Institute of Mathematics and Statistics, Faculty of Science and Technology

Lie-Algebraic Approaches to Highly Symmetric Geometries

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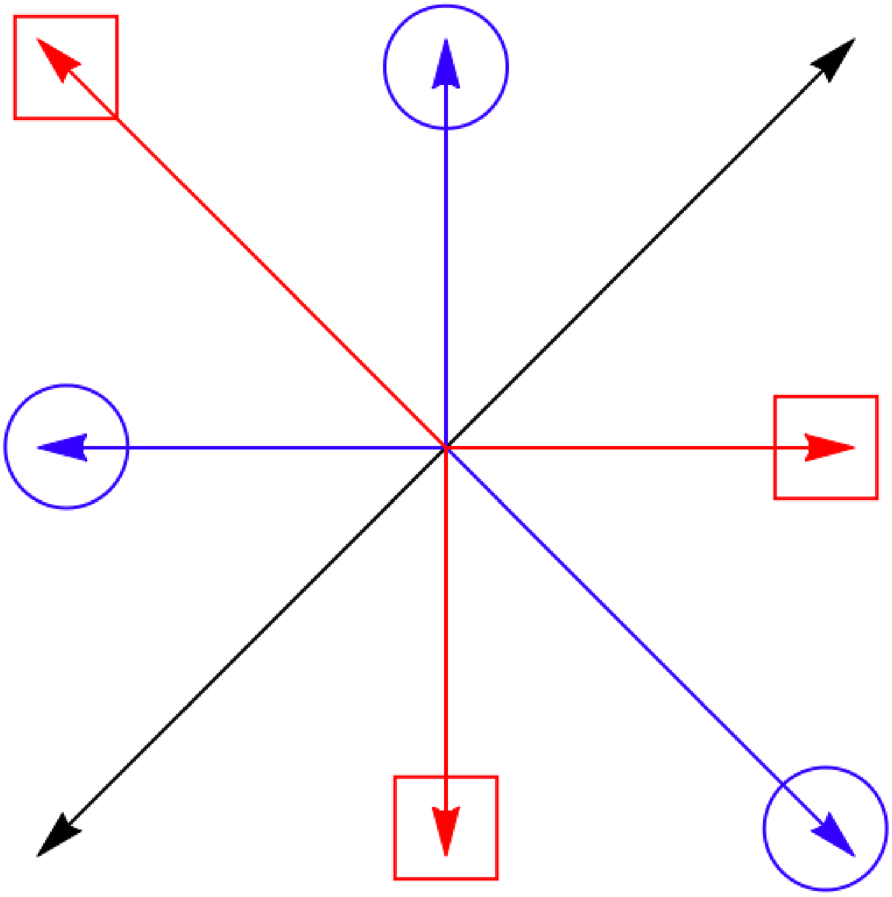

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Introduction

0.1 Symmetries: From Geometry to Algebra

When mathematicians decide to investigate some class of objects, a starting question could be what it means for two such objects to be equal. This usually leads to a class of maps which associate the objects to each other, meaning that in fact our objects are contained in a category, and two objects which are associated by maps can be considered equal.

In differential geometry, the objects of interest are smooth manifolds equipped with some geometric structure. The most straightforward class of transformations to consider is then the class of smooth maps with smooth inverses, or diffeomorphisms.

It is then a natural idea to consider the set of diffeomorphisms which map an object to itself. It is easy to see that this set forms a group, which we will call the symmetry group of the object. In some cases, we can parametrize the symmetry group (or some subgroup of this) by finitely many real numbers. In this case, the symmetry group (or its subgroup) will be a Lie group, and the number of parameters is its dimension.

Lie groups, as invented by the Norwegian mathematician Sophus Lie roughly in the 1870's, originated as a tool for integrating differential equations, and this unified the known methods at the time into a coherent framework. From our point of view, a differential equation is just an example of a geometric object, but the lesson we learn is that the symmetry group of an object encodes essential information about it. This remains true more generally. For instance, the existence, dimension and structure of the symmetry group are all invariant properties of the object.

When the symmetry group is large, in some sense, we often find that it may even contain complete information about the object, in the sense that we are able to essentially reproduce the object from pure symmetry information. When this is possible, it provides an algebraic approach to understanding the geometry, because dealing with group structures is in the domain of algebra. For instance this can be done when the symmetry group acts transitively on the underlying manifold of the object, so that the manifold consists of a single group orbit. This is the case of Klein geometries.

A key insight of Sophus Lie was that to understand the properties of Lie groups, which are quite complicated, it often suffices to understand related properties of their Lie algebras. The Lie algebra of a symmetry group is its set of vector fields of generators, equipped with the vector field commutators. This structure can be abstracted to a general Lie algebra, and this process gives a flexible set of tools which can be used in computing various quantities related to the geometry, such as tensor invariants or the values of invariant differential operators (De-Rham differential, adapted connections and more).

This leads to considering Lie algebras for their own sake, and to the theory of their representations. A representation is a realization of a Lie algebra as operators on a vector space, with respect to the operator commutator. Representation theory is of prime importance for making practical computations in the setting we have described.

0.2 Goals of the Project

The goal of this project is to facilitate the exploration and development of geometry by symmetry-based methods. To this end, we answer several natural questions that appear when considering symmetries, for particular examples of geometries. Such questions include:

- If some aspect of the symmetry algebras structure is specified, can we produce examples of geometric structures which realize this?
- When the symmetry is "large", how can we use this to calculate geometric invariants?
- Fixing the geometry dimension and type, how big can the symmetry dimension be?
- Given a (partial) algebraic input of a homogeneous geometry (isotropy data) how may we restore the geometric picture?

We elaborate on the last question, with the following observation: The most symmetric model of some type of geometric structure (for fixed dimension) is often unique, and there is a significant gap between the symmetry dimension of the maximal model and the so called sub-maximal model. It is interesting to determine the size of this gap.

We say that the *sub-maximal problem* is to compute the difference between maximal and sub-maximal symmetry dimension, and to realize a sub-maximal model, for some fixed geometry.

Resolving the sub-maximal problem for different geometries is a recurring theme in this thesis.

0.3 List of the Papers

The main portion of the thesis is composed of the following papers:

[1] D. V. Alekseevsky, B. S. Kruglikov, H. Winther, *Homogeneous almost complex structures in dimension 6 with semi-simple isotropy*, Ann. Glob. Anal. Geom. **46**, 361–387 (2014).

[2] B. Kruglikov, H. Winther, *Almost complex structures in 6D with non-degenerate Nijenhuis tensors and large symmetry groups*, Ann. Glob. Anal. Geom., doi:10.1007/s10455-016-9513-5 (2016)

[3] B. S. Kruglikov, Henrik Winther, Lenka Zalabová, *Submaximally Symmetric Almost Quaternionic Structures*, arXiv:1607.02025 (2016).

[4] Ioannis Chrysikos, Christian O'Cadiz Gustad, Henrik Winther, *Invariant connections and ∇-Einstein structures on isotropy irreducible spaces*, arXiv:1607.06774 (2016).

[5] B. Kruglikov, H. Winther, *Reconstruction from Representations: Jacobi via Cohomology*, arXiv:1611.05334 (2016).

[6] B. Kruglikov, H. Winther, *Nondegenerate para-complex structures in 6D with large symmetry*, arXiv:1611.05767 (2016).

0.4 Summary of the Papers

This thesis is built around several articles co-authored by me. All of these revolve around variations of the setting described in section 0.1, geometric structures which are equipped with symmetry algebras that are large, in some sense.

0.4.1 Homogeneous almost complex structures in dimension 6 with semi-simple isotropy

Joint with D. V. Alekseevsky and B. S. Kruglikov

In this paper, we investigate almost complex manifolds which come equipped with symmetry algebras of dimension at least 9, and such that the stabilizer of a point is a simple Lie algebra. Such spaces are classified, and moreover we exploit the symmetry in order to describe their invariant (pseudo-) Hermitian geometry completely, obtaining information such as the Nijenhuis tensor, realizeable restricted Gray-Hervella classes and the existence of SKT-metrics. In particular, an example of a homogeneous strictly nearly pseudo-Kähler manifold which does not correspond to a change of real forms of a compact nearly Kähler manifold is discovered.

0.4.2 Almost Complex Structures in 6D with Non-degenerate Nijenhuis Tensors and Large Symmetry Groups

Joint with B. S. Kruglikov

In this paper, we resolve the sub-maximal problem for almost complex structures in dimension 6 with non-degenerate Nijenhuis tensors. The Nijenhuis tensor itself is an invariant tensor, and this allows us to compute the possible isotropy algebras, and reconstruct the maximal and sub-maximal symmetry algebras from this data. A full global classification is obtained. Further, we use the symmetry algebra to determine that all sub-maximal models are also strictly nearly (pseudo-) Kähler manifolds.

0.4.3 Submaximally Symmetric Almost Quaternionic Structures

Joint with B. S. Kruglikov and Lenka Zalabová

In this paper, we resolve the sub-maximal problem for quaternionic and almost quaternionic geometry in dimension $4n > 4$. It turns out that the sub-maximal symmetry dimension can be realized in two different ways, both by a quaternionic model and by an almost quaternionic structure with nontrivial structure torsion. By employing tools from the theory of Parabolic Geometries, we produce models of these both in terms of abstract symmetry algebras, and as endomorphisms on the tangent bundle written in terms of local coordinates.

0.4.4 Invariant connections and ∇-Einstein structures on isotropy irreducible spaces

Joint with Ioannis Chrysikos and Christian O'Cadiz Gustad

In this paper, we consider a class of compact homogeneous spaces M which satisfy the algebraic property that their isotropy representation is (strongly) irreducible, and such that for the description $M = G/H$, (G, H) is not a symmetric pair. On these spaces the representation theory of the isotropy algebra completely controls the existence of invariant geometric objects. We use this to completely classify $G-$invariant affine- and metric connections on M, and determine how many of them satisfy the ∇-Einstein property, which is equivalent to having skew-symmetric torsion in this case.

0.4.5 Reconstruction from Representations: Jacobi via Cohomology

Joint with B. S. Kruglikov

In this paper, we develop the cohomology theory necessary to extract linear equations from, and otherwise simplify, the equations coming from the Jacobi identity when computing Lie algebra extensions of modules. These tools are applied them to several cases, and were also used by us explicitly in [6], and implicitly in [2].

0.4.6 Non-degenerate Para-Complex structures in 6D with Large Symmetry Groups

Joint with B. S. Kruglikov

In this paper, we resolve the sub-maximal problem for a special type of almost product structures, called non-degenerate para-complex structures, in dimension 6. These consist of two transversal tangent distributions of equal rank, which is 3 in our case, such that the curvature tensor of each distribution is surjective. We describe both all global maximal models and global sub-maximal models as homogeneous spaces. It turns out that these structures share analogous properties to those of almost complex structures with non-degenerate Nijenhuis tensor, for instance both the maximal and sub-maximal models turn out to be strictly nearly para-Kähler.