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Generating polytopes from Coxeter groups

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“To find out what one is fitted to do and to secure an
opportunity to do it is the key to happiness.”
–John Dewey

Abstract

The main goal of this thesis is to study finite reflection groups (Coxeter groups) W and to use these to generate polytopes in two and three dimensions. Such polytopes will be generated as the convex hull of the W-orbit through an initial point λ. We prove an efficient recipe for finding the stabilizer of λ, and examine several examples of such polytopes and illustrate how many vertices, edges and faces these polytopes have. At last we will illustrate how this information can be pictorially encoded on the marked Coxeter diagram for λ.

Acknowledgements

Five years of studies have come to an end, and when I arrived here for five years ago I did not know anyone. Through the years I have got a lot of good friends that will always have a special place in my heart. Without these people this had not been possible.

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Veronica Viken
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Introduction

In this thesis we will use finite reflection groups (Coxeter groups) to generate two and three-dimensional polytopes. Such groups act as the symmetries of these objects. Polytopes in two dimensions are polygons, which are geometrical figures such as squares, triangles, pentagons and so on. The polytopes in three dimensions are called polyhedra, and examples of polyhedra are the five Platonic solids:

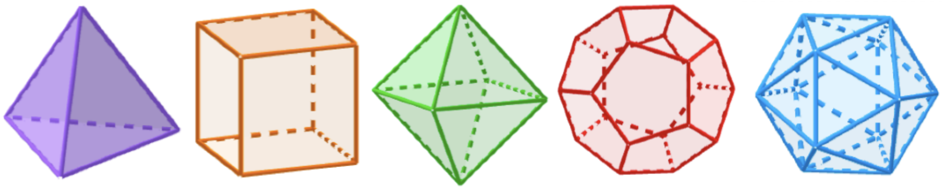


Figure 1.1: The five Platonic solids: tetrahedron, cube, octahedron, dodecahedron and icosahedron.

These are all well known, and pictorially and abstractly we want to figure out how many vertices, edges and faces that more general polyhedra consist of. For the three dimensional polyhedra we will use Zometool to build models of these geometrical figures [Hart, 2001]. Abstractly, we will examine the action of Coxeter groups on different initial points λ, and then get a recipe to encode the marked Coxeter diagrams.

In the Norwegian school geometric figures in two and three dimensions such as squares, triangles, parallelograms, rhombus, rectangles, cubes, tetrahedrons, prisms and pyramids are an important part of the curriculum. After upper elementary school, students shall be able to explore and describe different properties of such two and three dimensional figures. An important property of these figures is their symmetry group. They shall also be able to reflect points in $\mathbb{R}^2$ through the x-axis and the y-axis. So the geometrical figures we generate, how we generate them by reflections and the symmetry group are relevant for the Norwegian school.

1.1 Structure of the thesis

Chapter 2 introduces reflections, finite groups generated by reflections (Coxeter groups), and Coxeter diagrams and Dynkin diagrams. Chapter 3 we will get pictorial reflection recipes expressed in terms of Coxeter diagrams and the Dynkin diagrams. We also find a recipe for finding the stabilizer for a given initial point λ. Chapter 4 we first will define polytopes. Then we will apply Coxeter groups to an initial point λ to generate polytopes in two and three dimension. A last we will encode the marked Coxeter diagram to figure out what kind of polytope we have.

2

Coxeter/ Dynkin diagram

2.1 Reflections and rotations

Let $(V, \langle\cdot,\cdot\rangle)$ be a finite-dimensional (real) inner product space. A *reflection* in V is a linear transformation in V that sends any vector $\lambda \in V$ to its mirror image with respect to a hyperplane $\mathscr{P}$ (passing through the origin). Equivalently, given a nonzero vector $\alpha \in V$ (in the orthogonal complement $\mathscr{P}^{\perp}$) define a reflection in V by

$$S_\alpha(\lambda) = \lambda - 2\frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha. \tag{2.1.1}$$

Note that $S_\alpha(\lambda) = \lambda$ if $\lambda \in \mathscr{P}$ and $S_\alpha(\lambda) = -\lambda$ if $\lambda \in \mathscr{P}^{\perp}$. Since $S_\alpha^2 = 1$, then a reflection is its own inverse, so its order is two. Also note that $\langle S_\alpha(\lambda), S_\alpha(\mu)\rangle = \langle\lambda,\mu\rangle$ for all $\lambda,\mu \in V$, i.e. S_α is an *orthogonal transformation*.

The orthogonal transformations in V are denoted by

$$O(V) = \{T : V \longrightarrow V \text{ linear } : \langle Tv, Tw\rangle = \langle v,w\rangle \forall v,w \in V\},$$

and any $T \in O(V)$ satisfies $\det(T) = \pm 1$. Note that $O(V)$ is a group, and it contains the subgroup

$$SO(V) = \{T \in O(V) : \det(T) = 1\}.$$

We refer to any $T \in SO(V)$ as a *rotation*. When $V = \mathbb{R}^n$, we use the notation $O(n)$ and $SO(n)$ respectively. In particular, $O(n)$ consists of orthogonal matrices

A satisfying $AA^T = A^T A = I$, and the columns of A form an orthonormal basis of $\mathbb{R}^n$.

For example, consider a matrix in $O(2)$. Its first column is a unit vector $\left(\begin{smallmatrix}\cos\theta\\ \sin\theta\end{smallmatrix}\right)$, while the second column is unit and perpendicular to the first. Thus, we have two possibilities:

$$A = \begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix} \qquad\qquad B = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}.$$

Note that $\det(B) = 1$ and it represents a counterclockwise rotation of the plane about the origin through an angle of θ. On the other hand, $\det(A) = -1$, and A has eigenvalues 1 and -1 with eigenvectors $\left(\begin{smallmatrix}\cos(\frac{\theta}{2})\\ \sin(\frac{\theta}{2})\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}-\sin(\frac{\theta}{2})\\ \cos(\frac{\theta}{2})\end{smallmatrix}\right)$ respectively. Thus, A is a reflection in the line $\mathscr{P}$ corresponding to the $+1$ eigenspace.

We see that if $A \in O(2)$ with $\det(A) = -1$, then A corresponds to a reflection. When $n \geq 3$, having $A \in O(n)$ with $\det(A) = -1$ does not necessarily imply that A corresponds to a reflection, e.g. $\left(\begin{smallmatrix}-1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{smallmatrix}\right)$.

We will be interested in finite reflection groups, i.e. finite subgroups of $O(V)$ generated by reflections. Let us study the two-dimensional case first.

2.2 Dihedral group

Let G be a *finite* subgroup of $O(V)$. The set of rotations H in G forms a subgroup.

Let us first show that H is a cyclic subgroup. For any rotation $T \in H$, let $\theta(T) \in [0, 2\pi)$ denote its corresponding angle of (counterclockwise) rotation. Suppose $H \neq 1$. Since H is finite, we can choose $1 \neq R \in H$ with $\theta(R)$ minimal. Given $T \in H$, choose an integer k such that $k\theta(R) \leq \theta(T) < (k+1)\theta(R)$, so then

$$0 \leq \theta(T) - k\theta(R) < \theta(R).$$

But $\theta(T) - k\theta(R) = \theta(R^{-k}T)$ and $R^{-k}T \in H$ is a counterclockwise rotation through $\theta(T)$ followed by $-k$ rotations through $\theta(R)$, and since $\theta(R)$ is minimal we must have $R^{-k}T = 1$, i.e. $T = R^k$. Thus, H is a cyclic subgroup. Let $m = |H|$ be its order. Since $R^m = 1$, then $\theta(R) = \frac{2\pi}{m}$.

Now suppose that $H \neq G$. If $S, T \in G$ are two reflections, then $\det(ST) = \det(S)\det(T) = 1$, so $ST \in H$. This implies that the left cosets SH and TH

agree, and so H has index 2 in G. Given $H = \{1, R, \ldots R^{m-1}\}$, we obtain

$$G = \{1, R, \ldots, R^{m-1}, S, SR, \ldots SR^{m-1}\},$$

so $|G| = 2m$. Since RS is a reflection, then $(RS)^2 = 1$ and $RS = SR^{-1} = SR^{m-1}$. The group G is called the *dihedral group* (of order $2m$). It consists of m reflections and m rotations through integer multiples of $\frac{2\pi}{m}$. We also write $G = \langle R, S \rangle$ to denote that G is generated by R and S. Note that $T = SR$ is a reflection and $G = \langle T, S \rangle$, so G is a finite reflection group.

Proposition 2.2.1. *If* $\dim V = 2$*, then a finite subgroup of* $O(V)$ *is either a cyclic group* C_m *or a dihedral group* $\mathcal{D}_m$*. Any dihedral group is a reflection group.*

The dihedral group is the symmetry group of a regular m-gon. For example, if $m = 3$ we have an equilateral triangle, $m = 4$ a square, $m = 5$ an equiangular pentagon and so on.

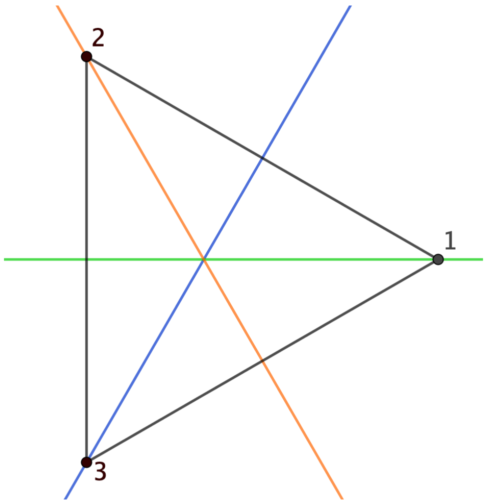


Figure 2.1: A triangle with its reflection lines.

Consider the $m = 3$ case as in Figure 2.1. Let R be counterclockwise rotation by $\frac{2\pi}{3}$ and let S be the reflection in the green line. Then SR is the reflection in the orange line, and SR^2 is the reflection in the blue line. Any element of $G = \mathcal{D}_3$ permutes the vertices, so if we label the vertices as above, we can express elements of G in permutation notation:

1	R	R^2	S	SR	SR^2
1	(123)	(132)	(23)	(13)	(12)

Thus, G is isomorphic to the permutation group $\mathfrak{S}_3$ on 3 symbols.

2.3 Root systems

We will be interested in finite subgroups of $O(V)$ that are generated by reflections. Since any reflection is determined by a hyperplane or, via $\langle\cdot,\cdot\rangle$, by some vector α, we are led to consider subsets $\Phi \subset V\backslash\{0\}$ that are invariant under S_α for all $\alpha \in \Phi$.

Definition 2.3.1. $\Phi \subset V\backslash\{0\}$ *is a* root system *if* $\forall\alpha \in \Phi$,

1. $-\alpha \in \Phi$,
2. $S_\alpha\Phi = \Phi$.

Any $\alpha \in \Phi$ is called a *root*. The *Weyl group* of Φ is the subgroup $W = W(\Phi)$ of $O(V)$ generated by $\{S_\alpha : \alpha \in \Phi\}$.

Remark 2.3.2. We do not make any assumption on the lengths of the roots. Also, given any root system Φ, note that $\widetilde{\Phi} := \{\frac{\alpha}{|\alpha|} : \alpha \in \Phi\}$ is a root system.

Example 2.3.3 (I_2^m root system)**.** *Let* $V = \mathbb{C}$. *Regard this as a two-dimensional real vector space with inner product* $\langle z, w\rangle = \Re(z\overline{w}) = \frac{1}{2}(z\overline{w} + \overline{z}w)$ *for* $z, w \in \mathbb{C}$. *Define* $\Phi = \{\zeta_k := e^{\frac{\pi i k}{m}} : k \in \mathbb{Z}/2m\mathbb{Z}\}$, *i.e. the* $2m$*-th roots of unity. Note that* $-\zeta_k = \zeta_{k+m} \in \Phi$, *while*

$$S_{\zeta_k}(\zeta_l) = \zeta_l - \frac{2\langle\zeta_l,\zeta_k\rangle\zeta_k}{\langle\zeta_k,\zeta_k\rangle} = \zeta_l - (\zeta_l\overline{\zeta_k} + \overline{\zeta_l}\zeta_k)\zeta_k = -\overline{\zeta_l}\zeta_k^2 = \zeta_{2k-l+m}.$$

Thus, Φ *is a root system. Its Weyl group is the dihedral group* $\mathcal{D}_m$. *Note that* I_2^3 *is also denoted* A_2. *This root system is shown is Figure 2.2*

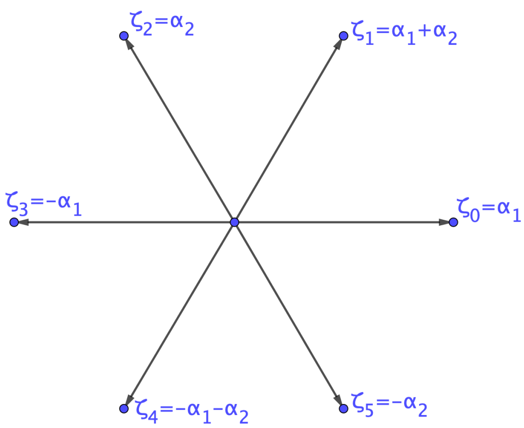


Figure 2.2: The A_2 root system.

Fix $t \in V$ such that $\langle t, \alpha \rangle \neq 0$ for all $\alpha \in \Phi$. We say that $\alpha \in \Phi$ is *positive* (and write $\alpha > 0$) if $\langle t, \alpha \rangle > 0$, or *negative* otherwise. The *positive system* is $\Pi = \{\alpha \in \Phi : \alpha > 0\}$, so that $\Phi = \Pi \cup (-\Pi)$.

Definition 2.3.4. *We say $\Delta = \{\alpha_1, \ldots, \alpha_n\} \subset \Phi$ is a* simple system *(and each α_i is a* simple root*) if Δ is a basis for* span(Φ)*, and for any $\alpha \in \Phi$, we have $\alpha = \sum_i m_i \alpha_i$ with either all $m_i \geq 0$ or all $m_i \leq 0$ (we do not require $m_i \in \mathbb{Z}$).*

Existence and uniqueness of simple systems is established in the theorem below [Humphreys, 1990, Thm 1.3].

Theorem 2.3.5. *If $\Delta \subset \Phi$ is a simple system, there is a unique positive system Π containing Δ. Conversely, every positive system $\Pi \subset \Phi$ contains a simple system Δ ; in particular, a simple system exists.*

Remark 2.3.6. For the root systems the rank is defined as the number of simple roots. The rank is independent of the choice of simple system.

Definition 2.3.7. *Let $\cos\theta_{ij} = \frac{\langle \alpha_i, \alpha_j \rangle}{|\alpha_i||\alpha_j|} \in (0, \pi)$ denote the angle between α_i and α_j.*

Then [Benson, 1985, Prop 5.1.1]

Proposition 2.3.8. *There exists an integer $p_{ij} \geq 1$ such that*

$$\theta_{ij} = \pi - \frac{\pi}{p_{ij}}. \tag{2.3.1}$$

Moreover, for $i \neq j$, $p_{ij} \geq 2$ is the order of $S_i S_j$ in W.

Example 2.3.9. *For the I_2^m root system, choose $\Pi = \{\zeta_k\}_{0 \leq k \leq m-1}$ and simple roots $\alpha_1 = \zeta_0$ and $\alpha_2 = \zeta_{m-1}$. Then*

$$\cos(\theta_{12}) = \frac{\langle \zeta_0, \zeta_{m-1} \rangle}{|\zeta_0||\zeta_{m-1}|} = \Re(\overline{\zeta_{m-1}}) = \cos\left(\frac{\pi(m-1)}{m}\right).$$

Thus, $\theta_{12} = \pi - \frac{\pi}{m}$, which confirms (2.3.1).

Example 2.3.10 (A_n root system)**.** *Consider $\mathbb{R}^{n+1}$ with the standard orthonormal basis $\{\epsilon_i\}_{i=1}^{n+1}$ and coordinates $\{x_i\}_{n=1}^{n+1}$ with respect to this basis. Let V be the hyperplane $x_1 + x_2 + \cdots + x_n = 0$. On V, define $\Phi = \{\epsilon_i - \epsilon_j\}_{1 \leq i \neq j \leq n+1}$ and take $\Pi = \{\epsilon_i - \epsilon_j\}_{1 \leq i < j \leq n+1}$ to be the positive system. The simple roots are*

$$\alpha_1 = \epsilon_1 - \epsilon_2, \quad \ldots, \quad \alpha_n = \epsilon_n - \epsilon_{n+1}.$$

We have $\theta_{i,i+1} = \frac{2\pi}{3}$ for $1 \leq i \leq n$ and $\theta_{ij} = \frac{\pi}{2}$ when $|i - j| \geq 2$. Note that all roots have the same length. In Figure 2.3 below we have the A_3 root system

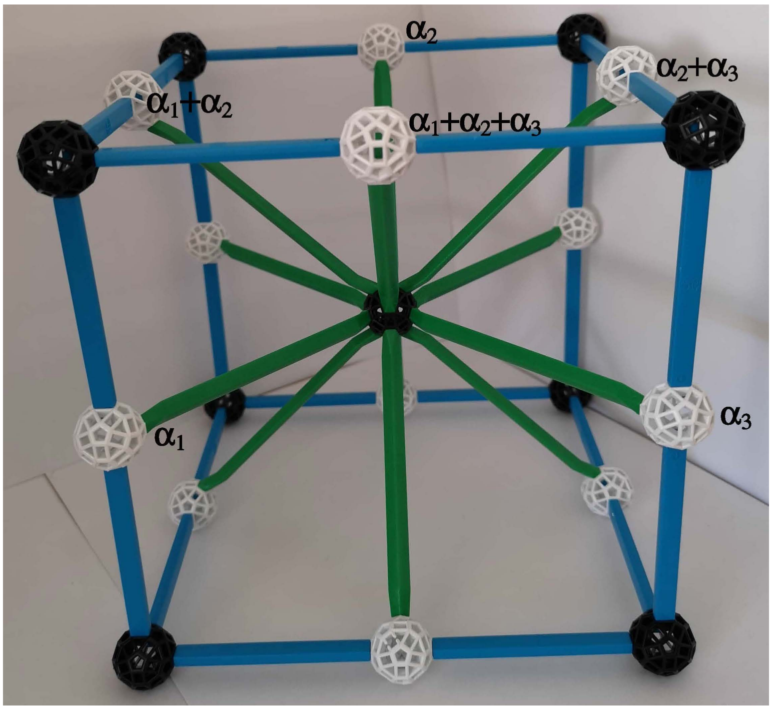


Figure 2.3: The A_3 root system with the white nodes as roots.

Example 2.3.11 (B_n root system)**.** *Let $V = \mathbb{R}^n$ with the standard orthonormal basis $\{\epsilon_i\}_{i=1}^n$. Define the root system $\Phi = \{\pm\epsilon_i\}_{i=1}^n \cup \{\pm\epsilon_i \pm \epsilon_j\}_{1\leq i<j\leq n}$ and take $\Pi = \{\epsilon_i\}_{i=1}^n \cup \{\epsilon_i \pm \epsilon_j\}_{1\leq i<j\leq n}$ as the positive system. The simple roots are*

$$\alpha_1 = \epsilon_1 - \epsilon_2, \quad \alpha_2 = \epsilon_2 - \epsilon_3, \quad \ldots, \quad \alpha_{n-1} = \epsilon_{n-1} - \epsilon_n, \quad \alpha_n = \epsilon_n.$$

We have $\theta_{ij} = \frac{\pi}{2}$ if $|i-j| \geq 2$, and

$$\theta_{12} = \theta_{23} = \cdots = \theta_{n-2,n-1} = \frac{2\pi}{3}, \quad \theta_{n-1,n} = \frac{3\pi}{4}.$$

Note that not all of the roots have unit length and $\sqrt{2}|\alpha_n| = |\alpha_{n-1}| = \cdots = |\alpha_1|$. In Figure 2.4 below we have the B_3 root system,

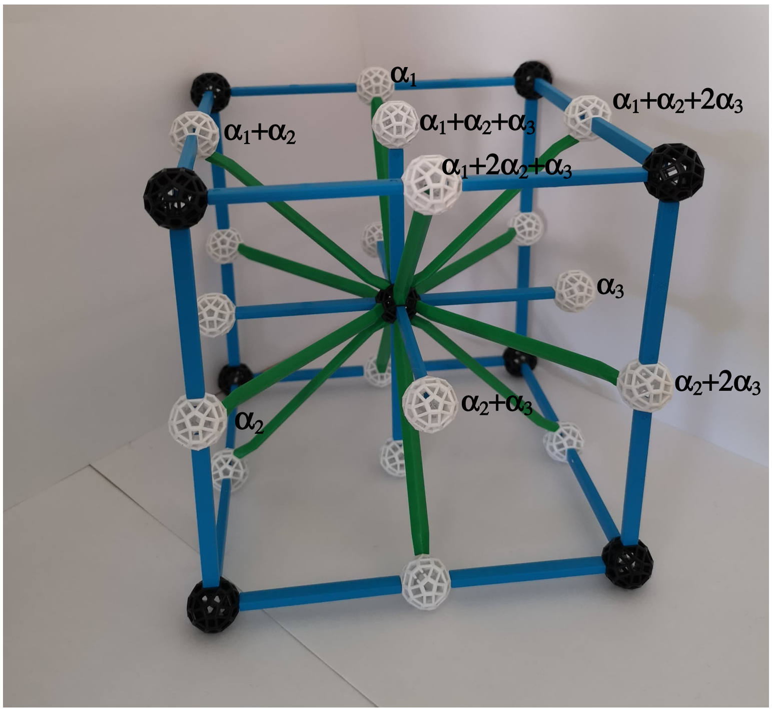


Figure 2.4: The B_3 root system with white nodes as roots.

Define the normalized root system for B_n *as* $\widetilde{B_n} = \{\frac{\alpha}{|\alpha|} : \alpha \in \Delta\}$. *Here, all roots have the same length.*

Example 2.3.12 (H_3 root system, [Humphreys, 1990, Section 2.13])**.** *Let* $\tau = 2\cos\left(\frac{\pi}{5}\right) = \frac{1+\sqrt{5}}{2}$, *which is better known as* the golden ratio. *Two quantities* a, b *satisfy the golden ratio if* $\frac{a}{b} = \frac{a+b}{a} = \tau$. *Solving for* τ *we get*

$$\tau^2 = \tau + 1 \tag{2.3.2}$$

Then let $V = \mathbb{R}^3$, *and we define the root system* $\Phi = \{\pm\epsilon_i\}_{1\le i\le 3} \cup \{(\pm\frac{\tau}{2}, \pm\frac{1}{2}, \pm\frac{\tau-1}{2})\} \cup \{$*all even permutations of each coordinates*$\}$. *Therefore* H_3 *will have* 30 *roots, where the simple roots are*

$$\alpha_1 = \left(\frac{\tau}{2}, \frac{-1}{2}, \frac{\tau-1}{2}\right), \quad \alpha_2 = \left(\frac{-\tau}{2}, \frac{1}{2}, \frac{\tau-1}{2}\right), \quad \alpha_3 = \left(\frac{1}{2}, \frac{\tau-1}{2}, -\frac{\tau}{2}\right).$$

We have

$$\theta_{12} = \frac{4\pi}{5}, \quad \theta_{23} = \frac{2\pi}{3}, \quad \theta_{13} = \frac{\pi}{2}.$$

It is easily checked that all the roots in H_3 *have unit length. The* H_3 *root system can be pictured on an icosadodecahedron like in Figure 2.5 where all the vertices correspond to roots in* H_3.

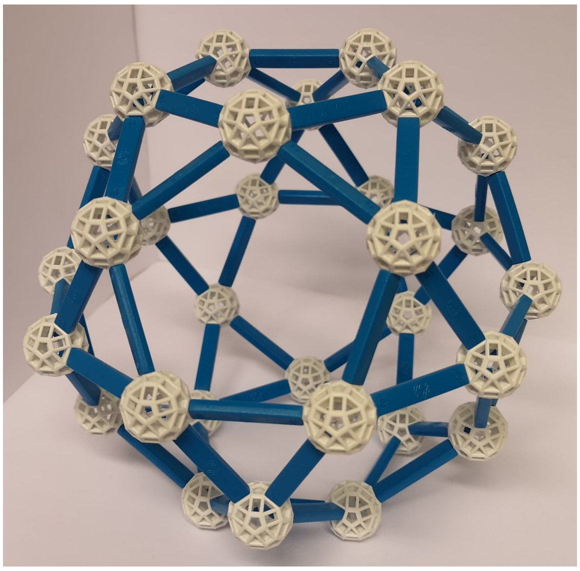


Figure 2.5: An icosadodecahedron where all vertices form the H_3 root system.

Root systems that arise from Lie theory [Humphreys, 1972] satisfy an additional crystallographic condition:

Definition 2.3.13. *A root system* Φ *is* crystallographic *if* $\forall \alpha, \beta \in \Phi$,

$$\langle \alpha, \beta^\vee \rangle \in \mathbb{Z}, \tag{2.3.3}$$

where $\beta^\vee$ *is the* coroot *of* β *, defined by*

$$\beta^\vee = \frac{2\beta}{\langle \beta, \beta \rangle}. \tag{2.3.4}$$

The A_n root system is crystallographic. The B_n root system as in Example 2.3.11 is also crystallographic, but the normalized root system $\widetilde{B_n}$ is not crystallographic. Also, H_3 is not crystallographic, and I_2^m is only crystallographic when $m = 3$.

2.4 Cartan matrix

Let Φ be a root system with the simple roots $\Delta = \{\alpha_i\}_{i=1}^n$.

Definition 2.4.1. *The* Cartan matrix C_{ij} *is defined by*

$$C_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle = \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle}.$$

By Definition 2.3.7 and Proposition 2.3.8, this can be rewritten as:

Proposition 2.4.2. *The Cartan matrix C_{ij} for (Φ, Δ) is computed by*

$$C_{ij} = -\frac{2|\alpha_i|}{|\alpha_j|}\cos\left(\frac{\pi}{p_{ij}}\right),$$

where $\theta_{ij} = \pi - \frac{\pi}{p_{ij}}$ is the angle between α_i and α_j.

Example 2.4.3 (Cartan matrices)**.** *In order to compute the Cartan matrices for I_2^m, A_n, B_n, $\widetilde{B_n}$ and H_3 we will use the lengths and the angles between the simple roots from the examples in Section 2.3. Therefore we get*

Root system	C_{ij}
I_2^m	$\begin{pmatrix} 2 & -2\cos\left(\frac{\pi}{m}\right) \\ -2\cos\left(\frac{\pi}{m}\right) & 2 \end{pmatrix}$
A_n	$\begin{pmatrix} 2 & -1 & 0 & & & & \\ -1 & 2 & -1 & 0 & & & \\ 0 & -1 & 2 & -1 & 0 & & \\ & & \ddots & \ddots & \ddots & & \\ & & 0 & -1 & 2 & -1 & 0 \\ & & & 0 & -1 & 2 & -1 \\ & & & & 0 & -1 & 2 \end{pmatrix}$
B_n	$\begin{pmatrix} 2 & -1 & 0 & & & & \\ -1 & 2 & -1 & 0 & & & \\ 0 & -1 & 2 & -1 & 0 & & \\ & & \ddots & \ddots & \ddots & & \\ & & 0 & -1 & 2 & -1 & 0 \\ & & & 0 & -1 & 2 & -2 \\ & & & & 0 & -1 & 2 \end{pmatrix}$
$\widetilde{B_n}$	$\begin{pmatrix} 2 & -1 & 0 & & & & \\ -1 & 2 & -1 & 0 & & & \\ 0 & -1 & 2 & -1 & 0 & & \\ & & \ddots & \ddots & \ddots & & \\ & & 0 & -1 & 2 & -1 & 0 \\ & & & 0 & -1 & 2 & -\sqrt{2} \\ & & & & 0 & -\sqrt{2} & 2 \end{pmatrix}$
H_3	$\begin{pmatrix} 2 & -\tau & 0 \\ -\tau & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$

Remark 2.4.4. For the crystallographic root systems, $m = C_{ij}C_{ji}$ can only take the values $0, 1, 2, 3$.

Proof.

$$\begin{aligned} m = C_{ij}C_{ji} = \langle \alpha_i, \alpha_j^\vee \rangle \langle \alpha_j, \alpha_i^\vee \rangle = \frac{4\langle \alpha_i, \alpha_j \rangle^2}{\langle \alpha_j, \alpha_j \rangle \langle \alpha_i, \alpha_i \rangle} = \frac{4|\alpha_i|^2|\alpha_j|^2(\cos(\theta_{ij}))^2}{|\alpha_i|^2|\alpha_j|^2} \\ = 4(\cos(\theta_{ij}))^2. \end{aligned}$$

By Definition 2.3.13 and 2.4.1 $\langle \alpha_i, \alpha_j^\vee \rangle = C_{ij} \in \mathbb{Z}$ and also C_{ji}, therefore $m = 4(\cos(\theta_{ij}))^2 \in \mathbb{Z}$ and $0 \leq m = 4(\cos(\theta_{ij}))^2 \leq 4$. But we cannot have an angle of 0 or π, since α_i and α_j are linearly independent so m only can take the value $0, 1, 2, 3$. □

2.5 Coxeter diagrams and Dynkin diagrams

Let Φ be a root system with simple roots $\Delta = \{\alpha_i\}_{i=1}^n$. Then:

Definition 2.5.1. *The Coxeter diagram of* (Φ, Δ) *is the graph with:*

- *nodes corresponding to simple roots;*
- *nodes corresponding to distinct* α_i *and* α_j *are connected by a bond if* $p_{ij} > 2$*. The bond is marked with* p_{ij} *underneath if* $p_{ij} > 3$*. (The* $p_{ij} = 3$ *occurs frequently, so by convention we omit this.)*

For the root systems mentioned in Section 2.3 and the rest of the connected Coxeter diagrams are given in Table 2.1 for details of the classification see [Benson, 1985, Ch. 5]:

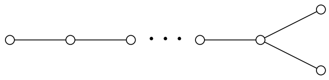
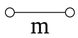
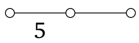
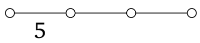
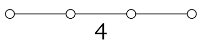
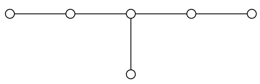
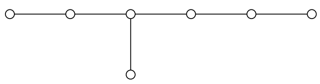
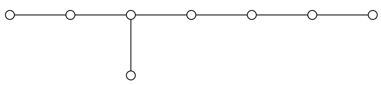
	Graph
A_n $(n \geq 1)$	
B_n $(n \geq 2)$	
D_n $(n \geq 4)$	
I_2^m	
H_3	
H_4	
F_4	
E_6	
E_7	
E_8	

Table 2.1: The complete classification of all connected Coxeter diagrams.

Note that the B_n and $\widetilde{B_n}$ root systems have the same Coxeter diagram.

If we are looking at the crystallographic root systems like A_n, B_n, they arise as the Weyl groups of simple Lie algebras. Here relative length of roots are important, and these can be encoded in Dynkin diagrams.

Definition 2.5.2. *The Dynkin diagram of a crystallographic root system* (Φ, Δ) *is the graph with:*

- *nodes corresponding to simple roots;*
- *nodes corresponding to distinct* α_i *and* α_j *are connected by a bond if and only if* $C_{ij} \neq 0$*. The bond has multiplicity* $C_{ij}C_{ji}$ *and it is directed towards the shorter root. (If* $C_{ij}C_{ji} = 1$*, then the two roots have same length and the bond is undirected.)*

Remark 2.5.3. From the Coxeter diagrams we only encode angles between the simple roots, but in the Dynkin diagrams we can encode angles and relative lengths.

For the crystallographic root systems mentioned in Section 2.3, and the rest of the connected Dynkin diagrams are given in Table 2.2. For details of the classification see [Humphreys, 1972, Ch. 11].

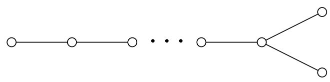
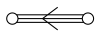
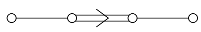
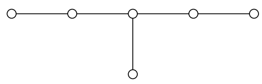
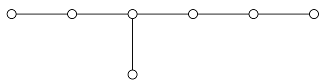
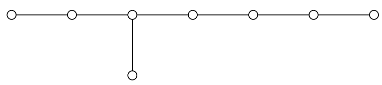
Graph	Dynkin diagram
A_n $(n \geq 1)$	
B_n $(n \geq 2)$	
C_n $(n \geq 3)$	
D_n $(n \geq 4)$	
G_2	
F_4	
E_6	
E_7	
E_8	

Table 2.2: The complete classification of all connected Dynkin diagrams.

2.6 Coxeter group

Definition 2.6.1 ([Benson, 1985][p. 37]). *A* Coxeter group *is a finite effective subgroup $W \leq O(V)$ that is generated by a set of reflections.*

Here effective means $\{x \in V : Tx = x \forall T \in W\} = 0$. Given a root system Φ, let $W = W(\Phi)$ be the subgroup of $O(V)$ generated by the reflections $\{S_\alpha : \alpha \in \Phi\}$. We refer to W as the *Weyl group* of Φ. This is an instance of a Coxeter group. Given a simple system $\Delta = \{\alpha_i\}_{i=1}^n$, refer to all $S_i := S_{\alpha_i}$ as *simple reflections*. These in fact determine the structure of W as the following two theorems show:

Theorem 2.6.2 ([Humphreys, 1990, p. 11]). *The simple reflections $S_1, \dots, S_n$ generate W.*

Theorem 2.6.3 ([Benson, 1985, Proposition 5.1.1]). *All the relations in W are generated by $S_i^2 = 1$ and $(S_iS_j)^{p_{ij}} = 1$, where $p_{ij} \geq 2$ for $i \neq j$ was defined in Proposition 2.3.8.*

If $p_{ij} = 2$, then $1 = (S_iS_j)^2 = S_iS_jS_iS_j$, i.e. $S_iS_j = S_jS_i$. Therefore, S_i and S_j commute when $p_{ij} = 2$.

Example 2.6.4. *For $W(A_2)$, we have $(S_1S_2)^3 = 1$, so*

$$W(A_2) = \langle S_1, S_2 \rangle = \{1,\ S_1,\ S_2,\ S_1S_2,\ S_2S_1,\ S_1S_2S_1 = S_2S_1S_2\}.$$

Therefore $|W(A_2)| = 6$.

Example 2.6.5. *For $W(B_2)$ we have $(S_1S_2)^4 = 1$, so*

$$\begin{aligned} W(B_2) &= \langle S_1, S_2 \rangle \\ &= \{1,\ S_1,\ S_2,\ S_1S_2,\ S_2S_1,\ S_1S_2S_1,\ S_2S_1S_2,\ S_1S_2S_1S_2 = S_2S_1S_2S_1\}. \end{aligned}$$

Therefore $|W(B_2)| = 8$.

Example 2.6.6. *Consider the symmetric group $\mathfrak{S}_{n+1}$. Given $\sigma \in \mathfrak{S}_{n+1}$, then we define a linear transformation such that $\sigma : \epsilon_i \mapsto \epsilon_{\sigma(i)}$, and extend it linearly $\sigma(x) = \sigma(\sum_i x_i\epsilon_i) = \sum_i x_i\sigma(\epsilon_i)$. It is known that $\mathfrak{S}_{n+1}$ is generated by transpositions $(i, i+1)$ for $1 \leq i \leq n$. Then $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ is fixed by every permutation. Let V be a hyperplane in $\mathbb{R}^{n+1}$ given by $x_1 + \cdots + x_n = 0$. This means V is the orthogonal complement to $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$. From Example 2.3.10 we have the simple roots of the A_n root system: $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \leq i \leq n$. Let us look at the action of reflection S_i:*

$$S_i(\alpha_i) = -\alpha_i = \epsilon_{i+1} - \epsilon_i$$

Note that the transposition $(i, i+1)$ indices a swap of i and $i+1$, and which is the same as sending α_i to its negative. Therefore $W(A_n)$ act as the symmetric group on $n+1$ symbols. Thus, $|W(A_n)| = (n+1)!$.

The order of the rest of the connected Coxeter groups is [Benson, 1985, p. 82]:

Φ	I_2^m	A_n	B_n	D_n	H_3
$\lvert W(\Phi)\rvert$	$2m$	$(n+1)!$	$2^n n!$	$2^{n-1} \cdot n!$	120

Φ	H_4	F_4	E_6	E_7	E_8
$\lvert W(\Phi)\rvert$	14400	1152	51 840	2903040	696729600

Table 2.3: The order of the connected Coxeter groups.

2.7 Fundamental domain

Let W be a finite subgroup of $O(V)$, and let $T \in W$, then a subset F of V is called a *fundamental domain* for W if and only if:

1. F is open.
2. $F \cap TF = \emptyset$ if $T \neq 1$.
3. $V = \cup_{T \in W} cl(TF)$, where $cl(TF)$ is the closure of TF.

The open sets TF are often called *open chambers*, and W acts *simply transitively* on the set of all open chambers, i.e. any (open) chamber has trivial stabilizer, and given any two chambers C_1 and C_2, we have $C_2 = w(C_1)$ for some $w \in W$. In particular, $|W|$ is the number of chambers.

In order to construct a fundamental domain for W, we can do the *Fricke-Klein construction* [Benson, 1985, Chapter 3]: Suppose $W = \{T_0 = 1, T_1, \ldots, T_{k-1}\}$, where $k \geq 1$. Choose a point $x_0 \in V$ that is only fixed by $T_0 \in W$, so the W-orbit through x_0 is $\mathrm{Orb}\,(x_0) := \{x_0, x_1, \ldots, x_{k-1}\}$, where $x_i = T_i x_0$. For $i \neq 0$, the line segment $[x_0 x_i] = \{x_0 + \gamma(x_i - x_0) : 0 \leq \gamma \leq 1\}$ has the perpendicular bisector

$$\mathscr{P}_i = (x_0 - x_i)^\perp = \{x \in V : |x - x_0| = |x - x_i|\}$$

passing through the midpoint $\frac{1}{2}(x_i + x_0)$. Consider the open half-spaces determined by $\mathscr{P}_i$, denoted L_i:

$$L_i = \{x \in V : |x - x_0| < |x - x_i|\}.$$

We then find that a fundamental domain is given by

$$F = \bigcap_{i=1}^{k-1} L_i. \tag{2.7.1}$$

Example 2.7.1. *Let us consider the B_2 root system Φ, where the roots are defined in Example 2.3.11.*

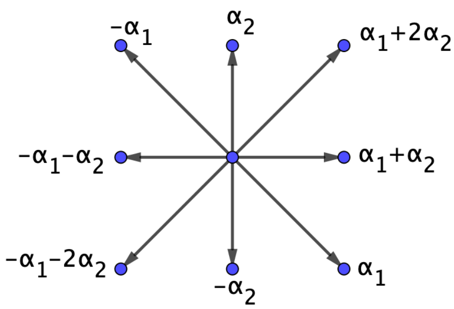


Figure 2.6: The B_2 root system.

Then from Example 2.6.5 we have the elements in $W(B_2)$. The reflection hyperplanes $\alpha^\perp$ (for $\alpha \in \Phi$) divide V into eight congruent chambers as shown in Figure 2.7. Choose x_0 to be off all the reflection hyperplanes. Then the W-orbit to x_0 as well the fundamental domain $F = L_1 \bigcap \cdots \bigcap L_7 = L_1 \bigcup L_2$ are shown in Figure 2.7.

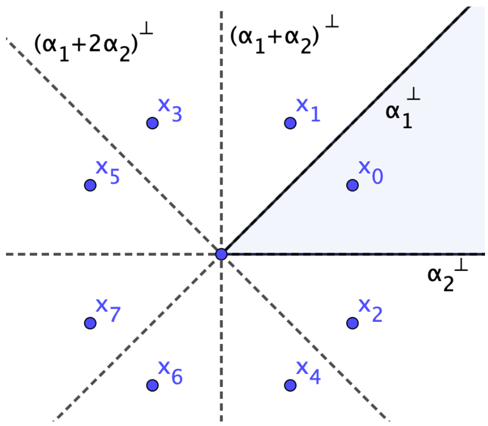


Figure 2.7: The B_2 root system with its reflection hyperplanes, the W-orbit to x_0 and the fundamental domain indicated in blue.

2.8 Fundamental weights

Fix a simple system $\Delta = \{\alpha_i\}_{i=1}^n$ for a root system Φ. Let $\{\lambda_i\}_{i=1}^n$ denote the *fundamental weights*, defined by

$$\langle \lambda_i, \alpha_j^\vee \rangle = \delta_{ij}, \tag{2.8.1}$$

for all i, j, where δ_{ij} is defined as

$$\delta_{ij} = \begin{cases} 1 \text{ for } i = j \\ 0 \text{ for } i \neq j. \end{cases}$$

This means α_i is perpendicular to λ_j when $i \neq j$. Given the Cartan matrix C_{ij}, from Definition 2.4.1 and its inverse C^{ij}, we have

Lemma 2.8.1. *$\alpha_i = \sum_j C_{ij}\lambda_j$ and $\lambda_i = \sum_j C^{ij}\alpha_j$*

Proof. Write $\alpha_i = \sum_{j=1}^{n} a_{ij}\lambda_j$. So we want to show that $a_{ij} = C_{ij}$. Take the inner product on both sides with $\alpha_k^\vee$.

$$C_{ik} = \langle \alpha_i, \alpha_k^\vee \rangle = \sum_{j=1}^{n} a_{ij} \langle \lambda_j, \alpha_k^\vee \rangle = \sum_{j=1}^{n} a_{ij}\delta_{jk} = a_{ik}$$

□

This means the Cartan matrix is the transition matrix between the basis of simple roots and the basis of fundamental weights.

Example 2.8.2. *For A_2 we get*

$$C_{ij} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad C^{ij} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Therefore

$$\begin{cases} \alpha_1 = 2\lambda_1 - \lambda_2 \\ \alpha_2 = -\lambda_1 + 2\lambda_2 \end{cases} \qquad \begin{cases} \lambda_1 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2 \\ \lambda_2 = \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2 \end{cases}$$

The A_2 root system and the fundamental weights for A_2 are shown in Figure 2.8 below.

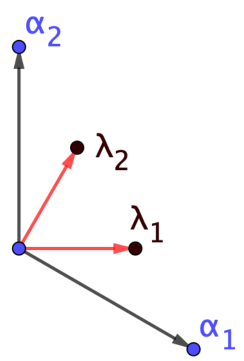


Figure 2.8: The simple roots and the fundamental weights in A_2.

Given a root system Φ with the simple system $\Delta = \{\alpha_i\}_{i=1}^n$, a fundamental domain F for $W = W(\Phi)$ is

$$F = \{x \in V : \langle x, \alpha_i \rangle > 0, \forall \alpha_i \in \Delta\}.$$

Note that F has boundary given by the *walls* $\alpha_i^\perp = \text{span}\{\lambda_j\}_{j \neq i}$. Moreover, the closure $\text{cl}(F)$ of F is the positive convex cone spanned by $\{\lambda_i\}_{i=1}^n$, i.e. any $x \in \text{cl}(F)$ has $x = \sum_i r_i \lambda_i$, where $r_i \geq 0$.

Example 2.8.3. *Consider the B_3 root system, then the fundamental domain is:*

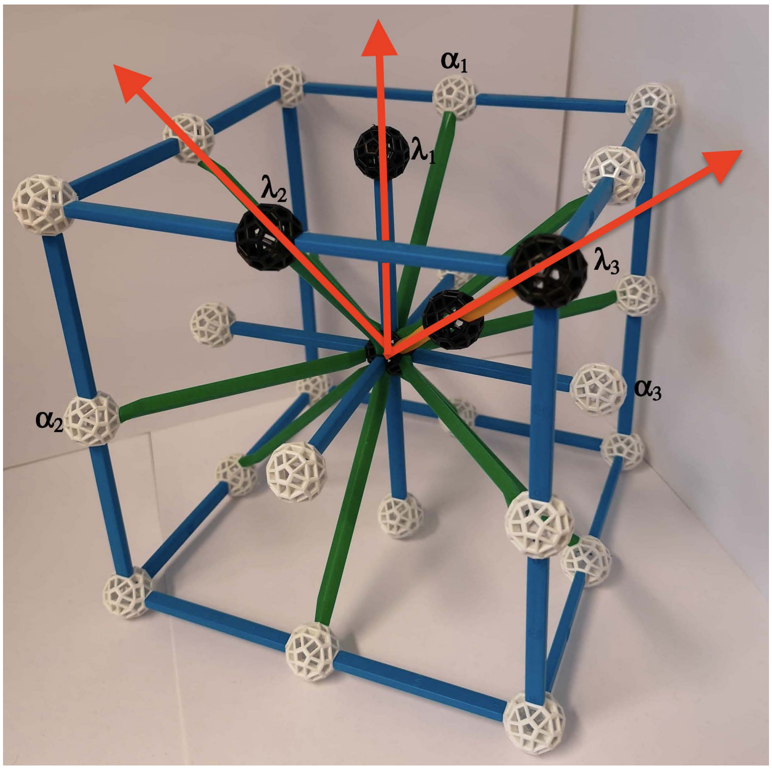


Figure 2.9: The B_3 root system with its simple roots, fundamental weights and fundamental domain indicated with the red arrows.

We see that if we divide the root system into such congruent chambers we get 24 *chambers, which is* $|W(B_3)|$.

3

Reflection recipes and stabilizers

3.1 Coxeter diagram and Dynkin diagram reflection recipes

Let Φ be a root system with the simple system $\Delta = \{\alpha_i\}_{i=1}^n$ and let the fundamental weights be $\{\lambda_i\}_{i=1}^n$.

Proposition 3.1.1. *If $\lambda = \sum_j r_j \lambda_j$, then $S_i(\lambda) = \sum_j \widehat{r_j} \lambda_j$, where*

$$\widehat{r_j} = r_j - r_i C_{ij} = r_j + 2r_i \frac{|\alpha_i|}{|\alpha_j|} \cos\left(\frac{\pi}{p_{ij}}\right).$$

Proof. We use (2.1.1), (2.8.1), and Lemma 2.8.1 to calculate:

$$\begin{aligned} S_i(\lambda) &= \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i \\ &= \sum_j r_j \lambda_j - \sum_j r_j \langle \lambda_j, \alpha_i^\vee \rangle \alpha_i \end{aligned}$$

$$
\begin{aligned}
&= \sum_j r_j\lambda_j - \sum_j r_j\delta_{ji}C_{ik}\lambda_k \\
&= \sum_j (r_j - r_iC_{ij})\lambda_j
\end{aligned}
$$

□

Let us encode $\lambda = \sum_j r_j\lambda_j$ by inscribing the coefficients r_j over corresponding nodes in the Coxeter diagram (or Dynkin diagram).

Example 3.1.2. *With respect to the H_3 root system $\lambda = \lambda_3$ corresponds to* $\overset{0}{\circ}\underset{5}{\text{———}}\overset{0}{\circ}\text{———}\overset{1}{\circ}$ *. Recall that $p_{12} = 5$, $p_{13} = 2$ and $p_{23} = 3$.*

Proposition 3.1.1 gives us a pictorial reflection recipe:

Corollary 3.1.3 (Coxeter diagram reflection recipe)**.** *Suppose all simple roots have the same length. We calculate $S_i(\lambda)$ from $\lambda = \sum_j r_j\lambda_j$ by:*

1. *Change the coefficient r_i over the i-th node to its negative.*
2. *If node j is connected to node i, we replace r_j by $r_j + 2\cos\left(\frac{\pi}{p_{ij}}\right)r_i$.*

If node j is not connected to node i, then the coefficient r_j is not affected under S_i.

Remark 3.1.4. If $r_i = 0$, then the reflection S_i does not change λ. Therefore it is convenient just to reflect in the nonzero positive nodes.

Example 3.1.5. *Recall from example 2.3.12 the H_3 root system and $\tau = 2\cos(\frac{\pi}{5})$, which satisfies $\tau^2 = \tau + 1$. The Cartan matrix for H_3 is found in Example 2.3.12. The W-orbit through* $\lambda = \overset{0}{\circ}\underset{5}{\text{———}}\overset{0}{\circ}\text{———}\overset{1}{\circ}$ *is shown in Figure 3.1.*

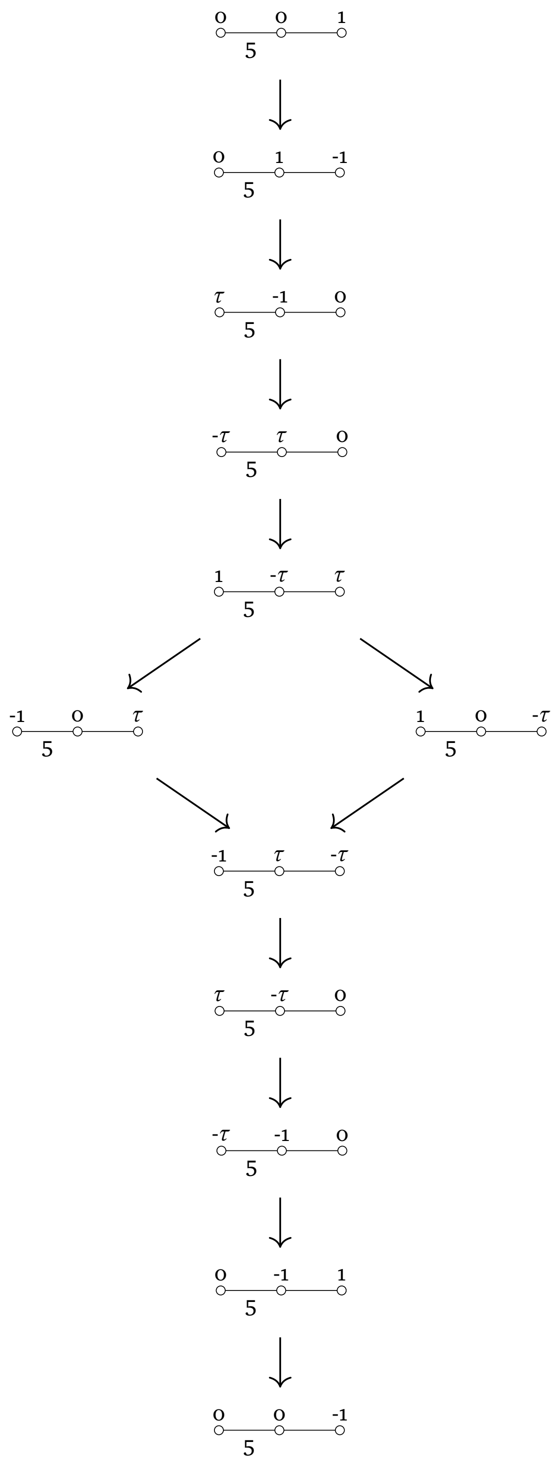


Figure 3.1: The $W(H_3)$-orbit through λ_3.

For the Dynkin diagram recipe we have to consider relative lengths. We have two cases consider for the Dynkin diagrams, when $i \neq j$ and either $|\alpha_i| < |\alpha_j|$ or $|\alpha_i| > |\alpha_j|$. For $|\alpha_i| < |\alpha_j|$ we know that $C_{ij} = -1$ so by Proposition 3.1.1

$$\widehat{r_j} = r_j - r_i C_{ij} = r_j + r_i.$$

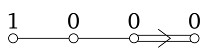
For $|\alpha_i| > |\alpha_j|$, C_{ij} has the values $-m = -1, -2, -3 \in \mathbb{Z}$, where $m = C_{ij}C_{ji}$ is the number of bonds in the Dynkin diagrams.

$$\widehat{r_j} = r_j - r_i C_{ij} = r_j + m r_i.$$

For the Dynkin diagrams we get the following reflection recipe:

Corollary 3.1.6 (Dynkin diagram reflection recipe)**.** *For Dynkin diagrams we calculate $S_i(\lambda)$ from $\lambda = \sum_j r_j \lambda_j$ by:*

1. *Change the coefficient r_i over the i-th node to its negative.*
2. *If node j is connected to node i, we either*
   (a) *we replace r_j by $r_j + r_i$ if the arrow is pointing towards node i*
   (b) *we replace r_j by $r_j + m r_i$, where $m = C_{ij}C_{ji}$, if the arrow is pointing towards node j or there are no arrow.*

Example 3.1.7 (B_3)**.** *The W-orbit through* $\lambda =$ 

is:

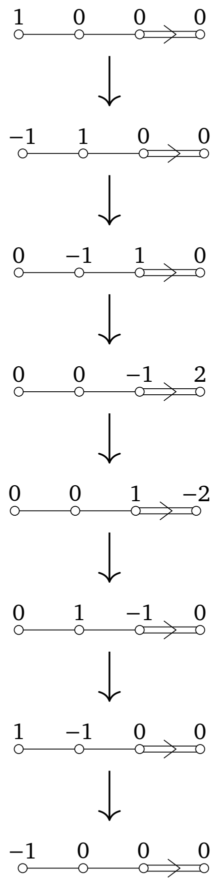


Figure 3.2: The $W(B_4)$-orbit through λ_1

3.2 Stabilizer

In this section we will describe the stabilizer and give the recipe for finding the stabilizer for λ.

Definition 3.2.1. *Given $W \leq O(V)$ and $\lambda \in V$, then the stabilizer* $\mathrm{Stab}(\lambda)$ *of λ is the subgroup given by* $\mathrm{Stab}(\lambda) = \{w \in W : w(\lambda) = \lambda\}$. *If $w \in \mathrm{Stab}(\lambda)$ we say w fixes λ.*

The following theorem and proof for stabilizers is based on [Benson, 1985, Thm 5.4.1]. The case there is for $\lambda = \lambda_j$, but we adapted it to the general case $\lambda = \sum_j r_j \lambda_j$.

Theorem 3.2.2. *Let $\Phi \subset V$ be a root system with* $\mathrm{span}\{\Phi\} = V$. *Let $\{\alpha_i\}_{i=1}^n$ be* *simple roots and* $\{\lambda\}_{i=1}^{n}$ *be fundamental weights. Let* $0 \neq \lambda = \sum_{j=1}^{n} r_j \lambda_j \in V$. *The stabilizer in* $W = W(\Phi)$ *of* λ *is*

$$\mathrm{Stab}(\lambda) = \langle S_j \rangle_{j \in I},$$

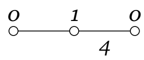
where $I = \{j : r_j = 0\} \subsetneq \{1, \ldots, n\}$.

Proof. If $I = \emptyset$, then λ is in the (open) fundamental domain see Section 2.7, so $\mathrm{Stab}(\lambda) = 1$. Suppose $I \neq \emptyset$. Set $K = \langle S_j \rangle_{j \in I}$ and $H = \mathrm{Stab}(\lambda)$. We want to show $K = H$. Clearly, $K \leq H$, since for $j \in I$, $\langle \lambda, \alpha_j^\vee \rangle = \langle \sum_{k=1}^{n} r_k \lambda_k, \alpha_j^\vee \rangle = \sum_{k=1}^{n} r_k \langle \lambda_k, \alpha_j^\vee \rangle = r_j = 0$. So $S_j(\lambda) = \lambda - \langle \lambda, \alpha_j^\vee \rangle \alpha_j = \lambda$. Now we want to show that $H \leq K$. Let $X_1 := \{x \in V : |x| = |\lambda|\}$ be a sphere. Note that $\lambda \in \bigcap_{j \notin I} L_j$, where L_j is the open half-space from Section 2.7. Since $\lambda \neq 0$, then $\bigcap_{j \notin I} L_j$ is non-empty and open. Therefore there exists $d > 0$ sufficiently small such that $X_2 := \{x \in V : |x - \lambda| = d\} \subset \bigcap_{j \notin I} L_j$. Since $H \leq O(V)$, then H preserves distances, so $HX_1 \subset X_1$ and $HX_2 \subset X_2$. Therefore $HX \subset X$, where $X = X_1 \cap X_2$ (since H also stabilizes λ).

Since $\bigcap_{j \in I} L_j \neq \emptyset$ is open, there exist $x_0 \in \bigcap_{j \in I} L_j$ not fixed by any non-identity element of H. The Fricke-Klein construction from Section 2.7 applied to $x_0 \in V$ gives fundamental domains $F(K)$ and $F(H)$ for K and H respectively. Since $K \leq H$ and $x_0 \in \bigcap_{j \in I} L_j$, then $F(H) \subseteq F(K) \subseteq \bigcap_{j \in I} L_j$. We also get fundamental domains for K and H in X, given by:

$$F_X(K) = F(K) \cap X, \quad F_X(H) = F(H) \cap X.$$

Since $X \subset \bigcap_{j \notin I} L_j$, then $F_X(K) \subseteq \left(\bigcap_{j \in I} L_j \right) \cap \left(\bigcap_{j \notin I} L_j \right) = L_1 \cap \cdots \cap L_n = F(W)$, which is the fundamental domain for W is by (2.7.1). If $F_X(H) \neq F_X(K)$, then pick $x \in F_X(K) \setminus F_X(H)$, $y \in F_X(H)$ and $T \in H$ such that $Tx = y$ and $x \neq y$. But since $x, y \in F_x(K) \subseteq F(W)$, then this is impossible since only the identity element in W preserves $F(W)$. Therefore $F_X(H) = F_X(K)$. Since the fundamental domains are the same, then H and K have the same number of chambers, so $|H| = |K|$. Therefore $H = K$. ◻

Example 3.2.3. *If we have* $\lambda =$ 

, then

$$\mathrm{Stab}(\lambda) = \langle S_1, S_3 \rangle = \{1,\ S_1,\ S_3,\ S_1S_3 = S_3S_1\} = W(A_1 \times A_1).$$

Recall that the size of the W-orbit $\mathrm{Orb}(\lambda)$ and the stabilizer $\mathrm{Stab}(\lambda)$ are related by:

Proposition 3.2.4. $|\operatorname{Orb}(\lambda)| = \frac{|W|}{|\operatorname{Stab}(\lambda)|}$

Example 3.2.5. *Consider* $\lambda = \overset{0}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{1}{\circ}\!\Rightarrow\!\overset{0}{\circ}$ *as in Example 3.2.3. Then using Table 2.3*

$$\begin{aligned}
|W(B_3)| &= 2^3 3! = 48 \\
|W(A_1 \times A_1)| &= 2 \cdot 2 = 4 \\
|\operatorname{Orb}(\lambda)| &= \frac{|W(B_3)|}{|\operatorname{Stab}(\lambda)|} = \frac{48}{4} = 12.
\end{aligned}$$

4

Polytopes

4.1 Definitions

A polytope can be in any dimension, but the most interesting polytopes for us are in dimension two and three. A polytope in two dimensions is called a polygon, or p-gon.

Definition 4.1.1. *A p-gon is a circuit of p line segments $l_1l_2, l_2l_3, \ldots, l_pl_1$, joining consecutive pairs of p points $l_1, l_2, \ldots, l_p$ such that no line segment is crossing over another line segment.*

The points $l_1, l_2, \ldots, l_p$ are called vertices, and the line segments $l_1l_2, l_2l_3, \ldots,$ l_pl_1 are called edges. For $p = 3$, we have a triangle, which has 3 vertices and 3 edges. The length of the edges and the angles are arbitrary as long as the triangle is closed. If the length of the edges are equal, then the polygon is *equilateral*, and if all interior angles between two edges are equal, then the polygon is *equiangular*. If $p = 1$ or $p = 2$, then we only get a vertex or an edge respectively. But if $p \geq 3$, then the polygon can be either equiangular, equilateral or both at the same time. If $p = 4$, a rhombus is only equilateral, a rectangle is only equiangular and a square is both.

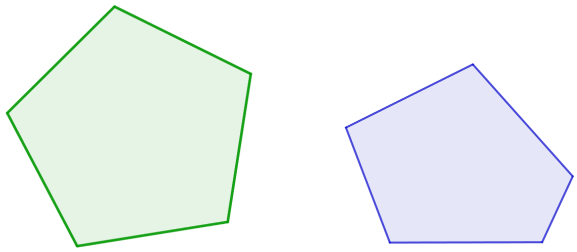
Definition 4.1.2. *[Coxeter, 1973, p. 2] A p-gon is regular if it is both equilateral and equiangular.*

A square is a regular polygon, and if all the angles in a triangle is $60°$ it is regular.

A set of points is convex, if for all points we can take the line segment between any two points and it is still in the set.

Definition 4.1.3. *A polytope is convex if it is a convex set of points.*

A consequence of Definition 4.1.2 is that all regular polygons are also convex polygons, but not vice versa.



(a) A regular and convex pentagon. **(b)** A convex, but not regular pentagon

Figure 4.1: Two convex polygons, where one of them is also regular.

As we see in Figure 4.1 both of them are convex, but (b) is not regular since it is not equilateral.

A polytope in dimension three is called a *polyhedron*.

Definition 4.1.4. *A polyhedron is a finite connected set of plane polygons, such that every edge of one polygon belongs to another polygon.*

An example of a polyhedron is the cube, which consists of six squares where three squares meet at each vertex. A polyhedron consists of vertices, edges and faces. Each polygon in the polyhedron is called a *face*.

Definition 4.1.5. *A regular polyhedron is a convex polyhedron, where all the faces are congruent regular polygons and all interior angles are the same.*

In fact there are only five regular polyhedra: the tetrahedron, the cube, the octahedron, the dodecahedron and the icosahedron. These are called the Platonic solids and are found in Figure 1.1. The tetrahedron consists of four regular triangles, the cube consists of six squares where three of them meet at each vertex, the octahedron consists of eight regular triangles where four of them meet at each vertex, the dodecahedron consists of twelve regular pentagons where three of them meet at each vertex, and the icosahedron consists of twenty regular triangles where five of them meet at each vertex.

The Euler characteristic is $\chi = v - e + f$, where v is the number of vertices, e is the number of edges and f is the number of faces. The Platonic solids and all other convex polyhedra satisfies $\chi = 2$.

Example 4.1.6. *The octahedron has* 6 *vertices,* 12 *edges and* 8 *faces, so* $\chi = 6 - 12 + 8 = 2$.

The number of edges, vertices and faces for the five Platonic solids are shown 4.1.

	Vertices	Edges	Faces
Tetrahedron	4	6	4
Cube	8	12	6
Octahedron	6	12	8
Dodecahedron	20	30	12
Icosahedron	12	30	20

Table 4.1: The number of vertices, edges and faces in the five Platonic solids.

From Table 4.1 we see that the cube has the same number of vertices as the number of faces of the octahedron and vice versa. We also have the same relation for the icosahedron and the dodecahedron. Therefore we say that the cube and octahedron, and the dodecahedron and icosahedron are *duals*. If we take the center of every face in a polyhedron and draw an edge from each center to the centers in the adjacent faces we get the dual polyhedron. If our starting polyhedron is a cube, then we get an octahedron inside the cube. If we do the same with the tetrahedron we only get another tetrahedron, therefore it is *self-dual*.

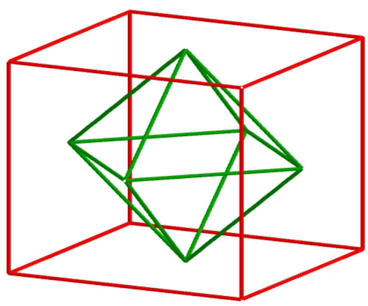


Figure 4.2: The cube and its dual, namely the octahedron.

Until now we have looked at polytopes in two and three dimensions, but in general a n-polytope is:

Definition 4.1.7 ([Coxeter, 1973, p. 126])**.** *A n-polytope is a finite convex region in n-dimensional space enclosed by a finite number of hyperplanes.*

As for polygons and polyhedra we will define regular.

Definition 4.1.8. *If P is a convex polytope in V with dimension n, then a sequence*

$$F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset F_n = P,$$

with F_i a face of P of dimension i is called a flag of faces of P.

Definition 4.1.9. *A convex polytope P in V is regular if the full symmetry group acts transitively on all flags of P.*

Therefore a polygon is regular if the symmetry group acts transitively on vertices and edges, and a polyhedra is regular if the symmetry group acts transitively on vertices, edges and faces. In the icosadodecahedron in Example 2.3.12 the symmetry group acts transitively on vertices and edges, but not on faces. Therefore it is not regular.

The *convex hull* $\operatorname{conv}(X)$ of the finite set of points X is the convex combination of all points. Here, a *convex combination* is a linear combination of points x_i where all coefficients r_i add up to 1. The convex hull of X can therefore be written as

$$\operatorname{conv}(X) = \left\{ \sum_{i=j}^{n} r_i x_i : r_i \geq 0 \,\forall i, \sum^{n} r_i = 1 \right\}$$

We can therefore generate a polytope by starting with a finite number of points, and then take the convex hull of these points.

4.2 Our construction of polytopes

In this section we will generate polytopes by starting with W-orbit through λ, and then take the convex hull of these points. We denote this polytope $P(\lambda)$. Using W, we may assume that λ lies in the closure of the fundamental domain. Thus, it suffices to take $\lambda = \sum_j r_j \lambda_j$ with $r_j \geq 0$. The rank two Coxeter groups generate polygons, the rank three Coxeter groups generate polyhedra and the rank n Coxeter groups generate n-dimensional polytopes.

4.2.1 Polygons

Since we will generate polygons from the rank two Coxeter groups we consider the action of $W(I_2^m)$ on λ. Recall that for $m = 3$ and $m = 4$ this is the same as $W(A_2)$ and $W(B_2)$ respectively. For polygons we need to generate vertices and edges. The vertices are generated by the $W(I_2^m)$-orbit through λ, which we get by using Proposition 3.1.3. This orbit gives us the coordinates of the vertices in terms of λ_i and $|\operatorname{Orb}(\lambda)|$ is the number of vertices. The edges of $P(\lambda)$ passing through λ is generated by:

Definition 4.2.1. *An edge in $P(\lambda)$ with endpoint λ are generated by reflections in the walls $\alpha_1^{\perp}, \alpha_2^{\perp}, \ldots, \alpha_n^{\perp}$.*

The edge e_i is a line segment between λ and $S_i(\lambda)$, where S_i do not fix λ. We denote these edges as $e_i = [\lambda, S_i(\lambda)]$. To get the other edges we apply $W(I_2^m)$ to e_i to reflect it around. What gives us a regular polygon will therefore depend on $\lambda = \overset{a}{\circ}\underset{\mathrm{m}}{\text{——}}\overset{b}{\circ}$, where $a, b \geq 0$. Rescaling λ induces a rescaling of $P(\lambda)$, which we will view as geometrically equivalent. Therefore we have four cases to consider, $\overset{1}{\circ}\underset{\mathrm{m}}{\text{——}}\overset{0}{\circ}$, $\overset{0}{\circ}\underset{\mathrm{m}}{\text{——}}\overset{1}{\circ}$, $\overset{a}{\circ}\underset{\mathrm{m}}{\text{——}}\overset{b}{\circ}$ where $a \neq b$, and $\overset{a}{\circ}\underset{\mathrm{m}}{\text{——}}\overset{b}{\circ}$ where $a = b$.

Definition 4.2.2. *Let $W \leq O(V)$. A convex polytope $P(\lambda)$ is W-*regular *if W acts transitively on all flags of P.*

When we are considering regular polytopes we have to distinguish between regular as in Definition 4.1.9 and W-regular. If a polytope is W-regular it is also regular as in Definition 4.1.9, but not vice versa.

Proposition 4.2.3. *Let $W = W(I_2^m)$. Then:*

1. *$P(\overset{1}{\circ}\underset{m}{\text{——}}\overset{0}{\circ})$ generates W-regular m-gon.*
2. *$P(\overset{1}{\circ}\underset{m}{\text{——}}\overset{1}{\circ})$ generates $2m$-gon that is not W-regular.*
3. *if $P(\overset{a}{\circ}\underset{m}{\text{——}}\overset{b}{\circ})$ is regular, then $a = b$.*

Proof. Consider $\lambda = \overset{1}{\circ}\underset{m}{\text{———}}\overset{0}{\circ}$, by Theorem 3.2.2 $\operatorname{Stab}(\lambda) = \langle S_2 \rangle$, and therefore by Proposition 3.2.4 and Table 2.3 $|\operatorname{Orb}(\lambda)| = \frac{|W(I_2^m)|}{|\operatorname{Stab}(\lambda)|} = \frac{2m}{2} = m$. From Section 2.2, $R = S_1S_2$ is a rotation by $\frac{2\pi}{m}$, and recall the order of R is m. Then $\operatorname{Orb}(\lambda) = \{R^k(\lambda_1) : k \in \mathbb{Z}\}$ and $P = \operatorname{conv}(\operatorname{Orb}(\lambda))$. The edges in P are between consecutive vertices

$$e_k = [R^k(\lambda_1),\, R^{k+1}(\lambda_1)],$$

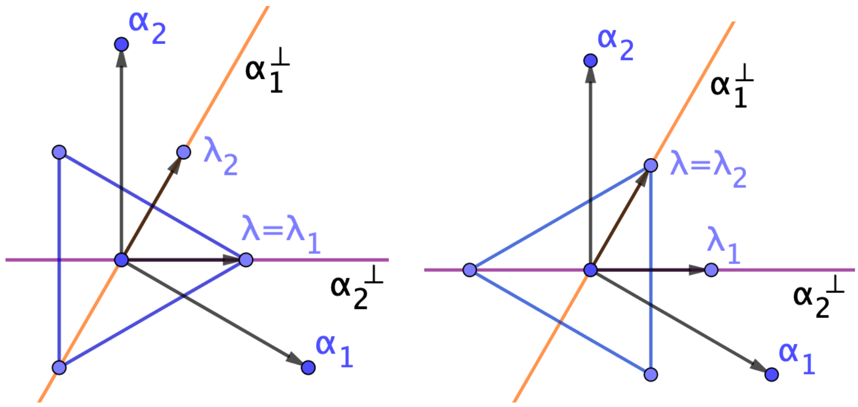
for $k = 0, \ldots, m-1$. Therefore we get m edges. Since $R(e_k) = e_{k+1}$, then W act transitively on vertices and edges and $P(\lambda)$ is W-regular. For $\lambda = \overset{0}{\circ}\underset{m}{\text{———}}\overset{1}{\circ}$ it is similar.

Let $\lambda = \overset{1}{\circ}\underset{m}{\text{———}}\overset{1}{\circ}$. By Theorem 3.2.2 $\operatorname{Stab}(\lambda) = e$, and by Proposition 3.2.4 and Table 2.3 $|\operatorname{Orb}(\lambda)| = \frac{|W(I_2^m)|}{\operatorname{Stab}\lambda} = \frac{2m}{1} = 2m$. But $P(\lambda)$ is a $2m$-gon with $2m$ edges. Then one edge is generated by $e_1 = [\lambda, S_1(\lambda)]$. Note that this edge has $\operatorname{Stab}(e_1) = \langle S_1 \rangle$, since the other elements of W do not fix e_1. By using Proposition 3.2.4 and Table 2.3 the size of the W-orbit through e_1 is $\frac{|W(I_2^m)|}{|\operatorname{Stab}(e_1)|} = m$. Since this only generates m edges, the edge $e_2 = [\lambda, S_2(\lambda)]$ generates the rest. Therefore $P(\overset{1}{\circ}\underset{m}{\text{———}}\overset{1}{\circ})$ is not W-regular.

If $\lambda = \overset{r_1}{\circ}\underset{m}{\text{———}}\overset{r_2}{\circ}$, then it is regular if the length of the edges is the same, so $|e_i| = |\lambda - S_i(\lambda)| = \lambda - (\lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i) = r_i$. This means the edges only have the same length if $r_1 = r_2$. Therefore $a \neq b$ does not generate a regular polygon. □

Let us consider three examples with $m = 3$:

Example 4.2.4. *Let* $\lambda = \overset{1}{\circ}\text{———}\overset{0}{\circ}$ *and* $\lambda = \overset{0}{\circ}\text{———}\overset{1}{\circ}$, *then*



(a) The regular polygon $P(\lambda_1)$ generated by $W(I_2^3)$.

(b) The regular polygon $P(\lambda_2)$ generated by $W(I_2^3)$.

Figure 4.3: Two regular triangles generated by $W(I_2^3)$.

Example 4.2.5. *Let* $\lambda = \overset{1}{\circ}\!\!-\!\!-\!\!-\!\!\overset{2}{\circ}$*, then:*

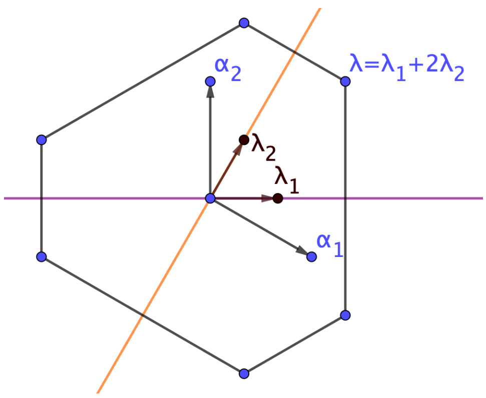


Figure 4.4: The non-regular polygon $P(\lambda_1 + 2\lambda_2)$ generated by $W(I_2^3)$.

This polygon we clearly see that is not regular, since the edges do not have the same length.

Example 4.2.6. *Let* $\lambda = \overset{1}{\circ} \quad \overset{1}{\circ}$*, then*

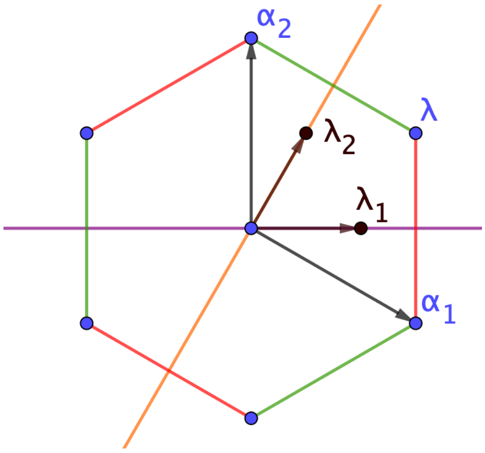


Figure 4.5: The regular polygon $P(\lambda_1 + \lambda_2)$ generated by $W(I_2^3)$.

When we apply W to λ we have two possibilities for edges. We could either reflect in the orange line $\alpha_1^\perp$ or in the purple line $\alpha_2^\perp$. Therefore we get two edges, the red edge and the green edge. By applying W to these two edges it is not possible to reflect the green edges and the red edges to each other, and therefore it is not W-regular.

We also have one more possibility to generate a regular polygon by considering $W(A_1 \times A_1)$, which has the Coxeter diagram $\circ \quad \circ$. For the $A_1 \times A_1$ root system the angles between the simple roots and the fundamental weights are the same, namely $\frac{\pi}{2}$. A consequence of that is if $\lambda = \overset{1}{\circ} \quad \overset{0}{\circ}$ or $\lambda = \overset{0}{\circ} \quad \overset{1}{\circ}$, then W just generates a single edge. But if we consider $\lambda = \overset{1}{\circ} \quad \overset{1}{\circ}$, we get

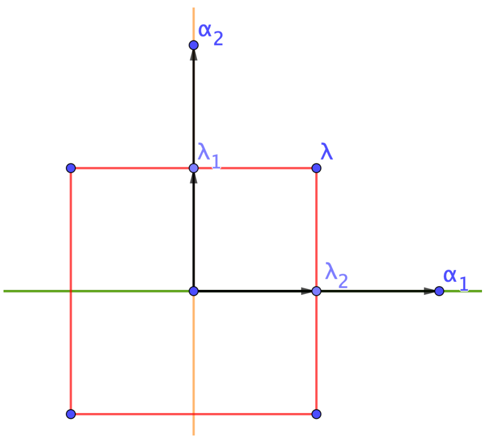


Figure 4.6: The regular polygon $P(\lambda_1 + \lambda_2)$ generated by $W(A_1 \times A_1)$.

We see that $P(\lambda_1 + \lambda_2)$ generates a regular square, but not a W-regular square.

4.2.2 Polyhedra

The rank three Coxeter groups generate the polyhedra. Therefore we will consider the action of $W(A_3)$, $W(B_3)$ and $W(H_3)$ on λ. Here, λ is either $\overset{a}{\circ}\!-\!\!-\!\overset{b}{\circ}\!-\!\!-\!\overset{c}{\circ}$, $\overset{a}{\circ}\!-\!\!-\!\overset{b}{\circ}\underset{4}{-\!\!-}\overset{c}{\circ}$ or $\overset{a}{\circ}\underset{5}{-\!\!-}\overset{b}{\circ}\!-\!\!-\!\overset{c}{\circ}$, where $a, b, c \geq 0$. We want all the edges to have the same length $a = b = c \geq 0$.

For polyhedra we need to generate vertices, edges and faces, and how we generate these will be explained with an example. Let us consider $\lambda = \overset{1}{\circ}\!-\!\!-\!\overset{0}{\circ}\!-\!\!-\!\overset{0}{\circ}$, where the W-orbit through λ is found in Figure A.1 in the appendix. This orbit has four elements, and as for polygons the W-orbit through λ is the vertices of $P(\lambda)$. From Table 4.1 we see that the tetrahedron has four vertices, six edges and four faces, so $P(\overset{1}{\circ}\!-\!\!-\!\overset{0}{\circ}\!-\!\!-\!\overset{0}{\circ})$ could be a tetrahedron. To get the edges we can compute the midpoint of the edges by taking the average of any two vertices in the W-orbit through λ. Then we will get all the elements of the W-orbit through λ_2 by a multiple. Therefore the second orbit in Figure A.1 generates the edges for $P(\lambda)$.

Example 4.2.7. *Consider the two first elements in the W-orbit through λ_1, $\overset{1}{\circ}\!-\!\!-\!\overset{0}{\circ}\!-\!\!-\!\overset{0}{\circ}$ and $\overset{-1}{\circ}\!-\!\!-\!\overset{1}{\circ}\!-\!\!-\!\overset{0}{\circ}$. Then the average is $\frac{1}{2}(\overset{0}{\circ}\!-\!\!-\!\overset{1}{\circ}\!-\!\!-\!\overset{0}{\circ})$. If we multiply by 2, then we get λ_2.*

To generate the faces we know that a face is two dimensional, and has to be spanned by two edges. Then we can compute the center of a face by taking the average of any three vertices in the W-orbit through λ. So we get four elements, and by looking at the W-orbit through λ_3 the centers are the elements of this orbit by a multiple. Therefore the W-orbit through λ_3 generates the faces for $P(\overset{1}{\circ}\!-\!\!-\!\overset{0}{\circ}\!-\!\!-\!\overset{0}{\circ})$.

Example 4.2.8. *Consider the three first elements in the W-orbit through λ, $\overset{1}{\circ}\!-\!\!-\!\overset{0}{\circ}\!-\!\!-\!\overset{0}{\circ}$, $\overset{-1}{\circ}\!-\!\!-\!\overset{1}{\circ}\!-\!\!-\!\overset{0}{\circ}$ and $\overset{0}{\circ}\!-\!\!-\!\overset{-1}{\circ}\!-\!\!-\!\overset{1}{\circ}$. Then the average is $\frac{1}{3}(\overset{0}{\circ}\!-\!\!-\!\overset{0}{\circ}\!-\!\!-\!\overset{1}{\circ})$, and if we multiply by 3 we get λ_3.*

For $P(\overset{1}{\circ}\!-\!\!-\!\overset{0}{\circ}\!-\!\!-\!\overset{0}{\circ})$ we now have four vertices, six edges and four faces, therefore this is a regular tetrahedron.

As for polygons we want to figure out what generates a regular polyhedron. We have already seen that $P(\overset{1}{\circ}\!-\!\!-\!\overset{0}{\circ}\!-\!\!-\!\overset{0}{\circ})$ generates a W-regular tetrahedron.

Which λ that generates the rest of the W-regular polyhedra are shown in Figure 4.2.

Platonic solid	$P(\lambda)$
Tetrahedron	$P(\overset{1}{\circ}\text{---}\overset{0}{\circ}\text{---}\overset{0}{\circ})$ and $P(\overset{0}{\circ}\text{---}\overset{0}{\circ}\text{---}\overset{1}{\circ})$
Cube	$P(\overset{0}{\circ}\text{---}\overset{0}{\circ}\underset{4}{\text{---}}\overset{1}{\circ})$
Octahedron	$P(\overset{1}{\circ}\text{---}\overset{0}{\circ}\underset{4}{\text{---}}\overset{0}{\circ})$
Dodecahedron	$P(\overset{1}{\circ}\underset{5}{\text{---}}\overset{0}{\circ}\text{---}\overset{0}{\circ})$
Icosahedron	$P(\overset{0}{\circ}\underset{5}{\text{---}}\overset{0}{\circ}\text{---}\overset{1}{\circ})$

Table 4.2: Platonic solids encoded by marked Coxeter diagrams.

Recall from Section 4.1 that the cube and the octahedron, and the dodecahedron and the icosahedron are duals, while the tetrahedron is self-dual. From the Table 4.2 we see that $W(B_3)$ generates the octahedron and the cube, $W(H_3)$ generates the dodecahedron and icosahedron, while $W(A_3)$ generates two triangles. These relations occurs since they are duals. To figure out what kind of polyhedra we get from the other λ, we will encode the marked Coxeter diagram for λ by considering equivalence classes for edges and faces.

4.3 Encode equivalence classes in terms of a recipe

Given W, we know all vertices in $P(\lambda)$ are equivalent under the action of W. But this may not be the case for edges and faces. By examine the marked Coxeter diagram (or Dynkin diagram) for λ we hope to determine the number of W inequivalent edge and face representatives. An element which belongs to a equivalence class is called a representative. For us the equivalence classes are the different orbits for vertices, edges and faces with respect to the Weyl group. Therefore if we have more than one representative for edges and/or faces, then the polytope is not W-regular. Even though it can be regular as in Definition 4.1.9, but that depends on if all the faces are congruent regular polygons.

4.3.1 Edges

To figure out how many equivalence classes there are for edges by encoding the marked Coxeter diagram for λ, we need to look at what generates an edge. By Definition 4.2.1 an edge is generated by reflecting in one of the nonzero nodes. We will show that the number of edge representatives are the same as the number of nonzero nodes. Let $e_i = [\lambda,\ S_i(\lambda)]$ be an edge, then we put a box around the i-th node to illustrate the representative. Therefore we encode the marked Coxeter diagram for λ by putting a box around the nodes where r_j is nonzero. Note that we only draw one box at the time, since we want to find the stabilizer for each representative.

Example 4.3.1. *Let* $\lambda = \overset{1}{\circ}\!\!-\!\!\!-\!\!\overset{1}{\circ}\!\!-\!\!\!-\!\!\overset{0}{\circ}$*, then the edge representatives are:*

$$\boxed{\overset{1}{\circ}}\!\!-\!\!\!-\!\!\overset{1}{\circ}\!\!-\!\!\!-\!\!\overset{0}{\circ} \qquad\qquad \overset{1}{\circ}\!\!-\!\!\!-\!\!\boxed{\overset{1}{\circ}}\!\!-\!\!\!-\!\!\overset{0}{\circ}$$

Example 4.3.2. *For* $\lambda = \overset{1}{\circ}\!\!-\!\!\!-\!\!\overset{0}{\circ}\!\!-\!\!\!-\!\!\overset{0}{\circ}$*, the initial edge is* $e_1 = [\lambda, S_1(\lambda)]$*. By looking at the midpoint of* e_1*, we get* $q = \frac{1}{2}(\overset{0}{\circ}\!\!-\!\!\!-\!\!\overset{1}{\circ}\!\!-\!\!\!-\!\!\overset{0}{\circ})$*. The midpoint must stabilize* e_1*, therefore by Theorem 3.2.2* $\mathrm{Stab}(e_1) = \mathrm{Stab}(q) = \langle S_1, S_3\rangle$*. The* W*-orbit through* e_1 *therefore generates by Proposition 3.2.4 and Table 2.3* $|\operatorname{Orb}(e_1)| = \frac{|W(A_3)|}{|\operatorname{Stab}(e_1)|} = \frac{24}{4} = 6$*. It is then clear that from the* W*-orbit through* λ *we generate three edges. The first one is* e_1*, the second is* $e_2 = [S_1(\lambda),\ S_2S_1(\lambda)]$ *and the third one is* $e_3 = [S_2S_1(\lambda),\ S_3S_2S_1(\lambda)]$*. The rest of the edges we get by using* $\mathrm{Stab}(\lambda)$ *on these edges:* $e_4 = S_2(e_1) = [\lambda,\ S_2S_1(\lambda)]$*,* $e_5 = S_3S_2(e_1) = [\lambda,\ S_3S_2S_1(\lambda)]$ *and* $e_6 = S_3(e_2) = [S_1(\lambda),\ S_3S_2S_1(\lambda)]$*. And we see that this is another way to find the edges than we did in Section 4.2.3.*

Example 4.3.3. *For* $\lambda = \overset{1}{\circ}\!\!-\!\!\!-\!\!\overset{0}{\circ}\!\!-\!\!\!-\!\!\overset{1}{\circ}$ *we could either apply* S_1 *or* S_3 *to* λ*, so by Theorem 3.2.2* $\mathrm{Stab}(\lambda) = \langle S_2\rangle$*. Therefore two edges will be* $e_1 = [\lambda,\ S_1(\lambda)]$ *and* $e_3 = [\lambda,\ S_3(\lambda)]$*. By applying* $\mathrm{Stab}(\lambda)$ *to* e_1 *and* e_3 *we get the two other edges passing through* λ*:*

$$e_1' = S_2(\lambda) = [\lambda,\ S_2S_1(\lambda)] \quad e_3' = S_3(\lambda) = [\lambda,\ S_2S_3(\lambda)]$$

So from each vertex in $P(\lambda)$ *there will be four edges. If we look at what stabilize* e_1 *we see it is only* $\langle S_1\rangle$*, and for* e_3 *it is only* $\langle S_3\rangle$*. Therefore the number of edges we get from* e_1 *and* e_3 *are:*

$$|\operatorname{Orb}(e_1)| = \frac{|W(A_3)|}{|\operatorname{Stab}(e_1)|} = \frac{24}{2} = 12 \qquad |\operatorname{Orb}(e_3)| = \frac{|W(A_3)|}{|\operatorname{Stab}(e_3)|} = \frac{24}{2} = 12$$

Therefore $P(\lambda)$ *has 24 edges.*

Proposition 4.3.4. *Given $0 \neq \lambda = \sum_{j=1}^{n} r_j \lambda_j$. Suppose $r_i \neq 0$, then consider the edge $e_i = [\lambda, S_i(\lambda)]$. The stabilizer in W of e_i is*

$$\mathrm{Stab}(e_i) = \langle S_j \rangle_{j \in E_i},$$

where $E_i = \{i\} \cup \{j : r_j = 0, C_{ij} = 0\}$.

Proof. Let q be the midpoint of e_i, then $\mathrm{Stab}(e_i) \subset \mathrm{Stab}(q)$ is clear. So let us calculate q to figure out $\mathrm{Stab}(q)$. To calculate q we take the average of λ and $S_i(\lambda)$. By (2.1.1) and Lemma 2.8.1 we get:

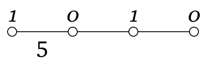
$$q = \frac{1}{2}(\lambda + S_i(\lambda)) = \lambda - \frac{1}{2}\langle \lambda, \alpha_i^\vee \rangle C_{ij} \lambda_j$$

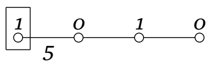
This means for the i-th node the average is zero. The average for the j-th node adjacent to the i-th becomes positive since for λ, $r_j \geq 0$ and for $S_1(\lambda)$, rj replaced by $r_j + 2r_i \cos\left(\frac{\pi}{m}\right)$, where $m \geq 3$. The average of the nodes not adjacent to the i-th node is $\frac{1}{2} r_j$ since r_j is not affected under S_i.

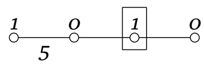
By Theorem 3.2.2 $\mathrm{Stab}(q)$ is consist of the nodes in q with $r_j = 0$. This means S_i and S_j where $r_j = 0$ and $C_{ij} = 0$ are the elements of $\mathrm{Stab}(\lambda)$. Therefore $\mathrm{Stab}(q) = \langle S_j \rangle_{j \in E_i}$.

To show that $\mathrm{Stab}\,(q) \subset \mathrm{Stab}(e_i)$ we apply $\mathrm{Stab}(q)$ to e_i. Since $S_i(e_i) = [S_i(\lambda),\ S_i S_i(\lambda)] = e_i$, then S_i is an element of $\mathrm{Stab}\,(e_i)$. Consider S_j, where $r_j = 0$ and $C_{ij} = 0$, then S_i and S_j commute so $S_j(e_i) = [S_j(\lambda),\ S_j S_i(\lambda)] = [\lambda,\ S_i S_j(\lambda)] = e_i$. Therefore S_j where $\{j : r_j = 0, C_{ij} = 0\}$ stabilize λ. Therefore $\mathrm{Stab}(e_i) = \mathrm{Stab}(q)$.

□

Example 4.3.5. *Consider*  *. Then the edge representatives are*





Then we get the edges $e_1 = [\lambda,\ S_1(\lambda)]$ and $e_3 = [\lambda,\ S_3(\lambda)]$, and the stabilizers of these are:

$$\mathrm{Stab}(e_1) = \langle S_1, S_4 \rangle \quad \mathrm{Stab}(e_3) = \langle S_3 \rangle$$

A pictorial way to find the stabilizer for edge representatives for a marked Coxeter diagram for λ is to use the following recipe.

Proposition 4.3.6 (Recipe for finding the stabilizer for the edge representatives). *Given any $\lambda = \sum_j r_j \lambda_j$, then the recipe for finding the stabilizer of the edge e_i, where S_i do not fix λ is : Remove nodes adjacent to the i-th node and remove the other nodes with a nonzero number over.*

Example 4.3.7. *Consider* $\lambda = \overset{1}{\circ}\text{---}\overset{0}{\circ}\underset{4}{\text{---}}\overset{1}{\circ}$ *, then λ has these representatives*

$\boxed{\overset{1}{\circ}}\text{---}\overset{0}{\circ}\underset{4}{\text{---}}\overset{1}{\circ}$ *and* $\overset{1}{\circ}\text{---}\overset{0}{\circ}\underset{4}{\text{---}}\boxed{\overset{1}{\circ}}$ *. The stabilizers of these are*

$$\mathrm{Stab}(\boxed{\overset{1}{\circ}}\text{---}\overset{0}{\circ}\underset{4}{\text{---}}\overset{1}{\circ}) = W(\circ) = \langle S_1 \rangle = W(A_1)$$

$$\mathrm{Stab}(\overset{1}{\circ}\text{---}\overset{0}{\circ}\underset{4}{\text{---}}\boxed{\overset{1}{\circ}}) = W(\circ) = \langle S_3 \rangle = W(A_1) \ .$$

The number of edges $\#e$ in $P(\overset{1}{\circ}\text{---}\overset{0}{\circ}\underset{4}{\text{---}}\overset{1}{\circ})$ *is by Proposition 3.2.4 and Table 2.3:*

$$\#e = \frac{|W(B_3)|}{|\,\mathrm{Stab}(\boxed{\overset{1}{\circ}}\text{---}\overset{0}{\circ}\underset{4}{\text{---}}\overset{1}{\circ})|} + \frac{|W(B_3)|}{|\,\mathrm{Stab}(\overset{1}{\circ}\text{---}\overset{0}{\circ}\underset{4}{\text{---}}\boxed{\overset{1}{\circ}})|} = \frac{48}{2} + \frac{48}{2} = 48$$

We will in the next section find the number of faces and what kind of faces this polytope consists of.

4.3.2 Faces

As for edges we also want to find face representatives and the stabilizer of these to compute the number of faces. From the face representatives we also want know what kind of polygon it generates.

Example 4.3.8. *Consider* $\lambda = \overset{1}{\circ}\text{---}\overset{0}{\circ}\text{---}\overset{0}{\circ}$*, and let f be a face and by applying W to λ we may assume that f contains the edge generated by the first reflection. Then we need two edges to span the face, since f is two dimensional. From Example 4.3.2, λ has one edge representative and the edges through λ are:*

$$\begin{aligned} e_1 &= [\lambda,\ S_1(\lambda)] \\ e_1' &= [\lambda,\ S_2S_1(\lambda)] \\ e_1'' &= [\lambda,\ S_3S_2S_1(\lambda)]. \end{aligned}$$

We see that $e_1'' = S_3(e_1')$ *and* $S_3(e_1) = e_1$, *therefore* S_3 *maps the face determined* e_1 *and* e_1' *to the face determined by* e *and* e''*. Therefore we get that*

$$f = [\lambda,\ S_1(\lambda),\ S_2S_1(\lambda)] = \operatorname{conv}(H \cdot \lambda),$$

where $H = \langle S_1,\ S_2\rangle$, *since* S_2 *stabilize* λ *we get*

$$f = \operatorname{conv}(\{\lambda,\ S_1(\lambda),\ S_2S_1(\lambda)\}).$$

Abstractly, f *is generated by* $W(\overset{1}{\circ}\!\!-\!\!\!-\!\!\overset{0}{\circ})$ *, so* f *is a triangle. This face is parallel to* $\operatorname{span}\{\alpha_1, \alpha_2\}$, *and the normal vector is* λ_3. *So* $\operatorname{Stab}(f) \subset \operatorname{Stab}(\lambda_3)$ *and by Theorem 3.2.2* $\operatorname{Stab}(\lambda) = \langle S_1, S_2\rangle$. *We can check if* $\operatorname{Stab}(\lambda_3)$ *stabilize* f *by applying* $\operatorname{Stab}(\lambda_3)$ *to* f*:*

$$\begin{aligned}
S_1(f) &= \operatorname{conv}\{S_1(\lambda),\ \lambda,\ S_1S_2S_1(\lambda) = S_2S_1S_2(\lambda) = S_2S_1(\lambda)\} = f \\
S_2(f) &= \operatorname{conv}\{\lambda,\ S_2S_1(\lambda),\ S_1(\lambda)\} = f
\end{aligned}$$

Therefore $\operatorname{Stab}(f) = \operatorname{Stab}(\lambda_3) = \langle S_1, S_2\rangle$. *By Proposition 3.2.4 and Table 2.3, the number faces* $\#f$ *of* $P(\overset{1}{\circ}\!\!-\!\!\!-\!\!\overset{0}{\circ}\!\!-\!\!\!-\!\!\overset{0}{\circ})$ *are:*

$$\begin{aligned}
|A_3| &= 24 \\
|\operatorname{Stab}(f)| &= |\langle S_1, S_2\rangle| = |I_2^3| = 6 \\
\#f &= \frac{|A_3|}{|\operatorname{Stab}(f)|} = \frac{24}{6} = 4.
\end{aligned}$$

Therefore $P(\overset{1}{\circ}\!\!-\!\!\!-\!\!\overset{0}{\circ}\!\!-\!\!\!-\!\!\overset{0}{\circ})$ *is regular tetrahedron. Since we have one representative for edges and one for faces it is also* W*-regular. Since* $\overset{1}{\circ}\!\!-\!\!\!-\!\!\overset{0}{\circ}$ *is a face representative, we actually can see this from the marked Coxeter diagram by drawing a box,* $\boxed{\overset{1}{\circ}\!\!-\!\!\!-\!\!\overset{0}{\circ}}\!\!-\!\!\!-\!\!\overset{0}{\circ}$ *. We have seen that the stabilizer of this representative is* $\langle S_1, S_2\rangle = W(\circ\!\!-\!\!\!-\!\!\circ)$*, so we could have got the stabilizer by removing the nodes not in the the box.*

Example 4.3.9. *Consider* $\lambda = \overset{1}{\circ}\!\!-\!\!\!-\!\!\overset{0}{\circ}\!\!-\!\!\!-\!\!\overset{1}{\circ}$, *then from Example 4.3.3,* λ *has two representatives for edges and the edges through* λ *are:*

$$\begin{aligned}
e_1 &= [\lambda,\ S_1(\lambda)] & e_3 &= [\lambda,\ S_3(\lambda)] \\
e_1' = S_2(e_1) &= [\lambda,\ S_2S_1(\lambda)] & e_3' = S_2(e_3) &= [\lambda,\ S_2S_3(\lambda)]
\end{aligned}$$

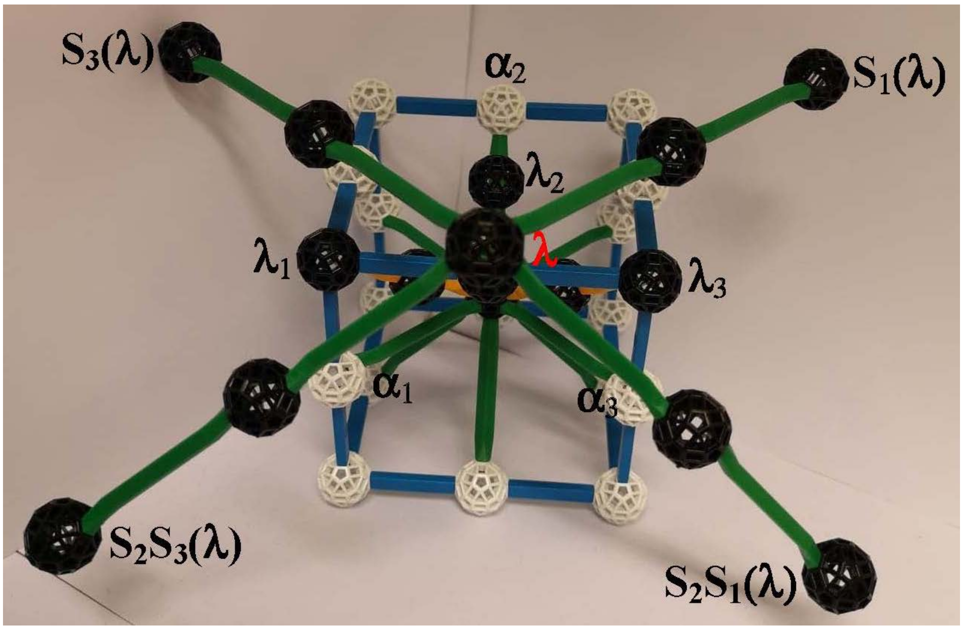


Figure 4.7: The four edges through λ in $P(\lambda_1 + \lambda_2)$ generated by $W(A_3)$.

We want a face to be generated by two adjacent edges. By looking at Figure 4.7, one face f_1 will be determined by e_1 and e_1'.

$$f_1 = \{\lambda, S_1(\lambda),\ S_2S_1(\lambda)\} = \operatorname{conv}(H_1 \cdot \lambda)$$

Where $H_1 = \langle S_1, S_2\rangle = \{1,\ S_1,\ S_2,\ S_1S_2,\ S_2S_1,\ S_2S_1S_2 = S_1S_2S_1\}$. We see that f_1 is spanned by three elements, and abstractly it is generated by $W(\overset{1}{\circ}\!\!-\!\!-\!\!\overset{0}{\circ})$. Therefore f_1 is a regular triangle. Since f_1 is parallel to $\operatorname{span}\{\alpha_1, \alpha_2\}$ *and the normal vector is λ_3, then* $\operatorname{Stab}(f_1) = \operatorname{Stab}(\lambda_3) = \langle S_1, S_2\rangle = W(A_2)$. *By Proposition 3.2.4 and Table 2.3 compute the order of the W-orbit through f_1:*

$$\# f_1 = \frac{|A_3|}{|\operatorname{Stab}(f_1)|} = \frac{24}{6} = 4$$

Similarly we get two other faces f_2 and f_3, f_2 will be determined by e_3 and $e_3{}'$ and f_3 is determined by e_1 and e_3, so:

$$f_2 = \operatorname{conv}\{\lambda,\ S_3(\lambda),\ S_2S_3(\lambda)\} \qquad f_3 = \operatorname{conv}(\{\lambda,\ S_1(\lambda),\ S_3(\lambda),\ S_3S_1\lambda\})$$

Abstractly, these two faces are generated by $W(\overset{0}{\circ}\!-\!\overset{1}{\circ})$ *and* $W(\overset{1}{\circ}\quad\overset{1}{\circ})$ *respectively, and therefore f_2 is a triangle and f_3 is a square. The stabilizers for these faces are*

$$\mathrm{Stab}\,(f_2) = \langle S_2, S_3\rangle = W(A_2) \qquad \mathrm{Stab}\,(f_3) = \langle S_1,\, S_3\rangle = W(A_1 \times A_1).$$

Therefore the total number of faces $\# f$ are:

$$\# f = 4 + \frac{24}{6} + \frac{24}{4} = 14$$

This polyhedron consists of six squares, and eight triangles, so this is a cubeoctahedron.

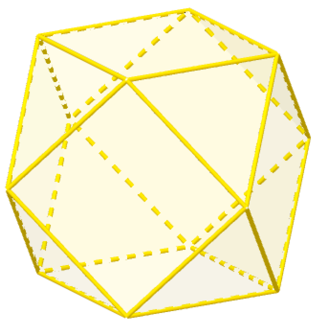


Figure 4.8: The polyhedron $P(\lambda_1 + \lambda_3)$ generated by $W(A_3)$.

Since we have three representatives for faces and $P(\lambda)$ consists of triangles and squares it is not regular. If we look at λ we can see these three representatives by drawing boxes: $\overset{1}{\circ}\!-\!\boxed{\overset{0}{\circ}\!-\!\overset{1}{\circ}}$, $\boxed{\boxed{\overset{1}{\circ}}\!-\!\overset{0}{\circ}\!-\!\boxed{\overset{1}{\circ}}}$ *and* $\boxed{\overset{1}{\circ}\!-\!\overset{0}{\circ}}\!-\!\overset{1}{\circ}$ *. For* $\overset{1}{\circ}\!-\!\boxed{\overset{0}{\circ}\!-\!\overset{1}{\circ}}$ *we have seen that the stabilizer is* $W(\circ\!-\!\circ)$ *, which is the same as removing the nodes that is not in the box representative face. And for the other two face representatives we can do the same.*

In more general, to find the face representatives by looking at the marked Coxeter diagram for λ, we will therefore look for $\overset{1}{\circ}\underset{m}{-}\overset{0}{\circ}$, $\overset{0}{\circ}\underset{m}{-}\overset{1}{\circ}$, $\overset{1}{\circ}\underset{m}{-}\overset{1}{\circ}$ and $\overset{1}{\circ}\quad\overset{1}{\circ}$ since these generate a regular polygons. Therefore we encode the marked Coxeter diagram for λ by drawing boxes around the representatives. Note that we only draw one box at the time.

Example 4.3.10. *Let* $\lambda = \overset{1}{\circ}\underset{5}{\text{——}}\overset{1}{\circ}\text{——}\overset{0}{\circ}$ *, then the face representatives are*

$$\boxed{\overset{1}{\circ}\underset{5}{\text{——}}\overset{1}{\circ}}\text{——}\overset{0}{\circ} \qquad \overset{1}{\circ}\underset{5}{\text{——}}\boxed{\overset{1}{\circ}\text{——}\overset{0}{\circ}}.$$

So $P(\lambda)$ *consists of decagons and triangles, and is a truncated dodecahedron.*

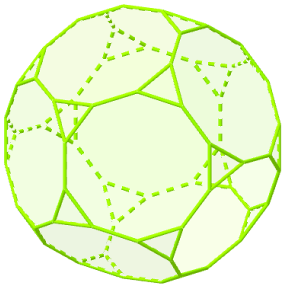


Figure 4.9: The polyhedron $P(\lambda_1 + \lambda_2)$ generated by $W(H_3)$.

In Example 4.3.8 and 4.3.9 we have seen that the stabilizer for face representatives we could have gotten by removing the nodes not in the box representative face. Suppose this is works for all the other face representatives, then we can use this to calculate the number faces.

Example 4.3.11. *Consider* $\lambda = \overset{1}{\circ}\text{——}\overset{0}{\circ}\underset{4}{\text{——}}\overset{1}{\circ}$ *. We already have looked at the edge representatives in Example 4.3.7, and for faces we have these representatives* $\boxed{\overset{1}{\circ}\text{——}\overset{0}{\circ}}\underset{4}{\text{——}}\overset{1}{\circ}$, $\overset{1}{\circ}\text{——}\boxed{\overset{0}{\circ}\underset{4}{\text{——}}\overset{1}{\circ}}$ *and* $\boxed{\boxed{\overset{1}{\circ}}\text{——}\overset{0}{\circ}\underset{4}{\text{——}}\boxed{\overset{1}{\circ}}}$*. The first representative generates triangles, the second representative generates squares, and the last one also generates squares, so this is not a regular polyhedron. The stabilizers for these representatives are*

$$\mathrm{Stab}\Big(\boxed{\overset{1}{\circ}\text{——}\overset{0}{\circ}}\underset{4}{\text{——}}\overset{1}{\circ}\Big) = W(\circ\text{——}\circ) = W(I_2^3)$$

$$\mathrm{Stab}\Big(\overset{1}{\circ}\text{——}\boxed{\overset{0}{\circ}\underset{4}{\text{——}}\overset{1}{\circ}}\Big) = W(\circ\underset{4}{\text{——}}\circ) = W(I_2^4)$$

$$\mathrm{Stab}\Big(\boxed{\boxed{\overset{1}{\circ}}\text{——}\overset{0}{\circ}\underset{4}{\text{——}}\boxed{\overset{1}{\circ}}}\Big) = W(\circ \qquad \circ) = W(A_1 \times A_1)\ .$$

By Proposition 3.2.4 and Table 2.3 the number of faces $\#f$ *are*

$$\#f = \frac{48}{8} + \frac{48}{6} + \frac{48}{4} = 26$$

Since it consists of six triangles and twenty squares this is a rhombicuboctahedron:

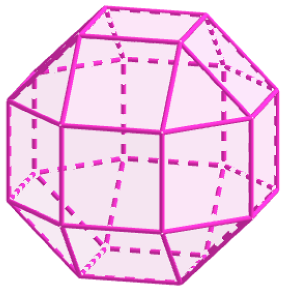


Figure 4.10: The polyhedron $P(\lambda_1 + \lambda_3)$ generated by $W(B_3)$.

We can also check the Euler characteristic since every convex polytope must satisfy this. We have not computed the number of vertices $\#v$, so let us do that by using Theorem 3.2.2, Proposition 3.2.4 and Table 2.3.

$$\mathrm{Stab}(\lambda) = W(A_1)$$
$$\#v = \frac{|W(B_3)|}{|W(A_1)|} = \frac{48}{2} = 24$$

Then $\chi = 24 - 48 + 26 = 2$*, so the Euler characteristic is satisfied.*

Instead considering the W-orbit through λ and then take the convex hull of these points to generate polytopes, we have seen that by encoding the marked Coxeter diagram for λ we can easily figure out the number vertices, edges and faces for $P(\lambda)$. By looking at the face representatives for a marked Coxeter diagram for λ we can also easily figure out what kind of faces $P(\lambda)$ consists of. Therefore when we have another λ than we have considered, we can easily figure out what kind of polytope it is.

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Appendices

A

Weyl group orbits

In Chapter 2 we considered Coxeter groups, which are finite groups generated by reflections. These we denoted $W = W(\Phi)$, where Φ is a root system as in Section 2.3. In Section 3.1 we gave a pictorial reflection recipe for the Coxeter diagrams and the Dynkin diagrams. Here we will give the W-orbits through λ_1, λ_2 and λ_3 for the rank three Coxeter groups.

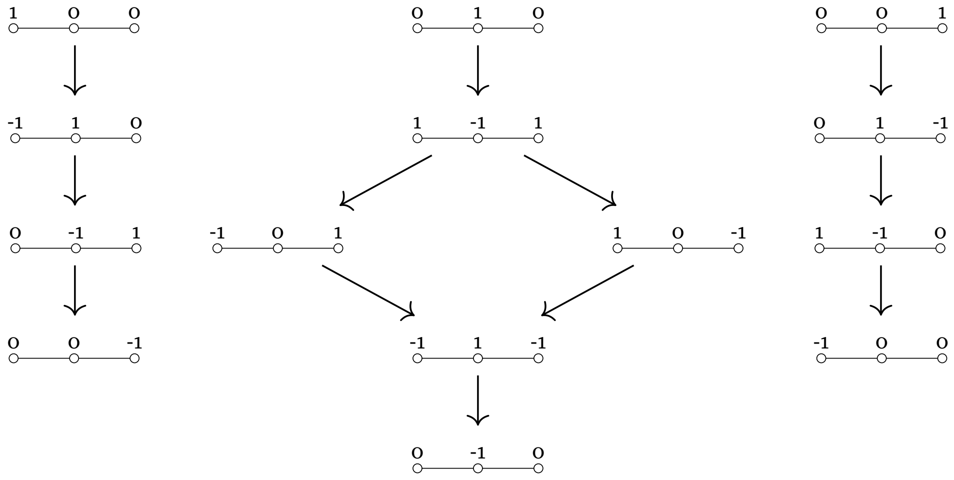

Figure A.1: The $W(A_3)$-orbits through λ_1, λ_2 and λ_3.

For the B_3 case we will use the Dynkin diagram:

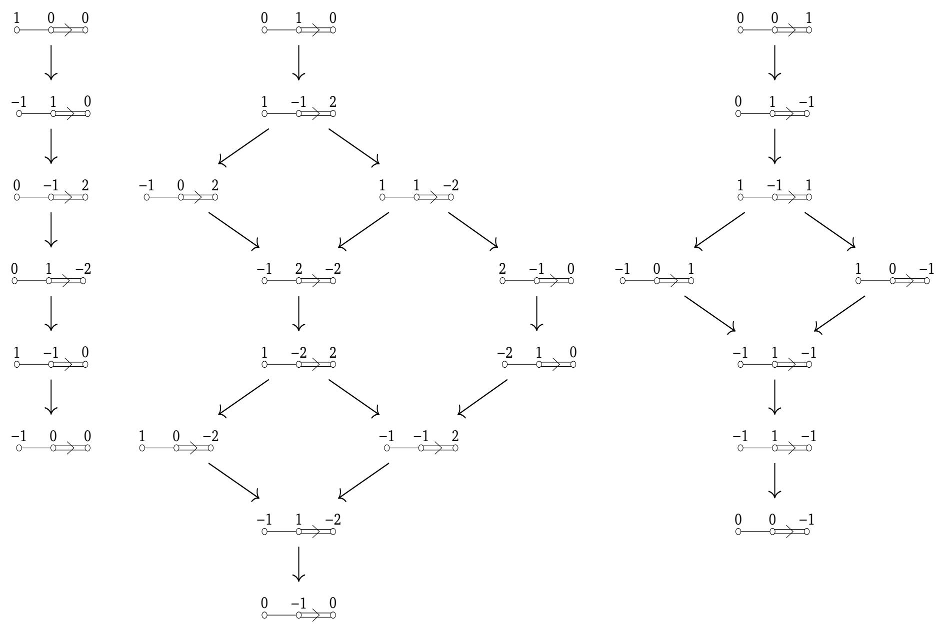

Figure A.2: The $W(B_3)$-orbits through λ_1, λ_2 and λ_3.

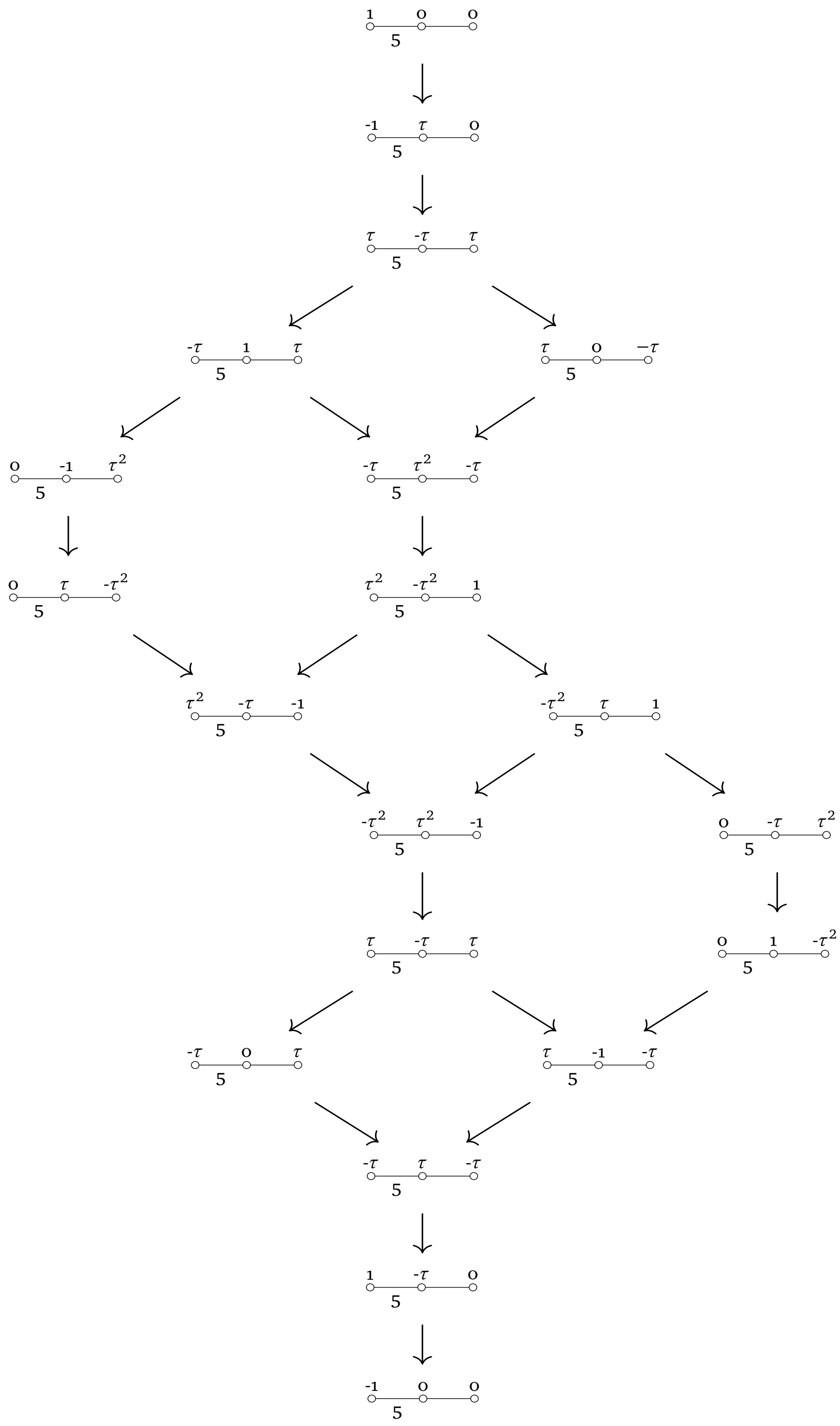

Figure A.3: The $W(H_3)$-orbit through λ_1.

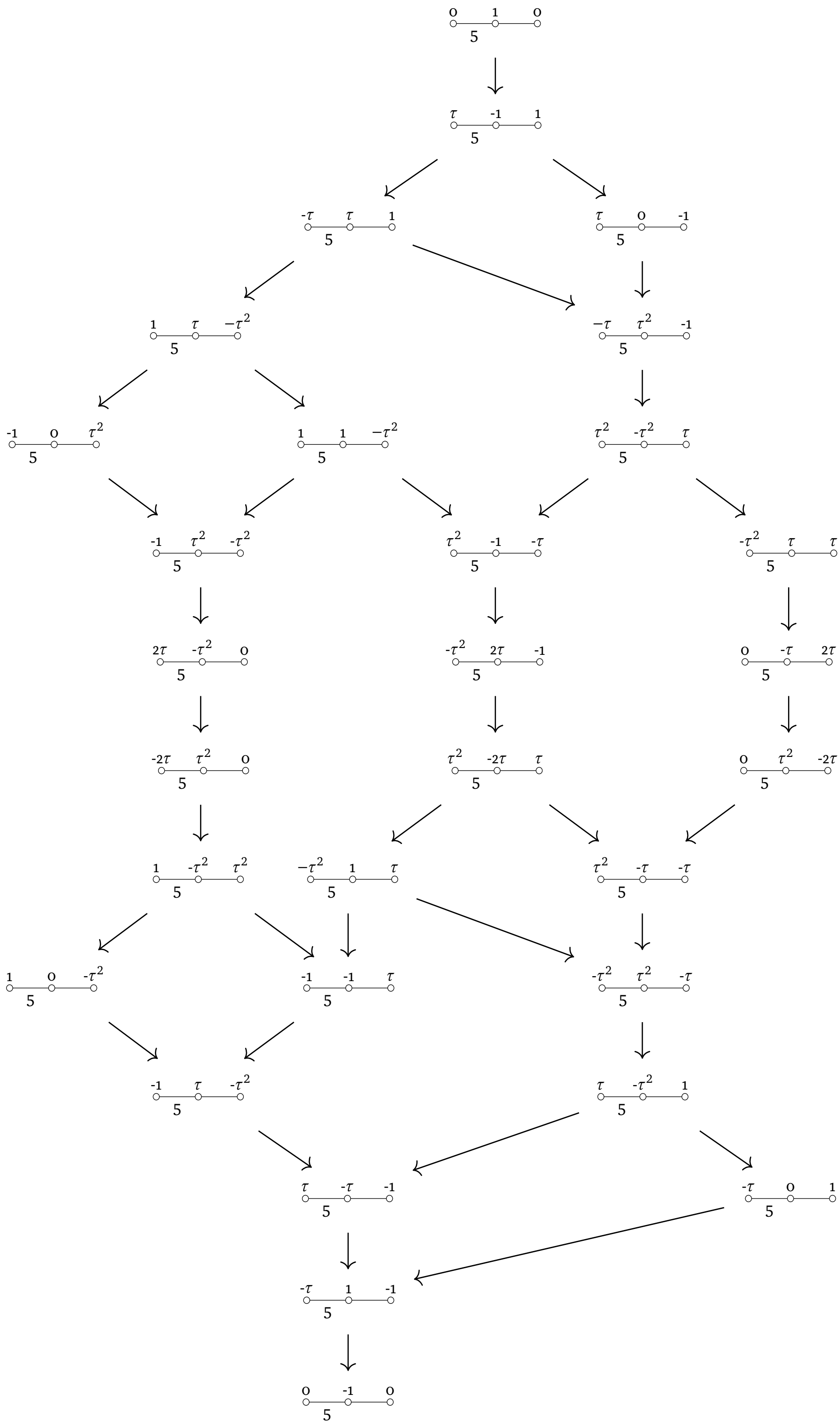

Figure A.4: The $W(H_3)$-orbit through λ_2.