfrak{m} = \mathfrak{sl}_2(\mathbb{C})$ – simple $\mathfrak{su}(1, 1) \oplus i\mathfrak{su}(1, 1)$	$N_J(x, y) = (1 + r^2 - t^2 + 2rti)[x, y]$	$N_J = 0$ for $J = \pm i$, NDG else
A4.3	$\mathfrak{m} = \mathfrak{su}(1, 1)_0 \oplus \mathfrak{su}(1, 1)_1$ $\mathfrak{su}(1, 1)_1$ Abelian	$N_J(x_0, y_0) = -(r^2 + 1)[x, y]_0 - 2rt[x, y]_1$	Graded NDG always
A4.4	$\mathfrak{m} = \mathfrak{su}(1, 1)_1 \oplus \mathfrak{su}(1, 1)_2$ 2-step nilpotent	$N_J(x_1, y_1) = \frac{2(r^3+r)}{t}[x, y]_1 + (3r^2 - 1)[x, y]_2$	Graded NDG always

Table A5: isotropy $\mathfrak{h} = \mathfrak{sl}_2(\mathbb{C})$, representation $V + \mathbb{C}$:

\mathfrak{m} as \mathfrak{h} -representation	$\mathfrak{m} = V \oplus \mathbb{C}$. $V \simeq \mathbb{C}^2$
vectors & parameters	$e \in \mathbb{C}$ is fixed. $x, y \in V$ are arbitrary. $\alpha, \beta, \gamma, r, \varepsilon \in \mathbb{R}$
structure J on \mathfrak{m}	$J = i$ on both submodules V and \mathbb{C} .
structure ω on \mathfrak{m}	Let ω_0 be the $\mathfrak{sl}_2(\mathbb{C})$ -invariant form 2-on V with \mathbb{C} -values. Then $\omega = \omega_V + \omega_{\mathbb{C}}$, $\omega_V = \operatorname{Re}(\lambda\omega_0)$, $\lambda \in \mathbb{C}$, $\omega_{\mathbb{C}}(e, ie) = 1$
compatibility of J and ω	ω is never compatible and never closed.

	Lie algebra structure of \mathfrak{g}	tensor N_J & condition for $d\omega = 0$	notes
A5.1	$[x, e] = \alpha x$ $[x, ie] = (\beta + \gamma i)x$ $[e, ie] = \varepsilon e$	$N_J = 0$	$(\varepsilon, \alpha) \in \{(0, 0), (0, 1), (1, 0)\}$ J is integrable
A5.2	$[x, y] = (\operatorname{Re} + r\operatorname{Im})\omega_0(x, y)e$	$N_J(x, y) = 2(1 - r) i \overline{\omega_0(x, y)}e$	DG ₂ unless $r = 1$

Table A6: isotropy $\mathfrak{h} = \mathfrak{sl}_2(\mathbb{C})$, representation \mathfrak{ad} :

The only possible structure is $\mathfrak{m} = \mathfrak{sl}_2(\mathbb{C})$ and J is integrable, see §5.
There are no \mathfrak{h} -invariant nonzero 2-forms on \mathfrak{m} .

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