Underactuated Waypoint Tracking of a Fixed-Wing UAV ⋆

Espen Oland Rune Schlanbusch Raymond Kristiansen

Narvik University College, Department of Technology, 8505 Narvik, Norway (e-mail: {eol},{runsch},{rayk},@hin.no)

Abstract: In this paper a new method of performing waypoint tracking is shown for underactuated fixed-wing UAVs. The position error can be mapped onto the desired axis using a desired rotation matrix, while the velocity error can be mapped to the desired axis using a desired angular velocity. With all errors defined along one axis, the tracking problem is easily solved using only one thruster. A velocity controller is derived which makes sure that the UAV tracks a desired total velocity moving towards the next waypoint, while a sliding surface attitude controller is designed to track the desired attitude. The impact of saturation on the attitude controller is also studied where it is shown that the actuators will desaturate in finite time, through a change in the reference trajectory. Using both controllers, a solution to the problem of waypoint tracking of an underactuated UAV is proposed, and simulations have been performed that support the theoretical results.

Keywords: Actuator Constraints, Aircraft Control, Attitude Control, Autonomous Control, Lyapunov Stability, Nonlinear Control, Sliding Surfaces.

1. INTRODUCTION

A fixed-wing unmanned aerial vehicle (UAV) has six degrees of freedom, and four actuators: thrust and three deflection angles. With fewer actuators than degrees of freedom this constitutes an underactuated control problem (*cf.* Reyhanoglu et al. (1999)). Generally it is only possible to track as many outputs as available control inputs, making two of the states unactuated. A mapping is therefore required to map the errors from the unactuated states onto the actuated ones. This is the principle of a guidance system that generates reference signals that can be tracked by the controller making the unactuated states go to zero. One guidance method is line-of-sight (LOS) guidance, where the vector between the vehicle and a waypoint can be represented by a pitch and yaw angle as well as a distance. This enables the attitude controller to track the desired attitude defined by the LOS-vector, while the thrust can be used to make the distance go to zero. Hence, only three actuators are required to reach any point in $\mathbb{R}^3$. This has made waypoint guidance very popular for underactuated rigid bodies such as e.g. fixed-wing UAVs, autonomous underwater vehicles (AUVs), ships, quadrotors and spacecraft (*cf.* Aguiar and Pascoal (2002), Børhaug and Pettersen (2005), Breivik and Fossen (2005), Fossen et al. (2003), Lee et al. (2010), Roberts and Tayebi (2009) and references therein).

The main focus in the literature is to parameterize the attitude using Euler angles, and to define the desired pitch and yaw angles to solve the waypoint tracking problem. By working directly on an angular level, the properties of the rotation matrix that are exploited in this paper, may be lost. Instead of using Euler angles, it is possible to define a desired rotation matrix such that the position errors become mapped onto one axis. Similarly, a desired angular velocity vector can be designed to map the velocity errors onto the same axis. An attitude controller can then be used to track the desired attitude, and by moving with a positive velocity, the vehicle will eventually reach its desired waypoint.

Roberts and Tayebi (2009) derived a desired rotation matrix by looking at the desired force vector in an inertial frame relative the constrained thrust in the body frame. By properly designing the rotation matrix using quaternions, they were able to solve the trajectory tracking problem for a vertical take-off and landing (VTOL) UAV. However, the method does have some critical issues; the thrust must be nonzero and it requires either an observer or the derivative of the force vector in order to find the desired angular velocity, resulting in very complex equations where the derivative of the aerodynamics and other forces must be taken into account.

Inspired by the results of Roberts and Tayebi (2009) we derive a waypoint guidance scheme which is applied to an underactuated fixed-wing UAV. Instead of defining the rotation matrix at the acceleration level, we define it at the position error level providing us with simple equations for calculating the desired rotation matrix and the desired angular velocity vector. A sliding surface controller is then derived to track the desired attitude, and a velocity controller is designed to make the UAV move with a desired velocity to the next waypoint.

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2. MODELING

2.1 Notation

In this paper the time derivative of a vector is denoted as $\dot{\mathbf{x}} = d\mathbf{x}/dt$ and the Euclidean length is written as $||\mathbf{x}|| = (\mathbf{x}^\top \mathbf{x})^{\frac{1}{2}}$. Superscript denotes the reference frame of the current vector, where n denotes the North East Down (NED) frame and b denotes the body frame. The NED frame, assumed to be inertial, has its $\mathbf{x}^n$-axis aligned towards the North, $\mathbf{y}^n$ is pointing East and $\mathbf{z}^n$ is pointing towards the center of the Earth. The body frame is fixed to the rigid body, where $\mathbf{x}^b$ is pointing along the fuselage of the body, $\mathbf{y}^b$ is pointing through the right wing, while $\mathbf{z}^b$ is pointing downwards. The rotation matrix is denoted as $\mathbf{R}_a^c \in SO(3)$ which rotates a vector from frame a to frame c, where its transpose $(\mathbf{R}_a^c)^\top = \mathbf{R}_c^a$, such that $(\mathbf{R}_c^a)^\top \mathbf{R}_c^a = \mathbf{R}_a^c \mathbf{R}_c^a = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix. The angular velocity vector is denoted $\boldsymbol{\omega}_{a,e}^c$ which represents the angular velocity of frame e relative frame a referenced in frame c, and angular velocities between different frames can be added as $\boldsymbol{\omega}_{a,d}^e = \boldsymbol{\omega}_{a,c}^e + \boldsymbol{\omega}_{c,d}^e$. The time derivative of the rotation matrix is found as $\dot{\mathbf{R}}_a^c = \mathbf{R}_a^c \mathbf{S}(\boldsymbol{\omega}_{c,a}^a)$ where the cross-product operator $\mathbf{S}(\cdot)$ is such that for two arbitrary vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^3$ we have that $\mathbf{S}(\mathbf{v}_1)\mathbf{v}_2 = \mathbf{v}_1 \times \mathbf{v}_2$. The cross-product operator also holds the properties that $\mathbf{S}(\mathbf{v}_1)\mathbf{v}_2 = -\mathbf{S}(\mathbf{v}_2)\mathbf{v}_1$, $\mathbf{S}(\mathbf{v}_1)\mathbf{v}_1 = \mathbf{0}$ and that $\mathbf{v}_1^\top \mathbf{S}(\mathbf{v}_2)\mathbf{v}_1 = 0$. Given $\mathbf{v}_1 = [v_1 \ v_2 \ v_3]^\top$, the cross-product operator is defined as

$$\mathbf{S}(\mathbf{v}_1) := \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}. \tag{1}$$

2.2 Translational Kinematics and Dynamics

The translational kinematics is defined as (*cf.* Stevens and Lewis (2003))

$$\dot{\mathbf{p}}^n = \mathbf{R}_b^n \mathbf{v}^b \tag{2}$$

$$\mathbf{v}_r^b = \mathbf{v}^b - \mathbf{R}_n^b \mathbf{v}_{wind}^n \tag{3}$$

where $\mathbf{v}^b$ is the velocity vector of the center of mass, $\mathbf{v}_r^b := [u \ v \ w]^\top$ is the velocity vector of the body ($\mathbf{v}^b$) relative the wind vector ($\mathbf{v}_{wind}^n$). The rotation matrix from body to wind is defined as

$$\mathbf{R}_b^w := \begin{bmatrix} \cos(\alpha)\cos(\beta) & \sin(\beta) & \sin(\alpha)\cos(\beta) \\ -\cos(\alpha)\sin(\beta) & \cos(\beta) & -\sin(\alpha)\sin(\beta) \\ -\sin(\alpha) & 0 & \cos(\alpha) \end{bmatrix} \tag{4}$$

where the angle of attack is defined as

$$\alpha := \tan^{-1}\left(\frac{w}{u}\right) \tag{5}$$

and the sideslip angle as

$$\beta := \sin^{-1}\left(\frac{w}{V_T}\right) \tag{6}$$

where V_T is the total velocity defined as

$$V_T := ||\mathbf{v}_r^b|| = \sqrt{(\mathbf{v}_r^b)^\top \mathbf{v}_r^b}. \tag{7}$$

The relative velocity vector in the wind frame is now found through the rotation

$$\mathbf{v}_r^w = \mathbf{R}_b^w \mathbf{v}_r^b = \begin{bmatrix} V_T \\ 0 \\ 0 \end{bmatrix} \tag{8}$$

which aligns the total velocity along the $\mathbf{x}^w$ axis.

Assuming that the wind velocity is constant or slowly varying the relative acceleration of the body frame is found using Newton's Second Law as

$$\dot{\mathbf{v}}_r^b = \frac{1}{m}\mathbf{f}_{thrust}^b + \frac{1}{m}\mathbf{R}_w^b \mathbf{f}_{aero}^w + \mathbf{R}_n^b \mathbf{f}_g^n - \mathbf{S}(\boldsymbol{\omega}_{n,b}^b)\mathbf{v}_r^b \tag{9}$$

where m is the mass, $\mathbf{f}_{thrust}^b = [T \ 0 \ 0]^\top$ is the thrust vector with T as the total thrust, $\mathbf{f}_{aero}^w$ is the aerodynamic force vector, $\mathbf{f}_g^n = [0 \ 0 \ g]^\top$ is the gravity vector where g is the gravitational acceleration and $\boldsymbol{\omega}_{n,b}^b$ is the angular velocity between the body and NED frame. The aerodynamic force vector can be defined as

$$\mathbf{f}_{aero}^w = \frac{1}{2}\rho S V_T^2 \begin{bmatrix} -(C_{D_0} + kC_L^2) \\ C_{Y_\beta}\beta \\ -(C_{L_0} + C_{L_\alpha}\alpha) \end{bmatrix} \tag{10}$$

where ρ is the air density, S is the wing area, $C_{(\cdot)}$ are aerodynamic coefficients, $C_L = C_{L_0} + C_{L_\alpha}\alpha$ and k is a constant scalar value dependent on the aircraft configuration.

The total velocity can be differentiated as

$$\dot{V}_T = \frac{1}{V_T}(\mathbf{v}_r^b)^\top \dot{\mathbf{v}}_r^b \tag{11}$$

and by inserting (9) into (11) the total acceleration becomes

$$\dot{V}_T = \frac{1}{V_T}(\mathbf{v}_r^b)^\top \Big(\frac{1}{m}\mathbf{f}_{thrust}^b + \frac{1}{m}\mathbf{R}_w^b \mathbf{f}_{aero}^w + \mathbf{R}_n^b \mathbf{f}_g^n - \mathbf{S}(\boldsymbol{\omega}_{n,b}^b)\mathbf{v}_r^b\Big) \tag{12}$$

where the thrust can be extracted, and by using the property that $(\mathbf{v}_r^b)^\top \mathbf{S}(\boldsymbol{\omega}_{n,b}^b)\mathbf{v}_r^b = 0$ it results in

$$\dot{V}_T = \frac{uT}{mV_T} + \frac{1}{V_T}(\mathbf{v}_r^b)^\top \Big(\frac{1}{m}\mathbf{R}_w^b \mathbf{f}_{aero}^w + \mathbf{R}_n^b \mathbf{f}_g^n\Big). \tag{13}$$

Note that in order to produce lift, an aircraft must have a positive velocity meaning that both V_T and u must be positive.

Remark 1. Even though an aircraft is underactuated since there are no direct control of the v and w velocity components, it is still controllable by controlling the total velocity and pointing the velocity vector in a desired direction.

2.3 Rotational Kinematics and Dynamics

The orientation of a fixed-wing UAV can be parameterized by using a unit quaternion as

$$\mathbf{q}_{n,b} := \left[\eta_{n,b} \ \boldsymbol{\epsilon}_{n,b}^\top\right]^\top \in \mathcal{S}^3 = \left[\cos\left(\frac{\vartheta_{n,b}}{2}\right) \ \mathbf{k}_{n,b}^\top \sin\left(\frac{\vartheta_{n,b}}{2}\right)\right]^\top \tag{14}$$

where $\eta_{n,b}$ is the scalar part and $\boldsymbol{\epsilon}_{n,b} \in \mathbb{R}^3$ is the vector part, and the subscript of $\mathbf{q}_{n,b}$ defines the orientation to be the body frame relative the NED frame. The quaternion performs a rotation of an angle $\vartheta_{n,b}$ around the unit vector $\mathbf{k}_{n,b}$, and the rotation matrix between the body and NED frame can be constructed as

$$\mathbf{R}_b^n = \mathbf{I} + 2\eta_{n,b}\mathbf{S}(\boldsymbol{\epsilon}_{n,b}) + 2\mathbf{S}^2(\boldsymbol{\epsilon}_{n,b}), \tag{15}$$

and it holds the property $\mathbf{q}_{n,b}^\top \mathbf{q}_{n,b} = \eta_{n,b}^2 + \boldsymbol{\epsilon}_{n,b}^\top \boldsymbol{\epsilon}_{n,b} = 1$. The rotational kinematics of the quaternion is given as (*cf.* Egeland and Gravdahl (2002))

$$\dot{\mathbf{q}}_{n,b} = \frac{1}{2}\mathbf{q}_{n,b} \otimes \begin{bmatrix} 0 \\ \boldsymbol{\omega}^b_{n,b} \end{bmatrix} = \frac{1}{2}\mathbf{T}(\mathbf{q}_{n,b}) \begin{bmatrix} 0 \\ \boldsymbol{\omega}^b_{n,b} \end{bmatrix} \tag{16}$$

where $\otimes$ denote the quaternion product and where

$$\mathbf{T}(\mathbf{q}_{n,b}) := \begin{bmatrix} \eta_{n,b} & -\boldsymbol{\epsilon}^\top_{n,b} \\ \boldsymbol{\epsilon}_{n,b} & \eta_{n,b}\mathbf{I} + \mathbf{S}(\boldsymbol{\epsilon}_{n,b}) \end{bmatrix}. \tag{17}$$

The angular dynamics of a fixed-wing UAV can be written as

$$\mathbf{J}\dot{\boldsymbol{\omega}}^b_{n,b} = -\mathbf{S}(\boldsymbol{\omega}^b_{n,b})\mathbf{J}\boldsymbol{\omega}^b_{n,b} + \boldsymbol{\tau}^b_{aero} \tag{18}$$

where

$$\mathbf{J} = \begin{bmatrix} J_{xx} & 0 & -J_{xz} \\ 0 & J_{yy} & 0 \\ -J_{xz} & 0 & J_{zz} \end{bmatrix} \tag{19}$$

is the inertia matrix and $J_{(\cdot)}$ are positive constants. The aerodynamic moments can be defined as

$$\boldsymbol{\tau}^b_{aero} := \mathbf{f}(\alpha, \beta) - \mathbf{D}\boldsymbol{\omega}^b_{n,b} + \mathbf{B}\mathbf{u} \tag{20}$$

where $\mathbf{u} = [\delta_a \ \delta_e \ \delta_r]^\top$ is a vector consisting of the deflection angles which are used for control, and an aerodynamic moment vector function is defined as

$$\mathbf{f}(\alpha, \beta) := \frac{1}{2}\rho S V_T^2 \begin{bmatrix} b(C_{l_0} + C_{l_\beta}\beta) \\ \bar{c}(C_{m_0} + C_{m_\alpha}\alpha) \\ b(C_{n_0} + C_{n_\beta}\beta) \end{bmatrix} \tag{21}$$

$$\mathbf{D} := -\frac{1}{2}\rho S V_T^2 \begin{bmatrix} \frac{b^2}{2V_T}C_{l_p} & 0 & \frac{b^2}{2V_T}C_{l_r} \\ 0 & \frac{\bar{c}^2}{2V_T}C_{m_q} & 0 \\ \frac{b^2}{2V_T}C_{n_p} & 0 & \frac{b^2}{2V_T}C_{n_r} \end{bmatrix} \tag{22}$$

where $\mathbf{D}$ is a positive definite matrix, b represents the wing span, $\bar{c}$ is the mean aerodynamic chord, and the control effectiveness matrix is defined as

$$\mathbf{B} := \frac{1}{2}\rho S V_T^2 \begin{bmatrix} bC_{l_{\delta_a}} & 0 & bC_{l_{\delta_r}} \\ 0 & \bar{c}C_{m_{\delta_e}} & 0 \\ bC_{n_{\delta_a}} & 0 & bC_{n_{\delta_r}} \end{bmatrix}. \tag{23}$$

The angular acceleration can now be written by inserting (20) into (18) as

$$\mathbf{J}\dot{\boldsymbol{\omega}}^b_{n,b} = -\mathbf{S}(\boldsymbol{\omega}^b_{n,b})\mathbf{J}\boldsymbol{\omega}^b_{n,b} + \mathbf{f}(\alpha, \beta) - \mathbf{D}\boldsymbol{\omega}^b_{n,b} + \mathbf{B}\mathbf{u}. \tag{24}$$

The control objective is to point the wind frame in a desired direction which solves the way-pointing tracking problem. Let an error quaternion between a desired frame and the wind frame be defined as the composite rotation

$$\mathbf{q}_{d,w} = \mathbf{q}_{d,n} \otimes \mathbf{q}_{n,b} \otimes \mathbf{q}_{b,w} \tag{25}$$

which has two equilibria at $\mathbf{q}^\star_{d,w} = \begin{bmatrix} \pm 1 & \mathbf{0}^\top \end{bmatrix}^\top$ which represents the same physical orientation, but mathematically they are different. From a control perspective it is more intuitive to control relative the origin, and based on the work by Kristiansen (2008) let an error function be defined as

$$\mathbf{e}_{q\pm} := \begin{bmatrix} 1 \mp \eta_{d,w} \\ \boldsymbol{\epsilon}_{d,w} \end{bmatrix} \tag{26}$$

which has the kinematics as

$$\begin{aligned} \dot{\mathbf{e}}_{q\pm} &= \mathbf{T}_e(\mathbf{e}_{q\pm})\boldsymbol{\omega}^w_{d,w} \\ &= \mathbf{T}_e(\mathbf{e}_{q\pm})(\mathbf{R}^w_b\boldsymbol{\omega}^b_{n,b} - \mathbf{R}^w_d\boldsymbol{\omega}^d_{n,d} + \boldsymbol{\omega}^w_{b,w}) \end{aligned} \tag{27}$$

with

$$\mathbf{T}_e(\mathbf{e}_{q\pm}) := \frac{1}{2}\begin{bmatrix} \pm\boldsymbol{\epsilon}^\top_{d,w} \\ \eta_{d,w}\mathbf{I} + \mathbf{S}(\boldsymbol{\epsilon}_{d,w}) \end{bmatrix} \tag{28}$$

where $\boldsymbol{\omega}^d_{n,d}$ is the desired angular velocity relative the NED frame and $\boldsymbol{\omega}^w_{b,w}$ is the angular velocity between the wind and body frame.

Assumption 1.
It is assumed that $\text{sign}(\eta_{d,w}(t)) = \text{sign}(\eta_{d,w}(t_0)) \ \forall t$.

This assumption makes it possible to focus only on one of the two equilibria during controller derivation. Note that this assumption can be relaxed using for example Hybrid switching (*cf.* Schlanbusch et al. (2011)).

Lemma 1. Under assumption 1 it can be shown that the following inequality holds

$$\mathbf{e}^\top_{q\pm}\mathbf{T}_e(\mathbf{e}_{q\pm})\mathbf{T}^\top_e(\mathbf{e}_{q\pm})\mathbf{e}_{q\pm} \geq \frac{1}{8}\mathbf{e}^\top_{q\pm}\mathbf{e}_{q\pm} \tag{29}$$

Proof 1. The proof is given in Kristiansen (2008) and Schlanbusch (2012). □

2.4 Wind Frame

The rotation from body frame to wind frame can be written using quaternions as

$$\mathbf{q}_{b,w} = \mathbf{q}_{b,s} \otimes \mathbf{q}_{s,w} = \mathbf{T}(\mathbf{q}_{b,s})\mathbf{q}_{s,w} \tag{30}$$

where subscript s denotes the stability frame which is an intermediate frame, and the two quaternions are defined as

$$\mathbf{q}_{b,s} := \begin{bmatrix} \cos\left(\frac{\alpha}{2}\right) & 0 & -\sin\left(\frac{\alpha}{2}\right) & 0 \end{bmatrix}^\top \tag{31}$$

$$\mathbf{q}_{s,w} := \begin{bmatrix} \cos\left(\frac{\beta}{2}\right) & 0 & 0 & \sin\left(\frac{\beta}{2}\right) \end{bmatrix}^\top. \tag{32}$$

Higher order derivatives of the angle of attack and sideslip can be estimated using linear filters which is often done for rotational control of aircraft (*cf.* Farrell et al. (2005), Sonneveldt et al. (2009)). A linear filter can be chosen as (*cf.* Fossen (2012))

$$\dot{\mathbf{x}}_d := \mathbf{A}_d\mathbf{x}_d + \mathbf{B}_d r \tag{33}$$

with

$$\mathbf{A}_d = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\omega_n^3 & -(2\zeta+1)\omega_n^2 & -(2\zeta+1)\omega_n \end{bmatrix} \tag{34}$$

$$\mathbf{B}_d = \begin{bmatrix} 0 & 0 & \omega_n^3 \end{bmatrix}^\top \tag{35}$$

where ζ is the relative damping ratio, ω_n is the natural frequency and r is the reference signal being either the angle of attack or sideslip. The state vector can be chosen as $\mathbf{x}_d := [\alpha \ \dot{\alpha} \ \ddot{\alpha}]^\top$ in the case of estimating the angle of attack and similarly for the sideslip angle. With the higher order derivatives of the angle of attack and sideslip, the angular velocity of the wind frame relative the body frame is found as (*cf.* Stevens and Lewis (2003))

$$\begin{aligned} \boldsymbol{\omega}^w_{b,w} &= \begin{bmatrix} 0 \\ 0 \\ \dot{\beta} \end{bmatrix} + \begin{bmatrix} \cos(\beta) & \sin(\beta) & 0 \\ -\sin(\beta) & \cos(\beta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -\dot{\alpha} \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -\dot{\alpha}\sin(\beta) \\ -\dot{\alpha}\cos(\beta) \\ \dot{\beta} \end{bmatrix} \end{aligned} \tag{36}$$

and the angular acceleration is found through direct differentiation as

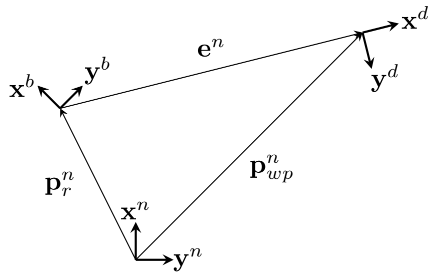

Fig. 1. Position vectors in the xy-plane relative

$$\dot{\boldsymbol{\omega}}_{b,w}^{w} = \begin{bmatrix} -\ddot{\alpha}\sin(\beta) - \dot{\alpha}\dot{\beta}\cos(\beta) \\ -\ddot{\alpha}\cos(\beta) + \dot{\alpha}\dot{\beta}\sin(\beta) \\ \ddot{\beta} \end{bmatrix}. \tag{37}$$

3. WAYPOINT GUIDANCE

To control the unactuated states, a guidance scheme is designed which maps the position error onto the desired attitude and the velocity error onto the desired angular velocity. The basic idea of this guidance scheme is to map the position and velocity errors onto the $\mathbf{x}^d$ axis, and then use the thrust to translate along this axis until the given waypoint is reached. In Fig. 1 the different reference frames and vectors are shown, where $\mathbf{p}_r^n$ is the position vector towards the body frame, $\mathbf{p}_{wp}^n$ represents the position of the waypoint and $\mathbf{e}^n$ is the error between the waypoint and the position of the UAV. To take the wind into account, the relative velocity can be defined in NED as $\dot{\mathbf{p}}_r^n := \mathbf{R}_b^n\mathbf{v}_r^b$. Let a tracking function be defined as

$$\mathbf{e}^d := \begin{bmatrix} ||\mathbf{p}_{wp}^n - \mathbf{p}_r^n|| \\ 0 \\ 0 \end{bmatrix} = \mathbf{R}_n^d(\mathbf{p}_{wp}^n - \mathbf{p}_r^n) \tag{38}$$

which projects the tracking error in NED frame onto the desired $\mathbf{x}^d$ axis. When the UAV comes close to a given waypoint, the guidance scheme switches to next waypoint ensuring that $||\mathbf{p}_{wp}^n - \mathbf{p}_r^n|| \geq d > 0\ \forall t$, where d is a constant. The rotation matrix between the NED frame and the desired frame can be constructed using quaternions. Let $\mathbf{e}^n := \mathbf{p}_{wp}^n - \mathbf{p}_r^n$ such that $\mathbf{e}^d = \mathbf{R}_n^d\mathbf{e}^n$. Then the angle of rotation can be found using the dot product while the axis of rotation can be found using the cross product as

$$\vartheta_{n,d} = \cos^{-1}\left(\frac{\mathbf{e}^d \cdot \mathbf{e}^n}{||\mathbf{e}^n||^2}\right) \tag{39}$$

$$\mathbf{k}_{n,d} = \frac{\mathbf{e}^d \times \mathbf{e}^n}{||\mathbf{e}^d \times \mathbf{e}^n||}. \tag{40}$$

The desired quaternion can now be constructed as

$$\begin{aligned} \mathbf{q}_{n,d} &= \begin{bmatrix} \eta_{n,d} & \boldsymbol{\epsilon}_{n,d}^\top \end{bmatrix}^\top \\ &= \begin{bmatrix} \cos\left(\frac{\vartheta_{n,d}}{2}\right) & \mathbf{k}_{n,d}^\top \sin\left(\frac{\vartheta_{n,d}}{2}\right) \end{bmatrix}^\top \end{aligned} \tag{41}$$

while the rotation matrix is found as

$$\mathbf{R}_d^n = \mathbf{I} + 2\eta_{n,d}\mathbf{S}(\boldsymbol{\epsilon}_{n,d}) + 2\mathbf{S}^2(\boldsymbol{\epsilon}_{n,d}). \tag{42}$$

To find the desired angular velocity, (38) can be differentiated resulting in

$$\dot{\mathbf{e}}^d = -\mathbf{S}(\boldsymbol{\omega}_{n,d}^d)\mathbf{e}^d - \mathbf{R}_b^d\mathbf{v}_r^b \tag{43}$$

where $\dot{\mathbf{p}}_r^n = \mathbf{R}_b^n\mathbf{v}_r^b$ has been used. The derivative of the tracking error in the desired frame is found as

$$\dot{\mathbf{e}}^d = \begin{bmatrix} \frac{d}{dt}||\mathbf{e}^n|| \\ 0 \\ 0 \end{bmatrix} \tag{44}$$

where by design, the velocity components along the $\mathbf{y}^d$ and $\mathbf{z}^d$ axes are zero. Equation (43) can now be solved with regards to the desired angular velocity by noting that $-\mathbf{S}(\boldsymbol{\omega}_{n,d}^d)\mathbf{e}^d = \mathbf{S}(\mathbf{e}^d)\boldsymbol{\omega}_{n,d}^d$ and that

$$\mathbf{S}(\mathbf{e}^d) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -||\mathbf{e}^n|| \\ 0 & ||\mathbf{e}^n|| & 0 \end{bmatrix}, \tag{45}$$

giving

$$\dot{\mathbf{e}}^d = \mathbf{S}(\mathbf{e}^d)\boldsymbol{\omega}_{n,d}^d - \mathbf{R}_b^d\mathbf{v}_r^b. \tag{46}$$

From (45) it is seen that it only has components along the $\mathbf{y}^d$ and $\mathbf{z}^d$ axes, meaning that when solving for the desired angular velocity, the $\mathbf{x}^d$ components can be ignored and therefore also $\dot{\mathbf{e}}^d$. This results in

$$\boldsymbol{\omega}_{n,d}^d = \mathbf{S}^\dagger(\mathbf{e}^d)\mathbf{R}_b^d\mathbf{v}_r^b \tag{47}$$

where $\dagger$ represents the pseudoinverse, and we obtain the desired pitch and yaw angular velocities required to align the total velocity in the desired direction. This is the result of projecting all the errors onto the $\mathbf{x}^d$ axis, which enables a simple expression for the desired angular velocity. Note that $\text{rank}(\mathbf{S}^\dagger(\mathbf{e}^d)) = 2$ such that it does not produce any roll motion, and the pseudoinverse must be used since $\mathbf{S}(\mathbf{e}^d)$ does not have an inverse. The desired angular acceleration in the desired frame can be found through differentiation of (47) or using for example linear filters (*cf.* Fossen (2012)). By following this desired attitude, angular velocity and acceleration, the tracking errors become mapped onto the $\mathbf{x}^d$ axis, and go to zero as the UAV moves between two waypoints with a positive total velocity.

4. TRANSLATIONAL CONTROL

The objective of the aircraft is to fly between two waypoints with a positive total velocity. Let a desired total velocity be a constant denoted as $V_{T,d} > 0$ and the velocity error defined as $\tilde{V} := V_T - V_{T,d}$, then a Lyapunov Function Candidate can be defined as

$$V_1 := \frac{1}{2}\tilde{V}^2, \tag{48}$$

and its derivative is found by inserting (13), resulting in

$$\begin{aligned} \dot{V}_1 =& \tilde{V}\Big(\frac{uT}{mV_T} + \frac{1}{V_T}(\mathbf{v}_r^b)^\top\Big(\frac{1}{m}\mathbf{R}_w^b\mathbf{f}_{aero}^w + \mathbf{R}_n^b\mathbf{f}_g^n\Big) \\ &- \dot{V}_{T,d}\Big). \end{aligned} \tag{49}$$

The thrust can now be chosen as

$$\begin{aligned} T =& \frac{mV_T}{u}\Big(\dot{V}_{T,d} - \frac{1}{V_T}(\mathbf{v}_r^b)^\top\Big(\frac{1}{m}\mathbf{R}_w^b\mathbf{f}_{aero}^w + \mathbf{R}_n^b\mathbf{f}_g^n\Big) \\ &- k_V\tilde{V}\Big) \end{aligned} \tag{50}$$

where $k_V > 0$ is a gain and by inserting the control law into (49) it results in

$$\dot{V}_1 = -k_V\tilde{V}^2. \tag{51}$$

The Lyapunov function (48) is positive definite, decrescent and radially unbounded, while its derivative (51) is negative definite. Hence by applying theorem 4.10 in Khalil (2002) it follows that the origin $\tilde{V} = 0$ is exponentially stable (ES). The actuator constraint of the thrust is not analyzed in this paper and is considered future work.

5. ROTATIONAL CONTROLLER

The attitude and angular velocity can be controlled using a sliding surface controller to make the UAV track its desired trajectory and follow its waypoints. Without loss of generality, let $\mathbf{e}_q := \mathbf{e}_{q+}$ and $\mathbf{T}_e := \mathbf{T}_e(\mathbf{e}_{q+})$ meaning that we focus on the positive equilibrium point during the controller derivation. Let a sliding surface variable be defined as (*cf.* Slotine and Li (1988))

$$\mathbf{s}^b := \boldsymbol{\omega}^b_{n,b} - \boldsymbol{\omega}^b_{n,r} \tag{52}$$

$$\boldsymbol{\omega}^b_{n,r} := \mathbf{R}^b_d\boldsymbol{\omega}^d_{n,d} - \mathbf{R}^b_w\boldsymbol{\omega}^w_{b,w} - \gamma\mathbf{R}^b_w\mathbf{T}_e^\top\mathbf{e}_q \tag{53}$$

where $\mathbf{s}^b$ is the sliding variable, $\boldsymbol{\omega}^b_{n,r}$ is reference angular velocity relative NED and γ is a positive gain. Premultiplying (52) by the inertia matrix, differentiating and inserting (24) and using that $\boldsymbol{\omega}^b_{n,b} = \mathbf{s}^b + \boldsymbol{\omega}^b_{n,r}$ we obtain

$$\mathbf{J}\dot{\mathbf{s}}^b = -\mathbf{S}(\boldsymbol{\omega}^b_{n,b})\mathbf{J}\boldsymbol{\omega}^b_{n,b} + \mathbf{f}(\alpha,\beta) - \mathbf{D}\mathbf{s}^b - \mathbf{D}\boldsymbol{\omega}^b_{n,r} + \mathbf{B}\mathbf{u} - \mathbf{J}\dot{\boldsymbol{\omega}}^b_{n,r} \tag{54}$$

where the reference angular acceleration becomes

$$\dot{\boldsymbol{\omega}}^b_{n,r} = \mathbf{R}^b_d\mathbf{S}(\boldsymbol{\omega}^d_{b,d})\boldsymbol{\omega}^d_{n,d} + \mathbf{R}^b_d\dot{\boldsymbol{\omega}}^d_{n,d} - \mathbf{R}^b_w\dot{\boldsymbol{\omega}}^w_{b,w} - \gamma\mathbf{R}^b_w\mathbf{S}(\boldsymbol{\omega}^w_{b,w})\mathbf{T}_e^\top\mathbf{e}_q - \frac{\gamma}{2}\mathbf{R}^b_w\dot{\boldsymbol{\epsilon}}_{d,w} \tag{55}$$

where the property that $\mathbf{T}_e^\top\mathbf{e}_q = \frac{1}{2}\boldsymbol{\epsilon}_{d,w}$ has been used and note that $\boldsymbol{\omega}^d_{b,d} = -\mathbf{R}^d_b\boldsymbol{\omega}^b_{n,b} + \boldsymbol{\omega}^d_{n,d}$.

A Lyapunov Function Candidate can now be chosen as

$$V_2 = \frac{1}{2}k_q\mathbf{e}_q^\top\mathbf{e}_q + \frac{1}{2}(\mathbf{s}^b)^\top\mathbf{J}\mathbf{s}^b \tag{56}$$

where k_q is a positive gain. By differentiating (56), inserting (54) and (27) it becomes

$$\dot{V}_2 = k_q\mathbf{e}_q^\top\mathbf{T}_e\boldsymbol{\omega}^w_{d,w} + (\mathbf{s}^b)^\top(-\mathbf{S}(\boldsymbol{\omega}^b_{n,b})\mathbf{J}\boldsymbol{\omega}^b_{n,b} + \mathbf{f}(\alpha,\beta) - \mathbf{D}\mathbf{s}^b - \mathbf{D}\boldsymbol{\omega}^b_{n,r} + \mathbf{B}\mathbf{u} - \mathbf{J}\dot{\boldsymbol{\omega}}^b_{n,r}) \tag{57}$$

which can be rewritten using the property that the angular velocity is $\boldsymbol{\omega}^w_{d,w} = \mathbf{R}^w_b(\mathbf{s}^b - \gamma\mathbf{R}^b_w\mathbf{T}_e^\top\mathbf{e}_q) = \mathbf{R}^w_b\mathbf{s}^b - \gamma\mathbf{T}_e^\top\mathbf{e}_q$ resulting in

$$\dot{V}_2 = -k_q\gamma\mathbf{e}_q^\top\mathbf{T}_e\mathbf{T}_e^\top\mathbf{e}_q + (\mathbf{s}^b)^\top(-\mathbf{S}(\boldsymbol{\omega}^b_{n,b})\mathbf{J}\boldsymbol{\omega}^b_{n,b} + \mathbf{f}(\alpha,\beta) - \mathbf{D}\mathbf{s}^b - \mathbf{D}\boldsymbol{\omega}^b_{n,r} + \mathbf{B}\mathbf{u} - \mathbf{J}\dot{\boldsymbol{\omega}}^b_{n,r} + k_q\mathbf{R}^b_w\mathbf{T}_e^\top\mathbf{e}_q). \tag{58}$$

The control signal can now be chosen as

$$\mathbf{u} = \mathbf{B}^{-1}(\mathbf{J}\dot{\boldsymbol{\omega}}^b_{n,r} + \mathbf{S}(\boldsymbol{\omega}^b_{n,b})\mathbf{J}\boldsymbol{\omega}^b_{n,b} - \mathbf{f}(\alpha,\beta) + \mathbf{D}\boldsymbol{\omega}^b_{n,r} - k_q\mathbf{R}^b_w\mathbf{T}_e^\top\mathbf{e}_q - \mathbf{K}_s\mathbf{s}^b) \tag{59}$$

where $\mathbf{K}_s$ is a positive definite gain matrix. The control law can now be inserting into (58) resulting in

$$\dot{V}_2 = -k_q\gamma\mathbf{e}_q^\top\mathbf{T}_e\mathbf{T}_e^\top\mathbf{e}_q - (\mathbf{s}^b)^\top(\mathbf{D}+\mathbf{K}_s)\mathbf{s}^b \tag{60}$$

$$\dot{V}_2 \leq -\frac{k_q\gamma}{8}||\mathbf{e}_q||^2 - \lambda_{min}(\mathbf{D}+\mathbf{K}_s)||\mathbf{s}^b||^2 \tag{61}$$

where Lemma 1 has been used and $\lambda_{min}(\mathbf{D}+\mathbf{K}_s)$ is the smallest eigenvalue of the resulting matrix. From (56) we see that the Lyapunov function is positive definite and decrescent, while its derivative (61) is negative definite. With $\mathbf{q}_{n,d}(t), \boldsymbol{\omega}^d_{n,d}(t), \dot{\boldsymbol{\omega}}^d_{n,d}(t) \in \mathcal{L}_\infty$ and using standard Lyapunov arguments (*cf.* Khalil (2002)), we conclude that the equilibrium point $(\mathbf{e}_{q+}, \mathbf{s}) = (\mathbf{0}, \mathbf{0})$ is uniformly exponentially stable (UES). Furthermore it follows as $\mathbf{s} \to \mathbf{0}$ that $\boldsymbol{\omega}^b_{n,b} \to \boldsymbol{\omega}^b_{n,r}$ and as $\mathbf{e}_{q+} \to \mathbf{0}$ it follows that $\boldsymbol{\omega}^w_{d,w} \to \mathbf{0}$ and hence all tracking errors will go to zero. A similar proof can be done for the negative equilibrium point.

5.1 Saturation

The control law in (59) has been derived under the assumption of infinite actuation, while the actuators are in fact saturated providing a limited amount of actuation. This is a challenging control problem that has received much attention the last decades. The common solution for flight control is to apply anti-windup where the difference between the desired control signal and the saturated control signal is integrated and fed back into the control law. Other approaches are by putting severe limitation on the feasible trajectories making the solution only applicable to a few scenarios. One of the major challenges with bounding the control law (59) is that it contains the vector (21) which can be very large during aggressive maneuvers forcing the system into saturation. In the following section we first show that the angular velocity is bounded. The control law is then differentiated and arranged in such a way that the reference trajectory becomes a second order differential equation exposed to a bounded disturbance which can be made arbitrarily small by increasing the gains.

Assumption 2. During the analysis of the saturation, it is assumed that the total velocity has converged to its desired value and is constant.

Assumption 3. It is assumed that the wind is constant or slowly varying, such that the angular velocity $\boldsymbol{\omega}^w_{n,w}$ is bounded.

Lemma 2. The angular velocity $\boldsymbol{\omega}^b_{n,b}$ of the system (24) is globally uniformly ultimately bounded for any $\mathbf{u}$.

Proof 2. The actuators are physically upper and lower bounded with a maximum and minimum deflection angle, such that $||\mathbf{u}|| \leq u_{max}$. Let a Lyapunov Function Candidate be defined as

$$V_3 = \frac{1}{2}(\boldsymbol{\omega}^b_{n,b})^\top\mathbf{J}\boldsymbol{\omega}^b_{n,b} \tag{62}$$

and by differentiation and inserting (24) it becomes

$$\dot{V}_3 = -(\boldsymbol{\omega}^b_{n,b})^\top\mathbf{D}\boldsymbol{\omega}^b_{n,b} + (\boldsymbol{\omega}^b_{n,b})^\top(\mathbf{f}(\alpha,\beta) + \mathbf{B}\mathbf{u}) \tag{63}$$

$$\leq -\lambda_{min}(\mathbf{D})||\boldsymbol{\omega}^b_{n,b}||^2 \quad \forall\ ||\boldsymbol{\omega}^b_{n,b}|| \geq \delta \tag{64}$$

where $\delta = \frac{||\mathbf{f}(\alpha,\beta)||+||\mathbf{B}||u_{max}}{\lambda_{min}(\mathbf{D})}$, and hence all the solutions are globally uniformly ultimately bounded (*cf.* Khalil (2002)). □

Lemma 3. The angular velocity $\boldsymbol{\omega}^w_{b,w}$ is bounded when the wind is slowly varying or constant.

Proof 3. The angular velocity between the body frame and wind frame can be written as

$$\boldsymbol{\omega}^w_{b,w} = \boldsymbol{\omega}^w_{n,w} - \boldsymbol{\omega}^w_{n,b} \tag{65}$$

where $\boldsymbol{\omega}^w_{n,w}$ is bounded using Assumption 3 and $\boldsymbol{\omega}^w_{n,b}$ is bounded as shown in Lemma 2. Hence, it follows that $\boldsymbol{\omega}^w_{b,w}$ must also be bounded. □

Lemma 4. The function $\dot{\mathbf{f}}(\alpha,\beta)$ is bounded.

Proof 4. Using Assumption 2 the function $\dot{\mathbf{f}}(\alpha,\beta)$ can be shown to be (*cf.* (21))

$$\dot{\mathbf{f}}(\alpha,\beta) = \frac{1}{2}\rho S V_T^2 \begin{bmatrix} bC_{l_\beta}\dot{\beta} \\ \bar{c}C_{m_\alpha}\dot{\alpha} \\ bC_{n_\beta}\dot{\beta} \end{bmatrix} \tag{66}$$

and since $\boldsymbol{\omega}^w_{b,w}$ is shown to be bounded in Lemma 3, it follows that $\dot{\alpha}, \dot{\beta}$ must also be bounded, and consequently also $\dot{\mathbf{f}}(\alpha,\beta)$. □

The angular velocity between body and wind frame can also be written as

$$\boldsymbol{\omega}^w_{b,w} = \mathbf{R}^w_d \boldsymbol{\omega}^d_{n,d} - \mathbf{R}^w_b \boldsymbol{\omega}^b_{n,r} - \frac{\gamma}{2}\boldsymbol{\epsilon}_{d,w} \tag{67}$$

where $\mathbf{T}_e^\top \mathbf{e}_q = \frac{1}{2}\boldsymbol{\epsilon}_{d,w}$ has been used. With $\dot{\boldsymbol{\epsilon}}_{d,w} = \frac{1}{2}(\eta_{d,w}\mathbf{I} + \mathbf{S}(\boldsymbol{\epsilon}_{d,w}))\boldsymbol{\omega}^w_{d,w}$ it follows that

$$\frac{k_q}{2}\mathbf{R}^b_w \dot{\boldsymbol{\epsilon}}_{d,w} = \frac{k_q}{4}\mathbf{R}^b_w(\eta_{d,w}\mathbf{I} + \mathbf{S}(\boldsymbol{\epsilon}_{d,w}))\boldsymbol{\omega}^w_{d,w} \tag{68}$$

where

$$\boldsymbol{\omega}^w_{d,w} = \mathbf{R}^w_b \boldsymbol{\omega}^b_{n,b} - \mathbf{R}^w_b \boldsymbol{\omega}^b_{n,r} - \frac{\gamma}{2}\boldsymbol{\epsilon}_{d,w}. \tag{69}$$

The control law (59) can now be differentiated as

$$\begin{aligned}\dot{\mathbf{u}} = \mathbf{B}^{-1}(&\mathbf{J}\ddot{\boldsymbol{\omega}}^b_{n,r} + (-\mathbf{S}(\mathbf{J}\boldsymbol{\omega}^b_{n,b}) + \mathbf{S}(\boldsymbol{\omega}^b_{n,b})\mathbf{J})\dot{\boldsymbol{\omega}}^b_{n,b} \\ &- \dot{\mathbf{f}}(\alpha,\beta) + \mathbf{D}\dot{\boldsymbol{\omega}}^b_{n,r} - \frac{k_q}{2}\mathbf{R}^b_w \mathbf{S}(\boldsymbol{\omega}^w_{b,w})\boldsymbol{\epsilon}_{d,w} \\ &- \frac{k_q}{2}\mathbf{R}^b_w \dot{\boldsymbol{\epsilon}}_{d,w} - \mathbf{K}_s(\dot{\boldsymbol{\omega}}^b_{n,b} - \dot{\boldsymbol{\omega}}^b_{n,r}))\end{aligned} \tag{70}$$

where the angular acceleration $\dot{\boldsymbol{\omega}}^b_{n,b}$ can be gathered, and by inserting (67)-(69) it becomes

$$\begin{aligned}\dot{\mathbf{u}} = \mathbf{B}^{-1}(&\mathbf{J}\ddot{\boldsymbol{\omega}}^b_{n,r} + (-\mathbf{K}_s - \mathbf{S}(\mathbf{J}\boldsymbol{\omega}^b_{n,b}) + \mathbf{S}(\boldsymbol{\omega}^b_{n,b})\mathbf{J})\dot{\boldsymbol{\omega}}^b_{n,b} \\ &- \dot{\mathbf{f}}(\alpha,\beta) + \mathbf{D}\dot{\boldsymbol{\omega}}^b_{n,r} + \mathbf{K}_s\dot{\boldsymbol{\omega}}^b_{n,r} \\ &+ \frac{k_q}{2}\mathbf{R}^b_w \mathbf{S}(\boldsymbol{\epsilon}_{d,w})(\mathbf{R}^w_d \boldsymbol{\omega}^d_{n,d} - \mathbf{R}^w_b \boldsymbol{\omega}^b_{n,r} - \frac{\gamma}{2}\boldsymbol{\epsilon}_{d,w}) \\ &- \frac{k_q}{4}\mathbf{R}^b_w(\eta_{d,w}\mathbf{I} + \mathbf{S}(\boldsymbol{\epsilon}_{d,w}))(\mathbf{R}^w_b \boldsymbol{\omega}^b_{n,b} - \mathbf{R}^w_b \boldsymbol{\omega}^b_{n,r} - \frac{\gamma}{2}\boldsymbol{\epsilon}_{d,w})),\end{aligned} \tag{71}$$

where $\frac{k_q}{2}\mathbf{R}^b_w \mathbf{S}(\boldsymbol{\omega}^w_{b,w})\boldsymbol{\epsilon}_{d,w} = -\frac{k_q}{2}\mathbf{R}^b_w \mathbf{S}(\boldsymbol{\epsilon}_{d,w})\boldsymbol{\omega}^w_{b,w}$ has been used. The expression $\frac{k_q}{4}\mathbf{R}^b_w(\eta_{d,w}\mathbf{I} + \mathbf{S}(\boldsymbol{\epsilon}_{d,w}))\mathbf{R}^w_b \boldsymbol{\omega}^b_{n,r} - \frac{k_q}{2}\mathbf{R}^b_w \mathbf{S}(\boldsymbol{\epsilon}_{d,w})\mathbf{R}^w_b \boldsymbol{\omega}^b_{n,r} = \frac{k_q}{4}\mathbf{R}^b_w(\eta_{d,w}\mathbf{I} - \mathbf{S}(\boldsymbol{\epsilon}_{d,w}))\mathbf{R}^w_b \boldsymbol{\omega}^b_{n,r}$, such that the (71) can be rewritten as

$$\begin{aligned}\dot{\mathbf{u}} = \mathbf{B}^{-1}(&\mathbf{J}\ddot{\boldsymbol{\omega}}^b_{n,r} + (-\mathbf{K}_s - \mathbf{S}(\mathbf{J}\boldsymbol{\omega}^b_{n,b}) + \mathbf{S}(\boldsymbol{\omega}^b_{n,b})\mathbf{J})\dot{\boldsymbol{\omega}}^b_{n,b} \\ &- \dot{\mathbf{f}}(\alpha,\beta) + (\mathbf{D} + \mathbf{K}_s)\dot{\boldsymbol{\omega}}^b_{n,r} \\ &+ \frac{k_q}{4}\mathbf{R}^b_w(\eta_{d,w}\mathbf{I} - \mathbf{S}(\boldsymbol{\epsilon}_{d,w}))\mathbf{R}^w_b \boldsymbol{\omega}^b_{n,r} \\ &+ \frac{k_q}{2}\mathbf{R}^b_w \mathbf{S}(\boldsymbol{\epsilon}_{d,w})(\mathbf{R}^w_d \boldsymbol{\omega}^d_{n,d} - \frac{\gamma}{2}\boldsymbol{\epsilon}_{d,w}) \\ &- \frac{k_q}{4}\mathbf{R}^b_w(\eta_{d,w}\mathbf{I} + \mathbf{S}(\boldsymbol{\epsilon}_{d,w}))(\mathbf{R}^w_b \boldsymbol{\omega}^b_{n,b} - \frac{\gamma}{2}\boldsymbol{\epsilon}_{d,w}).\end{aligned} \tag{72}$$

The terms can now be gathered by defining $\mathbf{K}_1 := \mathbf{J}$, $\mathbf{K}_2 := (\mathbf{D} + \mathbf{K}_s)$, $\mathbf{K}_3 := \frac{k_q}{4}\mathbf{R}^b_w(\eta_{d,w}\mathbf{I} - \mathbf{S}(\boldsymbol{\epsilon}_{d,w}))\mathbf{R}^w_b$, $\mathbf{K}_4 = \mathbf{B}^{-1}\mathbf{K}_s\mathbf{J}^{-1}\mathbf{B}$ which all are positive definite matrices, and

$$\begin{aligned}\boldsymbol{\mu} := &-(-\mathbf{K}_s - \mathbf{S}(\mathbf{J}\boldsymbol{\omega}^b_{n,b}) + \mathbf{S}(\boldsymbol{\omega}^b_{n,b})\mathbf{J})\mathbf{J}^{-1}(-\mathbf{S}(\boldsymbol{\omega}^b_{n,b})\mathbf{J}\boldsymbol{\omega}^b_{n,b} \\ &+ \mathbf{f}(\alpha,\beta) - \mathbf{D}\boldsymbol{\omega}^b_{n,b})) + \dot{\mathbf{f}}(\alpha,\beta) \\ &- \frac{k_q}{2}\mathbf{R}^b_w \mathbf{S}(\boldsymbol{\epsilon}_{d,w})(\mathbf{R}^w_d \boldsymbol{\omega}^d_{n,d} - \frac{\gamma}{2}\boldsymbol{\epsilon}_{d,w}) \\ &+ \frac{k_q}{4}\mathbf{R}^b_w(\eta_{d,w}\mathbf{I} + \mathbf{S}(\boldsymbol{\epsilon}_{d,w}))(\mathbf{R}^w_b \boldsymbol{\omega}^b_{n,b} - \frac{\gamma}{2}\boldsymbol{\epsilon}_{d,w})\end{aligned} \tag{73}$$

then (72) to be written as

$$\begin{aligned}\dot{\mathbf{u}} = &-\mathbf{K}_4\mathbf{u} + \mathbf{B}^{-1}(\mathbf{S}(\boldsymbol{\omega}^b_{n,b}) - \mathbf{S}(\mathbf{J}\boldsymbol{\omega}^b_{n,b})\mathbf{J}^{-1})\mathbf{B}\mathbf{u} \\ &+ \mathbf{B}^{-1}(\mathbf{K}_1\ddot{\boldsymbol{\omega}}^b_{n,r} + \mathbf{K}_2\dot{\boldsymbol{\omega}}^b_{n,r} + \mathbf{K}_3\boldsymbol{\omega}^b_{n,r} - \boldsymbol{\mu}).\end{aligned} \tag{74}$$

Now let a Lyapunov Function Candidate be defined as

$$V_4 = \frac{1}{2}\mathbf{u}^\top\mathbf{u} \tag{75}$$

which can be differentiated, and by inserting (74) it becomes

$$\begin{aligned}\dot{V}_4 = &\mathbf{u}^\top(-\mathbf{K}_4\mathbf{u} + \mathbf{B}^{-1}(\mathbf{S}(\boldsymbol{\omega}^b_{n,b}) - \mathbf{S}(\mathbf{J}\boldsymbol{\omega}^b_{n,b})\mathbf{J}^{-1})\mathbf{B}\mathbf{u} \\ &+ \mathbf{u}^\top\mathbf{B}^{-1}(\mathbf{K}_1\ddot{\boldsymbol{\omega}}^b_{n,r} + \mathbf{K}_2\dot{\boldsymbol{\omega}}^b_{n,r} + \mathbf{K}_3\boldsymbol{\omega}^b_{n,r} - \boldsymbol{\mu})).\end{aligned} \tag{76}$$

Let $\beta_1 := ||\mathbf{B}^{-1}(\mathbf{S}(\boldsymbol{\omega}^b_{n,b}) - \mathbf{S}(\mathbf{J}\boldsymbol{\omega}^b_{n,b})\mathbf{J}^{-1})\mathbf{B}||$ serve as an upper bound for the second term, then the Lyapunov derivative can be written as

$$\begin{aligned}\dot{V}_4 \leq &-(\lambda_{min}(\mathbf{K}_4) - \beta_1)||\mathbf{u}||^2 \\ &+ \mathbf{u}^\top\mathbf{B}^{-1}(\mathbf{K}_1\ddot{\boldsymbol{\omega}}^b_{n,r} + \mathbf{K}_2\dot{\boldsymbol{\omega}}^b_{n,r} + \mathbf{K}_3\boldsymbol{\omega}^b_{n,r} - \boldsymbol{\mu}))\end{aligned} \tag{77}$$

where the gain $\lambda_{min}(\mathbf{K}_4)$ must be larger than β_1 which can be done through $\mathbf{K}_s$. Desiring that the last term shall be zero we have

$$\mathbf{K}_1\ddot{\boldsymbol{\omega}}^b_{n,r} + \mathbf{K}_2\dot{\boldsymbol{\omega}}^b_{n,r} + \mathbf{K}_3\boldsymbol{\omega}^b_{n,r} = \boldsymbol{\mu} \tag{78}$$

which is a second order differential equation exposed to a disturbance $\boldsymbol{\mu}$ which can be shown to be bounded by applying Lemma 2-4 and noting that $|\alpha|, |\beta| \leq \pi/2$. By properly choosing the gains $\mathbf{K}_2$ and $\mathbf{K}_3$ which depend on k_q and $\mathbf{K}_s$ the system (78) can be made stable making $\boldsymbol{\omega}^b_{n,r}$ go to a small bounded set around the origin which can be made arbitrarily small by increasing the gains. When the reference trajectories enter this bounded set, the first term of (77) will dominate the reference trajectories, resulting in a desaturation of the actuators.

Remark 2. This result has several similarities to the work by Tandale and Vala (2005) who derive an adaptive quaternion tracking controller for spacecraft maneuvers in the presence of saturations. In their result, they must assume that the error between the actual and desired control signal is bounded which is a reasonable assumption considering that it is only natural to track feasible trajectories. Our result on the other hand has a disturbance vector that depend on the angular velocities, the quaternion error, angle of attack and sideslip angle, which all are bounded. The vector part of the quaternion is bounded to $||\boldsymbol{\epsilon}_{d,w}|| \leq 1$, $|\alpha|, |\beta| \leq \frac{\pi}{2}$ while the angular velocity vectors are bounded as shown in Lemma 2 and Lemma 3.

6. SUMMARY OF THE MAIN CONTRIBUTION

Theorem 1. Given an underactuated UAV described by the dynamics (2), (3), (9), (16) and (24) in closed loop with an attitude controller

$$\begin{aligned}\mathbf{u} = &\mathbf{B}^{-1}(\mathbf{J}\dot{\boldsymbol{\omega}}^b_{n,r} + \mathbf{S}(\boldsymbol{\omega}^b_{n,b})\mathbf{J}\boldsymbol{\omega}^b_{n,b} - \mathbf{f}(\alpha,\beta) \\ &+ \mathbf{D}\boldsymbol{\omega}^b_{n,r} - k_q\mathbf{R}^b_w\mathbf{T}_e^\top\mathbf{e}_q - \mathbf{K}_s\mathbf{s})\end{aligned} \tag{79}$$

$$\mathbf{s}^b = \boldsymbol{\omega}^b_{n,b} - \boldsymbol{\omega}^b_{n,r} \tag{80}$$

$$\boldsymbol{\omega}^b_{n,r} = \mathbf{R}^b_d\boldsymbol{\omega}^d_{n,d} - \mathbf{R}^b_w\boldsymbol{\omega}^w_{b,w} - \gamma\mathbf{R}^b_w\mathbf{T}_e^\top\mathbf{e}_q \tag{81}$$

that tracks the desired quaternion and angular velocity

$$\mathbf{q}_{n,d} = \left[\cos(\frac{\vartheta_{n,d}}{2})\ \mathbf{k}^\top_{n,d}\sin(\frac{\vartheta_{n,d}}{2})\right]^\top \tag{82}$$

$$\boldsymbol{\omega}^d_{n,d} = \mathbf{S}^\dagger(\mathbf{e}^d)\mathbf{R}^d_b\mathbf{v}^b_r \tag{83}$$

$$\vartheta_{n,d} = \cos^{-1}\left(\frac{\mathbf{e}^d \cdot \mathbf{e}^n}{||\mathbf{e}^n||^2}\right) \quad \mathbf{k}_{n,d} = \frac{\mathbf{e}^d \times \mathbf{e}^n}{||\mathbf{e}^d \times \mathbf{e}^n||} \tag{84}$$

$$\mathbf{e}^d = [||\mathbf{e}^n||\ 0\ 0]^\top \qquad \mathbf{e}^n = \mathbf{p}^n_{wp} - \mathbf{p}^n \tag{85}$$

with a translational controller

$$T = \frac{mV_T}{u}(\dot{V}_{T,d} - \frac{1}{V_T}(\mathbf{v}_r^b)^\top(\frac{1}{m}\mathbf{R}_w^b\mathbf{f}_{aero}^w + \mathbf{R}_n^b\mathbf{f}_g^n) - k_V\tilde{V}), \quad (86)$$

where $\tilde{V} = V_T - V_{T,d}$ and $V_{T,d} > 0$, then the UAV is able to reach any point in $\mathbb{R}^3$ described by $\mathbf{p}_{wp}^n$. The resulting equilibrium points $(\mathbf{e}_{q\pm}, \mathbf{s}) = (\mathbf{0}, \mathbf{0})$ are UES, $\tilde{V} = 0$ is ES. Furthermore, if the actuators go into saturation, they will become desaturated in finite time.

7. SIMULATION

Initial states of the fixed-wing UAV are chosen as $\mathbf{p}^n(0) = [0\ 0\ 0]^\top$, $\mathbf{v}^b(0) = [30\ 0\ 0]^\top$, $\mathbf{q}_{n,b}(0) = [1\ 0\ 0\ 0]^\top$ and $\boldsymbol{\omega}_{n,b}^b(0) = [0\ 0\ 0]^\top$, $\mathbf{v}_{wind}^n = [10\ 0\ 0]^\top$ and the following parameters have been used

$m = 20.64$	$J_{xx} = 1.607$	$J_{yy} = 7.51$
$J_{zz} = 7.18$	$J_{xz} = 0.59$	$b = 1.96$
$\bar{c} = 0.76$	$S = 1.37$	$k = 0.1$
$C_{L_0} = 0.1$	$C_{L_\alpha} = 0.25$	$C_{D_0} = 0.5$
$C_{Y_\beta} = -0.1$	$C_{l_0} = -0.001$	$C_{l_\beta} = -0.038$
$C_{l_p} = -0.213$	$C_{l_r} = 0.114$	$C_{l_{\delta_a}} = -0.056$
$C_{l_{\delta_r}} = 0.014$	$C_{m_0} = 0.022$	$C_{m_\alpha} = -0.473$
$C_{m_q} = -3.449$	$C_{m_{\delta_e}} = -0.364$	$C_{n_0} = 0$
$C_{n_\beta} = 0.036$	$C_{n_p} = -0.151$	$C_{n_r} = -0.195$
$C_{n_{\delta_a}} = -0.036$	$C_{n_{\delta_r}} = -0.055.$	

The gains were chosen as $\mathbf{K}_s = 2\mathbf{I}$, $k_q = 2$, $\gamma = 2, K_V = 2$ and the desired velocity as $V_{T,d} = 42$m/s. The matrix of waypoints was defined as

$$\mathbf{P}_{wp}^n = \begin{bmatrix} 2000 & 2000 & 0 & 0 & 0 & -2000 & 0 \\ 1000 & 4000 & 6000 & 7000 & 5000 & 6000 & 0 \\ 1000 & 1000 & 5000 & 10000 & 10000 & 5000 & 2000 \end{bmatrix}$$

which switches to the next waypoint whenever $||\mathbf{e}^n|| \leq d = 1$m. In Fig. 2 the attitude error is shown, where the top figure shows the initial convergence, where the tracking error goes quickly to zero. The bottom plot shows the first 100 seconds of the simulation, where the first switch of waypoints happens at about 58 seconds resulting in a spike in the attitude error, which quickly is handled by the attitude controller driving the errors to zero. Similarly the angular velocity error is shown in Fig. 3 where the top figure shows the initial convergence while the bottom figure shows the first 100 seconds where all the errors converge to zero. In Fig. 4 the deflection angles are shown which are upper and lower bounded by $\pm 20°$. The placement of the waypoints result in some sharp maneuvers, forcing the actuators into saturation for a little while until they become desaturated. The velocity tracking error is shown in Fig. 5 which exponentially converge to zero, and remain at zero throughout the simulation. A 3D plot of the simulation is shown in Fig. 6 where the waypoints are illustrated as red circles and the position tracking error is shown in Fig. 7.

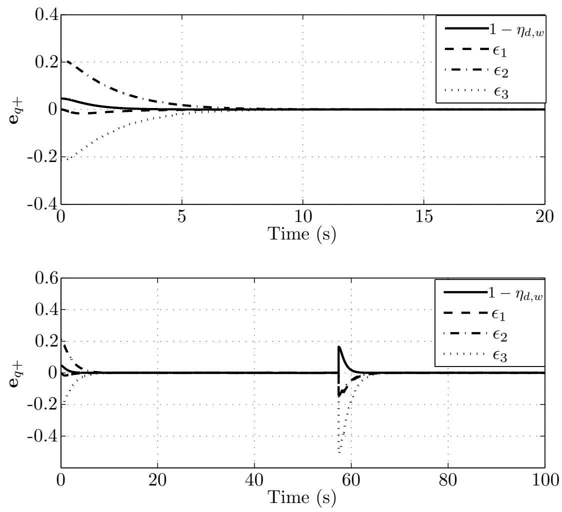

Fig. 2. Attitude and angular velocity tracking errors

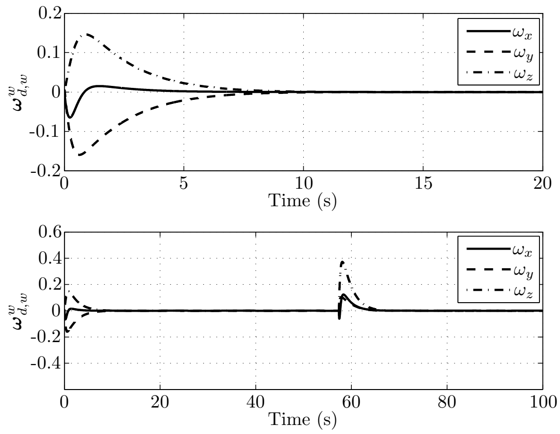

Fig. 3. Angular velocity error

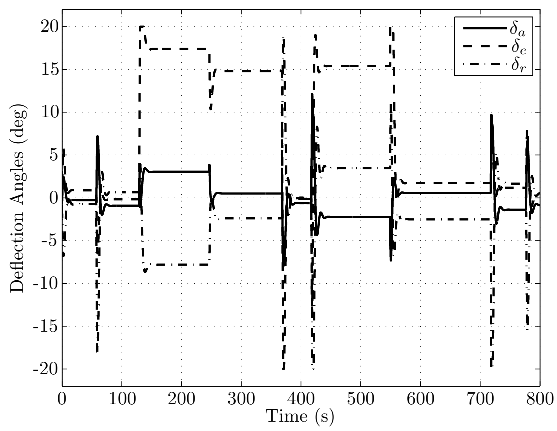

Fig. 4. Deflection angles

8. CONCLUSION

In this paper a solution to waypoint tracking for a fixed-wing UAV has been derived that is rather simple to implement, and through simulations it is shown to have good performance. Future work is to consider the actuator dynamics of the deflection angles in the translational dynamics which greatly complicates the control problem.

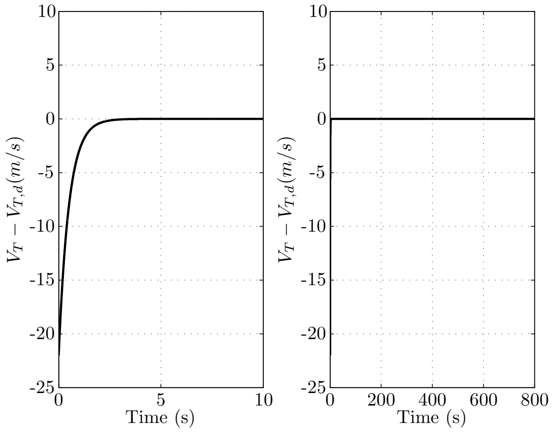

Fig. 5. Velocity tracking error

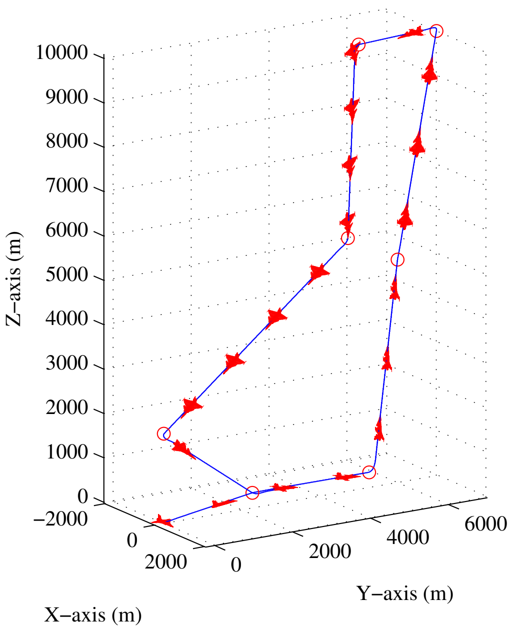

Fig. 6. Waypoint tracking

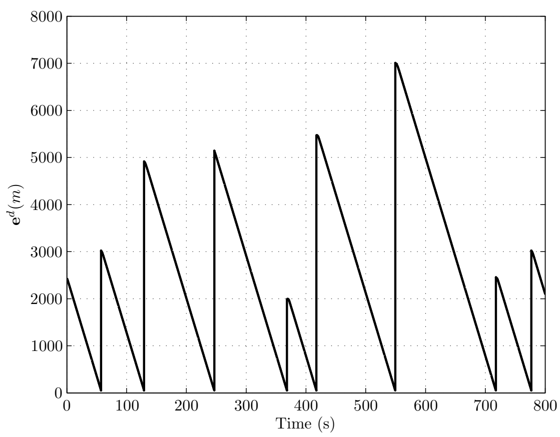

Fig. 7. Position tracking error

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