Faculty of Science and Technology
Department of Mathematics and Statistics

Differential invariants of Lie pseudogroups

Applications to equivalence problems and mathematical physics

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Eivind Schneider

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Introduction

A fundamental set of problems in mathematics concerns classifying and distinguishing objects with a given property under some kind of equivalence relation. A subclass of such problems arises when we restrict to objects that are defined through a system of differential equations. In general a system of differential equations comes "equipped" with a collection of symmetries, i.e. transformations that act on the space of solutions. This gives an equivalence relation on the solution space, since it is often natural to consider two solutions to be equivalent if they are related by a symmetry transformation.

This type of equivalence problem appears frequently in pure mathematics, but also in areas where the practical utility is more obvious. Examples can be found in such diverse fields as general relativity, fluid dynamics, thermodynamics and image recognition, to name a few, and it may come as a surprise to many that there exists a coherent mathematical theory for such a big class of problems. In this thesis we will see, with several examples, how this theory applies to classification and equivalence problems coming from both mathematics and physics. Our motivation for doing this is split into three main parts:

- The classification and recognition problems we consider are interesting in their own right, and they are worth solving. They appear naturally in relativity, mathematical physics, integrability theory, twistor theory and so forth.
- We want to investigate, and display, the power of the theory of scalar differential invariants. This theory has recently experienced an important change in fundamental ideas, after the appearance of the global Lie-Tresse theorem ([6]), and we are investigating the theoretical consequences of this.
- Every example we have looked at gave us some new insight on the practical implementation of the theory, thereby facilitating the development of computational methods.

In a big part of this thesis, our goal will be to recognize/distinguish and classify solutions of systems of partial differential equations (PDEs), under their Lie pseudogroup of symmetries. We will restrict our attention to smooth solutions, but in a quite general sense. Solutions can be functions on a manifold, maps from one manifold to another, sections of a bundle or, more generally, submanifolds of some given manifold. In particular we may consider the set of all submanifolds of a fixed manifold E (solutions of the trivial PDE), under a Lie pseudogroup action on E.

The solution spaces for PDEs are in general very complicated. In particular they are usually far from being finite-dimensional. The same is true for the Lie pseudogroups, which are also defined as solutions spaces to a system of PDEs. Thus it may come as a surprise that one of the main tools that helps us solve the equivalence and classification problems come from classical geometric invariant theory, where one studies actions of finite-dimensional algebraic groups acting on finite-dimensional algebraic varieties. The main purpose of this introduction is to outline, in simple terms, how and why this works. By doing that, we also get to fix some notation and definitions, in addition to recalling the main theorems.

In particular we will show how the general equivalence problem, as described above, induces group actions on an infinite number of finite-dimensional spaces, called the jet-spaces. Even though there is an infinite number of these spaces, the task of describing their corresponding orbit spaces turns out to be a finite process. By describing these orbit spaces (in terms of their algebra of differential invariants), we solve the original classification or recognition problem.

1 The geometry of differential equations

We will start by recalling some of the constructions needed in order to talk about invariant theory on differential equations. For a more detailed introduction to the geometric theory of PDEs, see for example [5, 8, 4].

1.1 Jets of sections

We fix a fiber bundle $\pi \colon E \to M$, and a point $x \in M$. For a section s of π we define its k-jet $[s]_x^k$ at the point $x \in M$ as the equivalence class of sections whose graphs are tangent to that of s up to order k at x. Denote by $J_x^k\pi$ the space of all k-jets of sections of π at the point x. We define the space of all k-jets as $J^k\pi = \cup_{x\in M} J_x^k\pi$. These are actually bundles over E, and we denote the projection $J^k\pi \to E$ by π_k. We also have projections $\pi_{k,l} \colon J^k\pi \to J^l\pi$ for $k > l$ defined by $\pi_{k,l}([s]_x^k) = [s]_x^l$. On some occasions we will use the notation $J^k(E) = J^k\pi$.

A section s of π is naturally prolonged to a section $j^k s$ of $J^k\pi$, defined by $j^k s(x) = [s]^k_x$. On the bundle we may choose local coordinates $x^1, ..., x^n, u^1, ..., u^m$, so that a section is given by m functions $u^i = f^i(x)$. The choice of coordinates on E uniquely gives a canonical choice of coordinates on $J^k\pi$:

$$x^i, \quad u^j, \quad u^j_\sigma, \qquad i \leq n, \quad j \leq m, \quad |\sigma| \leq k.$$

Here σ is a multi-index. The k-jet of s at x is then given by $u^i = f^i(x), u^i_\sigma = \partial_{x^\sigma}(f^i(x))$, with $|\sigma| \leq k$.

Remark 1. *General n-dimensional submanifolds of a manifold E correspond to singularities of sections, and can be handled by jet-space theory. However, in this thesis we will mostly work with bundles, so restricting to bundles already at this point seems natural. Note also that any n-dimensional submanifold of E, may locally be described as the section of some bundle.*

1.2 Differential equations

A partial differential equation (PDE) of order k is a submanifold $\mathcal{E}_k \subset J^k\pi$. Usually it is given by a set of equations

$$F_q(x^i, u^j, u^j_\sigma) = 0$$

where σ is a multi-index, $|\sigma| \leq k$ and $q = 1, ..., r$. We say that a section s of π is a solution to the PDE if its prolongation $j^k s$ is contained in $\mathcal{E}_k$.

The relationship between sections of π and solutions of $\mathcal{E}_k$ can be described geometrically in terms of the Cartan distribution $\mathcal{C}$. It is a distribution on $J^k\pi$ which can be defined at a point $\theta_k \in J^k\pi$ as the span of tangent planes of all graphs of prolonged sections $j^k s$ with the property $j^k s(\pi_k(\theta_k)) = \theta_k$. The solutions of the PDE correspond to n-dimensional integral manifolds of the Cartan distribution projecting diffeomorphically to M.

Note that if s is a (smooth) solution to $F_q = 0$, it will also be a solution to the differentiated equations $D_{x^i}(F_q) = 0$. So for a differential equation $\mathcal{E}_k \subset J^k\pi$ of order k, we can construct the differential equation $\mathcal{E}_{k+1} \subset J^{k+1}\pi$, defined as the set of solutions to the equations

$$F_q = 0, \quad D_{x^i}(F_q) = 0.$$

We call $\mathcal{E}_{k+1}$ the prolongation of $\mathcal{E}_k$. On some occasions we will use the notation $\mathcal{E}_k^{(1)}$ to denote this prolongation, and we also define inductively the

ith prolongation $\mathcal{E}_k^{(i)} = (\mathcal{E}_k^{(i-1)})^{(1)}$. Since this PDE is contained in $J^{k+i}\pi$ we find it most convenient to denote it by $\mathcal{E}_{k+i}$. Continuing this process, and using the notation $\mathcal{E}_l = \pi_{k,l}(\mathcal{E}_k)$ for $l < k$, we end up with a sequence of projections.

$$M \leftarrow \mathcal{E}_0 \leftarrow \cdots \leftarrow \mathcal{E}_{k-1} \leftarrow \mathcal{E}_k \leftarrow \mathcal{E}_{k+1} \leftarrow \cdots$$

We denote the inductive limit, which is a diffiety ([10]), by $\mathcal{E}_\infty$. The most important construction used in this thesis, the algebra of differential invariants, is an algebra of functions on this diffiety. In general we will refer to the PDE by $\mathcal{E}$ if the index is not essential.

1.3 Symmetries of PDEs and Lie pseudogroups

With the above interpretation of a PDE $\mathcal{E}$ as a submanifold in $J^k\pi$, we can use tools and ideas from differential geometry to study PDEs.

One natural question to ask is whether there exist transformations on $J^k\pi$ (or $J^\infty\pi$) preserving $\mathcal{E}_k$. What kind of transformations to allow can be widely discussed, as done in [10]. One natural choice is to consider fiber-preserving diffeomorphisms on E, a special class of point transformations. These transformations can be extended naturally to $J^k\pi$ since they act on sections of π. If the extended transformations preserve the equation $\mathcal{E}_k \subset J^k\pi$, i.e. take points in $\mathcal{E}_k$ to other points in $\mathcal{E}_k$, we say that they are symmetries of $\mathcal{E}_k$. The collection of symmetries makes up a Lie pseudogroup. To define Lie pseudogroups we need some terminology. See [6, 7] for details.

Inside of $J^k(E \times E)$, we have the jet-space for diffeomorphisms D^k consisting of k-jets of sections of the trivial bundle $E \times E$, projecting diffeomorphically to both factors. Its stabilizer D_a^k at $a \in E$ is an affine algebraic group and is called the differential group of order k.

Definition 1. A Lie pseudogroup of order l is given by a Lie equation, which is a collection of subbundles $G^j \subset D^j$, $0 < j \leq l$, such that the following properties are satisfied:

- For $\varphi_j, \psi_j \in G^j$ we have $\varphi_j \circ \psi_j \in G^j$ whenever defined.
- $G^j \subset (G^{j-1})^{(1)}$ and $\rho_{j,j-1} \colon G^j \to G^{j-1}$ is a bundle for every $j \leq l$.

In practice it is often more convenient to work with infinitesimal symmetries, rather than with finite ones. If X is a vector field on E, it can be lifted naturally to a vector field $X^{(k)}$ on $J^k\pi$, via its one-parameter group of transformations. We say that X is an infinitesimal symmetry of a kth order PDE $\mathcal{E}_k$ if $X^{(k)}$ is tangent to $\mathcal{E}_k$. The infinitesimal symmetries make up a Lie algebra (or Lie algebra sheaf) of vector fields on E, which may be of finite or infinite dimension.

2 Introduction to invariant theory

The main topic of this thesis is symmetry pseudogroups of differential equations. First and foremost our goal is to understand the quotient of solution spaces by symmetry pseudogroups. Before we consider pseudogroup actions on PDEs it will benefit our understanding to discuss group actions on manifolds and algebraic varieties, and the corresponding quotient spaces. This also allows us to introduce the theorem of Rosenlicht, which in [6] was proved to be very useful to the theory of differential invariants. For a more comprehensive treatment of this theory we refer to [9].

2.1 The problem with orbit spaces

Let G be a Lie group acting on a manifold M. We would like to get some understanding of the orbit space M/G. As a set, this space is always well-defined. The group action defines an equivalence relation on M, and M/G is the set of such equivalence classes. We then have the natural map $\pi\colon M \to M/G$, taking $x \in M$ to its equivalence class $[x] \in M/G$.

There is a natural topology on the set M/G, coming from the topology on M, called the quotient topology. The set V is open in M/G if the preimage $\pi^{-1}(V)$ is open in M. Hence M/G is not only a set, but also a topological space. However, it will in general not be a smooth manifold, even if M and G are. The following example illustrates this.

Example 1. *Let $M = \mathbb{R}^2$, and let $G = \mathbb{R}^+ = (0, \infty)$ act by scaling the vector space: $(t, (x, y)) \mapsto (tx, ty)$. The orbits are rays emanating from the origin, together with the point $(0, 0)$. The quotient space is $M/G = S^1 \cup \{(0, 0)\}$. We see that the only open set containing $(0, 0)$ (in the quotient topology) is the whole space M/G. This shows that, in particular, the quotient space is non-Hausdorff.*

Notice that if we consider the invariant submanifold $M_0 = M \setminus \{(0, 0)\}$, then the quotient $M_0/G = S^1$ is a manifold.

2.2 Invariants on algebraic varieties

In the case where we have an algebraic group acting algebraically on an algebraic variety we can always remove a Zariski closed set, so that the quotient becomes an algebraic variety.

Restricting to algebraic varieties may seem artificial when working in the field of differential geometry. However, it turns out that for our applications, namely to symmetries of PDEs, this restriction is completely natural. We will come back to this discussion later, after having taken a closer look on the general theory of rational invariants. The following theorem, first proved by Rosenlicht, will be essential for us.

Theorem 1 (Rosenlicht)**.** *Let G be an algebraic group acting rationally on an irreducible algebraic variety M. Then there exists a finite set of rational invariants that separate orbits in general position.*

Notice that this theorem does not apply to Example 1. There the field of rational invariants is generated by $I = x/y$, which does not separate two regular orbits lying on the same line in $\mathbb{R}^2$. An explanation for this is that $\mathbb{R}^+$ is not Zariski closed. Its closure is $\mathbb{R} \setminus \{0\}$, and for this group action, the invariant I does separate orbits in general position. If we allow the invariant to take the value ∞, it separates all orbits except for $\{(0,0)\}$. Thus, if we consider the $\mathbb{R} \setminus \{0\}$-action on $M_0 = M \setminus \{(0,0)\}$ we get $M_0/G = \mathbb{R}P^1$, which is an algebraic variety.

This is a general consequence of Rosenlicht's theorem. There exists a Zariski open set $M_0 \subset M$ such that M_0/G is an algebraic variety. Note that for the example above, the invariant $I = x/y$ solves the equivalence and classification problem on M (or, to be more precise, on M_0). Two points $p_1, p_2 \in M_0$ are equivalent if and only if $I(p_1) = I(p_2)$, and the equivalence classes are parametrized by the values of I.

Remark 2. *In practice we will usually find invariants by solving the PDE system $X(I) = 0$ for $X \in \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of vector fields on M corresponding to the action of G. For example, we see above that I satisfies this equation for $X = x\partial_x + y\partial_y$.*

Passing from a Lie group action on a manifold M to its Lie algebra of vector fields on M, one has to keep in mind that the Lie algebra only keeps information about the connected component of the Lie group. By "connected" we should, in the algebraic setting, understand the word in the context of Zariski topology. Rosenlicht's theorem guarantees that the field of rational invariants is finitely generated and separates orbits in general position. In particular this will hold for the (irreducible) Zariski connected component, but also for reducible groups containing it.

At first one may think that the above example shows a disadvantage of restricting to algebraic groups, since we cannot even treat such elementary Lie groups as $\mathbb{R}^+$. First of all, such cases can be handled by additional non-rational invariants, like $\operatorname{sgn}(x)$. Secondly, if the group action under consideration is algebraic, knowing that we need to look for rational invariants only may significantly simplify computations.

The last question is then whether the class of algebraic group actions is rich enough for our purpose. The answer seems to be yes, and the results of this thesis substantiate this claim. The main topic here is Lie pseudogroups which are symmetries of differential equations. Most popular differential equations are polynomial in derivatives, and this implies that the Lie pseudogroup of symmetries is algebraic in some special way. Thus, as was realized in [6], Rosenlicht's theorem is exactly what we need.

3 Invariants on differential equations

Let us now take a closer look at the equivalence and classification problem on solution spaces of PDEs. The general setting is the following. Let $\mathcal{E}_k \in J^k\pi$ be a kth order PDE, and let G be a Lie pseudogroup consisting of point symmetries of $\mathcal{E}$. Of main interest is the action of G on the solution space of $\mathcal{E}$. Ideally we would like to both describe the space of orbits of solutions, and also to be able to determine whether two given solutions are equivalent. However, for most PDEs we don't really know what the solution space looks like, and in general both the solution space and the Lie pseudogroup will be infinite-dimensional objects. One way to approach this problem is to describe the G-orbits on $\mathcal{E}_i \subset J^i\pi$ for every i. We do this by finding generators for the algebra of G-invariant functions on $\mathcal{E}_i$, the so-called differential invariants. It turns out that the algebra of invariant functions on $\mathcal{E}_\infty$ is finitely generated, as a differential algebra. In order to solve the equivalence problem, we need to find some generators for the algebra of differential invariants. Finding the (differential) syzygies among these generators solves the classification problem. Essentially they can be thought of as a system of differential equations whose solutions are equivalence classes of solutions of $\mathcal{E}$.

We will assume that the fibers of $\mathcal{E}_i$ over any point in E are irreducible algebraic varieties for every positive integer i. Most interesting PDEs are of this type. Note also that this type of algebraicity is well-defined: applying a diffeomorphism to E will preserve the algebraicity. We will also assume that $\mathcal{E}$ is formally integrable. The Lie pseudogroup G will be assumed transitive and algebraic, meaning that for every $a \in E$ the subgroups $G_a^j \subset D_a^j$ in the differential group of order k are algebraic subgroups (recall the notation from Section 1.3). The full symmetry pseudogroup (of point transformations) of an algebraic PDE will always be algebraic. If one is interested in a sub-pseudogroup proper care must be taken. For a more thorough treatment of these concepts we refer to [6].

3.1 Differential invariants

Given a PDE $\mathcal{E}$, we can for any k describe the G-orbits on $\mathcal{E}_k$ by finding generators for the algebra of G-invariant functions on $\mathcal{E}_k$. Assuming that both $\mathcal{E}$ and G are fixed, we make the following definition.

Definition 2. A differential invariant of order k is a function on $\mathcal{E}_k \subset J^k\pi$ which is constant on the orbits of G.

We may sometimes leave out the word "differential", since we will not talk about any other type of invariants. With this definition a differential invariant of order $k-1$ is also a differential invariant of order k. Thus we get a filtered algebra of differential invariants

$$\mathcal{A}_0 \subset \mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots \subset \mathcal{A} = \lim \mathcal{A}_k$$

where $\mathcal{A}_k$ is the algebra of differential invariants of order k.

In practice differential invariants of order k can often be found by solving the first-order Linear system of PDEs

$$(L_{X^{(k)}} I)|_{\mathcal{E}_k} = 0, \qquad X \in \mathfrak{g}.$$

The question about what kind of functions we allow, or require, our invariants to be (smooth, local, rational, etc.) is an interesting one. For a long time, the common approach was to look for differential invariants among functions defined locally on $J^k\pi$ (the microlocal approach).

Example 2. *For curves in $\mathbb{R}^2$ under the action of the Euclidean group $SO(2) \rtimes \mathbb{R}^2$, the classical curvature is defined as*

$$\kappa = \frac{y''(x)}{\sqrt{(1 + y'(x)^2)^3}}.$$

However, it is not really invariant. The reflection (or rotation) $(x, y) \mapsto (-x, -y)$ takes κ to $-\kappa$. Note that the rational invariant κ^2 does not have this problem.

See section 5.3 in [6] for a discussion on square roots in differential invariants. From a microlocal perspective the existence of local invariants on $J^k\pi$, or on some PDE $\mathcal{E}_k \subset J^k\pi$, is guaranteed by Frobenius' theorem (on integrable foliations). However, the domain on which the invariants are defined is not necessarily invariant.

In [6] Kruglikov and Lychagin realized that we can take advantage of the fact that algebraic geometry appear naturally on the fibers of jet bundles.

This allows to use Rosenlicht's theorem on the fibers of $\mathcal{E}_k$. Thus we get rational differential invariants, which are defined on a G-stable Zariski open set in $\mathcal{E}_k$, and we may consider $\mathcal{A}_k$ to be fields of rational invariants. The idea is the following:

Since G is assumed to act transitively on E, we may identify the orbit space on $\mathcal{E}_k$ with the orbit space on a fiber of $\mathcal{E}_k \to E$. More precisely $(\mathcal{E}_k \setminus S_k)/G$ is identified with $((\mathcal{E}_k \setminus S_k) \cap \pi_k^{-1}(a))/G_a^k$, where $S_k \subset \mathcal{E}_k$ is some Zariski closed subset. As in section 2.2 a Zariski closed subset needs to be removed. Thus we are in the situation where an algebraic group G_a^k acts algebraically on an algebraic variety $\mathcal{E}_a^k \setminus (S_k \cap \pi_k^{-1}(a))$, and there exists a geometric quotient if S_k is chosen appropriately. More precisely, we have the following theorem ([6]):

Theorem 2 (Kruglikov-Lychagin)**.** *Let G be an algebraic transitive pseudogroup of symmetries on a formally integrable irreducible algebraic differential equation $\mathcal{E}$. Then there exists an integer l and a Zariski closed invariant proper subset $S_l \subset \mathcal{E}^l$ such that $\mathcal{E}^k \setminus \pi_{k,l}^{-1}(S_l)$ admits a rational geometric quotient $Y^k \simeq (\mathcal{E}^k \setminus \pi_{k,l}^{-1}(S_l))/G^k$ for every $k \geq l$.*

3.2 Invariant Derivations and the Lie-Tresse theorem

Since the filtered algebra of rational differential invariants in general contains infinitely many independent functions, finding a generating set may seem difficult. What helps us here is that $\mathcal{A}$ is in fact a differential algebra: There exist derivations on $\mathcal{A}$.

Definition 3. An invariant derivation is a derivation on the algebra $\mathcal{A}$ of differential invariants which commutes with G.

In coordinates they take the form $\nabla = \alpha^i D_{x^i}$, where D_{x^i} are total derivatives and α^i are functions on $\mathcal{E}_k$ for some k. We will restrict our attention to derivations for which α^i are rational functions. The functions α^i must satisfy the system of differential equations coming from the condition $[\nabla, X^{(\infty)}] = 0$ for every $X \in \mathfrak{g}$.

It turns out that the algebra $\mathcal{A}$ of differential invariants can be generated by the field $\mathcal{A}_l$ of rational lth order differential invariants, for some positive integer l, together with invariant derivations. In fact it is sufficient to consider only polynomials of derivatives of invariants in $\mathcal{A}_l$ in order to separate orbits. Thus, we will in general be interested in the algebra of differential invariants that are rational on fibers of $\mathcal{E}_l \to E$, and polynomial on fibers of $\mathcal{E}_k \to \mathcal{E}_l$, for $k > l$.

Thus, given a PDE $\mathcal{E}$ with a Lie pseudogroup action G on it, we will by $\mathcal{A}$ mean the filtered differential algebra of rational-polynomial differential invariants.

The idea that the algebra of differential invariants is finitely generated dates back to Lie and Tresse, but the precise formulation of the theorem has evolved during the last century. See [6] for a summary of the history of the theorem. The most recent development is due to Kruglikov and Lychagin ([6]), where they give the global formulation of the theorem:

Theorem 3 (Kruglikov-Lychagin)**.** *Consider an algebraic action of a pseudogroup G on a formally integrable irreducible differential equation $\mathcal{E}$ over E. Suppose G acts transitively on E. Then there exists a number l and a Zariski closed invariant proper subset $S \subset \mathcal{E}_l$ such that the algebra of differential invariants separates the regular orbits from $\mathcal{E}_\infty \setminus \pi^{-1}_{\infty,l}(S)$ and is finitely generated in the following sense.*

There exists a finite number of functions $I_1, ..., I_t \in \mathcal{A}$ and a finite number of rational invariant derivations $\nabla_1, ..., \nabla_s \colon \mathcal{A} \to \mathcal{A}$ such that any function in $\mathcal{A}$ is a polynomial of differential invariants $\nabla_\sigma I_i$ where $\nabla_\sigma = \nabla_1^{i_1} \cdots \nabla_s^{i_s}$ for a multi-index $\sigma = (i_1, ..., i_s)$, with coefficients being rational functions of the invariants I_i.

This theorem lies behind most of the results in this thesis.

3.3 Solution of the recognition and classification problem

In order to explain how the differential invariants can be used to solve the recognition and classification problem, we find it convenient to introduce some special invariant derivations, called Tresse derivatives (see [7])

Let $\hat{d}$ denote the horizontal differential. It can be defined as the operator satisfying $(\hat{d}f) \circ j^k s = d(f \circ j^k s)$ for any function f on $J^\infty \pi$, where d is the exterior differential on the base manifold of π. In coordinates it is given by $\hat{d}f = D_{x^i}(f) dx^i$. Now, pick n differential invariants $I_1, ..., I_n$ satisfying $\hat{d}I_1 \wedge \cdots \wedge \hat{d}I_n \neq 0$. Then the Tresse derivatives $\hat{\partial}_i = \hat{\partial}_{I_i}$ are defined by

$$\hat{\partial}_i = \sum_j \left(D_{x^a}(f_b)\right)^{-1}_{ij} D_{x^j}.$$

The Tresse derivatives are commuting invariant derivations that satisfy $\hat{d}f = \sum \hat{\partial}_i(f) \hat{d}I_i$.

By using a finite set of differential invariants together with the Tresse derivatives constructed from n of them to generate the algebra of differential invariants it becomes clear how the algebra of rational scalar differential invariants lets us solve the equivalence problem of sections of π under the G-action. Denote the invariants generating the algebra by $I_1, ..., I_n, K_1, ..., K_q$. For a function f on $J^k\pi$, let us use the notation $f(s) = f \circ j^k s$, for a section s of π. Then $f(s)$ will be a function on the base of π.

Assuming that $(\hat{d}I_1 \wedge \cdots \wedge \hat{d}I_n) \circ j^k s = d(I_1(s)) \wedge \cdots \wedge d(I_n(s))$ is defined and nonzero (this puts minor restrictions on both I_i and s), the functions $I_i(s)$ can be taken as local coordinates on the base of π. Expressing $K_1(s), ..., K_q(s)$ in terms of these determines the equivalence class, i.e. two sections s_1, s_2 are locally equivalent if and only if the functions $K_j(s_i)(I(s_i), J(s_i))$ are equal in some neighborhood, for $i = 1, 2$.

More conveniently we may think of the functions $I_i(s), K_j(s)$ as defining a n-dimensional surface in $\mathbb{R}^{n+q}$. Then two sections are locally equivalent if their corresponding surfaces coincide in some neighborhood. These surfaces are not arbitrary surfaces, they are constrained by a system of PDEs. This system is called the quotient equation, and it is of great importance since its solutions are exactly the equivalence classes of sections of π under the G-action. It manifests itself as differential syzygies among the generating set of invariants and invariant derivations. In this way a description of the generators I_i, K_j and $\hat{\partial}_i$ solves the recognition problem, while the solution of the classification problem is given by the differential syzygies.

There is however one difficulty appearing. It is analogous to what happens in algebraic invariant theory, as discussed in Section 2, and it is clear that it will always make trouble for us as long as we try to describe the quotient by using the algebra of invariant functions. In order to get a good quotient space (or a good quotient PDE) we need to remove a Zariski closed set consisting of "singular" orbits.

After choosing the invariants I_i, K_j, the condition $\hat{d}I_1 \wedge \cdots \wedge \hat{d}I_n \neq 0$ determine an algebraic subset in $J^k\pi$, where k depends on the order of the differential invariants chosen. The invariants let us separate only sections whose k-jets does not intersect with this singular set.

From one viewpoint this is not a big problem. Since the Zariski closed set is of measure zero, we are still left with most equivalence classes. Others may argue that the solutions we remove, for example solutions of constant curvature, are the most interesting ones.

In any case it is obviously important to understand the space of generic solutions, and its quotient. And equivalence problems for special, singular sections can be considered separately by restricting to a sub-PDE and applying the same methods.

4 Four simple examples

The main part of this thesis contains many examples showing how to use the theory above. However, most of these examples are quite complicated. From one perspective this is a good thing, as it shows the strength of the theory. On the other hand, for a reader not yet completely comfortable with the theory it may be more appropriate to start with some simpler and more transparent examples. We give four such examples in this section. It is likely that some of the examples may provide new insight even to experts in the field.

We start by considering a few different Lie group actions on curves in the plane. Then we proceed to classify solutions of two well-known nonlinear PDEs.

4.1 Euclidean group on curves in $\mathbb{R}^2$

Consider the manifold $M = \mathbb{R}^2$ with coordinates x, u, and the Lie algebra $\mathfrak{g}$ spanned by $\partial_x, \partial_u, x\partial_u - u\partial_x$. There are two natural Lie groups actions with this infinitesimal action: $E(2)_+ = SO(2) \ltimes \mathbb{R}^2$ and $E(2) = O(2) \ltimes \mathbb{R}^2$. Both of these are algebraic, so the global Lie-Tresse theorem is applicable. It is easy to check that the two rational functions

$$\kappa = \frac{u_2^2}{(1+u_1^2)^3}, \qquad \kappa_1 = \frac{u_3}{(1+u_1^2)^2} - 3\frac{u_1 u_2^2}{(1+u_1^2)^3}$$

satisfy $X^{(2)}(\kappa) = 0$, $X^{(3)}(\kappa_1) = 0$ for every $X \in \mathfrak{g}$. The function κ_1 is not invariant with respect to $E(2)$, since the transformation $(x, u) \mapsto (-x, u)$ changes sign of κ_1. The function κ_1^2, on the other hand, is invariant under the $E(2)$-action. The algebra of rational-polynomial differential invariants of $E(2)_+$ (respectively $E(2)$) is generated by κ, κ_1 and the Tresse-derivative $\hat{\partial}_\kappa$ (respectively $\kappa, \kappa_1^2, \hat{\partial}_\kappa$).

This shows that it is possible, using rational differential invariants, to separate orbits for unconnected Lie groups, and not only for the Zariski connected component. The choice of Lie group action shows up already in the field of second-order differential invariants. The field generated by κ, κ_1 is a field extension of degree 2 of the field generated by κ, κ_1^2, and the Galois group of the field extension is $E(2)/E(2)_+ = \mathbb{Z}_2$.

4.2 Affine group on curves in $\mathbb{R}^2$

Consider the same manifold as above, but with the Lie algebra action $\mathfrak{g}$ spanned by the vector fields $\partial_x, \partial_u, x\partial_u, u\partial_x, x\partial_x, u\partial_u$. Again we consider two natural Lie groups with this infinitesimal action: $A(2) = GL(2) \ltimes \mathbb{R}^2$ and $A(2)_+ = GL(2)_+ \ltimes \mathbb{R}^2$, where $GL(2)_+$ denotes the subgroup of $GL(2)$ consisting of orientation preserving transformations. It is easy to check that the two rational functions

$$\kappa = \frac{(9u_2^2u_5 - 45u_2u_3u_4 + 40u_3^3)^2}{(3u_2u_4 - 5u_3^2)^3}, \quad \kappa_1 = \frac{9u_2^3u_6 - 63u_2^2u_3u_5 + 105u_2u_3^2u_4 - 35u_3^4}{(3u_2u_4 - 5u_3^2)^2}$$

satisfy $X^{(5)}(\kappa) = 0$, $X^{(6)}(\kappa_1) = 0$ for every $X \in \mathfrak{g}$. However, these are invariant also under the action $(x, u) \mapsto (-x, u)$ of $A(2)$, so they are not sufficient for separating $A_+(2)$-orbits. For $A(2)$, the algebra of rational-polynomial invariants is generated by κ, κ_1 and the Tresse-derivative $\hat{\partial}_\kappa$.

The fact that the global Lie-Tresse theorem doesn't hold in general for the topologically connected component of a Lie group may seem like an inconvenience, but passing to the connected component appears to have been often a result of (believed) necessity rather than desire.

In fact, if one is really interested in the topologically connected component, one can describe the orbits of $A_+(2)$ by using additional discrete data, for example the sign of the relative invariant $9u_2^2u_5 - 45u_2u_3u_4 + 40u_3^3$.

4.3 The Hunter-Saxton equation

The Hunter-Saxton equation is defined by

$$(u_t + uu_x)_x = \frac{1}{2}u_x^2.$$

It is an integrable PDE that arises in the study of liquid crystals. Its Lie algebra of symmetries is spanned by

$$\partial_t, \quad t\partial_t + x\partial_x, \quad x\partial_x + u\partial_u, \quad t^2\partial_t + 2tx\partial_x + 2x\partial_u, \quad f(t)\partial_x + f'(t)\partial_u$$

where f runs through all smooth locally defined functions. Note that even though the f is a general smooth function the pseudogroup is still algebraic, as this type of algebraicity is a property only of the vertical action of stabilizers in the fibers of the jet spaces.

The algebra of differential invariants of the corresponding Zariski connected Lie pseudogroup action is generated by

$$I = \frac{u_{xx}u_{xxxx}}{u_{xxx}^2}, \quad J = \frac{u_{xx}^2u_{xxxxx}}{u_{xxx}^3}, \quad H = \frac{u_{xx}^3u_{xxxxxx}}{u_{xxx}^4}$$

together with the Tresse-derivatives $\hat{\partial}_I, \hat{\partial}_J$. It is not difficult to check that the quotient PDE is given by

$$(4I - 7)H_I - (11I - 6J + 7)H_J = 8H - 25I - 16J,$$

where we use the simplified notation $H_I = \hat{\partial}_I(H), H_J = \hat{\partial}_J(H)$. This linear first-order PDE can be solved with the method of characteristics. We end up with the general solution

$$H = (4I-7)^2 F\left(\frac{(22I-4J-21)^2}{(4I-7)^3}\right) - \frac{63}{4}I + 8J + \frac{217}{32}.$$

Since each solution determines an equivalence class of solutions to the Hunter-Saxton equation, we see that the quotient of the solution space is parametrized by the function F. If we fix F and insert the expressions for I, J, H into the equation above, we get a new equation on J^6 which we may add to the HS equation. Doing this amounts to restricting to one equivalence class of solutions. For example, in the case $F \equiv 0$, we get the ODE

$$32u_{x^2}^3 u_{x^6} - 256u_{x^2}^2 u_{x^3} u_{x^5} + 504 u_{x^2} u_{x^3}^2 u_{x^4} - 217 u_{x^3}^4 = 0.$$

4.4 Burgers' equation

We compute the differential invariants and quotient PDE for Burgers' equation. In this case the topologically connected component of the symmetry group is different from the Zariski connected component.

Burgers' equation is defined by $\mathcal{B}_2 = \{u_{xx} = u_t + uu_x\} \subset J^2(\mathbb{R}^2 \times \mathbb{R})$. It appears in fields such as fluid mechanics and acoustics. Its symmetry algebra is spanned by the vector fields

$$\partial_x, \quad t\partial_x + \partial_u, \quad \partial_t, \quad 2t\partial_t + x\partial_x - u\partial_u, \quad t^2\partial_t + tx\partial_x + (x-tu)\partial_u.$$

Orbits in general position in $\mathcal{B}_k$ are five-dimensional for $k > 0$. The dimension of $\mathcal{B}_k$ is $2k+3$, meaning that there are $2(k-1)$ independent differential invariants of order k for $k > 1$. In particular we have the following three invariants:

$$I = \frac{(u_{tx} + u_x^2 + u(u_t + uu_x))^3}{(u_t + uu_x)^4},$$
$$J = \frac{(u_{tt} + 2uu_{tx} + (u^2 + 4u_x)(u_t + uu_x))(u_{xy} + u_x^2 + u(u_t + uu_x))}{(u_t + uu_x)^3}$$
$$H = \frac{u_{ttx} + 2uu_{tt} + 3(u^2 + 2u_x)u_{tx} + u_x(uu_t + 4u_x^2) + (u^3 + 7uu_x)(u_t + uu_x)}{(u_t + uu_x)^2}$$

A fourth invariant of order three can be generated from these:

$$(3J - 4I)H_I + (H - 3J + 4 + J^2/I)H_J + 2(H - 4)$$

The quotient equation is given by

$$\begin{aligned}0 = &I^2(4I-3J)^2H_{II} - 2I(4I-3J)(IH-3IJ+J^2+4I)H_{IJ}\\ &+(IH-3IJ+J^2+4I)^2H_{JJ} + I(4I^2-12IJ+6J^2+9I)H_I\\ &-I(2IH-2JH+2J^2-2I-11J)H_J + 2I^2H - 2I^2 - 15IJ.\end{aligned}$$

In [2] Hydon computed the discrete symmetries of Burgers' equation, the ones not contained in the topologically connected component. They are generated by $(t, x, u) \mapsto (-1/(4t), x/(2t), 2(tu - x))$ and form the cyclic group $\mathbb{Z}_4$. It is easy to check that the differential invariants above are invariant also under these transformations.

The connected component of the symmetry group above is not algebraic. Its Zariski closure (the Zariski connected component) contains four topologically connected components. The invariants above will not separate the orbits of the topologically connected component of the symmetry group.

5 The papers of this thesis

The remainder of the thesis consists of 6 papers. They all concern the classification or recognition problem for some mathematical structure. In all but one, these problems are solved by finding the algebra of rational differential invariants. The papers of the thesis are the following.

- E. Schneider, *Projectable Lie algebras of vector fields in 3D*, Journal of Geometry and Physics **132**, 222-229, (2018). https://doi.org/10.1016/j.geomphys.2018.05.025
- E. Schneider, *Differential invariants of surfaces.* (Close to submission)
- B. Kruglikov, E. Schneider, *Differential invariants of self-dual conformal structures*, Journal of Geometry and Physics **113**, 176-187, (2017). https://doi.org/10.1016/j.geomphys.2016.05.017
- B. Kruglikov, E. Schneider, *Differential invariants of Einstein-Weyl structures in 3D*, Journal of Geometry and Physics **131**, 160-169, (2018). https://doi.org/10.1016/j.geomphys.2018.05.011
- B. Kruglikov, D. McNutt, E. Schneider, *Differential invariants of Kundt waves*, arXiv:1901.02635. (Submitted)
- E. Schneider, *Differential invariants in thermodynamics.* (Submitted)

As a collection, the papers display the power of the theory of scalar differential invariants. They show the versatility of the global Lie-Tresse theorem and its underlying ideas, and in particular they corroborate the idea that restricting to rational differential invariants is appropriate in a very general setting. They also show the utility of computer algebra systems applied to some particular problems in pure mathematics since we, when computing differential invariants, rely heavily on Maple and, in particular, on the pdsolve procedure and Ian Anderson's DifferentialGeometry package.

At the same time, each of the papers contain results that are important and interesting by themselves. Three of the papers revolve around recognition and classification of special conformal and pseudo-Riemannian manifolds. One paper concerns recognition of surfaces in three dimensions under several different Lie group actions. The Lie group actions considered come from a special class of Lie groups consisting projectable transformations on the bundle $\mathbb{C}^2 \times \mathbb{C} \to \mathbb{C}^2$. The list of these Lie group actions was found in another paper of this thesis. We give a summary of the six papers.

Projectable Lie algebras of vector fields in 3D The classification of Lie group actions on three-dimensional space is of fundamental importance in differential geometry, as it also gives a classification of homogeneous spaces. The history of the problem dates back to Lie, who outlined how to make a complete classification. In this paper we lift the Lie group actions from $\mathbb{C}^2$ to $\mathbb{C}^2 \times \mathbb{C}$, and thereby obtain a complete list of a special type of Lie group actions on three-dimensional space. We also discuss a connection between some special lifts and Lie algebra cohomologies.

This first paper stands out in this thesis since it is the only one in which we do not consider differential invariants. However, it does concern a classification problem, and the results obtained are important for the next paper in which we consider the equivalence problem for surfaces in three-dimensional space.

Differential invariants of surfaces We find differential invariants for surfaces in three-dimensional space under the Lie group actions found in the previous paper. Our main motivation is to solve the equivalence problem, but algebra of differential invariants is also an important source for invariant differential equations. Geometrically we can think about this as recognizing surfaces in particular three-dimensional homogeneous spaces, and finding admissible PDEs for surfaces in these spaces.

In the context of a particular example of a Lie group action we discuss the notion of algebraic Lie group action, and for one of the algebraic actions we compute the differential syzygies. In the end we find an unconnected Lie group action, and discuss how its algebra of differential invariants is related to that of the connected component.

Differential invariants of self-dual conformal structures We consider the problem of recognizing and classifying four-dimensional self-dual conformal structures. These structures play an important role in the theory of dispersionless integrable systems and in twistor theory. In addition, they are central in Yang-Mills theory.

First we describe the scalar differential invariants in a coordinate free way. Then we use a result by Dunajski, Ferapontov and Kruglikov ([1]) in order to write the self-dual conformal structures in Plebański-Robinson form. We find the Lie pseudogroup preserving this form, and give generators of its algebra of differential invariants.

Differential invariants of Einstein-Weyl structures We treat the recognition problem for three-dimensional Einstein-Weyl structures. They are reductions of self-dual conformal structures, and are important in alternative theories of gravity. Using results from [1], the set of Einstein-Weyl structures are identified with solutions of a modified Manakov-Santini system, by bringing their metric and connection to a special form. We show that the Lie pseudogroup of symmetries to this system corresponds exactly to the Lie pseudogroup of diffeomorphisms preserving the form of the metric and connection. We find generators of the algebra of differential invariants. It can be generated by three invariant derivations and one single differential invariant. In the end we use the differential invariants to find some particular solutions to the modified Manakov-Santini system, and thereby produce some examples of Einstein-Weyl structures.

Differential invariants of Kundt waves Kundt waves are special Lorentzian spacetimes with vanishing polynomial scalar curvature invariants, meaning that they can not be distinguished by the "normal methods". The equivalence problem for Ricci-flat Kundt waves was already solved in [3] by using the Cartan-Karlhede algorithm, and part of our motivation was to compare that approach to the one used throughout this thesis, making this a good resource for researchers that are familiar with one of the methods and would like to understand the other approach.

We start by assuming that the Kundt waves are written down in special coordinates, so that the metric takes a particularly simple form, depending on one function of three variables. Then we compute differential invariants of the Lie pseudogroup preserving this form. The approach taken is thus similar to the one used for self-dual conformal structures and Einstein-Weyl structures. However the Lie pseudogroup consists of four (Zariski) connected components, and it acts intransitively, so extra care needs to be taken. To our knowledge this is the first time the algebra of differential invariants for an nonconnected Lie pseudogroup is found.

Differential invariants in thermodynamics In this paper we look at two Lie group actions appearing in thermodynamics. We compute differential invariants of the information gain function. They let us distinguish inequivalent thermodynamic states. In the end we take a closer look at our differential invariants in the context of ideal and van der Waals gases, and we show how they can be used to distinguish these gases.

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Projectable Lie algebras of vector fields in 3D

Eivind Schneider

Abstract

Starting with Lie's classification of finite-dimensional transitive Lie algebras of vector fields on $\mathbb{C}^2$ we construct transitive Lie algebras of vector fields on the bundle $\mathbb{C}^2 \times \mathbb{C}$ by lifting the Lie algebras from the base. There are essentially three types of transitive lifts and we compute all of them for the Lie algebras from Lie's classification. The simplest type of lift is encoded by Lie algebra cohomology.

1 Introduction

A fundamental question in differential geometry is to determine which transitive Lie group actions exist on a manifold. Sophus Lie considered this to be an important problem, in particular due to its applications in the symmetry theory of PDEs. In [13] (see also [14]) he gave a local classification of finite-dimensional transitive Lie algebras of analytic vector fields on $\mathbb{C}$ and $\mathbb{C}^2$. Lie never published a complete list of finite-dimensional Lie algebras of vector fields on $\mathbb{C}^3$, but he did classify primitive Lie algebras of vector fields on $\mathbb{C}^3$, those not preserving an invariant foliation, which he considered to be the most important ones and also some special imprimitive Lie algebras of vector fields.

Lie algebras of vector fields on $\mathbb{C}^3$ preserving a one-dimensional foliation are locally equivalent to projectable Lie algebras of vector fields on the total space of the fiber bundle $\pi\colon \mathbb{C}^2 \times \mathbb{C} \to \mathbb{C}^2$. Finding such Lie algebras amounts to extending Lie algebras of vector fields on the base (where they have been classified) to the total space. For the primitive Lie algebras of vector fields on the plane, this was completed by Lie [14]. Amaldi continued Lie's work by extending the imprimitive Lie algebras to three-dimensional space [2, 3] (see also [11]), but his obtained list of Lie algebras is incomplete. Nonsolvable Lie algebras of vector fields on $\mathbb{C}^3$ were recently classified in [5]. It was also showed there that a complete classification of finite-dimensional solvable Lie algebras of vector fields on $\mathbb{C}^3$ is hopeless, since it contains the subproblem of classifying left ideals of finite codimension in the universal enveloping algebra $U(\mathfrak{g})$ for the two-dimensional Lie algebras $\mathfrak{g}$, which is known to be a hard algebraic problem.

In this paper we consider Lie algebras of vector fields on the plane from Lie's classification, and extend them to the total space $\mathbb{C}^2 \times \mathbb{C}$. In order to avoid the issues discussed in [5] we only consider extensions that are of the same dimension as the original Lie algebra. The resulting list of Lie algebras has intersections with [14], [2, 3] and [5], but it also contains some additional solvable Lie algebras of vector fields in three-dimensional space which are missing from [2, 3].

We start in section 2 by reviewing the classification of Lie algebras of vector fields on $\mathbb{C}^2$, which will be our starting point. The lifting procedure is explained in section 3. We show that transitive lifts can be divided into three types, depending on how they act on the fibers of π. In section 4 we give a complete list of the lifted Lie algebras of vector fields, which is the main result of this paper. The relation between the simplest type of lift and Lie algebra cohomology is explained in section 5.

2 Classification of Lie algebras of vector fields on $\mathbb{C}^2$

Two Lie algebras $\mathfrak{g}_1 \subset \mathcal{D}(M_1), \mathfrak{g}_2 \subset \mathcal{D}(M_2)$ of vector fields on the manifolds M_1 and M_2, respectively, are locally equivalent if there exist open subsets $U_i \subset M_i$ and a diffeomorphism $f\colon U_1 \to U_2$ with the property $df(\mathfrak{g}_1|_{U_1}) = \mathfrak{g}_2|_{U_1}$. Recall that $\mathfrak{g}$ is transitive if $\mathfrak{g}|_p = T_pM$ at all points $p \in M$.

The classification of Lie algebras of vector fields on $\mathbb{C}$ and $\mathbb{C}^2$ is due to Lie [13] (see [1] for English translation). There are up to local equivalence only three finite-dimensional transitive Lie algebras of vector fields on $\mathbb{C}$ and they correspond to the the groups of metric, affine and projective transformations, respectively:

$$\langle \partial_u \rangle, \qquad \langle \partial_u, u\partial_u \rangle, \qquad \langle \partial_u, u\partial_u, u^2\partial_u \rangle \tag{1}$$

On $\mathbb{C}^2$ any finite-dimensional transitive Lie algebra of analytic vector fields is locally equivalent to one of the following:

Primitive

$\mathfrak{g}_1 = \langle \partial_x, \partial_y, x\partial_x, x\partial_y, y\partial_x, y\partial_y, x^2\partial_x + xy\partial_y, xy\partial_x + y^2\partial_y \rangle$

$\mathfrak{g}_2 = \langle \partial_x, \partial_y, x\partial_x, x\partial_y, y\partial_x, y\partial_y \rangle$

$\mathfrak{g}_3 = \langle \partial_x, \partial_y, x\partial_y, y\partial_x, x\partial_x - y\partial_y \rangle$

Imprimitive

$$\mathfrak{g}_4 = \langle \partial_x, e^{\alpha_i x}\partial_y, xe^{\alpha_i x}\partial_y, ..., x^{m_i-1}e^{\alpha_i x}\partial_y \mid i = 1, ..., s\rangle,$$
$$\text{where } m_i \in \mathbb{N}\setminus\{0\}, \alpha_i \in \mathbb{C}, \sum_{i=1}^{s} m_i + 1 = r \geq 2$$
$$\mathfrak{g}_5 = \langle \partial_x, y\partial_y, e^{\alpha_i x}\partial_y, xe^{\alpha_i x}\partial_y, ..., x^{m_i-1}e^{\alpha_i x}\partial_y \mid i = 1, ..., s\rangle,$$
$$\text{where } m_i \in \mathbb{N}\setminus\{0\}, \alpha_i \in \mathbb{C}, \sum_{i=1}^{s} m_i + 2 = r \geq 4$$
$$\mathfrak{g}_6 = \langle \partial_x, \partial_y, y\partial_y, y^2\partial_y\rangle$$
$$\mathfrak{g}_7 = \langle \partial_x, \partial_y, x\partial_x, x^2\partial_x + x\partial_y\rangle$$
$$\mathfrak{g}_8 = \langle \partial_x, \partial_y, x\partial_y, ..., x^{r-3}\partial_y, x\partial_x + \alpha y\partial_y\rangle,\ \alpha \in \mathbb{C},\ r \geq 3$$
$$\mathfrak{g}_9 = \langle \partial_x, \partial_y, x\partial_y, ..., x^{r-3}\partial_y, x\partial_x + \left((r-2)y + x^{r-2}\right)\partial_y\rangle,\ r \geq 3$$
$$\mathfrak{g}_{10} = \langle \partial_x, \partial_y, x\partial_y, ..., x^{r-4}\partial_y, x\partial_x, y\partial_y\rangle,\ r \geq 4$$
$$\mathfrak{g}_{11} = \langle \partial_x, x\partial_x, \partial_y, y\partial_y, y^2\partial_y\rangle$$
$$\mathfrak{g}_{12} = \langle \partial_x, x\partial_x, x^2\partial_x, \partial_y, y\partial_y, y^2\partial_y\rangle$$
$$\mathfrak{g}_{13} = \langle \partial_x, \partial_y, x\partial_y, ..., x^{r-4}\partial_y, x^2\partial_x + (r-4)xy\partial_y, x\partial_x + \tfrac{r-4}{2}y\partial_y\rangle,\ r \geq 5$$
$$\mathfrak{g}_{14} = \langle \partial_x, \partial_y, x\partial_y, ..., x^{r-5}\partial_y, y\partial_y, x\partial_x, x^2\partial_x + (r-5)xy\partial_y\rangle,\ r \geq 6$$
$$\mathfrak{g}_{15} = \langle \partial_x, x\partial_x + \partial_y, x^2\partial_x + 2x\partial_y\rangle$$
$$\mathfrak{g}_{16} = \langle \partial_x, x\partial_x - y\partial_y, x^2\partial_x + (1-2xy)\partial_y\rangle$$

In the list above (which is based on the one in [10]), and throughout the paper, r denotes the dimension of the Lie algebra. Our $\mathfrak{g}_{16}$ is by $y \mapsto \frac{1}{y-x}$ locally equivalent to $\langle \partial_x + \partial_y, x\partial_x + y\partial_y, x^2\partial_x + y^2\partial_y\rangle$, which often appears in these lists of Lie algebras of vector fields on the plane but has a singular orbit $y - x = 0$. We also refer to [14, 4, 6, 9] which treat transitive Lie algebras of vector fields on the plane.

3 Lifts of Lie algebras of vector fields on $\mathbb{C}^2$

In this section we describe how we lift the Lie algebras of vector fields from the base space to the total space of $\pi\colon \mathbb{C}^2 \times \mathbb{C} \to \mathbb{C}^2$.

Definition 4. Let $\mathfrak{g} \subset \mathcal{D}(\mathbb{C}^2)$ be a Lie algebra of vector fields on $\mathbb{C}^2$, and let $\hat{\mathfrak{g}} \subset \mathcal{D}(\mathbb{C}^2 \times \mathbb{C})$ be a projectable Lie algebra satisfying $d\pi(\hat{\mathfrak{g}}) = \mathfrak{g}$. The Lie algebra $\hat{\mathfrak{g}}$ is a lift of $\mathfrak{g}$ (on the bundle π) if $\ker(d\pi|_{\hat{\mathfrak{g}}}) = \{0\}$.

For practical purposes we reformulate this in coordinates. Throughout the paper (x, y, u) will be coordinates on $\mathbb{C}^2 \times \mathbb{C}$. If $X_i = a_i(x, y)\partial_x + b_i(x, y)\partial_y$ form a basis for $\mathfrak{g} \subset \mathcal{D}(\mathbb{C}^2)$, then a lift $\hat{\mathfrak{g}}$ of $\mathfrak{g}$ on the bundle π is spanned by vector fields of the form $\hat{X}_i = a_i(x, y)\partial_x + b_i(x, y)\partial_y + f_i(x, y, u)\partial_u$. The functions f_i are subject to differential constraints coming from the commutation relations of $\mathfrak{g}$. Finding lifts of $\mathfrak{g}$ amounts to solving these differential equations. We consider only transitive lifts.

3.1 Three types of lifts

The fibers of π are one-dimensional and, as is common in these type of calculations, we will use the classification of Lie algebras of vector fields on the line to simplify our calculations. Let $\mathfrak{g}$ be a finite-dimensional transitive Lie algebra of vector fields on $\mathbb{C}^2$ and $\hat{\mathfrak{g}}$ a transitive lift. For $p \in \mathbb{C}^2 \times \mathbb{C}$, let $a = \pi(p)$ be the projection of p and let $\mathfrak{st}_a \subset \mathfrak{g}$ be the stabilizer of $a \in \mathbb{C}^2$. Denote by $\hat{\mathfrak{st}}_a \subset \hat{\mathfrak{g}}$ the lift of $\mathfrak{st}_a$, i.e. $d\pi(\hat{\mathfrak{st}}_a) = \mathfrak{st}_a$. The Lie algebra $\hat{\mathfrak{st}}_a$ preserves the fiber $F_a = \pi^{-1}(a)$ over a, and thus induces a Lie algebra of vector fields on F_a by restriction to the fiber. Denote the corresponding Lie algebra homomorphism by

$$\varphi_a \colon \hat{\mathfrak{st}}_a \to \mathcal{D}(F_a).$$

In general this map will not be injective, and it is clear that as abstract Lie algebras $\varphi_a(\hat{\mathfrak{st}}_a)$ is isomorphic to $\mathfrak{h}_a = \hat{\mathfrak{st}}_a / \ker(\varphi_a)$.

Since $\hat{\mathfrak{g}}$ is transitive, the Lie algebra $\varphi_a(\hat{\mathfrak{st}}_a)$ is a transitive Lie algebra on the one-dimensional fiber F_a, and therefore it must be locally equivalent to one of the three Lie algebras (1). Transitivity of $\hat{\mathfrak{g}}$ also implies that for any two points $a, b \in \mathbb{C}^2$, the Lie algebras $\varphi_a(\hat{\mathfrak{st}}_a), \varphi_b(\hat{\mathfrak{st}}_b)$ of vector fields are locally equivalent. Since the Lie algebra structure of $\mathfrak{h}_a$ is independent of the point a, it will be convenient to define $\mathfrak{h}$ as the abstract Lie algebra isomorphic to $\mathfrak{h}_a$. Thus $\dim \mathfrak{h}$ is equal to 1, 2 or 3, which allows us to split the transitive lifts into three distinct types.

Definition 5. We say that the lift $\hat{\mathfrak{g}}$ of $\mathfrak{g} \subset \mathcal{D}(\mathbb{C}^2)$ is metric, affine or projective if $\mathfrak{h}$ is of dimension one, two or three, respectively.

Since the properties of the Lie algebras $\mathfrak{st}_a$ and $\mathfrak{h}$ are closely linked, we can immediately say something about existence of the different types of lifts.

Theorem 4. *If $\mathfrak{st}_a$ is solvable, then there are no projective lifts. If $\mathfrak{st}_a$ is abelian, then there are no projective or affine lifts.*

Proof. The map $\varphi_a \colon \hat{\mathfrak{st}}_a \to \mathfrak{h}_a \simeq \mathfrak{h}$ is a Lie algebra homomorphism, and the image of a solvable (resp. abelian) Lie algebra is solvable (resp. abelian). □

It follows from Lie's classification that only the primitive Lie algebras may have projective lifts.

The main goal of this section is to show that we can choose local coordinates in a neighborhood $U \subset \mathbb{C}^2 \times \mathbb{C}$ of any point such that $\varphi_a(\hat{\mathfrak{st}}_a)|_{U \cap F_a}$ takes one of the three normal forms from (1) for every $a \in \pi(U)$, simultaneously. This fact, together with theorem 4, simplifies computations. Before proving it we make the following observation.

Lemma 1. *Let $\mathfrak{g} \subset \mathcal{D}(\mathbb{C}^2)$ be a transitive Lie algebra of vector fields, and let $a \in \mathbb{C}^2$ be an arbitrary point. Then there exists a locally transitive two-dimensional subalgebra $\mathfrak{h} \subset \mathfrak{g}$, and a local coordinate chart $(U, (x, y))$ centered at a such that $\mathfrak{h} = \langle X_1, X_2 \rangle$ where $X_1 = \partial_x$ and either $X_2 = \partial_y$ or $X_2 = x\partial_x + \partial_y$.*

Proof. This is apparent from the list in section 2, but we also outline an independent argument. It is well known that a two-dimensional locally transitive Lie subalgebra can be brought to one of the above forms, so we only need to show that such exists.

Let $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}$ be the Levi-decomposition of $\mathfrak{g}$. Assume first that $\mathfrak{r}$ is a locally transitive Lie subalgebra and let

$$\mathfrak{r} \supset \mathfrak{r}_1 \supset \mathfrak{r}_2 \supset \cdots \supset \mathfrak{r}_k \supset \mathfrak{r}_{k+1} = \{0\}.$$

be its derived series. If $\mathfrak{r}_k$ is locally transitive, it contains an (abelian) two-dimensional transitive subalgebra and we are done. If $\mathfrak{r}_k$ is not locally transitive, then we take a vector field $X_i \in \mathfrak{r}_i$ for some $i < k$ which is transversal to those of $\mathfrak{r}_k$. Since we have $[\mathfrak{r}, \mathfrak{r}_k] \subset \mathfrak{r}_k$ (can be shown by induction on k), we get a map $\mathrm{ad}_{X_i} \colon \mathfrak{r}_k \to \mathfrak{r}_k$. Let $X_k \in \mathfrak{r}_k$ be an eigenvector of ad_{X_i}. Then X_i and X_k span a two-dimensional locally transitive subalgebra of $\mathfrak{g}$.

If $\mathfrak{s}$ is a transitive subalgebra, then $\mathfrak{s}$ is locally equivalent to the standard realization on $\mathbb{C}^2$ of either sl_2, $sl_2 \oplus sl_2$ or sl_3, all of which have a locally transitive two-dimensional Lie subalgebra.

If neither $\mathfrak{s}$ nor $\mathfrak{r}$ is locally transitive they both determine transversal one-dimensional foliations and $\mathfrak{s} \simeq sl_2$. Thus it is possible to choose coordinates such that $\mathfrak{s} = \langle \partial_x, x\partial_x, x^2\partial_x \rangle$ while $\mathfrak{r}$ is spanned by vector fields of the form $b_i(x, y)\partial_y$. Since $\mathfrak{r}$ is finite-dimensional we get $(b_i)_x = 0$, by computing Lie brackets with $x^2\partial_x$. Therefore $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r}$, and there exists a two-dimensional abelian locally transitive subalgebra. □

Example 3. *Let $X_1 = \partial_x$ and $X_2 = \partial_y$ be vector fields on $\mathbb{C}^2$ and consider the general lift $\hat{X}_1 = \partial_x + f_1(x, y, u)\partial_u, \hat{X}_2 = \partial_y + f_2(x, y, u)\partial_u$. We may change coordinates $u \mapsto A(x, y, u)$ such that $f_1 \equiv 0$. This amounts to solving*

$\hat{X}_1(A) = A_x + f_1 A_u = 0$ with $A_u \neq 0$, which can be done locally around any point. The commutation relation $[\hat{X}_1, \hat{X}_2] = (f_2)_x \partial_u = 0$ implies that f_2 is independent of x. Thus, in the same way as above, we may change coordinates $u \mapsto B(y, u)$ such that $f_2 \equiv 0$. A similar argument works if $X_2 = x\partial_x + \partial_y$.

The previous example is both simple and useful. Since all our Lie algebras of vector fields on $\mathbb{C}^2$ contain these Lie algebras as subalgebras, we can always transform our lifts to a simpler form by changing coordinates in this way. This idea is applied in the proof of the following theorem.

Theorem 5. *Let $\mathfrak{g} = \langle X_1, ..., X_r \rangle$ be a transitive Lie algebra of vector fields on $\mathbb{C}^2$ and let $\hat{\mathfrak{g}} = \langle \hat{X}_1, ..., \hat{X}_r \rangle$ be a transitive lift of $\mathfrak{g}$ on the bundle π, with $\hat{X}_i = X_i + f_i(x, y, u)\partial_u$.*

Then there exist local coordinates in a neighborhood $U \subset \mathbb{C}^2 \times \mathbb{C}$ of any point such that $f_i(x, y, u) = \alpha_i(x, y) + \beta_i(x, y)u + \gamma_i(x, y)u^2$ on U and $\varphi_a(\hat{\mathfrak{st}}_a)|_{U \cap F_a}$ is of the same normal form (1) for every $a \in \pi(U)$.

Proof. Let $p \in \mathbb{C}^2 \times \mathbb{C}$ be an arbitrary point, V an open set containing p, and $(V, (x, y, u))$ a coordinate chart centered at p. By lemma 1 we may assume that $X_1 = \partial_x$ and either $X_2 = \partial_y$ or $X_2 = x\partial_x + \partial_y$ and by example 3 we may set $f_1 \equiv 0 \equiv f_2$. We choose a basis of $\mathfrak{g}$ such that $\mathfrak{st}_0 = \langle X_3, ..., X_r \rangle$.

Since $\varphi_0(\hat{\mathfrak{st}}_0)$ is a transitive action on the line, we may in addition make a local coordinate change $u \mapsto A(u)$ on $U \subset V$ containing 0 so that $\varphi_0(\hat{\mathfrak{st}}_0)$ is of the form $\langle \partial_u \rangle$, $\langle \partial_u, u\partial_u \rangle$ or $\langle \partial_u, u\partial_u, u^2\partial_u \rangle$. Then for $i = 3, ..., r$, the functions f_i have the property

$$f_i(0, 0, u) = \tilde{\alpha}_i + \tilde{\beta}_i u + \tilde{\gamma}_i u^2.$$

We use the commutation relations of $\hat{\mathfrak{g}}$ to show that $f_i(x, y, u)$ will take this form for every $(x, y, u) \in U$.

If $[X_j, X_i] = c_{ji}^k X_k$ are the commutation relations for $\mathfrak{g}$, then the lift of $\mathfrak{g}$ obeys the same relations: $[\hat{X}_j, \hat{X}_i] = c_{ji}^k \hat{X}^k$. Thus

$$[\hat{X}_1, \hat{X}_i] = [X_1, X_i] + X_1(f_i)\partial_u = c_{1i}^k X_k + X_1(f_i)\partial_u$$

which implies that $X_1(f_i) = c_{1i}^k f_k$. In the same manner we get the equations $X_2(f_i) = c_{2i}^k f_k$. We can rewrite the equations as

$$\partial_x(f_i) = c_{1i}^k f_k, \qquad \partial_y(f_i) = \tilde{c}_{2i}^k(x) f_k.$$

The coefficients $\tilde{c}_{2i}^k(x)$ depend on whether $\langle X_1, X_2 \rangle$ is abelian or not, but in any case they are indepedent of u. We differentiate these equations three times with respect to u (denoted by $'$):

$$\partial_x(f_i''') = c_{1i}^k f_k''', \qquad \partial_y(f_i''') = \tilde{c}_{2i}^k(x) f_k'''$$

By the above assumption we have $f_i'''(0,0,u) = 0$, and by the uniqueness theorem for systems of linear ODEs it follows that we for every $(x,y,u) \in U$ have $f_i'''(x,y,u) = 0$, and therefore

$$f_i(x,y,u) = \alpha_i(x,y) + \beta_i(x,y)u + \gamma_i(x,y)u^2. \tag{2}$$

Note also that if f_i'' (or f_i') vanish on $(0,0,u)$, we may assume $\gamma_i \equiv 0$ (or $\gamma_i \equiv 0$ and $\beta_i \equiv 0$) for every i. The last statement of the theorem follows by the fact that $\dim \varphi_a(\hat{\mathfrak{st}}_a)$ is the same for every $a \in \pi(U)$. □

3.2 Coordinate transformations

When computing the lift of a Lie algebra we may choose coordinates so that the lift is of the special form indicated in theorem 5, and we may further simplify the expression for the lift by using transformations preserving this form. Thus after we have chosen such special coordinates, we consider metric lifts up to translations $u \mapsto u + A(x,y)$, affine lifts up to affine transformations $u \mapsto A(x,y)u + B(x,y)$ and projective lifts up to projective transformations $u \mapsto \frac{A(x,y)u+B(x,y)}{C(x,y)u+D(x,y)}$.

A geometric interpretation of theorem 5 is that we may choose a structure on the fibers, namely metric, affine or projective, and require the lift to preserve this structure. The following example shows the general procedure we use for finding lifts.

Example 4. *Consider the Lie algebra $\mathfrak{g}_6$ which is spanned by vector fields*

$$X_1 = \partial_x, \quad X_2 = \partial_y, \quad X_3 = y\partial_y, \quad X_4 = y^2\partial_y.$$

Since the stabilizer of 0 is solvable, we may by corollary 4 assume that the generators of a lift $\hat{\mathfrak{g}}_6$ is of the form $\hat{X}_i = X_i + f_i\partial_u$, where f_i are affine functions in u. All lifts are either metric og affine.

By example 3 we may assume that $f_1 \equiv 0 \equiv f_2$ after making an affine change of coordinates (or a translation if we consider metric lifts). The type of coordinate transformation was not specified in the example, but it is clear that the PDE in example 3 can be solved within our framework of metric and affine lifts, respectively.

The commutation relations $[X_1, X_3] = 0, [X_2, X_3] = X_2$ imply that f_3 is a function of u alone. The commutation relations $[X_1, X_4] = 0, [X_2, X_4] = 2X_3, [X_3, X_4] = X_4$ result in the differential equations

$$(f_4)_x = 0, \quad (f_4)_y = 2f_3, \quad y(f_4)_y + f_3(f_4)_u - f_4(f_3)_u = f_4.$$

The first two equations give $f_4 = 2yf_3(u) + b(u)$. After inserting this into the third equation, the equation simplifies to $f_3b_u - b(f_3)_u = b$.

Since the lift is either metric or affine, we may assume that $f_3 = A_0 + A_1 u$ *and* $b = B_0 + B_1 u$. *Then the equation above results in* $B_1 = 0$ *and* $B_0 A_1 = -B_0$. *Setting* $B_0 = 0$ *we get transitive lifts only when* $A_1 = 0$*:*

$$\partial_x, \quad \partial_y, \quad y\partial_y + A_0\partial_u, \quad y^2\partial_y + 2A_0 y\partial_u.$$

These are metric lifts. In the case $A_1 = -1$ *we get the affine lift spanned by*

$$\partial_x, \quad \partial_y, \quad y\partial_y - u\partial_u, \quad y^2\partial_y + (1 - 2yu)\partial_u$$

where A_0 *and* B_0 *have been normalized by a translation and scaling, respectively.*

Remark 3. *The family of metric lifts is also invariant under transformations of the form* $u \mapsto Cu + A(x, y)$, *where* C *is constant. However, we would like to restrict to* $C = 1$. *This will make the resulting list of lifts simpler, and it is always easy to see what a scaling transformation would do to the normal form. Geometrically this restriction makes sense if we think about the metric lift as one preserving a metric on the fibers. Another consequence of this choice is that we get a one-to-one correspondence between metric lifts and Lie algebra cohomology which will be discussed in section 5. The same cohomology spaces are treated in [8] where they are used for classifying Lie algebras of differential operators on* $\mathbb{C}^2$.

We also get a correspondence between metric lifts and "linear lifts", whose vector fields act as infinitesimal scaling transformations in fibers. Using the notation above they take the form $\hat{X} = X + f(x, y)u\partial_u$. *They make up an important type of lifts, but we do not consider them here due to their intransitivity. Since the transformation* $u \mapsto \exp(u)$ *takes metric lifts to linear lifts, the theories of these two types of lifts are analogous (given that we allow the right coordinate transformations). This makes many of the results in this paper applicable to linear lifts as well. As an example the classification of linear lifts under linear transformations* ($u \mapsto uA(x, y)$), *will be similar to that of metric lifts under translations* ($u \mapsto u + A(x, y)$).

4 List of lifts

This section contains the list of lifts of the Lie algebras from section 2 on the bundle $\pi\colon \mathbb{C}^2 \times \mathbb{C} \to \mathbb{C}^2$. For a Lie algebra $\mathfrak{g} \subset \mathcal{D}(\mathbb{C}^2)$ we will denote by $\hat{\mathfrak{g}}^m, \hat{\mathfrak{g}}^a, \hat{\mathfrak{g}}^p$ the metric, affine and projective lifts, respectively.

Theorem 6. *The following list contains all metric, affine and projective lifts of the Lie algebras from Lie's classification in section 2.*

$$
\begin{aligned}
\hat{\mathfrak{g}}_1^m &= \langle \partial_x, \partial_y, x\partial_y, x\partial_x - y\partial_y, y\partial_x, x\partial_x + y\partial_y + 2C\partial_u, \\
&\qquad x^2\partial_x + xy\partial_y + 3Cx\partial_u, xy\partial_x + y^2\partial_y + 3Cy\partial_u\rangle \\
\hat{\mathfrak{g}}_1^p &= \langle \partial_x, \partial_y, x\partial_y + \partial_u, x\partial_x - y\partial_y - 2u\partial_u, y\partial_x - u^2\partial_u, x\partial_x + y\partial_y, \\
&\qquad x^2\partial_x + xy\partial_y + (y - xu)\partial_u, xy\partial_x + y^2\partial_y + u(y - xu)\partial_u\rangle \\
\hat{\mathfrak{g}}_2^m &= \langle \partial_x, \partial_y, x\partial_y, x\partial_x - y\partial_y, y\partial_x, x\partial_x + y\partial_y + C\partial_u\rangle \\
\hat{\mathfrak{g}}_2^p &= \langle \partial_x, \partial_y, x\partial_y + \partial_u, x\partial_x - y\partial_y - 2u\partial_u, y\partial_x - u^2\partial_u, x\partial_x + y\partial_y\rangle \\
\hat{\mathfrak{g}}_3^p &= \langle \partial_x, \partial_y, x\partial_y + \partial_u, x\partial_x - y\partial_y - 2u\partial_u, y\partial_x - u^2\partial_u\rangle \\
\hat{\mathfrak{g}}_4^m &= \langle \partial_x, x^i e^{\alpha_j x}\partial_y + e^{\alpha_j x}\left(\sum_{k=0}^{i} \tbinom{i}{k} C_{j,k} x^{i-k}\right)\partial_u \mid C_{1,0} = 0\rangle \\
\hat{\mathfrak{g}}_5^m &= \langle \partial_x, y\partial_y + C\partial_u, x^i e^{\alpha_j x}\partial_y\rangle \\
\hat{\mathfrak{g}}_5^a &= \langle \partial_x, y\partial_y + u\partial_u, x^i e^{\alpha_j x}\partial_y + e^{\alpha_j x}\left(\sum_{k=0}^{i} \tbinom{i}{k} C_{j,k} x^{i-k}\right)\partial_u \mid C_{1,0} = 0\rangle \\
\hat{\mathfrak{g}}_6^m &= \langle \partial_x, \partial_y, y\partial_y + C\partial_u, y^2\partial_y + 2Cy\partial_u\rangle \\
\hat{\mathfrak{g}}_6^a &= \langle \partial_x, \partial_y, y\partial_y - u\partial_u, y^2\partial_y + (1 - 2yu)\partial_u\rangle \\
\hat{\mathfrak{g}}_7^m &= \langle \partial_x, \partial_y, x\partial_x + C\partial_u, x^2\partial_x + x\partial_y + 2Cx\partial_u\rangle \\
\hat{\mathfrak{g}}_7^a &= \langle \partial_x, \partial_y, x\partial_x - u\partial_u, x^2\partial_x + x\partial_y + (1 - 2xu)\partial_u\rangle \\
\hat{\mathfrak{g}}_8^m &= \langle \partial_x, \partial_y, x\partial_x + \alpha y\partial_y + A\partial_u, x\partial_y, ..., x^{s-1}\partial_y, \\
&\qquad x^{s+i}\partial_y + \tbinom{s+i}{s} B x^i\partial_u \mid i = 0, ..., r - 3 - s\rangle, \\
&\quad \text{where } B = 0 \text{ unless } \alpha = s \\
\hat{\mathfrak{g}}_8^a &= \langle \partial_x, \partial_y, x\partial_x + \alpha y\partial_y + (\alpha - s)u\partial_u, x\partial_y, ..., x^{s-1}\partial_y, \\
&\qquad x^{s+i}\partial_y + \tbinom{s+i}{s} x^i\partial_u \mid i = 0, ..., r - 3 - s\rangle, \alpha \neq s \\
\hat{\mathfrak{g}}_9^m &= \langle \partial_x, \partial_y, x\partial_x + ((r-2)y + x^{r-2})\partial_y + C\partial_u, x\partial_y, ..., x^{r-3}\partial_y\rangle \\
\hat{\mathfrak{g}}_9^a &= \langle x\partial_x + ((r-2)y + x^{r-2})\partial_y + \left(\tbinom{r-2}{s} x^{r-s-2} + (r - s - 2)u\right)\partial_u, \\
&\qquad \partial_x, \partial_y, x\partial_y, ..., x^{s-1}\partial_y, x^{s+i}\partial_y + \tbinom{s+i}{s} x^i\partial_u \mid i = 0, ..., r - 3 - s\rangle \\
\hat{\mathfrak{g}}_{10}^m &= \langle \partial_x, \partial_y, x\partial_x + A\partial_u, y\partial_y + B\partial_u, x\partial_y, ..., x^{r-4}\partial_y\rangle \\
\hat{\mathfrak{g}}_{10}^a &= \langle \partial_x, \partial_y, x\partial_x - su\partial_u, y\partial_y + u\partial_u, x\partial_y, ..., x^{s-1}\partial_y, \\
&\qquad x^{s+i}\partial_y + \tbinom{s+i}{s} x^i\partial_u \mid i = 0, ..., r - 4 - s\rangle \\
\hat{\mathfrak{g}}_{11}^m &= \langle \partial_x, \partial_y, x\partial_x + A\partial_u, y\partial_y + B\partial_u, y^2\partial_y + 2By\partial_u\rangle \\
\hat{\mathfrak{g}}_{11}^a &= \langle \partial_x, \partial_y, x\partial_x, y\partial_y - u\partial_u, y^2\partial_y + (1 - 2yu)\partial_u\rangle
\end{aligned}
$$

$$\hat{\mathfrak{g}}_{12}^{m} = \langle \partial_x, \partial_y, x\partial_x + A\partial_u, y\partial_y + B\partial_u, x^2\partial_x + 2Ax\partial_u, y^2\partial_y + 2By\partial_u \rangle$$

$$\hat{\mathfrak{g}}_{12}^{a1} = \langle \partial_x, \partial_y, x\partial_x - u\partial_u, y\partial_y, x^2\partial_x + (1-2xu)\partial_u, y^2\partial_y \rangle$$

$$\hat{\mathfrak{g}}_{12}^{a2} = \langle \partial_x, \partial_y, x\partial_x, y\partial_y - u\partial_u, x^2\partial_x, y^2\partial_y + (1-2yu)\partial_u \rangle$$

$$\hat{\mathfrak{g}}_{13}^{m1} = \langle \partial_x, \partial_y, x\partial_x + y\partial_y + A\partial_u, x\partial_y + B\partial_u, x^2\partial_y + 2Bx\partial_u, x^2\partial_x + 2xy\partial_y + (2xA + 2yB)\partial_u \rangle$$

$$\hat{\mathfrak{g}}_{13}^{m2} = \langle \partial_x, \partial_y, x\partial_x + \tfrac{r-4}{2}y\partial_y + C\partial_u, x\partial_y, ..., x^{r-4}\partial_y, x^2\partial_x + (r-4)xy\partial_y + 2Cx\partial_u \rangle$$

$$\hat{\mathfrak{g}}_{13}^{a1} = \langle \partial_x, \partial_y, x\partial_x + \tfrac{r-4}{2}y\partial_y - u\partial_u, x\partial_y, ..., x^{r-4}\partial_y, x^2\partial_x + (r-4)xy\partial_y + (1-2xu)\partial_u \rangle$$

$$\hat{\mathfrak{g}}_{13}^{a2} = \langle \partial_x, \partial_y, x^2\partial_x + (r-4)xy\partial_y + (x(r-6)u + (r-4)y)\partial_u, x\partial_x + \tfrac{r-4}{2}y\partial_y + \tfrac{r-6}{2}u\partial_u, x^i\partial_y + ix^{i-1}\partial_u \mid i = 1, ..., r-4 \rangle, r \neq 6$$

$$\hat{\mathfrak{g}}_{14}^{m} = \langle \partial_x, \partial_y, x\partial_x + A\partial_u, y\partial_y + B\partial_u, x\partial_y, ..., x^{r-5}\partial_y, x^2\partial_x + (r-5)xy\partial_y + (2A + (r-5)B)x\partial_u \rangle$$

$$\hat{\mathfrak{g}}_{14}^{a1} = \langle \partial_x, \partial_y, x\partial_x - u\partial_u, y\partial_y, x\partial_y, ..., x^{r-5}\partial_y, x^2\partial_x + (r-5)xy\partial_y + (1-2xu)\partial_u \rangle$$

$$\hat{\mathfrak{g}}_{14}^{a2} = \langle \partial_x, \partial_y, x^2\partial_x + (r-5)xy\partial_y + ((r-7)xu + (r-5)y)\partial_u, x\partial_x - u\partial_u, y\partial_y + u\partial_u, x^i\partial_y + ix^{i-1}\partial_u \mid i = 1, ..., r-5 \rangle$$

$$\hat{\mathfrak{g}}_{15}^{m} = \langle \partial_x, x\partial_x + \partial_y, x^2\partial_x + 2x\partial_y + Ce^y\partial_u \rangle$$

$$\hat{\mathfrak{g}}_{16}^{m} = \langle \partial_x, x\partial_x - y\partial_y + C\partial_u, x^2\partial_x + (1-2xy)\partial_y + 2Cx\partial_u \rangle$$

The proof of theorem 4 is a direct computation following the algorithm described above. The computations are not reproduced here, beyond example 4, but they can be found in the appendix of this thesis.

All capital letters in the list denote complex constants. For the metric lifts, one of the constants can always be set equal to 1 if we allow to rescale u, as discussed in remark 3. For example, this would let us identify the space of metric lifts of $\mathfrak{g}_{12}$ with $\mathbb{C}P^1$ instead of $\mathbb{C}^2 \setminus \{0\}$. In the affine lifts $\hat{\mathfrak{g}}_5^a$ one of the constants must be nonzero in order for the lift to be transitive, and it can be set equal to 1 by a scaling transformation. Notice also that even though $\mathfrak{g}_{15}$ is not locally equivalent to $\mathfrak{g}_{16}$, their lifts are locally equivalent. In addition the two affine lifts of $\mathfrak{g}_{12}$ are locally equivalent.

Most of this list already exist in the literature. The lifts of the three primitive Lie algebras can be found in [14]. The first attempt to give a complete list of imprimitive Lie algebras of vector fields on $\mathbb{C}^3$ was done by Amaldi in [2, 3]. Most of the Lie algebras we have found is contained in Amaldi's list of Lie algebras of "type A", but a few are missing. Examples of this are $\hat{\mathfrak{g}}_{10}^m, \hat{\mathfrak{g}}_{14}^m, \hat{\mathfrak{g}}_{14}^{a1}$ and $\hat{\mathfrak{g}}_8^a$ with general α and $B = 0$. There is also an error in the Lie algebra corresponding to $\hat{\mathfrak{g}}_{14}^{a2}$ which was noticed in [11, 12]. The lifts of nonsolvable Lie algebras are contained in [5], and the case of metric lifts was also considered in [15].

Remark 4. *We may endow the total space of π with the contact distribution defined by the vanishing of the 1-form $dy - u dx$, thereby identifying it with the space of 1-jets of functions on $\mathbb{C}$. One way to lift a Lie algebra $\mathfrak{g}$ of vector fields from the base space of π is to require the lift of $\mathfrak{g}$, which we in this case may call the contactization of $\mathfrak{g}$, to preserve this distribution. The contactization is uniquely defined and is locally equivalent to a lift in the above list. For example, the projective lifts of the primitive Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3$ preserve the contact distribution, and are thus equal to the contactizations of the three Lie algebras. The contactization of $\mathfrak{g}_6$ is a linear lift (see remark 3) and is locally equivalent to $\hat{\mathfrak{g}}_6^m$, through the transformation $u \mapsto C \log(u)$.*

5 Metric lifts and Lie algebra cohomology

We conclude this treatment by showing that there is a one-to-one correspondence between the space of metric lifts of $\mathfrak{g} \subset \mathcal{D}(\mathbb{C}^2)$ and the Lie algebra cohomology space $H^1(\mathfrak{g}, C^\omega(\mathbb{C}^2))$. The main result is analogous to [8, Theorem 2].

Due to theorem 5 the metric lift of a Lie algebra $\mathfrak{g} \subset \mathcal{D}(\mathbb{C}^2)$ may be given by a $C^\omega(\mathbb{C}^2)$-valued one-form ψ on $\mathfrak{g}$. For vector fields $X, Y \in \mathfrak{g}$ lifted to $\hat{X} = X + \psi_X \partial_u$ and $\hat{Y} = Y + \psi_Y \partial_u$ we have

$$[\hat{X}, \hat{Y}] = [X + \psi_X \partial_u, Y + \psi_Y \partial_u] = [X, Y] + (X(\psi_Y) - Y(\psi_X))\partial_u. \tag{3}$$

Consider the first terms of the Chevalley-Eilenberg complex

$$0 \longrightarrow C^\omega(\mathbb{C}^2) \xrightarrow{d} \mathfrak{g}^* \otimes C^\omega(\mathbb{C}^2) \xrightarrow{d} \Lambda^2 \mathfrak{g}^* \otimes C^\omega(\mathbb{C}^2)$$

where the differential d is defined by

$$df(X) = X(f), \quad f \in C^\omega(\mathbb{C}^2)$$
$$d\psi(X, Y) = X(\psi_Y) - Y(\psi_X) - \psi_{[X,Y]}, \quad \psi \in \mathfrak{g}^* \otimes C^\omega(\mathbb{C}^2).$$

This complex depends not only on the abstract Lie algebra, but also on its realization as a Lie algebra of vector fields. It is clear from (3) that $\psi \in \mathfrak{g}^* \otimes C^\omega(\mathbb{C}^2)$ corresponds to a metric lift if and only if $d\psi = 0$.

Two metric lifts are equivalent if there exists a biholomorphism

$$\phi\colon (x, y, u) \mapsto (x, y, u - U(x, y))$$

on $\mathbb{C}^2 \times \mathbb{C}$ that brings one to the other. A lift of X transforms according to

$$d\phi\colon X + \psi_X \partial_u \mapsto X + (\psi_X - dU(X))\partial_u$$

which shows that two lifts are equivalent if the difference between their defining one-forms is given by dU for some $U \in C^\omega(\mathbb{C}^2)$. Thus, if we include the intransitive trivial lift into the space of metric lifts we have the following theorem, relating the cohomology space

$$H^1(\mathfrak{g}, C^\omega(\mathbb{C}^2)) = \{\psi \in \mathfrak{g}^* \otimes C^\omega(\mathbb{C}^2) \mid d\psi = 0\}/\{dU \mid U \in C^\omega(\mathbb{C}^2)\},$$

to the space of metric lifts.

Theorem 7. *There is a one-to-one correspondence between the space of metric lifts of a Lie algebra $\mathfrak{g} \subset \mathcal{D}(\mathbb{C}^2)$ and the corresponding cohomology space $H^1(\mathfrak{g}, C^\omega(\mathbb{C}^2))$.*

Remark 5. *As discussed previously, we have the option of removing a free constant in the metric lifts by a scaling transformation. If we did this the space of metric lifts of $\mathfrak{g}$ would be $\mathbb{C}P^{n-1}$ in the case $H^1(\mathfrak{g}, C^\omega(\mathbb{C}^2)) = \mathbb{C}^n$.*

The theorem gives a transparent interpretation of metric lifts, while also showing a way to compute $H^1(\mathfrak{g}, C^\omega(\mathbb{C}^2))$, through example 4. This method is essentially the one that was used in [8], where the same cohomologies were found. There the authors extended Lie's classification of Lie algebras of vector fields to Lie algebras of first order differential operators on $\mathbb{C}^2$, and part of this work is equivalent to our classification of metric lifts.

Their results coincide with ours, with the exceptions $\mathfrak{g}_8$ which corresponds to case 5 and 20 in [8] and $\mathfrak{g}_{16}, \mathfrak{g}_{15}, \mathfrak{g}_7$ which correspond to cases 12, 13 and 14, respectively. For $\mathfrak{g}_8$ it seems like they have not considered the case corresponding to $\ker(d\pi|_{\hat{\mathfrak{g}}}) = 0$ which is the only case we consider. The realizations used in [8] for cases 12, 13 and 14 have singular orbits, while their cohomologies are computed after restricting to subdomains, avoiding singular orbits. The cohomology is sensitive to choice of realization as Lie algebra of vector fields, and will in general change by restricting to a subdomain. The following example, based on realizations of $sl(2)$, illustrates this.

Example 5. *The metric lift*

$$\hat{\mathfrak{g}}_{16}^m = \langle \partial_x, x\partial_x - y\partial_y + C\partial_u, x^2\partial_x + (1 - 2xy)\partial_y + 2Cx\partial_y \rangle$$

is parametrized by a single constant, and thus $H^1(\mathfrak{g}_{16}, C^\omega(\mathbb{C}^2)) = \mathbb{C}$*. Similarly, we see that* $H^1(\mathfrak{g}_{15}, C^\omega(\mathbb{C}^2)) = \mathbb{C}$*.*

The Lie algebra $\tilde{\mathfrak{g}}_{16} = \langle \partial_x, x\partial_x + y\partial_y, x^2\partial_x + y(2x+y)\partial_y \rangle$ *is related to [8, case 12] by the transformation* $y \mapsto x + y$*. It is also is locally equivalent to* $\mathfrak{g}_{16}$*, but it has a singular one-dimensional orbit,* $y = 0$*. Its metric lift is given by*

$$\langle \partial_x, x\partial_x + y\partial_y + A\partial_u, x^2\partial_x + y(2x+y)\partial_y + (2Ax + By)\partial_u \rangle$$

which implies $H^1(\tilde{\mathfrak{g}}_{16}, C^\omega(\mathbb{C}^2)) = \mathbb{C}^2$*.*

The Lie algebra $\tilde{\mathfrak{g}}_{15} = \langle y\partial_x, x\partial_y, x\partial_x - y\partial_y \rangle$ *is the standard representation on* $\mathbb{C}^2$*. If we split* $C^\omega(\mathbb{C}^2) = \oplus_{k=0}^\infty S^k(\mathbb{C}^2)^*$ *we get* $H^1(\tilde{\mathfrak{g}}_{15}, C^\omega(\mathbb{C}^2)) = \oplus_{k=0}^\infty H^1(\tilde{\mathfrak{g}}_{15}, S^k(\mathbb{C}^2)^*)$*. Since* $S^k(\mathbb{C}^2)^*$ *is a finite-dimensional module over* $\tilde{\mathfrak{g}}_{15}$*, the cohomologies* $H^1(\mathfrak{g}, S^k(\mathbb{C}^2)^*)$ *vanish by Whitehead's lemma, and we get* $H^1(\tilde{\mathfrak{g}}_{15}, C^\omega(\mathbb{C}^2)) = 0$*. Hence the cohomologies of the locally equivalent Lie algebras* $\mathfrak{g}_{15}$ *and* $\tilde{\mathfrak{g}}_{15}$ *are different. To summarize, we have two pairs of locally equivalent realizations of* $sl(2)$*, and their cohomologies are*

$$\begin{aligned} H^1(\mathfrak{g}_{16}, C^\omega(\mathbb{C}^2)) &= \mathbb{C}, \qquad & H^1(\tilde{\mathfrak{g}}_{16}, C^\omega(\mathbb{C}^2)) &= \mathbb{C}^2, \\ H^1(\mathfrak{g}_{15}, C^\omega(\mathbb{C}^2)) &= \mathbb{C}, \qquad & H^1(\tilde{\mathfrak{g}}_{15}, C^\omega(\mathbb{C}^2)) &= 0. \end{aligned}$$

The Lie algebra cohomologies considered in this paper are related to the relative invariants (and singular orbits) of the corresponding Lie algebras of vector fields [7]. A consequence of [7, Theorem 5.4] is that a locally transitive Lie algebra $\mathfrak{g}$ of vector fields has a scalar relative invariant if it has a nontrivial metric lift whose orbit-dimension is equal to that of $\mathfrak{g}$. The Lie algebra $\tilde{\mathfrak{g}}_{16}$ has two-dimensional orbits when $A = B$. Therefore there exists an absolute invariant, and it is given by e^u/y^A. The corresponding relative invariant of $\mathfrak{g}_{16}$ is y^A and it defines the singular orbit $y = 0$.

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Differential Invariants of Self-Dual conformal structures

Boris Kruglikov, Eivind Schneider

Abstract

We compute the quotient of the self-duality equation for conformal metrics by the action of the diffeomorphism group. We also determine Hilbert polynomial, counting the number of independent scalar differential invariants depending on the jet-order, and the corresponding Poincaré function. We describe the field of rational differential invariants separating generic orbits of the diffeomorphism pseudogroup action, resolving the local recognition problem for self-dual conformal structures.

Introduction

Self-duality is an important phenomenon in four-dimensional differential geometry that has numerous applications in physics, twistor theory, analysis, topology and integrability theory. A pseudo-Riemannian metric g on an oriented four-dimensional manifold M determines the Hodge operator $* : \Lambda^2 TM \to \Lambda^2 TM$ that satisfies the property $*^2 = \mathbf{1}$ provided g has the Riemannian or split signature. In this paper we restrict to these two cases, ignoring the Lorentzian signature.

The Riemann curvature tensor splits into $O(g)$-irreducible pieces $R_g = \mathrm{Sc}_g + \mathrm{Ric}_0 + W$, where the last part is the Weyl tensor [2] and $O(g)$ is the orthogonal group of g. In dimension 4, due to exceptional isomorphisms $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$, $\mathfrak{so}(2,2) = \mathfrak{so}(1,2) \oplus \mathfrak{so}(1,2)$, the last component splits further $W = W_+ + W_-$, where $*W_\pm = \pm W_\pm$. Metric g is called self-dual if $*W = W$, i.e. $W_- = 0$. This property does not depend on conformal rescalings of the metric $g \to e^{2\varphi} g$, and so is the property of the conformal structure $[g]$.

Since the space of W_- has dimension 5, and the conformal structure has 9 components in 4D, the self-duality equation appears as an underdetermined system of 5 PDE on 9 functions of 4 arguments. This is however a misleading count, since the equation is natural, and the diffeomorphism group acts as the symmetry group of the equation. Since $\mathrm{Diff}(M)$ is parametrized by 4 functions of 4 arguments, we expect to obtain a system of 5 PDE on $5 = 9-4$ functions of 4 arguments.

This 5×5 system is determined, but it has never been written explicitly. There are two approaches to eliminate the gauge freedom.

One way to fix the gauge is to pass to the quotient equation that is obtained as a system of differential relations (syzygies) on a generating set of differential invariants. By computing the latter for the self-dual conformal structures we write the quotient equation as a nonlinear 9×9 PDE system, which is determined but complicated to investigate.

Another approach is to get a cross-section or a quasi-section to the orbits of the pseudogroup $G = \mathrm{Diff}_{\mathrm{loc}}(M)$ action on the space $\mathcal{SD} = \{[g] : W_- = 0\}$ of self-dual conformal metric structures. This was essentially done in the recent work [5, III.A]: By choosing a convenient ansatz the authors of that work encoded all self-dual structures via a 3×3 PDE system $\mathcal{SDE}$ of the second order (this works for the neutral signature; in the Riemannian case use doubly biorthogonal coordinates to get self-duality as a 5×5 second-order PDE system [5, III.C] that can be investigated in a similar manner as the 3×3 system).

In this way almost all gauge freedom was eliminated, yet a part of symmetry remained shuffling the structures. This pseudogroup, denoted by $\mathcal{G}$, is parametrized by 5 functions of 2 arguments (and so is considerably smaller than G). We fix this freedom by computing the differential invariants of $\mathcal{G}$-action on $\mathcal{SDE}$ and passing to the quotient equation.

The differential invariants are considered in rational-polynomial form, as in [12]. This allows to describe the algebra of invariants in Lie-Tresse approach, and also using the principle of n-invariants of [1]. We count differential invariants in both approaches and organize the obtained numbers in the Hilbert polynomial and the Poincaré function.

1 Scalar invariants of self-dual structures

The first approach to compute the quotient of the self-duality equation by the local diffeomorphisms pseudogroup G action is via differential invariants of self-dual structures $\mathcal{SD}$. The signature of the metric g or conformal metric structure $[g]$ is either $(2,2)$ or $(4,0)$. In this and the following two sections we assume that g is a Riemannian metric on M for convenience. Consideration of the case $(2,2)$ is analogous.

To distinguish between metrics and conformal structures we will write $\mathcal{SD}_m$ for the former and $\mathcal{SD}_c$ for the latter. Denote the space of k-jets of such structures by $\mathcal{SD}_m^k$ and $\mathcal{SD}_c^k$ respectively. These clearly form a tower of bundles over M with projections $\pi_{k,l} : \mathcal{SD}^k_{\mathrm{x}} \to \mathcal{SD}^l_{\mathrm{x}}$, $\pi_k : \mathcal{SD}^k_{\mathrm{x}} \to M$, where x is either m or c.

1.1 Self-dual metrics: invariants

Consider the bundle $S^2_+T^*M$ of positively definite quadratic forms on TM and its space of jets $J^k(S^2_+T^*M)$. The equation $W_- = 0$ in 2-jets determines the submanifold $\mathcal{SD}^2_m \subset J^2$, and its prolongations are $\mathcal{SD}^k_m \subset J^k$ for $k > 2$.

Computation of the stabilizer of the action shows that the submanifolds $\mathcal{SD}^k_m$ are regular, meaning that generic orbits of the G-action in $\mathcal{SD}^k_m$ have the same dimension as in $J^k(S^2_+T^*M)$. This is based on a simple observation that generic self-dual metrics have no symmetry at all. Thus the differential invariants of the action on $\mathcal{SD}^k_m$ can be obtained from the differential invariants on the jet space J^k [9, 13].

These invariants can be constructed as follows. There are no invariants of order ≤ 1 due to existence of geodesic coordinates, the first invariants arise in order 2 and they are derived from the Riemann curvature tensor (as this is the only invariant of the 2-jet of g). Traces of the Ricci tensor $\mathrm{Tr}(\mathrm{Ric}^i)$, $1 \leq i \leq 4$, yield 4 invariants $I_1, \dots, I_4$ that in a Zariski open set of jets of metrics can be considered horizontally independent, meaning $\hat{d}I_1 \wedge \dots \wedge \hat{d}I_4 \neq 0$.

To get other invariants of order 2, choose an eigenbasis $e_1 \dots, e_4$ of the Ricci operator (in a Zariski open set it is simple), denote the dual coframe by $\{\theta^i\}$ and decompose $R_g = R^i_{jkl} e_i \otimes \theta^j \otimes \theta^k \wedge \theta^l$. These invariants include the previous I_i, and the totality of independent second-order invariants for self-dual metrics is

$$\dim\{R_g | W_- = 0\} - \dim O(g) = (20 - 5) - 6 = 9.$$

The invariants R^i_{jkl} are however not algebraic, but obtained as algebraic extensions via the characteristic equation. Then R^i_{jkl} (9 independent components) and e_i generate the algebra of invariants.

Alternatively, compute the basis of Tresse derivatives $\nabla_i = \hat{\partial}_{I_i}$ and express the metric in the dual coframe $\omega^j = \hat{d}I_j$: $g = G_{ij}\omega^i\omega^j$. Then the functions I_i, G_{kl} generate the space of invariants by the principle of n-invariants [1].

Remark . *There is a natural almost complex structure $\hat{J}$ on the twistor space of self-dual (M, g), i.e. on the bundle $\hat{M}$ over M whose fiber at a consists of the sphere of orthogonal complex structures on T_aM inducing the given orientation. The celebrated theorem of Penrose [15, 2] states that self-duality is equivalent to integrability of $\hat{J}$. Thus local differential invariants* *of g can be expressed through semi-global invariants of the foliation of the three-dimensional complex space $\hat{M}$ by rational curves. Similarly in the split signature one gets foliation by α-surfaces, and the geometry of this foliation of $\hat{M}$ yields the invariants on M.*

We explain how to get rid of non-algebraicity in the next subsection.

1.2 Self-dual conformal structures: invariants

Here the invariants of the second order are obtained from the Weyl tensor as the only conformally invariant part of the Riemann tensor R_g. For general conformal structures a description of the scalar invariants was given recently in [10]. In our case $W = W_+ + W_-$ the second component vanishes, and so we have only 5-dimensional space of curvature tensors $\mathcal{W}$, namely Weyl parts of R_g considered as $(3,1)$ tensors.

Let us fix a representative of the conformal structure $g_0 \in [g]$ by the requirement $\|W_+\|^2_{g_0} = 1$, this uniquely determines g_0 provided that W_+ is non-vanishing in a neighborhood (in the case of neutral signature we have to require $\|W_+\|^2_g \neq 0$ for some and hence any metric $g \in [g]$ and then we can fix g_0 up to $\pm$ by the requirement $\|W_+\|^2_{g_0} = \pm 1$). Use this representative to convert W_+ into a $(2,2)$-tensor, considered as a map $W_+ : \Lambda^2 T \to \Lambda^2 T$, where $T = T_a M$ for a fixed $a \in M$.

Recall [2] that the operator $W = W_+ + W_-$ is block-diagonal in terms of the Hodge $*$-decomposition $\Lambda^2 T = \Lambda^2_+ T \oplus \Lambda^2_- T$. Thus $W_+ : \Lambda^2_+ T \to \Lambda^2_+ T$ is a map of 3-dimensional spaces and it is traceless of norm 1. For the spectrum $\mathrm{Sp}(W_+) = \{\lambda_1, \lambda_2, \lambda_3\}$ this means $\sum \lambda_i = 0$, $\max |\lambda_i| = 1$. To conclude, we have only one scalar invariant of order 2, for which we can take $I = \mathrm{Tr}(W_+^2)$.

To obtain more differential invariants we proceed as follows. It is known that Riemannian conformal structure in 4D is equivalent to a quaternionic structure (split-quaternionic in the split-signature). In the domain, where $\mathrm{Sp}(W_+|\Lambda^2_+)$ is simple we even get a hyper-Hermitian structure (on the bundle TM pulled back to $\mathcal{SD}^2_c$, so no integrability conditions for the operators J_1, J_2, J_3) as follows.

Let $\sigma_i \in \Lambda^2_+$ be the eigenbasis of W_+ corresponding to eigenvalues λ_i, normalized by $\|\sigma_i\|^2_{g_0} = 1$ (this still leaves $\pm$ freedom for every σ_i). These 2-forms are symplectic (= nondegenerate, since again these are forms on a bundle over $\mathcal{SD}^2_c$) and g_0-orthogonal, so the operators $J_i = g_0^{-1}\sigma_i$ are anti-commuting complex operators on the space T, and they are in quaternionic relations up to the sign. We can fix one sign by requiring $J_3 = J_1 J_2$, but still have residual freedom $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Now we can fix a canonical (up to above residual symmetry) frame, depending on the 3-jet of $[g]$, as follows: $e_1 = g_0^{-1}\hat{d}I/\|g_0^{-1}\hat{d}I\|_{g_0}$, $e_2 = J_1 e_1$, $e_3 = J_2 e_1$, $e_4 = J_3 e_1$. The structure functions of this frame c_{ij}^k (given by $[e_i, e_j] = c_{ij}^k e_k$) together with I constitute the fundamental invariants of the conformal structure (we can fix, for instance, $I_1 = I$, $I_2 = c_{12}^1$, $I_3 = c_{13}^1$, $I_4 = c_{14}^1$ to be the basic invariants), and together with the invariant derivations $\nabla_j = \mathcal{D}_{e_j}$ (total derivative along e_j) they generate the algebra of scalar differential invariants micro-locally.

The micro-locality comes from non-algebraicity of the invariants. Indeed, since we used eigenvalues and eigenvectors in the construction, the output depends on an algebraic extension via some additional variables y. Notice though that this involves only 2-jet coordinates, i.e. the y-variables are in algebraic relations with the fiber variables of the projection $J^2 \to J^1$, and with respect to higher jets everything is algebraic. Thus we can eliminate the y-variables, as well as the residual freedom, and obtain the algebra of global rational invariants $\mathfrak{A}_l$.

Here l is the order of jet from which only polynomial behavior of the invariants can be assumed [12]. This yields the Lie-Tresse type description of the algebra $\mathfrak{A}_l$.

It is easy to see that the rational expressions occur at most on the level of 3-jets, so the generators of the rational algebra can be chosen polynomial in the jets of order > 3. Thus we conclude:

Theorem 15. *The algebra $\mathfrak{A}_3$ of rational-polynomial invariants as well as the field $\mathfrak{F}$ of rational differential invariants of self-dual conformal metric structures are both generated by a finite number of (the indicated) differential invariants I_i and invariant derivations ∇_j, and the invariants from this algebra/field separate generic orbits in $\mathcal{SD}_c^\infty$.*

A similar statement also holds true for metric invariants of $\mathcal{SD}_m^\infty$.

2 Stabilizers of generic jets

Our method to compute the number of independent differential invariants of order k follows the approach of [13]. We will use the jet-language from the formal theory of PDE, and refer the reader to [11].

Fix a point $a \in M$. Denote by $\mathbb{D}_k$ the Lie group of k-jets of diffeomorphisms preserving the point a. This group is obtained from $\mathbb{D}_1 = \mathrm{GL}(T)$ by successive extensions according to the exact 3-sequence

$$0 \to \Delta_k \longrightarrow \mathbb{D}_k \longrightarrow \mathbb{D}_{k-1} \to \{e\},$$

where $\Delta_k = \{[\varphi]^k_x : [\varphi]^{k-1}_x = [\mathrm{id}]^{k-1}_x\} \simeq S^kT^* \otimes T$ is Abelian ($k > 1$).

Denote by $\mathrm{St}_k \subset \mathbb{D}_{k+1}$ the stabilizer of a generic point $a_k \in \mathcal{SD}^k_{\mathrm{x}}$, and by St^0_k its connected component of unity.

2.1 Self-dual metrics: stabilizers

We refer to [13] for computations of stabilizers and note that even though the computation there is done for generic metrics, it applies to self-dual metrics as well. Thus in the metric case the stabilizers are the following: $\mathrm{St}_0 = \mathrm{St}_1 = O(g)$, and $\mathrm{St}^0_k = 0$ for $k \geq 2$.

Consequently the action of the pseudogroup G on jets of order $k \geq 2$ is almost free, meaning that $\mathbb{D}_{k+1}$ has a discrete stabilizer on $\mathcal{SD}^k_m|_a$.

2.2 Self-dual conformal structures: stabilizers

The stabilizers for general conformal structures were computed in [10]. In the self-dual case there is a deviation from the general result. Denote by $\mathcal{C}_M = S^2_+T^*M/\mathbb{R}_+$ the bundle of conformal metric structures.

Lemma 16. **([10])** *The following is a natural isomorphism:*

$$T_{[g]}(\mathcal{C}_M) = \mathrm{End}^{\mathrm{sym}}_0(T) = \{A : T \to T \,|\, g(Au, v) = g(u, Av), \mathrm{Tr}(A) = 0\}.$$

Denote $V_M = T_{[g]}(\mathcal{C}_M)$. The differential group $\mathbb{D}_{k+1}$ acts on $\mathcal{SD}^k_c$, in particular Δ_{k+1} acts on it. The next statement is obtained by a direct computation of the symbol of Lie derivative.

Lemma 17. *The tangent to the orbit $\Delta_{k+1}(a_k)$ is the image $\mathrm{Im}(\zeta_k) \subset T\mathcal{SD}^k_c$ of the map ζ_k that is equal to the following composition*

$$S^{k+1}T^* \otimes T \xrightarrow{\delta} S^kT^* \otimes (T^* \otimes T) \xrightarrow{\mathbf{1}\otimes\Pi} S^kT^* \otimes V_M.$$

Here δ is the Spencer operator and $\Pi : T^* \otimes T \to V_M \subset T^* \otimes T$ is the projection given by

$$\langle p, \Pi(B)u\rangle = \tfrac{1}{2}\langle p, Bu\rangle + \tfrac{1}{2}\langle u_\flat, Bp^\sharp\rangle - \tfrac{1}{n}\mathrm{Tr}(B)\langle p, u\rangle,$$

where $u \in T, p \in T^*, B \in T^* \otimes T$ are arbitrary, $\langle\cdot,\cdot\rangle$ denotes the pairing between T^* and T, and $u_\flat = g(u,\cdot)$, $p^\sharp = g^{-1}(p,\cdot)$ for some representative $g \in [g]$, on which the right-hand side does not depend.

Recall that i-th prolongation of a Lie algebra $\mathfrak{h} \subset \mathrm{End}(T)$ is defined by the formula $\mathfrak{h}^{(i)} = S^{i+1}T^* \otimes T \cap S^iT^* \otimes \mathfrak{h}$. As is well-known, for the conformal algebra of $[g]$ it holds: $\mathfrak{co}(g)^{(1)} = T^*$ and $\mathfrak{co}(g)^{(i)} = 0$, $i > 1$.

Lemma 18. *We have* $\mathrm{Ker}(\zeta_k) = 0$ *for* $k > 1$, *and therefore the projectors* $\rho_{k+1,k} : \mathbb{D}_{k+1} \to \mathbb{D}_k$ *induce the injective homomorphisms* $\mathrm{St}_k \to \mathrm{St}_{k-1}$ *and* $\mathrm{St}_k^0 \to \mathrm{St}_{k-1}^0$ *for* $k > 1$.

Proof. If $\zeta_k(\Psi) = 0$, then $\delta(\Psi) \in S^k T^* \otimes \mathfrak{co}(g)$, where $\mathfrak{co}(g) \subset \mathrm{End}(T)$ is the conformal algebra. This means that $\Psi \in \mathfrak{co}(g)^{(k+1)} = 0$, if $k > 1$. Thus we conclude injectivity of ζ_k: $\Delta_{k+1} \cap \mathrm{St}_k = \{e\}$, whence the second claim. □

The stabilizers of low order (for any $n \geq 3$) are the following. For any $a_0 \in \mathcal{C}_M$ its stabilizer is $\mathrm{St}_0 = CO(g) = (\mathrm{Sp}(1) \times_{\mathbb{Z}_2} \mathrm{Sp}(1)) \times \mathbb{R}_+$.

Next, the stabilizer $\mathrm{St}_1 \subset \mathbb{D}_2$ of $a_1 \in J^1(\mathcal{C}_M)$ is the extension (by derivations) of St_0 by $\mathfrak{co}(g)^{(1)} = T^* \stackrel{\iota}{\hookrightarrow} \Delta_2$, where $\iota : T^* \to S^2 T^* \otimes T$ is given by

$$\iota(p)(u, v) = \langle p, u \rangle v + \langle p, v \rangle u - \langle u_\flat, v \rangle p^\sharp,$$

for $p \in T^*$, $u, v \in T$. In other words, we have $\mathrm{St}_1 = CO(g) \ltimes T$.

Since for G-action on $\mathcal{SD}_c^2$ there is precisely 1 scalar differential invariant, we get $\dim \mathrm{St}_2 = (16 + 40 + 80) - (9 + 36 + 85 - 1) = 7$. This can be also seen as follows. Since $\mathrm{St}_2^0 \subset \mathrm{St}_1$ preserves the hyper-Hermitian structure determined by generic 2-jet $a_2 \in \mathcal{SD}_c^2$ (see Section 1) the $\mathbb{R}_+$ factor and one of the $\mathrm{Sp}(1)$ copies in St_0 disappears from the stabilizer of 2-jet, and we get $\mathrm{St}_2^0 \simeq \mathrm{Sp}(1) \ltimes T$.

Lemma 19. *For* $k \geq 3$ *we have:* $\mathrm{St}_k^0 = \{e\}$.

Proof. In Section 1 we constructed a canonical frame $e_1, \dots, e_4$ on T depending on (generic) jet a_3. In other words, we constructed a frame on the bundle $\pi_3^* TM$ over a Zariski open set in $\mathcal{SD}_c^3$.

The elements from St_3^0 shall preserve this frame, and so the last component $\mathrm{Sp}(1)$ from St_0 is reduced. But also the elements from St_3^0 shall preserve the 1-jet of the hyper-Hermitian structure and the invariant I determined by 2-jets, whence also the factor T is reduced, and St_3^0 is trivial (we take the connected component because of the undetermined signs $\pm$ in the normalizations). Hence the stabilizers St_k^0 for $k \geq 3$ are trivial as well. □

3 The Hilbert and Poincaré function for $\mathcal{SD}$

Now we can compute the number of independent differential invariants. Since G acts transitively on M the codimension of the orbit of G in $\mathcal{SD}_{\mathrm{x}}^k$ is equal to the codimension of the orbit of $\mathbb{D}_{k+1}$ in $\mathcal{SD}_{\mathrm{x}}^k|_a$ (where $a \in M$ is a fixed point and x is either m or c). Denoting the orbit through a generic k-jet a_k by $\mathcal{O}_k \subset \mathcal{SD}^k_{\mathrm{x}}|_a$ we have:

$$\dim(\mathcal{O}_k) = \dim \mathbb{D}_{k+1} - \dim \mathrm{St}_k \, .$$

Notice that

$$\mathrm{codim}(\mathcal{O}_k) = \dim \mathcal{SD}^k_{\mathrm{x}}|_a - \dim(\mathcal{O}_k) = \mathrm{trdeg}\, \mathfrak{F}_k$$

is the number of (functionally independent) scalar differential invariants of order k (here $\mathrm{trdeg}\, \mathfrak{F}_k$ is the transcendence degree of the field of rational differential invariants on $\mathcal{SD}^k_{\mathrm{x}}$).

The Hilbert function is the number of "pure order" k differential invariants $H(k) = \mathrm{trdeg}\, \mathfrak{F}_k - \mathrm{trdeg}\, \mathfrak{F}_{k-1}$. It is known to be a polynomial for large k, so we will refer to it as the Hilbert polynomial.

The Poincaré function is the generating function for the Hilbert polynomial, defined by $P(z) = \sum_{k=0}^{\infty} H(k) z^k$. This is a rational function with the only pole $z = 1$ of order equal to the minimal number of invariant derivations in the Lie-Tresse generating set [12].

3.1 Counting differential invariants

The results of Section 2 allow to compute the Hilbert polynomial and the Poincaré function.

Theorem 20. *The Hilbert polynomial for G-action on $\mathcal{SD}_m$ is*

$$H_m(k) = \begin{cases} 0 & \text{for } k < 2, \\ 9 & \text{for } k = 2, \\ \frac{1}{6}(k-1)(k^2 + 25k + 36) & \text{for } k > 2. \end{cases}$$

The corresponding Poincaré function is equal to

$$P_m(z) = \frac{z^2(9 + 4z - 30z^2 + 24z^3 - 6z^4)}{(1-z)^4}.$$

Notice that $H_m(k) \sim \frac{1}{3!} k^3$, meaning that the moduli of self-dual metric structures are parametrized by 1 function of 4 arguments. This function is the unavoidable rescaling factor.

Proof. As for the general metrics, there are no invariants of order < 2. Since $\mathrm{St}_2^0 = 0$, we have:

$$H_m(2) = \dim \mathcal{SD}^2_m|_a - \dim \mathbb{D}_3 = (10 + 40 + 95) - (16 + 40 + 80) = 9.$$

Alternatively, the only invariant of the 2-jet of a metric is the Riemann curvature tensor. Since $W_- = 0$, it has $20-5 = 15$ components and is acted upon effectively by the group $O(g)$ of dimension 6; hence the codimension of a generic orbit is $15 - 6 = 9$.

Starting from 2-jet we impose the self-duality constraint that, as discussed in the introduction, consist of 5 equations and is a determined system (mod gauge). In particular, there are no differential syzygies between these 5 equations, so that in "pure" order $k \geq 2$ the number of independent equations is $5 \cdot \binom{k+1}{3}$. Thus the symbol of the self-duality metric equation $W_- = 0$ on g, given by

$$\mathfrak{g}_k = \mathrm{Ker}(d\pi_{k,k-1} : T\mathcal{SD}_m^k \to T\mathcal{SD}_m^{k-1})$$

has dimension $\dim(S^kT^* \otimes S^2T^*) - \#$[independent equations].

Since the pseudogroup G acts almost freely on jets of order $k \geq 2$ (freely from some order k), we have:

$$H_m(k) = \dim \mathfrak{g}_k - \dim \Delta_{k+1} = 10 \cdot \binom{k+3}{3} - 5 \cdot \binom{k+1}{3} - 4 \cdot \binom{k+4}{3}$$

whence the claim for the Hilbert polynomial. The formula for the Poincaré function follows. □

Theorem 21. *The Hilbert polynomial for G-action on $\mathcal{SD}_c$ is*

$$H_c(k) = \begin{cases} 0 & \text{for } k < 2, \\ 1 & \text{for } k = 2, \\ 13 & \text{for } k = 3, \\ 3k^2 - 7 & \text{for } k > 3. \end{cases}$$

The corresponding Poincaré function is equal to

$$P_c(z) = \frac{z^2(1 + 10z + 5z^2 - 17z^3 + 7z^4)}{(1-z)^3}.$$

Notice that $H_c(k) \sim 6 \cdot \frac{1}{2!}\, k^2$, meaning that the moduli of self-dual conformal metric structures are parametrized by 6 function of 3 arguments. This confirms the count in [6, 5].

Proof. As for the general metrics, there are no invariants of order < 2. We already counted $H_c(2) = 1$. Since $\mathrm{St}_3^0 = 0$, we have:

$$\begin{aligned} H_c(3) &= \dim \mathcal{SD}_m^3|_a - \dim \mathbb{D}_4 - H_c(2) \\ &= (9 + 36 + 85 + 160) - (16 + 40 + 80 + 140) - 1 = 13. \end{aligned}$$

Starting from 2-jet we impose the self-duality constraint, and we computed in the previous proof that this yields $5\cdot\binom{k+1}{3}$ independent equations of "pure" order $k \geq 2$. Thus the symbol of the self-duality conformal equation $W_- = 0$ on $[g]$, given by

$$\mathfrak{g}_k = \mathrm{Ker}(d\pi_{k,k-1} : T\mathcal{SD}_c^k \to T\mathcal{SD}_c^{k-1}),$$

has dimension$=\dim(S^kT^* \otimes (S^2T^*/\mathbb{R}_+)) - \#$[independent equations].

Since the pseudogroup G acts almost freely on jets of order $k \geq 3$ (freely from some order k), we have:

$$H_c(k) = \dim \mathfrak{g}_k - \dim \Delta_{k+1} = 9\cdot\binom{k+3}{3} - 5\cdot\binom{k+1}{3} - 4\cdot\binom{k+4}{3}$$

whence the claim for the Hilbert polynomial. The formula for the Poincaré function follows. □

3.2 The quotient equation

Let $I_1, \dots, I_4$ be the basic differential invariants of self-dual conformal structures. For generic such structures c these invariant evaluated on c are independent. Thus we can fix the gauge by requiring $I_i = x_i$, $i = 1, \dots, 4$, to be the local coordinates on M. This adds 4 differential equations to 5 equations of self-duality on 9 components of c. Consequently, denoting

$$\Sigma_\infty = \{\theta \in \mathcal{SD}_c^\infty : \hat{d}I_1 \wedge \hat{d}I_2 \wedge \hat{d}I_3 \wedge \hat{d}I_4 \text{ is not defined at } \theta \text{ or vanishes}\},$$

the moduli space $(\mathcal{SD}_c^\infty \setminus \Sigma_\infty)/G$ is given as 9×9 PDE system

$$W_- = 0, I_1 = x_1, \dots, I_4 = x_4.$$

4 The self-duality equation

In the second approach we use a 3×3 PDE system from [5] which encodes all self-dual conformal structures. It was shown in loc.cit. that any anti-self-dual conformal structure in neutral signature $(2, 2)$ locally takes the form $[g]$ where

$$g = dtdx + dzdy + p\,dt^2 + 2q\,dtdz + r\,dz^2. \tag{1}$$

Here p, q, r are functions of (t, x, y, z) which satisfy the following three second-order PDEs:

$$p_{xx} + 2q_{xy} + r_{yy} = 0,$$

$$m_x + n_y = 0, \tag{2}$$

$$m_z - qm_x - rm_y + (q_x + r_y)m = n_t - pn_x - qn_y + (p_x + q_y)n,$$

where

$$m := p_z - q_t + pq_x - qp_x + qq_y - rp_y, \quad n := q_z - r_t + qr_y - rq_y + pr_x - qq_x.$$

Conversely, any such conformal structure is anti-self-dual. Therefore we can, instead of looking at arbitrary self-dual conformal structures, look at conformal structures $[g]$ where g is a metric of the Plebański-Robinson form (1) satisfying (2). So from now on we restrict to self-dual conformal structures in the neutral signature $(2,2)$.

Remark . *These equations are admittedly describing anti-self-dual metrics ($*W = -W$) instead of self-dual metrics ($*W = W$). However, in order to define the Hodge operator, one must specify an orientation. Change of orientation interchanges the equations, so from a local viewpoint self-dual and anti-self-dual structures are the same.*

Conformal structures of the form (1) are parametrized by sections of the bundle $\pi \colon \mathcal{C}_M^{\mathrm{PR}} = M \times \mathbb{R}^3(p,q,r) \to M$, where $M = \mathbb{R}^4(t,x,y,z)$. Self-dual conformal structures must, in addition, satisfy system (2), so they are described by a second-order PDE

$$\mathcal{SDE}_2 = \{\theta = [(p,q,r)]_x^2 : x \in M, \theta \text{ satisfies } (2)\} \subset J^2(\mathcal{C}_M^{\mathrm{PR}}).$$

We let $\mathcal{SDE}_k \subset J^k = J^k(\mathcal{C}_M^{\mathrm{PR}})$ denote the prolonged equation. From now on we will omit specification of the bundle over which the jet spaces are constructed, because it will always be $\mathcal{C}_M^{\mathrm{PR}}$ in what follows.

The prolonged equation $\mathcal{SDE}_k$ is given by $3\binom{k+2}{4}$ equations in J^k since the system (2) is determined. By subtracting this from the jet space dimension $\dim J^k = 4 + 3\binom{k+4}{4}$, we find

$$\dim \mathcal{SDE}_k = 4 + 3\binom{k+4}{4} - 3\binom{k+2}{4} = k^3 + \frac{9}{2}k^2 + \frac{13}{2}k + 7.$$

5 Symmetries of $\mathcal{SDE}$

Self-dual conformal structures locally correspond to sections of $\mathcal{C}_M^{\mathrm{PR}}$ that are solutions of $\mathcal{SDE}$. This correspondence is not 1-1 as there is some residual freedom left: two solutions of $\mathcal{SDE}$ can still be equivalent up to diffeomorphisms. The goal is to remove this freedom by factoring by diffeomorphisms that preserve the shape of the conformal structure $[g]$ where g is in Plebański-Robinson form (1).

These transformations form the symmetry pseudogroup $\mathcal{G}$ of the equation $\mathcal{SDE}$. We will study its Lie algebra $\mathfrak{g}$. By the Lie-Bäcklund theorem [8] for our equation all symmetries are (prolongations of) point transformations. It turns out that the Lie algebra of symmetries is the same as the Lie algebra of vector fields preserving the shape of $[g]$.

5.1 Symmetries of $\mathcal{SDE}$

A vector field X on J^0 is a symmetry of $\mathcal{SDE}$ if the prolonged vector field $X^{(2)}$ is tangent to $\mathcal{SDE}_2 \subset J^2$, i.e. if $X^{(2)}(F_i) = \lambda_i^j F_j$, where $F_1 = 0, F_2 = 0, F_3 = 0$ are the three equations (2). This gives an overdetermined system of PDEs that can be solved by the standard technique, and we obtain the following result:

Theorem 22. *The Lie algebra $\mathfrak{g}$ of symmetries of $\mathcal{SDE}$ is generated by the following five classes of vector fields $X_1(a)$, $X_2(b)$, $X_3(c)$, $X_4(d)$, $X_5(e)$, each of which depends on a function of (t,z):*

$$a\partial_t - xa_t\partial_x - xa_z\partial_y + (xa_{tt} - 2pa_t)\partial_p + (xa_{tz} - qa_t - pa_z)\partial_q + (xa_{zz} - 2qa_z)\partial_r,$$
$$b\partial_z - yb_t\partial_x - yb_z\partial_y + (yb_{tt} - 2qb_t)\partial_p + (yb_{tz} - qb_z - rb_t)\partial_q + (yb_{zz} - 2rb_z)\partial_r,$$
$$cx\partial_x + cy\partial_y + (cp - xc_t)\partial_p + (cq - \tfrac{1}{2}xc_z - \tfrac{1}{2}yc_t)\partial_q + (cr - yc_z)\partial_r,$$
$$d\partial_x - d_t\partial_p - \tfrac{1}{2}d_z\partial_q,$$
$$e\partial_y - \tfrac{1}{2}e_t\partial_q - e_z\partial_r.$$

The following table shows the commutation relations.

$[,]$	$X_1(g)$	$X_2(g)$	$X_3(g)$	$X_4(g)$	$X_5(g)$
$X_1(f)$	$X_1(fg_t - f_tg)$	$X_2(fg_t) - X_1(f_zg)$	$X_3(fg_t)$	$X_4((fg)_t) + X_5(f_zg)$	$X_5(fg_t)$
$X_2(f)$	$*$	$X_2(fg_z - f_zg)$	$X_3(fg_z)$	$X_4(fg_z)$	$X_4(f_tg) + X_5((fg)_z)$
$X_3(f)$	$*$	$*$	0	$-X_4(fg)$	$-X_5(fg)$
$X_4(f)$	$*$	$*$	$*$	0	0
$X_5(f)$	$*$	$*$	$*$	$*$	0

Notice that the Lie algebra is bi-graded $\mathfrak{g} = \oplus\mathfrak{g}_{i,j}$, meaning that we have $[\mathfrak{g}_{i_1,j_1}, \mathfrak{g}_{i_2,j_2}] \subset \mathfrak{g}_{i_1+i_2,j_1+j_2}$ with nontrivial graded pieces

$$\mathfrak{g}_{0,0} = \langle X_1, X_2\rangle, \qquad \mathfrak{g}_{0,1} = \langle X_3\rangle, \qquad \mathfrak{g}_{1,\infty} = \langle X_4, X_5\rangle.$$

5.2 Shape-preserving transformations

We say that a transformation $\varphi \in \mathrm{Diff}_{\mathrm{loc}}(M)$ preserves the PR-shape if for every $[g] \in \Gamma(\mathcal{C}_M^{\mathrm{PR}})$ we have $[\varphi_* g] \in \Gamma(\mathcal{C}_M^{\mathrm{PR}})$. A vector field X on $\mathbb{R}^4$ preserves the PR-shape if its flow does so.

Theorem 23. *The Lie algebra of vector fields preserving the PR-shape is generated by the five classes of vector fields*

$$a\partial_t - xa_t\partial_x - xa_z\partial_y, \quad b\partial_z - yb_t\partial_x - yb_z\partial_y, \quad cx\partial_x + cy\partial_y, \quad d\partial_x, \quad e\partial_y.$$

where a, b, c, d, e are arbitrary functions of (t, z).

Proof. In order to find the Lie algebra of vector fields preserving the shape of $[g]$, we let $X = f_1\partial_t + f_2\partial_x + f_3\partial_y + f_4\partial_z$ be a general vector field and take the Lie derivative $L_X g$. The vector field preserves the PR-shape of $[g]$ if

$$L_X g = \epsilon \cdot (dtdx + dzdy) + \tilde{p}\, dt^2 + 2\tilde{q}\, dtdz + \tilde{r}\, dz^2$$

for some functions $\epsilon, \tilde{p}, \tilde{q}, \tilde{r}$. This gives an overdetermined system of 6 PDEs on 4 unknowns with the solutions parametrized by 5 functions of 2 variables as indicated. □

5.3 Unique lift to J^0

The conformal metric (1) can also be considered as a horizontal (degenerate) symmetric tensor c_{PR} on $\mathcal{C}_M^{\mathrm{PR}}$. Namely, $c_{\mathrm{PR}} \in \Gamma(\pi^* S^2 T^* M/\mathbb{R}_+)$ is given at the point $(t, x, y, z, p, q, r) \in \mathcal{C}_M^{\mathrm{PR}}$ via its representative g by formula (1). The algebra of vector fields X preserving the shape of $[g]$ is naturally lifted to $\mathcal{C}_M^{\mathrm{PR}}$ by the requirement $L_{\hat{X}} c_{\mathrm{PR}} = 0$. This requirement algebraically restores the vertical components of the vector fields $X_1, \dots, X_5$ from Theorem 23 yielding the symmetry fields from Theorem 22. We conclude:

Theorem 24. *The Lie algebra of transformations preserving the PR-shape coincides with the Lie algebra $\mathfrak{g}$ of point symmetries of $\mathcal{SDE}$.*

Thus the conformal structure c_{PR} uniquely restores $\mathfrak{g} = \mathrm{sym}(\mathcal{SDE})$.

5.4 Conformal tensors invariant under $\mathfrak{g}$

The goal of this subsection is to show that the simplest conformally invariant tensor with respect to $\mathfrak{g}$ is c_{PR}, so that the conformal structure (of PR-shape) is in turn uniquely determined by $\mathfrak{g}$.

We aim to describe the horizontal conformal tensors on $\mathcal{C}_M^{\mathrm{PR}}$ that are invariant with respect to $\mathfrak{g}$. Since $\mathfrak{g}$ acts transitively on $\mathcal{C}_M^{\mathrm{PR}}$, we consider the stabilizer $\mathrm{St}_0 \subset \mathfrak{g}$ of the point given by $(t, x, y, z, p, q, r) = (0, 0, 0, 0, 0, 0)$ in $\mathcal{C}_M^{\mathrm{PR}}$. Denote by St_0^k the subalgebra of $\mathfrak{g}$ consisting of fields vanishing at 0 to order k, so that $\mathrm{St}_0 = \mathrm{St}_0^1$.

It is easy to see from formulae of Theorem 22 that the space $\mathrm{St}_0^1/\mathrm{St}_0^2$ is 18-dimensional, and 12 of the generators are vertical (they belong to $\langle \partial_p, \partial_q, \partial_r \rangle$). The complimentary linear fields have the horizontal parts

$$Y_1 = t\partial_t - x\partial_x, \qquad Y_2 = z\partial_t - x\partial_y, \qquad Y_3 = t\partial_z - y\partial_x,$$
$$Y_4 = z\partial_z - y\partial_y, \qquad Y_5 = x\partial_x + y\partial_y, \qquad Y_6 = z\partial_x - t\partial_y.$$

They form a 6-dimensional Lie algebra $\mathfrak{h}$ acting on the horizontal space $\mathbb{T} = T_0 M = T_0 \mathcal{C}_M^{\mathrm{PR}} / \operatorname{Ker}(d\pi)$. This Lie algebra is a semi-direct product of the reductive part $\mathfrak{h}_0 = \langle Y_1, Y_2, Y_3, Y_4, Y_5 \rangle$ and the nilpotent piece $\mathfrak{r} = \langle Y_6 \rangle$ (the nilradical is 2-dimensional). The reductive piece splits in turn $\mathfrak{h}_0 = \mathfrak{sl}_2 \oplus \mathfrak{a}$, where the semi-simple part is $\mathfrak{sl}_2 = \langle Y_1 - Y_4, Y_2, Y_3 \rangle$ and the Abelian part is $\mathfrak{a} = \langle Y_1 + Y_4, Y_5 \rangle$.

It is easy to see that the space $\mathbb{T}$ is $\mathfrak{h}_0$-reducible. In fact, with respect to $\mathfrak{h}_0$ it is decomposable $\mathbb{T} = \Pi_1 \oplus \Pi_2 = \langle \partial_t, \partial_z \rangle \oplus \langle \partial_x, \partial_y \rangle$, and Π_1, Π_2 are the standard $\mathfrak{sl}_2$-representations (denoted by Π in what follows). However $\mathfrak{r}$ maps Π_1 to Π_2 and Π_2 to 0. This $\Pi_2 \subset \mathbb{T}$ is an $\mathfrak{h}$-invariant subspace, but it does not have an $\mathfrak{h}$-invariant complement.

Moreover, Π_2 is the only proper $\mathfrak{h}$-invariant subspace, so there are no conformally invariant vectors (invariant 1-space) and covectors (invariant 3-space). We sumarize this as follows.

Lemma 25. *There are no horizontal 1-tensors on $\mathcal{C}_M^{PR}$ that are conformally invariant with respect to $\mathfrak{g}$.*

Now, let's consider conformally invariant horizontal 2-tensors. Since c_{PR} is $\mathfrak{g}$-invariant, we can lower the indices and consider $(0,2)$-tensors. We have the splitting $\mathbb{T}^* \otimes \mathbb{T}^* = \Lambda^2 \mathbb{T}^* \oplus S^2 \mathbb{T}^*$.

The symmetric part further splits $S^2(\Pi_1^* \oplus \Pi_2^*) = S^2\Pi_1^* \oplus (\Pi_1^* \otimes \Pi_2^*) \oplus S^2\Pi_2^*$. As an $\mathfrak{sl}_2$-representation, this is equal to $3 \cdot S^2\Pi \oplus \Lambda^2\Pi = 3 \cdot \mathfrak{ad} \oplus \mathbf{1}$, and the only one trivial piece $\mathbf{1} \subset \Pi_1^* \otimes \Pi_2^*$ (which is also $\mathfrak{h}$-invariant) is spanned by c_{PR}. Here $\Pi_1^* = \langle dt, dz \rangle$ and $\Pi_2^* = \langle dx, dy \rangle$. Thus there are no $\mathfrak{g}$-invariant symmetric conformal 2-tensors except c_{PR}.

The skew-symmetric part further splits $\Lambda^2(\Pi_1^* \oplus \Pi_2^*) = \Lambda^2\Pi_1^* \oplus (\Pi_1^* \otimes \Pi_2^*) \oplus \Lambda^2\Pi_2^*$, and as an $\mathfrak{sl}_2$-representation, this is equal to $S^2\Pi \oplus 3 \cdot \Lambda^2\Pi = \mathfrak{ad} \oplus 3 \cdot \mathbf{1}$. Thus there are three $\mathfrak{sl}_2$-trivial pieces, and they are $\mathfrak{h}_0$-invariant. However only one of them is $\mathfrak{r}$-invariant, namely $\Lambda^2\Pi_1^*$ that is spanned by $dz \wedge dt$. Thus we have proved the following statement.

Theorem 26. *The only conformally invariant symmetric 2-tensor is c_{PR}. The only conformally invariant skew-symmetric 2-tensor is $dz \wedge dt$.*

Since $dz \wedge dt$ is degenerate and does not define a convenient geometry, c_{PR} is the simplest $\mathfrak{g}$-invariant conformal tensor.

5.5 Algebraicity of $\mathfrak{g}$

We say that the Lie algebra $\mathfrak{g}$ is algebraic if its sheafification is equal to the Lie algebra sheaf of some algebraic pseudo-group $\mathcal{G}$ (see definition of an algebraic pseudo-group in [12]). Algebraicity of $\mathfrak{g}$ is important because it guarantees, through the global Lie-Tresse theorem [12], existence of rational differential invariants separating generic orbits (by [16] this yields rational quotient of the action on every finite jet-level).

Let $\mathbb{D}_k \subset J^k_{(\theta,\theta)}(\mathcal{C}_M^{\rm PR}, \mathcal{C}_M^{\rm PR})$ denote the differential group of order k at $\theta \in \mathcal{C}_M^{\rm PR}$. The stabilizer $\mathcal{G}_\theta \subset \mathcal{G}$ of θ can be viewed as a collection of subbundles $\mathcal{G}_\theta^k \subset \mathbb{D}_k$. The transitive Lie pseudo-group $\mathcal{G}$ is algebraic if $\mathcal{G}_\theta^k$ is an algebraic subgroup of $\mathbb{D}_k$ for every k. This is independent of the choice of θ since $\mathcal{G}$ is transitive, implying that subgroups $\mathcal{G}_\theta^k \subset \mathbb{D}_k$ are conjugate for different points $\theta \in \mathcal{C}_M^{\rm PR}$.

When determining whether $\mathfrak{g}$ is algebraic, there are essentially two approaches. One is to try to see it from the stabilizer $\mathfrak{g}_\theta$ alone, and the other is to integrate $\mathfrak{g}$ in order to investigate the pseudo-group $\mathcal{G}_\theta$. It turns out that the latter is more efficient in our case.

Consider the following pseudo-group $\mathcal{G}$ given via its action on $\mathcal{C}_M^{\rm PR}$.

$$
\begin{gathered}
t \mapsto T = A, \quad z \mapsto Z = B \\
x \mapsto X = C(B_z x - B_t y) + D, \quad y \mapsto Y = C(A_t y - A_z x) + E \\
p \mapsto P = \frac{C(B_z^2 p - 2B_tB_z q + B_t^2 r) + (CJ_{B,B_z} + B_z J_{B,C})x - (CJ_{B,B_t} + B_t J_{B,C})y + J_{B,D}}{J_{A,B}} \\
r \mapsto R = \frac{C(A_z^2 p - 2A_tA_z q + A_t^2 r) + (CJ_{A,A_z} + A_z J_{A,C})x - (CJ_{A,A_t} + A_t J_{A,C})y - J_{A,E}}{J_{A,B}} \\
q \mapsto Q = \frac{C(-A_zB_z p + (A_tB_z + A_zB_t)q - A_tB_t r) + (J_{B,E} - J_{A,D})/2}{J_{A,B}} \\
+ \frac{((J_{A_z,B} - J_{A,B_z})C - B_z J_{A,C} - A_z J_{B,C})x + ((J_{A,B_t} - J_{A_t,B})C + A_t J_{B,C} + B_t J_{A,C})y}{2J_{A,B}}
\end{gathered}
$$

Here we use the notation $J_{F,G} = F_t G_z - F_z G_t$ for two functions F, G of (t, z). The functions A, B, C, D, E are all (locally defined) functions depending on the variables (t, z). In addition A, B satisfy the requirement that $(t, z) \mapsto (A(t, z), B(t, z))$ is a local diffeomorphism of the plane, and $C \neq 0$ wherever it is defined.[1]

It is easy to check that this is a Lie pseudo-group (one should specify the differential equations defining $\mathcal{G}$, and they are $T_x = 0, \dots, T_r = 0, \dots, X_y + Z_t = 0, \dots$). Moreover it is easy to check that the Lie algebra sheaf of $\mathcal{G}$ coincides with the sheafification of $\mathfrak{g}$.

Theorem 27. *The Lie pseudo-group $\mathcal{G}$ and consequently the Lie algebra $\mathfrak{g}$ are algebraic.*

Proof. The subgroups $\mathcal{G}_\theta^k$ of $\mathbb{D}_k$ are constructed by repeated differentiation of $T, \dots, R$ by $t, \dots, r$ and evaluation at θ. The formulas for the group action

[1] The formulas above are corrections of the ones from the original paper.

make it clear that $\mathcal{G}_\theta^k$ will always be an algebraic subgroup of $\mathbb{D}_k$ (they provide a rational parametrization of it as a subvariety). Thus $\mathcal{G}$ is algebraic. The statement for $\mathfrak{g}$ follows. $\square$

Let us briefly explain how to read algebraicity from the Lie algebra $\mathfrak{g}$. Consider the Lie subalgebra $\mathfrak{f} \subset \mathfrak{gl}(T_0 J^0)$ obtained by linearization of the isotopy algebra at $0 \in J^0 = \mathcal{C}_M^{\text{PR}}$. As already noticed in §5.4, this is an 18-dimensional subalgebra admitting the following exact 3-sequence

$$0 \to \mathfrak{v} \longrightarrow \mathfrak{f} \longrightarrow \mathfrak{h} \to 0,$$

where $\mathfrak{v}$ is the vertical part and $\mathfrak{h}$ – the "horizontal" (that is the quotient). The explicit form of these vector fields come from Theorem 22:

$$\begin{aligned}
\mathfrak{v} &= \langle x\partial_p, x\partial_q, x\partial_r, y\partial_p, y\partial_q, y\partial_r, t\partial_p, t\partial_q, t\partial_r, z\partial_p, z\partial_q, z\partial_r \rangle, \\
\mathfrak{h} &= \mathfrak{sl}_2 + \mathfrak{a} + \mathfrak{r}, \quad \text{where} \quad \mathfrak{r} = \langle z\partial_x - t\partial_y \rangle, \\
\mathfrak{sl}_2 &= \langle z\partial_t - x\partial_y - p\partial_q - 2q\partial_r, t\partial_z - y\partial_x - 2q\partial_p - r\partial_q, \\
&\qquad t\partial_t - z\partial_z - x\partial_x + y\partial_y - 2p\partial_p + 2r\partial_r \rangle, \\
\mathfrak{a} &= \langle t\partial_t + z\partial_z - p\partial_p - q\partial_q - r\partial_r, x\partial_x + y\partial_y + p\partial_p + q\partial_q + r\partial_r \rangle.
\end{aligned}$$

By [4] the subalgebra $[\mathfrak{f}, \mathfrak{f}] \subset \mathfrak{gl}(T_0 J^0)$ is algebraic. Since $\mathfrak{f}$ is obtained from $[\mathfrak{f}, \mathfrak{f}] = \mathfrak{v} + \mathfrak{sl}_2 + \mathfrak{r}$ by extension by derivations $\mathfrak{a}$, and the semi-simple elements in the latter have no irrational ratio of spectral values, we conclude that $\mathfrak{f} \subset \mathfrak{gl}(T_0 J^0)$ is an algebraic Lie algebra [3]. The claim about algebraicity of $\mathfrak{g}$ follows by prolongations.

6 The Hilbert and Poincaré function for $\mathcal{SDE}$

Even though $\mathfrak{g}$ is just a PR-shape preserving Lie algebra, its prolongation to the space of 2-jets preserves $\mathcal{SDE}$ (this is an unexpected remarkable fact), and we consider the orbits of $\mathfrak{g}$ on this equation.

6.1 Dimension of generic orbits

We can compute the dimension of a generic orbit in $\mathcal{SDE}_k$ or J^k by computing the rank of the system of prolonged symmetry vector fields $X^{(k)}$ at a point in general position.

By prolonging the generators $X_1, ..., X_5$ and with the help of Maple we observe that the Lie algebra $\mathfrak{g}$ acts transitively on J^1. The dimension of a generic orbit on the Lie algebra acting on J^2 is 44, but the equation $\mathcal{SDE}_2 \subset J^2$ contains no generic orbits, and if we restrict to $\mathcal{SDE}_2$ a generic orbit of $\mathfrak{g}$ is of dimension 42. For higher jet-orders $k > 2$, the dimension of a generic orbit is the same on $\mathcal{SDE}_k$ as on J^k.

We are going to compute $\dim \mathcal{O}_k$ for $k \geq 3$ as follows. Since $\mathfrak{g}$ contains the translations ∂_t, ∂_z, all its orbits pass through the subset $S_k \subset J^k$ given by $t = 0, z = 0$. On S_k we can make the Taylor expansion of parametrizing functions a, b, c, d, e around $(t, z) = (0, 0)$.

We use $X_5(e)$ to show the idea. By varying the coefficients of the Taylor series $e(t, z) = e(0, 0) + e_t(0, 0)t + e_z(0, 0)z + \cdots$ we see that the vector fields $X_5(m, n) = z^m t^n \partial_y - \frac{n}{2} z^m t^{n-1} \partial_q - m z^{m-1} t^n \partial_r$ are contained in the symmetry algebra, with the convention that $t^{-1} = z^{-1} = 0$, and any vector field of the form $X_5(e)$ is tangent to a vector field in $\langle X_5(m, n) \rangle$. The prolongation of a vector field takes the form

$$X^{(k)} = \sum_i a_i \, \mathcal{D}_i^{(k+1)} + \sum_{|\sigma| \leq k} (\mathcal{D}_\sigma(\phi_p)\partial_{p_\sigma} + \mathcal{D}_\sigma(\phi_q)\partial_{q_\sigma} + \mathcal{D}_\sigma(\phi_r)\partial_{r_\sigma}) \qquad (3)$$

where $\mathcal{D}_\sigma$ is the iterated total derivative, $\mathcal{D}_i^{(k+1)}$ the truncated total derivative (the "restriction" to the space J^{k+1}, cf. [8, 11]), $a_i = dx_i(X)$ for $(x_1, x_2, x_3, x_4) = (t, x, y, z)$, and ϕ_p, ϕ_q, ϕ_r are the generating functions for X, i.e. $\phi_p = \omega_p(X), \phi_q = \omega_q(X), \phi_r = \omega_r(X)$ where

$$\begin{aligned}
\omega_p &= dp - p_t dt - p_x dx - p_y dy - p_z dz, \\
\omega_q &= dq - q_t dt - q_x dx - q_y dy - q_z dz, \\
\omega_r &= dr - r_t dt - r_x dx - r_y dy - r_z dz
\end{aligned}$$

In the case of $X_5(m, n)$, the generating functions are given by

$$\phi_p = -p_y z^m t^n, \ \phi_q = -\tfrac{n}{2} z^m t^{n-1} - q_y z^m t^n, \ \phi_r = -m z^{m-1} t^n - r_y z^m t^n.$$

We see that the restriction of $X_5(m, n)^{(k)}$ to the fiber over $0 \in \mathcal{C}_M^{\mathrm{PR}}$ is nonzero only when $m+n \leq k+1$. Hence we can parametrize $\langle X_5(m, n) \rangle^{(k)}$ by $J_0^{k+1}(\mathbb{R}^2(t, z), \mathbb{R}(e))$, and by extending this argument to the whole symmetry algebra we get (the vector fields $X_k(m, n)$ for $k = 1, \ldots, 4$, are defined similarly to the vector field $X_5(m, n)$ by simply substituting $a = z^m t^n$ etc into the formulae of Theorem 22)

$$\begin{aligned}
\mathfrak{g}^{(k)} &= \langle X_1(m, n), X_2(m, n), X_4(m, n), X_5(m, n) \rangle^{(k)} \oplus \langle X_3(m, n) \rangle^{(k)} \\
&= J_0^{k+1}(\mathbb{R}^2(t, z), \mathbb{R}^4(a, b, d, e)) \times J_0^k(\mathbb{R}^2(t, z), \mathbb{R}(c)).
\end{aligned}$$

Using formula (3) we verify that the Lie algebra $\mathfrak{g}^{(k)}$ acts freely on $\mathcal{SDE}_k$

for $k \geq 3$, whence

$$\begin{aligned}\dim \mathcal{O}_k &= \dim \left(J_0^{k+1}(\mathbb{R}^2,\mathbb{R}^4) \times J_0^k(\mathbb{R}^2,\mathbb{R})\right) \\ &= 4\dim\left(J_0^{k+1}(\mathbb{R}^2,\mathbb{R})\right) + \dim\left(J_0^k(\mathbb{R}^2,\mathbb{R})\right) \\ &= 4\binom{k+3}{2} + \binom{k+2}{2} = \frac{(k+2)(5k+13)}{2}.\end{aligned}$$

6.2 Counting the differential invariants

The number s_k of differential invariants of order k (as before, this is trdeg $\mathfrak{F}_k$) is equal to the codimension of a generic orbit of $\mathfrak{g}$ on $\mathcal{SDE}_k$. For the lowest orders, we have $s_0 = s_1 = 0$ and $s_2 = \dim \mathcal{SDE}_2 - \dim \mathcal{O}_2 = 46 - 42 = 4$. For higher jet-orders, the number of invariants of order k is given by

$$s_k = \text{codim}\mathcal{O}_k = \dim \mathcal{SDE}_k - \dim \mathcal{O}_k = k^3 + 2k^2 - 5k - 6, \quad k \geq 3.$$

The number of differential invariants of "pure order" k is then given by $H(k) = s_k - s_{k-1}$. The Poincaré function $P(z) = \sum_{k=0}^{\infty} H(k)z^k$ can now easily be computed, and we conclude:

Theorem 28. *The Hilbert polynomial for the action of $\mathfrak{g}$ on $\mathcal{SDE}$ is*

$$H(k) = \begin{cases} 0 & \text{for } k < 2, \\ 4 & \text{for } k = 2, \\ 20 & \text{for } k = 3, \\ 3k^2 + k - 6 & \text{for } k > 3. \end{cases}$$

The corresponding Poincaré function is equal to

$$P(z) = \frac{2z^2(2 + 4z - z^2 - 4z^3 + 2z^4)}{(1-z)^3}.$$

Notice that $H(k)$ in this statement has the same leading term as $H(k)$ in Theorem 21 for $k > 3$. The following table summarizes the counting results from the last two subsections for low order k.

k	0	1	2	3	4	5	6	7	...
$\dim \mathcal{SDE}_k$	7	19	46	94	169	277	424	616	...
$\dim \mathcal{O}_k$	7	19	42	70	99	133	172	216	...
$\text{codim}\, \mathcal{O}_k$	0	0	4	24	70	144	252	400	...
$H(k)$	0	0	4	20	46	74	108	148	...

7 The invariants of $\mathcal{SDE}$ and the quotient equation

From the global Lie-Tresse theorem [12] and Theorem 27 it follows that there exist rational differential invariants of $\mathfrak{g}$-action (or $\mathcal{G}$-action) on $\mathcal{SDE}$ that separate generic orbits.

7.1 Invariants of the second order

There are four independent differential invariants of the second order:

$$
\begin{aligned}
I_1 &= \frac{1}{K}\left(p_{yy}r_{xx} - p_{xx}r_{yy} + 2\,p_{xy}q_{xx} + 4\,q_{xy}^2 + 2\,q_{yy}r_{xy}\right)\\
I_2 &= \frac{1}{K^3}\Big(\left(q_{xy}r_{yy} - q_{yy}r_{xy}\right)p_{xx} + \left(p_{yy}r_{xy} - p_{xy}r_{yy}\right)q_{xx}\\
&\quad + \left(p_{xy}q_{yy} - p_{yy}q_{xy}\right)r_{xx}\Big)^2\\
I_3 &= \frac{1}{K^3}\Big(p_{yy}\left(q_{xx} - r_{xy}\right)^2 + r_{xx}\left(q_{yy} - p_{xy}\right)^2\\
&\quad - 2\,q_{xy}\left(p_{xy}q_{xx} + q_{yy}r_{xy} - p_{xy}r_{xy} - 2\,p_{yy}r_{xx} + 2\,q_{xy}^2 - q_{xx}q_{yy}\right)\Big)^2\\
I_4 &= \frac{1}{K^2}\Big(p_{xx}^2r_{yy}^2 + p_{yy}^2r_{xx}^2 - 2\,p_{xx}p_{yy}r_{xx}r_{yy} + 4\,p_{xx}p_{yy}r_{xy}^2\\
&\quad + 4\,p_{xy}^2r_{xx}r_{yy} - 4\,q_{xx}q_{yy}\left(p_{xx}r_{yy} - 4\,p_{xy}r_{xy} + p_{yy}r_{xx}\right)\\
&\quad + 4\,p_{xx}q_{xy}r_{yy}\left(p_{xx} + 4\,q_{xy} + r_{yy}\right) - 4\,p_{xy}r_{xy}\left(p_{xx}r_{yy} + p_{yy}r_{xx}\right)\\
&\quad + 4\,p_{xx}r_{xx}\left(q_{yy}^2 - p_{yy}q_{xy}\right) + 4\,p_{yy}r_{yy}\left(q_{xx}^2 - q_{xy}r_{xx}\right)\\
&\quad - 8\,p_{xy}q_{xy}\left(q_{xx}r_{yy} + q_{yy}r_{xx}\right) - 8\,q_{xy}r_{xy}\left(p_{xx}q_{yy} + p_{yy}q_{xx}\right)\Big)
\end{aligned}
$$

where

$$K = p_{xx}r_{yy} - 2\,p_{xy}r_{xy} + p_{yy}r_{xx} + 2\left(q_{xy}^2 - q_{xx}q_{yy}\right)$$

is a relative differential invariant.

7.2 Singular set

Let $\Sigma_2' \subset \mathcal{SDE}_2$ be the set of points θ where $\langle X_\theta^{(2)} : X \in \mathfrak{g}\rangle \subset T_\theta(\mathcal{SDE}_2)$ is of dimension less than 42. It's given by

$$\Sigma_2' = \{\theta \in \mathcal{SDE}_2 : \operatorname{rank}\left(\mathcal{A}|_\theta\right) < 4\}$$

where

$$\mathcal{A} = \begin{pmatrix} 0 & -2q_{xy} - 2r_{yy} & p_{xy} + q_{yy} & 0 \\ 0 & 2p_{xy} - 2q_{yy} & 2p_{yy} & p_{yy} \\ 4q_{xy} + r_{yy} & -r_{xx} & -2q_{xx} & -2q_{xx} \\ -p_{xy} + q_{yy} & q_{xx} - r_{xy} & 0 & -q_{xy} \\ -p_{yy} & 2q_{xy} - r_{yy} & q_{yy} & 0 \\ -2q_{xx} + 2r_{xy} & 0 & -2r_{xx} & -3r_{xx} \\ -2q_{xy} + r_{yy} & r_{xx} & -r_{xy} & -2r_{xy} \\ -2q_{yy} & 2r_{xy} & 0 & -r_{yy} \end{pmatrix}.$$

This set contains the singular points that can be seen from a local viewpoint on $\mathcal{SDE}_2$, but there may still be some singular (non-closed) orbits of dimension 42. We use the differential invariants I_i to filter out these. Let $\Sigma_3 \subset \mathcal{SDE}_3$ be the set of points where the 4-form

$$\hat{d}I_1 \wedge \hat{d}I_2 \wedge \hat{d}I_3 \wedge \hat{d}I_4$$

is not defined or is zero. Here $\hat{d}$ is the horizontal differential

$$\hat{d}f = \mathcal{D}_t(f)dt + \mathcal{D}_x(f)dx + \mathcal{D}_y(f)dy + \mathcal{D}_z(f)dz.$$

This defines the singular sets $\Sigma_k = (\pi_{k,3}|_{\mathcal{SDE}_k})^{-1}(\Sigma_3) \subset \mathcal{SDE}_k$ and $\Sigma_2 = \pi_{3,2}(\Sigma_3)$. The set Σ_2 of all singular points in $\mathcal{SDE}_2$ contains Σ_2'.

By using Maple, we can easily verify that $\{K = K_1 = K_2 = K_3 = K_4 = 0\}$ is contained in Σ_2', where K_i is the numerator of I_i for $i = 1, 2, 3, 4$. Notice also that 2-jets of conformally flat metrics are contained in Σ_2'.

7.3 Invariants of higher orders

The 1-forms $\hat{d}I_1, \hat{d}I_2, \hat{d}I_3, \hat{d}I_4$ determine an invariant horizontal coframe on $\mathcal{SDE}_3 \backslash \Sigma_3$. The basis elements of the dual frame $\hat{\partial}_{I_1}, \hat{\partial}_{I_2}, \hat{\partial}_{I_3}, \hat{\partial}_{I_4}$ are invariant derivations, the Tresse derivatives. We can rewrite metric (1) in terms of the invariant coframe:

$$g = \sum G_{ij} \hat{d}I_i \hat{d}I_j, \qquad \text{where} \qquad G_{ij} = g(\hat{\partial}_{I_i}, \hat{\partial}_{I_j}). \tag{4}$$

Since the $\hat{d}I_i$ are invariant, and $[g]$ is invariant, the map

$$\hat{G} = [G_{11} : G_{12} : G_{13} : G_{14} : G_{22} : G_{23} : G_{24} : G_{33} : G_{34} : G_{44}] \colon J^3 \to \mathbb{R}P^9$$

is invariant. Hence the functions G_{ij}/G_{44} are rational scalar differential invariants (of third order). This has been verified in Maple by differentiation of G_{ij}/G_{44} along the elements of $\mathfrak{g}$. It was also checked that these nine invariants are independent. By the principle of n-invariants [1], I_i and G_{ij}/G_{44} generate all scalar differential invariants.

Theorem 29. *The field of rational differential invariants of $\mathfrak{g}$ on $\mathcal{SDE}$ is generated by the differential invariants $I_k, G_{ij}/G_{44}$ and invariant derivations $\hat{\partial}_{I_k}$. The differential invariants in this field separate generic orbits in $\mathcal{SDE}_\infty$.*

7.4 The quotient equation

When restricted to a section g_0 of $\mathcal{C}_M^{\rm PR}$, the functions G_{ij} can be considered as functions of I_1, I_2, I_3, I_4. Two such nonsingular sections are equivalent if they determine the same map $\hat{G}(I_1, I_2, I_3, I_4)$. The quotient equation $(\mathcal{SDE}_\infty \setminus \Sigma_\infty)/\mathfrak{g}$ is given by

$$*W_g = W_g, \qquad \text{where} \qquad g = \sum G_{ij}(I_1, I_2, I_3, I_4)\hat{d}I_i\hat{d}I_j.$$

Here we consider $I_1, \ldots, I_4$ as coordinates on M. Equivalently, given local coordinates $(x_1, \ldots, x_4)$ on M the quotient equation is obtained by adding to $\mathcal{SDE}$ the equations $I_i = x_i$, $1 \le i \le 4$.

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Differential invariants of Einstein-Weyl structures in 3D

Boris Kruglikov, Eivind Schneider

Abstract

Einstein-Weyl structures on a three-dimensional manifold M are given by a system $\mathcal{E}$ of PDEs on sections of a bundle over M. This system is invariant under the Lie pseudogroup $\mathcal{G}$ of local diffeomorphisms on M. Two Einstein-Weyl structures are locally equivalent if there exists a local diffeomorphism taking one to the other. Our goal is to describe the quotient equation $\mathcal{E}/\mathcal{G}$ whose solutions correspond to nonequivalent Einstein-Weyl structures. The approach uses symmetries of the Manakov-Santini integrable system and the action of the corresponding Lie pseudogroup.

Introduction

A Weyl structure is a pair consisting of a conformal metric $[g]$ on a manifold M and a symmetric linear connection ∇ preserving the conformal structure. This means

$$\nabla g = \omega \otimes g \tag{1}$$

for some one-form ω on M [25]. The Einstein-Weyl condition says that the symmetrized Ricci tensor of ∇ belongs to the given conformal class:

$$Ric_{\nabla}^{\text{sym}} = \Lambda g \tag{2}$$

for some function Λ on M. We call the pair $([g], \nabla)$ an Einstein-Weyl structure if it satisfies this Einstein-Weyl equation.

In this paper we restrict to three-dimensional manifolds. This is the first non-trivial case, which is simultaneously the most interesting due to its relation with dispersionless integrable systems [5, 10]. In addition, in dimension 3 the Einstein equation is trivial, meaning that all Einstein manifolds are space forms, while the Einstein-Weyl equation is quite rich. The Einstein-Weyl equation has attracted a lot of attention due to its relations with twistor theory, Lax integrability of PDE and mathematical relativity [12, 13, 8]. It is worth mentioning that according to [6] the solution spaces of a third-order scalar ODE with vanishing Wünschmann and Cartan invariants carry a natural Einstein-Weyl structure. We aim to solve the local equivalence problem for Einstein-Weyl structures in 3D.

The Einstein-Weyl equation is invariant under the Lie pseudogroup of local diffeomorphisms of M. To construct the quotient of the action of this pseudogroup on the space of Einstein-Weyl structures we compute the algebra of differential invariants, thus following the approach to the equivalence problem as presented in [24, 1, 23].

We begin with general coordinate-free considerations in Section 1; this concerns conformal structures of any signature. Then in Section 2 we specialize to the normal form of the pair (g, ω) introduced in [7], which expresses Einstein-Weyl structures locally by solutions of the modified Manakov-Santini system [22]; this is specific for the Lorentzian signature. It will be demonstrated in Section 2 that the symmetry algebra of this PDE system coincides with the algebra of shape preserving transformations for the metric in normal form (3). Consequently, the problem is equivalent to computing differential invariants of the modified Manakov-Santini system with respect to its symmetry pseudogroup.

In both cases we compute generators of the algebra of scalar rational differential invariants and derive the Poincaré function counting the local moduli of the problem. Section 1 and Sections 2-3, supporting two different approaches to the same problem, can be read independently, and the reader interested in geometry of the Manakov-Santini system can proceed straightforwardly to the latter sections. Section 4 provides some particular solutions of the Manakov-Santini system, yielding several families of non-equivalent Einstein-Weyl spaces parametrized by one or two functions of one argument. We stress that these explicit Einstein-Weyl structures are non-homogeneous and not obtained by any symmetry reduction. Appendix 4 is devoted to the proof of a general theorem on algebraicity of the symmetry pseudogroup.

1 Invariants of Einstein-Weyl structures

In this section we discuss the general coordinate-free approach to computation of differential invariants of Einstein-Weyl structures in 3D. The conformal structure can be both of Riemannian and Lorentzian signature. We refer to [23, 16, 17] for the basics of jet-theory, Lie pseudogroups and differential invariants.

1.1 Setup of the problem

Let a Lie pseudogroup G act on the space of jets $\mathcal{J}$ or a differential equation considered as a co-filtered submanifold in it (also know as diffiety); we keep the same notation for the latter in this setup.

A differential invariant of order k is a smooth function I on $\mathcal{J}_k$ that is constant on orbits of the G-action. If the pseudogroup G is topologically connected (the same as path-connected), then the definition of differential invariant is equivalent to the constraint $L_{X^{(k)}}I = 0$ for every X in the Lie algebra sheaf $\mathfrak{g}$ corresponding to G, where $X^{(k)}$ denotes the prolongation of the vector field X to k-jets.

It turns out that in our problem, the pseudogroup G, the space $\mathcal{J}$ and the action are algebraic in the sense of [18] (for the data in this section this follows from the definition, and for the objects in the following sections it follows from a general theorem in the appendix). Moreover, the action of G is transitive on the base and $\mathcal{J}$ is irreducible. Under these conditions, the global Lie-Tresse theorem [18] implies that the space of rational differential invariants is finitely generated as a differential field, i.e. there exist a finite number of differential invariants and invariant derivations that algebraically generate all other invariants. In addition, the theorem states that differential invariants separate orbits in general position, thus solving the local equivalence problem for generic structures.

In our work the pseudogroup G (and later $\mathcal{G}$) will be connected in the Zariski topology. In this case the condition that a rational function I is a differential invariant is equivalent to the constraint $L_{X^{(k)}}I = 0$ for every X from the Lie algebra sheaf $\mathfrak{g}$ of G.

Weyl structures are given by triples (g, ∇, ω) satisfying relation (1). Let us note that essentially two of the structures are enough to recover the third one. Indeed, g and ∇ give ω by (1). Also, g and ω give $\nabla = \nabla^g + \rho(\omega)$, where $2\rho(\omega)(X,Y) = \omega(X)Y + \omega(Y)X - g(X,Y)\omega_g^\sharp$. In coordinates this relates the Christoffel symbols of ∇ and the Levi-Civita connection ∇^g:

$$\Gamma_{ij}^k = \gamma_{ij}^k + \tfrac{1}{2}(\omega_i\delta_j^k + \omega_j\delta_i^k - g_{ij}\omega^k).$$

Finally, the same formula expresses ∇^g from ∇ and ω. It is known that if (M, g) is holonomy irreducible and admits no parallel null distribution, then ∇^g determines g up to homothety. This recovers $[g]$ in this generic case.

It is not true though that k-jet of one pair correspond to k-jet of another representative pair, the jets are staggered in this correspondence. In what follows we will restrict to equivalence classes of pairs (g, ω): when the representative of $[g]$ is changed $g \mapsto f^2 g$, the one-form also changes $\omega \mapsto \omega + 2df/f$.

Remark 9. *Note that we multiply g by f^2 and not by $f \neq 0$ because we want to preserve signature of the metric: multiplication by -1 changes the signature in 3D. Restricting to conformal structures of fixed signature (an open subset in $\mathcal{J}_0$) we thus will be able to separate orbits by algebraic invariants (next sections). In [19] we studied self-dual conformal structures. For split signature $(2,2)$ a modification of scaling $g \mapsto f\,g$ is required, then the separation is also guaranteed.*

Thus the space of moduli of Weyl structures can be considered as the space $\mathcal{W}$ of pairs (g, ω) modulo the pseudogroup $G = \mathrm{Diff}_{\mathrm{loc}}(M) \times C^\infty_{\neq 0}(M)$ consisting of pairs (φ, f) of a local diffeomorphism φ and a nonzero function f. The action is clearly algebraic.

1.2 Weyl structures

The G-action has order 1, i.e. for any point $a \in M$ the stabilizer subgroup in $(k+1)$-jets G_a^k acts on the space $\mathcal{W}_a^k$ of k-jets of the structures at a. For a point $a_k \in \mathcal{W}_a^k$ denote $\mathrm{St}_{a_k}^{k+1}$ its stabilizer in G_a^{k+1}. Let also $g_k = \mathrm{Ker}(d\pi_{k,k-1} : T_{a_k}\mathcal{W}_a^k \to T_{a_{k-1}}\mathcal{W}_a^{k-1})$ denote the symbol of the space of Weyl structures. The differential group G has the following co-filtration:

$$0 \to \Delta_k \longrightarrow G_a^k \longrightarrow G_a^{k-1} \to 1,$$

where $\Delta_k = S^k T_a^* \otimes T_a \oplus S^k T_a^*$ for $k > 1$, and we abbreviate $T_a = T_a M$. For $k = 1$, $G_a^1 = \Delta_1 = \mathrm{GL}(T_a) \oplus T_a^* \oplus \mathbb{R}^\times$.

The 0-jet a_0 is the evaluation (g_a, ω_a). By G_a^1-action the second component can be made zero, and the first component rescaled. The action of $GL(T_a)$ on the conformal class $[g_a]$ yields $\mathrm{St}_{a_0}^1 = CO(g_a)$.

The group $\Delta_2 = S^2 T_a^* \otimes T_a \oplus S^2 T_a^*$ acts on the symbol $g_1 = T_a^* \otimes S^2 T_a^* \oplus T_a^* \otimes T_a^*$ of $\mathcal{W}$. This action is free and $g_1/\Delta_2 = \Lambda^2 T_a^*$. This is the space where $Ric_\nabla^{\mathrm{skew}} = \frac{3}{2} d\omega$ [13] lives. The stabilizer from the previous jet-level $CO(g_a)$ acts with an open orbit, i.e. there are no scalar invariants. There are however the following vector and tensor invariants: $L^1 = \mathrm{Ker}(d\omega)$, $\Pi^2 = L^1_\perp$ (generically L^1 is non-null and so transversal to Π^2) and a complex structure $J = g^{-1} d\omega$ on Π, where the representative g is normalized so that $\|d\omega\|_g^2 = 1$. The stabilizer $\mathrm{St}_{a_1}^2$ is either $SO(2) \times \mathbb{Z}_2$ or $SO(1,1) \times \mathbb{Z}_2$.

Starting from $k \geq 2$ the action of G_a^{k+1} on a Zariski open subset of $\mathcal{W}_a^k$ is free, i.e. the stabilizer is resolved: $\mathrm{St}_{a_k}^{k+1} = 0$ for generic $a_k \in \mathcal{W}_a^k$. This can be seen by the exact sequences approach as in [20], and can be verified directly. The metric g chosen with the above normalization is the unique conformal representative, then ω is defined uniquely as well, and we can have the following canonical frame on M, defined by a Zariski generic a_2:

$e_1 \in L^1$ normalized by $\omega(e_1) = 1$, $e_2 = \pi(\omega_g^\sharp)$ with $\pi : T_a \to \Pi^2$ being the orthogonal projection along L^1, and $e_3 = Je_2$. Coefficients of the structure (g, ω) written in this frame give a complete set of scalar rational differential invariants.

The count of them is as follows. Let s_k be the number of independent differential invariants of order $\leq k$, which coincides with the transcendence degree of the field of order $\leq k$ rational differential invariants. Let $h_k = s_k - s_{k-1}$ be the number of "pure" order k invariants. Then $h_0 = h_1 = 0$ and $h_2 = \dim g_2 - \dim \Delta_3 - \dim SO(2) = 54 - 40 - 1 = 13$ and $h_k = \dim g_k - \dim \Delta_{k+1} = 9\binom{k+2}{2} - 4\binom{k+3}{2} = \frac{1}{2}(5k^2 + 7k - 6)$ for $k > 2$. These numbers are encoded by the Poincaré function

$$P(z) = \sum_{k=0}^{\infty} h_k z^k = \frac{(13 - 9z + z^3)z^2}{(1-z)^3}.$$

1.3 Einstein-Weyl structures

The Einstein-Weyl equation (2) is a set of 5 equations on 8 unknowns, which looks like an underdetermined system. However its $\mathrm{Diff}_{\mathrm{loc}}(M)$-invariance reduces the number of unknowns to 8-3=5 and makes it a determined system – formally this follows from the normalization of [7].

Denote this equation by $\mathcal{EW}$. The number of its determining equations of order k is $5\binom{k}{2}$. Let $\tilde{g}_k = \mathrm{Ker}(d\pi_{k,k-1} : T_{a_k}\mathcal{EW}_a^k \to T_{a_{k-1}}\mathcal{EW}_a^{k-1})$ be the symbol of the system. Its dimension is $\dim \tilde{g}_k = \dim g_k - 5\binom{k}{2}$.

The action of G_a^{k+1} on $\mathcal{EW}_a^k$ is still free starting from $k \geq 2$ and this implies that the number of "pure order" k invariants is: $\bar{h}_0 = \bar{h}_1 = 0$, $\bar{h}_2 = 13-5 = 8$, and $\bar{h}_k = h_k - 5\binom{k}{2} = 3(2k-1)$ for $k > 2$. The corresponding Poincaré function is equal to

$$\bar{P}(z) = \sum_{k=0}^{\infty} \bar{h}_k z^k = \frac{(8 - z - z^2)z^2}{(1-z)^2}.$$

We again have the canonical frame (e_1, e_2, e_3), and this yields all scalar rational differential invariants of $\mathcal{EW}$.

2 Einstein-Weyl structures via an integrable system

In this section we study the Lie algebra $\mathfrak{g}$ of point symmetries of the modified Manakov-Santini system $\mathcal{E}$, defined by (4), which describes three-dimensional Einstein-Weyl structures of Lorentzian signature. We calculate the dimensions of generic orbits of $\mathfrak{g}$. The Einstein-Weyl structures corresponding to solutions of $\mathcal{E}$ are of special shape (3), and we compute the Lie algebra $\mathfrak{h}$ of vector fields preserving this shape (ansatz). It turns out that the lift of $\mathfrak{h}$ to the total space E is exactly $\mathfrak{g}$, whence $\mathfrak{h} \simeq \mathfrak{g}$.

2.1 A modified Manakov-Santini equation and its symmetry

By [7] any Lorentzian signature Einstein-Weyl structure is locally of the form

$$\begin{aligned} g &= 4dtdx + 2udtdy - (u^2 + 4v)dt^2 - dy^2 \\ \omega &= (uu_x + 2u_y + 4v_x)dt - u_x dy \end{aligned} \tag{3}$$

where u and v are functions of (t, x, y) satisfying

$$\begin{aligned} F_1 &= (u_t + uu_y + vu_x)_x - (u_y)_y = 0, \\ F_2 &= (v_t + vv_x - uv_y)_x - (v_y - 2uv_x)_y = 0. \end{aligned} \tag{4}$$

This system, derived in the proof of Theorem 1 in [7], is related to the Manakov-Santini system [22] by the change of variables $(u, v) \mapsto (v_x, u - v_y)$ and potentiation. We will refer to it as the modified Manakov-Santini system.

Note that normalization of the coefficient of dy^2 in g to be -1 gives a representative of the conformal class $[g]$, reducing the $C^\infty_{\neq 0}(M)$-component of the pseudogroup G from the previous section.

Let $M = \mathbb{R}^3(t, x, y)$. We treat the pair (g, ω) as a section of the bundle

$$\pi \colon E = M \times \mathbb{R}^2(u, v) \to M.$$

This is a subbundle of $S^2T^*M \oplus T^*M$, considered in Section 1.

Einstein-Weyl structures correspond to sections of π satisfying (4). Consider the system (4) as a nonlinear subbundle $\mathcal{E}_2 = \{F_1 = F_2 = 0\}$ of the jet bundle $J^2\pi$, and denote its prolongation by $\mathcal{E}_k \subset J^k\pi$. The notation $\mathcal{E}_0 = J^0\pi = E$, $\mathcal{E}_1 = J^1\pi$ will be used. Let $\mathcal{E} \subset J^\infty\pi$ denote the projective limit of $\mathcal{E}_k$.

The dimension of $J^k\pi$ is $3 + 2\binom{k+3}{3}$, while the number of equations determining $\mathcal{E}_k$ is $2\binom{k+1}{3}$. The system $\mathcal{E}$ is determined, so these equations are independent, whence

$$\dim \mathcal{E}_k = \dim J^k\pi - 2\tbinom{k+1}{3} = 3 + 2(k+1)^2, \ k \geq 2.$$

For $k = 0, 1$ we have $\dim \mathcal{E}_0 = 5$, $\dim \mathcal{E}_1 = 11$.

$[\,,\,]$	$X_1(g)$	$X_2(g)$	$X_3(g)$	$X_4(g)$	$X_5(g)$
$X_1(f)$	0	0	0	$X_1(-g\dot{f})$	$X_1(2fg)$
$X_2(f)$	$*$	0	$X_1(fg)$	$X_2(\frac{f\dot{g}}{2} - g\dot{f})$	$X_2(fg) + X_3(2f\dot{g})$
$X_3(f)$	$*$	$*$	0	$X_3(-g\dot{f} - \frac{f\dot{g}}{2})$	$X_3(fg)$
$X_4(f)$	$*$	$*$	$*$	$X_4(f\dot{g} - g\dot{f})$	$X_5(f\dot{g})$
$X_5(f)$	$*$	$*$	$*$	$*$	0

Table 1: The structure of the symmetry Lie algebra $\mathfrak{g}$.

A vector field X on E is an (infinitesimal point) symmetry of $\mathcal{E}$ if its prolongation $X^{(2)}$ to $J^2\pi$ is tangent to $\mathcal{E}_2$, in other words if it satisfies the Lie equation

$$(L_{X^{(2)}}F_i)|_{\mathcal{E}_2} = 0 \text{ for } i = 1, 2.$$

Decomposing this by the fiber coordinates of $\mathcal{E}_2 \to E$, we get an overdetermined system of linear PDEs on the coefficients of X. This system can be explicitly solved, and the result is as follows.

Theorem 30. *The Lie algebra $\mathfrak{g}$ of symmetries of $\mathcal{E}$ has the following generators, involving five arbitrary functions $a = a(t), \ldots, e = e(t)$:*

$$\begin{aligned}
X_1(a) &= a\partial_x + \dot{a}\partial_v \\
X_2(b) &= b\partial_y + \dot{b}\partial_u \\
X_3(c) &= yc\partial_x - 2c\partial_u + (uc + y\dot{c})\partial_v \\
X_4(d) &= d\partial_t + \frac{1}{2}\dot{d}y\partial_y + \frac{1}{2}(y\ddot{d} - u\dot{d})\partial_u - \dot{d}v\partial_v \\
X_5(e) &= (y^2\dot{e} + 2xe)\partial_x + ye\partial_y + (ue - 3y\dot{e})\partial_u + (y^2\ddot{e} + 2yu\dot{e} + 2ve + 2x\dot{e})\partial_v
\end{aligned}$$

Table 1 shows the commutation relations of $\mathfrak{g}$.

It follows from the table that $\mathfrak{g}$ is a perfect Lie algebra: $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. We also see that the splitting $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$, with $\mathfrak{g}_0 = \langle X_4, X_5 \rangle, \mathfrak{g}_1 = \langle X_2, X_3 \rangle, \mathfrak{g}_2 = \langle X_1 \rangle$, gives a grading of $\mathfrak{g}$, i.e. $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ ($\mathfrak{g}_i = 0$ for $i \notin \{0, 1, 2\}$).

Integration gives the action of the Lie pseudogroup $\mathcal{G}_{\text{top}}$ on E defined by

$$
\begin{aligned}
t &\mapsto D(t),\\
x &\mapsto E(t)^2 x + E(t)E'(t)y^2 + C(t)\,y + A(t),\\
y &\mapsto \sqrt{D'(t)}E(t)\,y + B(t),\\
u &\mapsto \frac{E(t)}{\sqrt{D'(t)}}\,u - \frac{y}{E(t)^2}\frac{d}{dt}\frac{E(t)^3}{\sqrt{D'(t)}} + \frac{B'(t)}{D'(t)} - \frac{2C(t)}{E(t)\sqrt{D'(t)}},\\
v &\mapsto \frac{E(t)^2}{D'(t)}\,v + \frac{C(t) + 2E(t)E'(t)\,y}{D'(t)}\,u + \frac{E(t)E''(t) - 3E'(t)^2}{D'(t)}\,y^2\\
&\quad + \frac{E(t)^4}{D'(t)}\frac{d}{dt}\frac{C(t)}{E(t)^4}\,y + \frac{2E(t)E'(t)}{D'(t)}\,x + \frac{E(t)^2A'(t) - C(t)^2}{D'(t)E(t)^2},
\end{aligned}
$$

where $D \in \mathrm{Diff}^+_{\mathrm{loc}}(\mathbb{R})$ is an orientation-preserving local diffeomorphism of $\mathbb{R}$ and A, B, C, E are smooth functions with the same domain as D and $E(t) > 0$ for every t in its domain.

This Lie pseudogroup is topologically connected and has $\mathfrak{g}$ as its Lie algebra of vector fields. However $\mathcal{G}_{\mathrm{top}}$ is not algebraic. Since the global Lie-Tresse theorem holds for algebraic Lie pseudogroups, we consider the Zariski closure of $\mathcal{G}_{\mathrm{top}}$, denoted by $\mathcal{G}_Z$. The subgroup $\mathcal{G}_{\mathrm{top}}$ is normal in $\mathcal{G}_Z$ and $\mathcal{G}_Z/\mathcal{G}_{\mathrm{top}} = \mathbb{Z}_2 \times \mathbb{Z}_2$ is generated by reflections $(t, x, y) \mapsto (-t, -x, -y)$ and $(y, u) \mapsto (-y, -u)$. Thus it can be argued, also from a geometric viewpoint, that it is more natural to consider $\mathcal{G}_Z$ instead of $\mathcal{G}_{\mathrm{top}}$. In fact, the Lie pseudogroup $\mathcal{G}_Z$ is the full pseudogroup of symmetries, and so we simply denote it by $\mathcal{G}$.

This pseudogroup $\mathcal{G}$ can be also parametrized by five functions of one variable:

$$
\begin{aligned}
t &\mapsto D(t),\\
x &\mapsto \frac{E(t)^2}{D'(t)}\,x + \frac{d}{dt}\frac{E(t)^2}{D'(t)}\frac{y^2}{2} + C(t)\,y + A(t),\\
y &\mapsto E(t)\,y + B(t),\\
u &\mapsto \frac{E(t)}{D'(t)}\,u - \frac{D'(t)}{E(t)^2}\frac{d}{dt}\frac{E(t)^3}{D'(t)^2}\,y + \frac{B'(t)}{D'(t)} - \frac{2C(t)}{E(t)},\\
v &\mapsto \frac{E(t)^2}{D'(t)^2}\,v + \frac{C(t) + \frac{d}{dt}\frac{E(t)^2}{D'(t)}\,y}{D'(t)}\,u - \frac{E(t)^4}{D'(t)^3}\frac{d^2}{dt^2}\frac{D'(t)}{E(t)^2}\frac{y^2}{2}\\
&\quad + \frac{E(t)^4}{D'(t)^3}\frac{d}{dt}\frac{C(t)D'(t)^2}{E(t)^4}\,y + \frac{x}{D'(t)}\frac{d}{dt}\frac{E(t)^2}{D'(t)} + \frac{A'(t)}{D'(t)} - \frac{C(t)^2}{E(t)^2},
\end{aligned}
$$

but now $D \in \mathrm{Diff}_{\mathrm{loc}}(\mathbb{R})$, $E(t) \neq 0$ and A, B, C are arbitrary.

2.2 Dimension of generic orbits

Denote by $\mathcal{O}_k$ a generic orbit of the $\mathcal{G}$-action on $\mathcal{E}_k$. Its topologically-connected component is an orbit of the prolongation $\mathfrak{g}^{(k)}$ of $\mathfrak{g}$, and so we consider the action of the latter.

The Lie algebra $\mathfrak{g}$ acts transitively on $J^0\pi$ and $\mathfrak{g}^{(1)}$ acts locally transitively on $J^1\pi$ (the hyperplane given by $u_x = 0$ is invariant). A generic orbit of $\mathfrak{g}^{(2)}$ on both $\mathcal{E}_2$ and $J^2\pi$ has dimension 18. The next theorem describes the orbit dimensions for every k.

Proposition 31. *A generic orbit $\mathcal{O}_k$ of the $\mathfrak{g}^{(k)}$-action on $\mathcal{E}_k$ satisfies:*

$$\dim \mathcal{O}_0 = 5, \qquad \dim \mathcal{O}_1 = 11, \qquad \dim \mathcal{O}_k = 5k+8, \quad k \geq 2.$$

Proof. Consider the point $(t, x, y, u, v) = (0, 0, 0, 0, 0) \in E$, and denote its fiber under the projection $\mathcal{E}_k \to E$ by S_k. Since $\mathfrak{g}$ acts transitively on E, every orbit of $\mathfrak{g}^{(k)}$ in $\mathcal{E}_k$ intersects S_k at some point $\theta_k \in S_k$. Denote by $\mathcal{O}_{\theta_k}$ the $\mathfrak{g}^{(k)}$-orbit through $\theta_k \in S_k$. We have $T_{\theta_k}\mathcal{O}_{\theta_k} = \text{span}\{X_i^{(k)}(f_i)_{\theta_k} : f_i \in C^\infty(\mathbb{R}), i = 1, ..., 5\}$. Here and below $X_i^{(k)}(f)_{\theta_k}$ denotes the prolongation of the vector field $X_i(f)$ to $J^k\pi$, evaluated at the point θ_k.

The k-th prolongation of a vector field X has the coordinate form

$$X^{(k)} = \sum_{i=1}^{3} \alpha^i \, \mathcal{D}_i^{(k+1)} + \sum_{|\sigma| \leq k} \left(\mathcal{D}_\sigma(\phi_u)\partial_{u_\sigma} + \mathcal{D}_\sigma(\phi_v)\partial_{v_\sigma}\right). \tag{5}$$

Here σ is a multi-index, $\mathcal{D}_\sigma$ is the iterated total derivative, $\mathcal{D}_i^{(k+1)}$ is the truncated total derivative as a derivation on k-jets[1], $\alpha^i = dx^i(X)$ with the notation $(x^1, x^2, x^3) = (t, x, y), u_\sigma = u_{x^\sigma}, v_\sigma = v_{x^\sigma}$, and the functions $\phi_u = \omega_u(X)$, $\phi_v = \omega_v(X)$ are components of the generating section $\phi = (\phi_u, \phi_v)$ for X, where

$$\omega_u = du - u_t dt - u_x dx - u_y dy, \qquad \omega_v = dv - v_t dt - v_x dx - v_y dy.$$

Below we denote by $Y_i^k(m) = \frac{1}{m!} X_i^{(k)}(t^m)$ for $i = 1, ..., 5$, the vector fields on $\mathcal{E}_k$. Consider first the vector field $X_1(a)$. Its generating section is

$$\phi_1 = (-u_x a(t),\ \dot{a}(t) - v_x a(t)).$$

This together with (5) implies that the vector $X_1^{(k)}(a)_{\theta_k}$ depends only on $a(0), ..., a^{(k+1)}(0)$. Therefore $\text{span}\{X_1^{(k)}(a)_{\theta_k} : a \in C^\infty(\mathbb{R})\} = \text{span}\{Y_1^k(m)_{\theta_k} : m = 0, \ldots, k+1\}$.

[1]The truncated total derivative is given by $\mathcal{D}_i^{(k+1)} = \partial_{x^i} + \sum_{|\sigma| \leq k}(u_{\sigma i}\partial_{u_\sigma} + v_{\sigma i}\partial_{v_\sigma})$.

Repeating this argument for $X_2(b), ..., X_5(e)$ we conclude that the subspace $T_{\theta_k}\mathcal{O}_{\theta_k} \subset T_{\theta_k}\mathcal{E}_k$ is spanned by

$$V_k = \{Y_1^k(m), Y_2^k(m), Y_3^k(n), Y_4^k(m), Y_5^k(n) : m \leq k+1, n \leq k\} \tag{6}$$

evaluated at θ_k. This gives the upper bound $5k+8 = |V_k|$ for $\dim \mathcal{O}_k$. (For $k = 0, 1$ the orbit dimension is bounded even more by $\dim \mathcal{E}_0 = 5$ and $\dim \mathcal{E}_1 = 11$.)

We use induction to show that there exist orbits of dimension $5k+8$ for $k \geq 2$. Due to lower semicontinuity of matrix rank, an orbit in general position will then also have the same dimension. We choose θ_k to be given by $u_x = 1, u_{xx} = 1$ and all other jet-variables set to 0. For the induction step assume that all vectors in the set V_k are independent, and hence $\dim \mathcal{O}_{\theta_k} = 5k+8$. For $k = 2$ this is easily verified in Maple. The five vectors

$$Y_1^{k+1}(k+2)_{\theta_{k+1}} = \partial_{v_{t^{k+1}}}, \qquad Y_2^{k+1}(k+2)_{\theta_{k+1}} = \partial_{u_{t^{k+1}}},$$
$$Y_3^{k+1}(k+1)_{\theta_{k+1}} = \partial_{v_{t^k y}} - 2\partial_{u_{t^{k+1}}}, \qquad Y_4^{k+1}(k+2)_{\theta_{k+1}} = \frac{1}{2}\partial_{u_{t^k y}},$$
$$Y_5^{k+1}(k+1)_{\theta_{k+1}} = -3\partial_{u_{t^k y}} + 2\partial_{v_{t^k x}} + 2\partial_{v_{t^{k-1} y^2}}$$

are independent and tangent to the fiber of S_{k+1} over $\theta_k \in S_k$. Therefore they are independent with the prolonged vector fields from V_k at θ_{k+1}. Thus $\dim \mathcal{O}_{\theta_{k+1}} = 5k+8+5 = 5(k+1)+8$, completing the induction step and the proof. □

2.3 Shape-preserving transformations

The ansatz (3) for Einstein-Weyl structures on M is not invariant under arbitrary local diffeomorphisms of M, and we want to determine the pseudogroup preserving this shape of (g, ω). Its Lie algebra sheaf is given as follows.

Theorem 32. *The Lie algebra $\mathfrak{h}$ of vector fields preserving shape (3) of (g, ω) has the following generators, involving five arbitrary functions $a = a(t), ..., e = e(t)$:*

$$a\partial_x, \quad b\partial_y, \quad yc\partial_x, \quad d\partial_t + \frac{1}{2}\dot{d}y\partial_y, \quad (y^2\dot{e} + 2xe)\partial_x + ye\partial_y. \tag{7}$$

Proof. Let $X = \alpha(t,x,y)\partial_t + \beta(t,x,y)\partial_x + \gamma(t,x,y)\partial_y$ be a vector field on M preserving the shape of (g, ω), and φ_τ its flow. The pullback of g through φ_τ has the same shape, up to a conformal factor f^τ, so that

$$\varphi_\tau^* g = f^\tau(4dtdx + 2u^\tau dtdy - ((u^\tau)^2 + 4v^\tau)dt^2 - dy^2),$$

where f^τ, u^τ, v^τ are τ-parametric functions of t, x, y with $f^0 = 1, u^0 = u, v^0 = v$. Denote $\chi = \frac{d}{d\tau}\big|_{\tau=0} f^\tau, \mu = \frac{d}{d\tau}\big|_{\tau=0} u^\tau, \nu = \frac{d}{d\tau}\big|_{\tau=0} v^\tau$. Then the Lie derivative is

$$L_X g = \chi g + 2\mu dt dy - (2u\mu + 4\nu) dt^2.$$

Similarly, from $\varphi_\tau^* \omega = \omega^\tau + d \log f^\tau$, we obtain the formula

$$L_X \omega = (u_x \mu + u \mu_x + 2\mu_y + 4\nu_x) dt - \mu_x dy + d\chi.$$

These restrictions yield an overdetermined system of differential equations on α, β and γ whose solutions give exactly the vector fields (7). □

The Lie algebra $\mathfrak{h}$ of vector field on M can be naturally lifted to the Lie algebra $\hat{\mathfrak{h}}$ on the total space E. Let $X \in \mathfrak{h}$. Its lift $\hat{X} = X + A\partial_u + B\partial_v \in \hat{\mathfrak{h}}$ is computed as follows. The pullback of g to E is a horizontal symmetric two-form $\hat{g}$. Then the condition $L_{\hat{X}} \hat{g} = \chi \hat{g}$ uniquely determines the coefficients A, B.

Applying this to the general vector field $X = 2d\partial_t + (a + yc + 2xe + y^2\dot{e})\partial_x + (b + y\dot{d} + ye)\partial_y \in \mathfrak{h}$ we get $\chi = 2(e(t) + \dot{d}(t))$. Moreover for the pullback $\hat{\omega}$ of ω and the prolongation of the vector field $\hat{X}$ we get $L_{\hat{X}^{(1)}} \hat{\omega} = d\chi$. Comparing the resulting A and B with the vector fields in Theorem 30, we conclude:

Corollary 1. *The lift $\hat{\mathfrak{h}}$ of the Lie algebra $\mathfrak{h}$ of shape-preserving vector fields is exactly the Lie algebra $\mathfrak{g}$ of point symmetries of $\mathcal{E}$.*

Let us reformulate our lift of the algebra $\mathfrak{h}$ using integrability of system (4). Its Lax pair is given by a rank 2 distribution $\tilde{\Pi}^2 = \operatorname{span}\{\partial_y - \lambda\partial_x + n\partial_\lambda, \partial_t - (\lambda^2 - u\lambda - v)\partial_x + m\partial_\lambda\}$ on $\mathbb{P}^1$-bundle $\tilde{M}$ over M, which is Frobenius-integrable in virtue of (4) (the form of m, n is not essential here, see [7]). The fiber can be identified with the projectivized null-cone of g. The coordinate λ along it is called the spectral parameter. The action of $\mathfrak{h}$ on M induces the action on $\tilde{M}$ and hence on $\tilde{\Pi}^2$. Since the plane $\tilde{\Pi}^2_{(t,x,y,\lambda)}$ is projected to the plane $\Pi^2 = \operatorname{Ann}(dx + \lambda dy + (\lambda^2 - u\lambda - v)dt)$, this in turn gives the action on u, v, i.e. the required lift.

3 Differential invariants of $\mathcal{E}$

In this section we determine generators of the field of scalar rational differential invariants of the equation $\mathcal{E}$ with respect to its symmetry pseudogroup $\mathcal{G}$. We also compute the Poincaré function of the $\mathcal{G}$-action, counting moduli of the problem, and discuss solution of the equivalence problem for Einstein-Weyl structures written in form (3).

3.1 Hilbert polynomial and Poincaré function

The number s_k of independent differential invariants of order k is equal to the codimension of a generic orbit $\mathcal{O}_k \subset \mathcal{E}_k$. Since, as in Section 1.1, rational differential invariants of $\mathcal{G}$ coincide with those of $\mathfrak{g}^{(k)}$, we can compute s_k using the results from Section 2.2:

$$s_k = \dim \mathcal{E}_k - \dim \mathcal{O}_k = 2k^2 - k - 3, \qquad k \geq 2$$

Due to local transitivity $s_0 = s_1 = 0$.

The difference $h_k = s_k - s_{k-1}$ counts the number of invariants of "pure" order k. It is given as follows: $h_0 = h_1 = 0$, $h_2 = 3$ and $h_k = 4k - 3$ for $k > 2$. The Hilbert polynomial is the stable value of h_k: $H(k) = 4k - 3$.

These numbers can be compactified into the Poincaré function:

$$P(z) = \sum_{k=0}^{\infty} h_k z^k = \frac{(3 + 3z - 2z^2)z^2}{(1-z)^2}.$$

3.2 Invariant derivations and differential invariants

All objects we treat in this section will be written in terms of ambient coordinates on $J^k\pi \supset \mathcal{E}_k$.

From the previous section, we know that there exist three independent rational differential invariants of order two. The second-order invariants are generated by

$$I_1 = \frac{u_{xy} + v_{xx}}{u_x^2}, \quad I_2 = \frac{u_x^2 u_{xy} + u_x u_{xx} v_x + u_{xx} u_{yy} - u_{xy}^2}{u_x^4},$$
$$I_3 = \frac{u_x^2 v_{xx} - u_x u_{xx} v_x + u_{xx} v_{xy} - u_{xy} v_{xx}}{u_x^4}.$$

In order to generate all differential invariants, we also need invariant derivations. These are derivations on the algebra of differential invariants commuting with $\mathcal{G}$. It is easily checked that

$$\nabla_1 = \frac{u_x}{u_{xx}} D_x, \qquad \nabla_2 = \frac{1}{u_x}\left(\frac{u_{xy}}{u_{xx}} D_x - D_y\right),$$
$$\nabla_3 = \frac{1}{u_x^3}\Big(u_{xx} D_t + ((v u_x)_x + u_{yy})\, D_x + (u u_x - 2u_y)_x\, D_y\Big)$$

are three independent invariant derivations. Their commutation relations are given by

$$[\nabla_1, \nabla_2] = -\nabla_2, \qquad [\nabla_1, \nabla_3] = -K_3 \nabla_1 + (K_1 - 2K_2)\nabla_2 + K_1 \nabla_3,$$
$$[\nabla_2, \nabla_3] = K_4 \nabla_1 + K_3 \nabla_2 + K_2 \nabla_3,$$

where

$$
\begin{aligned}
K_1 &= \frac{u_x u_{xxx}}{u_{xx}^2} - 3 = \nabla_1(\log(u_{xx})) - 3,\\
K_2 &= \frac{u_{xy}u_{xxx} - u_{xx}u_{xxy}}{u_x u_{xx}^2} = \nabla_2(\log(u_{xx})),\\
K_3 &= K_2\left(1 - 2\frac{u_{xy}}{u_x^2}\right) - 2\frac{u_{xx}}{u_x^3}\nabla_2(u_y) + \frac{2}{u_x^2}\nabla_2(u_{xy}),\\
K_4 &= \frac{1}{u_x^4}\left(u_{xx}\nabla_2(2u_{yy} - u_x u_y) - \nabla_2(u_{xy}/u_{xx})u_{xx}(2u_{xy} - u_x^2) - \nabla_2(u_{xy}^2)\right)
\end{aligned}
$$

are independent differential invariants of the third order.

The nine third-order differential invariants $\nabla_j(I_i)$ are independent, and together with I_1, I_2, I_3 they generate all differential invariants of order three. In particular, $K_1, \dots, K_4$ can be expressed through them.

Moreover, I_1, I_2, I_3 and $\nabla_1, \nabla_2, \nabla_3$ generate all rational scalar differential invariants of the $\mathcal{G}$-action on $\mathcal{E}$.

3.3 EW structure written in invariant coframe

The invariant derivations $\nabla_1, \nabla_2, \nabla_3$ constitute a horizontal frame on an open subset in $\mathcal{E}_2$. Let $\alpha^1, \alpha^2, \alpha^3$ be the dual horizontal coframe. The 1-forms α^i are defined at all points where $u_{xx} \neq 0$. Since $\alpha^1 \wedge \alpha^2 \wedge \alpha^3 = -u_x^3 dt \wedge dx \wedge dy$, they determine a horizontal coframe outside the singular set $\Sigma_2 = \{u_x = 0, u_{xx} = 0\} \subset \mathcal{E}_2$.

In $\mathcal{E}_2 \setminus \Sigma_2$ we can rewrite g and ω in terms of the coframe $\alpha_1, \alpha_2, \alpha_3$. Then $g = g_{ij}\alpha^i\alpha^j$ and $\omega = \omega_i\alpha^i$, where $g_{ij} = g(\nabla_i, \nabla_j)$ and $\omega_i = \omega(\nabla_i)$. After rescaling the metric by a factor of u_x^2, we get the following expression.

$$
\begin{aligned}
g' &= 4\alpha^1\alpha^3 - \alpha^2\alpha^2 + 2\alpha^2\alpha^3 + (4I_2 - 1)\alpha^3\alpha^3,\\
\omega' &= 2\alpha^1 + \alpha^2 + (4I_2 - 1)\alpha^3.
\end{aligned}
$$

Thus, given any Einstein-Weyl structure whose 2-jet is in the complement of Σ_2 we may rewrite it in the form (g', ω'), and we see that this expression only depends on $\alpha^1, \alpha^2, \alpha^3$ and I_2. A consequence of these computations is the following theorem.

Theorem 33. *The field of rational $\mathfrak{g}$-differential invariants on $\mathcal{E}$ is generated by the differential invariant I_2 together with the invariant derivations $\nabla_1, \nabla_2, \nabla_3$.*

The reason that we are able to generate the rest of the second-order differential invariants from these is that some algebraic combinations of the higher-order invariants will be of lower order. In particular, we have the following identities relating I_1, I_3 to the invariants K_i from the commutation relations of the invariant derivations.

$$I_1 = \nabla_1(I_2) + \frac{K_2 + K_3}{2} - I_2 K_1,$$
$$I_3 = (\nabla_1 - \nabla_2)(I_2) + \frac{K_2 + 3K_3 + 2K_4}{4} + I_2(K_2 - K_1 - 1).$$

3.4 The equivalence-problem of Einstein-Weyl structures

By Theorem 34 from the appendix and the global Lie-Tresse theorem [18] the field of differential invariants separates generic orbits on $\tilde{\mathcal{E}} = \mathcal{E}_\infty \setminus \pi_{\infty,\ell}^{-1}(S)$ for some Zariski closed invariant subset $S \subset \mathcal{E}_\ell$. Therefore, the description of the field of differential invariants is sufficient for describing the quotient equation $\tilde{\mathcal{E}}/\mathcal{G}$.

In order to finish a description of the field of differential invariants one must find the (differential) syzygies in the differential field of scalar invariants. Since all invariants are rational this can be done by brute force. Using $\nabla_1, \nabla_2, \nabla_3, I_1, I_2, I_3$ as the generating set of the field of invariants, a simple computation with the DifferentialGeometry package of Maple shows that the twelve invariants $I_k, \nabla_i(I_j)$ are functionally independent, so there are no syzygies on this level. There are five polynomial relations between $I_i, \nabla_j(I_i), \nabla_k\nabla_j(I_i)$. Due to their length the expressions are not reproduced here, but they can be found in the Maple file ancillary to the arXiv version of this paper.

There is another way to describe the quotient equation in our case, using the same approach as [20] and [19]. Take three independent differential invariants J_1, J_2, J_3 of order k (for instance I_1, I_2, I_3). Their horizontal differentials $\hat{d}J_1, \hat{d}J_2, \hat{d}J_3$ determine a horizontal coframe on $\mathcal{E}_\ell \setminus S$ for some Zariski closed subset $S \subset \mathcal{E}_\ell$, $\ell > k$. It is then possible, in the same way as in Section 3.3, to rewrite the Einstein-Weyl structure in terms of this coframe:

$$g' = \sum G_{ij} \hat{d}J_i \hat{d}J_j, \qquad \omega' = \sum \Omega_i \hat{d}J_i.$$

For one of the nonzero coefficients G_{ij} we may, after rescaling the metric, assume that $G_{ij} = 1$. The quotient equation $(\mathcal{E}_\infty \setminus \pi_{\infty,\ell}^{-1}(S))/\mathcal{G}$ is obtained by adding to the Einstein-Weyl equation on $\mathbb{R}^3(x_1, x_2, x_3)$ the equations $\{J_i = x_i\}_{i=1}^3$.

For practical purposes the following approach solves the local equivalence problem for Einstein-Weyl structures of the form (3), using the idea of a signature manifold [4]. Let I_1, I_2, I_3 be the basic invariants and $I_{ij} = \nabla_j(I_i)$

their derivations. For a section $s \in \Gamma(\pi)$ let $\mathcal{S}_s \subset \mathbb{R}^{12}(z)$ be the image of the map

$$M \ni x \mapsto \big(z_1 = I_1(j_2(s))(x), \dots, z_4 = I_{11}(j_3(s))(x), \dots, z_{12} = I_{33}(j_3(s))(x)\big).$$

For generic s the manifold $\mathcal{S}_s$ is three-dimensional; it is called the signature of s. If, in addition, the Einstein-Weyl structure s is given by algebraic functions, then $\mathcal{S}_s$ is an algebraic manifold and it can be defined by polynomial equations.

Let us call a section s I-regular if $\hat{d}I_i|_s$ are defined and $(\hat{d}I_1 \wedge \hat{d}I_2 \wedge \hat{d}I_3)|_s \neq 0$. The invariant derivations ∇_j can be reconstructed from the twelve invariants I_k, I_{ij}, which in turn determine all other differential invariants. Therefore two I-regular sections s_1, s_2 of π are equivalent if and only if their signatures coincide. In the algebraic case this is equivalent to equality of the corresponding polynomial ideals, and so this can be decided algorithmically.

4 Some particular Einstein-Weyl spaces

Symmetries can be used to find invariant solutions of differential equations. They can be also used to obtain explicit non-symmetric solutions: use a differential constraint consisting of several differential invariants and solve the arising overdetermined system. In this setup the solutions come in a family, invariant under the symmetry group action, so in examples below we normalize them using $\mathcal{G}$ to simplify the expressions. Since use of symmetry gives a differently looking solution, but an equivalent Einstein-Weyl space, the generality does not suffer.

1. We begin with the only relative invariant of order 1: $u_x = 0$. This coupled with equation $F_1 = 0$ gives $u_{yy} = 0$, so $u = a(t)y + b(t)$. This can be transformed to $u = 0$ by our pseudogroup $\mathcal{G}$. Then the second equation $F_2 = 0$ becomes the dispersionless Kadomtsev-Petviashvili (dKP), also known as the Khokhlov-Zabolotskaya equation in 1+2 dimensions [15, 14]:

$$v_{tx} + v_x^2 + v v_{xx} - v_{yy} = 0.$$

This equation is integrable and has been extensively studied, see e.g. [22, 8].

Note that the orbit in $\mathcal{E}_2$ of lowest dimension, given by $\{u_x = 0, u_{tx} = 0, u_{xx} = 0, u_{xy} = 0, v_{xx} = 0, v_{xy} = 0\}$, leads to the solution

$$u = f_1(t) + f_2(t)y,\ v = f_3(t) + f_4(t)x + f_5(t)y + \tfrac{1}{2}(f_4(t)^2 + 2f_2(t)f_4(t) + \dot{f}_4(t))y^2$$

which is $\mathcal{G}$-equivalent to $(u, v) \equiv (0, 0)$.

2. Consider the special value of the first invariant $I_1 = 0$. The arising system $u_{xy} + v_{xx} = 0$ has a solution $u = w_x, v = -w_y$. Substitution of this into the modified Manakov-Santini system reduces it to the prolongation of the first equation from the universal hierarchy of Martínez Alonso and Shabat [21]:

$$w_{tx} + w_x w_{xy} - w_y w_{xx} - w_{yy} = 0. \tag{8}$$

In fact, the equations $F_1 = 0$ and $F_2 = 0$ are x- and y-derivatives of the left-hand side F of (8), so we get the PDE $F = f(t)$ and the function $f(t)$ can be eliminated by a point transformation.

Equation (8) possesses a Lax pair and so is integrable by the inverse scattering transform. Its hierarchy carries an involutive $GL(2)$-structure [11], and so is also integrable by twistor methods. The method of hydrodynamic reductions [9] can be exploited to obtain solutions w involving arbitrary functions of one argument.

3. Consider a stronger ansatz for the modified Manakov-Santini equation: $I_1 = 0$, $I_2 = 0$, $I_3 = 0$, in addition to $F_1 = F_2 = 0$. This overdetermined system can be analyzed by the rifsimp package of Maple. The main branch is equivalent to the constraint $u_{xx} = 0, u_{xy} = 0, v_{xx} = 0$. This can be explicitly solved.

Modulo the pseudogroup $\mathcal{G}$ the general solution to this system is

$$u = x + e^y, \qquad v = f(t) + h(t)e^{-y}.$$

Degenerations include the solution

$$u = 0, \qquad v = \tfrac{1}{12}y^4 + xy + h(t)$$

which is a partial solution to the dKP.

4. Finally, consider an ansatz obtained by the requirement that all structure coefficients $K_1, \dots, K_4$ of the frame ∇_i on $\mathcal{E}_\infty$ and the coefficient I_2, arising in the expression of (g, ω), are constants.

By the last formulae in §3.3 this case corresponds to constancy of all differential invariants obtained from I_1, I_2, I_3 by ∇_i-derivations. Also note that in this case $(\nabla_1, \nabla_2, \nabla_3)$ form a 3-dimensional Lie algebra $\mathfrak{s}$.

The obtained system $F_1 = 0, F_2 = 0, K_1 = k_1, K_2 = k_2, K_3 = k_3, K_4 = k_4, I_2 = c$ is inconsistent for generic parameters in the right-hand sides. Using the differential syzygies between the invariants and derivations, we further constrain those values. The obtained system can be solved in Maple.

Let us restrict to the case, when the corresponding algebra $\mathfrak{s} = \mathfrak{sl}(2, \mathbb{R})$ (otherwise $\mathfrak{s}$ is solvable). This corresponds to very particular values of the parameters: $I_1 = -\frac{3}{25}$, $I_2 = \frac{21}{100}$, $I_3 = -\frac{147}{500}$, $K_1 = 1$, $K_2 = 0$, $K_3 = \frac{9}{50}$, $K_4 = -\frac{9}{500}$.

Modulo the pseudogroup $\mathcal{G}$ the general solution to this system is

$$u = y^{2/3} - \tfrac{10}{3}xy^{-1}, \quad v = \tfrac{2}{5}xy^{-1/3} - \tfrac{7}{3}x^2y^{-2} + \tfrac{21}{25}y^{4/3} + (f(t)y^{1/3} + h(t))y^2.$$

A degeneration of this family gives the following family of solutions

$$u = -\tfrac{10}{3}xy^{-1}, \quad v = -\tfrac{7}{3}x^2y^{-2} + (f(t)y^{1/3} + h(t))y^2.$$

It shall be noted that we have essentially quotiented out the pseudogroup $\mathcal{G}$ (only the translation by t remains in the latter cases) because we integrated $\mathfrak{g}$ explicitly and have found a convenient cross-section of the action.

In the general case, when we impose an invariant differential constraint, the family of solutions can keep $\mathcal{G}$-invariance and the separation of generic solutions can be done using the differential invariants obtained in §3.4.

Appendix: Symmetry of algebraic PDEs

Let $\mathcal{E} \subset J^\infty\pi$ be a differential equation. It is called algebraic if for every $a \in E = J^0\pi$ and every $k \in \mathbb{N}$ the fiber $\mathcal{E}_a^k \subset J_a^k\pi$ is an algebraic variety (maybe reducible). Here we use the natural algebraic structure in the fibers $J^k\pi \to E$.

Note that the definition of algebraic pseudogroup in [18] used an assumption that $\mathcal{G}$ acts transitively on $J^0\pi$. For instance, this is the case if the bundle is trivial $\pi : E = \mathbb{R}^n(x) \times \mathbb{R}^m(u) \to \mathbb{R}^n(x)$ and the defining equations of $\mathcal{E}$ do not depend on x, u. It is also the case for the modified Manakov-Santini system (4). We will not however rely on it in the proof below.

Theorem 34. *The symmetry pseudogroup $\mathcal{G}$ of an algebraic differential equation $\mathcal{E}$ is algebraic. This means that the defining Lie equations of $\mathcal{G}$ are algebraic.*

In this formulation, by symmetries we mean either point or contact symmetries. The statement holds true also for mixed point-contact symmetries, as the ones appearing in the Bäcklund type theorem in [3], and can be extended for generalized symmetries as those considered in [2, 16].

Proof. Without loss of generality we can assume $\mathcal{E}$ to be formally integrable, because addition of compatibility conditions does not change the symmetry. The differential ideal $I_\mathcal{E}$ of the equation $\mathcal{E}$ is filtered by ideals $I_\mathcal{E}^i$ of functions on $J^i\pi$, and it is completely determined by $I^k_{\mathcal{E}}$ for some k. By the assumption there exist generators $F_1, \dots, F_r$ of $I^k_{\mathcal{E}}$ that are algebraic in jet-variables u_σ, $|\sigma| > 0$, over any point $a = (x, u) \in E$, and from now on we restrict to a single point $a \in E$.

Let $\varphi : E \to E$ be a local diffeomorphism (point transformation) with $\varphi(a) = a$. It is a symmetry if $\varphi^* F_i \in I^k_{\mathcal{E}}$ for every $i = 1, \dots, r$, where we tacitly omitted the notation for prolongation. For each i, the membership problem is algorithmically solvable by the Gröbner basis method, and the condition for membership is a set of algebraic relations. Unite those by i. Decompose each relation by all jet-variables u_σ, $|\sigma| > 0$ and collect the coefficients. This gives a finite number of algebraic differential equations on the components of φ. Their orders do not exceed the maximal order of F_i, because of the prolongations of φ involved. This is the set of Lie equations defining $\mathcal{G}$, and the claim follows.

In the case of a contact diffeomorphism $\varphi : J^1\pi \to J^1\pi$, when u is one-dimensional, the decomposition has to be done with respect to u_σ, $|\sigma| > 1$. The rest of arguments is the same. □

Remark 10. *The algebraic property involves only behavior with respect to the jet-variables and shall not be confused with total algebraicity. For instance, the linear equation $y''(x) = y(x)$ has only one algebraic solution $y = 0$. The symmetry group is 8-dimensional and it contains shifts by solutions $y \mapsto y + a\, e^{-x} + b\, e^x$ that are not algebraic in x. The symmetry pseudogroup $G = \{x \mapsto A(x, y), y \mapsto B(x, y)\}$ has the following defining equations for G^2_o at $o = (0, 0)$, which are manifestly algebraic:*

$$A_x B_{xx} - B_x A_{xx} = A_x^3 B, \quad A_y B_{xx} + 2A_x B_{xy} - B_y A_{xx} - 2B_x A_{xy} = 3A_x^2 A_y B,$$
$$A_y B_{yy} - B_y A_{yy} = A_y^3 B, \quad 2A_y B_{xy} + A_x B_{yy} - 2B_y A_{xy} - B_x A_{yy} = 3A_x A_y^2 B.$$

Similar situation is also with other algebraic differential equations, like Painlevé transcendents, hypergeometric equation etc. In fact, according to a theorem of Sophus Lie, all linear second order ODEs are locally equivalent, in particular, they have an 8-dimensional point symmetry algebra. For the hyper-geometric equation, the generators of this algebra express by the solutions of the equation, yet the defining equation is algebraic.

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Differential invariants of Kundt waves

Boris Kruglikov, David McNutt, Eivind Schneider

Abstract

Kundt waves belong to the class of spacetimes which are not distinguished by their scalar curvature invariants. We address the equivalence problem for the metrics in this class via scalar differential invariants with respect to the equivalence pseudo-group of the problem. We compute and finitely represent the algebra of those on the generic stratum and also specify the behavior for vacuum Kundt waves. The results are then compared to the invariants computed by the Cartan-Karlhede algorithm.

Introduction

The Kundt waves can be written in local coordinates as follows

$$g = dx^2 + dy^2 - du\Big(dv - \tfrac{2v}{x}\,dx + \big(8xh - \tfrac{v^2}{4x^2}\big)\,du\Big), \qquad (1)$$

where $h = h(x, y, u)$ is an arbitrary function. In order for g to be vacuum, h must be harmonic in x, y. These metrics were originally defined by Kundt [1] in 1961, as a special class of pure radiation spacetimes of Petrov type III or higher, admitting a non-twisting, non-expanding shear-free null congruence ℓ [2]: $g(\ell, \ell) = 0$, $\mathrm{Tr}_g(\nabla\ell) = 0$, $\|\nabla\ell\|_g^2 = 0$.

All Weyl curvature invariants [3], i.e. scalars constructed from tensor products of covariant derivatives of the Riemann curvature tensor by complete contractions, vanish for these spacetimes. Thus, these plane-fronted metrics belong to the collection of VSI spacetimes, where all polynomial scalar curvature invariants vanish [4]. These spaces have been extensively explored in the literature [5, 6].

Since it is impossible to distinguish Kundt waves from Minkowski spacetime by Weyl curvature invariants, other methods have been applied. In [7] Cartan invariants have been computed for vacuum Kundt waves and the maximum iteration steps in Cartan-Karlhede algorithm was determined. Cartan invariants allow to distinguish all metrics, but initially they are functions on the Cartan bundle, also known as the orthonormal frame bundle, not on the original spacetime.

Cartan invariants are polynomials in structure functions of the canonical frame (Cartan connection) and their derivatives along the frame [8]. Thus they are obtained from the components of the Riemann curvature tensor and its covariant derivatives without complete contractions. Absolute invariants are chosen among those that are invariant with respect to the structure group of the Cartan bundle. This is usually achieved by a normalization of the group parameters [8, 9].

When the frame is fixed (the structure group becomes trivial) the Cartan invariants descend to the base of the Cartan bundle, i.e. the spacetime (in some cases, which we do not consider, the frame cannot be completely fixed but then the form of the curvature tensor and its covariant derivatives are unaffected by the frame freedom). The Cartan-Karlhede algorithm [10, 2] specifies when the normalization terminates and how many derivatives of the curvature along the frame are involved in the final list of invariants.

In this paper we propose another approach, which originates from the works of Sophus Lie. Namely we distinguish spacetimes by scalar differential invariants of their metrics. The setup is different: we first determine the equivalence group of the problem that is the group preserving the class of metrics under consideration. It is indeed infinite-dimensional and local, so it is more proper to talk of a Lie pseudogroup, or its Lie algebra sheaf. Then we compute invariants of this pseudogroup and its prolonged action. The invariants live on the base of the Cartan bundle, i.e. the spacetime, but they are allowed to be rational rather than polynomial in jet-variables (derivatives of the metric components). We recall the setup in Section 1.

Recently [12] it was established that the whole infinite-dimensional algebra of invariants can be finitely generated in Lie-Tresse sense. This opens up an algebraic approach to the classification, and that is what we implement here. We compute explicitly the generating differential invariants and invariant derivations, organize their count in Poincaré series, and resolve the equivalence problem for generic metrics within the class. We also specify how this restricts to vacuum Kundt waves. This is done in Sections 2-3. More singular spaces can be treated in a manner analogous to our computations.

Since vacuum Kundt waves have already been investigated via the Cartan method [7], we include a discussion on the correspondence of the invariants in this case. This correspondence does not preserve the order of invariants, because the approaches differ, and we include a general comparison of the two methods. This is done in Section 4.

1 Setup of the problem: actions and invariants

Metrics of the form (1) are defined on an open subset of the manifold $M = (\mathbb{R}\setminus\{0\})\times\mathbb{R}^3 \subset \mathbb{R}^4$. Thus a metric g can be identified as a (local) section of the bundle $\pi\colon M\times\mathbb{R}\to M$ with the coordinates x, y, u, v, h. We denote the total space of the bundle by E. The Kundt waves then satisfy the condition $h_v = 0$. This partial differential equation (PDE) determines a hypersurface $\mathcal{E}_1$ in $J^1\pi$.

Here $J^k\pi$ denotes the k-th order jet bundle. This space is diffeomorphic to $M \times \mathbb{R}^N$, where $N = \binom{k+4}{4}$, and we will use the standard coordinates $h, h_x, h_y, ..., h_{uv^{k-1}}, h_{v^k}$ on $\mathbb{R}^N$. Function $h = h(x, y, u, v)$ determines the section $j^k h$ of $J^k\pi$ in which those standard coordinates are the usual partial derivatives of h.

The space $J^k\pi$ comes equipped with a distribution (a sub-bundle of the tangent bundle), called the Cartan distribution. A PDE of order k is considered as a submanifold of $J^k\pi$, and its solutions correspond to maximal integral manifolds of the Cartan distribution restricted to the PDE. For a detailed review of jets, we refer to [9, 11]. The prolongation $\mathcal{E}_k \subset J^k\pi$ is the locus of differential corollaries of the defining equation of $\mathcal{E}_1$ up to order k. We also let $\mathcal{E}_0 = J^0\pi = E$.

The vanishing of the Ricci tensor is equivalent to the condition $h_{xx} + h_{yy} = 0$. This yields a sub-equation $\mathcal{R}_2 \subset \mathcal{E}_2 \subset J^2\pi$, whose prolongations we denote by $\mathcal{R}_k \subset J^k\pi$. Since this case of vacuum Kundt waves was considered thoroughly in [7] we will focus here mostly on general Kundt waves. However, after finding the differential invariants in the general case it is not difficult to describe the differential invariants in the vacuum case. This will be done in Section 3.

1.1 Lie pseudogroup

The Lie pseudogroup of transformations preserving the shape (i.e. form of the metric) can be found by pulling back g from (1) through a general transformation $(\tilde{x}, \tilde{y}, \tilde{u}, \tilde{v}) \mapsto (x, y, u, v)$, and then requiring that the obtained metric is of the same shape:

$$d\tilde{x}^2 + d\tilde{y}^2 - d\tilde{u}\left(d\tilde{v} - \tfrac{2\tilde{v}}{\tilde{x}}\, d\tilde{x} + \left(8\tilde{x}\tilde{h} - \tfrac{\tilde{v}^2}{4\tilde{x}^2}\right) d\tilde{u}\right).$$

This requirement can be given in terms of differential equations on x, y, u, v as functions of $\tilde{x}, \tilde{y}, \tilde{u}, \tilde{v}$, with the (invertible) solutions described below. The obtained differential equations are independent of whether the Kundt wave is Ricci-flat or not, so the shape-preserving Lie pseudogroup is the same for both general and Ricci-flat Kundt waves.

A pseudogroup preserving shape (1) contains transformations of the form (we also indicate their lift to $J^0\pi = E$)

$$x \mapsto x, \quad y \mapsto y + C, \quad u \mapsto F(u), \quad v \mapsto \frac{v}{F'(u)} - 2\frac{F''(u)}{F'(u)^2}x^2, \tag{2}$$

$$h \mapsto \frac{h}{F'(u)^2} + \frac{2F'''(u)F'(u) - 3F''(u)^2}{8F'(u)^4}x, \tag{3}$$

where F is a local diffeomorphism of the real line, i.e. $F'(u) \neq 0$. This Lie pseudogroup was already described in [4], formula (A.37).

Transformations (2)-(3) form the Zariski connected component $\mathcal{G}_0$ of the entire Lie pseudogroup $\mathcal{G}$ of shape-preserving transformations. (Note that $\mathcal{G}_0$ differs from the topologically connected component of unity given by $F'(u) > 0$.) The pseudogroup $\mathcal{G}$ is generated, in addition to transformations (2)-(3), by the maps $y \mapsto -y$ and $(x, h) \mapsto (-x, -h)$ preserving shape (1). Note that $\mathcal{G}/\mathcal{G}_0 = \mathbb{Z}_2 \times \mathbb{Z}_2$.

The Lie algebra sheaf $\mathfrak{g}$ of vector fields corresponding to $\mathcal{G}$ (and $\mathcal{G}_0$) is spanned by the vector fields

$$X = \partial_y, \quad Y(f) = 4f\partial_u - (4vf' + 8x^2f'')\partial_v + (xf''' - 8hf')\partial_h \tag{4}$$

where $f = f(u) \in C^\infty_{\text{loc}}(\mathbb{R})$ is an arbitrary function.

When looking for differential invariants, it is important to distinguish between $\mathcal{G}$ and $\mathcal{G}_0$. Firstly, differential $\mathcal{G}_0$-invariants need not be $\mathcal{G}$-invariant. Secondly, a set of differential invariants that separates $\mathcal{G}$-orbits as a rule will not separate $\mathcal{G}_0$-orbits. We will restrict our attention to the $\mathcal{G}$-action while outlining the changes needed to be made for the other choices of the Lie pseudogroup.

1.2 Differential invariants and the global Lie-Tresse theorem

A differential invariant of order k is a function on $\mathcal{E}_k$ which is constant on orbits of $\mathcal{G}$. In accordance with [12] we consider only invariants that are rational in the fibers of $\pi_k : \mathcal{E}_k \to E$ for every k.

The global Lie-Tresse theorem states that for algebraic transitive Lie pseudogroups, rational differential invariants separate orbits in general position in $\mathcal{E}_\infty$ (i.e. orbits in the complement of a Zariski-closed subset), and the field of rational differential invariants is generated by a finite number of differential invariants and invariant derivations. In fact it suffices to consider the (sub)algebra of invariants that are rational on fibers of $\pi_\ell : \mathcal{E}_\ell \to E$ and polynomial on fibers of $\pi_{k,\ell} : \mathcal{E}_k \to \mathcal{E}_\ell$ for some ℓ. In the case of Kundt waves we will show that $\ell = 2$. For simplicity we will mostly discuss the field of rational invariants in what follows.

We refer to [12] for the details of the theory which holds for transitive Lie pseudogroups. The Lie pseudogroup we consider is not transitive: the $\mathcal{G}$-orbit foliation of E is $\{x = \text{const}\}$. Let us justify validity of a version of the Lie-Tresse theorem for our Lie pseudogroup action.

For every $a \in E$ the action of the stabilizer of a in $\mathcal{G}_0$ is algebraic on the fiber $\pi_{\infty,0}^{-1}(a)$, and so for every k and a we have an algebraic action of a Lie group on the algebraic manifold of $\pi_{k,0}^{-1}(a)$. By Rosenlicht's theorem rational invariants separate orbits in general position. It is important that the dependence of the action on a is algebraic.

From the description of the $\mathcal{G}_0$ action on E it is clear that orbits in general position intersect with the fiber over $a(x) = (x, 0, 0, 0, 1)$ for a unique $x \in \mathbb{R} \setminus \{0\}$. A $\mathcal{G}$-orbit in $\mathcal{E}_\infty$ intersecting with the fiber of $a(x)$ intersects $a(-x)$ as well. Thus we can separate orbits with scalar differential invariants, in addition to the invariant x or x^2, for $\mathcal{G}_0$ or $\mathcal{G}$ respectively. It is not difficult to see, following [12], that in our case the field of differential invariants is still finitely generated. We skip the details because this will be apparent from our explicit description of the generators of this field in what follows.

1.3 The Hilbert and Poincaré functions

The transcendence degree of the field of rational differential invariants of order k (that is the minimal number of generators of this field, possibly up to algebraic extensions) is equal to the codimension of the $\mathfrak{g}$-orbits in general position in $\mathcal{E}_k$. The results in this section are valid for both $\mathcal{G}_0$ and $\mathcal{G}$ and all intermediate Lie pseudogroups (there are three of them since the quotient $\mathcal{G}/\mathcal{G}_0$ is the Klein four-group).

For $k \geq 0$, the dimension of $J^k\pi$ is given by

$$\dim J^k\pi = 4 + \binom{k+4}{4}.$$

The number of independent equations defining $\mathcal{E}_k$ is $\binom{k+3}{4}$ which yields

$$\dim \mathcal{E}_k = \dim J^k\pi - \binom{k+3}{4} = 4 + \binom{k+3}{3}, \quad k \geq 0.$$

For small k, the dimension of a $\mathfrak{g}$-orbit in $J^k\pi$ in general position may be found by computing the dimension of the span of $\mathfrak{g}|_{\theta_k} \subset T_{\theta_k} J^k\pi$ for a general point $\theta_k \in J^k\pi$. It turns out that the equation $\mathcal{E}_k$ intersects with regular orbits, so we get the same results by choosing $\theta_k \in \mathcal{E}_k$.

Theorem 35. *The dimension of a $\mathfrak{g}$-orbit in general position in $\mathcal{E}_k$ is 4 for $k=0$ and it is equal to $k+5$ for $k>0$.*

Proof. We need to compute the dimension of the span of $X^{(k)}$ and $Y(f)^{(k)}$ at a point in general position in $\mathcal{E}_k$. The k-th prolongation of the vector field $Y(f)$ is given by

$$Y(f)^{(k)} = 4f\,\mathcal{D}_u^{(k+1)} - (4vf' + 8x^2 f'')\,\mathcal{D}_v^{(k+1)} + \sum_{|\sigma|\le k} \mathcal{D}_\sigma(\phi)\partial_{h_\sigma} \tag{5}$$

where $\sigma = (i_1, \dots, i_t)$ is a multi-index of length $|\sigma| = t$ (i_j corresponds to one of the base coordinates x, y, u, v), $\mathcal{D}_\sigma = \mathcal{D}_{i_1} \cdots \mathcal{D}_{i_t}$ is the iterated total derivative, $\mathcal{D}_i^{k+1}$ is the truncated total derivative as a derivation on $J^k\pi$, and

$$\begin{aligned}\phi &= Y(f) \lrcorner (dh - h_x dx - h_y dy - h_u du - h_v dv)\\ &= xf''' - 8hf' - 4f\,h_u + (4vf' + 8x^2 f'')\,h_v\end{aligned}$$

is the generating function for $Y(f)$; we refer to Section 1.5 in [11]. We see that the k-th prolongation depends on $f, f', ..., f^{(k+3)}$.

We can without loss of generality assume that the u-coordinate of our point in general position is 0, since ∂_u is contained in $\mathfrak{g}$. At $u=0$ the vector field $Y(f)^{(k)}$ depends only on the $(k+3)$-degree Taylor polynomial of f at $u=0$, which implies that there are at most $k+4$ independent vector fields among these. Adding the vector field $X^{(k)}$ to them gives $k+5$ as an upper bound of the dimension of an orbit.

Let $\theta_k \in \mathcal{E}_k$ be the point defined by $x=1, h=1$, with all other jet-variables set to 0 and let $Z_m = Y(u^m)$. It is clear from (5) that the k-th prolongations of $X, Z_0, ..., Z_{k+3}$ span a $(k+5)$-dimensional subspace of $T_{\theta_k}\mathcal{E}_k$, implying that $k+5$ is also a lower bound for the dimension of an orbit in general position and verifying the claim of the theorem. □

Let $s_k^{\mathcal{E}}$ denote the codimension of an orbit in general position inside of $\mathcal{E}_k$, i.e. the number of independent differential invariants of order k. It is given by

$$s_0^{\mathcal{E}} = 1 \quad \text{and} \quad s_k^{\mathcal{E}} = \frac{k}{6}(k+5)(k+1) \text{ for } k \ge 1.$$

The Hilbert function $H_k^{\mathcal{E}} = s_k^{\mathcal{E}} - s_{k-1}^{\mathcal{E}}$ is given by

$$H_0^{\mathcal{E}} = H_1^{\mathcal{E}} = 1 \quad \text{and} \quad H_k^{\mathcal{E}} = \frac{k(k+3)}{2} \text{ for } k \ge 2.$$

This counts the number of independent differential invariants of “pure” order k. For small k the results are summed up in the following table.

k	0	1	2	3	4	5	6
$\dim J^k\pi$	5	9	19	39	74	130	214
$\dim \mathcal{E}_k$	5	8	14	24	39	60	88
$\dim \mathcal{O}_k$	4	6	7	8	9	10	11
$s_k^{\mathcal{E}}$	1	2	7	16	30	50	77
$H_k^{\mathcal{E}}$	1	1	5	9	14	20	27

The corresponding Poincaré function $P_{\mathcal{E}}(z) = \sum_{k=0}^{\infty} H_k^{\mathcal{E}} z^k$ is given by

$$P_{\mathcal{E}}(z) = \frac{1 - 2z + 5z^2 - 4z^3 + z^4}{(1-z)^3}.$$

2 Differential invariants of Kundt waves

We give a complete description of the field of rational differential invariants. We will focus on the action of the entire Lie pseudogroup $\mathcal{G}$ (with four Zariski connected components), while also describing what to do if one wants to consider only one (or two) connected components.

2.1 Generators

The second order differential invariants of the $\mathcal{G}$-action are generated by the following seven functions

$$I_0 = x^2, \qquad I_1 = \frac{(xh_x - h)^2}{h_y^2}, \qquad I_{2a} = \frac{h_{xx}}{xh_x - h},$$
$$I_{2b} = \frac{xh_{xy}}{h_y}, \qquad I_{2c} = \frac{h_{yy}}{xh_x - h}, \qquad I_{2d} = \frac{(x^2h_{yu} - vh_y)^2}{x(xh_x - h)^3},$$
$$I_{2e} = \frac{(x^3h_{xu} - vxh_x - x^2h_u + vh)(xh_x - h)}{(x^2h_{yu} - vh_y)h_y}$$

and these invariants separate orbits of general position in $\mathcal{E}_2$. They are independent as functions on $\mathcal{E}_2$, and one verifies that the number of invariants agrees with the Hilbert function $H_k^{\mathcal{E}}$ for $k = 0, 1, 2$.

Note that $\sqrt{I_0} = x$ and $\sqrt{I_1} = \frac{xh_x - h}{h_y}$ are not invariant under the discrete transformations $(x, h) \mapsto (-x, -h)$ and $y \mapsto -y$. They are however invariant under the Zariski connected pseudogroup $\mathcal{G}_0$ and should be used for generating the field of differential $\mathcal{G}_0$-invariants, since the invariants above do not separate $\mathcal{G}_0$-orbits on $\mathcal{E}_2$.

Remark 11. *If $\mathcal{A}_2$ denotes the field of second order differential $\mathcal{G}$-invariants and $\mathcal{B}_2$ the field of second order differential $\mathcal{G}_0$-invariants, then $\mathcal{B}_2$ is an algebraic field extension of $\mathcal{A}_2$ of degree 4 and its Galois group is $\mathcal{G}/\mathcal{G}_0 = \mathbb{Z}_2 \times \mathbb{Z}_2$. Intermediate pseudogroups lying between $\mathcal{G}_0$ and $\mathcal{G}$ are in one-to-one correspondence with subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$ that, by Galois theory, are in one-to-one correspondence with algebraic field extensions of $\mathcal{A}_2$ that are contained in $\mathcal{B}_2$.*

Including $\mathcal{B}_2$ there are four such nontrivial algebraic extensions of $\mathcal{A}_2$, and they are the splitting fields of the polynomials $t^2 - I_0$, $t^2 - I_1$, $t^2 - I_0 I_1$ and $(t^2 - I_0)(t^2 - I_1)$ over $\mathcal{A}_2$, respectively.

Higher-order invariants are generated by second-order invariants and invariant derivations, so the field of all differential invariants depends solely on the chosen field extension of $\mathcal{A}_2$.

In order to generate higher-order differential invariants we use invariant derivations, i.e. derivations on $\mathcal{E}_\infty$ commuting with the $\mathcal{G}$-action. It is not difficult to check that the following derivations are invariant.

$$\nabla_1 = xD_x + 2vD_v, \quad \nabla_2 = \frac{xh_x - h}{h_y}\, D_y, \quad \nabla_4 = \frac{x^2 h_{yu} - v h_y}{h_y}\, D_v,$$
$$\nabla_3 = \frac{h_y}{x^2 h_{yu} - v h_y}\left(D_u - \Big(8x^2 h_x - \frac{v^2}{4x^2}\Big) D_v\right).$$

Theorem 36. *The field of rational scalar differential invariants of $\mathcal{G}$ is generated by the second-order invariants $I_0, I_1, I_{2a}, I_{2b}, I_{2c}, I_{2d}, I_{2e}$ together with the invariant derivations $\nabla_1, \nabla_2, \nabla_3, \nabla_4$.*

The algebra of rational differential invariants, which are polynomial starting from the jet-level $\ell = 2$, over $\mathcal{A}_2$, $\mathcal{B}_2$ or an intermediate field, depending on the choice of Lie pseudogroup, is generated by the above seven second-order invariants (with possible passage from I_0 to $\sqrt{I_0}$ and from I_1 to $\sqrt{I_1}$) and the above four invariant derivations.

Proof. We shall prove that the field generated by the indicated differential invariants and invariant derivations for every $k > 2$ contains $H_k^{\mathcal{E}} = \frac{k(k+3)}{2}$ functionally independent invariants, and moreover that their symbols are quasilinear and independent. This together with the fact that the indicated invariants generate all differential invariants of order ≤ 2 implies the statement of the theorem.

We demonstrate by induction in k a more general claim that there are $H_k^{\mathcal{E}}$ quasilinear differential invariants of order k with the symbols at generic $\theta_{k-1} \in J^{k-1}\pi$ proportional to $h_{x^i y^j u^l}$, where $i+j+l = k$ and $0 \leq l < k$. The number of such k-jets is indeed equal to the value of the Hilbert function $H_k^{\mathcal{E}}$.

The base $k = 3$ follows by direct computation of the symbols of

$$\nabla_1 I_{2a}, \nabla_1 I_{2b}, \nabla_1 I_{2c}, \nabla_1 I_{2d}, \nabla_1 I_{2e}, \nabla_2 I_{2c}, \nabla_2 I_{2d}, \nabla_3 I_{2d}, \nabla_3 I_{2e}.$$

Assuming the k-th claim, application of ∇_1 gives $k(k+3)/2$ differential invariants of order $k+1$, and ∇_2 adds k additional differential invariants, covering the symbols $h_{x^i y^j u^l}$ with $i + j + l = k + 1$ and $0 \leq l < k$. Further application of ∇_3 gives 2 more differential invariants with symbols h_{xu^k}, h_{yu^k}. Thus the invariants are independent and the calculation

$$\frac{k(k+3)}{2} + k + 2 = \frac{(k+1)(k+4)}{2}$$

completes the induction step.

For the algebra of invariants it is enough to note that our generating set produces invariants that are quasi-linear in jets of order $\ell = 2$ or higher, and so any differential invariant can be modified by elimination to an element in the base field $\mathcal{A}_2$, $\mathcal{B}_2$ or an intermediate field. □

Remark 12. *As follows from the proof it suffices to have only derivations $\nabla_1, \nabla_2, \nabla_3$. Yet ∇_4 is obtained from those by commutators.*

It is possible to give a more concise description of the field/algebra of differential invariants than that of Theorem 36. Let α_i denote the horizontal coframe dual to the derivations ∇_i, i.e.

$$\alpha_1 = \frac{1}{x}\, dx, \quad \alpha_2 = \frac{h_y}{x h_x - h}\, dy, \quad \alpha_3 = \frac{x^2 h_{yu} - v h_y}{h_y}\, du,$$
$$\alpha_4 = \frac{h_y}{x^2 h_{yu} - v h_y}\left(dv - \frac{2v}{x}\, dx + \Bigl(8x^2 h_x - \frac{v^2}{4x^2}\Bigr) du\right).$$

Then we have:

$$\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 = (I_0 I_1)^{-1/2} dx \wedge dy \wedge du \wedge dv.$$

Metric (1) written in terms of this coframe has coefficients $g_{ij} = g(\nabla_i, \nabla_j)$ and therefore has the form

$$g = I_0 \alpha_1^2 + I_1 \alpha_2^2 + 8(I_1 I_{2d})^{-1} \alpha_3^2 - \alpha_3 \alpha_4.$$

This suggests that ∇_i and I_0, I_1, I_{2d} generate the field of differential invariants. This is indeed true, and can be demonstrated as follows.

The differential invariants appearing as nonzero coefficients in the commutation relations $[\nabla_i, \nabla_j] = K_{ij}^k \nabla_k$ are given by

$$K_{12}^2 = (I_0 I_{2a} - I_{2b}),\ K_{13}^3 = -(I_0 \nabla_3(I_{2b}) + 2),\ K_{13}^4 = -\frac{8 I_0 I_{2a}}{I_1 I_{2d}},$$
$$K_{23}^2 = -\frac{\nabla_3(I_1)}{2I_1},\ K_{23}^3 = I_{2c}(I_1 - I_{2e}) - I_0 I_1 \nabla_3(I_{2c}) = -K_{24}^4,\ K_{34}^3 = -1,$$
$$K_{14}^4 = I_0 \nabla_3(I_{2b}),\ K_{23}^4 = -\frac{8 I_{2b}}{I_1 I_{2d}},\ K_{34}^4 = \frac{I_{2e}}{2 I_0 I_1} - \frac{I_1 I_{2d}}{2} \nabla_3\Big(\frac{1}{I_1 I_{2d}}\Big).$$

In particular we can get the differential invariants $I_{2a}, I_{2b}, I_{2c}, I_{2e}$ from K_{13}^4, $\nabla_1(I_1)$, $\nabla_2(I_1)$, $\nabla_3(I_1)$ thereby verifying that I_0, I_1, I_{2d} are in fact sufficient to be a generating set of differential invariants.

Remark 13. *For the $\mathcal{G}_0$-action, the invariant derivations $D_x + \frac{2v}{x} D_v$ and D_y should be used instead of ∇_1, ∇_2 (they are not invariant under the reflections). In this case only one coefficient of g is nonconstant, suggesting that one differential invariant and four invariant derivations are sufficient for generating the field of differential invariants.*

2.2 Syzygies

Differential relations among the generators of the algebra of differential invariants are called differential syzygies. They enter the quotient equation, describing the equivalence classes $\mathcal{E}_\infty/\mathcal{G}$.

To simplify notations let us rename the generators $a = I_0, b = I_1, c = I_1 I_{2d}$ and use the iterated derivatives $f_{i_1 \dots i_r} = (\nabla_{i_r} \circ \cdots \circ \nabla_{i_1})(f)$ for $f = a, b, c$. We can generate all differential invariants of order k by using only these and $\nabla_1^{k-2}(K_{13}^4)$. The syzygies coming from the commutation relations of ∇_i have been described in the previous section. Thus it is sufficient to only consider iterated derivatives that satisfy $i_1 \leq \cdots \leq i_r$.

These are generated by some simple syzygies

$$a_1 = 2a, \quad a_2 = 0, \quad a_3 = 0, \quad a_4 = 0, \quad b_4 = 0, \quad c_4 = -2c$$

and by two more complicated syzygies that involve differentiation of b, c with respect to ∇_1, ∇_2, ∇_3 up to order three:

$$\begin{aligned} 0 =& 2a^2c^2(2b^2b_3b_{233} - 2b^2b_{23}b_{33} - 3bb_3^2b_{23} + 3b_2b_3^3) - ab(4b^2b_3cc_{13} \\ & -4b^2b_3cc_{23} - 4b^2b_3c_1c_3 + 4b^2b_3c_2c_3 + 8b^2b_{33}c^2 - 4b^2b_{33}cc_1 \\ & +4b^2b_{33}cc_2 - 2bb_1b_{33}c^2 - 4bb_3^2c^2 + 2bb_3^2cc_1 - 4bb_3^2cc_2 + 2bb_3b_{13}c^2 \\ & +2bb_3b_{23}c^2 - b_1b_3^2c^2 - 3b_2b_3^2c^2) - b^2b_3c(4bc - 2bc_1 + 2bc_2 - b_1c), \end{aligned}$$

$$
\begin{aligned}
0 =&8ab^2c^2(b_3b_{123} - b_3b_{223} - b_{13}b_{23} + b_{23}^2)\\
&+4abc^2(b_2b_3b_{13} - b_2b_3b_{23} - 2b_3^2b_{12} + 4b_3^2b_{22})\\
&+ac^2(4b_1b_2b_3^2 - 12b_2^2b_3^2) + 16b^3c^2(b_{23} - b_{13} - b_3)\\
&+8b^3c((2c_1 - 2c_2 - c_{11} + 2c_{12} - c_{22})b_3 + (b_{13} - b_{23})(c_1 - c_2))\\
&+b^3(4b_3c_1^2 - 8b_3c_1c_2 + 4b_3c_2^2) + bc^2(b_1^2b_3 + 2b_1b_2b_3)\\
&+b^2c^2(16b_1b_3 + 4b_1b_{13} - 4b_1b_{23} - 24b_2b_3 - 4b_3b_{11} + 4b_3b_{12})\\
&+b^2c(-8b_1b_3c_1 + 12b_1b_3c_2 + 12b_2b_3c_1 - 12b_2b_3c_2).
\end{aligned}
$$

2.3 Comparing Kundt waves

In order to compare two Kundt waves of the form (1) choose four independent differential invariants $J_1, ..., J_4$ of order k such that $\hat{d}J_1 \wedge \hat{d}J_2 \wedge \hat{d}J_3 \wedge \hat{d}J_4 \neq 0$, where $\hat{d}$ is the horizontal differential defined by $(\hat{d}f) \circ j^k h = d(f \circ j^k h)$ for a function f on $\mathcal{E}_k$. Then rewrite the metric in terms of the obtained invariant coframe, similar to what we did in Section 2.1:

$$
g = G_{ij}\hat{d}J_i\hat{d}J_j
$$

where G_{ij} are differential invariants of order $k+1$. For a given Kundt wave metric g the ten invariants G_{ij}, expressed as functions of J_i, determine its equivalence class.

In practice one can proceed as follows. Let $\hat{\partial}_i$ be the horizontal frame dual to the coframe $\hat{d}J_j$. These are commuting invariant derivations, called Tresse derivatives. In terms of them $G_{ij} = g(\hat{\partial}_i, \hat{\partial}_j)$. Together the 14 functions (J_a, G_{ij}) determine a map $\sigma_g : M^4 \to \mathbb{R}^{14}$ (for a Zariski dense set of g) whose image, called the signature manifold, is the complete invariant of a generic Kundt wave g.

In particular, we can take the four *second-order* differential invariants I_0, I_1, I_{2d}, I_{2e} that are independent for generic Kundt waves. Then G_{ij} are differential invariants of third order, implying that third order differential invariants are sufficient for classifying generic Kundt waves.

Remark 14. *The four-dimensional submanifold $\sigma_g(M^4) \subset \mathbb{R}^{14}$ is not arbitrary. Indeed, the differential syzygies of the generators (J_a, G_{ij}) can be interpreted as a system of PDE (the quotient equation) with independent J_a and dependent G_{ij}. The signature manifolds, encoding the equivalence classes of Kundt waves, are solutions to this system.*

2.4 Example

Consider the class of Kundt waves parametrized by two functions of two variables:

$$h = E(u) - \tfrac{1}{4}\,\mathcal{S}\big(F(u)\big)x + F''(u)^2(x^3 \pm y), \tag{6}$$

where $\mathcal{S}(F) = \frac{F'''}{F'} - \frac{3}{2}\big(\frac{F''}{F'}\big)^2$ is the Schwartz derivative. This class is $\mathcal{G}$-invariant and using the action (2)-(3) the pseudogroup is almost fully normalized in passing from this class to

$$h(x, y, u) = A(u) + x^3 + y. \tag{7}$$

The metric g corresponding to this h was found by Skea in [15] as an example of class of spacetimes whose invariant classification requires the fifth covariant derivative of the Riemann tensor (so up to order seven in the metric coefficients g_{ij} equivalently given by j^7h). However with our approach they can be classified via third order differential invariants, and we will demonstrate how to do it for this simple example.

The transformations from $\mathcal{G}_0$ preserving (7) form the two-dimensional non-connected group $\mathcal{G}_0'$: $(x, y, u, A) \mapsto (x, y+c, \pm u+b, A-c)$, and those of $\mathcal{G}$ form the group $\mathcal{G}'$ extending $\mathcal{G}_0'$ by the map $(x, y, u, A) \mapsto (-x, -y, u, -A)$. Distinguishing the Kundt waves given by (6) with respect to pseudogroup $\mathcal{G}$ (or $\mathcal{G}_0$) is equivalent to distinguishing the Kundt waves given by (7) with respect to group $\mathcal{G}'$ (or $\mathcal{G}_0'$).

The differential invariants from Section 2.1 can be used for this purpose. However the normalization of (6) to (7) allows for a reduction from 4-dimensional signature manifolds to signature curves as follows. The metrics with $A_{uu} \equiv 0$ are easy to classify, so assume $A_{uu} \neq 0$.

The invariants $\sqrt{I_0} = x$, $\sqrt{I_1} = \frac{xh_x - h}{h_y}$, I_{2d}, I_{2e} are basic for the action of $\mathcal{G}_0$, and their combination gives simpler invariants $J_1 = x$, $J_2 = A + y$, $J_3 = v^2$, $J_4 = A_u/v$ with $\frac{\hat{d}J_1 \wedge \hat{d}J_2 \wedge \hat{d}J_3 \wedge \hat{d}J_4}{dx \wedge dy \wedge du \wedge dv} = -2A_{uu}$. The nonzero coefficients G_{ij} are given by

$$G_{11} = 1 = G_{22}, \quad G_{13} = \frac{J_4}{2J_1 A_{uu}}, \quad G_{14} = \frac{J_3}{J_1 A_{uu}}, \quad G_{23} = -\frac{J_4^2}{2A_{uu}},$$

$$G_{33} = -\frac{J_4(32J_1^6 J_4 - 4J_1^2 J_3 J_4^3 + 32J_1^3 J_2 J_4 + 4J_1^2 A_{uu} - J_3 J_4)}{16 J_3 A_{uu}^2 J_1^2},$$

$$G_{34} = \frac{-32J_1^6 J_4 - 32J_4 J_2 J_1^3 + (4J_3 J_4^3 - 2A_{uu})J_1^2 + J_4 J_3}{8A_{uu}^2 J_1^2},$$

$$G_{24} = -\frac{J_3 J_4}{A_{uu}}, \quad G_{44} = \frac{-32J_1^6 J_3 + 4J_1^2 J_3^2 J_4^2 - 32J_1^3 J_2 J_3 + J_3^2}{4A_{uu}^2 J_1^2}.$$

There are five functionally independent invariants, and they are expressed by J_1, J_2, J_3, J_4, A_{uu}. Restricted to the specific Kundt wave (7), only four of them are independent yielding one dependence. This can be interpreted as a relation between the invariants A_u^2 and A_{uu}, giving a curve in the plane due to constraints $A_x = A_y = A_v = 0$, and completely determining the equivalence class. In addition, $A + y$ is a $\mathcal{G}_0$-invariant of order 0.

Consequently, two Skea metrics given by (7) are $\mathcal{G}_0$-equivalent if their signatures $\{(A_u(u)^2, A_{uu}(u))\} \subset \mathbb{R}^2$ coincide as unparametrized curves. Indeed, let $A_{uu} = f(A_u^2)$ be a signature curve (no restrictions but, for simplicity, we consider the one that projects injectively to the first components). Viewed as an ODE on $A = A(u)$ it has a solution uniquely given by the initial data $(A(0), A_u(0))$. This can be arbitrarily changed using the freedom $(u, y) \mapsto (u+b, y+c)$ of $\mathcal{G}_0'$ whence the data encoding g is restored uniquely.

For the $\mathcal{G}$-action, we combine the invariants $I_0, I_1 I_{2a}, I_{2d}, I_{2e}$ to construct a simpler base $J_1 = x^2, J_2 = (A + y)x, J_3 = v^2, J_4 = xA_u/v$ of invariants. In this case we again get $\frac{\hat{d}J_1 \wedge \hat{d}J_2 \wedge \hat{d}J_3 \wedge \hat{d}J_4}{dx \wedge dy \wedge du \wedge dv} = -4x^3 A_{uu} \neq 0$, and basic order 0, 1 and 2 differential invariants for the dimension reduction are $(A+y)^2$, A_u^2, $A_{uu}/(A+y)$. Proceeding as before we obtain a signature curve $\{(A_u(u)^2, A_{uu}(u)^2)\} \subset \mathbb{R}^2$ that, as an unparametrized curve, is a complete $\mathcal{G}$-invariant of the Kundt waves of Skea type (7).

3 Specification to the vacuum case

It was argued in Section 1.1 that the Lie pseudogroup preserving vacuum Kundt waves of the form (1) is the same as the one preserving general Kundt waves of the same form. The PDE $\mathcal{R}_k = \{h_{xx} + h_{yy} = 0\}^{(k-2)} \cup \mathcal{E}_k$ defining vacuum Kundt waves contains some orbits in $\mathcal{E}_k$ of maximal dimension. This follows from the proof of Theorem 35, since the point $\theta_k \in \mathcal{E}_k$ chosen there belongs also to $\mathcal{R}_k$.

This implies that orbits in general position in $\mathcal{R}_k$ are also orbits in general position in $\mathcal{E}_k$. Generic vacuum Kundt waves are separated by the invariants found in Section 2, and all previous results are easily adapted to the vacuum case.

3.1 Hilbert and Poincaré function

For vacuum Kundt waves we have additional $\binom{k+1}{3}$ independent differential equations of order k defining $\mathcal{R}_k \subset \mathcal{E}_k$, so the dimension of $\mathcal{R}_k$ is $4 + (k+1)^2$ for $k \geq 0$. The codimension of orbits in general position in $\mathcal{R}_k$ is thus given by

$$s_0^{\mathcal{R}} = 1 \quad \text{and} \quad s_k^{\mathcal{R}} = k(k+1) \text{ for } k \geq 1.$$

Consequently the Hilbert function $H_k^{\mathcal{R}} = s_k^{\mathcal{R}} - s_{k-1}^{\mathcal{R}}$ is given by

$$H_0^{\mathcal{R}} = H_1^{\mathcal{R}} = 1 \quad \text{and} \quad H_k^{\mathcal{R}} = 2k \text{ for } k \geq 2.$$

The corresponding Poincaré function $P_{\mathcal{R}}(z) = \sum_{k=0}^{\infty} H_k^{\mathcal{R}} z^k$ is equal to

$$P_{\mathcal{R}}(z) = \frac{1 - z + 3z^2 - z^3}{(1-z)^2}.$$

3.2 Differential invariants

The differential invariants of second order from Section 2.1 are still differential invariants in the vacuum case. The only difference is that two second order invariants I_{2a}, I_{2c} become dependent since the vacuum condition implies $I_{2a} + I_{2c} = 0$; in higher order we add differential corollaries of this relation. It follows that we can generate all $\mathcal{G}$-invariants of higher order by using the differential invariants I_0, I_1, I_{2d} and invariant derivations ∇_i above.

The differential syzygies found in Section 2.2 will still hold, but we get some new ones obtained by ∇_i differentiations of the Ricci-flat condition $I_{2a} + I_{2c} = 0$. In terms of the differential invariants a, b, c, K_{13}^4 from Section 2.2, the syzygy on $\mathcal{R}_2$ takes the form

$$K_{13}^4 bc(a+b) + 4a(2b + b_1 + b_2) = 0.$$

The case of $\mathcal{G}_0$-invariants is treated similarly.

3.3 Comparing vacuum Kundt waves

For the basis of differential invariants we can take the same second-order invariants as for the general Kundt waves: I_0, I_1, I_{2d}, I_{2e}. Then we express the metric coefficients G_{ij} in terms of this basis of invariants.

The corresponding four-dimensional signature manifold $\sigma_g(M^4)$ is restricted by differential syzygies of the general case plus the vacuum constraint. Considered as an unparametrized submanifold in $\mathbb{R}^{14}$ it completely classifies the metric g.

4 The Cartan-Karlhede algorithm

Next, we would like to compare the Lie-Tresse approach to differential invariants with Cartan's equivalence method. We outline the Cartan-Karlhede algorithm for finding differential invariants. The general description of the algorithm can be found in [10]. Its application to vacuum Kundt waves has been recently treated in [7].

4.1 The algorithm for vacuum Kundt waves

Consider the null-coframe

$$\ell = du, \quad n = \frac{1}{2}dv - \frac{v}{x}dx + \left(4xh - \frac{v^2}{8x^2}\right)du, \qquad \begin{aligned} m &= \tfrac{1}{\sqrt{2}}(dx + idy), \\ \bar{m} &= \tfrac{1}{\sqrt{2}}(dx - idy), \end{aligned}$$

in which the metric (1) has the form $g = 2m \odot \bar{m} - 2\ell \odot n$ (as before we have $h_v = 0 = h_{xx} + h_{yy}$). Let $\Delta, D, \delta, \bar{\delta}$ be the frame dual to coframe $\ell, n, m, \bar{m}$:

$$\Delta = \partial_u - \left(8xh - \frac{v^2}{4x^2}\right)\partial_v, \quad D = 2\partial_v, \qquad \begin{aligned} \delta &= \tfrac{1}{\sqrt{2}}(\partial_x - i\partial_y) + \tfrac{v\sqrt{2}}{x}\,\partial_v, \\ \bar{\delta} &= \tfrac{1}{\sqrt{2}}(\partial_x + i\partial_y) + \tfrac{v\sqrt{2}}{x}\,\partial_v. \end{aligned}$$

There is a freedom in choosing the (co)frame, encoded as the Cartan bundle. The general orthonormal frame bundle $\tilde{\rho} : \tilde{\mathcal{P}} \to M$ is a principal bundle with the structure group $O(1,3)$. For Kundt waves the non-twisting non-expanding shear-free null congruence ℓ is up to scale unique, and this reduces the structure group to the stabilizer $H \subset O(1,3)$ of the line direction $\mathbb{R} \cdot \ell$, yielding the reduced frame bundle $\rho : \mathcal{P} \to M$, which is a principal H-subbundle of $\tilde{\mathcal{P}}$.

This so-called parabolic subgroup H has dimension four and the H-action on our null (co)frame is given by boosts $(\ell, n) \mapsto (B\ell, B^{-1}n)$, spins $m \mapsto e^{i\theta}m$ and null rotations $(n, m) \mapsto (n + cm + \bar{c}\bar{m} + |c|^2\ell, m + \bar{c}\ell)$ about ℓ, where parameters B, θ are real and the parameter c is complex.

Let ∇ denote the Levi-Civita connection of g, and let R be the Riemann curvature tensor. Written in terms of the frame, the components of R and its covariant derivatives are invariant functions on $\mathcal{P}$, but they are not invariants on M. The structure group H acts on them and their H-invariant combinations are absolute differential invariants.

In practice H is used to set as many components of $\nabla^k R$ as possible to constants, as this is a coordinate independent condition for the parameters of H. In the Newman-Penrose formalism [14], the Ricci (Φ) and Weyl (Ψ) spinors for the Kundt waves are given by

$$\Phi_{22} = 2x(h_{xx} + h_{yy}), \qquad \Psi_4 = 2x(h_{xx} - h_{yy} - 2ih_{xy}).$$

A boost and spin transform Ψ_4 to $B^{-2}e^{-2i\theta}\Psi_4$. Thus if $\Psi_4 \neq 0$ it can be made equal to 1 by choosing $B^2 = 4x\sqrt{h_{xx}^2 + h_{xy}^2}$ and $e^{2i\theta} = \frac{h_{xx} - ih_{xy}}{\sqrt{h_{xx}^2 + h_{xy}^2}}$.

This reduces the frame bundle and the new structure group H is two-dimensional. In the next step of the Cartan-Karlhede algorithm we use the null-rotations to normalize components of the first covariant derivative of the Weyl spinor. The benefit of setting $\Psi_4 = 1$ is that components of the Weyl spinor and its covariant derivatives can be written in terms of the spin-coefficients and their derivatives. For example, the nonzero components of the first derivative of the Weyl spinor are

$$(D\Psi)_{50} = 4\alpha, \quad (D\Psi)_{51} = 4\gamma, \quad (D\Psi)_{41} = \tau.$$

The null-rotations, with complex parameter c, sends γ to $\gamma + c\alpha + \frac{5}{4}\bar{c}\tau$, but leaves α and τ unchanged. Assuming that $|\alpha| \neq \frac{5}{4}|\tau|$ it is possible to set $\gamma = 0$, and this fixes the frame. In this case there will be four Cartan invariants of first order in curvature components, namely the real and imaginary parts of α and τ. They can be expressed in terms of differential invariants as follows:

$$\alpha = \frac{-\sqrt{2i}}{8\sqrt{I_0}} \frac{J_-^{1/4}}{J_+^{5/4}} \left(i\sqrt{I_0 I_1}(2I_0 I_{2a}^2 - I_{2a} + 2\nabla_1 I_{2a}) + 2I_{2b}^2 - 3I_{2b} + 2\nabla_1 I_{2b} \right)$$

$$\tau = \frac{1}{\sqrt{2iI_0}} \frac{J_+^{1/4}}{J_-^{1/4}}, \qquad \text{where} \qquad J_\pm = I_{2b} \pm i\sqrt{I_0 I_1} I_{2a}.$$

These give four independent invariant functions on $\mathcal{R}_\infty$, but when restricted to a vacuum Kundt wave metric (to the section $j_M^\infty g \subset \mathcal{R}_\infty$) at most three of them are independent:

$$\hat{d}(\alpha + \bar{\alpha}) \wedge \hat{d}(\alpha - \bar{\alpha}) \wedge \hat{d}(\tau + \bar{\tau}) \wedge \hat{d}(\tau - \bar{\tau}) = 0.$$

The generic stratum of this case corresponds to the invariant branch (0,3,4,4) of the Cartan-Karlhede algorithm in [7].

At the next step of this algorithm the derivatives of the three Cartan invariants from the last step are computed, resulting in the invariants $\Delta|\tau|, \bar{\delta}\alpha, \mu, \nu$ (the latter again complex-valued). One more derivative gives the invariant $\Delta(\Delta|\tau|)$ as a component of the third covariant derivative of the curvature tensor. Further invariants (when restricted to $j_M^\infty g$) will depend on those already constructed, so only 12 real-valued Cartan invariants are required to classify vacuum Kundt waves.

Remark 15. *In Section 2.3 it was stated that 14 differential invariants* (J_a, G_{ij}) *are sufficient for classifying Kundt waves, but choosing* $J_1 = I_0$, $J_2 = I_1$, $J_3 = I_{2d}$, $J_4 = I_{2e}$ *it turns out that we get precisely 12 functionally independent differential invariants among them.*

4.2 Cartan invariants vs. absolute differential invariants

Let us take a closer look at the relationship between the Cartan invariants and the differential invariants from Section 2.

Differential invariants are functions on $J^\infty\pi$, or on a PDE therein, which are constant on orbits of the Lie pseudogroup $\mathcal{G}$. Cartan invariants, on the other hand, are components of the curvature tensor and its covariant derivatives. These components are dependent on the point in M and the frame.

If we normalize the group parameters and hence fix the frame, i.e. a section of the Cartan bundle, then the Cartan invariants restricted to this section are invariant functions on $J^\infty\pi$. The following commutative diagram explains the situation.

$$\begin{array}{ccc} \mathcal{P} & \longleftarrow & \pi_\infty^*\mathcal{P} \\ \downarrow{\scriptstyle \rho} & & \downarrow \\ M & \underset{\pi_\infty}{\longleftarrow} & \mathcal{E}_\infty \subset J^\infty\pi \end{array}$$

Initially the Cartan invariants are functions on

$$\pi_\infty^*\mathcal{P} = \{(\omega, g_\infty) \in \mathcal{P} \times \mathcal{E}_\infty \mid \rho(\omega) = \pi_\infty(g_\infty)\}$$

and they suffice to solve the equivalence problem because $\mathcal{P}$ is equipped with an absolute parallelism Ω (Cartan connection) whose structure functions generate all invariants on the Cartan bundle. Indeed, an equivalence of two Lorentzian spaces (M_1, g_1) and (M_2, g_2) lifts to an equivalence between $(\mathcal{P}_1, \Omega_1)$ and $(\mathcal{P}_2, \Omega_2)$ and vise versa the equivalence upstairs projects to an equivalence downstairs.

Projecting the algebra of invariants on the Cartan bundle to the base we obtain the algebra of absolute differential invariants consisting of $\mathcal{G}$-invariant functions on $\mathcal{E}_\infty$. This is achieved by invariantization of the invariants on $\mathcal{P}$ with respect to the structure group.

This is done in steps by normalizing the group parameters, effecting in further reduction of the structure group. When the frame is fully normalized (or normalized to a group acting trivially on invariants) the Cartan bundle is reduced to a section of $\mathcal{P}$, restriction to which of the $\nabla^k R$ components gives scalar differential invariants on M. Often these functions and their algebraic combinations that are absolute differential invariants, evaluated on the metric, are called Cartan invariants.

4.3 A comparison of the two methods

The definite advantage of Cartan's invariants is their universality. A basic set of invariants can be chosen for almost the entire class of metrics simultaneously. The syzygies are also fully determined by the commutator relations, the Bianchi and Ricci identities in the Newman-Penrose formalism [14]. Yet this basic set is large and algebraically dependent invariants should be removed, resulting in splitting of the class into different branches of the Cartan-Karlhede algorithm. See the invariant-count tree for the class of vacuum Kundt waves in [7].

The normalization of group parameters however usually introduces algebraic extensions into the algebra of invariants. The underlying assumption at the first normalization step in Section 4.1 is that Ψ_4 is nonzero. This means that also for Cartan invariants we must restrict to the complement of a Zariski-closed set in $\mathcal{E}_k$.

Setting Ψ_4 to 1 introduces radicals into the expressions of Cartan invariants. A sufficient care with this is to be taken in the real domain, because the square root is not everywhere defined and is multi-valued. At this stage it is the choice of the $\pm$ sign, but the multi-valuedness becomes more restrictive with further invariants. For instance, the expressions for α and τ contain radicals of $J_\pm$ depending on $\sqrt{I_0 I_1}$.

Recall that even though the invariant I_0 and I_1 are squares, the extraction of the square root cannot be made $\mathcal{G}$-equivariantly and is related to a choice of domain for the pseudogroup $\mathcal{G}_0$. Changing the sign of $\sqrt{I_0 I_1}$ results in interchange $J_- \leftrightarrow J_+$ modifying the formula for α and τ (which, as presented, is also subject to some sign choices). The complex radicals carry more multi-valued issues: choosing branch-cuts and restricting to simply connected domains.

Thus Cartan's invariants computed via the normalization technique are only locally defined. In addition, the domains where they are defined are not Zariski open, in particular they are not dense.

In contrast, elements of the algebra of rational-polynomial differential invariants described in Section 2 are defined almost everywhere, on a Zariski-open dense set. The above radicals are avoidable because we know from Section 1.2 that generic Kundt waves, as well as vacuum Kundt waves, can be separated by rational invariants.

Another aspects of comparison is coordinate independence. The class of metrics (1) is given in specific Kundt coordinates, from which we derived the pseudogroup $\mathcal{G}$. Changing the coordinates does not change the pseudogroup, but only its coordinate expression. In other words, this is equivalent to a conjugation of $\mathcal{G}$ in the pseudogroup $\mathrm{Diff}_{\mathrm{loc}}(M)$.

The Cartan-Karlhede algorithm is manifestly coordinate independent, i.e. the invariants are computed independently of the form in which a Kundt wave is written. However a normalization of parameters is required to get a canonical frame. It is a simple integration to derive from this Kundt coordinates. It is also possible to skip integration with the differential invariants approach as abstractly jets are coordinate independent objects. This would give an equivalent output.

5 Conclusion

In this paper we discussed Kundt waves, a class of metrics that are not distinguished by Weyl's scalar curvature invariants. We computed the algebra of scalar differential invariants that separate generic metrics in the class and showed that this algebra is finitely generated in Lie-Tresse sense globally. These invariants also separate the important sub-class of vacuum Kundt waves.

The latter class of metrics was previously investigated via Cartan's curvature invariants in [7] and we compared the two approaches. In particular, we pointed out that normalization in the Cartan-Karlhede algorithm leads to multi-valuedness of invariants. Moreover, the obtained Cartan's invariants are local even in jets-variables (derivatives of the metric components). This leads to restriction of domains of definitions, which in general may not be even invariant with respect to the equivalence group, see [12].

With the differential invariant approach the signature manifold can be reduced in dimension, as we saw in Section 2.4. For the general class of Kundt waves where $h_v = 0$, the v-variable can be removed from consideration and furthermore it is not difficult to remove the y-variable too. This dimension reduction leads to a much simpler setup and the classification algorithm. We left additional independent variables to match the traditional approach via curvature invariants.

The two considered approaches are not in direct correspondence and each method has its own specifications. For instance, the invariant-count tree in the Cartan-Karlhede algorithm ideologically has a counter-part in the Poincaré function for the Lie-Tresse approach. However orders of the invariants in the two methods are not related, obstructing to align the filtrations on the algebras of invariants.

For simplicity in this paper we restricted to generic metrics in the class of Kundt waves. This manifests in a choice of four functionally independent differential invariants, which is not always possible. For instance, metrics admitting a Killing vector never admit four independent invariants. With the Cartan-Karlhede approach this corresponds to invariant branches like (0,1,3,3) ending not with 4, and for the vacuum case all such possibilities were classified in [7].

With the differential invariants approach we treated metrics specified by explicit inequalities: $h_y \neq 0$, $I_0 I_1 \neq 0$, $\dots$, such that the basic invariants and derivations are defined. It is possible to restrict to the singular strata, and find the algebra of differential invariants with respect to the restricted pseudogroup. Thus differential invariants also allow to distinguish more special metrics in the class of Kundt waves.

To summarize, the classical Lie-Tresse method of differential invariants is a powerful alternative to the Cartan equivalence method traditionally used in relativity applications.

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Differential invariants in thermodynamics

Eivind Schneider

Abstract

Due to the first and second law of thermodynamics the state of a thermodynamic system is described by a Legendrian manifold. This Legendrian manifold is locally determined by the information gain function. We describe the algebra of rational differential invariants of the information gain function under the action of two different Lie groups appearing naturally as a result of measuring random vectors, and we discuss our results in the context of ideal and van der Waals gases.

1 Introduction

The fundamental thermodynamic relation can be formulated as $dE-(TdS-\sum_{i=2}^{n} p_i dq^i) = 0$, where E is the internal energy, S the entropy and T the temperature while $q^i$and p_i are additional extensive and intensive variables, respectively. In terms of information gain I which is up to an additive constant equal to $-S$, it can be written as $dE + TdI + \sum p_i dq^i = 0$. Geometrically we should interpret this to mean that a thermodynamic state is a Legendrian manifold, integral to a contact distribution. Let V be a vector space. We consider the contact one-form given by $du - \sum_{i=1}^{n} \lambda_i dx^i$ on $V \times \mathbb{R} \times V^*$ (with coordinates x^i, u, λ_i). By taking $\lambda_1 = -T^{-1}, x^1 = E, \lambda_i = -T^{-1}p_i, x^i = q^i$ we can relate this to the one-form above, which means that the Legendrian manifold is given locally over a neighborhood $D \subset V$ by

$$\{u = I(x), \lambda_i = \partial_{x^i} I(x) \mid x \in D\} \subset V \times \mathbb{R} \times V^*.$$

In Section 3 we describe two Lie group actions that appear naturally on the space $V \times \mathbb{R} \times V^*$. They arise from the fact that we may change basis in V and change units of information. After that, in Section 4, we approach the equivalence problem of thermodynamic states under these Lie group actions by computing differential invariants of the information gain function. In section 5 we discuss our results in the context of gases. But first we outline how these Legendrian manifolds naturally appear in the context of measuring random vectors (following [5]), as this will help us find the natural Lie group actions acting on them.

2 The geometry of thermodynamics

The process of measuring random vectors in a vector space V can be thought about as a map $X\colon (\Omega, \mathcal{A}, q) \to V$ from a probability space to V. This will depend on the probability measure q. By changing the measure, we get a different expectation. Assume that the expected value in V is 0, i.e. $E(X) = \int_\Omega X dq = 0$. We restrict to finite-dimensional vector spaces here, even though the Bochner integral lets us treat more general Banach spaces (see [5]). Note however that the linear structure on V is important, and this will play a role later when we consider Lie group actions on V.

If we want to measure a vector $x \in V$, we choose a measure p different from q, but equivalent to it. Applying the Radon-Nikodym theorem tells us that there is a function ρ such that $dp = \rho dq$ and

$$\int_\Omega \rho dq = 1, \qquad \int_\Omega \rho X dq = x.$$

These conditions do not determine ρ uniquely. We define the information gain $I(p,q) = \int_\Omega \rho \ln \rho dq$, and add the requirement that ρ minimizes $I(p,q)$. This is the principle of minimal information gain.

As a result (see [5]) we get $\rho = \frac{1}{Z(\lambda)} e^{\langle \lambda, X\rangle}$ with $\lambda \in V^*$ where $Z(\lambda) = \int_\Omega e^{\langle \lambda, X\rangle} dq$ is called the partition function. Due to $\int_\Omega \rho X dq = x$ we get $d_\lambda Z = Z(\lambda) x$. Thus, if we define $H(\lambda) = -\ln Z(\lambda)$ we end up with $x = -d_\lambda H$. And we also get $I(p,q) = H(\lambda) - \langle \lambda, d_\lambda H\rangle = H(\lambda) + \langle \lambda, x\rangle$. Assuming that $x = -d_\lambda H$ has a unique solution $\lambda(x)$, we may write $I = I(x) = H(\lambda(x)) + \langle \lambda(x), x\rangle$. Then we get $d_x I = \lambda$, where the functions H and I are related by $H = I + \langle \lambda, x\rangle$.

Now let x^i be coordinates on the vector space V and and λ_i be the dual coordinates on V^*. On the space $V \times V^*$ we have the natural symplectic form $\omega = \sum_{i=1}^n d\lambda_i \wedge dx^i$. The function $H(\lambda)$ determines a submanifold $L_H = \{x^i = -\frac{\partial H}{\partial \lambda_i}\} \subset V \times V^*$ which is Lagrangian with respect to ω, meaning that $\omega|_{L_H} = 0$.

The Lagrangian manifold $L_H \subset V \times V^*$ can be extended to a manifold $\tilde{L}_H \subset V \times \mathbb{R} \times V^*$. Let u be the coordinate on $\mathbb{R}$. Then we get a submanifold in $V \times \mathbb{R} \times V^*$ which is locally described by the function I over a neighbourhood $D \subset V$:

$$\tilde{L}_H = \{u = I(x), \lambda_i = \partial_{x^i} I(x) \mid x \in D\} \in V \times \mathbb{R} \times V^*.$$

Thus the principle of minimal information gain leads to a submanifold in $V \times \mathbb{R} \times V^*$ which is Legendrian with respect to the one-form $\theta = du - \sum \lambda_i dx^i$.

Conversely, one can ask whether it is possible to reconstruct I, H and Z from any Legendrian manifold $\tilde{L} \subset V \times \mathbb{R} \times V^*$. This is the case if the symmetric 2-form $(\sum d\lambda_i dx^i)|_{\tilde{L}}$ is positive definite. By starting with such a Legendrian manifold $\tilde{L}$, one can recover the functions

$$I = u|_{\tilde{L}}, \quad H = I - \langle \lambda, x \rangle|_{\tilde{L}}, \quad Z = e^{-H}.$$

If $\tilde{L}$ is given by I as above, the symmetric form can be written as

$$\sum \frac{\partial^2 I}{\partial x^i \partial x^j} dx^i dx^j.$$

3 Equivalence of thermodynamical systems

We start by describing two different Lie group actions arising naturally from the set-up above, and thereby defining what it could mean for two thermodynamic states to be equivalent.

Affine action Since the choice of basis on V is arbitrary, we consider the Legendrian manifold $\tilde{L}$ up to linear transformations on $V \times V^*$. In the treatment above we started by assuming $E(X) = 0$ with respect to the initial measure. This was done because of convenience, not necessity, so we add translations to $GL(V)$ and obtain the affine group $\mathrm{Aff}(V) = V \rtimes GL(V)$ on V. After choosing a basis, the affine action on $V \times \mathbb{R} \times V^*$ is given by $(x^i, u, \lambda_k) \mapsto (\sum a^i_j x^j + c_i, u, \sum b^l_k \lambda_l)$ where the matrix (b^i_j) is the transpose of $(a^i_j)^{-1}$. The corresponding Lie algebra of vector fields is spanned by $x^i \partial_{x^j} - \lambda_j \partial_{\lambda_i}$ and ∂_{x^i}. Notice that this action does not alter the value of I (or H) at a point.

Scaling In addition to changing basis, we may change the unit of information. The unit of information appears as the base of the logarithm we use, which in Section 2 was chosen to be e. Let $Z_a(\lambda_a), H_a(\lambda_a), I_a(x_a)$ be defined in a way similar to $Z(\lambda), H(\lambda), I(x)$ above but with a as the base of the logarithm instead of e, and let $a = e^b$. The functions in the new units are related to the old ones in the following way.

$$\begin{aligned} Z_a(\lambda_a) &= \int_\Omega a^{\langle \lambda_a, v\rangle} dq = \int_\Omega e^{\langle b\lambda_a, v\rangle} dq = Z(b\lambda_a) \\ H_a(\lambda_a) &= -\log_a Z_a(\lambda_a) = -\log_a Z(b\lambda_a) = -\ln Z(b\lambda_a)/b = \tfrac{H(b\lambda_a)}{b} \\ I_a(x_a) &= H_a(\lambda_a) + \langle \lambda_a, x_a \rangle = \tfrac{H(b\lambda_a) + \langle b\lambda_a, x_a\rangle}{b} = \tfrac{I(x_a)}{b} \end{aligned}$$

In other words, we have the scaling transformation $(x, u, \lambda) \mapsto (x, bu, b\lambda)$ on $V \times \mathbb{R} \times V^*$. Denote by G_0 the Lie group of such transformations. The corresponding infinitesimal action is given by $\sum \lambda_i \partial_{\lambda_i} + u\partial_u$.

Remark 16. *A natural question is where we allow b to take its values. We could restrict to $b > 0$, or to $b \neq 0$. We will discuss this in more detail when we compute the differential invariants of these Lie group actions.*

4 Differential invariants

The Legendrian manifold $\tilde{L}$ (the thermodynamic state) is locally determined by the information gain function I on V. We will compute differential invariants for I under the two Lie group actions $\text{Aff}(V)$ and $G_0 \times \text{Aff}(V)$. We consider I as local a section of the trivial bundle $V \times \mathbb{R}$ on which we continue to use coordinates $x^1, ..., x^n, u$. In order to study the orbit space of such sections under the action of these Lie groups, we look at their prolonged action on the jet bundles $J^k(V) \to V \times \mathbb{R} = J^0(V)$ (we will use the simplified notation J^k).

Let x^i, u_σ be canonical coordinates on J^k where $0 \leq |\sigma| \leq k$ for the multi-index $\sigma = (i_1, ..., i_n)$, $i_j \geq 0$. For example, when $n = 2$ we have coordinates x^i, u_{ij} on J^2, with $0 \leq i + j \leq 2$ and $i, j > 0$. For $|\sigma| = 0$, we will also use the notation $u_\sigma = u$. The section $u = I(x)$ on $V \times \mathbb{R}$ prolongs to a section on J^k given by $u = I(x), u_\sigma = \frac{\partial^{|\sigma|}}{\partial x^\sigma} I(x)$. We denote this prolongation by $j^k(I)$. Since diffeomorphisms on $V \times \mathbb{R}$ transform sections of $V \times \mathbb{R}$, they lift naturally to J^k. Thus we can consider the action of the aforementioned Lie groups on J^k. In fact we already described their action on J^1 since J^1 can be naturally identified with $V \times \mathbb{R} \times V^*$.

Differential invariants are functions on J^k that are constant on the orbits of the Lie group actions. For transitive and algebraic Lie group actions the global Lie-Tresse theorem [4] guarantees that the algebra of rational differential invariants separates orbits in general position in J^∞, and that it is finitely generated. Since $\text{Aff}(V)$ acts transitively on the base V, and not at all along the fiber, it is clear that we also in this intransitive case can separate orbits by rational invariants (the algebra of differential invariants for the $\text{Aff}(V)$-action can be gotten from that of $G_0 \times \text{Aff}(V)$ by adding the $G_0 \times \text{Aff}(V)$-invariant u). For more thorough treatments of jet bundles and differential invariants we refer to [2, 3, 6].

For the Lie groups $\text{Aff}(V)$ and $G_0 \times \text{Aff}(V)$ we give a complete description of their algebras of rational differential invariants.

4.1 Differential invariants under Aff(V)

In order to describe the field of differential invariants we follow [1], where differential invariants under the $GL(V)$-action are found.

Theorem 37. *The horizontal symmetric forms* $\alpha_k = \sum_{|\sigma|=k} \frac{u_\sigma}{\sigma!} dx^\sigma$ *are Aff*(V)*-invariant, for* $k \geq 0$*.*

From these symmetric forms we may construct (rational) scalar differential invariants in the following way. First, $\alpha_0 = u$ is a scalar differential invariant. The symmetric two-form α_2 is nondegenerate for points in general position in J^2, so we may use it to construct the vector $v_1 = \alpha_2^{-1}(\alpha_1)$. By using this vector, we may construct a new symmetric two-form $\alpha_{1,3} = i_{v_1}\alpha_3$. We can use α_2 to turn $\alpha_{1,3}$ into an operator $A\colon T \to T$. For a point in J^3 in general position, the vectors $v_k = A^{k-1}(v_1)$ for $k = 1, ..., n$ are independent and thus define a frame. Expressing α_k in terms of v_k gives us scalar differential invariants from their coefficients. In other words, the functions $\alpha_k(v_{i_1}, ..., v_{i_k})$ are differential invariants. Remark that all these differential invariants will be rational functions on J^k, and affine on the fibers of $J^k \to J^3$. In particular we get only two independent differential invariants on J^2, given by α_0 and $\alpha_1(v_1)$.

This way of generating differential invariants may not be the most convenient one. Another way to generate the field is to use invariant derivations and a finite number of differential invariants, in accordance with the Lie-Tresse theorem. As invariant derivations we can take $v_1, ..., v_n$. They are of the form $\sum \alpha_i D_{x^i}$, where D_{x^i} are total derivatives and α_i are functions on J^3 (and on J^2 for v_1).

Theorem 38. *The field of differential invariants are generated by the invariant derivations* $v_1, ..., v_n$ *together with the first-order invariant* $\alpha_0 = u$*, the second-order invariant* $\alpha_1(v_1)$*, the third-order invariants* $\alpha_3(v_{i_1}, v_{i_2}, v_{i_3})$ *and the fourth order invariants* $\alpha_4(v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4})$*.*

It is not difficult to see that this set of invariants is sufficient for generating all differential invariants of higher order. However, we do not necessarily need all of them.

Example 7 (The two-dimensional case)**.** *We take a closer look at the case when* V *is two-dimensional, as they are particular important when we will consider gases. We have the first-order invariant* $\alpha_0 = u$ *and the second-order invariant*

$$\alpha_1(v_1) = \frac{u_{10}^2 u_{02} - 2u_{10}u_{01}u_{11} + u_{01}^2 u_{20}}{(u_{20}u_{02} - u_{11}^2)}.$$

The invariant derivations are given by

$$
\begin{aligned}
v_1 &= \tfrac{1}{u_{20}u_{02}-u_{11}^2}\left((u_{10}u_{02}-u_{01}u_{11})D_{x^1}+(u_{01}u_{20}-u_{10}u_{11})D_{x^2}\right),\\
v_2 &= \tfrac{1}{(u_{20}u_{02}-u_{11}^2)^3}\Big((\\
&-(u_{10}u_{02}-u_{01}u_{11})(3u_{10}u_{11}u_{02}-2u_{01}u_{20}u_{02}-u_{01}u_{11}^2)u_{21}\\
&+(u_{10}u_{11}-u_{01}u_{20})(3u_{10}u_{11}u_{02}-u_{01}u_{20}u_{02}-2u_{01}u_{11}^2)u_{12}\\
&+u_{02}(u_{10}u_{02}-u_{01}u_{11})^2u_{30}-u_{11}(u_{10}u_{11}-u_{01}u_{20})^2u_{03})D_{x^1}\\
&+((u_{10}u_{02}-u_{01}u_{11})(u_{10}u_{20}u_{02}+2u_{10}u_{11}^2-3u_{01}u_{20}u_{11})u_{21}\\
&-(u_{10}u_{11}-u_{01}u_{20})(2u_{10}u_{20}u_{02}+u_{10}u_{11}^2-3u_{01}u_{20}u_{11})u_{12}\\
&-u_{11}(u_{10}u_{02}-u_{01}u_{11})^2u_{30}+u_{20}(u_{10}u_{11}-u_{01}u_{20})^2u_{03})D_{x^2}\Big).
\end{aligned}
$$

In this case the four third-order differential invariants $\alpha_3(v_{i_1}, v_{i_2}, v_{i_3})$ are independent, and together with v_1, v_2 and $\alpha_1(v_1)$ they generate the algebra of differential invariants.

Note that when $\tilde{L}$ is the Legendrian manifold corresponding to the information gain function I we have, in coordinates,

$$\left(\sum d\lambda_i dx^i\right)\Big|_{\tilde{L}} = \sum I_{x^i x^j} dx^i dx^j = \alpha_2|_I.$$

In particular, since we require α_2 to be a definite symmetric two-form, we may find differential invariants from the curvature tensor of this two-form. For example, the Ricci scalar is a third order differential invariant. Such invariants are, however, invariant under much more general transformations, so they don't generate all $\mathrm{Aff}(V)$-invariants. What's more, they will not be invariant under $G_0 \times \mathrm{Aff}(V)$.

4.2 Differential invariants under $G_0 \times \mathrm{Aff}(V)$

Now we consider the action by the Lie group $G_0 \times \mathrm{Aff}(V)$. It acts on $V \times \mathbb{R}$ by $(b, A) \cdot (x, u) \mapsto (Ax, bu)$. We mentioned previously that we have a choice for the G_0-parameter b, since we may take it from either $\mathbb{R} \setminus \{0\}$ or $(0, \infty)$. This corresponds to $a = e^b$ in $(0, \infty) \setminus \{1\}$ or $(1, \infty)$, respectively. Here we will stick to the first choice $\mathbb{R} \setminus \{0\}$, with the main reason that this gives the Zariski-closure of the other option. The theorems we have for existence of rational invariants separating orbits hold for algebraic (Zariski-closed) groups. However the structure of both orbit spaces on J^k should be clear as soon as we understand one of them. Also, the obtained orbit space will be the same of that of positive I's under the action of the topologically connected component of the Lie group (not containing $u \mapsto -u$).

In order to describe the field of differential invariants we reuse ideas from the previous section. The symmetric forms α_k are scaled by the $\mathbb{R}^*$-action.

If we modify them to $\beta_k = \alpha_k/\alpha_0$, for $k \geq 1$, we obtain $G_0 \times \mathrm{Aff}(V)$-invariant symmetric forms. By using these instead of α_k we can generate the algebra of differential invariants in exactly the same way as we did in the previous section. Invariant vectors can be constructed from β_k in exactly the same way as above (from α_k), and they will in fact be the exact same vectors v_i as before.

Theorem 39. *The field of differential invariants are generated by the invariant derivations $v_1, ..., v_n$ together with the second-order invariant $\beta_1(v_1)$, the third-order invariants $\beta_3(v_{i_1}, v_{i_2}, v_{i_3})$ and the fourth order invariants $\beta_4(v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4})$.*

Example 8 (The two-dimensional case)**.** *When V is two-dimensional we have the second-order invariant*

$$\beta_1(v_1) = \frac{u_{10}^2 u_{02} - 2u_{10}u_{01}u_{11} + u_{01}^2 u_{20}}{(u_{20}u_{02} - u_{11}^2)u}.$$

The invariant derivations from the previous section are still invariant under the current Lie group action. We have the relations $\beta_3(v_{i_1}, v_{i_2}, v_{i_3}) = \alpha_3(v_{i_1}, v_{i_2}, v_{i_3})/\alpha_0$. These four invariants are thus also independent, and together with v_1, v_2 and $\beta_1(v_1)$ they generate the algebra of differential invariants.

5 Application to gases

We explain how we can use differential invariants in order to distinguish gases under the Lie group actions considered above. We use the ideal gas, and the van der Waals gases as examples. In order to keep this chapter concise, we do not go into the detailed physics but instead refer to [5] for more details about gases.

The simplest gases can be described as Legendrian manifolds of the contact form $\theta = du - (-T^{-1})d\varepsilon - (-pT^{-1})dv$ where T is temperature, p is pressure, v is specific volume and ε is specific energy. We consider this as a contact form on $V \times \mathbb{R} \times V^*$ where V is a two-dimensional vector space. In order to relate it to our formulas above we let $x^1 = \varepsilon, x^2 = v$ (and $\lambda_1 = -T^{-1}, \lambda_2 = -pT^{-1}$).

5.1 Distinguishing gases

Integral manifolds of θ are locally determined by the information gain function I. We can use the differential invariants from above to determine when two different Legendrian manifolds of this type are equivalent under the groups $\mathrm{Aff}(V)$ and $G_0 \times \mathrm{Aff}(V)$, respectively.

We outline first how to do it for the $\mathrm{Aff}(V)$-action. For a function f on J^k, we denote by $f|_I$ the restriction of f to the section $u = I(x)$, i.e. $f|_I = f \circ j^k(I)$. For a point in general position in J^3, the differential invariants $\xi = \alpha_1(v_1)$ and $\eta = \alpha_3(v_1, v_1, v_1)$ will be horizontally independent, meaning that $\hat{d}\xi \wedge \hat{d}\eta \neq 0$. Here $\hat{d}$ denotes the horizontal differential, which can be defined in coordinates by $\hat{d}f = D_{x^1}(f)dx^1 + D_{x^2}(f)dx^2$, so that $(\hat{d}f)|_I = d(f|_I)$. Thus for generic I we have $d(\xi|_I) \wedge d(\eta|_I) \neq 0$, so $\xi|_I$ and $\eta|_I$ can be taken as local coordinates on V. The four invariants $h_{ij} = \alpha_3(v_i, v_j, v_2)$ and $h_0 = \alpha_0$ may also be restricted to I, and the functions $h_0|_I, h_{ij}|_I$ on V may be written in terms of $\xi|_I$ and $\eta|_I$. The four functions $h_0|_I(\xi|_I, \eta|_I), h_{ij}|_I(\xi|_I, \eta|_I)$ determine the equivalence class of the Legendrian manifold given by I.

To check equivalence under the $G_0 \times \mathrm{Aff}(V)$-action we use the invariants $\tilde{\xi} = \xi/h_0, \tilde{\eta} = \eta/h_0, \tilde{h}_{ij} = h_{ij}/h_0$ instead, and the equivalence class of I is determined by the three functions $\tilde{h}_{ij}|_I(\tilde{\xi}|_I, \tilde{\eta}|_I)$.

Remark 17. *The functions $\tilde{h}_{ij}|_I(\tilde{\xi}|_I, \tilde{\eta}|_I)$ are not arbitrary functions. They must satisfy a system of differential equations defined by the differential syzygies in the algebra of differential invariants.*

5.2 Ideal gas

As our first example we take the ideal gas, which is defined by the state equations $pv = RT$ and $\varepsilon = \frac{n}{2}RT$ where n counts the degrees of freedom. As shown in [5] they give the following information gain function:

$$I = -R\ln\left(\ln(v) + \frac{n}{2}\ln(\varepsilon)\right) + C = -R\ln\left(\ln(x^2) + \frac{n}{2}\ln(x^1)\right) + C$$

where C is some constant. We look at the differential invariants found above, restricted to the section $u = I$.

We first consider the $\mathrm{Aff}(V)$-invariants. We have $\xi|_I = R(n+2)/2, \eta|_I = R(n+2)$. These are constant, so the ideal gas lies in fibers over the singular set in J^3 determined by the equation $\hat{d}\xi \wedge \hat{d}\eta = 0$. The functions $h_{ij}|_I$ are also constant multiples of $R(n+2)$. The derivations v_1 and v_2 are both constant multiples of $x^1\partial_{x^1} + x^2\partial_{x^2}$ on the ideal gas. The only nonconstant function we get from the invariants is $h_0|_I$.

If we consider $G_0 \times \mathrm{Aff}(V)$-invariants instead we still have $(\hat{d}\tilde{\xi} \wedge \hat{d}\tilde{\eta})|_I = 0$, even though $\tilde{\xi}|_I, \tilde{\eta}|_I$ are not constant. Thus the ideal gas is a singular Legendrian manifold, also with respect to this Lie group action.

5.3 Van der Waals gas

The state equations for van der Waals gases are $\left(p+\frac{a}{v^2}\right)(v-b)=RT$ and $\varepsilon=\frac{n}{2}RT-\frac{a}{v}$, and their information gain function is given by

$$I=-R\left((v-b)\left(\frac{a}{v}+\varepsilon\right)^{n/2}\right)+C=-R\left((x^2-b)\left(\frac{a}{x^2}+x^1\right)^{n/2}\right)+C.$$

For van der Waals gases we have $(\hat{d}\xi\wedge\hat{d}\eta)|_I\neq 0$, so we can use the differential invariants above to distinguish them, and it is not difficult to find the functions $h_{ij}|_I(\xi|_I,\eta|_I)$ and $h_0|_I(\xi|_I,\eta|_I)$:

$$\begin{aligned}
h_{11} &= \tfrac{1}{4R^3n(Rn-2\,\xi)}\Big(n^2\,(2-n)\,(n\xi-2\,\xi+4\,\eta)\,R^4\\
&+8\,n\left(\left(\left(n^2-n-1\right)\xi+2\,n\eta\right)\xi-(\xi-\eta)^2\right)R^3\\
&-8\,\left(\left(3\,n^2+2\,n+2\right)\xi+2\,n\eta\right)\xi^2R^2+32\,\xi^4\,(n+1)\,R-16\,\xi^5\Big)\\
h_{12} &= \tfrac{1}{4nR^4(Rn-2\,\xi)^2}\Big(\big(30\,n^3\xi^3-20\,n^2\,(2\,\xi-\eta)\,\xi^2\\
&-8\,n\left(4\,\xi^2-\xi\eta+4\,\eta^2\right)\xi-16\,\xi^3+24\,\xi^2\eta-32\,\xi\eta^2+8\,\eta^3\big)R^4\\
&+\left(-160\,n^3\xi^4+32\,n\left(2\,\xi^2+\xi\eta+2\,\eta^2\right)\xi^2+64\,\xi^3\eta\right)R^3\\
&+16\,\left(15\,\xi n^2+5\,\left(2\,\xi-\eta\right)n+4\,\xi-8\,\eta\right)\xi^4R^2\\
&-64\,\left(3\,n\xi^6+2\,\xi^6-\xi^5\eta\right)R+64\,\xi^7\Big)\\
h_{22} &= \tfrac{1}{16n^2R^5(Rn-2\,\xi)^3}\Big(\,(2-n)^3\,n^5\,(3\,n\xi-6\,\xi+8\,\eta)\,R^8\\
&+24\,\left(n-2\right)n^4\big(2\,n^3\xi^2-n^2\,(7\,\xi-3\,\eta)\,\xi+n\,(2\,\xi+\eta)\,(2\,\xi-3\,\eta)\\
&+4\,\xi^2-4\,\xi\eta+2\,\eta^2\big)R^7-16\,n^3\big(21\,n^4\xi^3-n^3\,(73\,\xi-12\,\eta)\,\xi^2\\
&+n^2\left(42\,\xi^3-18\,\xi^2\eta-33\,\xi\eta^2\right)+\left(36\,\xi^3+12\,\xi\eta^2+12\,\eta^3\right)n\\
&+8\,\xi^3-24\,\xi^2\eta+36\,\xi\eta^2-8\,\eta^3\big)R^6\\
&+32\,n^2\big(42\,n^4\xi^4-n^3\,(81\,\xi+5\,\eta)\,\xi^3+n^2\left(-18\,\xi^4-12\,\xi^3\eta-42\,\xi^2\eta^2\right)\\
&+12\,n\,(2\,\eta+\xi)\,\xi\left(2\,\xi^2-2\,\xi\eta+\eta^2\right)+40\,\xi^3\eta-36\,\xi^2\eta^2+24\,\xi\eta^3-4\,\eta^4\big)R^5\\
&-96\,n\big(35\,n^4\xi^3-10\,n^3\,(3\,\xi+2\,\eta)\,\xi^2-4\,n^2\left(7\,\xi^2+8\,\xi\eta+3\,\eta^2\right)\xi\\
&-8\,\left(\xi+\eta\right)(2\,\xi-\eta)\,\eta n+8\,\xi\,(\xi-\eta)^2\,\big)\xi^2R^4\\
&+128\,\xi^4\big(42\,n^4\xi^2-11\,\xi\,(\xi+3\,\eta)\,n^3-3\,n^2\left(10\,\xi^2+18\,\xi\eta-\eta^2\right)\\
&-12\,n\left(\xi^2+3\,\xi\eta-\eta^2\right)-8\,\xi^2\big)R^3\\
&-256\,\xi^5\left(21\,n^3\xi^2-n^2\,(3\,\xi+16\,\eta)\,\xi-3\,n\left(6\,\xi^2+6\,\xi\eta-\eta^2\right)-12\,\xi^2\right)R^2\\
&+1536\,\left(2\,\xi n^2-n\,(\xi+\eta)-2\,\xi\right)\xi^7R-256\,\left(-4+3\,n\right)\xi^9\Big)\\
h_0 &= -R\Big(\tfrac{n}{2}\ln\left(\tfrac{nF^2G}{(Rn-2\xi)^5((n+2)R-2\xi)^4}\right)+\ln\left(\tfrac{F}{G}\right)\\
&+\ln\left(\tfrac{a^{n/2}}{b^{n/2-1}}\right)+(n+1)\ln(2)-\tfrac{3\ln(3)}{2}n\Big)+C
\end{aligned}$$

where

$$\begin{aligned}
F &= n(n+3\,n+2)R^3-(6\,\xi n^2+12\,n\xi+12\,\xi-4\,\eta)R^2+12\,(n+1)\xi^2R-8\,\xi^3,\\
G &= n(n^2-4)R^3-(6\,n^2\xi-24\,\xi+8\,\eta)R^2+12\,n\xi^2R-8\,\xi^3.
\end{aligned}$$

We suppressed the notation signifying restriction to I in order to simplify the equations. The functions $h_{ij}|_I(\xi|_I,\eta|_I)$ are rational functions, and we notice that they do not depend on the constants a, b, C. The expression for $h_0|_I(\xi|_I, \eta|_I)$ shows that changing a and b will only affect the constant C under the Aff(V)-action.

The $G_0 \times \mathrm{Aff}(V)$-invariants are more difficult to handle. In order to find the functions $\tilde{h}_{ij}|_I(\tilde{\xi}|_I, \tilde{\eta}|_I)$ we can in theory make the substitutions

$$\xi|_I = \tilde{\xi}|_I \cdot h_0|_I, \qquad \eta|_I = \tilde{\eta}|_I \cdot h_0|_I, \qquad h_{ij}|_I = \tilde{h}_{ij}|_I \cdot h_0|_I$$

and eliminate $h_0|_I$ in order to get three equations determining $\tilde{h}_{ij}|_I(\xi|_I, \eta|_I)$. However, this seems unmanageable in practice. The first three equations are polynomial in $h_0|_I$, but with degrees up to 18, while the fourth equation is not even algebraic.

It is well known that we can use $G_0 \times \mathrm{Aff}(V)$ to normalize the constants a, b, R. In the "critical variables", in which the critical point is given by $(p, v, T) = (1, 1, 1)$, the constants are normalized to $a = 3, b = 1/3, R = 8/3$. Thus every van der Waals gas is, under the $G_0 \times \mathrm{Aff}(V)$-action, equivalent to the one given by the equations

$$\left(p + \frac{3}{v^2}\right)\left(v - \frac{1}{3}\right) = \frac{8}{3}T, \qquad \varepsilon = \frac{4n}{3}T - \frac{3}{v}$$

and the information gain function

$$I = -\frac{8}{3}\ln\left(\left(v - \frac{1}{3}\right)\left(\frac{3}{v} + \varepsilon\right)^{n/2}\right) + C.$$

Notice that normalizing a, b, R in this way will affect the value of C.

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Appendix

Lifts of Lie algebras of vector fields in the plane

Introduction

These notes contain the computations resulting in [2, Theorem 2]. We go through all Lie algebras of vector fields on $\mathbb{C}^2$ that are listed in Section 2 of that paper, and lift them to to $\mathbb{C}^2 \times \mathbb{C}$. Combined with the results in Section 3, the computations are cumbersome but straightforward. For every Lie algebra of vector fields the first step will be to rectify the subalgebra spanned by ∂_x, ∂_y or $\partial_x, x\partial_x + \partial_y$. Since this is similar for all cases, we will usually start the computations here with the assumption that the lifts of these vector fields are already straightened out. See examples 1 and 2 in [2] for details. Throughout, greek and capital letters will in general denote complex constants, while lowercase letters will denote functions.

1 Lifts of the primitive Lie algebras

We will find the lifts of the primitive Lie algebras. These computations were done in [1], but we redo them here for completeness. Since we have the inclusions $\mathfrak{g}_3 \subset \mathfrak{g}_2 \subset \mathfrak{g}_1$ it will be convenient to start with the smallest one.

The following example will be useful.

Example 9. *Let $X = A(u-u_1)(u-u_2)\partial_u$. The coordinate transformation $U = \frac{1}{u-u_2}$ takes X to $A((u_1-u_2)U-1)\partial_U$, which can be brought to the form $C\partial_U$ or $CU\partial_U$, depending on whether $u_1 = u_2$ or not, for a constant C. Thus any vector field of the form $(a+bu+cu^2)\partial_u$, can be transformed to either $C\partial_u$ or a $Cu\partial_u$ by a projective transformation.*

1.1 $\mathfrak{g}_3$

The lift of the smallest primitive Lie algebra is spanned by vector fields of the form $X_1 = \partial_x, X_2 = \partial_y, X_3 = x\partial_y + a(x,y,u)\partial_u, X_4 = y\partial_x + b(x,y,u)\partial_u, X_5 = x\partial_x - y\partial_y + c(x,y,u)\partial_u$. The six commutation relations resulting from taking commutators of X_1, X_2 with X_3, X_4, X_5 imply that a, b, c are functions of u only, and we may assume them to be polynomials of (at most) second order.

The commutation relations

$$[X_3, X_4] = X_5, \quad [X_3, X_5] = -2X_3, \quad [X_4, X_5] = 2X_4$$

are equivalent to the equations

$$ab' - ba' = c, \quad ac' - ca' = -2a, \quad bc' - cb' = 2b,$$

respectively.

Following example 9 it is possible to change coordinates so that $a(u) \equiv A$ or $a(u) = Au$. The latter option is inconsistent with the second equation when $A \neq 0$ (and when $A = 0$ we get only the trivial, intransitive, lift). If we assume that $a \equiv A$, the equations have solutions $c = -2(u - C), b = -(u - C)^2/A$. The constants may be removed by an affine transformation, leaving us with only one transitive lift.

$$\boxed{\partial_x, \quad \partial_y, \quad x\partial_y + \partial_u, \quad x\partial_x - y\partial_y - 2u\partial_u, \quad y\partial_x - u^2\partial_u.} \qquad (1)$$

1.2 $\mathfrak{g}_2$

This Lie algebra contains $\mathfrak{g}_3$ as a subalgebra. In addition to $X_1, ..., X_5$ from above, we now have $X_6 = x\partial_x + y\partial_y + d(x, y, u)\partial_u$.

The computations here are simplified by using the results from the section above, but we can not necessarily assume that the Lie subalgebra $\mathfrak{g}_3$ is transitive. There are two cases to consider.

The subalgebra $\mathfrak{g}_3$ lifts nontrivially

For the same reasons as in the previous section, we may assume that $a = 1, b = -u^2, c = -2u$. The commutation relations $[X_i, X_6] = X_i$ for $i = 1, 2$ imply that d is a function of u only. The commutation relation $[X_3, X_6] = 0$ implies that d is constant, and this constant vanishes due to $[X_4, X_6] = 0$. The result is a projective lift.

$$\boxed{\partial_x, \quad \partial_y, \quad x\partial_y + \partial_u, \quad x\partial_x - y\partial_y - 2u\partial_u, \quad y\partial_x - u^2\partial_u, \quad x\partial_x + y\partial_y.} \qquad (2)$$

The subalgebra $\mathfrak{g}_3$ lifts trivially

Now $a = b = c = 0$. Then the commutation relations put no restrictions on d, but since the lift should be transitive, d is constant. This results in a metric lift.

$$\boxed{\partial_x, \quad \partial_y, \quad x\partial_y, \quad x\partial_x - y\partial_y, \quad y\partial_x, \quad x\partial_x + y\partial_y + C\partial_u.} \tag{3}$$

1.3 $\mathfrak{g}_1$

The Lie algebra $\mathfrak{g}_1$ contain both $\mathfrak{g}_2$ and $\mathfrak{g}_3$ as subalgebras. In addition to the elements of $\mathfrak{g}_2$, we have $X_7 = x^2\partial_x + xy\partial_y + e(x, y, u)\partial_u$ and $X_8 = xy\partial_x + y^2\partial_y + f(x, y, u)\partial_u$. Again, there are two cases.

The subalgebra $\mathfrak{g}_3$ lifts nontrivially

The arguments used for $\mathfrak{g}_2$ still hold, and thus $a = 1, b = -u^2, c = -2u, d = 0$. It is a simple exercise to check that the commutation relations involving X_7, X_8 yield $e = y - ux$ and $f = u(y - xu)$.

$$\boxed{\begin{array}{c} \partial_x, \quad \partial_y, \quad x\partial_y + \partial_u, \quad x\partial_x - y\partial_y - 2u\partial_u, \quad y\partial_x - u^2\partial_u, \\ x\partial_x + y\partial_y, \;\; x^2\partial_x + xy\partial_y + (y - xu)\partial_u, \;\; xy\partial_x + y^2\partial_y + u(y - xu)\partial_u. \end{array}} \tag{4}$$

The subalgebra $\mathfrak{g}_3$ lifts trivially

Now $a = b = c = 0$, and $d = C$ or $d = Cu$ after a change of coordinates, in accordance with example 9.

When $d = C$ commutation relations between X_1, X_2 and X_7, X_8 imply that e and f are functions of u alone. Using the remaining commutation relations, we end up with a metric lift.

$$\boxed{\begin{array}{c} \partial_x, \quad \partial_y, \quad x\partial_y, \quad x\partial_x - y\partial_y, \quad y\partial_x, \quad x\partial_x + y\partial_y + 2C\partial_u, \\ x^2\partial_x + xy\partial_y + 3Cx\partial_u, \quad xy\partial_x + y^2\partial_y + 3Cy\partial_u. \end{array}} \tag{5}$$

The other option, $d = Cu$, only results in a intransitive (linear) lift.

2 Lifts of the imprimitive Lie algebras

2.1 The Lie algebra $\mathfrak{r} = \langle \partial_x, \partial_y, x\partial_y, ..., x^r\partial_y \rangle$

Let us consider the Lie algebra $\mathfrak{r} = \langle \partial_x, \partial_y, x\partial_y, ..., x^r\partial_y \rangle$. This is a subalgebra of several of the Lie algebras of vector fields on $\mathbb{C}^2$ from Lie's classification, and is therefore useful to consider on its own. It is also a special case of $\mathfrak{g}_4$.

After a coordinate change, the lifted basis elements will take the form

$$X_1 = \partial_x, \quad X_2 = \partial_y, \quad Y_i = x^i \partial_y + c_i(x, y, u)\partial_u$$

and since the stabilizer of 0 is abelian, we may assume that $c_i(x, y, u) = (\alpha+\beta u+\gamma u^2)b_i(x, y)$, where the greek constants are independent of the index i.[1] The functions c_i (and thus also b_i) are restricted by the commutation relations of the Lie algebra.

$$[X_2, Y_i] = (c_i)_y \partial_u = 0 \qquad \Rightarrow \qquad (c_i)_y = 0 \tag{6}$$

$$[X_1, Y_i] = ix^{i-1}\partial_y + (c_i)_x \partial_u = \begin{cases} X_2, & i = 1 \\ iY_{i-1}, & i = 2, 3, ... \end{cases}$$

This implies

$$(c_1)_x = 0, \qquad (c_i)_x = ic_{i-1}. \tag{7}$$

Equation (6) implies that $b_i(x, y)$ is a function of x only.

Theorem 40. *The coefficients b_k are of the following form:*

$$b_k = \sum_{i=1}^{k} \binom{k}{k-i} C_i x^{k-i}$$

Proof. We need to solve the equations $(b_k)_y = 0, (b_1)_x = 0, (b_k)_x = kb_{k-1}$. It is clear that the above functions are solutions, and since we have k linear first-order ODEs (wrt. x) with constant coefficients (with one-dimensional solution space), they must be all solutions. □

The general lift is spanned by

$$\boxed{X_1 = \partial_x,\ X_2 = \partial_y,\ Y_k = x^k \partial_y + (\alpha + \beta u + \gamma u^2) \left(\sum_{i=1}^{k} \tbinom{k}{k-i} C_i x^{k-i} \right) \partial_u,} \tag{8}$$

where $k = 0, 1, ..., r - 2$.

Remark 18. *If this was the Lie algebra we wanted to lift, we could also assume that $\alpha = 1, \beta = \gamma = 0$ in order to get no singular points. Since this is usually just a subalgebra of a transitive Lie algebra, we should not make this assumption yet. However, all the Lie algebras from Lie's classification that contains this Lie algebra as a subalgebra has a solvable stabilizer of* 0*, so we can assume that $\gamma = 0$ wherever it appears above.*

[1] This can also be seen from the commutation relations $[Y_i, Y_j] = 0$.

Note that if $b_i = 0$ for $i < s$ and we relabel the constants so that $b_s = D_0$, we get

$$Y_k = x^k \partial_y + (\alpha + \beta u + \gamma u^2) \left(\sum_{i=s}^{k} \binom{s+k}{s+k-i} D_{i-s} x^{s+k-i} \right) \partial_u \qquad (9)$$

which can also be written as

$$Y_{s+k} = x^{s+k} \partial_y + (\alpha + \beta u + \gamma u^2) \left(\sum_{j=0}^{k} \binom{s+k}{j} D_{k-j} x^j \right) \partial_u \qquad (10)$$

which may be useful later.

2.2 $\mathfrak{g}_4$

For $\mathfrak{g}_4$, the stabilizer of the 0-fiber is abelian, so we need only look for metric lifts. The lift of $\mathfrak{g}_4$ is spanned by ∂_x and $X_{j,i} = x^i e^{\alpha_j x} \partial_y + b_{j,i}(x, y) \partial_u$ after a coordinate change. We may also set $b_{1,0} \equiv 0$.

The commutation relations

$$[X_{1,0}, X_{j,i}] = e^{\alpha_j x} (b_{j,i})_y \partial_u = 0$$

imply that $b_{j,i}$ are independent of y. The relations

$$[\partial_x, X_{j,i}] = (i x^{i-1} e^{\alpha_j x} + \alpha_j x^i e^{\alpha_j x}) \partial_y + (b_{j,i})_x \partial_u = i X_{j,i-1} + \alpha_j X_{j,i}$$

imply $(b_{j,i})_x = i b_{j,i-1} + \alpha_j b_{j,i}$. It is easily verified that the solutions are given by

$$b_{j,i} = e^{\alpha_j x} \sum_{k=0}^{i} \binom{i}{k} B_{j,k} x^{i-k}$$

with $B_{1,0} = 0$. The general lift is thus spanned by

$$\boxed{\partial_x, \quad x^i e^{\alpha_j x} \partial_y + e^{\alpha_j x} \left(\sum_{k=0}^{i} \binom{i}{k} B_{j,k} x^{i-k} \right) \partial_u, \quad \text{with } B_{1,0} = 0.} \qquad (11)$$

2.3 $\mathfrak{g}_5$

Since $\mathfrak{g}_5 \supset \mathfrak{g}_4$, we may use the results from the previous section. We add $Y = y\partial_y + a\partial_u$ to the lifted basis (11), with one difference in that the constants $B_{j,k}$ now are affine functions in u (since we cannot assume that the lift of the subalgebra is transitive), and they are all proportional to each other as functions of u. The relations $[\partial_x, Y] = 0$ and $[X_{1,0}, Y] = X_{1,0}$ imply that a is independent of x and y, respectively. If $B_{j,i} \equiv 0$ for every i and j, we may assume that a is constant, and we get a metric lift.

$$\boxed{\partial_x, \quad x^i e^{\alpha_j x}\partial_y, \quad y\partial_y + C\partial_u} \tag{12}$$

Assume that $B_{j,i} \neq 0$ while $B_{j,k} \equiv 0$ for $k < i$ (or $i = 0$ and $B_{j,0} \neq 0$). Then $b_{j,i} = e^{\alpha_j x} B_{j,i}$, and the commutation relation

$$[X_{j,i}, Y] = X_{j,i}$$

implies that $B_{j,i}$ is constant. It also gives $a_u = 1$, so that $a = u + A$. A translation in u-direction lets us set $A = 0$. The rest of the commutation relations give no new restrictions. Observe that we may use a scaling in u in order to normalize one of the constants $B_{j,k}$, so we get the affine lift

$$\boxed{(11), \quad y\partial_y + u\partial_u, \qquad B_{j,k} = 1 \text{ for some } (j,k).} \tag{13}$$

2.4 $\mathfrak{g}_6$

This computation is done in detail in [2, Example 2], so we do not repeat it here. There is a metric lift and an affine lift.

$$\boxed{\partial_x, \quad \partial_y, \quad y\partial_y + C\partial_u, \quad y^2\partial_y + 2Cy\partial_u} \tag{14}$$

$$\boxed{\partial_x, \quad \partial_y, \quad y\partial_y - u\partial_u, \quad y^2\partial_y + (1 - 2yu)\partial_u} \tag{15}$$

2.5 $\mathfrak{g}_7$

The lift of $\mathfrak{g}_7$ is spanned by $X_1 = \partial_x, X_2 = \partial_y, X_3 = x\partial_x + a\partial_u, X_4 = x^2\partial_x + x\partial_y + b\partial_u$. The function a must be independent of x and y due to the commutation relations $[X_1, X_3] = X_1$ and $[X_2, X_3] = 0$.

$$\begin{aligned}
&[X_1, X_4] = 2x\partial_x + \partial_y + b_x\partial_u = 2X_3 + X_2 \quad \Rightarrow \quad b_x = 2a \\
&[X_2, X_4] = b_y\partial_y = 0 \quad \Rightarrow \quad b_y = 0 \\
&[X_3, X_4] = x^2\partial_x + x\partial_y + (xb_x + ab_u - ba_u) = X_4 \quad \Rightarrow \quad xb_x + ab_u - ba_u = b
\end{aligned}$$

The equations are the same as for $\mathfrak{g}_6$ (after switching x, y-coordinates), so the lifts can be found in exactly the same way.

$$\boxed{\partial_x, \quad \partial_y, \quad x\partial_x + C\partial_u, \quad x^2\partial_x + x\partial_y + 2Cx\partial_u} \tag{16}$$

$$\boxed{\partial_x, \quad \partial_y, \quad x\partial_x - u\partial_u, \quad x^2\partial_x + x\partial_y + (1 - 2xu)\partial_u} \tag{17}$$

2.6 $\mathfrak{g}_8$

The lift of $\mathfrak{g}_8$ is spanned by vector fields of the form

$$X_1 = \partial_x, \quad X_2 = \partial_y, \quad X_3 = x\partial_x + \lambda y\partial_y + a_3\partial_u, \quad Y_i = x^i\partial_y + b_i\partial_u,$$

for $i = 1, ..., r-3$. Since the stabilizer of 0 is solvable, we may assume that a_3, b_i are affine functions in u. The commutation relations $[X_1, X_3] = X_1$ and $[X_2, X_3] = \lambda X_2$ imply that $a_3 = A + Bu$, with $A, B \in \mathbb{C}$. From section 2.1 we have

$$b_k = (\alpha + \beta u) \sum_{i=1}^{k} \binom{k}{k-i} C_i x^{k-i}.$$

In the case $\beta \neq 0$ we may by a u-translation set $\alpha = 0$, and we can without loss of generality assume $\beta = 1$. Let $b_i = D_i u$ be the first nonzero coefficient. Then

$$(i-\lambda)Y_i = [X_3, Y_i] = (i-\lambda)x^i\partial_y + AD_i\partial_u$$

which can be consistent only if $A = 0$ or if the dimension is 3. In the first case the lift is intransitive ($u = 0$ is a singular orbit). The lift of the three dimensional Lie algebra will be considered later.

The other option is that $\alpha = 1$ and $\beta = 0$. The commutation relation $[X_3, Y_i] = (i-\lambda)Y_i$ implies $x(b_i)_x = (i-\lambda+B)b_i$ which gives $xib_{i-1} = (i-\lambda+B)b_i$ when combined with $(b_i)_x = ib_{i-1}$ (from $[X_1, Y_i] = iY_{i-1}$). Assume first that $B = \lambda - k$ for some $k \in \{1, ..., r-3\}$. Then b_k is constant, $b_i = 0$ for $i < k$, and $b_{k+l} = \binom{k+l}{l} b_k x^l$.

If $B \neq 0$, we may remove A by a translation, and set $b_k = 1$ by a scaling transformation, to get the affine lift spanned by

$$\boxed{\partial_x, \;\; \partial_y, \;\; x\partial_x + \lambda y\partial_y + (\lambda - k)u\partial_u, \;\; x^i\partial_y + b_i\partial_u, \;\; k \in \{1, ..., r-3\}} \quad (18)$$

with $b_i \equiv 0$ for $i = 1, ..., k-1$ and $b_{k+l} = \binom{k+l}{l} x^l$ for $l = 0, ..., r-3-k$.

If $B = 0$, then $\lambda = k$, and we have the metric lift

$$\boxed{\partial_x, \quad \partial_y, \quad x\partial_x + ky\partial_y + A\partial_u, \quad x^i\partial_y + b_i\partial_u, \quad k \in \{1, ..., r-3\}} \quad (19)$$

with $b_i \equiv 0$ for $i = 1, ..., k-1$ and $b_{k+l} = \binom{k+l}{l} Cx^l$ for $l = 0, ..., r-3-k$.

Now assume that $B \neq \lambda - k$ for every $k \in \{1, ..., r-3\}$. Then $b_i = \frac{xi}{i-\lambda+B} b_{i-1}$. Either $b_i \equiv 0$ for every i or $r \leq 4$ or $b_2 = \frac{2x}{2-\lambda+B} b_1$. The last case is inconsistent with $(b_i)_x = ib_{i-1}$ for $i = 2$ with the current assumptions on B.

In the first case, we must have $B = 0$ in order for the lift to be transitive, and we get a metric lift.

$$\boxed{\partial_x, \quad \partial_y, \quad x\partial_x + \lambda y\partial_y + A\partial_u, \quad x^i\partial_y.} \tag{20}$$

This is also the lift we get when $r = 3$. When $r = 4$ we have $0 = x(b_1)_x = (1 - \lambda + B)b_1$ which yields $b_1 = 0$, resulting in the metric lift already considered.

2.7 $\mathfrak{g}_9$

A lift of $\mathfrak{g}_9$ is spanned by $X_1 = \partial_x, X_2 = \partial_y, X_3 = x\partial_x + ((r-2)y + x^{r-2})\partial_y + b(x, y, u)\partial_u$ and $Y_i = x^i\partial_y + c_i\partial_u$ for $i = 1, ..., r - 3$. The subalgebra $\mathfrak{r}$ from section 2.1 is of the form (8). The commutation relations $[X_1, X_3] = X_1 + (r - 2)Y_{r-3}$ and $[X_2, X_3] = (r - 2)X_2$ give

$$b_x = (r - 2)c_{r-3}, \qquad b_y = 0. \tag{21}$$

There are two cases to consider.

The subalgebra $\mathfrak{r}$ lifts trivially

If $\mathfrak{r}$ lifts trivially (or $r = 3$), then $b_x = b_y = 0$, and hence $b = b(u)$. We may assume that the lift is metric so we get

$$\boxed{\partial_x, \quad \partial_y, \quad x\partial_x + ((r-2)y + x^{r-2})\partial_y + C\partial_u, \quad x\partial_y, \quad ..., \quad x^{r-3}\partial_y.} \tag{22}$$

The subalgebra $\mathfrak{r}$ lifts nontrivially

From section 2.1 we know that if s is the lowest integer such that Y_s lifts nontrivially, then

$$Y_{s+k} = x^{s+k}\partial_y + (\alpha + \beta u)\left(\sum_{j=0}^{k}\binom{s+k}{j}C_{k-j}x^j\right)\partial_u$$

for $k \geq 0$ with $C_0 \neq 0$. From $b_x = (r - 2)c_{r-3}$ we get that

$$b = (\alpha + \beta u)\sum_{j=0}^{r-3}\binom{r-2}{j+1}C_{r-s-j-3}x^{j+1} + A_0 + A_1u.$$

The commutation relation $[Y_s, X_3] = (r-2-s)Y_i$ is only consistent if $\beta = 0$ (and we assume without loss of generality that $\alpha = 1$), and it also implies that $A_1 = r - s - 2$. A u-translation lets us set $A_0 = 0$. The commutation relations $[Y_{s+k}, X_3] = (r-2-s-k)Y_{s+k}$ imply that $x(c_{s+k})_x = kc_{s+k}$, so that $c_{s+k} = \binom{s+k}{k} C_0 x^k$. A scaling transformation lets us set $C_0 = 1$.

Thus the general lift is spanned by

$$\boxed{\begin{aligned} \partial_x, \quad \partial_y, \quad x^i\partial_y + c_i\partial_u, \\ x\partial_x + ((r-2)y + x^{r-2})\partial_y + \left(\binom{r-2}{s} x^{r-s-2} + (r-s-2)u\right)\partial_u, \end{aligned}} \tag{23}$$

where $c_i = 0$ for $i < s$ and $c_{s+k} = \binom{s+k}{s} x^k$ for $k = 0, ..., r-3-s$.

2.8 $\mathfrak{g}_{10}$

A general lift is spanned by the vector fields $X_1 = \partial_x, X_2 = \partial_y, X_3 = x\partial_x + b_1(x,y,u)\partial_u, X_4 = y\partial_y + b_2(x,y,u)\partial_u, Y_i = x^i\partial_y + c_i\partial_u$. The commutation relations

$$[X_1, X_3] = X_1, \quad [X_2, X_3] = 0, \quad [X_1, X_4] = 0, \quad [X_2, X_4] = X_2$$

imply that b_1 and b_2 are functions of u only. The relation

$$[X_3, X_4] = (b_1(b_2)_u - b_2(b_1)_u)\partial_u = 0$$

tells us that b_1 is proportional to b_2.

The subalgebra $\mathfrak{r}$ is lifted nontrivially

For some s we have $Y_s = x^s\partial_y + (\alpha + \beta u)D_0\partial_u$. Assuming the lift is affine, the commutation relation $[Y_s, X_3] = -sY_s$ implies that $\beta = 0$, and we can assume that $\alpha = 1$. We get the following conditions on b_1, b_2:

$$[Y_s, X_3] = -sx^s\partial_y + (b_1)_u\partial_u = -sY_s \qquad \Rightarrow \qquad (b_1)_u = -s \tag{24}$$

$$[Y_s, X_4] = x^s\partial_y + (b_2)_u\partial_u = Y_s \qquad \Rightarrow \qquad (b_2)_u = 1 \tag{25}$$

Hence $b_1 = -su + B_1$ and $b_2 = u + B_2$. Since b_1 must be proportional to b_2 we get $B_1 = -sB_2$. A translation in u-direction lets us remove the constants so that $b_1 = -su, b_2 = u$.

We use the rest of the commutation relations:

$$\begin{aligned} [x\partial_x - su\partial_u, x^{s+i}\partial_y + c_{s+i}\partial_u] &= (s+i)x^{s+i}\partial_y + (x(c_{s+i})_x + sc_{s+i})\partial_u \\ &= (s+i)(x^{s+i}\partial_y + c_{s+i}\partial_u) \end{aligned}$$

This gives $x(c_{s+i})_x = ic_{s+i}$ which implies that $c_{s+i} = C_i x^i$. The equation $(c_{s+i})_x = (s+i)c_{s+i-1}$ (coming from commutator of ∂_x and Y_{s+i}) gives the relation $iC_i = (s+i)C_{i-1}$. Thus $C_i = \frac{s+i}{i}C_{i-1} = \frac{(s+i)(s+i-1)}{i(i-1)}C_{i-2} = \binom{s+i}{i}C_0$. We get an affine lift, and may remove the constant by rescaling.

$$\boxed{\begin{array}{r}\partial_x, \quad \partial_y, \quad x\partial_x - su\partial_u, \quad y\partial_y + u\partial_u, \quad x\partial_y, \quad ..., \quad x^{s-1}\partial_y, \\ x^{s+i}\partial_y + \binom{s+i}{i}x^i\partial_y, \quad i = 0, ..., r-4-s.\end{array}} \tag{26}$$

The subalgebra $\mathfrak{r}$ is lifted trivially

Now $Y_i = x^i\partial_y$ for every i. The commutation relations $[Y_i, X_3] = -iY_i$ and $[Y_i, X_4] = Y_i$ give nothing new. But we may assume that the lift is metric, so it's spanned by

$$\boxed{\partial_x, \quad \partial_y, \quad x\partial_x + A\partial_u, \quad y\partial_y + B\partial_u, \quad x^i\partial_y.} \tag{27}$$

This is also the lift we get for $r = 4$.

2.9 $\mathfrak{g}_{11}$

A lift of $\mathfrak{g}_{11}$ is spanned by vector fields of the form $X_1 = \partial_x, X_2 = \partial_y, X_3 = x\partial_x + (C_0 + C_1u)\partial_u, X_4 = y\partial_y + (A_0 + A_1u)\partial_u, X_5 = y^2\partial_y + (2y(A_0 + A_1u) + B_0 + B_1u)\partial_u$. We have used some computations from $\mathfrak{g}_6$ (the example in the paper) and the relations $[X_1, X_3] = X_1, [X_2, X_3] = 0$. The constants satisfy $B_1 = 0, B_0(A_1 + 1) = 0$ and $A_0C_1 - A_1C_0 = 0, B_0C_1 = 0$.

If $C_1 = 0$, then either $A_1 = 0$ and $B_0 = 0$, and we get a metric lift

$$\boxed{\partial_x, \quad \partial_y, \quad x\partial_x + A\partial_u, \quad y\partial_y + B\partial_u, \quad y^2\partial_y + 2By\partial_u} \tag{28}$$

or $C_0 = 0$. In the latter case we have either $B_0 = 0$, in which case $A_1 = 0$ (which is already considered), or $A_1 = -1$ which means that we can set $A_0 = 0$ by a translation, and $B_0 = 1$ by a scaling transformation.

$$\boxed{\partial_x, \quad \partial_y, \quad x\partial_x, \quad y\partial_y - u\partial_u, \quad y^2\partial_y + (1 - 2yu)\partial_u.} \tag{29}$$

If $B_0 = 0$, assume first that A_1 is nonzero. A u-translation lets us set $A_0 = 0$, implying that $C_0 = 0$. This gives an intransitive lift. Let now $B_0 = A_1 = 0$. Either $C_1 = 0$ (which results in the metric lift already considered) or $A_0 = 0$, and $C_1 = 0$ for the lift to be transitive. This lift is already given in (28).

2.10 $\mathfrak{g}_{12}$

A lift of $\mathfrak{g}_{12}$ is spanned by vector fields of the form $X_1 = \partial_x, X_2 = \partial_y, X_3 = x\partial_x + a(u)\partial_u, X_4 = y\partial_y + b(u)\partial_u, X_6 = x^2\partial_x + d(x, y, u)\partial_u, X_5 = y^2\partial_y + c(x, y, u)\partial_u$ where all functions can be assumed to be affine in u.

We still have the same equations as for $\mathfrak{g}_{11}$. In addition the commutation relations give

$$d_y = 0, \quad d_x = 2a, \quad bd_u - db_u = 0, \quad xd_x + ad_u - da_u = d$$

and

$$[X_6, X_5] = (x^2c_x - y^2d_y + dc_u - cd_u)\partial_u = (dc_u - cd_u)\partial_u = 0.$$

Now b and c are proportional to both a and d, as functions of u.

Assume first that $a = A$ and $b = B$ are constants (one or both may be zero). Then we see from the equations that $c = 2By + e(u)$ and $d = 2Ax + f(u)$. The equation $xd_x + ad_u - da_u = d$ implies that $Af_u = f$ which implies $f = 0$ since f is an affine function. In the same way we see that $e = 0$. We then get the metric lift

$$\boxed{\partial_x, \ \partial_y, \ x\partial_x + A\partial_u, \ y\partial_y + B\partial_u, \ y^2\partial_y + 2By\partial_u, \ x^2\partial_x + 2Ax\partial_u.} \quad (30)$$

Now assume that $a = Au$ and $b = Bu$ (the constant parts may be removed by a u-translation). Then we see from the equations that $c = 2Byu + e(u)$ and $d = 2Axu + f(u)$. The equation $xd_x + ad_u - da_u = d$ implies that $Auf' = f(1 + A)$, so that $f = 0$ or $A = -1$ and $f = F$ is constant. In the same way, $e = 0$ or $B = -1$ while $e = E$ is constant. If both f and e vanish, we have a singular lift, so assume that $A = -1$ and that $f = F$ is nonzero.

The equation $db_u - bd_u = 0$ implies that $FB = 0$. This means that $B = 0$, and thus $c = e(u)$. Then the equation $dc_u - cd_u = 0$ implies that $(F - 2xu)e_u + 2xe = 0$. Thus $Fe_u = 0$ and $e - ue_u = 0$. These equations imply that $e = 0$, so we get

$$\boxed{\partial_x, \quad \partial_y, \quad x\partial_x - u\partial_u, \quad y\partial_y, \quad y^2\partial_y, \quad x^2\partial_x + (1 - 2xu)\partial_u.} \quad (31)$$

If we assume that $B = -1$ and $e = E$ we get the lift

$$\boxed{\partial_x, \quad \partial_y, \quad x\partial_x, \quad y\partial_y - u\partial_u, \quad y^2\partial_y + (1 - 2yu)\partial_u, \quad x^2\partial_x.} \quad (32)$$

They are locally equivalent.

2.11 $\mathfrak{g}_{13}$

We consider a lift spanned by $X_1 = \partial_x, X_2 = \partial_y, X_3 = x^2\partial_x + (r-4)xy\partial_y + a\partial_u, X_4 = x\partial_x + \frac{r-4}{2}y\partial_y + b\partial_u, Y_i = x^i\partial_y + c_i\partial_u$ where $r > 4$. Since the stabilizer of 0 is solvable, we need only consider affine lifts. We assume that the subalgebra $\mathfrak{r}$ is lifted as in (8).

We have the commutation relations

$$\begin{aligned}
&[X_1, X_3] = 2x\partial_x + (r-4)y\partial_y + a_x\partial_u = 2X_4 \quad \Rightarrow \quad a_x = 2b \\
&[X_2, X_3] = (r-4)x\partial_y + a_y\partial_u = (r-4)Y_1 \quad \Rightarrow \quad a_y = (r-4)c_1 \\
&[X_1, X_4] = \partial_x + b_x\partial_u = X_1 \quad \Rightarrow \quad b_x = 0 \\
&[X_2, X_4] = \frac{r-4}{2}\partial_y + b_y\partial_u = \frac{r-4}{2}X_2 \quad \Rightarrow \quad b_y = 0
\end{aligned}$$

so b is a function of u only. The first equation implies $a = 2xb(u) + d(y,u)$, while the second one gives $d = (r-4)yc_1(u) + e(u)$, so that $a(x,y,u) = 2xb(u) + (r-4)yc_1(u) + e(u)$. Since the lift is metric or affine, we may assume that $b = B_1 + B_2u, c_1 = (\alpha + \beta u)C_1, e = E_1 + E_2u$. Now, the commutation relation $[X_3, X_4] = -X_3$ gives

$$\begin{aligned}
E_2 = 0, \quad C_1\beta(r-4)(r-6) = 0, \quad E_1(1+B_2) = 0 \\
C_1((r-6)\alpha + 2B_1\beta - 2B_2\alpha)(r-4) = 0
\end{aligned}$$

Let $Y_s = x^s\partial_y + (\alpha+\beta u)C_s\partial_u$ be the first element of $\mathfrak{r}$ that lifts nontrivially (if such exists, if else $C_s = 0$). The commutation relation $[Y_s, X_4] = (r/2 - s - 2)Y_s$ gives

$$\beta C_s(r-2s-4) = 0, \quad C_s((r-2s-4)\alpha + 2B_1\beta - 2B_2\alpha) = 0$$

The commutation relation $[Y_{r-4}, X_3] = 0$ gives the equations

$$\begin{aligned}
0 &= (C_1(r-4)x^{r-4} - x^2c'_{r-4}(x))\beta, \\
0 &= \alpha(C_1(r-4)x^{r-4} - x^2c'_{r-4}(x)) \\
&\quad + 2((B_2x + E_2/2)\alpha - (B_1x + E_1/2)\beta)c_{r-4}(x).
\end{aligned}$$

The subalgebra $\mathfrak{r}$ lifts trivially

If $C_s = 0$, then $\mathfrak{r}$ lifts trivially. Then the only restrictions we have are

$$E_2 = 0, \quad E_1(1+B_2) = 0.$$

If $B_2 = -1$, we can set $B_1 = 0$ by using a translation. We end up with the following lift after scaling u so that $E_1 = 1$.

$$\boxed{\partial_x, \partial_y, x^2\partial_x + (r-4)xy\partial_y + (1-2xu)\partial_u, x\partial_x + \frac{r-4}{2}y\partial_y - u\partial_u, x^i\partial_y} \qquad (33)$$

If $E_1 = 0$, the constants B_1 and B_2 are free, but if $B_2 \neq 0$, the equation $B_1 + B_2 u = 0$ determines a singular orbit, which we do not allow. Thus we get a metric lift.

$$\boxed{\partial_x, \partial_y, x^2\partial_x + (r-4)xy\partial_y + 2B_1 x\partial_u, x\partial_x + \frac{r-4}{2}y\partial_y + B_1\partial_u, x^i\partial_y} \qquad (34)$$

The subalgebra $\mathfrak{r}$ lifts nontrivially

Now $C_s \neq 0$. We can change coordinates (by translation) so that either $\alpha = 0$ or $\beta = 0$, but not both. We may also assume that the other one is equal to 1, since the constants C_i preserve this freedom.

$\alpha = 0$ and $\beta = 1$: We see that $B_1 = E_2 = 0$ and that $s = (r-4)/2$, so $r \geq 6$ is even. The last commutation relation above implies that $B_1 = E_1 = 0$ since $\mathfrak{r}$ lifts nontrivially (and therefore $c_{r-4} \neq 0$). The final condition we get is that either $C_1 = 0$ or $r = 6$. But since $B_1 = E_1 = \alpha = 0$, all possible lifts will have a singular orbit given by $u = 0$.

$\alpha = 1$ and $\beta = 0$: We have $E_2 = 0$ and $B_2 = r/2 - s - 2$. Now, either $C_1 = 0$ or $s = 1$.

If $C_1 = 0$, then $c_{r-4} = Fx^{r-2s-4}$, which means that $c_{r-5} = \frac{c'_{r-4}}{r-4}$ and $c_{r-6} = \frac{c''_{r-4}}{(r-4)(r-5)}, \ldots$, and

$$c_{r-4-(r-2s-4)} = c_{2s} = \frac{c_{r-4}^{(r-4-2s)}}{(r-4)(r-5)\cdots 2s} = \frac{(r-4-2s)!(2s)!}{(r-4)!}F = F/\binom{r-4}{2s}.$$

But c_s is the first (constant) nonzero term, so c_{2s} can not be constant. $C_1 = 0$ is not possible.

Then we must have $s = 1$. This implies that $E_1 = 0$, and that $c_{r-4} = (C_1(r-4)x + D)x^{r-6}$. The relation $[X_4, Y_{r-4}] = \frac{r-4}{2}Y_{r-4}$ implies that $D = 0$. Thus we get the lift

$$\boxed{\begin{array}{r} \partial_x, \partial_y, x\partial_x + \dfrac{r-4}{2}y\partial_y + (B_1 + (r/2-3)u)\partial_u, x^i\partial_y + iC_1x^{i-1}\partial_u, \\ x^2\partial_x + (r-4)xy\partial_y + (2x(B_1 + (r/2-3)u) + (r-4)yC_1)\partial_u. \end{array}} \qquad (35)$$

As long as $r \neq 6$ we may set $B_1 = 0$ by a translation and $C_1 = 1$ by a scaling transformation. If $r = 6$, we get a metric lift.

$$\boxed{\begin{array}{r} \partial_x, \quad \partial_y, \quad x^2\partial_x + 2xy\partial_y + (2xB_1 + 2yC_1)\partial_u, \\ x\partial_x + y\partial_y + B_1\partial_u, \quad x\partial_y + C_1\partial_u, \quad x^2\partial_y + 2C_1x\partial_u \end{array}} \tag{36}$$

2.12 $\mathfrak{g}_{14}$

The lift is of the form $X_1 = \partial_x, X_2 = \partial_y, X_3 = x\partial_x + Af(u)\partial_u, X_4 = y\partial_y + Bf(u)\partial_u, X_5 = x^2\partial_x + (r-5)xy\partial_y + b(x,y,u)\partial_u, Y_i = x^i\partial_y + c_i(x)(\alpha+\beta u)\partial_u$.

The subalgebra $\mathfrak{r}$ lifts trivially

The commutation relation $[X_2, X_5] = (r-5)Y_1$ implies that $b_y = 0$. The relation $[X_1, X_5] = 2X_3 + (r-5)X_4$ implies that $b(x,u) = f(u)((r-5)B + 2A)x + g(u)$. If we let $f(u) = F_1 + F_2u$ and $g(u) = G_1 + G_2u$, we get from $[X_3, X_5] = X_5$ and $[X_4, X_5] = 0$ that

$$G_2 = 0, \quad BF_2G_1 = 0, \quad G_1(AF_2 + 1) = 0.$$

There are two possibilities: either $G_1 = 0$ or both $B = 0$ and $AF_2 + 1 = 0$.

If $G_1 = 0$, we may assume (after a translation) that either $F_2 = 0$ or $F_1 = 0$. If $F_1 = 0$ we get a singular lift. If $F_2 = 0$ we may without loss of generality assume that $F_1 = 1$ (the constant is absorbed by A and B). We get a metric lift

$$\boxed{\begin{array}{r} \partial_x, \quad \partial_y, \quad x\partial_x + A\partial_u, \quad y\partial_y + B\partial_u, \\ x^2\partial_x + (r-5)xy\partial_y + ((r-5)B + 2A)x\partial_u, \quad x^i\partial_y. \end{array}} \tag{37}$$

If $G_1 \neq 0$ then $B = 0$, $A \neq 0$ and $F_2 = -1/A$ and $F_1 = 0$ after a translation. A scaling transformation ($u \mapsto u/G_1$) lets us set $G_1 = 1$.

$$\boxed{\begin{array}{r} \partial_x, \quad \partial_y, \quad x\partial_x - u\partial_u, \quad y\partial_y, \\ x^2\partial_x + (r-5)xy\partial_y + (1 - 2ux)\partial_u, \quad x^i\partial_y. \end{array}} \tag{38}$$

The subalgebra $\mathfrak{r}$ lifts nontrivially

Let $Y_s = x^s\partial_y + (\alpha + \beta u)C_s\partial_u$ be the lift of the first basis element of the subalgebra $\mathfrak{r}$ which lifts nontrivially and let $f(u) = F_1 + F_2u$. The commutation relation $[Y_s, X_3] = -sY_s$ implies that $\beta = 0$ and $A = -s/F_2$. In order for $\mathfrak{r}$ to lift nontrivially we must have $\alpha \neq 0$ and we may set $\alpha = 1$. The relation $[Y_s, X_4] = Y_s$ implies that $B = 1/F_2$.

To sum up what we got so far, $Af(u) = A(F_1 + F_2u) = A_1 - su$ and $Bf(u) = B(F_1 + F_2u) = B_1 + u$ where $A_1 = -sB_1$.

$$x\partial_x - s(B_1 + u)\partial_u, \quad y\partial_y + (B_1 + u)\partial_u$$

A translation lets us set $B_1 = 0$.

The commutation relations $[X_2, X_5] = (r-5)Y_1$ and $[X_1, X_5] = 2X_3 + (r-5)X_4$ imply that $b(x, y, u) = g(u) + (r-5-2s)xu + (r-5)yC_1$, where C_1 is the constant in $Y_1 = x\partial_y + C_1\partial_u$. Then $[X_4, X_5] = 0$ implies $g(u) = Gu$. And $[X_3, X_5] = X_5$ implies $G = 0$. So we get

$$X_5 = x^2\partial_x + (r-5)xy\partial_y + ((r-5-2s)xu + (r-5)C_1y)\partial_u.$$

Here we have that $s = 1$ and $C_1 \neq 0$, or alternatively that $s > 1$ and $C_1 = 0$.

Assume that $s = 1$. The commutation relation given by $[X_5, x^{r-5}\partial_y + c_{r-5}(x)\partial_u] = 0$ implies that

$$c_{r-5}(x) = C_1(r-5)x^{r-6} + Dx^{r-7}.$$

The commutation relation $[X_3, Y_{r-5}] = (r-5)Y_{r-5}$ implies that $D = 0$ and we get the following lift.

$$\boxed{\begin{aligned} \partial_x, \quad \partial_y, \quad x\partial_x - u\partial_u, \quad y\partial_y + u\partial_u, \\ x^2\partial_x + (r-5)xy\partial_y + ((r-7)xu + (r-5)y)\partial_u, \quad x^i\partial_y + ix^{i-1}\partial_u. \end{aligned}} \tag{39}$$

Now assume that $s > 1$. Then $C_1 = 0$. The commutation $[X_3, Y_{r-5}] = (r-5)Y_{r-5}$ implies that $\mathfrak{r}$ must lift trivially.

2.13 $\mathfrak{g}_{15}$

Let $X_1 = \partial_x, X_2 = x\partial_x + \partial_y, X_3 = x^2\partial_x + 2x\partial_y + a(x, y, u)\partial_u$. The commutation relations are

$$\begin{aligned} [X_1, X_3] &= 2x\partial_x + 2\partial_x + a_x\partial_u = 2X_2 + 2X_1 \quad \Rightarrow \quad a_x = 0 \\ [X_2, X_3] &= x^2\partial_x + 2x\partial_x + (xa_x + a_y)\partial_u = X_3 \quad \Rightarrow \quad xa_x + a_y = a \end{aligned}$$

Hence $a(x, y, u) = b(u)e^y$. In order to get a nonsingular lift we may assume that b is constant. The nontrivial lift thus becomes

$$\boxed{\partial_x, \quad x\partial_x + \partial_y, \quad x^2\partial_x + 2x\partial_y + Ce^y\partial_u.} \tag{40}$$

2.14 $\mathfrak{g}_{16}$

Since the dimension is 3, we may assume that the lift is metric. Let $X_1 = \partial_x, X_2 = x\partial_x - y\partial_y + b(x,y)\partial_u, X_3 = x^2\partial_x + (1-2xy)\partial_y + c(x,y)\partial_u$. Two of the commutation relations give $b_x = 0, c_x = 2b$. Thus $b = b(y)$ and $c = 2b(y)x + f(y)$. The last commutation relation gives $b(y) = A - yf(y)$, so that $c = 2xA + f(y)(1-2xy)$.

After applying the transformation $u \mapsto u - \int f(y)dy$, the lifted vector fields take the form

$$\boxed{\partial_x, \qquad x\partial_x - y\partial_y + A\partial_u, \qquad x^2\partial_x + (1-2xy)\partial_y + 2Ax\partial_u.} \tag{41}$$

Alternatively we can take a singular version of $\mathfrak{g}_{16}$ (which we use in the example in section 5) and look for metric lifts. Let $X_1 = \partial_x, X_2 = x\partial_x + y\partial_y + b(x,y)\partial_u, X_3 = x^2\partial_x + y(2x+y)\partial_y + c(x,y)\partial_u$. This is $\langle \partial_x + \partial_y, x\partial_x + y\partial_y, x^2\partial_x + y^2\partial_y \rangle$ after transformation $Y = y - x$.

$$[X_1, X_2] = \partial_x + b_x\partial_u \quad \Rightarrow \quad b_x = 0$$

Let $b(y) = b_0 + yf(y)$. The transformation $u \mapsto u - \int f(y)dy$ turns X_2 into $x\partial_x + y\partial_y + b_0\partial_u$

$$[X_1, X_3] = 2x\partial_x + 2y\partial_y + c_x\partial_u \quad \Rightarrow \quad c_x = 2b_0$$
$$[X_2, X_3] = x^2\partial_x + y(2x+y)\partial_y + (xc_x + yc_y)\partial_u \quad \Rightarrow \quad xc_x + yc_y = c$$

We end up with the lift

$$\partial_x, \quad x\partial_x + y\partial_y + A\partial_u, \quad x^2\partial_x + y(2x+y)\partial_y + (2Ax + By)\partial_u. \tag{42}$$

The transformation $(x,y) \mapsto (-x, 1/y)$ takes this lift to

$$\partial_x, \quad x\partial_x - y\partial_y + A\partial_u, \quad x^2\partial_x + (1-2xy)\partial_y + (2Ax - B/y)\partial_u \tag{43}$$

so we see that $B = 0$ gives the first metric lift.

References

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