

# On geometric construction of some power means

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**Abstract.** In the homogenization theory, there are many examples where the effective conductivities of composite structures are power means of the local conductivities. The main aim of this paper is to initiate research concerning geometric construction of some power means of three or more variables. We contribute by giving methods for the geometric construction of the harmonic mean  $P_{-1}$  and the arithmetic mean  $P_1$  of three variables  $a, b$  and  $c$

## 1 Introduction

As it is well known, homogenization theory, as well as PDEs, plays pretty important roles in the study of many applied problems, see [7]. The use of power means is of certain interest in homogenization theory. This is natural since the this theory is mainly handling differential equations with rapidly oscillating coefficients. These equations can be replaced with a homogenized equation where the coefficients can be interpreted as special means. Therefore many research papers are devoted to development of methods, say, tools for such theories. Sometimes just a simple inequality or correctly discovered relation between some parameters can be extremely useful for solving problems, where it was unclear how one can find an appropriate approach.

For instance, in the study of a scale of two-component composite structures of equal proportions with infinitely many micro-levels, it was found that their effective conductivities are power means of the local conductivities, see [8] and [9].

As regarding to such a basic concept as power means and its relation with important Jensen's inequality, we refer e.g. to the new book [10], see also references therein.

Power means have fascinated mathematicians during many centuries. In the simplest form they can be described as follows: For  $n$  positive numbers,  $(a_1, a_2, \dots, a_n)$ , the power mean  $P_k^n$  of order  $k$  with equal weights is defined as:

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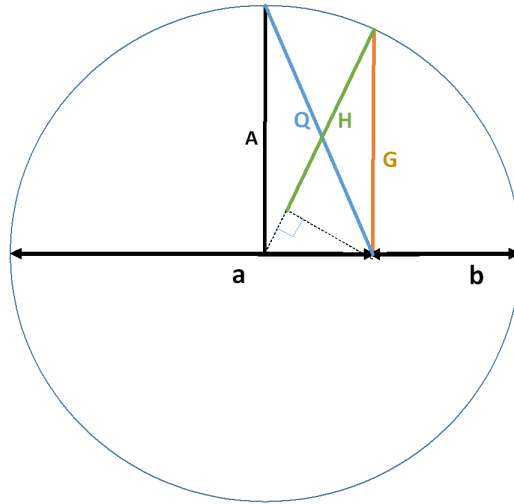
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$$P_k^n = \left[ \frac{a_1^k + a_2^k + \dots + a_n^k}{n} \right]^{\frac{1}{k}}, \text{ if } k \neq 0,$$

and

$$P_0^n = [a_1 a_2 \dots a_n]^{\frac{1}{n}}, \text{ if } k = 0.$$

There is a substantial literature on the subject of power means, see e.g. [1], [3], [4], [11] and [12]. The close connection between convexity and power means is described e.g. in the book [10]. In particular, it is well known that  $P_k^n < P_l^n$  if  $k < l$  if all the  $a_i$  are not identical, and that  $P_k^n$  converges towards  $(a_i)_{\max}$  when  $k \rightarrow +\infty$  and towards  $(a_i)_{\min}$  when  $k \rightarrow -\infty$ . The most commonly used power means are the arithmetic mean  $A = P_1^n$ , the geometric mean  $G = P_0^n$  and the harmonic mean  $H = P_{-1}^n$ . These three means for two variables were explored by the classic Greek mathematicians because of their importance in the study of geometry and music. They are today called *Pythagorean means*. The Greek mathematicians constructed the *Pythagorean means* for two variable lengths  $a$  and  $b$  as shown in Figure 1, see e.g. [14]. The quadratic mean  $Q = P_2(a, b)$ , also known as the *Root Mean Square*, is also included in the figure.

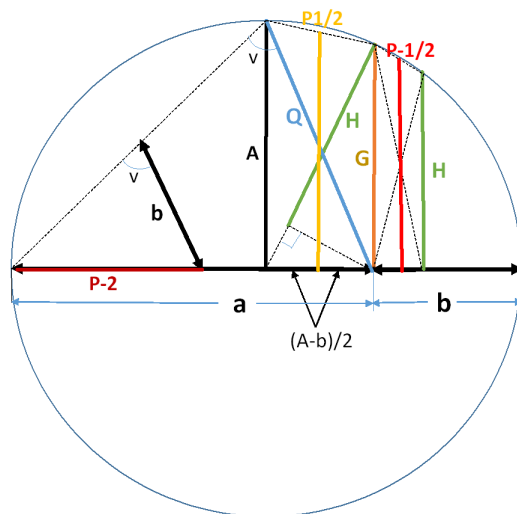


**Fig. 1** Classic Greek construction of Pythagorean means of two variables  $a$ ,  $b$ .  $A$  is the arithmetic mean,  $Q$  is the quadratic mean,  $H$  is the harmonic mean and  $G$  is the geometric mean.

Power means have throughout history mostly been analyzed and calculated on the basis of numeric variables. R. Høibakk and D. Lukkassen have studied the properties of certain power means based on geometric variables, and have shown that  $P_{-2}$ ,  $P_{-1}$ ,  $P_{-\frac{1}{2}}$ ,  $P_0$ ,  $P_{\frac{1}{2}}$ ,  $P_1$  and  $P_2$  for two variables can be constructed in a basic geometric structure different from the one employed by the Greek mathematicians [5]. Also other papers (e.g. [2] and [13]) considered geometric construction of power means of two variables.

**Remark 1.1.** The classic Greek method of constructing the *Pythagorean means*, as shown in Figure 1, may also be extended to construct  $P_{-2}$ ,  $P_{-1}$ ,  $P_{-\frac{1}{2}}$ ,  $P_0$ ,  $P_{\frac{1}{2}}$ ,  $P_1$  and  $P_2$  for two variables, as shown

in Figure 1. To accomplish this, we use the facts shown in [5]:  $P_{\frac{1}{2}}(a, b) = P_1(P_0(a, b), P_1(a, b))$ ,  $P_{-\frac{1}{2}}(a, b) = P_{-1}(P_0(a, b), P_{-1}(a, b))$  and  $P_2(a, b) \times P_{-2}(a, b) = ab$ . The construction methods are demonstrated in Figure 1.

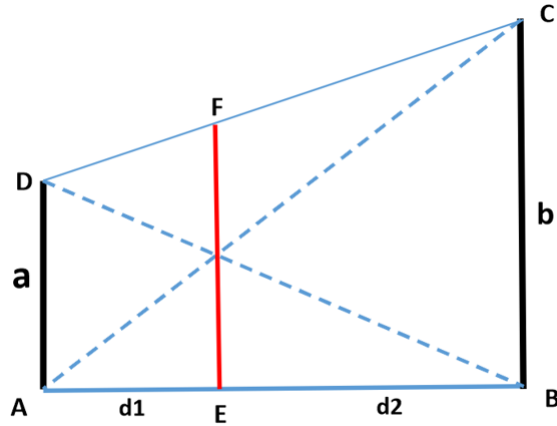


**Fig. 2** Geometric construction of  $P_{-2}$ ,  $P_{-1}$ ,  $P_{-\frac{1}{2}}$ ,  $P_0$ ,  $P_{\frac{1}{2}}$ ,  $P_1$  and  $P_2$  of two variables.

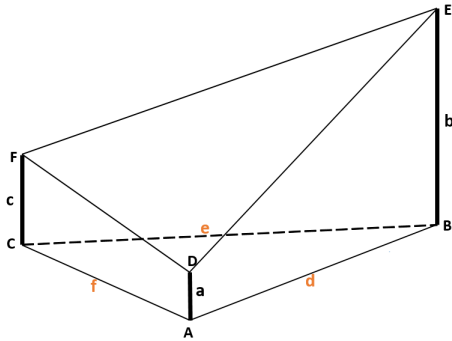
The basic two-dimensional structure which was used by R. Høibakk and D. Lukkassen [5] for the geometric construction of  $P_{-2}$ ,  $P_{-1}$ ,  $P_{-\frac{1}{2}}$ ,  $P_0$ ,  $P_{\frac{1}{2}}$ ,  $P_1$  and  $P_2$  for two variables,  $a$  and  $b$ , is shown in Figure 1. The source for this structure can be found in another work by the same authors, [6]. In the trapezoid  $ABCD$ , in Figure 1, the variables  $a = AD$  and  $b = BC$  are constructed vertical to the "floor"  $AB$ . Independent of the width of the "floor"  $AB$ , the length of the vertical line  $EF$  through the intersection of the diagonals is equal to the harmonic power mean of the two variables  $a$  and  $b$ , *i.e.*,  $EF = P_{-1}(a, b) = \frac{2ab}{a+b}$ . The arithmetic mean can be found by bisecting the "floor"  $AB$  and draw the corresponding vertical line from the "floor" to the "roof"  $DC$ . If, in addition,  $(d_1, d_2) = (a, b)$ , the "roof"  $DC$  equals the double of the root mean square, *i.e.*,  $DC = 2P_2(a, b)$ , see Figure 1.

The main purpose of this paper is to initiate the study of geometric construction of some power means of three or more variables. Guided by the discussion above we also contribute by presenting the construction of the arithmetic and the harmonic power means of three variables,  $P_{-1}(a, b, c)$  and  $P_1(a, b, c)$ , see Sections 2 and 3 below. (Note that the geometric mean of three variables  $a, b,$  and  $c$ ,  $P_0 = \sqrt[3]{abc}$  cannot be constructed geometrically.)

## 2 3-D Construction of the harmonic mean $P_{-1}(a, b, c)$

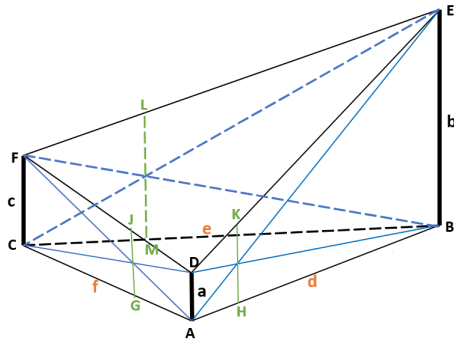


**Fig. 3** Geometric construction of the harmonic mean of two variables  $a$  and  $b$ .



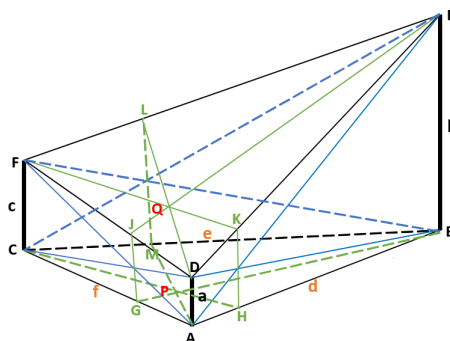
**Fig. 4** An arbitrary triangle  $ABC$  and the triangular roof,  $DEF$ . The vertical lines are formed by the variables  $a, b$  and  $c$ .

Consider first Figure 2, which is constructed by drawing an arbitrary triangle  $ABC$  and raising the three variables  $a, b$  and  $c$  as vertical lines from each of the three corners. The top end of the vertical lines form a triangular "roof",  $DEF$ .



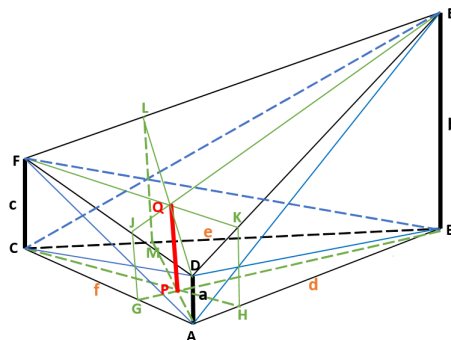
**Fig. 5** The diagonals in the trapezoidal walls,  $ABED$ ,  $ADFC$  and  $BEFC$ , are drawn.

In Figure 2 the diagonals in the trapezoidal walls,  $ABED$ ,  $ADFC$  and  $BEFC$ , are drawn. The vertical lines through the intersection points of these diagonals touch the "floor" and the "roof" at  $H$  and  $K$ , at  $G$  and  $J$  and at  $M$  and  $L$ , respectively.



**Fig. 6** The lines connecting the points  $H, G, \text{ and } M$  and  $K, L, \text{ and } J$  are drawn. The lines intersect at  $P$  and  $Q$ .

The lines connecting the points  $H, G, \text{ and } M$  and  $K, L, \text{ and } J$  with the opposite corners in the "floor" triangle  $ABC$  and the "roof" triangle  $DEF$  cross at  $P$  ("floor") and  $Q$  ("roof"), respectively, are shown in Figure 2.



**Fig. 7** The connecting line,  $PQ$  is drawn.

**Theorem 2.1.** The connecting line  $PQ$  in Figure 2, will, since it is the intersection line between three vertical planes, be vertical, and equal to the harmonic power mean of  $a, b$  and  $c$ , *i.e.*,

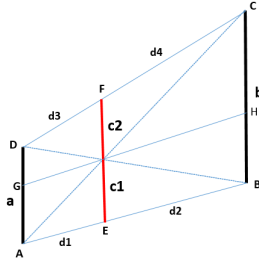
$$PQ = P_{-1}(a, b, c) = \frac{3abc}{ab + ac + bc}$$

We first prove two lemmas of independent interest.

**Lemma 2.1. (a)** In Figure 1 the angles between the "floor"  $AB$  and the variables  $a$  and  $b$ , are right-angled. For the construction of the arithmetic and the harmonic mean for two variables, this is a convenience, not a requirement. The only requirement is that the variables are parallel.

(b) In Figure 2 the variables  $a = AD$  and  $b = BC$  are parallel but not vertical to the "floor"  $AB$  or to the "roof"  $DC$ . The "floor"  $AB$  can be of an arbitrary length. The line  $EF$  through the intersection of the diagonals is parallel to the variables. The intersection of the diagonals split  $EF$  in two parts  $c_1$  and  $c_2$ . The line  $EF$  separates  $AB$  in two parts,  $d_1$  and  $d_2$  and  $DC$  in  $d_3$  and  $d_4$ . The line  $GH$  through the intersection of the diagonals is drawn parallel to the "floor"  $AB$ .

**Proof.** From similar triangles in Figure 2 we have that :



**Fig. 8** Geometric construction of the harmonic mean of  $a$  and  $b$  where the variables are parallel, but not vertical to the floor or roof.

$$\frac{d_2}{c_1} = \frac{d_1}{a - c_1} \text{ and } \frac{d_2}{b - c_1} = \frac{d_1}{c_1},$$

giving that

$$c_1 = \frac{ab}{a + b}.$$

By drawing a line through the intersection of the diagonals that is parallel to  $DC$  and using the corresponding similar triangles we also find that

$$c_2 = \frac{ab}{a + b},$$

*i.e.*,

$$c_1 = c_2,$$

and that

$$EF = c_1 + c_2 = \frac{2ab}{a + b} = P_{-1}(a, b).$$

Hence, both statements (a) and (b) in Lemma 3 follows.

**Remark 2.1.** From similar triangles in Figure 2 we also have that

$$\frac{d_2}{c_1} = \frac{d_1 + d_2}{a} \text{ and } \frac{d_1}{c_1} = \frac{d_1 + d_2}{b},$$

resulting in

$$d_1 = \frac{a(d_1 + d_2)}{a + b} = \frac{aAB}{a + b},$$

and

$$d_2 = \frac{b(d_1 + d_2)}{a + b} = \frac{bAB}{a + b}.$$

Correspondingly, we also find that

$$d_3 = \frac{a(d_3 + d_4)}{a + b} = \frac{aDC}{a + b},$$

and

$$d_4 = \frac{b(d_3 + d_4)}{a + b} = \frac{bDC}{a + b}.$$

We also need the following result:

**Lemma 2.2.** The three lines in the "floor"  $BG$ ,  $CH$  and  $AM$  in Figure 2 intersect in a single point  $P$  within the triangle  $ABC$ . Clearly, then also the lines  $DL$ ,  $EJ$  and  $FK$  intersect in a single point  $Q$  in the "roof" triangle  $DEF$ .

**Proof.** The bottom triangle,  $ABC$ , is placed in a coordinate system, as shown in Figure 2.

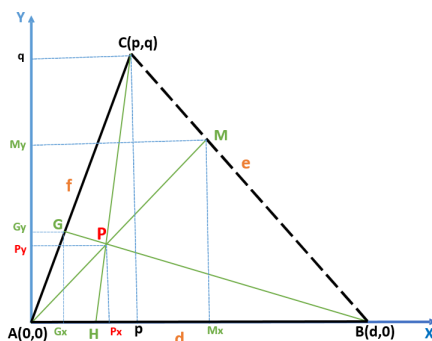


Fig. 9 The bottom triangle from Figure 2 placed in a coordinate system.

Vertical above the points  $A, B$  and  $C$  (i.e. in the  $z$ -direction), although not visible in Figure 2, stand the three variables  $a, b$  and  $c$ , see Figure 2. Their existence is used in the calculation of  $AH, AG$  and  $BM$ . According to Lemma 2.1 and Remark 2.1 we have that

$$AH = \frac{da}{a + b},$$

$$BM = \frac{eb}{b + c},$$

and

$$AG = \frac{fa}{a + c},$$

where  $d = AB$ ,  $e = BC$  and  $f = AC$ . The coordinates of  $A, B$  and  $C$  are  $(0,0)$ ,  $(d,0)$  and  $(p,q)$ , respectively. From these formulas and from similar triangles in Figure 2 we can determine the other set of coordinates (see Figure 2):

$$\begin{aligned}
H_x &= \frac{da}{a+b}, \\
H_y &= 0, \\
G_x &= \frac{pa}{a+c}, \\
G_y &= \frac{qa}{a+c} \\
M_x &= \frac{pb+cd}{b+c},
\end{aligned} \tag{2.1}$$

and

$$M_y = \frac{qb}{b+c}.$$

Next, we formulate the equations for the intersecting lines in Figure 2 using the coordinates above:

$$BG : y = -\frac{qa}{d(a+c)-pa}x + \frac{qad}{d(a+c)-pa},$$

and

$$CH : y = \frac{q(a+b)}{p(a+b)-ad}x - \frac{qad}{p(a+b)-ad}.$$

The intersection point  $P = (P_{x_1}, P_{x_2})$  between  $BG$  and  $CH$  is decided by

$$P_{x_1} = \frac{acd + pab}{ab + ac + bc}, \tag{2.2}$$

and

$$P_{y_1} = \frac{qab}{ab + ac + bc}.$$

The line through  $A$  and  $M$  is

$$AM : y = \frac{qb}{pb + cd}x.$$

The intersecting point  $(P_{x_2}, P_{y_2})$  between  $BG$  and  $AM$  is defined by:

$$P_{x_2} = \frac{acd + pab}{ab + ac + bc},$$

and

$$P_{y_2} = \frac{qab}{ab + ac + bc}.$$

Hence, the pair of coordinates,  $(P_{x_1}, P_{y_1})$  and  $(P_{x_2}, P_{y_2})$  are identical. This means that the three lines intersect in a single point. This completes the proof.



**Proof (Proof of Theorem 2).** From similar triangles in Figure 2 we find that

$$\frac{PG}{P_x - G_x} = \frac{BP}{d - P_x},$$

*i.e.*,

$$\frac{PG}{BP} = \frac{P_x - G_x}{d - P_x}.$$

We insert the values for  $P_x$  and  $G_x$  from (2.2) and (2.1), which give that

$$\frac{PG}{BP} = \frac{ac}{b(a+c)}$$

*i.e.*

$$PG = kac, \tag{2.3}$$

and

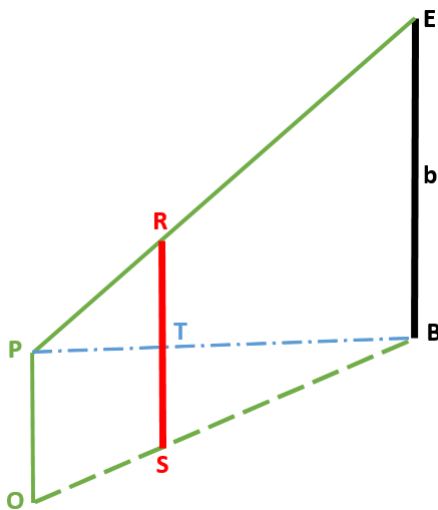
$$BP = kb(a+c). \tag{2.4}$$

Hence,

$$BG = PG + BP = k(ab + ac + bc), \tag{2.5}$$

where  $k$  is an undetermined constant.

We consider the trapezoid  $GBEJ$  from Figure 2:



**Fig. 10** The trapezoid  $GBEJ$  from Figure 2.

From similar triangles in Figure 2 we see that

$$PT = GJ \frac{BP}{BG} \tag{2.6}$$

and

$$TQ = b \frac{PG}{BG}. \tag{2.7}$$

Moreover, from Figure 2, Lemma 2.1 and Remark 2.1 we know that

$$GJ = \frac{2ac}{a+c}. \tag{2.8}$$

We insert the values from (2.8), (2.3), (2.4) and (2.5) in (2.6) and in (2.7), which give that

$$PT = \frac{2abc}{ab+ac+bc}$$

and

$$TQ = \frac{abc}{ab+ac+bc}.$$

We then find

$$PQ = PT + TQ = \frac{3abc}{ab+ac+bc}.$$

Hence, we can conclude that

$$P_{-1}(a,b,c) = PQ = \frac{3abc}{ab+ac+bc}.$$

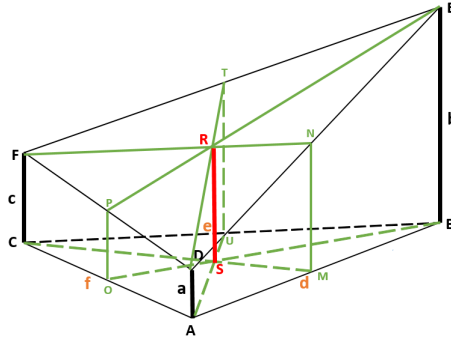
The proof is complete.

### 3 3-D Construction of the arithmetic mean, $P_1(a,b,c)$

First we note that the construction of the arithmetic mean of three variables

$$P_1(a,b,c) = \frac{a+b+c}{3}$$

can easily be done by constructing a line of the length  $a+b+c$  and trisecting it with standard method. In this paper we want to show that it also can be constructed using the three-dimensional structure described above.



**Fig. 11** Construction of an arbitrary triangle  $ABC$  with the variables  $a, b$  and  $c$  as vertical lines in the triangle corners.

Figure 2 is, as before, constructed by drawing an arbitrary triangle  $ABC$  and raising the three variables  $a, b$  and  $c$  as vertical lines from each of the three corners. The top end of the vertical lines

form a triangular "roof",  $DEF$ . We draw the lines from the corners of the "floor" triangle  $ABC$  and the "roof" triangle  $DEF$  to the mid point of the opposite triangle sides. The lines in the "floor" triangle,  $BO$ ,  $CM$  and  $AU$ , intersect at  $S$ , and the corresponding lines in the "roof" triangle,  $EP$ ,  $FN$  and  $DT$ , intersect at  $R$ . The trapezoids  $ADTU$ ,  $BEPO$  and  $CFNM$  are all vertical to the ground "floor". The intersecting line  $RS$  is therefore also vertical to the "floor" triangle.

Our main result in this section reads:

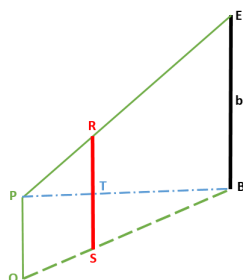
**Theorem 3.1.** The line  $RS$  is equal to the arithmetic mean of the lengths  $a, b, c$  in Figure 2, *i.e.*,

$$P_1(a, b, c) = RS = \frac{a+b+c}{3}.$$

**Proof.** It is a well established fact that the lines from the corners in an arbitrary triangle to the mid point of the opposite side intersect in a single point, and that the intersection point divides each such line in two segments in proportion 2 : 1, see e.g. [15].

Since  $O$  and  $P$  are at the mid point of  $AC$  and  $DF$ , respectively, the line  $OP$  is the arithmetic mean of  $a$  and  $c$ , *i.e.*,  $OP = \frac{a+c}{2}$ . Correspondingly we also have that  $MN = \frac{a+b}{2}$  and  $UT = \frac{b+c}{2}$ .

We consider the trapezoid  $BEPO$  in Figure 2, which can be seen in detail in Figure 2.



**Fig. 12** The trapezoid  $BEPO$  from Figure 2.

We know that

$$OP = \frac{a+c}{2} \tag{3.1}$$

and that

$$\frac{SO}{BS} = \frac{1}{2}.$$

From similar triangles in Figure 2 we find that

$$\frac{TR}{b} = \frac{SO}{BO} = \frac{1}{3},$$

*i.e.*,

$$TR = \frac{b}{3}$$

and

$$\frac{ST}{OP} = \frac{BS}{BO} = \frac{2}{3}. \tag{3.2}$$

We insert  $OP$  from (3.1) into (3.2) and obtain that

$$ST = \frac{a+c}{3}.$$

Therefore, we have that

$$RS = ST + TR = \frac{a+b+c}{3},$$

*i.e.*,

$$P_1(a, b, c) = RS.$$

The proof is complete.

### Acknowledgement

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