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ON SOME POWER MEANS AND THEIR GEOMETRIC CONSTRUCTIONS

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Abstract

The main aim of this paper is to further develop the recently initiated research concerning geometric construction of some power means where the variables are appearing as line segments. It will be demonstrated that the arithmetic mean, the harmonic mean and the quadratic mean can be constructed for any number of variables and that all power means where the number of variables are $n = 2^m$, $m \geq 1 \in \mathbb{N}$ for all powers $k = \pm 2^{-q}$ and $k = \pm 2^q$, $q \in \mathbb{N}$ can be geometrically constructed.

Mathematics Subject Classification: 26E60, 51M15

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1 Introduction

Means and averages have been studied and used since antiquity. The biblical story about the Egyptian Pharao's dream about seven fat and seven skinny cows coming up from the Nile, and the interpretation by Joseph, lead to detailed measuring of the rise and fall of the river and of the avearaging of the use of the yearly crops.

The Greek mathematicians explored what is now called *the Pythagorean means,* the arithmetic, the geometric and the harmonic means, because of their importance in the study of geometry and music.

Power means have found many applications in modern mathematics. Let us just mention that in homogenization theory, there are examples where the effective conductivities of composite structures are power means of the local conductivities, see [11] and [13]. Other recent studies have investigated the requrements of integer variables for the power mean also to be integer valued, see [7].

For n positive numbers, $a_1, a_2, ..., a_n$, the power mean P_k^n of order k, with equal weights, is defined as follows,

$$P_k^n = \left[\frac{a_1^k + a_2^k + ... + a_n^k}{n}\right]^{\frac{1}{k}}, \text{ if } k \neq 0,$$

and

$$P_0^n = [a_1 a_2 .... a_n]^{\frac{1}{n}}, \text{ if } k = 0.$$

There is a substantial literature on the subject of power means see [1], [5], [10], [18] and [22]. The close connection between convexity and power means is described e.g. in the new book [14].

The Greek mathematicians constructed the *Pythagorean means* of two variable line segments a and b as showed in Figure 1, see e.g. [3]. The quadratic mean, $Q = P_2^2$, also known as the *Root Mean Square*, is also included in the figure.

Power means have throughout history mostly been analyzed and calculated on the basis of numeric variables. In [6] the authors studied the properties of certain power means based on variable line segments, and showed that P_{-2}^2, P_{-1}^2, $P_{-1/2}^2$, P_0^2, $P_{1/2}^2$, P_1^2 and P_2^2 for two variables can be constructed in a basic geometric structure different from the one employed by the Greek mathematicians. In a recent work by the authors of this paper it has been shown that

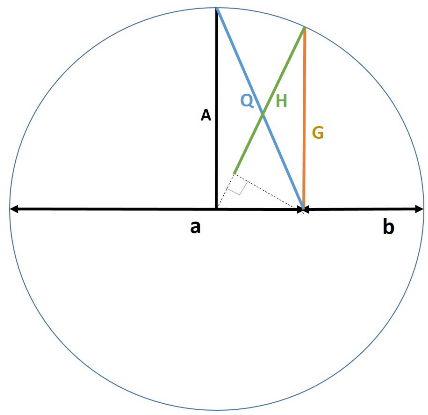

Figure 1: Classic Greek construction of Pythagorean means of the line segments a and b. A is the arithmetic mean, Q is the quadratic mean, H is the harmonic mean and G is the geometric mean.

the arithmetic and the harmonic mean of three variables can be constructed in a three dimensional structure [9]. Other works on geometric constructions of power means are [2], [3], [4], [15], [16], [17],[18], [19], [20], [21] and [23].

Most of the works so far has been concerned with geometric constructions of special power means of two variables. In the recent paper [9] we raised the questions to more general situations e.g. involving three or more variables and more general power means. However, in [9] only the case with three variables was considered. In this paper we continue this research by considering the more general case with n variables and also more general power means involved.

This paper is organized as follows: In Sections 2 and 3 we demonstrate how the arithmetic, harmonic and the quadratic mean can be constructed for any number of variables. In Section 4 we show that it is possible to construct the geometric mean and also in P^n_{-2}, P^n_{-1}, $P^n_{-1/2}$, $P^n_{1/2}$, P^n_1 and P^n_2 for $n = 2^m$ variables, where m is any positive integer. In Section 5 we discuss and illustrate the fact that all power means for $n = 2^m$ variables, where the power is $k = \pm 2^{-q}$, can be geometrically constructed (here q is any positive integer). Finally, in Section 6 we show that the findings in Sections 4 and 5 allow the construction of power means for all variables $n = 2^m$, where the power is $k = \pm 2^q$ (and again m and q are arbitrary positive integers).

Remark 1 *The classic Greek method of constructing the Pythagorean means, as shown in Figure 1, may also be extended to construct P^2_{-2}, P^2_{-1}, $P_{-1/2}$, P^2_0, $P^2_{1/2}$, P^2_1 and P^2_2 for two variables. To accomplish this, we use the facts*

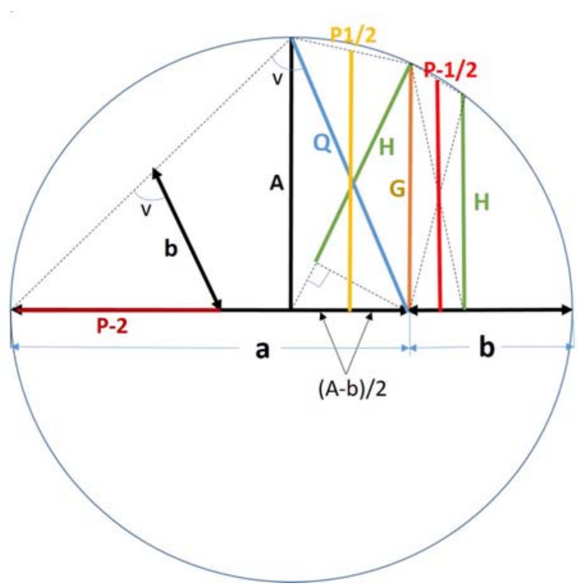

Figure 2: Geometric construction of power means $P_{-2}^2, P_{-1}^2, P_{-1/2}^2, P_0^2, P_{1/2}^2, P_1^2$ and P_2^2.

described in [6]:

$$\begin{aligned} P_{1/2}^2(a,b) &= P_1^2(P_1^2(a,b), P_0^2(a,b)), \\ P_{-1/2}^2(a,b) &= P_{-1}(P_{-1}^2(a,b), P_0^2(a,b)) \end{aligned}$$

and

$$P_2^2(a,b) \times P_{-2}^2(a,b) = ab.$$

The construction method is illustrated in Figure 2.

2 Harmonic means for n variables

The basic structure which was used in [6] for the geometric construction of P_{-2}^2, P_{-1}^2, $P_{-1/2}^2$, P_0^2, $P_{1/2}^2$, P_1^2 and P_2^2 for two variables a_1 and a_2, is shown in Figure 3. This structure can be found in [8]. Independent of the width of the "floor" AB, the length of the vertical line EF through the intersection of the diagonals, is equal to the harmonic power mean of the two variables a_1 and a_2, *i.e.*,

$$EF = P_{-1}^2(a_1, a_2) = \frac{2a_1a_2}{a_1 + a_2}.$$

The arithmetic mean is found by bisecting the "floor" AB and constructing the vertical line between the "floor" and the "roof". If, in addition, $(d_1, d_2) = (a_1, a_2)$, the "roof" DC equals $2Q = 2P_2^2(a_1, a_2)$.

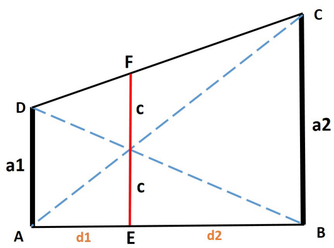

Figure 3: Construction of the harmonic mean $P_{-1}^{2}(a_1, a_2)$.

We will show that the harmonic mean of three and more variables can be constructed using the same basic structure. First we state the following lemma (see [9]).

Lemma 1 *In Figure 4 we consider a more general structure than that presented in Figure 3. The only requirement is that the lines AD and BC are parallel. Let EF be the line through the intersection of the diagonals AC and BD, parallel to AD and BC. Then, it holds that EF is equal to the harmonic mean of AD and BC. Moreover, $c_1 = c_2 = (a_1 a_2) / (a_1 + a_2)$.*

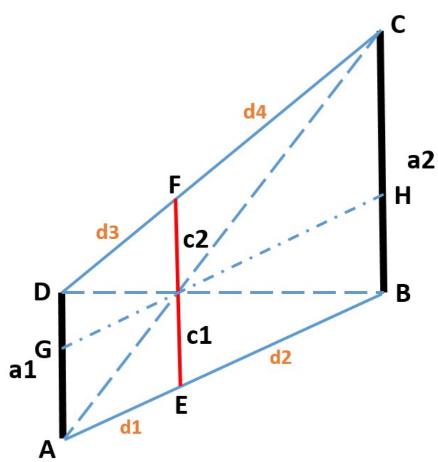

Figure 4: Alternative construction of the harmonic mean $P_{-1}^{2}(a_1, a_2)$.

The following iterative Theorem is useful for our purposes.

Theorem 2 *Let $n = 3, 4, 5, ...$ It holds that*

$$P_{-1}^{n}(a_1, ..., a_n) = \frac{n}{2} P_{-1}^{2}(a_1, \frac{1}{n-1} P_{-1}^{n-1}(a_2, ..., a_n)). \tag{1}$$

Proof. We have that

$$\frac{n}{2}P_{-1}^{2}(a_1, \frac{1}{n-1}P_{-1}^{n-1}(a_2, ..., a_n)) =$$

$$\frac{n}{2}(\frac{2a_1 \times \frac{1}{n-1}P_{-1}(a_2, ..., a_n)}{a_1 + \frac{1}{n-1}P_{-1}(a_2, ..., a_n)}) = \frac{n}{2}(\frac{2a_1 \times \frac{1}{n-1}\frac{(n-1)a_2a_3.....a_n}{a_2a_3....a_{n-1}+....+a_3a_4...a_n}}{a_1 + \frac{1}{n-1}\frac{(n-1)a_2a_3.....a_n}{a_2a_3....a_{n-1}+....+a_3a_4...a_n}}) =$$

$$\frac{n}{2}(\frac{2a_1a_2....a_n}{a_1 \times (a_2a_3....a_{n-1} + ... + a_3a_4...a_n) + a_2a_3.....a_n}) =$$

$$\frac{na_1a_2....a_n}{a_1a_2.....a_{n-1} + ... + a_2a_3.....a_n} = P_{-1}^{n}(a_1, ..., a_n).$$

The proof is complete. ■

Remark 2 *Iterative use of (1) implies that*

$$P_{-1}^{n}(a_1, ..., a_n) = \frac{n}{2}P_{-1}^{2}(a_1, \frac{1}{2}P_{-1}^{2}(a_2, \frac{1}{2}P_{-1}^{2}(a_3, ..., \frac{1}{2}P_{-1}^{2}(a_{n-1}, a_n)))...). \quad (2)$$

This formula is particularly suitable for the geometric construction of harmonic mean for n variables.

2.1 Three variables

Consider now the case $n = 3$. The means P_{-1}^{3} and P_{1}^{3} are constructed as shown in Figure 5. The variables a_1, a_2 and a_3 are organized vertically in ascending order on a horizontal floor AC (of an arbitrary width), under a "roof" line FD, connecting the top of the smallest variable a_1 and the top of the largest variable a_3.

From Lemma 1 we know that

$$GH = \frac{1}{2}P_{-1}^{2}(a_2, a_3) = \frac{a_2a_3}{a_2 + a_3}$$

is the vertical line through the intersection of the diagonals of the trapezoid $BCDE$. Moreover, JK is the corresponding vertical line through the intersection of the diagonals in the trapezoid $AHGF$. The length of JK is then equal to

$$\begin{aligned} JK &= \frac{1}{2}P_{-1}^{2}(a_1, GH) = \frac{1}{2}P_{-1}^{2}(a_1, \frac{a_2a_3}{a_2 + a_3}) \quad (3) \\ &= \frac{a_1a_2a_3}{a_1a_2 + a_1a_2 + a_2a_3} = \frac{1}{3}P_{-1}^{3}(a_1, a_2, a_3). \end{aligned}$$

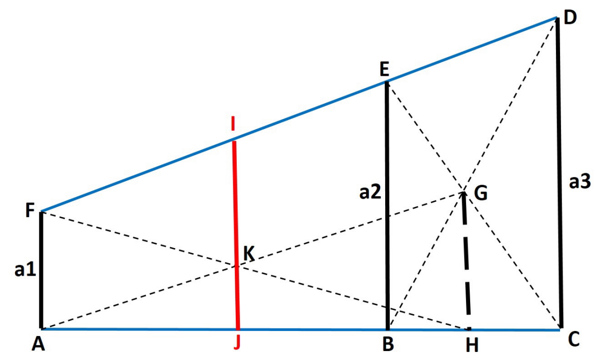

Figure 5: Construction of the harmonic mean $P^3_{-1}(a_1,a_2,a_3)$.

By using Lemma 1 it holds that $IJ = 3KJ$, *i.e.*,

$$P^3_{-1}(a_1, a_2, a_3) = IJ.$$

In order to see this, we consider the three trapezoids $BCDE$, $AHGF$ and $AGMF$ in Figure 6. From the fact that $c_1 = c_2$ in Lemma 1, we know that

$$HG = GM.$$

Moreover, the same lemma yields the relations

$$JN = P^2_{-1}(a_1, GH) = IK = P^2_{-1}(a_1, GM),$$

and

$$JK = KN = IN.$$

From (3) we then have that

$$P^3_{-1}(a_1, a_2, a_3) = IJ = JK + KN + IN = \frac{3a_1a_2a_3}{a_1a_2 + a_1a_2 + a_2a_3}$$

This confirms that $IJ = 3KJ$.

The arithmetic mean, $P^3_1(a_1, a_2, a_3)$, may be constructed in the same structure by letting the width of the "floor" AC in Figure 5 be equal to the sum of the variables, trisect it with a standard method.

2.2 Four variables

To construct the harmonic mean of 4 variables one may use the formula (1) for this case

$$P^4_{-1}(a_1, ..., a_4) = \frac{4}{2}P^2_{-1}(a_1, \frac{1}{3}P^3_{-1}(a_2, a_3, a_4)),$$

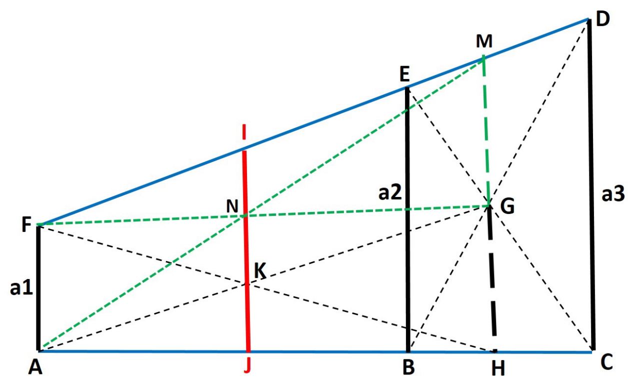

Figure 6: Verification that $IJ = P_{-1}^3(a_1, a_2, a_3)$.

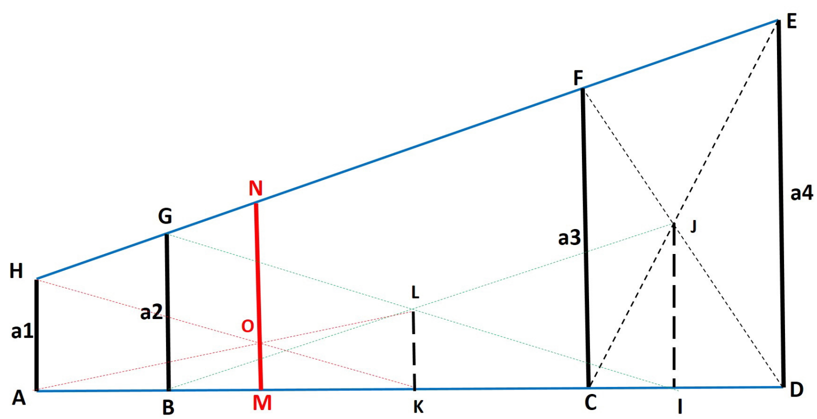

Figure 7: Nested construction of $P_{-1}^4(a_1, a_2, a_3, a_4)$.

or, as written in (2),

$$P_{-1}^4(a_1, ..., a_4) = \frac{4}{2}P_{-1}^2(a_1, \frac{1}{3}P_{-1}^3(a_2, a_3, a_4)) = \frac{4}{2}P_{-1}^2(a_1, \frac{1}{2}P_{-1}^2(a_2, \frac{1}{2}P_{-1}^2(a_3, a_4))).$$

The construction is shown in Figure 7.

Figure 7 shows that

$$IJ = \frac{1}{2}P_{-1}^2(a_3, a_4) \text{ and } KL = \frac{1}{2}P_{-1}^2(IJ, a_2) = \frac{1}{3}P_{-1}(a_2, a_3, a_4)$$

and

$$MO = \frac{1}{2}P_{-1}^2(KL, a_1) = \frac{1}{4}P_{-1}^4(a_1, a_2, a_3, a_4).$$

By recursive use of Lemma 1, it can then be deduced that

$$MN = 4 \times MO,$$

i.e.,

$$P_{-1}^{4}(a_1, a_2, a_3, a_4) = MN = \frac{4a_1a_2a_3a_4}{a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4}. \qquad (4)$$

2.3 n variables

The nested version for P_{-1}^{n} for n variables (see (2))

$$P_{-1}^{n}(a_1, ..., a_n) = \frac{n}{2}(a_1, \frac{1}{2}P_{-1}^{2}(a_2, \frac{1}{2}P_{-1}^{2}(a_3, ..., \frac{1}{2}P_{-1}^{2}(a_{n-1}, a_n)))....),$$

can now be used for the geometric construction of the harmonic mean for any number of n variables using the iterative methods presented above. In particular, in this case formula (4) reads

$$P_{-1}^{n}(a_1, ..., a_n) = \frac{n\Pi_{i=1}^{n} a_i}{\Sigma_{i=1}^{n}\Pi_{j=1, j\neq i}^{n} a_j}.$$

3 Quadratic means for n variables

The quadratic mean for n variables $a_1, ...a_n$,

$$P_2^n = \sqrt{\frac{1}{n}(a_1^2 + ... + a_n^2)},$$

can geometrically be constructed for any number of variables. To show this we use a property deducted from the crossed ladders diagram, see Figure 8. From Figure 8 and Lemma 1 we find that

$$r = a - c = a - \frac{ab}{a+b} = \frac{a^2}{a+b} \qquad (5)$$

and

$$s = b - c = b - \frac{ab}{a+b} = \frac{b^2}{a+b}. \qquad (6)$$

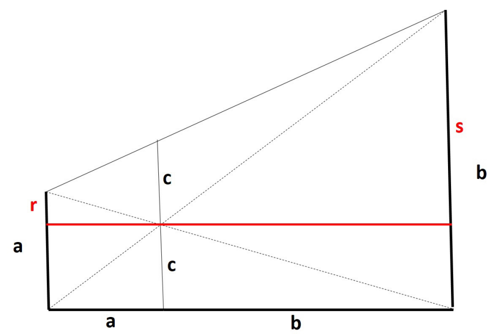

Figure 8: The crossed ladders diagram.

Setting

$$a = \sqrt{a_1^2 + ... + a_n^2} \tag{7}$$

and

$$b = (\sqrt{n} - 1)\sqrt{a_1^2 + ... + a_n^2} \tag{8}$$

this gives that

$$r = \frac{(\sqrt{a_1^2 + ... + a_n^2})^2}{\sqrt{a_1^2 + ... + a_n^2} + (\sqrt{n} - 1)\sqrt{a_1^2 + ... + a_n^2}} =$$
$$\sqrt{\frac{1}{n}(a_1^2 + ... + a_n^2)} = P_2^n(a_1, ...a_n).$$

We can easily construct (7) and (8) for any number of variables. In Figure 9 we have shown this for three variables a_1, a_2 and a_3. The resulting crossed ladders diagram with

$$a = \sqrt{a_1^2 + a_2^2 + a_3^2}, \quad b = (\sqrt{3} - 1)\sqrt{a_1^2 + a_2^2 + a_3^2}$$

and the corresponding r equal to

$$P_2^3 = \sqrt{\frac{1}{3}(a_1^2 + a_2^2 + a_3^2)}$$

is also shown in the figure.

The same procedure can obviously be used for the construction of the quadratic mean of any number of variables.

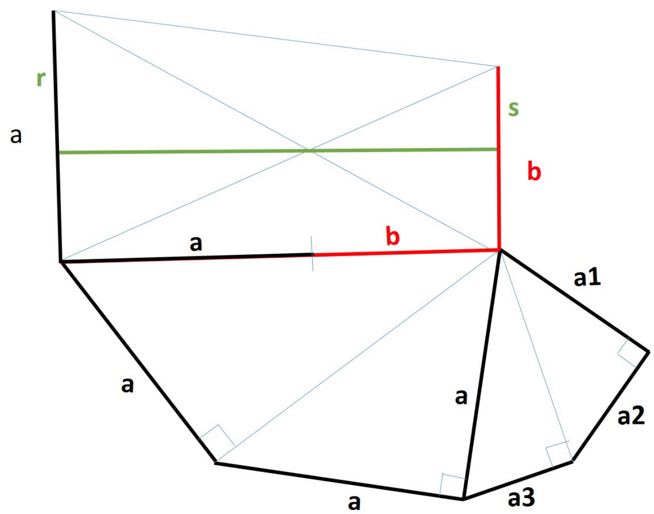

Figure 9: Construction of $a = \sqrt{a_1^2 + a_2^2 + a_3^2}$ and $b = (\sqrt{3} - 1)\sqrt{a_1^2 + a_2^2 + a_3^2}$ and of $r = P_2^3(a_1, a_2, a_3)$.

4 Power means for $n = 2^m$ variables

For $n = 2^m$, where $m \geq 1$ is any integer, another formula can be used for the geometric construction of the harmonic mean.

We first consider the case $m = 2$, i.e., $n = 4$.

4.1 The case $n = 4$

We need the following result:

Lemma 3 *For all real k we have that*

$$P_k^4(a_1, a_2, a_3, a_4) = P_k^2(P_k^2(a_1, a_2), P_k^2(a_3, a_4)). \tag{9}$$

Proof. *It yields that*

$$\begin{aligned} P_k^4(a_1, a_2, a_3, a_4) &= P_k^2(P_k^2(a_1, a_2), P_k^2(a_3 a_4)) = \\ P_k^2((\frac{a_1^k + a_2^k}{2})^{\frac{1}{k}}, (\frac{a_3^k + a_4^k}{2})^{\frac{1}{k}}) &= \left[\frac{((\frac{a_1^k + a_2^k}{2})^{\frac{1}{k}})^k + ((\frac{a_3^k + a_4^k}{2})^{\frac{1}{k}})^k}{2}\right]^{\frac{1}{k}} = \\ (\frac{a_1^k + a_2^k + a_1^k + a_2^k}{4})^{\frac{1}{k}} &= P_k^4(a_1, a_2, a_{3,} a_4) \end{aligned}$$

so the proof is complete. ■

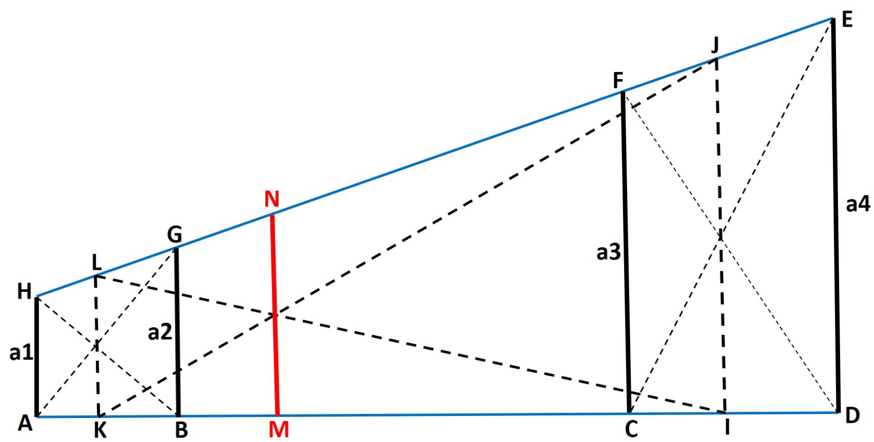

Figure 10: Alternative construction of $P^4_{-1}(a_1, a_2, a_3, a_4)$.

Figure 10 shows the geometric construction of

$$P^4_{-1}(a_1, a_2, a_3, a_4) = \frac{4a_1a_2a_3a_4}{a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4}$$

using (9) in the case $k = -1$.

The variables are, as before, organized vertically on the "floor" AD of arbitrary width, each touching the "roof" HE connecting the top of the smallest and the largest variable. $P^2_{-1}(a_1, a_2) = KL$ and $P^2_{-1}(a_3, a_4) = IJ$ are constructed using the crossing diagonals of the trapezoids $ABGH$ and $CDEF$, respectively. Then $P^4_{-1}(a_1, a_2, a_3, a_4) = MN$ is the vertical line between the "floor" AD and the "roof" HE through the intersection of the diagonals of the trapezoid $KIJL$.

The verification of the construction follows easily by using similar arguments as presented earlier in this paper.

To construct the arithmetic mean $P^4_1(a_1, a_2, a_3, a_4)$ in the same structure, the width of the "floor", AD, would be chosen equal to the sum of the variables and then quadrisect with standard method.

Remark 3 *In addition it is also possible to construct* $P^4_{-2}, P^4_{-1/2}, P^4_0, P^4_{1/2}$ *and* P^4_2 *for 4 variables. One may then use the methods presented in [6], or the ones described in Remark 1 in the Introduction of this paper. These methods allow the construction of* $P^2_{-2}, P^2_{-1}, P^2_{-1/2}, P^2_0, P^2_{1/2}, P^2_1$ *and* P^2_2 *for* (a_1, a_2) *and for* (a_3, a_4)*, respectively. Then, by using our iterative formula (9) the corresponding values of* P^4_k *in the cases* $k = -2, -1, -1/2, 0, 1/2, 1, 2$ *can easily be constructed.*

4.2 The case $n = 2^m$

For this case we have the following useful result:

Theorem 4 *Let $m = 2, 3, ...$ Then*

$$P_k^{2^m}(a_1, ..., a_{2^m}) = P_k^2(P_k^{2^{m-1}}(a_1, ..., a_{2^{m-1}}), P_k^{2^{m-1}}(a_{2^{m-1}+1}, ..., a_{2^m})). \quad (10)$$

Proof. We have that

$$P_k^2(P_k^{2^{m-1}}(a_1, ..., a_{2^{m-1}}), P_k^{2^{m-1}}(a_{2^{m-1}+1}, ..., a_{2^m})) =$$

$$P_k^2\left[(\frac{a_1^k + ... + a_{2^{m-1}}^k}{2^{m-1}})^{\frac{1}{k}}, (\frac{a_{2^{m-1}+1}^k + ... + a_{2^m}^k}{2^{m-1}})^{\frac{1}{k}}\right] =$$

$$\left[\frac{((\frac{a_1^k+...+a_{2^{m-1}}^k}{2^{m-1}})^{\frac{1}{k}})^k + ((\frac{a_{2^{m-1}+1}^k+...+a_{2^m}^k}{2^{m-1}})^{\frac{1}{k}})^k}{2}\right]^{\frac{1}{k}} =$$

$$\left(\frac{a_1^k + a_2^k + ... + a_{2^m}^k}{2^m}\right)^{\frac{1}{k}} = P_k^{2^m}(a_1, ..., a_{2^m}).$$

The proof is complete. ■

Remark 4 *The formula (10) can again be written nested as follows (see (2)):*

$$P_k^{2^m}(a_1, ..., a_{2^m}) = \quad (11)$$
$$P_k^2(P_k^2(...P_k^2(P_k^2(a_{2^{m-1}-3}, a_{2^{m-1}-2}), P_k^2(a_{2^{(m-1)}-1}, a_{2^{(m-1)}}))...),$$
$$(P_k^2(...P_k^2(P_k^2(a_{2^m-3}, a_{2^m-2}), P_k^2(a_{2^m-1}, a_{2^m}))...)).$$

This formulation will, by recursive use of the methods shown for $n = 4$, allow geometric construction of $P_{-2}^n, P_{-1}^n, P_{-1/2}^n, P_0^n, P_{1/2}^n, P_1^n$ and P_2^n for $n = 2^m$ variables for all integer values of $m \geq 1$.

5 Power means where the power $k = \pm 2^{-q}$

5.1 The two variables case

In the Introduction we presented the formulas

$$P^2_{1/2}(a,b) = P^2_1(P^2_1(a,b), P^2_0(a,b)),$$
$$P^2_{-1/2}(a,b) = P_{-1}(P^2_{-1}(a,b), P^2_0(a,b)),$$

and

$$P^2_2(a,b) \times P^2_{-2}(a,b) = ab.$$

This can be generalized. It is in fact well known that (see [13])

$$P^2_k(a,b) \times P^2_{-k}(a,b) = ab,$$

for any real k and also that

$$\begin{aligned} P^2_{2^{-q}}(a,b) &= P^2_{2^{-(q-1)}}(P^2_{2^{-(q-1)}}(a,b), P^2_0(a,b)) \text{ and} \\ P^2_{-2^{-q}}(a,b) &= P^2_{-2^{-(q-1)}}(P^2_{-2^{-(q-1)}}(a,b), P^2_0(a,b)). \end{aligned} \tag{12}$$

The latter formulas can be used for geometric construction of all power means of two variables, where the power $k = \pm 2^{-q}$ and q is a positive integer. In particular, for $q = 2$ we have that

$$P^2_{\pm 1/4}(a,b) = P^2_{\pm 1/2}(P^2_{\pm 1/2}(a,b), P^2_0(a,b)).$$

In the introduction we have shown how to construct $P^2_{\pm 1/2}(a,b)$ and $P^2_0(a,b)$. Using $a_1 = P^2_{\pm 1/2}(a,b)$ and $b_1 = P^2_0(a,b)$,the same method can be used to construct $P^2_{\pm 1/4}(a,b) = P^2_{\pm 1/2}(a_1,b_1)$. Moreover, by recursive use of the same method, all power means of two variables where the power $k = \pm 2^{-q}$ and q is a positive integer, can be geometrically constructed.

5.2 The case with $n = 2^m$ variables

Using the formulas (10), (11) and (12) we can construct all power means of the type $P^{2^m}_{\pm 2^{-q}}(a_1, ..., a_{2^m})$. We will show this for $P^4_{1/4}(a,b,c,d)$.

From (10), (11) and (12) we can write

$$P^4_{1/4}(a,b,c,d) =$$
$$P^2_{1/4}(P^2_{1/4}(a,b), P^2_{1/4}(c,d)) =$$
$$P^2_{1/4}(P^2_{1/2}(P^2_{1/2}(a,b), P^2_0(a,b)), P^2_{1/2}(P^2_{1/2}(c,d), P^2_0(c,d))).$$

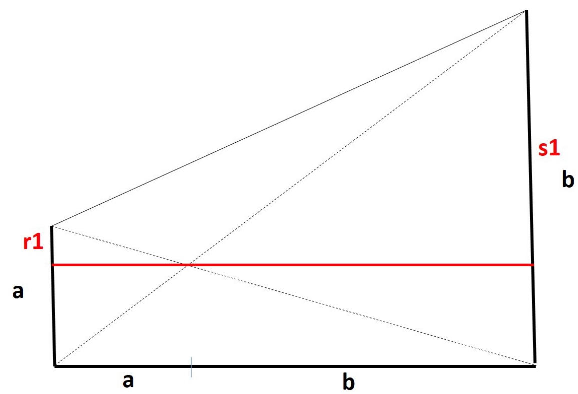

Figure 11: The Crossed ladders diagram.

We have earlier shown the construction of

$$A = P^2_{1/2}(P^2_{1/2}(a, b), P^2_0(a, b))$$

and of

$$B = P^2_{1/2}(P^2_{1/2}(c, d), P^2_0(c, d)).$$

We then have that

$$P^4_{1/4}(a, b, c, d) = P^2_{1/4}(A, B) = P^2_{1/2}(P^2_{1/2}(A, B), P^2_0(A, B)),$$

which can be geometrically constructed using the method shown i Section 4.

By recursive use of the methods described in Section 4 we clearly can construct all power mean of the type $P^{2^m}_{\pm 2^{-q}}(a_1, ..., a_{2^m})$, where the number of variables $n = 2^m$ where $m \geq 1$ is an integer, and where the power $k = \pm 2^{-q}$ (q is a positive integer).

6 Power means where the power is $k = \pm 2^q$

By sequential use of the properties of the Crossed ladders diagram we can construct $P^n_{\pm 2^q}(a_1, ..., a_n)$ for any number of variables of the type $n = 2^m$, $n \in \mathbb{N}$, for all powers $k = \pm 2^q$, $q \in \mathbb{N}$.

6.1 The case with 2 variables

It is known that $P^2_{\pm 2^q}(a_1, a_2)$ is geometrically constructable, see e.g. [12]. Here we present the following alternative proof of this theorem:.

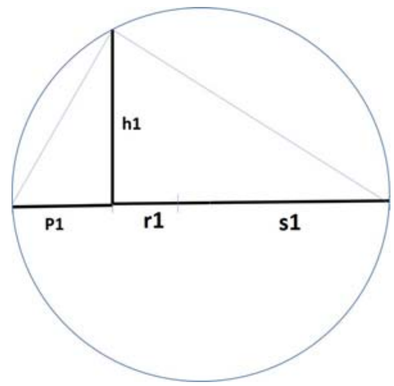

Figure 12: Construction of $h_1 = P_2^2(a, b)$.

Proof. In Section 3 we showed that r_1 and s_1 in the Crossed ladders diagram, see Figure 11, have the values

$$(r_1, s_1) = (\frac{a^2}{a+b}, \frac{b^2}{a+b}).$$

We then have

$$r_1 + s_1 = \frac{a^2 + b^2}{a+b} = \frac{(P_2^2(a,b))^2}{P_1^2(a,b)}.$$

By using $r_1 + s_1$ and $P_1^2(a, b)$ as adjoining parts of the hypotenuse in a right-angle triangle, see Figure 12, the height h_1 from the hypotenuse to the right angle is

$$h_1^2 = (r_1 + s_1)P_1^2(a, b) = (P_2^2(a, b))^2,$$

i.e. we have that

$$h_1 = P_2^2(a, b).$$

Next we construct a Crossed ladders diagram with

$$(a_1, b_1) = (r_1, s_1),$$

which leads to

$$(r_2, s_2) = (\frac{a_1^2}{a_1 + b_1}, \frac{b_1^2}{a_1 + b_2}) = (\frac{a^4}{(a+b)(a^2+b^2)}, \frac{b^4}{(a+b)(a^2+b^2)})$$

and

$$r_2 + s_2 \quad = \quad \frac{a^4 + b^4}{(a+b)(a^2+b^2)} = \frac{(P_4^2(a,b))^4}{2P_1^2(a,b)(P_2^2(a,b))^2}.$$

Having constructed $P_2^2(a,b)$ we can now construct $P_4^2(a,b)$ by using the above method twice. First we use r_2+s_2 and $2P_1^2(a,b)$ as the adjoining parts of the hypotenuse giving

$$h_1=\frac{(P_4^2(a,b))^2}{P_2^2(a,b)}.$$

Next, we use h_1 and $P_2^2(a,b)$ as the adjoining parts of the hypotenuse to construct P_4^2 :

$$h_2=P_4^2(a,b).$$

If

$$(a_{q-1},b_{q-1})=(r_{q-1},s_{q-1})=$$

$$\left(\frac{a^{2^{q-1}}}{(a+b)(a^2+b^2)...(a^{2^{q-1}-1}+b^{2^{q-1}-1})},\frac{b^{2^{q-1}}}{(a+b)(a^2+b^2)...(a^{2^{q-1}-1}+b^{2^{q-1}-1})}\right),$$

then

$$r_q=\frac{(a_{q-1})^2}{a_{q-1}+b_{q-1}}=\frac{\left(\frac{a^{2^{q-1}}}{(a+b)(a^2+b^2)...(a^{2^{q-1}-1}+b^{2^{q-1}-1})}\right)^2}{\frac{a^{2^{q-1}}}{(a+b)(a^2+b^2)...(a^{2^{q-1}-1}+b^{2^{q-1}-1})}+\frac{b^{2^{q-1}}}{(a+b)(a^2+b^2)...(a^{2^{q-1}-1}+b^{2^{q-1}-1})}}=$$

$$\frac{a^{2^q}}{(a+b)(a^2+b^2)...(a^{2^{q-1}}+b^{2^{q-1}})}=\frac{a^{2^q}}{2^qP_1^2(a,b)(P_2^2(a,b))^2...(P_{2^q-1}^2(a,b))^{2^q-1}}$$

and, respectively,

$$s_q=\frac{(b_{q-1})^2}{a_{q-1}+b_{q-1}}=\frac{b^{2^q}}{2^qP_1^2(a,b)(P_2^2(a,b))^2...(P_{2^q-1}^2(a,b))^{2^q-1}}.$$

Hence, iterative use of the Crossed ladders diagram based on $(a_{q-1},b_{q-1})=(r_{q-1},s_{q-1})$ lead to

$$(r_q,s_q)=\left(\frac{a^{2^q}}{2^qP_1^2(a,b)(P_2^2(a,b))^2...(P_{2^q-1}^2(a,b))^{2^q-1}},\frac{b^{2^q}}{2^qP_1^2(a,b)(P_2^2(a,b))^2...(P_{2^q-1}^2(a,b))^{2^q-1}}\right)$$

and

$$r_q+s_q=\frac{(P_{2^q}^2(a,b))^{2^q}}{2^{q-1}P_1^2(a,b)(P_2^2(a,b))^2...(P_{2^q-1}^2(a,b))^{2^q-1}}.$$

Having constructed $P_1^2(a,b), P_2^2(a,b), ...$ and $P_{2^q-1}^2(a,b)$ we can now construct $P_{2^q}^2(a,b)$ by q sequential use of the right-angel triangel method shown above. The proof is complete. ■

In particular, knowing that $P_{2^q}^2(a,b)P_{-2^q}^2(a,b) = ab$ we can easily construct $P_{-2^q}^2(a,b)$ once we have constructed $P_{2^q}^2(a,b)$.

6.2 The case with $n = 2^m$ variables

By using formulas (10) and (11) we can write

$$P_{\pm 4}^4(a_1, a_2, a_3, a_4) = P_{\pm 4}^2(P_{\pm 4}^2(a_1, a_2), P_{\pm 4}^2(a_3, a_4)).$$

By iterative use of these formulas and of the methods shown earlier in this paper we can construct all power means of the type $P_{\pm 2^q}^n(a_1, ..., a_n)$, where the number of variables is $n = 2^m$, $m \in \mathbb{N}$, and for all powers $k = \pm 2^q$, $q \in \mathbb{N}$.

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