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Some inequalities for Cesàro means of double Vilenkin–Fourier series

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Abstract

In this paper, we state and prove some new inequalities related to the rate of L^p approximation by Cesàro means of the quadratic partial sums of double Vilenkin–Fourier series of functions from L^p.

MSC: 42C10; 42B25

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1 Introduction

Let N_+ denote the set of positive integers, and let $N := N_+ \cup \{0\}$. Let $m := (m_0, m_1, \ldots)$ be a sequence of positive integers not less than 2. Denote by $Z_{m_k} := \{0, 1, \ldots, m_k - 1\}$ the additive group of integers modulo m_k. Define the group G_m as the complete direct product of the groups Z_{m_j} with the product of the discrete topologies of Z_{m_j}.

The direct product of the measures

$$\mu_k(\{j\}) := \frac{1}{m_k} \quad (j \in Z_{m_k})$$

is the Haar measure on G_m with $\mu(G_m) = 1$. If the sequence m is bounded, then G_m is called a bounded Vilenkin group. In this paper, we consider only bounded Vilenkin groups. The elements of G_m can be represented by sequences $x := (x_0, x_1, \ldots, x_j, \ldots)$ $(x_j \in Z_{m_j})$. The group operation $+$ in G_m is given by

$$x + y = \bigl((x_0 + y_0) \bmod m_0, \ldots, (x_k + y_k) \bmod m_k, \ldots\bigr)$$

for $x := (x_0, \ldots, x_k, \ldots)$ and $y := (y_0, \ldots, y_k, \ldots) \in G_m$. The inverse of $+$ will be denoted by $-$.

It is easy to give a base for the neighborhoods of G_m:

$$I_0(x) := G_m,$$
$$I_n(x) := \{y \in G_m | y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\}$$

for $x \in G_m$ and $n \in N$. Define $I_n := I_n(0)$ for $n \in N_+$. Set $e_n := (0, \ldots, 0, 1, 0, \ldots) \in G_m$, where the nth coordinate of which is 1, and the rest are zeros $(n \in N)$.

We define the so-called generalized number system based on m as follows: $M_0 := 1$, $M_{k+1} := m_k M_k$ $(k \in N)$. Then every $n \in N$ can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j}$ $(j \in N_+)$, and only a finite number of n_j differ from zero. We also use the following notation: $|n| := \max\{k \in N : n_k \neq 0\}$ (i.e., $M_{|n|} \leq n < M_{|n|+1}$, $n \neq 0$). For $x \in G_m$, we denote $|x| := \sum_{j=0}^{\infty} \frac{x_j}{M_{j+1}}$ $(x_j \in Z_{m_j})$.

Next, we introduce on G_m an orthonormal system, which is called the Vilenkin system. First, we define the complex-valued functions $r_k(x) : G_m \to C$, the generalized Rademacher functions, as follows:

$$r_k(x) := \exp \frac{2\pi i x_k}{m_k} \quad \bigl(i^2 = -1, x \in G_m, k \in N\bigr).$$

Now we define the Vilenkin system $\psi := (\psi_n : n \in N)$ on G_m as

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \quad (n \in N).$$

In particular, if $m = 2$, then we call this system the Walsh–Paley system. Each ψ_n is a character of G_m, and all characters of G_m are of this norm. Moreover, $\psi_n(-x) = \bar{\psi}_n(x)$.

The Dirichlet kernels are defined by

$$D_n := \sum_{k=0}^{n-1} \psi_k \quad (n \in N_+).$$

Recall that (see [20] or [23])

$$D_{M_n}(x) = \begin{cases} M_n & \text{if } x \in I_n, \\ 0 & \text{if } x \notin I_n. \end{cases} \tag{1}$$

The Vilenkin system is orthonormal and complete in $L^1(G_m)$ (see [1]).

Next, we introduce some notation from the theory of two-dimensional Vilenkin system. Let $\tilde{m}$ be a sequence like m. The relation between the sequences $(\tilde{m}_n)$ and $(\tilde{M}_n)$ is the same as between sequences (m_n) and (M_n). The group $G_m \times G_{\tilde{m}}$ is called a two-dimensional Vilenkin group. The normalized Haar measure is denoted by μ as in the one-dimensional case. We also suppose that $m = \tilde{m}$ and $G_m \times G_{\tilde{m}} = G_m^2$.

The norm of the space $L^p(G_m^2)$ is defined by

$$\|f\|_p := \left(\int_{G_m^2} \bigl|f(x,y)\bigr|^p \, d\mu(x,y) \right)^{1/p} \quad (1 \leq p < \infty).$$

Denote by $C(G_m^2)$ the class of continuous functions on the group G_m^2 endowed with the supremum norm.

For brevity in notation, we write $L^\infty(G_m^2)$ instead of $C(G_m^2)$.

The two-dimensional Fourier coefficients, the rectangular partial sums of the Fourier series, and the Dirichlet kernels with respect to the two-dimensional Vilenkin system are defined as follows:

$$\widehat{f}(n_1,n_2) := \int_{G_m^2} f(x,y)\bar{\psi}_{n_1}(x)\bar{\psi}_{n_2}(y)\,d\mu(x,y),$$

$$S_{n_1,n_2}(x,y,f) := \sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2-1}\widehat{f}(k_1,k_2)\psi_{k_1}(x)\psi_{k_2}(y),$$

$$D_{n_1,n_2}(x,y) := D_{n_1}(x)D_{n_2}(y),$$

Denote

$$S_n^{(1)}(x,y,f) := \sum_{l=0}^{n-1}\widehat{f}(l,y)\bar{\psi}_l(x),$$

$$S_m^{(2)}(x,y,f) := \sum_{r=0}^{m-1}\widehat{f}(x,r)\bar{\psi}_r(y),$$

where

$$\widehat{f}(l,y) = \int_{G_m} f(x,y)\psi_l(x)\,d\mu(x)$$

and

$$\widehat{f}(x,r) = \int_{G_m} f(x,y)\psi_r(y)\,d\mu(y).$$

The $(C,-\alpha)$ means of the double Vilenkin–Fourier series are defined as follows:

$$\sigma_n^{-\alpha}(f,x,y) = \frac{1}{A_{n-1}^{-\alpha}}\sum_{j=1}^{n} A_{n-j}^{-\alpha-1}S_{j,j}(f,x,y),$$

where

$$A_0^{\alpha} = 1, \qquad A_n^{\alpha} = \frac{(\alpha+1)\cdots(\alpha+n)}{n!}.$$

It is well known that (see [28])

$$A_n^{\alpha} = \sum_{k=0}^{n} A_k^{\alpha-1}, \tag{2}$$

$$A_n^{\alpha} - A_{n-1}^{\alpha} = A_n^{\alpha-1}, \tag{3}$$

and

$$c_1(\alpha)n^{\alpha} \le A_n^{\alpha} \le c_2(\alpha)n^{\alpha}, \tag{4}$$

where positive constants c_1 and c_2 depend on α.

The dyadic partial moduli of continuity of a function $f \in L^p(G_m^2)$ in the L^p-norm are defined by

$$\omega_1\left(f, \frac{1}{M_n}\right)_p = \sup_{u \in I_n} \left\| f(\cdot + u, \cdot) - f(\cdot, \cdot)\right\|_p$$

and

$$\omega_2\left(f, \frac{1}{M_n}\right)_p = \sup_{v \in I_n} \left\| f(\cdot, \cdot + v) - f(\cdot, \cdot)\right\|_p,$$

whereas the dyadic mixed modulus of continuity is defined as follows:

$$\begin{aligned}
&\omega_{1,2}\left(f, \frac{1}{M_n}, \frac{1}{M_m}\right)_p \\
&\quad = \sup_{(u,v) \in I_n \times I_m} \left\| f(\cdot + u, \cdot + v) - f(\cdot + u, \cdot) - f(\cdot, \cdot + v) + f(\cdot, \cdot)\right\|_p.
\end{aligned}$$

It is clear that

$$\omega_{1,2}\left(f, \frac{1}{M_n}, \frac{1}{M_m}\right)_p \leq \omega_1\left(f, \frac{1}{M_n}\right)_p + \omega_2\left(f, \frac{1}{M_m}\right)_p.$$

The dyadic total modulus of continuity is defined by

$$\omega\left(f, \frac{1}{M_n}\right)_p = \sup_{(u,v) \in I_n \times I_n} \left\| f(\cdot + u, \cdot + v) - f(\cdot, \cdot)\right\|_p.$$

The problems of summability of partial sums and Cesàro means for Walsh–Fourier series were studied in [2, 13–19, 21, 22, 25, 26].

The convergence issue of Fejér (and Cesàro) means on the Walsh and Vilenkin groups for unbounded case were studied in [3–11].

In his monograph [27], Zhizhinashvili investigated the behavior of Cesàro (C, α)-means for double trigonometric Fourier series in detail. Goginava [18] studied the analogous question in the case of the Walsh system. In particular, the following theorems were proved.

Theorem A *Let f belong to $L^p(G_2)$ for some $p \in [1, \infty]$ and $\alpha \in (0,1)$. Then, for any $2^k \leq n < 2^{k+1}$ $(k, n \in N)$, we have the inequality*

$$\begin{aligned}
\left\| \sigma_{2^k}^{-\alpha}(f) - f\right\|_p \leq c(\alpha) \Bigg\{ & 2^{k\alpha} \omega_1\left(f, 1/2^{k-1}\right)_p + 2^{k\alpha} \omega_2\left(f, 1/2^{k-1}\right)_p \\
& + \sum_{r=0}^{k-2} 2^{r-k} \omega_1\left(f, 1/2^r\right)_p + \sum_{s=0}^{k-2} 2^{s-k} \omega_2\left(f, 1/2^s\right)_p \Bigg\}.
\end{aligned}$$

Theorem B *Let f belong to $L^p(G_2)$ for some $p \in [1,\infty]$ and $\alpha \in (0,1)$. Then, for any $2^k \leq n < 2^{k+1}$ $(k, n \in N)$, we have the inequality*

$$\begin{aligned}\left\|\sigma_n^{-\alpha}(f)-f\right\|_p \leq c(\alpha)\Bigg\{&2^{k\alpha}k\omega_1\left(f,1/2^{k-1}\right)_p+2^{k\alpha}k\omega_2\left(f,1/2^{k-1}\right)_p\\&+\sum_{r=0}^{k-2}2^{r-k}\omega_1\left(f,1/2^r\right)_p+\sum_{s=0}^{k-2}2^{s-k}\omega_2\left(f,1/2^s\right)_p\Bigg\}.\end{aligned}$$

In this paper, we state and prove analogous results in the case of double Vilenkin–Fourier series. Our main results are the following theorems.

Theorem 1 *Let f belong to $L^p(G_m^2)$ for some $p \in [1,\infty]$ and $\alpha \in (0,1)$. Then, for any $M_k \leq n < M_{k+1}$ $(k, n \in N)$, we have the inequality*

$$\begin{aligned}\left\|\sigma_{M_k}^{-\alpha}(f)-f\right\|_p \leq c(\alpha)\Bigg(&\omega_1(f,1/M_{k-1})_pM_k^\alpha+\omega_2(f,1/M_{l-1})_pM_k^\alpha\\&+\sum_{r=0}^{k-2}\frac{M_r}{M_k}\omega_1(f,1/M_r)_p+\sum_{s=0}^{k-2}\frac{M_s}{M_k}\omega_2(f,1/M_s)_p\Bigg).\end{aligned}$$

Theorem 2 *Let f belong to $L^p(G_m^2)$ for some $p \in [1,\infty]$ and $\alpha \in (0,1)$. Then, for any $M_k \leq n < M_{k+1}$ $(k, n \in N)$, we have the inequality*

$$\begin{aligned}&\left\|\sigma_n^{-\alpha}(f)-f\right\|_p\\&\quad\leq c(\alpha)\Bigg(\omega_1(f,1/M_{k-1})_pM_k^\alpha\log n+\omega_2(f,1/M_{l-1})_pM_k^\alpha\log n\\&\quad+\sum_{r=0}^{k-2}\frac{M_r}{M_k}\omega_1(f,1/M_r)_p+\sum_{s=0}^{k-2}\frac{M_s}{M_k}\omega_2(f,1/M_s)_p\Bigg).\end{aligned}$$

To make the proofs of these theorems clearer, we formulate some auxiliary lemmas in Sect. 2. Some of these lemmas are new and of independent interest. Detailed proofs can be found in Sect. 3.

2 Auxiliary lemmas

To prove Theorems 1 and 2, we need the following three lemmas (see [1, 12], and [8], respectively)

Lemma 1 *Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be real numbers. Then*

$$\frac{1}{n}\int_G\left|\sum_{k=1}^n\alpha_kD_k(x)\right|d\mu(x) \leq \frac{c}{\sqrt{n}}\left(\sum_{k=1}^n\alpha_k^2\right)^{1/2}.$$

Lemma 2 *Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be real numbers. Then*

$$\frac{1}{n}\int_{G_m^2}\left|\sum_{k=1}^n\alpha_kD_k(x)D_k(y)\right|d\mu(x,y) \leq \frac{c}{\sqrt{n}}\left(\sum_{k=1}^n\alpha_k^2\right)^{1/2}.$$

Lemma 3 *Let* $0 \le j < n_s M_s$ *and* $0 \le n_s < m_s$. *Then*

$$D_{n_s M_s - j} = D_{n_s M_s} - \psi_{n_s M_s - 1} \bar{D}_j.$$

We also need the following new nemmas of independent interest.

Lemma 4 *Let* f *belong to* $L^p(G_m^2)$ *for some* $p \in [1,\infty]$. *Then, for every* $\alpha \in (0,1)$, *we have the inequality*

$$I := \frac{1}{A_n^{-\alpha}} \left\| \int_{G_m^2} \sum_{i=1}^{M_{k-1}} A_{n-i}^{-\alpha-1} D_i(u) D_i(v) \bigl[f(\cdot - u, \cdot -) - f(\cdot,\cdot) \bigr] \, d\mu(u,v) \right\|_p \le \sum_{r=0}^{k-2} \frac{M_r}{M_k} \omega_1(f, 1/M_r)_p + \sum_{s=0}^{k-2} \frac{M_s}{M_k} \omega_2(f, 1/M_s)_p,$$

where $M_k \le n < M_{k+1}$.

Lemma 5 *Let* $\alpha \in (0,1)$ *and* $p = M_k, M_k + 1, \ldots$. *Then*

$$II := \int_{G_m^2} \left| \sum_{i=1}^{M_k} A_{p-i}^{-\alpha-1} D_i(u) D_i(v) \right| d\mu(u,v) \le c(\alpha) < \infty, \quad k = 1, 2, \ldots.$$

Lemma 6 *We have the inequality*

$$III := \int_{G_m^2} \left| \sum_{i=1}^{n} A_{n-i}^{-\alpha-1} D_i(u) D_i(v) \right| d\mu(u,v) \le c(\alpha) \log n$$

3 The detailed proofs

Proof of Lemma 3 Applying Abel's transformation, from (2) we get

$$\begin{aligned} I &\le \frac{1}{A_n^{-\alpha}} \left\| \int_{G_m^2} \sum_{i=1}^{M_{k-1}-1} A_{n-i}^{-\alpha-2} \sum_{l=1}^{i} D_i(u) D_i(v) \bigl[f(\cdot - u, \cdot - v) - f(\cdot,\cdot) \bigr] \, d\mu(u,v) \right\|_p \\ &\quad + \frac{1}{A_n^{-\alpha}} \left\| \int_{G_m^2} A_{n-M_{k-1}}^{-\alpha-1} \sum_{i=1}^{M_{k-1}} D_i(u) D_i(v) \bigl[f(\cdot - u, \cdot - v) - f(\cdot,\cdot) \bigr] \, d\mu(u,v) \right\|_p \\ &:= I_1 + I_2, \end{aligned} \tag{5}$$

where the first and second terms on the right side of inequality (5) are denoted by I_1 and I_2, respectively.

For I_2, we have the estimate

$$\begin{aligned}
I_2 &\leq \frac{1}{A_n^{-\alpha}} \left\| \int_{G_m^2} A_{n-M_{k-1}}^{-\alpha-1} \sum_{r=1}^{k-2} \sum_{i=M_r}^{M_{r+1}-1} D_i(u)D_i(v) \right. \\
&\quad \left. \times \left[f(\cdot - u, \cdot - v) - f(\cdot, \cdot) \right] \right\|_p d\mu(u,v) \\
&\leq \frac{1}{A_n^{-\alpha}} \left\| \int_{G_m^2} A_{n-M_{k-1}}^{-\alpha-1} \sum_{r=1}^{k-2} \sum_{i=M_r}^{M_{r+1}-1} D_i(u)D_i(v) \right. \\
&\quad \left. \times \left[f(\cdot - u, \cdot - v) - S_{M_r,M_r}(\cdot - u, \cdot - v, f) \right] d\mu(u,v) \right\|_p \\
&\quad + \frac{1}{A_n^{-\alpha}} \left\| \int_{G_m^2} A_{n-M_{k-1}}^{-\alpha-1} \sum_{r=1}^{k-2} \sum_{i=M_r}^{M_{r+1}-1} D_i(u)D_i(v) \right. \\
&\quad \left. \times \left[S_{M_r,M_r}(\cdot - u, \cdot - v, f) - S_{M_r,M_r}(\cdot, \cdot, f) \right] d\mu(u,v) \right\|_p \\
&\quad + \frac{1}{A_n^{-\alpha}} \left\| \int_{G_m^2} A_{n-M_{k-1}}^{-\alpha-1} \sum_{r=1}^{k-2} \sum_{i=M_r}^{M_{r+1}-1} D_i(u)D_i(v) \right. \\
&\quad \left. \times \left[S_{M_r,M_r}(\cdot, \cdot, f) - f(\cdot, \cdot) \right] d\mu(u,v) \right\|_p \\
&:= I_{21} + I_{22} + I_{23},
\end{aligned} \tag{6}$$

where the first, second, and third terms on the right side of inequality (6) are denoted by I_{21}, I_{22}, and I_{23}, respectively.

It is evident that

$$\begin{aligned}
&\int_{G_m^2} \sum_{i=M_r}^{M_{r+1}-1} D_i(u)D_i(v) \left[S_{M_r,M_r}(\cdot - u, \cdot - v, f) - S_{M_r,M_r}(\cdot, \cdot, f) \right] d\mu(u,v) \\
&\quad = \sum_{i=M_r}^{M_{r+1}-1} \left(\int_{G_m^2} D_i(u)D_i(v) S_{M_r,M_r}(\cdot - u, \cdot - v, f)\, d\mu(u,v) - S_{M_r,M_r}(\cdot, \cdot, f) \right) \\
&\quad = \sum_{i=M_r}^{M_{r+1}-1} \left(S_i\left(\cdot, \cdot, S_{M_r,M_r}(f)\right) - S_{M_r,M_r}(\cdot, \cdot, f) \right) \\
&\quad = \sum_{i=M_r}^{M_{r+1}-1} \left(S_{M_r,M_r}(\cdot, \cdot, f) - S_{M_r,M_r}(\cdot, \cdot, f) \right) = 0.
\end{aligned}$$

Hence

$$I_{22} = 0. \tag{7}$$

Moreover, by the generalized Minkowski inequality, Lemma 2, and by (1) and (4) we obtain

$$
\begin{aligned}
I_{21} &\le \frac{1}{A_n^{-\alpha}} \left|A_{n-M_{k-1}}^{-\alpha-1}\right| \sum_{r=1}^{k-2} \int_{G_m^2} \left| \sum_{i=M_r}^{M_{r+1}-1} D_i(u) D_i(v) \right| \\
&\quad \times \left\| f(\cdot - u, \cdot - v) - S_{M_r,M_r}(\cdot - u, \cdot - v, f) \right\|_p \, d\mu(u,v) \\
&\le \frac{c(\alpha)}{M_k} \sum_{r=1}^{k-2} \bigl(\omega_1(f, 1/M_r)_p + \omega_2(f, 1/M_r)_p \bigr) \\
&\quad \times \int_{G_m^2} \left| \sum_{i=M_r}^{M_{r+1}-1} D_i(x) D_i(y) \right| d\mu(u,v) \\
&\le c(\alpha) \sum_{r=1}^{k-2} \frac{M_r}{M_k} \bigl(\omega_1(f, 1/M_r)_p + \omega_2(f, 1/M_r)_p \bigr).
\end{aligned}
\tag{8}
$$

The estimation of I_{23} is analogous to that of I_{21}:

$$
I_{23} \le c(\alpha) \sum_{r=1}^{k-2} \frac{M_r}{M_k} \bigl(\omega_1(f, 1/M_r)_p + \omega_2(f, 1/M_r)_p \bigr).
\tag{9}
$$

Analogously, we can estimate I_1 as follows:

$$
\begin{aligned}
I_1 &\le \frac{1}{A_n^{-\alpha}} \sum_{r=1}^{k-2} \left\| \int_{G_m^2} \sum_{i=M_r}^{M_{r+1}-1} A_{n-i}^{-\alpha-2} \sum_{l=1}^{i} D_l(u) D_l(v) \right. \\
&\quad \left. \times \bigl[f(\cdot - u, \cdot - v) - S_{M_r,M_r}(\cdot - u, \cdot - v, f) \bigr] \, d\mu(u,v) \right\|_p \\
&\quad + \frac{1}{A_n^{-\alpha}} \sum_{r=1}^{k-2} \left\| \int_{G_m^2} \sum_{i=M_r}^{M_{r+1}-1} A_{n-i}^{-\alpha-2} \sum_{l=1}^{i} D_l(u) D_l(v) \right. \\
&\quad \left. \times \bigl[S_{M_r,M_r}(\cdot - u, \cdot - v, f) - S_{M_r,M_r}(\cdot, \cdot, f) \bigr] \right\|_p \, d\mu(u,v) \\
&\quad + \frac{1}{A_n^{-\alpha}} \sum_{r=1}^{k-2} \left\| \int_{G_m^2} \sum_{i=M_r}^{M_{r+1}-1} A_{n-i}^{-\alpha-2} \sum_{l=1}^{i} D_l(u) D_l(v) \right. \\
&\quad \left. \times \bigl[S_{M_r,M_r}(\cdot, \cdot, f) - f(\cdot, \cdot) \bigr] \, d\mu(u,v) \right\|_p \\
&\le \frac{1}{A_n^{-\alpha}} \sum_{r=1}^{k-2} \int_{G_m^2} \left| \sum_{i=M_r}^{M_{r+1}-1} A_{n-i}^{-\alpha-2} \sum_{l=1}^{i} D_l(u) D_l(v) \right| \\
&\quad \times \left\| f(\cdot - u, \cdot - v) - S_{M_r,M_r}(\cdot - u, \cdot - v, f) \right\|_p \, d\mu(u,v)
\end{aligned}
$$

$$+\frac{1}{A_n^{-\alpha}}\sum_{r=1}^{k-2}\int_{G_m^2}\left|\sum_{i=M_r}^{M_{r+1}-1}A_{n-i}^{-\alpha-2}\sum_{l=1}^{i}D_l(u)D_l(v)\right|$$

$$\times\left\|S_{M_r,M_r}(\cdot,\cdot,f)-f(\cdot,\cdot)\right\|_p\,d\mu(u,v)$$

$$\leq c(\alpha)M_k^{\alpha}\sum_{r=1}^{k-2}\sum_{i=M_r}^{M_{r+1}-1}(n-i)^{-\alpha-2}i\big(\omega_1(f,1/M_r)_p+\omega_2(f,1/M_r)_p\big)$$

$$\leq c(\alpha)M_k^{\alpha}\sum_{r=1}^{k-2}\sum_{i=M_r}^{M_{r+1}-1}(n-M_{r+1}-1)^{-\alpha-2}i\big(\omega_1(f,1/M_r)_p+\omega_2(f,1/M_r)_p\big)$$

$$\leq c(\alpha)\sum_{r=0}^{k-2}\frac{M_r}{M_k}\big(\omega_1(f,1/M_r)_p+\omega_2(f,1/M_r)_p\big). \tag{10}$$

By combining (7)–(9) with (10) for I we find that

$$I\leq c(\alpha)\sum_{r=0}^{k-2}\frac{M_r}{M_k}\big(\omega_1(f,1/M_r)_p+\omega_2(f,1/M_r)_p\big). \tag{11}$$

The proof of Lemma 3 is complete. □

Proof of Lemma 4 It is evident that

$$II\leq\int_{G_m^2}\left|\sum_{i=1}^{M_k-1}A_{p-M_k+i}^{-\alpha-1}D_{M_k-i}(u)D_{M_k-i}(v)\right|d\mu(u,v)$$

$$+\left|A_{p-M_k}^{-\alpha-1}\right|\int_{G_m^2}D_{M_k}(u)D_{M_k}(v)\,d\mu(u,v)$$

$$:=II_1+II_2, \tag{12}$$

where the first and second terms on the right side of inequality (12) are denoted by II_1 and II_2, respectively.

From (1) by $|A_{p-M_k}^{-\alpha-1}|\leq 1$ we get that

$$II_2\leq 1. \tag{13}$$

Moreover, by Lemma 3 we have that

$$II_1\leq\int_{G_m^2}\left|\sum_{i=1}^{M_k-1}A_{p-M_k+i}^{-\alpha-1}\bar{D}_i(u)\bar{D}_i(v)\right|d\mu(u,v)$$

$$+\int_{G_m^2}D_{M_k}(u)\left|\sum_{i=1}^{M_k-1}A_{p-M_k+i}^{-\alpha-1}\bar{D}_i(v)\right|d\mu(u,v)$$

$$+\int_{G_m^2}D_{M_k}(v)\left|\sum_{i=1}^{M_k-1}A_{p-M_k+i}^{-\alpha-1}\bar{D}_i(u)\right|d\mu(u,v)$$

$$
\begin{aligned}
&+ \left|\sum_{i=1}^{M_k-1} A_{p-M_k+i}^{-\alpha-1}\right| \int_{G_m^2} D_{M_k}(u) D_{M_k}(v)\, d\mu(u,v) \\
&:= II_{11} + II_{12} + II_{13} + II_{14},
\end{aligned}
\tag{14}
$$

where the first, second, third, and fourth terms on the right side of inequality (14) are denoted by II_{11}, II_{12}, II_{13}, and II_{14} respectively.

From (1) and (4) it follows that

$$
II_{14} \le c(\alpha) \sum_{v=1}^{\infty} v^{-\alpha-1} < \infty. \tag{15}
$$

By Applying Abel's transformation, in view of Lemma 2, we have that

$$
\begin{aligned}
II_{11} &\le \int_{G_m^2} \left| \sum_{i=1}^{M_k-2} A_{p-M_k+i}^{-\alpha-2} \sum_{l=1}^{i} \bar{D}_l(u) \bar{D}_l(v) \right| d\mu(u,v) \\
&\quad + \int_{G_m^2} \left| A_{p-1}^{-\alpha-1} \sum_{i=1}^{M_k-1} \bar{D}_i(u) \bar{D}_i(v) \right| d\mu(u,v) \\
&\le c(\alpha) \left\{ \sum_{v=1}^{M_k-2} (p - M_k + i)^{-\alpha-2} i + (p-1)^{-\alpha-1} M_k \right\} \\
&\le c(\alpha) \left\{ \sum_{i=1}^{\infty} i^{-\alpha-1} + M_k^{-\alpha} \right\} < \infty.
\end{aligned}
\tag{16}
$$

The estimation of II_{12} and II_{13} are analogous to the estimation of II_{11}. Applying Abel's transformation, in view of Lemma 1, we find that

$$
\begin{aligned}
II_{12} &\le \int_{G_m^2} D_{M_k}(u) \left| \sum_{i=1}^{M_k-2} A_{p-M_k+i}^{-\alpha-2} \sum_{l=1}^{i} \bar{D}_l(v) \right| d\mu(u,v) \\
&\quad + \int_{G_m^2} D_{M_k}(u) \left| A_{p-1}^{-\alpha-1} \sum_{i=1}^{M_k-1} \bar{D}_i(v) \right| d\mu(u,v) \\
&\le c(\alpha) \left\{ \sum_{v=1}^{M_k-2} (p - M_k + i)^{-\alpha-2} i + (p-1)^{-\alpha-1} M_k \right\} \\
&\le c(\alpha) \left\{ \sum_{i=1}^{\infty} i^{-\alpha-1} + M_k^{-\alpha} \right\} < \infty
\end{aligned}
\tag{17}
$$

and

$$
\begin{aligned}
III_{12} &\le \int_{G_m^2} D_{M_k}(v) \left| \sum_{i=1}^{M_k-2} A_{p-M_k+i}^{-\alpha-2} \sum_{l=1}^{i} \bar{D}_l(u) \right| d\mu(u,v) \\
&\quad + \int_{G_m^2} D_{M_k}(v) \left| A_{p-1}^{-\alpha-1} \sum_{i=1}^{M_k-1} \bar{D}_i(u) \right| d\mu(u,v)
\end{aligned}
$$

$$
\begin{aligned}
&\leq c(\alpha)\left\{\sum_{\nu=1}^{M_k-2}(p-M_k+i)^{-\alpha-2}i+(p-1)^{-\alpha-1}M_k\right\} \\
&\leq c(\alpha)\left\{\sum_{i=1}^{\infty}i^{-\alpha-1}+M_k^{-\alpha}\right\}<\infty. \qquad (18)
\end{aligned}
$$

The proof is complete by combining (12)–(18). □

Proof of Lemma 5 Let

$$n=n_{k_1}M_{k_1}+\cdots+n_{k_s}M_{k_s}, \quad k_1>\cdots>k_s\geq 0.$$

Denote

$$n^{(i)}=n_{k_i}M_{k_i}+\cdots+n_{k_s}M_{k_s}, \quad i=1,2,\ldots,s.$$

Since (see [20])

$$D_{j+n_AM_A}=D_{n_AM_A}+\psi_{n_AM_A}D_j, \qquad (19)$$

we find that

$$
\begin{aligned}
III &\leq \int_{G_m^2}\left|\sum_{i=1}^{n_{k_1}M_{k_1}}A_{n-i}^{-\alpha-1}D_i(u)D_i(v)\right|d\mu(u,v) \\
&\quad+\int_{G_m^2}\left|\sum_{i=1}^{n^{(2)}}A_{n^{(2)}-i}^{-\alpha-1}D_i(u)D_i(v)\right|d\mu(u,v) \\
&\quad+\int_{G_m^2}D_{n_{k_1}M_{k_1}}(u)D_{n_{k_1}M_{k_1}}(v)\left|\sum_{i=1}^{n^{(2)}}A_{n^{(2)}-i}^{-\alpha-1}\right|d\mu(u,v) \\
&\quad+\int_{G_m^2}D_{n_{k_1}M_{k_1}}(u)\left|\sum_{i=1}^{n^{(2)}}A_{n^{(2)}-i}^{-\alpha-1}D_i(v)\right|d\mu(u,v) \\
&\quad+\int_{G_m^2}D_{n_{k_1}M_{k_1}}(v)\left|\sum_{i=1}^{n^{(2)}}A_{n^{(2)}-i}^{-\alpha-1}D_i(u)\right|d\mu(u,v) \\
&:=III_1+III_2+III_3+III_4+III_5, \qquad (20)
\end{aligned}
$$

where the first, second, third, fourth, and fifth terms on the right side of inequality (20) are denoted by III_1, III_2, III_3, III_4, and III_5, respectively.

By (1) we have that

$$III_3\leq c(\alpha). \qquad (21)$$

Moreover, since (see [24])

$$\left|\sum_{i=1}^{n}A_{n-i}^{-\alpha-1}D_i(u)\right|=O\bigl(|u|^{\alpha-1}\bigr), \qquad (22)$$

for III_4, we get that

$$\begin{aligned} III_4 &\le \int_{G_m^2} D_{n_{k_1}M_{k_1}}(u)|v|^{\alpha-1}\,d\mu(u,v) \\ &\le \int_{G_m} |v|^{\alpha-1}\,d\mu(v) = \frac{1}{\alpha} < \infty. \end{aligned} \tag{23}$$

Analogously, we find that

$$\begin{aligned} III_5 &\le \int_{G_m^2} D_{n_{k_1}M_{k_1}}(v)|u|^{\alpha-1}\,d\mu(u,v) \\ &\le \int_{G_m} |u|^{\alpha-1}\,d\mu(v) = \frac{1}{\alpha} < \infty. \end{aligned} \tag{24}$$

For $r \in \{0,\dots m_A - 1\}$ and $0 \le j < M_A$ (see [20]), this yields that

$$D_{j+rM_A} = \left(\sum_{q=0}^{r-1} \psi_{M_A}^q\right) D_{M_A} + \psi_{M_A}^r D_j.$$

Thus we have

$$\begin{aligned} &\int_{G_m^2} \sum_{i=1}^{n_{k_1}M_{k_1}-1} A_{n-i}^{-\alpha-1} D_i(u)D_i(v)\,d\mu(u,v) \\ &\le \int_{G_m^2} \sum_{r=0}^{n_{k_1}-1} \sum_{i=0}^{M_{k_1}-1} A_{n-i-rM_{k_1}}^{-\alpha-1} D_{i+rM_{k_1}}(u)D_{i+rM_{k_1}}(v)\,d\mu(u,v) \\ &\le \int_{G_m^2} \sum_{r=0}^{n_{k_1}-1} \sum_{i=0}^{M_{k_1}-1} A_{n-i-rM_{k_1}}^{-\alpha-1} \left(\sum_{q=0}^{r-1} \psi_{M_{k_1}}^q\right) D_{M_{k_1}}(u) \\ &\quad \times \left(\sum_{q=0}^{r-1} \psi_{M_{k_1}}^q\right) D_{M_{k_1}}(v)\,d\mu(u,v) \\ &\quad + \int_{G_m^2} \sum_{r=0}^{n_{k_1}-1} \sum_{i=0}^{M_{k_1}-1} A_{n-i-rM_{k_1}}^{-\alpha-1} \left(\sum_{q=0}^{r-1} \psi_{M_{k_1}}^q\right) D_{M_{k_1}}(u)\psi_{M_A}^r D_i(v)\,d\mu(u,v) \\ &\quad + \int_{G_m^2} \sum_{r=0}^{n_{k_1}-1} \sum_{i=0}^{M_{k_1}-1} A_{n-i-rM_{k_1}}^{-\alpha-1} \psi_{M_A}^r D_i(u) \left(\sum_{q=0}^{r-1} \psi_{M_{k_1}}^q\right) D_{M_{k_1}}(v)\,d\mu(u,v) \\ &\quad + \int_{G_m^2} \sum_{r=0}^{n_{k_1}-1} \sum_{i=0}^{M_{k_1}-1} A_{n-i-rM_{k_1}}^{-\alpha-1} \psi_{M_A}^r D_i(u)\psi_{M_A}^r D_i(v)\,d\mu(u,v). \end{aligned}$$

On the other hand, by (1) and (4) we obtain that

$$\int_{G_m^2} A_{n-n_{k_1}M_{k_1}}^{-\alpha-1} D_{n_{k_1}M_{k_1}}(u)D_{n_{k_1}M_{k_1}}(v)\,d\mu(u,v) \le c(\alpha).$$

Consequently, for III_1, we have the estimate

$$\begin{aligned} III_1 &\le \int_{G_m^2} D_{M_{k_1}}(u) D_{M_{k_1}}(v) \left| \sum_{r=0}^{n_{k_1}-1} \sum_{i=1}^{M_{k_1}} A_{n-i-rM_{k_1}}^{-\alpha-1} \right| d\mu(u,v) \\ &\quad + \int_{G_m^2} D_{M_{k_1}}(u) \left| \sum_{r=0}^{n_{k_1}-1} \sum_{i=1}^{M_{k_1}} A_{n-i-rM_{k_1}}^{-\alpha-1} D_i(v) \right| d\mu(u,v) \\ &\quad + \int_{G_m^2} D_{M_{k_1}}(v) \left| \sum_{r=0}^{n_{k_1}-1} \sum_{i=1}^{M_{k_1}} A_{n-i-rM_{k_1}}^{-\alpha-1} D_i(u) \right| d\mu(u,v) \\ &\quad + \int_{G_m^2} \left| \sum_{r=0}^{n_{k_1}-1} \sum_{i=1}^{M_{k_1}} A_{n-i-rM_{k_1}}^{-\alpha-1} D_i(u) D_i(v) \right| d\mu(u,v) + c(\alpha) \\ &:= III_{11} + III_{12} + III_{13} + III_{14} + c(\alpha), \end{aligned} \tag{25}$$

where the first, second, third, and fourth terms on the right side of inequality (25) are denoted by III_{11}, III_{12}, III_{13}, and III_{14}, respectively.

From Lemma 4 we have that

$$III_{14} \le c(\alpha). \tag{26}$$

The estimation of III_{11} is analogous to that of III_3, and we find that

$$III_{11} \le c(\alpha). \tag{27}$$

The estimation of III_{12} and III_{13} is analogous to that of III_4, and we obtain that

$$III_{12} < \infty \tag{28}$$

and

$$III_{13} < \infty. \tag{29}$$

After substituting (21) and (23)–(29) into (20), we conclude that

$$\begin{aligned} &\int_{G_m^2} \left| \sum_{i=1}^{n} A_{n-i}^{-\alpha-1} D_i(u) D_i(v) \right| d\mu(u,v) \\ &\quad \le \int_{G_m^2} \left| \sum_{i=1}^{n^{(2)}} A_{n^{(2)}-i}^{-\alpha-1} D_i(u) D_i(v) \right| d\mu(u,v) + c(\alpha) \\ &\quad \le \cdots \le \int_{G_m^2} \left| \sum_{i=1}^{n^{(s)}} A_{n^{(s)}-i}^{-\alpha-1} D_i(u) D_i(v) \right| d\mu(u,v) + c(\alpha)s \\ &\quad \le c(\alpha) + c(\alpha)s \le c(\alpha) \log n. \end{aligned}$$

The proof is complete. □

Now we are ready to prove the main results.

Proof of Theorem 1 It is evident that

$$\begin{aligned}
&\left\|\sigma_{M_k}^{-\alpha}(f)-f\right\|_p\\
&\quad\leq \frac{1}{A_{M_k-1}^{-\alpha}}\left\|\int_{G_m^2}\sum_{i=1}^{M_{k-1}}A_{M_k-i}^{-\alpha-1}D_i(u)D_i(v)\big[f(\cdot-u,\cdot-v)-f(\cdot,\cdot)\big]\,d\mu(u,v)\right\|_p\\
&\qquad+\frac{1}{A_{M_k-1}^{-\alpha}}\left\|\int_{G_m^2}\sum_{i=M_{k-1}+1}^{M_k}A_{M_k-i}^{-\alpha-1}D_i(u)D_i(v)\big[f(\cdot-u,\cdot-v)-f(\cdot,\cdot)\big]\,d\mu(u,v)\right\|_p\\
&\quad:=I+II.
\end{aligned}\tag{30}$$

From Lemma 5 it follows that

$$I\leq c(\alpha)\sum_{r=0}^{k-2}\frac{M_r}{M_k}\big(\omega_1(f,1/M_r)_p+\omega_2(f,1/M_r)_p\big).\tag{31}$$

Moreover, for II, we have the estimate

$$\begin{aligned}
II&\leq \frac{1}{A_{M_k-1}^{-\alpha}}\left\|\int_{G_m^2}\sum_{i=M_{k-1}+1}^{M_k}A_{M_k-i}^{-\alpha-1}D_i(u)D_i(v)\right.\\
&\qquad\left.\times\big[f(\cdot-u,\cdot-v)-S_{M_{k-1}}^{(1)}(\cdot-u,\cdot-v,f)\big]\,d\mu(u,v)\right\|_p\\
&\qquad+\frac{1}{A_{M_k-1}^{-\alpha}}\left\|\int_{G_m^2}\sum_{i=M_{k-1}+1}^{M_k}A_{M_k-i}^{-\alpha-1}D_i(u)D_i(v)\right.\\
&\qquad\left.\times\big[S_{M_{k-1}}^{(1)}(\cdot-u,\cdot-v,f)-f(\cdot,\cdot)\big]\,d\mu(u,v)\right\|_p\\
&:=II_1+II_2,
\end{aligned}\tag{32}$$

where the first and second terms on the right side of inequality (32) are denoted by II_1 and II_2, respectively.

In view of the generalized Minkowski inequality, by (4) and Lemma 5 we get that

$$\begin{aligned}
II_1&\leq \frac{1}{A_{M_k-1}^{-\alpha}}\int_{G_m^2}\left|\sum_{i=M_{k-1}+1}^{M_k}A_{M_k-i}^{-\alpha-1}D_i(u)D_i(v)\right|\\
&\qquad\times\left\|f(\cdot-u,\cdot-v)-S_{M_{k-1}}^{(1)}(\cdot-u,\cdot-v,f)\right\|_p\,d\mu(u,v)\\
&\leq c(\alpha)M_k^{\alpha}\omega_1(f,1/M_{k-1})_p.
\end{aligned}\tag{33}$$

The estimation of II_2 is analogous to that of II_1, and we find that

$$II_2\leq c(\alpha)M_k^{\alpha}\omega_2(f,1/M_{k-1})_p.\tag{34}$$

Combining (30)–(34), we obtain the proof of Theorem 1. □

Proof of Theorem 2 It is evident that

$$\begin{aligned}
&\left\|\sigma_n^{-\alpha}(f)-f\right\|_p \\
&\leq \frac{1}{A_{n-1}^{-\alpha}}\left\|\int_{G_m^2}\sum_{i=1}^{M_{k-1}} A_{n-i}^{-\alpha-1}D_i(u)D_i(v)\bigl[f(\cdot-u,\cdot-v)-f(\cdot,\cdot)\bigr]\,d\mu(u,v)\right\|_p \\
&\quad+\frac{1}{A_{n-1}^{-\alpha}}\left\|\int_{G_m^2}\sum_{i=M_{k-1}+1}^{M_k} A_{n-i}^{-\alpha-1}D_i(u)D_i(v)\bigl[f(\cdot-u,\cdot-v)-f(\cdot,\cdot)\bigr]\,d\mu(u,v)\right\|_p \\
&\quad+\frac{1}{A_{n-1}^{-\alpha}}\left\|\int_{G_m^2}\sum_{i=M_k+1}^{n} A_{n-i}^{-\alpha-1}D_i(u)D_i(v)\bigl[f(\cdot-u,\cdot-v)-f(\cdot,\cdot)\bigr]\,d\mu(u,v)\right\|_p \\
&:= I+II+III,
\end{aligned} \tag{35}$$

where the first, second, and third terms on the right side of inequality (35) are denoted by I, II, and III, respectively.

From Lemma 4 it follows that

$$I\leq c(\alpha)\sum_{r=0}^{k-2}\frac{M_r}{M_k}\bigl(\omega_1(f,1/M_r)_p+\omega_2(f,1/M_r)_p\bigr). \tag{36}$$

Next, we repeat the arguments just in the same way as in the proof of Theorem 1 and find that

$$II\leq c(\alpha)M_k^{\alpha}\bigl(\omega_1(f,1/M_{k-1})_p+\omega_2(f,1/M_{k-1})_p\bigr). \tag{37}$$

On the other hand, for III, we have

$$\begin{aligned}
III &\leq \frac{1}{A_{n-1}^{-\alpha}}\left\|\int_{G_m^2}\sum_{i=M_k+1}^{n} A_{n-i}^{-\alpha-1}D_i(u)D_i(v)\right. \\
&\quad\left.\times\bigl[f(\cdot-u,\cdot-v)-f(\cdot,\cdot)\bigr]\right\|_p\,d\mu(u,v) \\
&\leq \frac{1}{A_{n}^{-\alpha}}\left\|\int_{G_m^2}\sum_{i=M_k+1}^{n} A_{n-i}^{-\alpha-1}D_i(u)D_i(v)\right. \\
&\quad\left.\times\bigl[f(\cdot-u,\cdot-v)-S_{M_k,M_k}(\cdot-u,\cdot-v,f)\bigr]\,d\mu(u,v)\right\|_p \\
&\leq \frac{1}{A_{n}^{-\alpha}}\left\|\int_{G_m^2}\sum_{i=M_k+1}^{n} A_{n-i}^{-\alpha-1}D_i(u)D_i(v)\right. \\
&\quad\left.\times\bigl[S_{M_k,M_k}(\cdot-u,\cdot-v,f)-S_{M_k,M_k}(\cdot,\cdot,f)\bigr]\,d\mu(u,v)\right\|_p
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{A_n^{-\alpha}} \left\| \int_{G_m^2} \sum_{i=M_k+1}^{n} A_{n-i}^{-\alpha-1} D_i(u) D_i(v) \right. \\
&\quad \left. \times \left[S_{M_k,M_k}(\cdot,\cdot,f) - f(\cdot,\cdot) \right] d\mu(u,v) \right\|_p \\
&:= III_1 + III_2 + III_3, \qquad (38)
\end{aligned}
$$

where the first, second, and third terms on the right side of inequality (38) are denoted by III_1, III_2, and III_3, respectively.

It is easy to show that

$$III_2 = 0. \qquad (39)$$

By the generalized Minkowski inequality and Lemma 5, for III_1, we obtain that

$$
\begin{aligned}
III_1 &\le \frac{1}{A_n^{-\alpha}} \int_{G_m^2} \left| \sum_{i=M_k+1}^{n} A_{n-i}^{-\alpha-1} D_i(u) D_i(v) \right| \\
&\quad \times \left\| f(\cdot - u, \cdot - v) - S_{M_r,M_r}(\cdot - u, \cdot - v, f) \right\|_p d\mu(u,v) \\
&\le c(\alpha) M_k^{\alpha} \left(\omega_1(f, 1/M_{k-1})_p + \omega_2(f, 1/M_{k-1})_p \right) \\
&\quad \times \int_{G_m^2} \left| \sum_{v=M_k+1}^{n} A_{n-v}^{-\alpha-1} D_v(u) D_v(v) \right| d\mu(u,v) \\
&\le c(\alpha) M_k^{\alpha} \log n \left(\omega_1(f, 1/M_{k-1})_p + \omega_2(f, 1/M_{k-1})_p \right). \qquad (40)
\end{aligned}
$$

The estimation of III_3 is analogous to that of III_2, and we find that

$$III_3 \le c(\alpha) M_k^{\alpha} \log n \left(\omega_1(f, 1/M_{k-1})_p + \omega_2(f, 1/M_{k-1})_p \right). \qquad (41)$$

After substituting (36)–(37) and (41) into (35), we obtain the proof of Theorem 2. □

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