

MULTI-DIMENSIONAL HARDY TYPE INEQUALITIES IN HÖLDER SPACES

EVGENIYA BURTSEVA, STAFFAN LUNDBERG, LARS-ERIK PERSSON AND
NATASHA SAMKO

(Communicated by J. Pečarić)

Abstract. Most Hardy type inequalities concern boundedness of the Hardy type operators in Lebesgue spaces. In this paper we prove some new multi-dimensional Hardy type inequalities in Hölder spaces.

1. Introduction

The original Hardy inequality from 1925 (see [2]) reads:

$$\int_0^{\infty} \left(\frac{1}{x} \int_0^x f(y) dy \right)^p dx \leq \left(\frac{p}{p-1} \right)^p \int_0^{\infty} f^p(x) dx, \quad p > 1.$$

Since the constant $\left(\frac{p}{p-1}\right)^p$ is sharp this means that the Hardy operator H defined by $Hf(x) := \frac{1}{x} \int_0^x f(y) dy$ maps L_p into L_p with the operator norm $p' := \frac{p}{p-1}$.

After this fundamental discovery by Hardy it was an almost unbelievable development of this area which today usually is referred to as Hardy type inequalities. A great number of papers and even books have been published on the subject and the research in this area is still very intensive. One important reason for that is that Hardy type inequalities are especially useful for various types of applications within different parts of Mathematics but also in other Sciences, see e.g. the books [5], [6] and [7] and the references therein.

Most of the developments described above are devoted to study the boundedness of Hardy type operators between weighted Lebesgue spaces and most of the results are for the one-dimensional case. But for applications it is also often required to consider the boundedness between other function spaces. Unfortunately, there exist not so many results concerning the boundedness of Hardy type operators in other function spaces. However, some results of this type can be found in Chapter 11 of the

Mathematics subject classification (2010): 46E15, 26D15, 47B38.

Keywords and phrases: Inequalities, Hardy type inequalities, Hölder spaces, Hardy type operators, boundedness, compactification.

book [6], where it is reported on Hardy type inequalities in Orlicz, Lorentz and rearrangement invariant spaces and also on some really first not complete results in general Banach function spaces. Moreover, in [15] some corresponding Hardy type inequalities in weighted Morrey spaces were proved; in [13] the weighted estimates for multi-dimensional Hardy type operators were proved in generalized Morrey spaces; in [1] was proved the weighted boundedness of some multi-dimensional Hardy type operators from generalized Morrey to Orlicz-Morrey spaces. For more information concerning Hardy type inequalities in Morrey type spaces and their applications we refer to [1], [9], [10], [12], [16] and references therein.

In this paper we continue this research by investigating Hardy type inequalities in Hölder spaces in the multi-dimensional case. Hölder spaces on unbounded sets can be defined with compactification at infinity (see Definition 3.1) or without.

We study multi-dimensional Hardy operators of order $\alpha \in [0, 1)$ as defined in (1.1). We refer to the paper [19] where a version of Hardy operators of the order $\alpha = 0$ was studied within the frameworks of Triebel-Lizorkin spaces. This version may be regarded as a one-dimensional Hardy type operator in a given direction $\frac{x}{|x|}$ of a function f of many variables. Multi-dimensional Hardy operators in our paper are of different nature.

By $C^\lambda(\Omega)$, $0 < \lambda \leq 1$, where Ω is an open set in \mathbb{R}^n , $\Omega \subseteq \mathbb{R}^n$, $n \geq 1$, we denote the class of bounded Hölder continuous functions, defined by the seminorm

$$[f]_\lambda := \sup_{\substack{x, x+h \in \Omega \\ |h| < 1}} \frac{|f(x+h) - f(x)|}{|h|^\lambda} < \infty.$$

Equipped with the norm

$$\|f\|_{C^\lambda} = \sup_{x \in \Omega} |f(x)| + [f]_\lambda$$

$C^\lambda(\Omega)$ is a Banach space. We shall deal with the case $\Omega = B_R$, where $B_R = B(0, R) := \{x \in \mathbb{R}^n : |x| < R\}$, $0 < R \leq \infty$.

We consider the Hardy type operators

$$H^\alpha f(x) = |x|^{\alpha-n} \int_{|y| < |x|} f(y) dy \quad \text{and} \quad \mathcal{H}^\alpha f(x) = |x|^\alpha \int_{|y| > |x|} \frac{f(y)}{|y|^n} dy, \quad \alpha \geq 0, \quad (1.1)$$

where $x \in B_R$, $0 < R \leq \infty$ for the operator H^α , and $R = \infty$ for the operator \mathcal{H}^α . We write $H = H^\alpha$ and $\mathcal{H} = \mathcal{H}^\alpha$ in the case $\alpha = 0$.

The operator H^α , $\alpha = 0$, may be considered in both with and without compactification settings, but a consideration of \mathcal{H} requires the compactification due to the needed convergence of integrals at infinity. We provide details for the operator H^α , $\alpha \geq 0$, without compactification, and for both the operators H and \mathcal{H} with compactification. We also show that in the setting of the spaces with compactification we may consider only the case $\alpha = 0$.

In Sections 2 and 3 we present and prove our new results on the boundedness of the Hardy type operator H^α in Hölder spaces without compactification (Theorem 2.2),

and for the operators H and \mathcal{H} in the case with compactification (Theorems 3.5 and 3.6).

2. Boundedness of the Hardy type operator H^α in a Hölder type space

Denote

$$C_0^\lambda(B_R) = \{f \in C^\lambda(B_R) : f(0) = 0\}.$$

For the Hardy operator H^α defined by

$$H^\alpha f(x) := |x|^{\alpha-n} \int_{|y|<|x|} f(y)dy, \quad \alpha \geq 0,$$

we show that it maps Hölder space into itself in the case $\alpha = 0$ and we prove a boundedness result of the type $C^\lambda \rightarrow C^{\lambda+\alpha}$ in the case $\alpha > 0$ provided that $\lambda + \alpha \leq 1$, see Theorem 2.2.

In the case $\alpha > 0$ we will need the following Lemma:

LEMMA 2.1. *Let*

$$g(r) = \frac{1}{r^n} \int_{|y|<r} f(y)dy, \quad 0 < r < R,$$

where $f \in C^\lambda(B_R)$, $0 < \lambda \leq 1$, $0 < R \leq \infty$. Then

$$|g'(r)| \leq C_{n,\lambda} \frac{[f]_\lambda}{r^{1-\lambda}}, \quad 0 < r < R, \tag{2.1}$$

where $C_{n,\lambda}$ depends only on n and λ .

Proof. Passing to polar coordinates, we have

$$g(r) = \frac{1}{r^n} \int_0^r t^{n-1} \Phi(t)dt, \quad \Phi(t) = \int_{\mathbb{S}^{n-1}} f(t\sigma)d\sigma.$$

Hence,

$$g'(r) = -\frac{n}{r^{n+1}} \int_0^r t^{n-1} \Phi(t)dt + \frac{\Phi(r)}{r} = \frac{n}{r^{n+1}} \int_0^r t^{n-1} [\Phi(r) - \Phi(t)]dt.$$

Therefore,

$$|g'(r)| \leq \frac{n}{r^{n+1}} \int_0^r t^{n-1} |\Phi(r) - \Phi(t)|dt.$$

It is easily seen that

$$|\Phi(r) - \Phi(t)| \leq [f]_\lambda |\mathbb{S}^{n-1}|(r-t)^\lambda.$$

Consequently,

$$|g'(r)| \leq \frac{n|\mathbb{S}^{n-1}|[f]_\lambda}{r^{n+1}} \int_0^r t^{n-1}(r-t)^\lambda dt = \frac{n|\mathbb{S}^{n-1}|[f]_\lambda}{r^{1-\lambda}} \int_0^1 s^{n-1}(1-s)^\lambda ds,$$

and we arrive at (2.1). The proof is complete. \square

In the following theorem we deal also with the space $\tilde{C}_0^\lambda(\Omega)$ consisting of functions f for which $[f]_\lambda < \infty$ and $f(0) = 0$. This space contains functions which are unbounded in the case Ω is unbounded. Note that $[f]_\lambda$ is a norm in this space.

Now we are in a position to prove the following theorem:

THEOREM 2.2. *Let $\alpha \geq 0$, $\lambda > 0$ and $\lambda + \alpha \leq 1$. In the case $\alpha = 0$ the Hardy operator H^α is bounded in $C^\lambda(B_R)$ and $[H^\alpha f|_{\alpha=0}]_\lambda \leq C[f]_\lambda$. In the case $\alpha > 0$ the operator H^α is bounded from $\tilde{C}_0^\lambda(B_R)$ into $\tilde{C}_0^{\lambda+\alpha}(B_R)$, $0 < R \leq \infty$.*

Proof. Let first $\alpha = 0$. For $Hf = H^\alpha f|_{\alpha=0}$ we have

$$Hf(x) = |x|^{-n} \int_{|y|<|x|} f(y)dy = \int_{B(0,1)} f(|x|y)dy$$

so that

$$\begin{aligned} |Hf(x+h) - Hf(x)| &\leq \int_{B(0,1)} |f(|x+h|y) - f(|x|y)| dy \\ &\leq [f]_\lambda \int_{B(0,1)} ||x+h| - |x||^\lambda |y|^\lambda dy =: A. \end{aligned}$$

Since, by triangle inequality $||x+h| - |x||^\lambda \leq |h|^\lambda$, $\lambda > 0$, for all $x, x+h \in \mathbb{R}^n$, we obtain that

$$A \leq [f]_\lambda \int_{B(0,1)} |h|^\lambda |y|^\lambda dy \leq [f]_\lambda |h|^\lambda \int_{B(0,1)} |y|^\lambda dy = C|h|^\lambda [f]_\lambda.$$

Thus, $|Hf(x+h) - Hf(x)| \leq C|h|^\lambda [f]_\lambda$ and therefore $[Hf]_\lambda \leq C[f]_\lambda$, with C not depending on x and h .

Since the inequality $\sup_{x \in \Omega} |Hf(x)| \leq c \sup_{x \in \Omega} |f(x)|$ is obvious, the proof is complete for $\alpha = 0$.

Let now $\alpha > 0$ and $f \in \tilde{C}_0^\lambda(B_R)$. We have

$$H^\alpha f(x) = |x|^\alpha g(|x|), \quad g(r) = \frac{1}{r^n} \int_{B(0,r)} f(y)dy = \int_{B(0,1)} f(ry)dy. \tag{2.2}$$

Hence, by the triangle inequality,

$$\begin{aligned} |H^\alpha f(x+h) - H^\alpha f(x)| &\leq ||x+h|^\alpha - |x|^\alpha| |g(|x+h|)| + |g(|x+h|) - g(|x|)| |x|^\alpha \\ &\leq C[f]_\lambda ||x+h|^\alpha - |x|^\alpha| |x+h|^\lambda + |g(|x+h|) - g(|x|)| |x|^\alpha \\ &=: \Delta_1 + \Delta_2, \end{aligned}$$

where we used the fact that $f(0) = 0$ and consequently

$$|g(|x+h|)| = |Hf(|x+h|)| \leq C|x+h|^\lambda [f]_\lambda \tag{2.3}$$

according to the case $\alpha = 0$ in the last passage.

We consider separately the cases $|x+h| \leq 2|h|$ and $|x+h| \geq 2|h|$.

The case $|x+h| \leq 2|h|$.

In this case we also have $|x| \leq 3|h|$.

Thus, by (2.3),

$$\Delta_1 \leq C[f]_\lambda |h|^\alpha |x+h|^\lambda \leq C_1 [f]_\lambda |h|^{\lambda+\alpha}$$

and

$$\Delta_2 \leq C[g]_\lambda |h|^\lambda |x|^\alpha \leq C_1 [f]_\lambda |h|^{\lambda+\alpha}.$$

The case $|x+h| \geq 2|h|$.

We have

$$\Delta_1 \leq C[f]_\lambda |x+h|^{\lambda+\alpha} \left| 1 - \left(\frac{|x|}{|x+h|} \right)^\alpha \right|.$$

Since, $|1-t^\alpha| \leq |1-t|$ for all $0 < t \leq 1$, $0 < \alpha \leq 1$, we obtain

$$\Delta_1 \leq C[f]_\lambda \frac{||x+h| - |x||}{|x+h|^{1-\lambda-\alpha}} \leq C[f]_\lambda |h|^{\lambda+\alpha}.$$

For Δ_2 we use the mean value theorem and find that

$$\Delta_2 \leq C |g'(\xi)| ||x+h| - |x|| |x|^\alpha \leq C |g'(\xi)| |h| |x|^\alpha$$

with ξ between $|x|$ and $|x+h|$.

If $|x| \leq |x+h|$, then, by Lemma 2.1, we get

$$\Delta_2 \leq C \frac{[f]_\lambda}{|\xi|^{1-\lambda}} |x|^\alpha |h| \leq C \frac{[f]_\lambda}{|x|^{1-\lambda-\alpha}} |h| \leq C[f]_\lambda |h|^{\lambda+\alpha}$$

because $|x| \geq |x+h| - |h| \geq |h|$. Finally, when $|x| \geq |x+h|$, we have

$$\Delta_2 \leq C \frac{[f]_\lambda}{|\xi|^{1-\lambda}} |x|^\alpha |h| \leq C \frac{[f]_\lambda}{|x+h|^{1-\lambda}} |x|^\alpha |h| = C \frac{[f]_\lambda}{|x+h|^{1-\lambda-\alpha}} \left(\frac{|x|}{|x+h|} \right)^\alpha |h|,$$

where $\frac{|x|}{|x+h|} \leq \frac{|h|}{|x+h|} + \frac{|x+h|}{|x+h|} \leq \frac{3}{2}$. Therefore,

$$\Delta_2 \leq C[f]_\lambda |h|^{\lambda+\alpha}.$$

It remains to gather the estimates for Δ_1 and Δ_2 .

In view of (2.2), the equality $H^\alpha f(0) = 0$ is obvious, so the proof is complete. \square

We define the generalized Hölder space $C^{\omega(\cdot)}(\Omega)$ as the set of functions continuous in Ω having the finite norm

$$\|f\|_{C^{\omega(\cdot)}} = \sup_{x \in \Omega} |f(x)| + [f]_{\omega(\cdot)}$$

with the seminorm

$$[f]_{\omega(\cdot)} = \sup_{\substack{x, x+h \in \Omega \\ |h| < 1}} \frac{|f(x+h) - f(x)|}{\omega(|h|)},$$

where $\omega : [0, 1] \rightarrow \mathbb{R}_+$ is a non-negative increasing function in $C([0, 1])$ such that $\omega(0) = 0$ and $\omega(t) > 0$ for $0 < t \leq 1$. Such spaces are known in the literature, see for instance [8], [14], [17, Section 13.6], [18].

Let also $C_0^{\omega(\cdot)}(B_R) := \{f \in C^{\omega(\cdot)}(B_R) : f(0) = 0\}$.

As usual, by saying that a function φ is almost decreasing, we mean that $\varphi(t) \leq C\varphi(s)$ for some $C \geq 1$ and for all $t \geq s$.

Following the same lines as in proof of Theorem 2.2 one can prove the following generalization of Theorem 2.2:

THEOREM 2.3. *Let $\omega \in C([0, 1])$ be positive on $(0, 1]$, increasing and such that $\omega(0) = 0$ and $\frac{\omega(t)}{t^{1-\alpha}}$ is almost decreasing. In the case $\alpha = 0$ the operator $H^\alpha|_{\alpha=0}$ is bounded in $C^{\omega(\cdot)}(B_R)$. When $\alpha > 0$, it is bounded from $\tilde{C}_0^{\omega(\cdot)}(B_R)$ into $\tilde{C}_0^{\omega_\alpha(\cdot)}(B_R)$, where $\omega_\alpha(t) = t^\alpha \omega(t)$.*

3. Boundedness of Hardy type operators in Hölder type spaces with compactification

Let \mathbb{R}^n denote the compactification of \mathbb{R}^n by a single infinite point.

DEFINITION 3.1. Let $0 \leq \lambda < 1$. We say that f belongs to $C^\lambda(\mathbb{R}^n)$, for all $x, y \in \mathbb{R}^n$, if

$$|f(x) - f(y)| \leq C \frac{|x - y|^\lambda}{(1 + |x|)^\lambda (1 + |y|)^\lambda}.$$

The set $C^\lambda(\mathbb{R}^n)$ is a Banach space with respect to the norm

$$\|f\|_{C^\lambda(\mathbb{R}^n)} = \|f\|_{C(\mathbb{R}^n)} + \sup_{x, y \in \mathbb{R}^n} |f(x) - f(y)| \left(\frac{(1 + |x|)(1 + |y|)}{|x - y|} \right)^\lambda.$$

It may be shown that $C^\lambda(\mathbb{R}^n)$ is a subspace of $C^\lambda(\mathbb{R}^n)$, which is invariant with respect to the inversion change of variables $x_* = \frac{x}{|x|^2}$, i.e.

$$C^\lambda(\mathbb{R}^n) = \{f : f \in C^\lambda(\mathbb{R}^n) \text{ and } f_* \in C^\lambda(\mathbb{R}^n)\},$$

where $f_* = f(x_*)$.

In the setting of the spaces $C^\lambda(\mathbb{R}^n)$ we consider only the case $\alpha = 0$, see Remark 3.4 below.

3.1. Hardy operator H

Our main result in this case reads:

THEOREM 3.2. *Let $0 \leq \lambda < 1$. Then the operator H is bounded in $C^\lambda(\mathbb{R}^n)$.*

Proof. We note that

$$Hf(x) - Hf(y) = \int_{B(0,1)} [f(|x|z) - f(|y|z)] dz.$$

Hence,

$$\begin{aligned} |Hf(x) - Hf(y)| &\leq c \int_{B(0,1)} \frac{||x| - |y|| |z|^\lambda}{(1 + |x||z|)^\lambda (1 + |y||z|)^\lambda} dz \\ &\leq c ||x| - |y||^\lambda \int_{B(0,1)} \frac{|z|^\lambda}{(1 + |x||z|)^\lambda (1 + |y||z|)^\lambda} dz =: A \end{aligned} \tag{3.1}$$

Let $|x| > 1, |y| > 1$. Then

$$\begin{aligned} A &\leq c ||x| - |y||^\lambda \int_{B(0,1)} \frac{|z|^\lambda}{(|x||z|)^\lambda (|y||z|)^\lambda} dz = c \frac{||x| - |y||^\lambda}{|x|^\lambda |y|^\lambda} \int_{B(0,1)} \frac{dz}{|z|^\lambda} \\ &\leq C \left[\frac{||x| - |y||}{(1 + |x|)(1 + |y|)} \right]^\lambda \int_{B(0,1)} \frac{dz}{|z|^\lambda} \leq C_1 \frac{|x - y|^\lambda}{(1 + |x|)^\lambda (1 + |y|)^\lambda}, \end{aligned} \tag{3.2}$$

since $\frac{1}{|x|} < \frac{2}{1+|x|}$.

Let $|x| < 1, |y| < 1$. Then

$$\begin{aligned} A &\leq c |x - y|^\lambda \int_{B(0,1)} |z|^\lambda dz = c_1 |x - y|^\lambda \\ &\leq C \frac{|x - y|^\lambda}{(1 + |x|)^\lambda (1 + |y|)^\lambda}, \end{aligned} \tag{3.3}$$

since $1 < \frac{2}{1+|x|}$.

Let $|x| < 1, |y| > 1$. Then

$$A \leq c |x - y|^\lambda \int_{B(0,1)} \frac{|z|^\lambda}{(|y||z|)^\lambda} dz \leq C_1 \frac{|x - y|^\lambda}{(1 + |x|)^\lambda (1 + |y|)^\lambda}.$$

Let $|x| > 1$, $|y| < 1$. Then

$$A \leq c|x-y|^\lambda \int_{B(0,1)} \frac{|z|^\lambda}{(|x||z|)^\lambda} dz \leq C_1 \frac{|x-y|^\lambda}{(1+|x|)^\lambda(1+|y|)^\lambda}.$$

Since the inequality $\|Hf\|_{C(\mathbb{R}^n)} \leq c\|f\|_{C(\mathbb{R}^n)}$ is obvious, the proof is complete. \square

3.2. Hardy operator \mathcal{H}

To formulate the corresponding result for the operator \mathcal{H} we need to consider the following subspaces:

$$C_0^\lambda(\mathbb{R}^n) = \{f \in C^\lambda(\mathbb{R}^n) : f(0) = 0\}, \quad C_\infty^\lambda(\mathbb{R}^n) = \{f \in C^\lambda(\mathbb{R}^n) : f(\infty) = 0\}$$

and

$$C_{\infty,0}^\lambda = C_\infty^\lambda \cap C_0^\lambda.$$

THEOREM 3.3. *Let $0 < \lambda < 1$. Then the operator \mathcal{H} is bounded from $C_{\infty,0}^\lambda(\mathbb{R}^n)$ to $C_\infty^\lambda(\mathbb{R}^n)$*

Proof. Let $f \in C_{\infty,0}^\lambda(\mathbb{R}^n)$ and denote $g(x) = \mathcal{H}f(x)$. Clearly, $g(\infty) = 0$, and

$$\begin{aligned} |g(x) - g(y)| &= \left| \int_{|z|>1} [f(|x|z) - f(|y|z)] \frac{dz}{|z|^n} \right| \\ &\leq C|x-y|^\lambda \int_{|z|>1} \frac{|z|^{\lambda-n} dz}{(1+|x||z|)^\lambda (1+|y||z|)^\lambda} =: \Delta. \end{aligned} \tag{3.4}$$

Let $|x| > 1$, $|y| > 1$. Then

$$\Delta \leq C \frac{|x-y|^\lambda}{|x|^\lambda |y|^\lambda} \int_{|z|>1} \frac{dz}{|z|^{n+\lambda}} \leq C_1 \frac{|x-y|^\lambda}{(1+|x|)^\lambda (1+|y|)^\lambda}.$$

Hence $g(x) \in C_{\infty,0}^\lambda(\mathbb{R}^n)$.

Let $|y| < |x| < 2$.

Since $f(0) = 0$, we have $|f(z)| \leq C|z|^\lambda$ and then

$$|g(x) - g(y)| = \left| \int_{|y|}^{|x|} \frac{f(z)}{|z|^n} dz \right| \leq C \int_{|y|}^{|x|} |z|^{\lambda-n} dz = C_1 (|x|^\lambda - |y|^\lambda) \leq C_2 \frac{|x-y|^\lambda}{(1+|x|)^\lambda (1+|y|)^\lambda},$$

since $a^\lambda - b^\lambda \leq (a-b)^\lambda$, $a > b > 0$, $0 \leq \lambda \leq 1$.

Let now $|y| < 1, |x| > 2$. $\mathcal{H}f$ is bounded. Indeed,

$$|g(x) - g(y)| \leq \int_{\mathbb{R}^n} \frac{f(z)}{|z|^n} dz.$$

As already shown, for each function $f \in C_{\infty,0}^\lambda$ we have that $|f(z)| \leq c|z|^\lambda, 0 < |z| < 1$ and $|f(z)| \leq \frac{c}{|z|^\lambda}, |z| > 1$. Therefore

$$|g(x)| \leq c_1 \int_0^1 \frac{1}{|z|^{n-\lambda}} dz + c_2 \int_1^\infty \frac{1}{|z|^{n+\lambda}} dz = C < \infty,$$

for $0 \leq \lambda < 1$, and then

$$|g(x) - g(y)| \leq C.$$

It is easily checked that

$$1 \leq 6 \frac{|x-y|}{(1+|x|)(1+|y|)}, \text{ when } |y| < 1, |x| > 2. \tag{3.5}$$

Consequently,

$$|g(x) - g(y)| \leq C \leq C_2 \frac{|x-y|^\lambda}{(1+|x|)^\lambda (1+|y|)^\lambda},$$

which proves that $g(x) \in C_{\infty,0}^\lambda(\mathbb{R}^n)$ also in this case.

The case $|x| < 1, |y| > 2$ can be similarly treated.

Similarly as in Theorem 3.2 we note that the boundedness of the operator \mathcal{H} in $C(\mathbb{R}^n)$ is obvious, so the proof is complete. \square

REMARK 3.4. When $\alpha > 0$. Theorems 3.2 and 3.3 may not be extended to the setting $C^\lambda(\mathbb{R}^n) \rightarrow C^{\lambda+\alpha}(\mathbb{R}^n)$, in which we require the Hölder behavior of functions also at the infinite point, in contrast to Theorem 2.2. In fact, the function $f_0 = \frac{1}{(1+x)^\lambda} \in C_\infty^\lambda(\mathbb{R}_+)$ provides a corresponding counterexample for both the operators H^α and \mathcal{H}^α . For example, for the operator H^α we have

$$H^\alpha f_0(x) = \frac{x^{\alpha-1}}{1-\lambda} [(1+x)^{1-\lambda} - 1].$$

Hence, when $x \rightarrow \infty$ we obtain that $H^\alpha f_0(x) \sim cx^{\alpha-\lambda}$, while the inclusion $H^\alpha f_0(x) \in C_{\infty,+}^{\lambda+\alpha}(\mathbb{R}_+)$ requires the behavior $|H^\alpha f_0(x)| \leq c(1+x)^{-\alpha-\lambda}$.

Corresponding generalizations of Theorems 3.2 and 3.3 may be also formulated in terms of the generalized Hölder spaces $C^\omega(\mathbb{R}^n), C_\infty^\omega(\mathbb{R}^n), C_0^\omega(\mathbb{R}^n)$ and $C_{\infty,0}^\omega(\mathbb{R}^n)$ defined below.

DEFINITION 3.5. Let $\omega = \omega(h)$ be an increasing function. The generalized Hölder space $C^\omega(\mathbb{R}^n)$ is defined as consisting of all functions satisfying the condition

$$|f(x) - f(y)| \leq C\omega\left(\frac{|x-y|}{(1+|x|)(1+|y|)}\right), \quad x, y \in \mathbb{R}^n.$$

The subspaces $C_\infty^\omega(\mathbb{R}^n)$, $C_0^\omega(\mathbb{R}^n)$ and $C_{\infty,0}^\omega(\mathbb{R}^n)$ of the space $C^\omega(\mathbb{R}^n)$ are defined by the conditions $f(\infty) = 0$, $f(0) = 0$ and $f(\infty) = f(0) = 0$, respectively.

REFERENCES

- [1] E. BURTSEVA AND N. SAMKO, *Weighted Adams type theorem for the Riesz fractional integral in generalized Morrey space*, *Fract. Calc. Appl. Anal.* **19** (4): 954–972, 2016.
- [2] G. H. HARDY, *Notes on some points in the integral calculus, LX. An inequality between integrals*, *Messenger of Math.* **54**: 150–156, 1925.
- [3] G. H. HARDY, *Notes on some points in the integral calculus, LXIV*, *Messenger of Math.* **57**: 12–16, 1928.
- [4] G. H. HARDY, J. E. LITTLEWOOD, AND G. POLYA, *Inequalities*, Cambridge Univ. Press, 1934.
- [5] V. KOKILASHVILI, A. MESKHI, AND L.E. PERSSON, *Weighted Norm Inequalities for Integral Transforms with Product Weights*, Nova Scientific Publishers, Inc., New York, 2010.
- [6] A. KUFNER, L. MALIGRANDA, AND L.E. PERSSON, *The Hardy Inequality - About its History and Some Related Results*, Vydavatel'sky Servis Publishing House, Pilsen, 2007.
- [7] A. KUFNER, L.E. PERSSON, AND N. SAMKO, *Weighted Inequalities of Hardy Type*, second edition, World Scientific Publishing Co. Inc., River Edge, NY, 2017.
- [8] L. LEINDLER, *A note on embedding of classes H^ω* , *Analysis Mathematica* **27**: 71–76, 2001.
- [9] D. LUKKASSEN, L. E. PERSSON, AND N. SAMKO, *Hardy type operators in local vanishing Morrey spaces on fractal sets*, *Fract. Calc. Appl. Anal.* **18** (5): 1252–1276, 2015.
- [10] S. LUNDBERG AND N. SAMKO, *On some hyperbolic type equations and weighted anisotropic Hardy operators*, *Math. Methods Appl. Sci.*, doi : 10.1002/mma.4062, 2016.
- [11] S. M. NIKOL'SKI, *Priblizhenie funktsii mnogikh peremennykh i teoremy vložheniya*, (Russian) [Approximation of functions of several variables and imbedding theorems] second edition, revised and supplemented, Nauka, Moscow, 1977.
- [12] L. E. PERSSON AND N. SAMKO, *Some remarks and new developments concerning Hardy-type inequalities*, *Rend. Circ. Mat. Palermo, serie I*, **82** (4): 1–29, 2010.
- [13] L. E. PERSSON AND N. SAMKO, *Weighted Hardy and potential operators in the generalized Morrey spaces*, *J. Math. Anal. Appl.* **377**: 792–806, 2011.
- [14] N. SAMKO, *On compactness of Integral Operators with a Generalized Weak Singularity in Weighted Spaces of Continuous Functions with a Given Continuity Modulus*, *Proc. A. Razmadze Math. Inst.* **136**: 91–113, 2004.
- [15] N. SAMKO, *Weighted Hardy and singular operators in Morrey spaces*, *J. Math. Anal. Appl.* **350**: 56–72, 2009.
- [16] N. SAMKO, *Weighted Hardy operators in the local generalized vanishing Morrey spaces*, *Positivity* **17** (3): 683–706, 2013.
- [17] S. G. SAMKO, A. A. KILBAS, AND O. I. MARICHEV, *Fractional Integrals and Derivatives*, Theory and Applications, London-New York: Gordon & Breach, Sci. Publ., (Russian edition – *Fractional Integrals and Derivatives and some of their Applications*, Minsk: Nauka i Tekhnika, 1987.), 1993, 1012 pages.

- [18] S. B. STECHKIN, *On the order of the best approximations of continuous functions*, Izv. Akad. Nauk SSSR Ser. Mat. **15** (3): 219–242, 1951.
- [19] C. TANG AND R. ZHOU, *Boundedness of weighted Hardy operator and its adjoint on Triebel-Lizorkin-type spaces*, J. Funct. Spaces Appl. **2012**, Article ID 610649.

(Received January 15, 2017)

Evgeniya Burtseva
Luleå University of Technology
SE 971 87 Luleå, Sweden
e-mail: Evgeniya.Burtseva@ltu.se

Staffan Lundberg
Luleå University of Technology
SE 971 87 Luleå, Sweden
e-mail: Staffan.Lundberg@ltu.se

Natasha Samko
Luleå University of Technology
SE 971 87 Luleå, Sweden
e-mail: Natasha.Samko@ltu.se

Lars-Erik Persson
Luleå University of Technology
SE 971 87 Luleå, Sweden
and
UiT, The Arctic University of Norway
NO 8505, Narvik, Norway
e-mail: Lars-Erik.Persson@ltu.se