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Some new Hardy-type inequalities for Riemann-Liouville fractional q-integral operator

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Abstract

We consider the q-analog of the Riemann-Liouville fractional q-integral operator of order $n \in N$. Some new Hardy-type inequalities for this operator are proved and discussed.

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1 Introduction

In 1910 FH Jackson defined q-derivative and definite q-integral [1] (see also [2]). It was the starting point of q-analysis. Today the interest in the subject has exploded. The q-analysis has numerous applications in various fields of mathematics, *e.g.*, dynamical systems, number theory, combinatorics, special functions, fractals and also for scientific problems in some applied areas such as computer science, quantum mechanics and quantum physics (see, *e.g.*, [3–7]). For further development and recent results in q-analysis, we refer to the books [2, 3] and [5] and the references given therein. The first results concerning integral inequalities in q-analysis were proved in 2004 by Gauchman [8]. Later on some further q-analogs of the classical inequalities have been proved (see [9–12]). Moreover, in 2014 Maligranda *et al.* [13] derived a q-analog of the classical Hardy inequality. Further development of Hardy's original inequality from 1925 (see [14] and [15]) has been enormous. Some of the most important results and applications have been presented and discussed in the books [16, 17] and [18]. Hence, it seems to be a huge new research area to investigate which of these so-called Hardy-type inequalities have their q-analogs.

The aim of this paper is to obtain some q-analogs of Hardy-type inequalities for the Riemann-Liouville fractional integral operator of order $n \in \mathbf{N}$ and to find necessary and sufficient conditions of the validity of these inequalities for all non-negative real functions (see Theorems 3.1 and 3.2).

The paper is organized as follows. In order not to disturb our discussions later on, some preliminaries are presented in Section 2. The main results can be found in Section 3, while the detailed proofs are given in Section 4.

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2 Preliminaries

First we recall some definitions and notations in q-analysis from the recent books [2, 3] and [5].

Let $q \in (0, 1)$. Then a q-real number $[\alpha]_q$ is defined by

$$[\alpha]_q := \frac{1 - q^\alpha}{1 - q}, \quad \alpha \in \mathrm{R},$$

where $\lim_{q \to 1} \frac{1 - q^\alpha}{1 - q} = \alpha$.

The q-analog of the binomial coefficients is defined by

$$[n]_q! := \begin{cases} 1 & \text{if } n = 0, \\ [1]_q \times [2]_q \times \cdots \times [n]_q & \text{if } n \in N, \end{cases} \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[n-k]_q![k]_q!}.$$

We introduce the q-analog of a polynomial in the following way:

$$(x - a)_q^n := \begin{cases} 1 & \text{if } n = 0, \\ (x - a)(x - qa) \cdots (x - q^{n-1}a) & \text{if } n \in N, \end{cases} \tag{1}$$

$$(x - a)_q^{n+m} = (x - a)_q^m (x - q^m a)_q^n, \quad n, m = 0, 1, 2, \ldots. \tag{2}$$

The q-gamma function Γ_q is defined by

$$\Gamma_q(n + 1) := [n]_q!, \quad n \in N.$$

For $f : [0, b) \to R$, $0 < b \leq \infty$, we define the q-derivative as follows:

$$D_q f(x) := \frac{f(x) - f(qx)}{(1 - q)x}, \quad x \in [0, b).$$

Clearly, if the function $f(x)$ is differentiable at a point $x \in (0, 1)$, then $\lim_{q \to 1} D_q f(x) = f'(x)$.

Let $0 < a \leq b < \infty$. The definite q-integral (also called the q-Jackson integral) of a function $f(x)$ is defined by the formulas

$$\int_0^a f(x) \, d_q x := (1 - q)a \sum_{k=0}^\infty q^k f(q^k a). \tag{3}$$

Moreover, the improper q-integral of a function $f(x)$ is defined by

$$\int_0^\infty f(x) \, d_q x := (1 - q) \sum_{k=-\infty}^\infty q^k f(q^k), \tag{4}$$

provided that the series on the right-hand sides of (3) and (4) converge absolutely.

Suppose that $f(x)$ and $g(x)$ are two functions which are defined on $(0, \infty)$. Then

$$\int_0^\infty f(x) D_q(g(x)) = \sum_{j=0}^\infty f(q^j)(g(q^j) - g(q^{j+1})). \tag{5}$$

Let Ω be a subset of $(0, \infty)$ and $\mathcal{X}_\Omega(t)$ denote the characteristic function of Ω. For all z: $0 < z < \infty$, we have that

$$\int_0^\infty \mathcal{X}_{(0,z]}(t) f(t) \, d_q t = (1-q) \sum_{q^i \le z} q^i f(q^i) \tag{6}$$

and

$$\int_0^\infty \mathcal{X}_{[z,\infty)}(t) f(t) \, d_q t = (1-q) \sum_{q^i \ge z} q^i f(q^i). \tag{7}$$

Al-Salam (see [19] and also [3]) introduced the fractional q-integral of the Riemann-Liouville operator $\mathcal{I}_{q,n}$ of order $n \in \mathbf{N}$ by

$$\mathcal{I}_{q,n} f(x) := \frac{1}{\Gamma_q(n)} \int_0^x K_{n-1}(x,s) f(s) \, d_q s,$$

where $K_{n-1}(x,s) = (x - qs)_q^{n-1}$.

Next we will present a lemma (Lemma 2.1) concerning discrete Hardy-type inequalities which are proved in [20]. In this paper all authors studied inequalities of the form

$$\left(\sum_{j=1}^\infty u_j^r \left((S_n f)_j \right)^r \right)^{\frac{1}{r}} \le C \left(\sum_{i=1}^\infty v_i^p f_i^p \right)^{\frac{1}{p}}, \quad \forall f \ge 0 \tag{8}$$

for the n-multiple discrete Hardy operator with weights of the form

$$(S_n f)_j = \sum_{k_1=j}^\infty \omega_{1,k_1} \sum_{k_2=1}^{k_1} \omega_{2,k_2} \sum_{k_3=1}^{k_2} \omega_{3,k_3} \cdots \sum_{k_{n-1}=1}^{k_{n-2}} \omega_{n-1,k_{n-1}} \sum_{i=1}^{k_{n-1}} f_i = \sum_{i=j}^\infty A_{n-1}(i,j) f_i,$$

where $u = \{u_i\}_{i=1}^\infty$, $v = \{v_i\}_{i=1}^\infty$, $\omega_i = \{\omega_{i,k}\}_{k=1}^\infty$ are positive sequences of real numbers (*i.e.*, weight sequences). She also studied inequality (8) for the operator S_n^* defined by

$$\left(S_n^* f \right)_i := \sum_{j=1}^i f_i A_{n-1,1}(i,j),$$

which is the conjugate to the operator S_n, where $A_{n-1,1}(i,j) \equiv 1$ for $n = 1$ and

$$A_{n-1,1}(i,j) = \sum_{k_{n-1}=j}^i \omega_{n-1,k_{n-1}} \sum_{k_{n-2}=k_{n-1}}^i \omega_{n-2,k_{n-2}} \cdots \sum_{k_1=k_2}^i \omega_{1,k_1}$$

for $n \ge 2$.

We consider the following Hardy-type inequalities:

$$\left(\sum_{j=-\infty}^\infty u_j^r \left((S_n f)_j \right)^r \right)^{\frac{1}{r}} \le C \left(\sum_{i=-\infty}^\infty v_i^p f_i^p \right)^{\frac{1}{p}} \tag{9}$$

and

$$\left(\sum_{i=-\infty}^\infty u_i^r \left((S_n^* f)_i \right)^r \right)^{\frac{1}{r}} \le C^* \left(\sum_{i=-\infty}^\infty v_i^p f_i^p \right)^{\frac{1}{p}}. \tag{10}$$

In the sequel, for any $p > 1$, the conjugate number p' is defined by $p' := p/(p-1)$, and the considered functions are assumed to be non-negative. Moreover, the symbol $M \ll K$ means that there exists $\alpha > 0$ such that $M \le \alpha K$, where α is a constant which may depend only on parameters such as p, q, r. Similarly, the case $K \ll M$. If $M \ll K \ll M$, then we write $M \approx K$.

Lemma 2.1

(i) *Let $1 < p \le r < \infty$ and $n \ge 1$. Then inequality* (9) *holds if and only if*
$\mathfrak{A}(n) = \max_{0 \le m \le n-1} \mathfrak{A}_m(n) < \infty$, *where*

$$\mathfrak{A}_m(n) = \sup_{k \in \mathbb{Z}} \left(\sum_{j=k}^{\infty} A_{m,1}^{p'}(j,k) v_j^{-p'} \right)^{\frac{1}{p'}} \left(\sum_{i=-\infty}^{k} A_{n-1,m+1}^{r}(k,i) u_i^{r} \right)^{\frac{1}{r}}, \quad n \in \mathbb{N}.$$

Moreover, $\mathfrak{A}(n) \approx C$, where C is the best constant in (9).

(ii) *Let $1 < p \le r < \infty$ and $n \ge 1$. Then inequality* (10) *holds if and only if*
$\mathfrak{A}^*(n) = \max_{0 \le m \le n-1} \mathfrak{A}^*_m(n) < \infty$, *where*

$$\mathfrak{A}^*_m(n) = \sup_{k \in \mathbb{Z}} \left(\sum_{i=k}^{\infty} A_{m,1}^{r}(i,k) u_i^{r} \right)^{\frac{1}{r}} \left(\sum_{j=-\infty}^{k} A_{n-1,m+1}^{p'}(k,j) v_j^{-p'} \right)^{\frac{1}{p'}}, \quad n \in \mathbb{N}.$$

Moreover, $\mathfrak{A}^*(n) \approx C$, where C is the best constant in (10).

We also need the corresponding result for the case $1 < r < p < \infty$, which was proved in [21].

Lemma 2.2

(i) *Let $1 < r < p < \infty$ and $n \ge 1$. Then inequality* (9) *holds if and only if*
$\mathfrak{B}(n) = \max_{0 \le m \le n-1} \mathfrak{B}_m(n) < \infty$, *where*

$$\mathfrak{B}_m(n) = \left\{ \sum_{i=-\infty}^{\infty} \left(\sum_{j=i}^{\infty} A_{m,1}^{p'}(j,i) v_j^{-p'} \right)^{\frac{p(r-1)}{p-r}} \left(\sum_{k=-\infty}^{i} A_{n-1,m+1}^{r}(i,k) u_k^{r} \right)^{\frac{p}{p-r}} \right.$$

$$\left. \times \triangle^+ \left(\sum_{j=i}^{\infty} A_{m,1}^{p'}(j,i) v_j^{-p'} \right) \right\}^{\frac{p-r}{pr}}, \quad \triangle^+ E_{i,j} = E_{i,j} - E_{i,j+1}, n \in \mathbb{N}.$$

Moreover, $\mathfrak{B}(n) \approx C$, where C is the best constant in (9).

(ii) *Let $1 < r < p < \infty$ and $n \ge 1$. Then inequality* (10) *holds if and only if*
$\mathfrak{B}^*(n) = \max_{0 \le m \le n-1} \mathfrak{B}^*_m(n) < \infty$, *where*

$$\mathfrak{B}^*_m(n) = \left\{ \sum_{i=-\infty}^{\infty} \left(\sum_{j=i}^{\infty} A_{m,1}^{r}(j,i) u_j^{r} \right)^{\frac{r}{p-r}} \left(\sum_{k=-\infty}^{i} A_{n-1,m+1}^{p'}(i,k) v_k^{-p'} \right)^{\frac{r(p-1)}{p-r}} \right.$$

$$\left. \times \triangle^+ \left(\sum_{j=i}^{\infty} A_{m,1}^{r}(j,i) u_j^{r} \right) \right\}^{\frac{p-r}{pr}}, \quad \triangle^+ E_{i,j} = E_{i,j} - E_{i,j+1}, \forall n \in \mathbb{N}.$$

Moreover, $\mathfrak{B}^*(n) \approx C^*$, where C^* is the best constant in (10).

Let $(a_{i,j}^{(n)})$ be a matrix whose elements are non-negative and non-increasing in the second index for all i, j: $\infty > i \geq j > -\infty$, and the entries of the matrix $a_{i,j}^{(n)}$ satisfy the following (so-called discrete Oinarov condition):

$$a_{i,j}^{(n)} \approx \sum_{\gamma=0}^{n} a_{i,k}^{(\gamma)} d_{k,j}^{n,\gamma}, \quad \gamma = 1, 2, \ldots, n-1, n \in \mathbb{N} \tag{11}$$

for all $\infty > i \geq k \geq j > -\infty$.

Remark 2.3 Note that the matrices $(d_{k,j}^{\gamma,m})$, $\gamma = 0, 1, \ldots, m$, $m \geq 0$, are arbitrary non-negative matrices which satisfy (11) (see [22]).

Moreover, in [22] necessary and sufficient conditions for inequalities (9) and (10) were proved for matrix operators with a matrix $(a_{i,j}^{(n)})$ which satisfies (11). For our purposes we need such characterization on the following form.

Lemma 2.4

(i) *Let $1 < p \leq r < \infty$ and the entries of the matrix $(a_{i,j}^{(n)})$ satisfy condition (11). Then inequality (9) for the operator $(\mathcal{A}^- f)_j := \sum_{i=j}^{\infty} a_{i,j}^{(n)} f_i$, $j \in Z$, holds if and only if at least one of the conditions $\mathfrak{B}^+ < \infty$ or $\mathfrak{B}^- < \infty$ holds, where*

$$\mathfrak{B}^- = \sup_{k \in Z} \left(\sum_{i=k}^{\infty} v_i^{-p'} \left(\sum_{j=-\infty}^{k} (a_{i,j}^{(n)})^r u_j^r \right)^{\frac{p'}{r}} \right)^{\frac{1}{p'}},$$

$$\mathfrak{B}^+ = \sup_{k \in Z} \left(\sum_{j=-\infty}^{k} u_j^r \left(\sum_{i=k}^{\infty} (a_{i,j}^{(n)})^{p'} v_i^{-p'} \right)^{\frac{r}{p'}} \right)^{\frac{1}{r}}.$$

Moreover, $\mathfrak{B}^+ \approx \mathfrak{B}^- \approx C$, where C is the best constant in (9).

(ii) *Let $1 < p \leq r < \infty$. Let the entries of the matrix $(a_{i,j}^{(n)})$ satisfy condition (11). Then inequality (10) for the operator $(\mathcal{A}^+ f)_i := \sum_{j=-\infty}^{i} a_{i,j}^{(n)} f_j$, $i \in Z$, holds if and only if at least one of the conditions $\mathfrak{A}^+ < \infty$ or $\mathfrak{A}^- < \infty$ holds, where*

$$\mathfrak{A}^- = \sup_{k \in Z} \left(\sum_{i=k}^{\infty} u_i^r \left(\sum_{j=-\infty}^{k} (a_{i,j}^{(n)})^{p'} v_j^{-p'} \right)^{\frac{r}{p'}} \right)^{\frac{1}{r}},$$

$$\mathfrak{A}^+ = \sup_{k \in Z} \left(\sum_{j=-\infty}^{k} v_j^{-p'} \left(\sum_{i=k}^{\infty} (a_{i,j}^{(n)})^r u_i^r \right)^{\frac{p'}{r}} \right)^{\frac{1}{p'}}.$$

Moreover, $\mathfrak{A}^+ \approx \mathfrak{A}^- \approx C$, where C is the best constant in (10).

3 The main results

Let $1 < r, p \leq \infty$. Then the q-analog of the two-weighted inequality for the operator $\mathcal{I}_{q,n}$ of the form

$$\left(\int_0^\infty u^r(x) \left(\mathcal{I}_{q,n} f(x) \right)^r d_q x \right)^{\frac{1}{r}} \leq C \left(\int_0^\infty v^p(x) f^p(x) d_q x \right)^{\frac{1}{p}} \tag{12}$$

has several applications in various fields of science. In the classical analysis two-weighted estimates for the Riemann-Liouville fractional operator were derived by Stepanov for the case with parameters greater than one (see [23, 24]).

We consider the operator $I_{q,n}$ of the following form:

$$I_{q,n}f(x) = \frac{1}{\Gamma_q(n)} \int_0^\infty \mathcal{X}_{(0,x]}(s) K_{n-1}(x,s) f(s) \, d_q s,$$

which is defined for all $x > 0$. Although it does not coincide with the operator $\mathcal{I}_{q,n}$ (they coincide at the points $x = q^k$, $k \in \mathbb{Z}$), we have the equality

$$\int_0^\infty u^r(x) \left(\mathcal{I}_{q,n}f(x) \right)^r d_q x = \int_0^\infty u^r(x) \left(I_{q,n}f(x) \right)^r d_q x.$$

Therefore, inequality (12) can be rewritten as

$$\left(\int_0^\infty u^r(x) \left(I_{q,n}f(x) \right)^r d_q x \right)^{\frac{1}{r}} \le C \left(\int_0^\infty v^p(x) f^p(x) \, d_q x \right)^{\frac{1}{p}}. \tag{13}$$

Its conjugate operator $I_{q,n}^*$ can be defined by

$$I_{q,n}^* f(s) := \frac{1}{\Gamma_q(n)} \int_0^\infty \mathcal{X}_{[s,\infty)}(x) K_{n-1}(x,s) f(x) \, d_q x,$$

with the same kernel. The dual inequality of inequality (13) reads as follows:

$$\left(\int_0^\infty u^r(x) \left(I_{q,n}^* f(x) \right)^r d_q x \right)^{\frac{1}{r}} \le C^* \left(\int_0^\infty v^p(x) f^p(x) \, d_q x \right)^{\frac{1}{p}}, \tag{14}$$

where C and C^* are positive constants independent of f and $u(\cdot)$, $v(\cdot)$ are positive real-valued functions on $(0, \infty)$, *i.e.*, weight functions. In what follows we investigate inequalities (13) and (14).

Let $\mathbf{N_0} = \mathbf{N} \cup \{0\}$. Then, for $0 \le m \le n-1$, $m, n \in \mathbf{N_0}$, we use the following notations:

$$\begin{aligned}
Q_m^{n-1} = \Bigg\{ & \int_0^\infty \left(\int_0^\infty \mathcal{X}_{(0,z]}(s) K_m^{p'}(z,s) v^{-p'}(s) \, d_q s \right)^{\frac{p(r-1)}{p-r}} \\
& \times \left(\int_0^\infty \mathcal{X}_{[z,\infty)}(x) K_{n-m-1}^r(x,z) u^r(x) \, d_q x \right)^{\frac{p}{p-r}} \\
& \times D_q \left(\int_0^\infty \mathcal{X}_{(0,z]}(s) K_m^{p'}(z,s) v^{-p'}(s) \, d_q s \right) \Bigg\}^{\frac{p-r}{pr}},
\end{aligned}$$

$$\begin{aligned}
\mathcal{Q}_m^{n-1} = \Bigg\{ & \int_0^\infty \left(\int_0^\infty \mathcal{X}_{(0,z]}(s) K_m^r(z,s) u^r(s) \, d_q s \right)^{\frac{r}{p-r}} \\
& \times \left(\int_0^\infty \mathcal{X}_{[z,\infty)}(x) K_{n-m-1}^{-p'}(z,x) v^{-p'}(x) \, d_q x \right)^{\frac{r(p-1)}{p-r}} \\
& \times D_q \left(\int_0^\infty \mathcal{X}_{(0,z]}(s) K_m^r(z,s) u^r(x) \, d_q s \right) \Bigg\}^{\frac{p-r}{pr}},
\end{aligned}$$

$$H_m^{n-1} = \sup_{z>0} \left(\int_0^\infty \mathcal{X}_{[z,\infty)}(x) K_{n-m-1}^r(x,z) u^r(x)\, d_q x \right)^{\frac{1}{r}} \left(\int_0^\infty \mathcal{X}_{(0,z]}(s) K_m^{p'}(z,s) v^{-p'}(s)\, d_q s \right)^{\frac{1}{p'}},$$

$$\mathcal{H}_m^{n-1} = \sup_{z>0} \left(\int_0^\infty \mathcal{X}_{(0,z]}(x) K_m^r(z,x) u^r(x)\, d_q x \right)^{\frac{1}{r}} \left(\int_0^\infty \mathcal{X}_{[z,\infty)}(s) K_{n-m-1}^{p'}(s,z) v^{-p'}(s)\, d_q s \right)^{\frac{1}{p'}},$$

$$A^+(z) = \left(\int_0^\infty \mathcal{X}_{[z,\infty)}(x) u^r(x) \left(\int_0^\infty \mathcal{X}_{(0,z]}(t) K_{n-1}^{p'}(x,t) v^{-p'}(t)\, d_q t \right)^{\frac{r}{p'}} d_q x \right)^{\frac{1}{r}},$$

$$A^-(z) = \left(\int_0^\infty \mathcal{X}_{(0,z]}(t) v^{-p'}(t) \left(\int_0^\infty \mathcal{X}_{[z,\infty)}(x) K_{n-1}^r(x,t) u^r(x)\, d_q x \right)^{\frac{p'}{r}} d_q t \right)^{\frac{1}{p'}}.$$

$$\mathcal{A}^+(z) = \left(\int_0^\infty \mathcal{X}_{[z,\infty)}(t) v^{-p'}(t) \left(\int_0^\infty \mathcal{X}_{(0,z]}(x) K_{n-1}^r(t,x) u^r(x)\, d_q x \right)^{\frac{p'}{r}} d_q t \right)^{\frac{1}{p'}},$$

$$\mathcal{A}^-(z) = \left(\int_0^\infty \mathcal{X}_{(0,z]}(x) u^r(x) \left(\int_0^\infty \mathcal{X}_{[z,\infty)}(t) K_{n-1}^{p'}(t,x) v^{-p'}(t)\, d_q t \right)^{\frac{r}{p'}} d_q x \right)^{\frac{1}{r}},$$

$$H_{n-1} = \max_{0 \le k \le n-1} H_k^{n-1}, \qquad \mathcal{H}_{n-1} = \max_{0 \le k \le n-1} \mathcal{H}_k^{n-1},$$

$$A_q^+ = \sup_{z>0} A^+(z), \qquad A_q^- = \sup_{z>0} A^-(z), \qquad \mathcal{A}_q^+ = \sup_{z>0} \mathcal{A}^+(z), \qquad \mathcal{A}_q^- = \sup_{z>0} \mathcal{A}^-(z),$$

$$Q_{n-1} = \max_{0 \le k \le n-1} Q_k^{n-1} \quad \text{and} \quad \mathcal{Q}_{n-1} = \max_{0 \le k \le n-1} \mathcal{Q}_k^{n-1}.$$

Our main results read as follows.

Theorem 3.1

  (i) *Let $1 < r < p < \infty$. Then inequality (13) holds if and only if $Q_{n-1} < \infty$. Moreover, $Q_{n-1} \approx C$, where C is the best constant in (13).*

  (ii) *Let $1 < p \le r < \infty$. Then inequality (13) holds if and only if at least one of the conditions $H_{n-1} < \infty$ or $A_q^+ < \infty$ or $A_q^- < \infty$ holds. Moreover, $H_{n-1} \approx A_q^+ \approx A_q^- \approx C$, where C is the best constant in (13).*

Theorem 3.2

  (i) *Let $1 < r < p < \infty$. Then inequality (14) holds if and only if $\mathcal{Q}_{n-1} < \infty$. Moreover, $\mathcal{Q}_{n-1} \approx C^*$, where C^* is the best constant in (14).*

  (ii) *Let $1 < p \le r < \infty$. Then inequality (14) holds if and only if at least one of the conditions $\mathcal{H}_{n-1} < \infty$ or $\mathcal{A}_q^+ < \infty$, or $\mathcal{A}_q^- < \infty$ holds. Moreover, $\mathcal{H}_{n-1} \approx \mathcal{A}_q^+ \approx \mathcal{A}_q^- \approx C$, where C is the best constant in (14).*

For the proofs of these results, we need the following lemmata of independent interest.

Lemma 3.3 *Let x, t, s: $0 < s \le t \le x < \infty$. Then*

$$\max_{0 \le m \le n-1} K_{n-m-1}(x,t) K_m(t,s) \le K_{n-1}(x,s) \quad \le \sum_{m=0}^{n-1} \begin{bmatrix} n-1 \\ m \end{bmatrix}_q K_{n-m-1}(x,t) K_m(t,s) \tag{15}$$

for m: $0 \le m \le n-1$, $n, m-1 \in \mathbb{N}$ and where $K_{n-1}(x,s) = (x - qs)_q^{n-1}$.

Lemma 3.4 *Let f and g be non-negative functions on $(0, \infty)$, $\alpha, \beta \in \mathbb{R}$ and*

$$I(z) := \left(\int_0^\infty \mathcal{X}_{(0,z]}(t) f(t)\, d_q t \right)^\alpha \left(\int_0^\infty \mathcal{X}_{[z,\infty)}(x) g(x)\, d_q x \right)^\beta.$$

Then

$$\sup_{z>0} I(z) = (1-q)^{\alpha+\beta} \sup_{k\in\mathbb{Z}} \left(\sum_{j=k}^{\infty} q^j f(q^j) \right)^{\alpha} \left(\sum_{i=-\infty}^{k} q^i g(q^i) \right)^{\beta}, \tag{16}$$

where at least one of α and β is non-negative.

This result was proved in [25], but for the readers' convenience we will include in Section 4 a proof which is slightly simpler than that in the Russian version given in [25].

Lemma 3.5 *Let $\alpha, \beta \in \mathbb{R}^+$, $K(\cdot, \cdot)$ be a non-negative function and*

$$\mathcal{I}^+(z) := \left(\int_0^\infty \mathcal{X}_{[z,\infty)}(x) g(x) \left(\int_0^\infty \mathcal{X}_{(0,z]}(t) K(x,t) f(t) \, d_q t \right)^\alpha d_q x \right)^\beta,$$

$$\mathcal{I}^-(z) := \left(\int_0^\infty \mathcal{X}_{(0,z]}(t) f(t) \left(\int_0^\infty \mathcal{X}_{[z,\infty)}(x) K(x,t) g(x) \, d_q x \right)^\alpha d_q t \right)^\beta.$$

Then

$$\sup_{z>0} \mathcal{I}^+(z) = \sup_{k\in\mathbb{Z}} \left((1-q) \sum_{j=-\infty}^{k} q^j g(q^j) \left((1-q) \sum_{i=k}^{\infty} q^i K(q^j, q^i) f(q^i) \right)^\alpha \right)^\beta \tag{17}$$

and

$$\sup_{z>0} \mathcal{I}^-(z) = \sup_{k\in\mathbb{Z}} \left((1-q) \sum_{j=k}^{\infty} q^j f(q^j) \left((1-q) \sum_{j=-\infty}^{k} q^i K(q^j, q^i) g(q^i) \right)^\alpha \right)^\beta. \tag{18}$$

Lemma 3.6 *Let $Q_m^{n-1}, \mathcal{Q}_m^{n-1} < \infty$ for $0 < m \leq n-1$. Then*

$$Q_m^{n-1} = \left\{ \sum_{i=-\infty}^{\infty} \left((1-q) \sum_{t=i}^{\infty} q^t K_m^{p'}(q^i, q^t) v^{-p'}(q^t) \right)^{\frac{p(r-1)}{p-r}} \right.$$

$$\times \left((1-q) \sum_{j=-\infty}^{i} q^j K_{n-m-1}^r(q^j, q^i) u^r(q^j) \right)^{\frac{p}{p-r}}$$

$$\times \left. \triangle^+ \left(\sum_{n=i}^{\infty} (1-q) q^n K_m^r(q^i, q^n) v^{-p'}(q^n) \right) \right\}^{\frac{p-r}{pr}}$$

and

$$\mathcal{Q}_m^{n-1} = \left\{ \sum_{i=-\infty}^{\infty} \left((1-q) \sum_{t=i}^{\infty} q^t K_m^r(q^i, q^t) u^r(q^t) \right)^{\frac{r}{p-r}} \right.$$

$$\times \left((1-q) \sum_{j=-\infty}^{i} q^j K_{n-m-1}^{p'}(q^j, q^i) v^{-p'}(q^j) \right)^{\frac{r(p-1)}{p-r}}$$

$$\times \left. \triangle^+ \left(\sum_{n=i}^{\infty} (1-q) q^n K_m^r(q^i, q^n) u^r(q^n) \right) \right\}^{\frac{p-r}{pr}},$$

where $\triangle^+ E_{n,i} = E_{n,i} - E_{n,i+1}$.

4 Proofs

Proof of Lemma 3.3 Let $0 < s \leq t \leq x < \infty$. First we prove the lower estimate. By using (2) we find that

$$
\begin{aligned}
K_{n-m-1}(x,t)K_m(t,s) &= (x - qt)_q^{n-m-1}(t - qs)_q^m \\
&\leq (x - qs)_q^{n-m-1}(x - qs)_q^m \\
&\leq (x - qs)_q^{n-m-1}\left(x - q^{n-m}s\right)_q^m \\
&= (x - qs)_q^{n-1} = K_{n-1}(x,s)
\end{aligned}
$$

for $0 < s \leq t \leq x < \infty$ and $0 \leq m \leq n - 1$, $m - 1, n \in \mathbb{N}$. Hence,

$$
\max_{0 \leq m \leq n-1} K_{n-m-1}(x,t)K_m(t,s) \leq K_{n-1}(x,s),
$$

and the lower estimate in (15) is proved.

According to (1) we get that $K_0(x,t)K_0(t,s) = K_0(x,s) \equiv 1$ for $n = 1$. Moreover, we have that

$$
\begin{aligned}
K_1(x,s) &= (x - qs)_q < (x - qt)_q + (t - qs)_q \\
&= \sum_{m=0}^{1} \begin{bmatrix} 1 \\ m \end{bmatrix}_q K_{2-m-1}(x,t)K_m(t,s)
\end{aligned}
$$

for $n = 2$.

This means that the inequality

$$
K_{n-2}(x,s) < \sum_{m=0}^{n-2} \begin{bmatrix} n - 2 \\ m \end{bmatrix}_q K_{n-m-2}(x,t)K_m(t,s) \tag{19}
$$

holds for $n = 3$. Our aim is now to use induction, and we assume that (19) holds for $n = l - 1$, $l \geq 3$, and we will prove that it then holds also for $n = l$.

We use our induction assumption, make some calculations and obvious estimates and find that

$$
\begin{aligned}
K_{l-1}(x,s) &= K_{l-2}(x,s)\left(x - q^{l-1}s\right) \\
&< \left(\sum_{m=0}^{l-2} \begin{bmatrix} l - 2 \\ m \end{bmatrix}_q K_{l-m-2}(x,t)K_m(t,s)\right)\left(x - q^{l-1}s\right) \\
&< \sum_{m=0}^{l-2} \begin{bmatrix} l - 2 \\ m \end{bmatrix}_q K_{l-m-2}(x,t)K_m(t,s)\left(x - q^{l-m-1}t + q^{l-m-2}t - q^{l-1}s\right) \\
&= \sum_{m=0}^{l-2} \begin{bmatrix} l - 2 \\ m \end{bmatrix}_q K_{l-m-2}(x,t)K_m(t,s)\left(x - q^{l-m-1}t\right) \\
&\quad + \sum_{m=0}^{l-2} \begin{bmatrix} l - 2 \\ m \end{bmatrix}_q K_{l-m-2}(x,t)K_m(t,s)q^{l-m-2}\left(t - q^{m+1}s\right)
\end{aligned}
$$

$$= \begin{bmatrix} l-2 \\ 0 \end{bmatrix}_q K_{l-1}(x,t)K_0(t,s) + \sum_{m=1}^{l-2} \begin{bmatrix} l-2 \\ m \end{bmatrix}_q K_{l-m-1}(x,t)K_m(t,s)$$

$$+ \sum_{m=1}^{l-2} q^{l-m-2} \begin{bmatrix} l-2 \\ m-1 \end{bmatrix}_q K_{l-m-1}(x,t)K_m(t,s) + \begin{bmatrix} l-2 \\ l-2 \end{bmatrix}_q K_0(x,t)K_{l-1}(t,s)$$

$$= \begin{bmatrix} l-1 \\ 0 \end{bmatrix}_q K_{l-1}(x,t)K_0(t,s)$$

$$+ \sum_{m=1}^{l-2} \left(q^{l-m-2} \begin{bmatrix} l-2 \\ m-1 \end{bmatrix}_q + \begin{bmatrix} l-2 \\ m \end{bmatrix}_q \right) K_{l-m-1}(x,t)K_m(t,s)$$

$$+ \begin{bmatrix} l-1 \\ l-1 \end{bmatrix}_q K_0(x,t)K_l(t,s).$$

Since, for any $m \geq 1$ ($q^{l-m-2} \begin{bmatrix} l-2 \\ m-1 \end{bmatrix}_q + \begin{bmatrix} l-2 \\ m \end{bmatrix}_q = \begin{bmatrix} l-1 \\ m \end{bmatrix}_q$), we get that

$$K_{l-1}(x,s) < \sum_{m=0}^{l-1} \begin{bmatrix} l-1 \\ m \end{bmatrix}_q K_{l-m-1}(x,t)K_m(t,s).$$

Hence, (19) holds also with $n = l$ which, by the induction axiom, means that also the upper estimate in (15) is proved. The proof is complete. $\qquad\square$

Proof of Lemma 3.4 From (6) and (7) it follows that

$$I(z) = (1-q)^{\alpha+\beta} \left(\sum_{q^j \leq z} q^j f(q^j) \right)^{\alpha} \left(\sum_{q^i \geq z} q^i g(q^i) \right)^{\beta}.$$

If $z = q^k$, then, for $k \in \mathbb{Z}$,

$$I(z) = (1-q)^{\alpha+\beta} \left(\sum_{j=k}^{\infty} q^j f(q^j) \right)^{\alpha} \left(\sum_{i=-\infty}^{k} q^i g(q^i) \right)^{\beta}.$$

If $q^k < z < q^{k-1}$, then, for $k \in \mathbb{Z}$,

$$I(z) = (1-q)^{\alpha+\beta} \left(\sum_{j=k}^{\infty} q^j f(q^j) \right)^{\alpha} \left(\sum_{i=-\infty}^{k-1} q^i g(q^i) \right)^{\beta}.$$

Hence, for $k \in \mathbb{Z}$ and $\beta > 0$, we find that

$$\sup_{q^k \leq z < q^{k-1}} I(z) = (1-q)^{\alpha+\beta} \left(\sum_{j=k}^{\infty} q^j f(q^j) \right)^{\alpha} \left(\sum_{i=-\infty}^{k} q^i g(q^i) \right)^{\beta}.$$

Therefore

$$\sup_{z>0} I(z) = \sup_{k \in \mathbb{Z}} \sup_{q^k \leq z < q^{k-1}} I(z)$$

$$= (1-q)^{\alpha+\beta} \sup_{k \in \mathbb{Z}} \left(\sum_{j=k}^{\infty} q^j f(q^j) \right)^{\alpha} \left(\sum_{i=-\infty}^{k} q^i g(q^i) \right)^{\beta}. \tag{20}$$

We have proved that (16) holds wherever $\beta > 0$.

Next we assume that $\alpha > 0$. Let $q^{k+1} < z < q^k$, $k \in \mathbb{Z}$. Then we get that

$$I(z) = (1-q)^{\alpha+\beta} \sup_{k \in \mathbb{Z}} \left(\sum_{j=k+1}^{\infty} q^j f(q^j) \right)^{\alpha} \left(\sum_{i=-\infty}^{k} q^i g(q^i) \right)^{\beta},$$

and analogously as above we find that

$$\sup_{q^{k+1} < z \leq q^k} I(z) = (1-q)^{\alpha+\beta} \left(\sum_{j=k}^{\infty} q^j f(q^j) \right)^{\alpha} \left(\sum_{i=-\infty}^{k} q^i g(q^i) \right)^{\beta},$$

and (16) holds also for the case $\alpha > 0$. The proof is complete. □

Proof of Lemma 3.5 Let $z = q^k$, $k \in \mathbb{Z}$. By using (6) and (7) we have that

$$\mathcal{I}^+(z) = \left((1-q) \sum_{j=-\infty}^{k} q^j g(q^j) \left((1-q) \sum_{i=k}^{\infty} q^i K(q^j,q^i) f(q^i) \right)^{\alpha} \right)^{\beta}.$$

For the cases $q^{k+1} < z < q^k$, $k \in Z$ and $q^k < z < q^{k-1}$, $k \in Z$, we find that

$$\mathcal{I}^+(z) = \left((1-q) \sum_{j=-\infty}^{k} q^j g(q^j) \left((1-q) \sum_{i=k+1}^{\infty} q^i K(q^j,q^i) f(q^i) \right)^{\alpha} \right)^{\beta}$$

and

$$\mathcal{I}^+(z) = \left((1-q) \sum_{j=-\infty}^{k-1} q^j g(q^j) \left((1-q) \sum_{i=k}^{\infty} q^i K(q^j,q^i) f(q^i) \right)^{\alpha} \right)^{\beta},$$

respectively.

Hence, we conclude that

$$\sup_{q^{k+1} < z < q^{k-1}} \mathcal{I}^+(z) = \left((1-q) \sum_{j=-\infty}^{k} q^j g(q^j) \left((1-q) \sum_{i=k}^{\infty} q^i K(q^j,q^i) f(q^i) \right)^{\alpha} \right)^{\beta}.$$

Since $\sup_{z>0} \mathcal{I}^+(z) = \sup_{k \in Z} \sup_{q^{k+1} < z < q^{k-1}} \mathcal{I}^+(z)$, we find that (17) holds. The identity (18) can be proved in a similar way as (17). The proof is complete. □

Proof of Lemma 3.6 Without loss of generality we may assume that $Q_m^{n-1} < \infty$. By using (5), (6) and (7) we can deduce that

$$Q_m^{n-1} = \left\{ \sum_{i=-\infty}^{\infty} \left(\int_0^{\infty} \mathcal{X}_{(0,q^i]}(s) K_m^{p'}(q^i,s) v^{-p'}(s) \, d_q s \right)^{\frac{p(r-1)}{p-r}} \right.$$

$$\times \left(\int_0^{\infty} \mathcal{X}_{[q^i,\infty)}(x) K_{n-m-1}^r(x,q^i) u^r(x) \, d_q x \right)^{\frac{p}{p-r}}$$

$$\times \left(\int_0^{\infty} \mathcal{X}_{(0,q^i]}(s) K_m^{p'}(q^i,s) v^{-p'}(s) \, d_q s \right.$$

$$- \int_0^\infty \mathcal{X}_{(0,q^{i+1}]}(s) K_m^{p'}(q^{i+1},s) v^{-p'}(s)\, d_q s \Bigg) \Bigg\}^{\frac{p-r}{pr}}$$

$$= \Bigg\{ \sum_{i=-\infty}^{\infty} \Bigg((1-q) \sum_{t=i}^{\infty} q^t K_m^{p'}(q^i,q^t) v^{-p'}(q^t) \Bigg)^{\frac{p(r-1)}{p-r}}$$

$$\times \Bigg((1-q) \sum_{j=-\infty}^{i} q^j K_{n-m-1}^r(q^j,q^i) u^r(q^j) \Bigg)^{\frac{p}{p-r}}$$

$$\times \triangle^+ \Bigg(\sum_{n=i}^{\infty} (1-q) q^n K_m^r(q^i,q^n) v^{-p'}(q^n) \Bigg) \Bigg\}^{\frac{p-r}{pr}},$$

and the first equality in Lemma 3.6 is proved.

The second inequality can be proved in a similar way, so we leave out the details. The proof is complete. □

Proof of Theorem 3.1　By using formulas (4) and (6) we find that inequality (13) can be rewritten as

$$\Bigg(\sum_{j=-\infty}^{\infty} (1-q)^{r+1} q^j u^r(q^j) \Bigg(\sum_{i=j}^{\infty} q^i f(q^i) K_{n-1}(q^j,q^i) \Bigg)^r \Bigg)^{\frac{1}{r}}$$

$$\leq C \Bigg(\sum_{i=-\infty}^{\infty} (1-q) q^i f^p(q^i) v^p(q^i) \Bigg)^{\frac{1}{p}}. \tag{21}$$

Let

$$u_j^r = (1-q)^{r+1} q^j u^r(q^j), \qquad f_i = q^i f(q^i),$$

$$v_i^p = (1-q) q^{i(1-p)} v^p(q^i), \qquad W^{(n)}(i,j) = K_{n-1}(q^j,q^i). \tag{22}$$

Then we get that inequality (21) can be rewritten as the discrete weighted Hardy-type inequality (see, *e.g.*, [26])

$$\Bigg(\sum_{j=-\infty}^{\infty} u_j^r \Bigg(\sum_{i=j}^{\infty} W^{(n)}(i,j) f_i \Bigg)^r \Bigg)^{\frac{1}{r}} \leq C \Bigg(\sum_{i=-\infty}^{\infty} v_i^p a_i^p \Bigg)^{\frac{1}{p}}. \tag{23}$$

Hence, inequality (13) is equivalent to inequality (23), where $(W^{(n)}(i,j))$ is the non-negative triangular matrix which has entries $W^{(n)}(i,j) \geq 0$ for $j \leq i$ and $W^{(n)}(i,j) \equiv 0$ for $j > i$ and is non-decreasing in the first index for all $i \geq j > -\infty$.

First we will prove that, for $n \in N$,

$$(1-q)^{n-1} \sum_{k_{n-1}=j}^{i} [n-1]_q q^{k_{n-1}} \sum_{k_{n-2}=k_{n-1}}^{i} [n-2]_q q^{k_{n-2}} \cdots \sum_{k_1=k_2}^{i} [1]_q q^{k_1} = W^{(n)}(i,j). \tag{24}$$

We will use induction and first we note that $W^{(0)}(i,j) = (q^j - q^i)_q^0 \equiv 1$ for $n = 1$. If $n = 2$, then

$$(1-q) \sum_{k_1=j}^{i} q^{k_1} = \sum_{k_1=j}^{i} [1]_q \big(q^{k_1} - q^{k_1+1} \big) = \big(q^j - q^{i+1} \big)_q^1 = W^{(1)}(i,j).$$

Assume now that formula (24) holds for $n - 1 \in N$, *i.e.*, that

$$(1 - q)^{n-2} \sum_{k_{n-2}=j}^{i} [n - 2]_q q^{k_{n-2}} \sum_{k_{n-3}=k_{n-2}}^{i} [n - 3]_q q^{k_{n-3}} \cdots \sum_{k_1=k_2}^{i} [1]_q q^{k_1} = W^{(n-1)}(i, j).$$

By using this induction assumption we find that

$$(1 - q)^{n-1} \sum_{k_{n-1}=j}^{i} [n - 1]_q q^{k_{n-1}} \sum_{k_{n-2}=k_{n-1}}^{i} [n - 2]_q q^{k_{n-2}} \cdots \sum_{k_1=k_2}^{i} [1]_q q^{k_1}$$

$$= (1 - q)[n - 1]_q \sum_{k_{n-1}=j}^{i} q^{k_{n-1}} W^{(n-1)}(i, j)$$

$$= (1 - q^{n-1}) \sum_{k_{n-1}=j}^{i} q^{k_{n-1}} (q^{k_{n-1}} - q^{i+1})_q^{n-2}.$$

Since $(q^{k_{n-1}} - q^{i+1})_q^{n-1} - (q^{k_{n-1}+1} - q^{i+1})_q^{n-1} = (1 - q^{n-1}) q^{k_{n-1}} (q^{k_{n-1}} - q^{i+1})_q^{n-2}$, we get that (24) holds also for n. Hence, by the induction axiom, we conclude that (24) holds for each $n \in N$.

Let $w_{m,k} = [m]_q (q^{k_m} - q^{k_m+1})$, $m = 1, 2, 3, \ldots, n - 1$. Then, by using (24), we have that

$$W^{(n)}(i, j) = \sum_{k_{n-1}=j}^{i} w_{n-1,k_{n-1}} \sum_{k_{n-2}=k_{n-1}}^{i} w_{n-2,k_{n-2}} \cdots \sum_{k_1=k_2}^{i} w_{1,k_1}. \tag{25}$$

Therefore, we see that the matrix operator in (23), defined by

$$(\widetilde{S}f)_j := \sum_{i=j}^{\infty} W^{(n)}(i, j) f_i, \quad j \in Z,$$

is an n-multiple discrete Hardy operator with weights (see (25)).

Therefore, Lemma 2.1 and Lemma 2.2 can be used.

(i) Let $1 < r < p < \infty$. Then, based on Lemma 2.2, it follows that inequality (23) holds if and only if $\widetilde{Q}_{n-1} = \max_{0 \leq m \leq n-1} \widetilde{Q}_m^{n-1} < \infty$, where

$$\widetilde{Q}_m^{n-1} = \left\{ \sum_{i=-\infty}^{\infty} \left(\sum_{j=i}^{\infty} (W^{(m+1)}(j, i))^{p'} v_j^{-p'} \right)^{\frac{p(r-1)}{p-r}} \right.$$

$$\times \left(\sum_{k=-\infty}^{i} (W^{(n-m)}(i, k))^{r} u_k^{r} \right)^{\frac{p}{p-r}}$$

$$\left. \times \triangle^{+} \left(\sum_{j=i}^{\infty} (W^{(m+1)})^{p'} (j, i) v_j^{-p'} \right) \right\}^{\frac{p-r}{pr}}.$$

Since inequality (23) is equivalent to inequality (13), we conclude that the condition $\widetilde{Q}_{n-1} < \infty$ is a necessary and sufficient condition for the validity of inequality (13). Moreover, $\widetilde{Q}_{n-1} \approx C$.

By using the definitions (22) in $\widetilde{Q}_m^{n-1}$, we get that

$$\widetilde{Q}_m^{n-1} = \left\{ \sum_{i=-\infty}^{\infty} \left((1-q) \sum_{j=i}^{\infty} q^j K_m^{p'}(q^i, q^j) v^{-p'}(q^j) \right)^{\frac{p(r-1)}{p-r}} \right.$$

$$\times \left((1-q) \sum_{k=-\infty}^{i} q^k K_{n-m-1}^r(q^k, q^i) u^r(q^k) \right)^{\frac{p}{p-r}}$$

$$\left. \times \triangle^+ \left(\sum_{j=i}^{\infty} (1-q) q^j K_m^{p'}(q^i, q^j) v^{-p'}(q^j) \right) \right\}^{\frac{p-r}{pr}}.$$

By using Lemma 3.6, we find that

$$\widetilde{Q}_m^{n-1} = \left\{ \int_0^{\infty} \left(\int_0^{\infty} \mathcal{X}_{(0;z]}(s) K_m^{p'}(z,s) v^{-p'}(s) \, d_q s \right)^{\frac{p(r-1)}{p-r}} \right.$$

$$\times \left(\int_0^{\infty} \mathcal{X}_{[z,\infty)}(x) K_{n-m-1}^r(x,z) u^r(x) \, d_q x \right)^{\frac{p}{p-r}}$$

$$\left. \times D_q \left(\int_0^{\infty} \mathcal{X}_{(0;z]}(s) K_m^{p'}(z,s) v^{-p'}(s) \, d_q s \right) \right\}^{\frac{p-r}{pr}},$$

i.e., that $\widetilde{Q}_{n-1} = Q_{n-1}$. Then we find that inequality (13) holds if and only if $Q_{n-1} < \infty$. Moreover, $Q_{n-1} \approx C$, where C is the best constant in (13). Thus the proof of the statement (i) of Theorem 3.1 is complete.

(ii) Let $1 < p \le r < \infty$. Then from Lemma 2.1 it follows that inequality (23) holds if and only if $\widetilde{H}_{n-1} = \max_{0 \le m \le n-1} \widetilde{H}_m^{n-1} < \infty$ holds, where

$$\widetilde{H}_m^{n-1} = \sup_{k \in \mathbb{Z}} \left(\sum_{i=k}^{\infty} \left(W^{(m+1)}(i,k) \right)^{p'} v_i^{-p'} \right)^{\frac{1}{p'}}$$

$$\times \left(\sum_{j=-\infty}^{k} \left(W^{(n-m)}(k,j) \right)^r u_j^r \right)^{\frac{1}{r}}, \quad n \in \mathbb{N}.$$

If $x = q^j$, $s = q^i$, $t = q^k$, for $j \le k \le i$, then, by Lemma 3.3 and (22), we obtain that (recall that $W^{(n)}(i,j) = K_{n-1}(q^j, q^i)$)

$$W^{(n)}(i,j) \le \sum_{m=0}^{n-1} \begin{bmatrix} n-1 \\ m \end{bmatrix}_q W^{(n-m)}(k,j) W^{(m+1)}(i,k). \tag{26}$$

Since, again by Lemma 3.3, $\max_{0 \le m \le n-1} W^{(m+1)}(i,k) W^{(n-m)}(k,j) \le W^{(n)}(i,j)$, it follows that

$$W^{(n)}(i,j) \ge h(n) \sum_{m=0}^{n-1} \begin{bmatrix} n-1 \\ m \end{bmatrix}_q W^{(m+1)}(i,k) W^{(n-m)}(k,j), \tag{27}$$

where $h(n) = \left(\sum_{m=0}^{n-1} \begin{bmatrix} n-1 \\ m \end{bmatrix}_q \right)^{-1}$.

According to (26) and (27) we have that

$$W^{(n)}(i,j) \approx \sum_{m=0}^{n-1} W^{(m)}(i,k)\mathcal{W}^{(n,m)}(k,j),$$

where $\mathcal{W}^{(n,m)}(k,j) = \begin{bmatrix} n-1 \\ m \end{bmatrix}_q W^{(n-m-1)}(k,j)$.

Therefore, we have proved that the matrix $(W^{(n)}(i,j))$ in (23) satisfies the Oinarov condition (11) and Lemma 2.4 can be used.

Hence, we have the following necessary and sufficient conditions for the validity of inequality (23):

$$\widetilde{A}^+ = \sup_{k\in Z}\left(\sum_{j=-\infty}^{k} u_j^r \left(\sum_{i=k}^{\infty}\left(W^{(n)}(i,j)\right)^{p'}v_i^{-p'}\right)^{\frac{r}{p'}}\right)^{\frac{1}{r}} < \infty$$

or

$$\widetilde{A}^- = \sup_{k\in Z}\left(\sum_{i=k}^{\infty} v_i^{-p'} \left(\sum_{j=-\infty}^{k}\left(W^{(n)}(i,j)\right)^{r}u_j^{r}\right)^{\frac{p'}{r}}\right)^{\frac{1}{p'}} < \infty.$$

Since, again by Lemma 3.3, $\max_{0\le m\le n-1} W^{(m+1)}(i,k)W^{(n-m)}(k,j) \le W^{(n)}(i,j)$, we get that

$$\widetilde{H}_{n-1} = \sup_{k\in Z}\left(\sum_{j=-\infty}^{k} u_j \left(\sum_{i=k}^{\infty}\left(\max_{0\le m\le n-1}\left(W^{(n-m)}(k,j)W^{(m+1)}(i,k)\right)\right)^{p'}v_i^{-p'}\right)^{\frac{r}{p'}}\right)^{\frac{1}{r}}$$

$$\le \left(\sum_{j=-\infty}^{k} u_j \left(\sum_{i=k}^{\infty}\left(W^{(n)}(i,j)\right)^{p'}v_i^{-p'}\right)^{\frac{r}{p'}}\right)^{\frac{1}{r}} = \widetilde{A}^+.$$

Moreover, by (26) we have that

$$\widetilde{A}^+ \le \sup_{k\in Z}\left(\sum_{j=-\infty}^{k} u_j \left(\sum_{i=k}^{\infty}\left(\sum_{m=0}^{n-1}\begin{bmatrix} n-1 \\ m \end{bmatrix}_q W^{(n-m)}(k,j)W^{(m+1)}(i,k)\right)^{p'}v_i^{-p'}\right)^{\frac{r}{p'}}\right)^{\frac{1}{r}}$$

$$\le \sum_{m=0}^{n-1}\begin{bmatrix} n-1 \\ m \end{bmatrix}_q \widetilde{H}_{n-1} \ll \widetilde{H}_{n-1}.$$

Hence, $\widetilde{A}^+ \approx \widetilde{H}_{n-1}$. In a similar way it can be proved that $\widetilde{H}_{n-1} \approx \widetilde{A}^-$.

Since inequality (13) is equivalent to inequality (23), we get that inequality (13) holds if and only if at least one of the conditions $\widetilde{A}^+ < \infty$ or $\widetilde{A}^- < \infty$, or $\widetilde{H}_{n-1} < \infty$ holds.

Now, using notations (22) in $\widetilde{H}_m^{n-1}$, we obtain that

$$\widetilde{H}_m^{n-1} = (1-q)^{\frac{1}{r}+\frac{1}{p'}} \sup_{k\in\mathbb{Z}}\left(\sum_{j=-\infty}^{k} q^j u^r(q^j)K_{n-m-1}^r(q^j,q^k)\right)^{\frac{1}{r}}$$

$$\times \left(\sum_{i=k}^{\infty} q^t v^{-p'}(q^i)K_m^{p'}(q^k,q^i)\right)^{\frac{1}{p'}}.$$

The parameters and functions in $\widetilde{H}_m^{n-1}$ satisfy all the conditions of Lemma 3.4. Therefore, we find that

$$
\widetilde{H}_m^{n-1} = \sup_{z>0} \left(\int_0^\infty \mathcal{X}_{[z,\infty)}(x) K_{n-m-1}^r(x,z) u^r(x) \, d_q x \right)^{\frac{1}{r}}
$$
$$
\times \left(\int_0^\infty \mathcal{X}_{(0,z]}(s) K_m^{p'}(z,s) v^{-p'}(s) \, d_q s \right)^{\frac{1}{p'}},
$$

i.e., that $\widetilde{H}_m^{n-1} = H_m^{n-1}$ and $\widetilde{H}_{n-1} = H_{n-1} = \max_{0 \le m \le n-1} H_m^{n-1} < \infty$.

In a similar way as above, by using Lemma 3.5 and (22), we get that $A_q^+ = \widetilde{A}^+$ and $A_q^- = \widetilde{A}^-$.

Hence, we obtain that inequality (13) holds if and only if at least one of the conditions $H_{n-1} < \infty$ or $A_q^+ < \infty$, or $A_q^- < \infty$ holds. Moreover, $A_q^+ \approx A_q^- \approx H_{n-1} \approx C$, where C is the best constant in (13). Also the proof of the statement (ii) of Theorem 3.1 is complete. □

Proof of Theorem 3.2 In a similar way as in the proof of Theorem 3.1, by using (4), (7) and (22), we can prove that we have the following discrete Hardy-type inequality:

$$
\left(\sum_{i=-\infty}^\infty u_i^r \left(\sum_{j=-\infty}^i W_{n-1,1}^{(n)}(i,j) f_i \right)^r \right)^{\frac{1}{r}} \le C \left(\sum_{i=-\infty}^\infty v_i^p a_i^p \right)^{\frac{1}{p}}, \tag{28}
$$

which is equivalent to inequality (14).

(i) Let $1 < r < p < \infty$. By using Lemma 2.2 and Lemma 3.6, we can in a similar way as in the proof of Theorem 3.1(i) derive that inequality (14) holds if and only if $\mathcal{Q}_{n-1} < \infty$ holds. Moreover, $\mathcal{Q}_{n-1} \approx C^*$, where C^* is the best constant in (14). The proof of part (i) is complete.

(ii) Let $1 < p \le r < \infty$. By using Lemma 2.4, Lemma 2.1, Lemma 3.4 and Lemma 3.5, we can, analogously as in the proof of the (ii)-part, prove that inequality (14) holds if and only if at least one of the conditions $\mathcal{H}_{n-1} < \infty$ or $\mathcal{A}_q^+ < \infty$, or $\mathcal{A}_q^- < \infty$ holds. Moreover, $\mathcal{H}_{n-1} \approx \mathcal{A}_q^- \approx \mathcal{A}_q^+ \approx C^*$, where C^* is the best constant in (14). The proof of part (ii) is complete. □

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