Integrable systems in 4D associated with sixfolds in $\mathbf{Gr}(4,6)$

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Abstract

Let $\mathbf{Gr}(d, n)$ be the Grassmannian of d-dimensional linear subspaces of an n-dimensional vector space V. A submanifold $X \subset \mathbf{Gr}(d, n)$ gives rise to a differential system $\Sigma(X)$ that governs d-dimensional submanifolds of V whose Gaussian image is contained in X. We investigate a special case of this construction where X is a sixfold in $\mathbf{Gr}(4, 6)$. The corresponding system $\Sigma(X)$ reduces to a pair of first-order PDEs for 2 functions of 4 independent variables. Equations of this type arise in self-dual Ricci-flat geometry. Our main result is a complete description of *integrable* systems $\Sigma(X)$. These naturally fall into two subclasses.

- Systems of Monge-Ampère type. The corresponding sixfolds X are codimension 2 linear sections of the Plücker embedding $\mathbf{Gr}(4, 6) \hookrightarrow \mathbb{P}^{14}$.
- General linearly degenerate systems. The corresponding sixfolds X are the images of quadratic maps $\mathbb{P}^6 \dashrightarrow \mathbf{Gr}(4, 6)$ given by a version of the classical construction of Chasles.

We prove that integrability is equivalent to the requirement that the characteristic variety of system $\Sigma(X)$ gives rise to a conformal structure which is self-dual on every solution. In fact, all solutions carry hyper-Hermitian geometry.

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Contents

1 Introduction

1.1 Formulation of the problem

Let $u(\mathbf{x})$ and $v(\mathbf{x})$ be functions of the 4 independent variables $\mathbf{x} = (x^1, \dots, x^4)$. In this paper we investigate integrability of first-order systems of the form

$$F(u_1, \dots, u_4, v_1, \dots, v_4) = 0, \quad H(u_1, \dots, u_4, v_1, \dots, v_4) = 0, \tag{1}$$

where F and H are (nonlinear) functions of the partial derivatives $u_i = \frac{\partial u}{\partial x^i},\ v_i = \frac{\partial v}{\partial x^i}$. The geometry behind systems (1) is as follows. Let V be a 6-dimensional vector space with coordinates $x^1, \dots, x^4, u, v$. Solutions to system (1) correspond to 4-dimensional submanifolds of V defined as $u = u(\mathbf{x}),\ v = v(\mathbf{x})$. Their 4-dimensional tangent spaces, specified by the equations $du = u_i dx^i,\ dv = v_i dx^i$, are parametrised by 2×4 matrices

$$U = \begin{pmatrix} u_1 & \dots & u_4 \\ v_1 & \dots & v_4 \end{pmatrix},$$

whose entries are restricted by equations (1). Thus, equations (1) can be interpreted as the defining equations of a sixfold X in the Grassmannian $\mathbf{Gr}(4,6)$. Solutions to system (1) correspond to 4-dimensional submanifolds of V whose Gaussian images (tangent spaces translated to the origin) are contained in X. There exist two types of integrable systems (1).

Systems of Monge-Ampère type have the form

$$\begin{aligned} a^{ij}(u_i v_j - u_j v_i) + b^i u_i + c^i v_i + m = 0, \\ \alpha^{ij}(u_i v_j - u_j v_i) + \beta^i u_i + \gamma^i v_i + \mu = 0, \end{aligned} \tag{2}$$

where each equation is a constant-coefficient linear combination of the minors of U. These systems were introduced in [2] in the context of 'complete exceptionality'. Geometrically, the associated sixfolds X are linear sections of the Plücker embedding $\mathbf{Gr}(4,6) \hookrightarrow \mathbb{P}^{14}$. A typical example is the system

$$u_2 - v_1 = 0, \quad u_3 v_4 - u_4 v_3 - 1 = 0, \tag{3}$$

which reduces to the first heavenly equation of Plebanski [24], $w_{13}w_{24} - w_{14}w_{23} - 1 = 0$, under the substitution $w_1 = u,\ w_2 = v$. It governs self-dual Ricci-flat 4-manifolds; see Section 2.1 for further details on Monge-Ampère systems.

General linearly degenerate systems correspond to sixfolds X resulting as images of quadratic maps $\mathbb{P}^6 \dashrightarrow \mathbf{Gr}(4,6)$ (we refer to [7] for a discussion of the concept of linear degeneracy, see also Section 1.5). As an example, let us consider the system

$$\alpha u_2 v_1 - u_1 v_2 = 0, \quad u_4 v_1 - u_1 v_3 = 0,$$

$\alpha \neq 0, 1$ is a parameter. Note that this system does not belong to the Monge-Ampère class (2). The elimination of v leads to the second-order equation for u,

$$(\partial_3 - \partial_4)\frac{u_2}{u_1} = (\alpha^{-1} - 1)\partial_2 \frac{u_4}{u_1},$$

here $\partial_i = \partial_{x^i}$. Similarly, the elimination of u leads to the second-order equation for v,

$$(\partial_4 - \partial_3)\frac{v_2}{v_1} = (\alpha - 1)\partial_2 \frac{v_3}{v_1}.$$

Thus, one can speak of a four-dimensional Bäcklund transformation. This example can be viewed as a 4D generalisation of the Bäcklund transformation for the Veronese web equation constructed in [28]. We refer to Section 2.3 for further examples and classification results.

The main goal of this paper is to prove that systems of the above two types exhaust the list of non-degenerate integrable systems (1).

1.2 Non-degeneracy, conformal structure and self-duality

We will assume that system (1) is non-degenerate in the sense that the corresponding characteristic variety,

$$\det\left[\sum_{i=1}^{4} p_i \begin{pmatrix} F_{u_i} & F_{v_i} \\ H_{u_i} & H_{v_i} \end{pmatrix}\right] = 0,$$

defines an irreducible quadric of rank 4. This is the case for all examples of physical/geometric relevance. Explicitly, the characteristic variety can be represented in the form $g^{ij}p_ip_j = 0$ where

$$g^{ij} = \frac{1}{2}(F_{u_i}H_{v_j} + F_{u_j}H_{v_i} - F_{v_i}H_{u_j} - F_{v_j}H_{u_i}).$$

The characteristic variety gives rise to the conformal structure $g = g_{ij}dx^idx^j$ where g_{ij} is the inverse matrix of g^{ij}; note that non-degeneracy is equivalent to $\det g \neq 0$. Let $[g]$ denote the corresponding conformal class. Remarkably, integrability of system (1) has a natural interpretation in terms of the conformal geometry of $[g]$. In 4D, the key invariant of a conformal structure is its Weyl tensor W. It has self-dual and anti-self-dual parts,

$$W_+ = \frac{1}{2}(W + *W) \quad \text{and} \quad W_- = \frac{1}{2}(W - *W),$$

respectively. Here the Hodge star operator is defined as $*W^i_{jkl} = \frac{1}{2}\sqrt{\det g}\ g^{ia}g^{bc}\epsilon_{ajbd}W^d_{ckl}$. A conformal structure is said to be self-dual if, with a proper choice of orientation, we have

$$W_- = 0. \tag{4}$$

The integrability of conditions of self-duality by the twistor construction is due to Penrose [23], see also [10] for a direct demonstration. We will prove in Section 3 that integrability of 4D equations (1) is equivalent to the requirement that the conformal structure $[g]$ defined by the characteristic variety must be self-dual on every solution. Thus, *solutions to integrable systems carry integrable conformal geometry.* More precisely, with a suitable choice of orientation, it will be shown that the conditions of self-duality, $W_- = 0$, lead to Monge-Ampère systems. Similarly, the conditions of anti-self-duality, $W_+ = 0$, characterise general linearly degenerate systems associated with quadratic maps $\mathbb{P}^6 \dashrightarrow \mathbf{Gr}(4, 6)$. The intersection of these two classes consists of linearisable systems characterised by the conformal flatness of g.

For example, the conformal structure of system (3) is given by

$$g = u_3dx^1dx^3 + u_4dx^1dx^4 + v_3dx^2dx^3 + v_4dx^2dx^4.$$

A direct calculation shows that $[g]$ is self-dual on every solution, which means that (4) holds identically modulo (3). System (3) possesses the Lax representation $[X, Y] = 0$ where X, Y are parameter-dependent vector fields,

$$X = u_3\partial_4 - u_4\partial_3 + \lambda\partial_1, \quad Y = -v_3\partial_4 + v_4\partial_3 - \lambda\partial_2,$$

$\partial_i = \partial_{x^i}$. Projecting integral surfaces of the distribution spanned by X, Y from the extended space of variables $\mathbf{x}, \lambda$ (correspondence space) to the space of independent variables $\mathbf{x}$ one obtains a three-parameter family of totally null surfaces (α-surfaces) of the conformal structure $[g]$. According to [23], the existence of such surfaces is necessary and sufficient for self-duality. We refer to [1, 20, 21] for a novel version of the inverse scattering transform based on commuting parameter-dependent vector fields.

1.3 Dispersionless integrability in 4D

Integrability of multi-dimensional dispersionless PDEs can be approached based on the method of hydrodynamic reductions [17, 12, 11, 13]. In the most general set-up (for definiteness, we restrict to the 4D case), it applies to quasilinear systems of the form

$$A_1(\mathbf{u})\mathbf{u}_1 + A_2(\mathbf{u})\mathbf{u}_2 + A_3(\mathbf{u})\mathbf{u}_3 + A_4(\mathbf{u})\mathbf{u}_4 = 0, \tag{5}$$

where $\mathbf{u} = (u^1, ..., u^m)^t$ is an m-component column vector of the dependent variables, $\mathbf{u}_i = \frac{\partial \mathbf{u}}{\partial x^i}$, and A_i are $l \times m$ matrices where the number l of equations is allowed to exceed the number m of unknowns. Note that nonlinear system (1) can be brought to quasilinear form (5) by choosing u_i, v_i as the new dependent variables and writing out all possible consistency conditions among them, see Section 3. The method of hydrodynamic reductions consists of seeking multi-phase solutions in the form

$$\mathbf{u} = \mathbf{u}(R^1, ..., R^N)$$

where the phases $R^i(\mathbf{x})$, whose number N is allowed to be arbitrary, are required to satisfy a triple of consistent $(1+1)$-dimensional systems

$$R^i_{x^2} = \mu^i(R) R^i_{x^1}, \quad R^i_{x^3} = \eta^i(R) R^i_{x^1}, \quad R^i_{x^4} = \lambda^i(R) R^i_{x^1}, \tag{6}$$

known as systems of hydrodynamic type. The corresponding characteristic speeds must satisfy the commutativity conditions [27],

$$\frac{\partial_j \mu^i}{\mu^j - \mu^i} = \frac{\partial_j \eta^i}{\eta^j - \eta^i} = \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i}, \tag{7}$$

here $i \neq j$, $\partial_j = \partial_{R^j}$. Multi-phase solutions of this type originate from gas dynamics, and are known as nonlinear interactions of planar simple waves. Equations (6) are said to define an N-component hydrodynamic reduction of the original system (5). System (5) is said to be *integrable* if, for every N, it possesses infinitely many N-component hydrodynamic reductions parametrised by $2N$ arbitrary functions of one variable [13]. This requirement imposes strong constraints (integrability conditions) on the matrix elements of $A_i(\mathbf{u})$, see Section 3 for details.

The method of hydrodynamic reductions has been successfully applied to a whole range of systems in 3D, leading to extensive classification results. The corresponding submanifolds X are generally transcendental, parametrised by generalised hypergeometric functions [22]. The results of this paper are based on a direct application of the method of hydrodynamic reductions to 4D systems of type (1). The 4D situation turns out to be far more restrictive, in particular, the integrability conditions force X to be algebraic.

1.4 Equivalence group $\mathbf{SL}(6)$

All constructions described in the previous sections are equivariant with respect to the group $\mathbf{SL}(6)$ acting by linear transformations on the space V with coordinates $x^1, \dots, x^4, u, v$. The extension of this action to $\mathbf{Gr}(4,6)$ is given by the formula

$$U \to (AU + B)(CU + D)^{-1} \tag{8}$$

where A, B, C, D are 2×2, 2×4, 4×2 and 4×4 matrices, respectively; note that the extended action is no longer linear. Transformation law (8) suggests that the action of $\mathbf{SL}(6)$ preserves the class of equations (1). Furthermore, transformations (8) preserve the integrability, so that $\mathbf{SL}(6)$ can be viewed as a natural *equivalence group* of the problem: all our classification results will be formulated modulo this equivalence. In coordinates u_i, v_i, the infinitesimal generators corresponding to equivalence transformations (8) are as follows:

8 translations:

$$\mathbf{U}_i = \frac{\partial}{\partial u_i}, \quad \mathbf{V}_i = \frac{\partial}{\partial v_i},$$

19 linear generators (note the relation $\sum \mathbf{X}_{ii} = \mathbf{L}_{11} + \mathbf{L}_{22}$):

$$\mathbf{X}_{ij} = u_i \frac{\partial}{\partial u_j} + v_i \frac{\partial}{\partial v_j}, \quad \mathbf{L}_{11} = u_k \frac{\partial}{\partial u_k}, \quad \mathbf{L}_{12} = u_k \frac{\partial}{\partial v_k}, \quad \mathbf{L}_{21} = v_k \frac{\partial}{\partial u_k}, \quad \mathbf{L}_{22} = v_k \frac{\partial}{\partial v_k}.$$

8 projective generators:

$$\mathbf{P}_i = u_i u_k \frac{\partial}{\partial u_k} + v_i u_k \frac{\partial}{\partial v_k}, \quad \mathbf{Q}_i = u_i v_k \frac{\partial}{\partial u_k} + v_i v_k \frac{\partial}{\partial v_k}.$$

Let us represent system (1) in evolutionary form,

$$u_4 = f(u_1, u_2, u_3, v_1, v_2, v_3), \quad v_4 = h(u_1, u_2, u_3, v_1, v_2, v_3), \tag{9}$$

and consider the induced action of the equivalence group $\mathbf{SL}(6)$ on the space $J^1(\mathbb{R}^6, \mathbb{R}^2)$ of 1-jets of functions f, h of variables $u_1, u_2, u_3, v_1, v_2, v_3$. This is a 20-dimensional space with coordinates $u_i, v_i, f, h, f_{u_i}, f_{v_i}, h_{u_i}, h_{v_i}$, $i = 1, 2, 3$. One can show that the action of $\mathbf{SL}(6)$ on $J^1(\mathbb{R}^6, \mathbb{R}^2)$ has a unique Zariski open orbit (its complement consists of 1-jets of degenerate systems), see Section 3.1. This property allows one to assume that all sporadic factors depending on first-order derivatives of f and h that arise in the process of Gaussian elimination in the proofs of our main results in Section 3, are nonzero. This considerably simplifies the arguments by eliminating unessential branching. Furthermore, in the verification of polynomial identities involving first- and second-order partial derivatives of f and h one can, without any loss of generality, give the first-order derivatives any 'generic' numerical values: this often renders otherwise impossible computations manageable.

1.5 Linearly degenerate systems

The definition of linear degeneracy is inductive: a multi-dimensional system is said to be *linearly degenerate* (completely exceptional [2]) if such are all its traveling wave reductions to two dimensions. Thus, it is sufficient to define this concept in the 2D case,

$$u_2 = f(u_1, v_1), \quad v_2 = h(u_1, v_1).$$

Setting $u_1 = a,\ v_1 = p$ and differentiating by x^1 one can rewrite this system in two-component quasilinear form,

$$a_2 = f(a,p)_1, \quad p_2 = h(a,p)_1,$$

or, in matrix notation,

$$\begin{pmatrix} a \\ p \end{pmatrix}_2 = A \begin{pmatrix} a \\ p \end{pmatrix}_1, \quad A = \begin{pmatrix} f_a & f_p \\ h_a & h_p \end{pmatrix}.$$

Recall that the matrix A is said to be linearly degenerate if its eigenvalues (assumed real and distinct) are constant in the direction of the corresponding eigenvectors. Explicitly, $L_{r^i}\lambda^i = 0$, no summation, where L_{r^i} denotes Lie derivative in the direction of the eigenvector r^i, and $Ar^i = \lambda^i r^i$. For quasilinear systems, the property of linear degeneracy is known to be related to the impossibility of breakdown of smooth initial data [25]. In terms of the original functions $f(u_1, v_1)$ and $h(u_1, v_1)$, the conditions of linear degeneracy reduce to a pair of second-order differential constraints [7],

$$(f_{u_1} - h_{v_1})f_{u_1u_1} + 2h_{u_1}f_{u_1v_1} + h_{u_1}h_{v_1v_1} + f_{v_1}h_{u_1u_1} = 0,$$

$$(h_{v_1} - f_{u_1})h_{v_1v_1} + 2f_{v_1}h_{u_1v_1} + f_{v_1}f_{u_1u_1} + h_{u_1}f_{v_1v_1} = 0.$$

Requiring that all traveling wave reductions of a multi-dimensional system to 2D are linearly degenerate in the above sense, we obtain differential characterisation of linear degeneracy:

Proposition 1 [7]. System (9) is linearly degenerate if and only if the functions f and h satisfy the relations

$$\begin{aligned} &Sym_{\{i,j,k\}}\left((f_{u_k} - h_{v_k})f_{u_iu_j} + h_{u_k}(f_{u_iv_j} + f_{u_jv_i}) + f_{v_k}h_{u_iu_j} + h_{u_k}h_{v_iv_j}\right) = 0, \\ &Sym_{\{i,j,k\}}\left((h_{v_k} - f_{u_k})h_{v_iv_j} + f_{v_k}(h_{u_iv_j} + h_{u_jv_i}) + h_{u_k}f_{v_iv_j} + f_{v_k}f_{u_iu_j}\right) = 0, \end{aligned} \tag{10}$$

where Sym denotes complete symmetrisation over $i, j, k \in \{1, 2, 3\}$. Note that conditions (10) are invariant under the equivalence group $\mathbf{SL}(6)$.

The key observation is that second-order overdetermined system (10) is *not in involution*: its differential prolongation results in the two branches characterised by additional second-order differential constraints. The first branch leads to Monge-Ampère systems (10 additional second-order constraints). The second branch corresponds to general linearly degenerate systems (4 additional second-order constraints), see Section 3.2 for the details of this analysis.

1.6 Summary of the main results

Our results imply that several seemingly different approaches to integrability described above lead to one and the same class of systems (1).

Theorem 1 *Under the non-degeneracy assumption, the following conditions are equivalent:*

(a) System (1) is integrable by the method of hydrodynamic reductions.

(b) Conformal structure $[g]$ defined by the characteristic variety of system (1) is self-dual on every solution.

(c) System (1) is linearly degenerate.

(d) The associated sixfold $X \subset \mathbf{Gr}(4,6)$ is either a codimension two linear section of the Plücker embedding $\mathbf{Gr}(4,6) \hookrightarrow \mathbb{P}^{14}$, or the image of a quadratic map $\mathbb{P}^6 \dashrightarrow \mathbf{Gr}(4,6)$.

Theorem 1 and the results of [3] imply that any integrable system (1) possesses a Lax representation in parameter-dependent commuting vector fields. Integral surfaces of these vector fields give rise to α-surfaces of the conformal structure $[g]$.

Examples of integrable systems (1) are discussed in Section 2. The proof of Theorem 1 is given in Section 3. All calculations are based on computer algebra systems Mathematica and Maple (these only utilise symbolic polynomial algebra over $\mathbb{Q}$, so the results are rigorous). The programmes are available from the arXiv supplement to this paper.

2 Examples and classification results

In this section we discuss examples of 4D systems which, as will be demonstrated in Section 3, exhaust the list of all integrable systems of type (1).

2.1 Monge-Ampère systems

Systems of Monge-Ampère type correspond to sixfolds $X \subset \mathbf{Gr}(4,6)$ that can be obtained as codimension two linear sections of the Plücker embedding of the Grassmannian. Recall that $\mathbf{Gr}(4,6)$ is an 8-dimensional algebraic variety of degree 14 embedded into $\mathbb{P}^{14}$. All 2-component systems of Monge-Ampère type are integrable. They were classified in our recent paper [8].

Proposition 2 [8]. *In four dimensions, any non-degenerate system of Monge-Ampère type is* $\mathbf{SL}(6)$*-equivalent to one of the following normal forms:*

1. $u_2 - v_1 = 0, \quad u_3 + v_4 = 0,$
2. $u_2 - v_1 = 0, \quad u_3 + v_4 + u_1v_2 - u_2v_1 = 0,$
3. $u_2 - v_1 = 0, \quad u_3v_4 - u_4v_3 - 1 = 0,$
4. $u_2 - v_1 = 0, \quad u_1 + v_2 + u_3v_4 - u_4v_3 = 0.$

All these systems can be reduced to various heavenly-type equations. Introducing the potential w such that $w_1 = u,\ w_2 = v$ one obtains the linear ultrahyperbolic equation $w_{13} + w_{24} = 0$, the second heavenly equation $w_{13} + w_{24} + w_{11}w_{22} - w_{12}^2 = 0$ [24], the first heavenly equation $w_{13}w_{24} - w_{14}w_{23} - 1 = 0$ [24], and the Husain equation $w_{11} + w_{22} + w_{13}w_{24} - w_{14}w_{23} = 0$ [18], respectively. All of them originate from self-dual Ricci-flat geometry. Their integrability by the method of hydrodynamic reductions was established in [12, 13].

Representing system (1) in evolutionary form (9) one obtains a differential characterisation of the Monge-Ampère property.

Proposition 3 [8]. *The necessary and sufficient conditions for system (9) to be of Monge-Ampère type are equivalent to the following second-order relations for f and h,*

$$f_{u_iu_i} = \frac{2h_{u_i}}{h_{v_i}-f_{u_i}} f_{u_iv_i}, \qquad f_{v_iv_i} = \frac{2f_{v_i}}{f_{u_i}-h_{v_i}} f_{u_iv_i},$$

$$f_{u_iu_j} = \frac{h_{u_j}}{h_{v_i}-f_{u_i}} f_{u_iv_i} + \frac{h_{u_i}}{h_{v_j}-f_{u_j}} f_{u_jv_j}, \qquad f_{v_iv_j} = \frac{f_{v_j}}{f_{u_i}-h_{v_i}} f_{u_iv_i} + \frac{f_{v_i}}{f_{u_j}-h_{v_j}} f_{u_jv_j}, \tag{11}$$

$$f_{u_iv_j} + f_{u_jv_i} = \frac{f_{u_j}-h_{v_j}}{f_{u_i}-h_{v_i}} f_{u_iv_i} + \frac{f_{u_i}-h_{v_i}}{f_{u_j}-h_{v_j}} f_{u_jv_j},$$

where $i, j = 1, 2, 3$*. Equations for* h *can be obtained by the simultaneous substitution* $f \leftrightarrow h$ *and* $u \leftrightarrow v$ *(30 second-order relations altogether).*

Table 1 below contains the (Lie algebra) structure of the stabilisers of Monge-Ampère systems under the action of the equivalence group $\mathbf{SL}(6)$ (note that different cases are distinguished by the dimensions of the stabilisers).

Table 1: types of isotropy algebras $\mathfrak{s} \subset \mathfrak{sl}_6$ of Monge-Ampère systems in 4D

System of equations	dim($\mathfrak{s}$)	Levi decomposition of the algebra $\mathfrak{s}$
1: linear ultrahyperbolic $u_2 - v_1 = 0$ $u_3 + v_4 = 0$	13	$\mathfrak{s} = \mathfrak{s}_0 \oplus \mathfrak{s}_1$ graded by $r \in \mathfrak{z}(\mathfrak{gl}_2)$ $\mathfrak{s} = (\mathfrak{sl}_2 \oplus \mathfrak{gl}_2) \ltimes (\mathbb{R}^2 \otimes \mathbb{R}^3)$ $\mathfrak{s}$ is self-normalizing
2: 2nd heavenly $u_2 - v_1 = 0$ $u_3 + v_4 + u_1 v_2 - u_2 v_1 = 0$	11	$\mathfrak{s} = \mathfrak{s}_0 \oplus \mathfrak{s}_1 \oplus \mathfrak{s}_2$ graded by $r \in \mathfrak{z}(\mathfrak{gl}_2)$ $\mathfrak{s} = \mathfrak{gl}_2 \ltimes ((\mathbb{R}^1 + \mathbb{R}^3) \ltimes \mathbb{R}^3)$ $\mathfrak{s}$ is self-normalizing
3: 1st heavenly $u_2 - v_1 = 0$ $u_3 v_4 - u_4 v_3 - 1 = 0$	10	$\mathfrak{s} = \mathfrak{s}_0 \oplus \mathfrak{s}_1$ graded by $r \in \mathfrak{z}(\mathfrak{gl}_2)$ $\mathfrak{s} = \mathfrak{sl}_2 \oplus (\mathfrak{gl}_2 \ltimes \mathbb{R}^3)$ $\mathfrak{s}$ is not self-normalizing
4: Husain system $u_2 - v_1 = 0$ $u_1 + v_2 + u_3 v_4 - u_4 v_3 = 0$	9	semi-simple $\mathfrak{s} = \mathfrak{sl}_2 \oplus \mathfrak{sl}(2, \mathbb{C})_{\mathbb{R}}$ $\mathfrak{s}$ is not self-normalizing

Notes:
(1) The factors $\mathbb{R}^2, \mathbb{R}^3$ are irreducible representations of the corresponding $\mathfrak{sl}_2$ (same for the $\mathfrak{sl}_2$ factor in $\mathfrak{gl}_2 = \mathfrak{sl}_2 \oplus \mathbb{R}$) in cases 1-3.
(2) Lie algebra structure of the nilradical $\mathbb{R}^1 + \mathbb{R}^3_a + \mathbb{R}^3_b$ of $\mathfrak{s}$ in case 2: $[\mathbb{R}^1, \mathbb{R}^3_a] = \mathbb{R}^3_b$, $[\mathbb{R}^3_a, \mathbb{R}^3_a] = \mathbb{R}^3_b$ ($\mathfrak{sl}_2$-equivariance fixes the brackets uniquely).
(3) We indicate real forms of the equations in the left-hand side. Since the classification is over $\mathbb{C}$, the corresponding complex forms should be taken, e.g. $(\mathfrak{sl}_2^{\mathbb{C}})^{\oplus 3}$ in case 4.
(4) Normalizers of $\mathfrak{s} \subset \mathfrak{sl}_6$ in cases 3, 4 both have dimensions 11 (extension of the $\mathfrak{sl}_2$ factor to $\mathfrak{gl}_2$ in case 3 and of $\mathfrak{s}$ to the trace-free part of $\mathfrak{gl}_2 \oplus \mathfrak{gl}(2, \mathbb{C})_{\mathbb{R}}$ in case 4).

2.2 Linearisable systems

In this section we characterise systems (1) which can be linearised by a transformation from the equivalence group $\mathbf{SL}(6)$. Note that linearisable systems are necessarily of Monge-Ampère type.

Theorem 4. *Under the non-degeneracy assumption, the following conditions are equivalent:*
(a) System (1) is linearisable by a transformation from the equivalence group $\mathbf{SL}(6)$*.*
(b) System (1) is invariant under a 13*-dimensional subgroup of* $\mathbf{SL}(6)$*.*
(c) The characteristic variety of system (1) defines a conformal structure $[g]$ *which is flat on every solution:* $W = 0$*.*

Proof. Equivalence $(a) \iff (b)$: Consider a non-degenerate linear system, say $u_2 - v_1 = 0$, $u_3 + v_4 = 0$ (note that all non-degenerate linear systems of type (1) are $\mathbf{SL}(6)$-equivalent). This system is invariant under a 13-dimensional subgroup of $\mathbf{SL}(6)$ with the following infinitesimal generators (we use the notations of Section 1.4):

$$\begin{gathered}\mathbf{U}_1, \quad \mathbf{U}_4, \quad \mathbf{V}_2, \quad \mathbf{V}_3, \quad \mathbf{U}_2+\mathbf{V}_1, \quad \mathbf{U}_3-\mathbf{V}_4,\\ \mathbf{X}_{11}+\mathbf{X}_{22}, \quad \mathbf{X}_{33}+\mathbf{X}_{44}, \quad \mathbf{X}_{14}-\mathbf{X}_{23}, \quad \mathbf{X}_{41}-\mathbf{X}_{32},\\ \mathbf{X}_{12}-\mathbf{X}_{43}+\mathbf{L}_{12}, \quad \mathbf{X}_{21}-\mathbf{X}_{34}+\mathbf{L}_{21}, \quad \mathbf{X}_{22}+\mathbf{X}_{33}+\mathbf{L}_{22}.\end{gathered} \tag{12}$$

This Lie algebra is isomorphic to the semi-direct product $(V_1 \otimes V_2) \rtimes (\mathfrak{gl}_2 \times \mathfrak{sl}_2)$, where $V_1 \otimes V_2 \simeq \mathbb{R}^6$ is the tensor product of the standard representation V_1 of $\mathfrak{gl}_2 = \mathfrak{sl}_2 \oplus \mathbb{R}$, and the representation V_2 of $\mathfrak{sl}_2$. Here $\mathfrak{gl}_2$ (resp. $\mathfrak{sl}_2$) acts on the first (resp. second) factor of $V_1 \otimes V_2$.

To establish the converse, let G be the symmetry group of system (1). We can always assume that the point o, specified by $u_i = v_i = 0$, belongs to the sixfold $X \subset \mathbf{Gr}(4,6)$ corresponding to our system. Let G_o be the stabiliser of this point in G. Note that $\dim G - \dim G_o \leq 6$, as G takes X to itself. The stabiliser P of the point o is spanned by infinitesimal generators $\mathbf{X}_{ij}, \ \mathbf{L}_{ij}, \ \mathbf{P}_i, \ \mathbf{Q}_i$. Since the system is non-degenerate, we can bring it to a canonical form

$$u_2 = v_1 + o(u_i, v_i), \quad u_3 = -v_4 + o(u_i, v_i). \tag{13}$$

This form (together with the point o) is stabilised by 7 elements of P listed in the last two lines of (12). Thus, $\dim G_o \leq 7$ so that $\dim G \leq 13$. The equality holds only if $\dim G_o = 7$. However, the generator $\mathbf{X}_{11} + \mathbf{X}_{22} + \mathbf{X}_{33} + \mathbf{X}_{44}$ acts by non-trivial rescalings on terms of order 2 and higher in (13). Hence, for $\dim G_o = 7$, all higher-order terms must vanish identically, leading to a linear system.

Equivalence $(a) \iff (c)$: Let us represent system (1) in evolutionary form (9) and take the corresponding conformal structure $[g]$. Conformal flatness is equivalent to the vanishing of the Weyl tensor

$$W_{ijkl} = R_{ijkl} - w_{ik}g_{jl} - w_{jl}g_{ik} + w_{jk}g_{il} + w_{il}g_{jk} = 0, \tag{14}$$

where $R_{ijkl} = g_{is}R^s_{jkl}$ is the curvature tensor, $w_{ij} = \frac{1}{2}R_{ij} - \frac{R}{12}g_{ij}$ is the Schouten tensor, R_{ij} is the Ricci tensor, and R is the scalar curvature. Calculating (14) and using equations (9) along with their differential consequences to eliminate all higher-order partial derivatives of u and v containing differentiation by x^4, we obtain expressions that have to vanish identically in the remaining higher-order derivatives (no more than third-order derivatives are involved in this calculation). In particular, equating to zero coefficients at the remaining third-order derivatives of u and v we obtain 34 second-order relations for f and h that contain 30 relations (11) governing Monge-Ampère systems, plus 4 extra (more complicated) relations. The easiest way to finish the proof is to note that according to Proposition 2 of Section 2.1, any 4D system of Monge-Ampère type is $\mathbf{SL}(6)$-equivalent to one of the four normal forms, and direct verification shows that conformal structures defined by characteristic varieties of the last three (non-linearisable) normal forms are not flat on generic solutions. Thus, the above 34 second-order relations are nothing but the linearisability conditions. This finishes the proof of Proposition 4.

2.3 Systems associated with quadratic maps $\mathbb{P}^6 \dashrightarrow \mathbf{Gr}(4,6)$

In this section we classify integrable systems (1) which correspond to sixfolds $X \subset \mathbf{Gr}(4,6)$ resulting as images of quadratic maps $\mathbb{P}^6 \dashrightarrow \mathbf{Gr}(4,6)$. These maps come from the following geometric construction.

Consider two vector spaces V and W. Let $A \in \mathrm{Hom}(W,V)$ and $B \in \mathrm{Hom}(W,V)$ be two linear maps. The collection of 2-planes $Ax \wedge Bx$, $x \in W$, defines a subvariety of $\mathbf{Gr}(2,V)$, the image of a quadratic map $\mathbb{P}(W) \dashrightarrow \mathbf{Gr}(2,V)$. In the particular case $V=W$ this construction goes back to Chasles [4] who considered the locus of lines spanned by an argument and the value of a projective transformation; see also [5], p. 556. Quadratic maps $\mathbb{P}^6 \dashrightarrow \mathbf{Gr}(2,6)$ result from the above construction when $\dim V = 6$, $\dim W = 7$. This gives a map $\mathbb{P}(W) = \mathbb{P}^6 \dashrightarrow \mathbf{Gr}(2,V)$, leading by duality to a quadratic map $\mathbb{P}^6 \dashrightarrow \mathbf{Gr}(4,V^*) = \mathbf{Gr}(4,6)$.

In coordinates, this reads as follows. Consider projective space $\mathbb{P}(W) = \mathbb{P}^6$ with homogeneous coordinates $\xi = (\xi^1 : \xi^2 : \xi^3 : \xi^4 : \xi^5 : \xi^6 : \xi^7)$. Let A and B be two 7×6 matrices representing the corresponding linear maps. Introduce the 2×6 matrix of linear forms on W,

$$\begin{pmatrix} \eta^1 & \eta^2 & \eta^3 & \eta^4 & \eta^5 & \eta^6 \\ \tau^1 & \tau^2 & \tau^3 & \tau^4 & \tau^5 & \tau^6 \end{pmatrix},$$

where $\eta = \xi A$ and $\tau = \xi B$. The Plücker coordinates $p^{ij} = \eta^i\tau^j - \eta^j\tau^i$ define a quadratic map $\mathbb{P}^6 \dashrightarrow \mathbf{Gr}(2,6) \subset \mathbb{P}^{14}$. By duality, this gives a sixfold $X \subset \mathbf{Gr}(4,6)$, and the corresponding system (1). Explicit parametric formulae can be obtained from the factorised representation,

$$\begin{pmatrix} \eta^1 & \eta^2 & \eta^3 & \eta^4 & \eta^5 & \eta^6 \\ \tau^1 & \tau^2 & \tau^3 & \tau^4 & \tau^5 & \tau^6 \end{pmatrix} = \begin{pmatrix} \eta^5 & \eta^6 \\ \tau^5 & \tau^6 \end{pmatrix} \begin{pmatrix} u_1 & u_2 & u_3 & u_4 & 1 & 0 \\ v_1 & v_2 & v_3 & v_4 & 0 & 1 \end{pmatrix},$$

which gives $u_i = p^{i6}/p^{56}$, $v_i = p^{i5}/p^{65}$, $i = 1, \dots, 4$. Eliminating ξ's, we obtain two relations among u_i, v_i, which constitute the required system $\Sigma(X)$.

Tables 2–6 below comprise a complete list of resulting systems (1) labelled by Jordan-Kronecker normal forms [16] of the matrix pencil A, B (see the end of this section for an illustrative calculation leading to the first case of Table 2). Note that A and B are defined up to transformations $A \to PAQ$, $B \to PBQ$, where the 7×7 matrix P is responsible for a change of basis in W and the 6×6 matrix Q corresponds to the action of the equivalence group $\mathbf{SL}(6)$. Modulo these transformations, A and B must have exactly one Kronecker block of the size $(n+1)\times n$, for $n = 2, \dots, 6$ (the cases of a single 2×1 Kronecker block, as well as of more than one Kronecker blocks, lead to either degenerate or linear systems). We group systems according to the size of the Kronecker block. Within each table, systems are labelled by Serge types of the remaining Jordan block. In all cases (with the exception of the most generic system from Table 6) we have chosen canonical forms which, via elimination of u, imply second-order equations for v. We also present the associated dispersionless Lax pairs in the form of two commuting λ-dependent vector fields, $[X,Y]=0$.

Table 2: canonical forms with one 3×2 Kronecker block

Segre type	Canonical form	Equation for v	Lax pair
[1111]	$\alpha u_2 v_1 = u_1 v_2$ $u_4 v_1 = u_1 v_3$	$\left(\frac{v_2}{v_1}\right)_4 - \left(\frac{v_2}{v_1}\right)_3 = (\alpha-1)\left(\frac{v_3}{v_1}\right)_2$	$X = \partial_1 + \frac{\lambda-\alpha}{1-\lambda}\frac{v_1}{v_2}\partial_2$ $Y = \partial_4 - \lambda\partial_3 + (\lambda-\alpha)\frac{v_3}{v_2}\partial_2$
[211]	$u_2 v_1 - u_1 v_2 = v_1 v_2$ $u_4 v_1 - u_1 v_4 = v_1 v_3$	$\left(\frac{v_2}{v_1}\right)_3 = \left(\frac{v_4}{v_1}\right)_2$	$X = \partial_1 + (\lambda-1)\frac{v_1}{v_2}\partial_2$ $Y = \partial_4 - \lambda\partial_3 + (\lambda-1)\frac{v_4}{v_2}\partial_2$
[22]	$u_2 v_1 - u_1 v_2 = v_1^2$ $u_4 v_1 - u_1 v_4 = v_1 v_3$	$\left(\frac{v_2}{v_1}\right)_3 = \left(\frac{v_4}{v_1}\right)_1$	$X = \partial_2 - \left(\lambda + \frac{v_2}{v_1}\right)\partial_1$ $Y = \partial_4 - \lambda\partial_3 - \frac{v_4}{v_1}\partial_1$
[31]	$u_2 = -v_1 v_2$ $u_4 = v_3 - v_1 v_4$	$v_{23} + v_2 v_{14} - v_4 v_{12} = 0$	$X = \partial_2 + \lambda v_2 \partial_1$ $Y = \partial_4 - \lambda\partial_3 + \lambda v_4 \partial_1$
[4]	$u_1 = v_2 - v_1^2$ $u_4 = v_3 - v_1 v_4$	$v_{24} - v_{13} + v_4 v_{11} - v_1 v_{14} = 0$	$X = \partial_2 - (v_1+\lambda)\partial_1$ $Y = \partial_3 - v_4\partial_1 - \lambda\partial_4$

Table 3: canonical forms with one 4×3 Kronecker block

Segre type	Canonical form	Equation for v	Lax pair
[111]	$u_3 v_1 = \alpha(v_2 - v_3) u_1$ $u_4 v_1 = \alpha(v_3 - v_4) u_1$	$m_4 + \alpha m n_1 = n_3 + \alpha n m_1$ $m = \frac{v_2 - v_3}{v_1},\ n = \frac{v_3 - v_4}{v_1}$	$X = \partial_2 - c(m + \lambda n)\partial_1 - \lambda^2 \partial_4$ $Y = \partial_3 - c n \partial_1 - \lambda \partial_4$ $c = 1 + \alpha - \lambda\alpha$
[21]	$u_3 v_1 - u_1 v_3 = (v_2 - \alpha v_3) v_1$ $u_4 v_1 - u_1 v_4 = (v_3 - \alpha v_4) v_1$	$(\partial_2 - \alpha\partial_3)\frac{v_4}{v_1}$ $= (\partial_3 - \alpha\partial_4)\frac{v_3}{v_1}$	$X = \partial_2 + (\lambda - \alpha)\frac{\lambda v_4 + v_3}{v_1}\partial_1 - \lambda^2 \partial_4$ $Y = \partial_3 + (\lambda - \alpha)\frac{v_4}{v_1}\partial_1 - \lambda \partial_4$
[3]	$u_3 = v_2 - v_1 v_3$ $u_4 = v_3 - v_1 v_4$	$v_{24} - v_{33} = v_3 v_{14} - v_4 v_{13}$	$X = \partial_2 - (\lambda v_4 + v_3)\partial_1 - \lambda^2 \partial_4$ $Y = \partial_3 - v_4 \partial_1 - \lambda \partial_4$

Table 4: canonical forms with one 5×4 Kronecker block

Segre type	Canonical form	Equation for v	Lax pair
[11]	$u_3(v_2 - v_1) = u_2(v_3 - v_2)$ $u_4(v_2 - v_1) = u_2(v_4 - v_3)$	$m_3 + m n_1 = n_2 + n m_1$ $m = \frac{v_3 - v_2}{v_2 - v_1},\ n = \frac{v_4 - v_3}{v_2 - v_1}$	$X = \partial_3 - (\lambda + m)\partial_2 + \lambda m \partial_1$ $Y = \partial_4 - (\lambda^2 + \lambda m + n)\partial_2 + (\lambda^2 m + \lambda n)\partial_1$
[2]	$v_3(u_2 - v_1) = v_2(u_3 - v_2)$ $v_4(u_2 - v_1) = v_2(u_4 - v_3)$	$m_3 + m n_1 = n_2 + n m_1$ $m = \frac{v_3}{v_2},\ n = \frac{v_4}{v_2}$	$X = \partial_3 - (\lambda + m)\partial_2 + \lambda m \partial_1$ $Y = \partial_4 - (\lambda^2 + \lambda m + n)\partial_2 + (\lambda^2 m + \lambda n)\partial_1$

Table 5: canonical form with one 6×5 Kronecker block

Segre type	Canonical form	Equation for v and Lax pair
[1]	$\frac{u_2 - u_1 v_1}{v_2 - v_1^2} = \frac{u_3 - u_1 v_2}{v_3 - v_1 v_2} = \frac{u_4 - u_1 v_3}{v_4 - v_1 v_3}$	$m_3 + m n_1 = n_2 + n m_1$ $X = \partial_3 - (\lambda + m)\partial_2 + (\lambda m - a)\partial_1$ $Y = \partial_4 - (\lambda^2 + \lambda m + n)\partial_2 + (\lambda^2 m + \lambda n - \lambda a - b)\partial_1$ $m = \frac{v_3 - v_1 v_2}{v_2 - v_1^2},\ n = \frac{v_4 - v_1 v_3}{v_2 - v_1^2},\ a = \frac{v_2^2 - v_1 v_3}{v_2 - v_1^2},\ b = \frac{v_2 v_3 - v_1 v_4}{v_2 - v_1^2}$

Table 6: canonical form with one 7×6 Kronecker block

Segre type	Canonical form	Lax pair
[0]	$\frac{u_2 - u_1 v_1}{v_2 - u_1 - v_1^2} = \frac{u_3 - u_1 v_2}{v_3 - u_2 - v_1 v_2}$ $= \frac{u_4 - u_1 v_3}{v_4 - u_3 - v_1 v_3}$	note that there is no equation for v in this case $X = \partial_3 - (\lambda + m)\partial_2 + (\lambda m - a)\partial_1$ $Y = \partial_4 - (\lambda^2 + \lambda m + n)\partial_2 + (\lambda^2 m + \lambda n - \lambda a - b)\partial_1$ $m = \frac{u_3 - u_1 v_2}{u_2 - u_1 v_1},\ n = \frac{u_4 - u_1 v_3}{u_2 - u_1 v_1},\ a = \frac{u_2 v_2 - u_3 v_1}{u_2 - u_1 v_1},\ b = \frac{u_2 v_3 - u_4 v_1}{u_2 - u_1 v_1}$

Remark. Note that both systems from Table 4 are related to (one and the same!) quasilinear system for the corresponding variables m, n, namely

$$m_4 - n_3 + m n_2 - n m_2 = 0, \quad m_3 - n_2 + m n_1 - n m_1 = 0 \tag{15}$$

(indeed, in terms of these variables their Lax pairs are identically the same). Thus, although the original systems are not equivalent under the natural equivalence group $\mathbf{SL}(6)$, the corresponding equations for v are related by a Bäcklund transformation. System (15) can be viewed as a travelling wave reduction of the 6D integrable system

$$m_6 - n_5 + m n_4 - n m_4 = 0, \quad m_3 - n_2 + m n_1 - n m_1 = 0$$

discussed in [13].

Similarly, the coincidence of Lax pairs from Tables 5 and 6 indicates that the corresponding systems can be considered as (nonlinear) reductions of one and the same first-order 4-component system for the variables a, b, m, n resulting from the commutativity condition $[X, Y] = 0$. This 4-component system can be viewed as yet another equivalent form of the equations governing hyper-Hermitian conformal structures in 4D [10].

Notice that the absence of terms with ∂_λ in the Lax representations from Tables 2-6 means that all solutions of the above systems carry hyper-Hermitian geometry [9], which is associated to the canonical conformal structure.

Example. Let us give details of calculations in the case when the pair A, B contains one 3×2 Kronecker block (upper left) and a 4×4 Jordan block of Segre type [1111], explicitly,

$$A = \begin{pmatrix} 1 & 0 & & & & \\ 0 & 1 & & & & \\ 0 & 0 & & & & \\ & & 1 & 0 & 0 & 0 \\ & & 0 & 1 & 0 & 0 \\ & & 0 & 0 & 1 & 0 \\ & & 0 & 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & & & & \\ 1 & 0 & & & & \\ 0 & 1 & & & & \\ & & \alpha & 0 & 0 & 0 \\ & & 0 & \beta & 0 & 0 \\ & & 0 & 0 & \gamma & 0 \\ & & 0 & 0 & 0 & \delta \end{pmatrix}.$$

The corresponding 2×6 matrix of linear forms is

$$\begin{pmatrix} \xi^1 & \xi^2 & \xi^4 & \xi^5 & \xi^6 & \xi^7 \\ \xi^2 & \xi^3 & \alpha\xi^4 & \beta\xi^5 & \gamma\xi^6 & \delta\xi^7 \end{pmatrix} = \begin{pmatrix} \xi^6 & \xi^7 \\ \gamma\xi^6 & \delta\xi^7 \end{pmatrix} \begin{pmatrix} u_1 & u_2 & u_3 & u_4 & 1 & 0 \\ v_1 & v_2 & v_3 & v_4 & 0 & 1 \end{pmatrix},$$

so that

$$u_1 = \frac{\delta\xi^1 - \xi^2}{(\delta-\gamma)\xi^6}, \quad u_2 = \frac{\delta\xi^2 - \xi^3}{(\delta-\gamma)\xi^6}, \quad u_3 = \frac{(\delta-\alpha)\xi^4}{(\delta-\gamma)\xi^6}, \quad u_4 = \frac{(\delta-\beta)\xi^5}{(\delta-\gamma)\xi^6},$$
$$v_1 = \frac{\gamma\xi^1 - \xi^2}{(\gamma-\delta)\xi^7}, \quad v_2 = \frac{\gamma\xi^2 - \xi^3}{(\gamma-\delta)\xi^7}, \quad v_3 = \frac{(\gamma-\alpha)\xi^4}{(\gamma-\delta)\xi^7}, \quad v_4 = \frac{(\gamma-\beta)\xi^5}{(\gamma-\delta)\xi^7}.$$

The elimination of ξ's leads to the following relations:

$$u_3v_4 = \frac{(\delta-\beta)(\gamma-\alpha)}{(\delta-\alpha)(\gamma-\beta)}u_4v_3, \qquad u_4(v_2 - \delta v_1) = \frac{\delta-\beta}{\gamma-\beta}v_4(u_2 - \gamma u_1).$$

Modulo equivalence transformations, this system is reducible to the first case of Table 2.

2.4 Symmetries of general linearly degenerate systems

The equivalence group $\mathbf{SL}(6)$ preserves both the class of Monge-Ampère equations and the class given by the Chasles construction. The stabilizer of an equation is its *linear* symmetry group (the full group of point symmetries of an integrable system is normally infinite-dimensional).

For Monge-Ampère systems, the Lie algebras $\mathfrak{s}$ corresponding to these groups were indicated in Table 1. Below we provide some data on the isotropy algebras for general linearly degenerate systems from Tables 2-6. We denote by $\mathfrak{c}(\mathfrak{s}) = \{g \in \mathfrak{sl}_6 : [g, \mathfrak{s}] = 0\}$ the centralizer of $\mathfrak{s}$, and by $\mathfrak{n}(\mathfrak{s}) = \{g \in \mathfrak{sl}_6 : [g, \mathfrak{s}] \subset \mathfrak{s}\}$ the normalizer of $\mathfrak{s}$.

Table 7: types of isotropy algebras $\mathfrak{s} \subset \mathfrak{sl}_6$ for general linearly degenerate systems in 4D

Segre type	$\dim \mathfrak{s}$	$\dim \mathfrak{c}(\mathfrak{s})$	$\dim \mathfrak{n}(\mathfrak{s})$	Lie algebra type	dim. derived ser.
[1111]	8	0	8	solvable	(8,4,0)
[211]	8	0	8	solvable	(8,4,0)
[22]	9	0	9	solvable	(9,6,2,0)
[31]	9	0	9	solvable	(9,6,2,0)
[4]	10	0	10	solvable	(10,8,5,1,0)
[111]	6	2	8	solvable	(6,3,0)
[21]	7	1	8	solvable	(7,4,1,0)
[3]	8	0	8	solvable	(8,6,3,0)
[11]	5	3	7	solvable	(5,2,0)
[2]	6	0	6	solvable	(6,4,1,0)
[1]	4	0	4	solvable	(4,2,0)
[0]	3	0	3	simple: $\mathfrak{sl}_2$	(3)

The listed dimensions do not separate types [1111] and [211], as well as [22] and [31]. Yet, the symmetry algebras do distinguish between them. To see this let $\mathbf{z} = \sum_{i=1}^{8} z_i e_i$ be a general element of $\mathfrak{s} = \langle e_1, \dots, e_8 \rangle$ in the first two cases. Denote by $\mathrm{ad}_{\mathbf{z}} \in \mathrm{End}(\mathfrak{s})$ the adjoint operator. For the Segre type [1111] its spectrum is $\mathrm{Sp}(\mathrm{ad}_{\mathbf{z}}) = \{0(\times 4), z_1, z_2, z_3, z_4\}$, while for the Segre type [211] it is $\mathrm{Sp}(\mathrm{ad}_{\mathbf{z}}) = \{0(\times 4), z_1(\times 2), z_2, z_3\}$. Thus multiplicities of the eigenvalues for general $\mathbf{z}$ distinguish these cases.

However the other two types are not distinguished by the multiplicities. Here $\dim \mathfrak{s} = 9$, so let $\mathbf{z} = \sum_{i=1}^{9} z_i e_i$. For the Segre type [22] we have $\mathrm{Sp}(\mathrm{ad}_{\mathbf{z}}) = \{0(\times 3), \pm i z_1, z_2, z_3, z_1 + z_2, z_3 - z_1\}$, and for the Segre type [31], $\mathrm{Sp}(\mathrm{ad}_{\mathbf{z}}) = \{0(\times 3), z_1, z_2, z_3, 2z_2, z_1 + z_2, z_1 + 2z_2\}$. But since linear relations among the eigenvalues in these two cases are different, these types are also distinguished by the symmetry algebras.

3 Proofs of the main results

After a short remark on the action of $\mathbf{SL}(6)$, we investigate the differential prolongation of conditions of linear degeneracy (10). The main feature of this second-order PDE system is its non-involutivity, manifesting itself in additional (hidden) second-order differential constraints. These constraints are obtained by differentiations and linear combinations of the equations in the original system. Afterwards, we complete the proof of Theorem 1.

3.1 Action of the equivalence group

While the action of $\mathbf{SL}(6)$ on the Grassmannian $\mathbf{Gr}(4,6)$ is transitive, the action on its tangent space $T\mathbf{Gr}(4,6)$ has orbits distinguished by the rank of the corresponding 2×4 matrices. We will need the action on the space of 1-jets $J^1_6\mathbf{Gr}(4,6)$ of submanifolds $X \subset \mathbf{Gr}(4,6)$ of dimension 6, which can be identified with the space $\mathbf{Gr}_6(T\mathbf{Gr}(4,6))$ locally isomorphic to $J^1(\mathbb{R}^6, \mathbb{R}^2)$.

Lemma. *The equivalence group* $\mathbf{SL}(6)$ *has a unique Zariski open orbit in the space* $J^1_6\mathbf{Gr}(4,6)$ *(its complement consists of 1-jets of degenerate systems).*

Proof. The stabilizer in $\mathbf{SL}(6)$ of a point $o \in \mathbf{Gr}(4,6)$ is the parabolic subgroup $P_o = S(\mathbf{GL}(2) \times \mathbf{GL}(4)) \ltimes (\mathbb{R}^2 \otimes \mathbb{R}^4)$ of upper-triangular block matrices of the size 2+4. The summand $\mathbb{R}^2 \otimes \mathbb{R}^4$ acts trivially on $T_o\mathbf{Gr}(4,6)$, so the effective action is only supported by the subgroup $S(\mathbf{GL}(2)\times\mathbf{GL}(4))$. It is easy to check that this action is transitive on 6-planes corresponding to non-degenerate 1-jets of X characterised by $\det g\neq 0$ where g denotes a metric representative of the canonical conformal structure $[g]$ (see Section 1.2).

At the level of Lie algebra $\mathfrak{sl}(6)$, the prolongation of the 35 infinitesimal generators $\mathbf{U}_i,\mathbf{V}_j$, $\mathbf{X}_{ij},\mathbf{L}_{ij}$, $\mathbf{P}_i,\mathbf{Q}_j$ (see Section 1.4) to $J^1(\mathbb{R}^6,\mathbb{R}^2)$ has full rank in the Zariski open set of non-degenerate 1-jets. Indeed, the 35×20 matrix of coefficients of these vector fields drops rank precisely on the submanifold $\det g=0$.

Remark. The next Sections contain details of calculations assisted with symbolic packages Maple and Mathematica. However, even these packages cannot resolve the large linear systems that arise after a prolongation to higher (third, forth and fifth) jets. To handle this difficulty we used the following trick: since $\mathbf{SL}(6)$ acts on $J^1(\mathbb{R}^6,\mathbb{R}^2)$ with an open orbit consisting precisely of admissible 1-jets, and since the prolongation, involutivity and integrability are $\mathbf{SL}(6)$-equivariant properties, we can substitute any numerical non-degenerate 1-jet into all prolonged equations; we used $(f_1,f_2,f_3,f_4,f_5,f_6)=(0,1,0,1,0,0)$, $(h_1,h_2,h_3,h_4,h_5,h_6)=(0,0,1,0,0,0)$. This allows to resolve the arising systems, and to compute their ranks without any loss of generality.

3.2 Prolongation of the conditions of linear degeneracy

To describe the result we will exploit the language of formal theory of differential equations, cf. [19]. Recall that a system of PDEs of order k on sections of a bundle ν over a manifold X can be respresented as a submanifold $\mathcal{E}_k\subset J^k(\nu)$ in the space of jets. In our case, $X\subset\mathbf{Gr}(4,6)$ is the sixfold encoding the system, and $\nu=T_X\mathbf{Gr}(4,6)/TX$ is its normal bundle. Locally, in the affine chart we can identify $X=\mathbb{R}^6(u_1,u_2,u_3,v_1,v_2,v_3)$ and $\nu=X\times\mathbb{R}^6(u_4,v_4)$ with sections given by (9). Thus, an affine chart of $J^k(\nu)$ is the space $J^k(\mathbb{R}^6,\mathbb{R}^2)$ of jets of maps $(f,h):\mathbb{R}^6\to\mathbb{R}^2$, and we will further denote this space by J^k.

Let us consider the system $\mathcal{E}_2\subset J^2$ given by 20 PDEs (10) (note that these equations, $E_l=0$, are quadratic expressions that are linear in 2-jets with coefficients being linear in 1-jets). Its prolongation $\mathcal{E}_3=\mathcal{E}_2^{(1)}\subset J^3$ is given by adding $20\cdot 6=120$ equations obtained by differentiating (10) (note that higher-order terms of these equations, $D_iE_l=0$, are linear in 3-jets with coefficients being linear in 1-jets).

These equations however are not in the Frobenius (closed) form, meaning that not all 3-jets, which are fibre variables of the bundle $\pi_{3,2}:J^3\to J^2$ of rank $2\cdot\binom{6+2}{3}=112$, can be expressed in terms of lower-order jets. In fact, the number of free 3-jets at this step is 17 (invariantly, this means that the symbol $g_3=\mathrm{Ker}(d\pi_{3,2}:T\mathcal{E}_3\to T\mathcal{E}_2)\subset S^3\mathbb{R}^{6*}\otimes\mathbb{R}^2$ has codimension 17), whence $120-(112-17)=25$ combinations of our equations have vanishing 3-symbols. These equations of order 2 define a proper locus $\tilde{\mathcal{E}}_2:=\pi_{3,2}(\mathcal{E}_3)\subset\mathcal{E}_2$ given by a quadratic ideal in 2-jet variables.

Proposition 5. *The system $\tilde{\mathcal{E}}_2=\tilde{\mathcal{E}}_2'\cup\tilde{\mathcal{E}}_2''$ is a reducible algebraic (sub-)variety in J^2 with an irreducible component $\tilde{\mathcal{E}}_2'$ of codimension 24 and an irreducible component $\tilde{\mathcal{E}}_2''$ of codimension 30. The intersection $\tilde{\mathcal{E}}_2'\cap\tilde{\mathcal{E}}_2''$ is an irreducible algebraic variety of codimension 34.*

Proof. This is obtained by prime ideal decomposition. Indeed, the substitution of a non-degenerate 1-jet $x_1=\{(f_a,h_b)\}$ into the equations (see Remark in Section 3.1) splits the system into 20 linear, and a bunch of quadratic equations in the variables f_{ab},h_{ab}, $1\le a\le b\le 6$. The quadratic ideal is then seen to be generated by products of linear expressions (from the set of 4 and 10 equations respectively), so that its locus in every $\pi_{2,1}^{-1}(x_1)$ is the union of two subspaces that are linear in 2-jet variables (but polynomial in $x_1 \in J^1$), and this implies the claim.

The second prolongation $\mathcal{E}_4 = \mathcal{E}_2^{(2)} = \mathcal{E}_3^{(1)} \subset J^4$ (obtained by adding equations $D_i D_j E_l = 0$ whose higher-order terms are linear in 4-jets with coefficients being linear in 1-jets) is already in the Frobenius form (all 4-jets are expressed in terms of lower-order jets).

Yet the system generated by $\tilde{\mathcal{E}}_2$ is not in involution: the prolongation $\tilde{\mathcal{E}}_2^{(1)} \subset J^3$ is not in closed form – the number of free 3-jets is 8. Even the system $\pi_{4,3}(\mathcal{E}_4) \subset J^3$ is not closed – the number of free 3-jets at this step is 3. We have to do one more prolongation: for the system $\mathcal{E}_5 = \mathcal{E}_2^{(3)} = \mathcal{E}_4^{(1)} \subset J^5$ (obtained by adding equations $D_i D_j D_k E_l = 0$) the projection $\tilde{\mathcal{E}}_3 = \pi_{5,3}(\mathcal{E}_5) \subset J^3$ is Frobenius (all 3-jets can be expressed, or equivalently the symbol $\tilde{g}_3 = 0$).

Consequently, we obtain a PDE system $\tilde{\mathcal{E}}$ given by the second-order equation-manifold $\tilde{\mathcal{E}}_2$, the third-order locus $\tilde{\mathcal{E}}_3$ (obtained by adding 112 third-order PDE), and its prolongations.

Proposition 6. *The system $\tilde{\mathcal{E}}$ is involutive.*

Proof: Due to Proposition 5 this system splits as $\tilde{\mathcal{E}} = \tilde{\mathcal{E}}' \cup \tilde{\mathcal{E}}''$ into the union of systems that are linear in jets of order > 1. The symbols $\tilde{g}_k = \mathrm{Ker}(d\pi_{k,k-1} : T\tilde{\mathcal{E}}_k \to T\tilde{\mathcal{E}}_{k-1})$ of the new systems satisfy: $\dim \tilde{g}_0 = 2$, $\dim \tilde{g}_1 = 2\cdot 6 = 12$, $\dim \tilde{g}_2' = 42 - 24 = 18$, $\dim \tilde{g}_2'' = 42 - 30 = 12$, $\dim \tilde{g}_k = 0$ for $k > 2$. Thus, the solution spaces of these equations have dimensions that are bounded by $\dim \tilde{g}_0 + \dim \tilde{g}_1 + \dim \tilde{g}_2' = 32$ and $\dim \tilde{g}_0 + \dim \tilde{g}_1 + \dim \tilde{g}_2'' = 26$, respectively.

To ensure involutivity we have to check that for every point $o \in X$ and every ∞-jet admissible by the system $\tilde{\mathcal{E}}_o = \tilde{\mathcal{E}} \cap \pi_{\infty,0}^{-1}(o)$ over it, there is a solution to (10) with this jet at o.

Let us start with the system $\tilde{\mathcal{E}}'$. We claim that all its solutions are given by the Chasles construction. The latter have normal forms specified in Tables 2-6. The most general solution has Segre type [0] and since its stabilizer in $\mathbf{SL}(6)$ is 3-dimensional, the space of solutions of the Chasles type has dimension $35 - 3 = 32$.

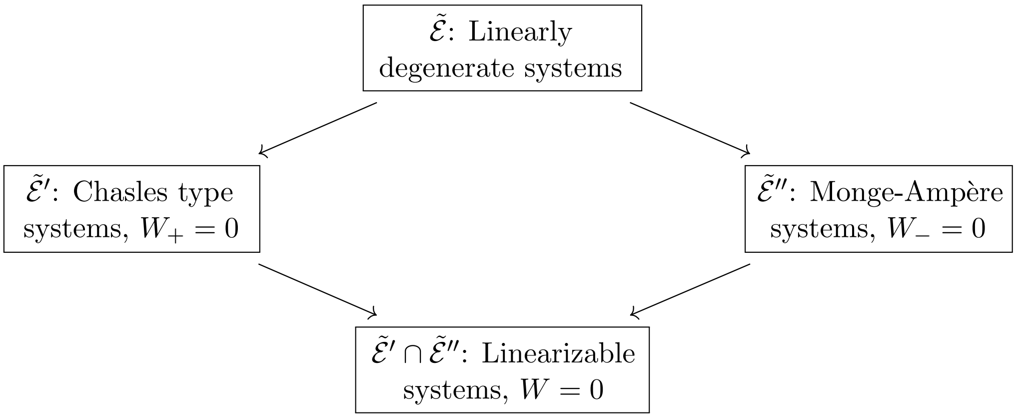
Another way to see this is as follows. The general solution of the Chasles type is given by the 2-planes $\langle A, B\rangle \in \mathbf{Gr}(2, U)$, where $U = \mathrm{Hom}(W, V) \simeq \mathbb{R}^6 \otimes \mathbb{R}^{7*}$ is the space of 6×7 matrices: $X_{A,B} = \{Ax \wedge Bx : x \in W\} \subset \mathbf{Gr}(2, V)$. Reparametrization $(A, B) \sim (PA, PB)$ yields the same solution for $P \in \mathbf{SL}(7)$ (more general equivalence $(A, B) \sim (PAQ, PBQ)$ yields equivalent manifolds $X_{A,B}$). Thus the space of solutions of the Chasles type has dimension $80 - 48 = 32$.

Moreover, the map $\langle A, B\rangle \mapsto j_o^\infty(X_{AB}) \in \tilde{\mathcal{E}}_o'$ from the projective variety $\mathbf{Gr}(2, U)$ to the irreducible variety $\tilde{\mathcal{E}}_o'$ has an open image (by what we have already computed) and therefore must be epimorphic. This proves the claim about $\tilde{\mathcal{E}}'$.

For the system $\tilde{\mathcal{E}}''$ we claim that all solutions are sixfolds X of the Monge-Ampère type. The normal forms are collected in Table 1 and the most general of those is the Husain equation. Since its stabilizer with respect to $\mathbf{SL}(6)$ is 9-dimensional, the space of 2-component Monge-Ampère systems has dimension $35 - 9 = 26$. We can show that all solutions of $\tilde{\mathcal{E}}''$ are Monge-Ampère by an approach similar to the case of $\tilde{\mathcal{E}}'$, but it is easier to conclude the claim by observing that 30 second-order equations specifying $\tilde{\mathcal{E}}_2''$ are exactly the PDEs from Proposition 3.

Finally, the intersection $\tilde{\mathcal{E}}' \cap \tilde{\mathcal{E}}''$ consists of linearizable systems. Indeed, the stabilizer of a linear system is a 13-dimensional subgroup of $\mathbf{SL}(6)$, so that the space of such systems has dimension $35 - 13 = 22$, which coincides with $\dim \tilde{g}_0 + \dim \tilde{g}_1 + \dim(\tilde{g}_2' \cap \tilde{g}_2'') = 2 + 12 + 8$.

We can summarize the prolongation-projection of the conditions of linear degeneracy in the following diagram.



Note that the two irreducible components can be characterised in terms of the Weyl tensor of the canonical conformal structure as self-dual and anti-self-dual systems (up to the change of orientation).

3.3 Proof of Theorem 1

Implication (a)$\Longrightarrow$(c). Our strategy is to derive a set of constraints for the right-hand sides f and h in (9) that are necessary and sufficient for integrability. As outlined in [7], in three dimensions this leads to an involutive system of third-order integrability conditions for f and h. The crucial difference occuring in the 4D case is the appearance, along with third-order constraints, of a whole set of second-order integrability conditions that turn out to be equivalent to relations (10) characterising linearly degenerate systems. This shows that the requirement of integrability in higher dimensions is far more rigid. Here are the details of calculations. Based on evolutionary representation (9) we introduce the notation

$$u_1 = a,\ u_2 = b,\ u_3 = c,\ v_1 = p,\ v_2 = q,\ v_3 = r,\ u_4 = f(a,b,c,p,q,r),\ v_4 = h(a,b,c,p,q,r).$$

This results in the equivalent quasilinear representation of type (5),

$$\begin{gathered} a_2 = b_1, \quad a_3 = c_1, \quad a_4 = f(a,b,c,p,q,r)_1, \\ b_3 = c_2, \quad b_4 = f(a,b,c,p,q,r)_2, \quad c_4 = f(a,b,c,p,q,r)_3, \\ p_2 = q_1, \quad p_3 = r_1, \quad p_4 = h(a,b,c,p,q,r)_1, \\ q_3 = r_2, \quad q_4 = h(a,b,c,p,q,r)_2, \quad r_4 = h(a,b,c,p,q,r)_3. \end{gathered} \tag{16}$$

Following the method of hydrodynamic reductions let us look for multi-phase solutions where a, b, c, p, q, r are sought as functions of N phases $R^1, ..., R^N$ that are required to satisfy a triple of consistent $(1+1)$-dimensional systems (6),

$$R^i_{x^2} = \mu^i(R) R^i_{x^1}, \quad R^i_{x^3} = \eta^i(R) R^i_{x^1}, \quad R^i_{x^4} = \lambda^i(R) R^i_{x^1}.$$

Here the characteristic speeds μ^i, η^i and λ^i satisfy the commutativity conditions (7),

$$\frac{\partial_j \lambda^i}{\lambda^j - \lambda^i} = \frac{\partial_j \mu^i}{\mu^j - \mu^i} = \frac{\partial_j \eta^i}{\eta^j - \eta^i}, \tag{17}$$

$i \neq j$, $\partial_j = \partial_{R^j}$. The substitution into (16) implies the relations

$$\partial_i b = \mu^i \partial_i a, \quad \partial_i c = \eta^i \partial_i a, \quad \partial_i q = \mu^i \partial_i p, \quad \partial_i r = \eta^i \partial_i p, \tag{18}$$

as well as

$$\begin{aligned} (\lambda^i - f_a - \mu^i f_b - \eta^i f_c)\partial_i a &= (f_p + \mu^i f_q + \eta^i f_r)\partial_i p, \\ (\lambda^i - h_p - \mu^i h_q - \eta^i h_r)\partial_i p &= (h_a + \mu^i h_b + \eta^i h_c)\partial_i a. \end{aligned} \tag{19}$$

The last two equations imply the dispersion relation connecting λ^i, μ^i and η^i,

$$(\lambda^i - f_a - \mu^i f_b - \eta^i f_c)(\lambda^i - h_p - \mu^i h_q - \eta^i h_r) = (f_p + \mu^i f_q + \eta^i f_r)(h_a + \mu^i h_b + \eta^i h_c).$$

In what follows we assume that the dispersion relation defines a non-degenerate quadric in the (λ, μ, η)-space: this is equivalent to the requirement of non-degeneracy from Section 1.2. Setting in (19) $\partial_i a = \varphi^i \partial_i p$ we can parametrise μ^i and λ^i in the form

$$\mu^i = -\frac{f_p + (f_a - h_p)\varphi^i - h_a {\varphi^i}^2 + \eta^i (f_r + (f_c - h_r)\varphi^i - h_c {\varphi^i}^2)}{f_q + (f_b - h_q)\varphi^i - h_b {\varphi^i}^2},$$

$$\lambda^i = \frac{(f_q + f_b\varphi^i)(h_p + h_a\varphi^i) - (f_p + f_a\varphi^i)(h_q + h_b\varphi^i) + \eta^i[(f_q + f_b\varphi^i)(h_r + h_c\varphi^i) - (f_r + f_c\varphi^i)(h_q + h_b\varphi^i)]}{f_q + (f_b - h_q)\varphi^i - h_b {\varphi^i}^2}.$$

Substituting these expressions into commutativity conditions (17), and using the relations

$$\partial_i a = \varphi^i \partial_i p, \quad \partial_i b = \mu^i \varphi^i \partial_i p, \quad \partial_i c = \eta^i \varphi^i \partial_i p, \quad \partial_i q = \mu^i \partial_i p, \quad \partial_i r = \eta^i \partial_i p, \tag{20}$$

we obtain $\partial_j \varphi^i$ and $\partial_j \eta^i$ in the form $\partial_j \varphi^i = (\dots)\partial_j p$, $\partial_j \eta^i = (\dots)\partial_j p$, $i \neq j$, where dots denote rational expressions in φ^i, φ^j, η^i, η^j whose coefficients depend on second-order partial derivatives of f and h. Calculating consistency conditions for relations (20) we obtain (one and the same!) expression for $\partial_i \partial_j p$ in the form $\partial_i \partial_j p = (\dots)\partial_i p \partial_j p$, $i \neq j$, where, again, dots denote terms rational in φ^i, φ^j, η^i, η^j. Ultimately, N-phase solutions are governed by the relations

$$\partial_j \varphi^i = (\dots)\partial_j p, \quad \partial_j \eta^i = (\dots)\partial_j p, \quad \partial_i \partial_j p = (\dots)\partial_i p \partial_j p, \tag{21}$$

$i \neq j$. Direct calculation of the compatibility conditions based on (20) and (21) results in

$$\partial_k \partial_j \varphi^i - \partial_j \partial_k \varphi^i = (\dots)\partial_j p \partial_k p, \quad \partial_k \partial_j \eta^i - \partial_j \partial_k \eta^i = (\dots)\partial_j p \partial_k p,$$

$$\partial_k \partial_j \partial_i p - \partial_j \partial_k \partial_i p = (\dots)\partial_i p \partial_j p \partial_k p,$$

where dots denote complicated rational expressions in $\varphi^i, \varphi^j, \varphi^k$ and η^i, η^j, η^k, whose coefficients depend on partial derivatives of f and h up to the third order. To ensure the solvability of equations (21) we set all these coefficients equal to zero. Without any loss of generality we can set $(i, j, k) = (1, 2, 3)$. In particular, the coefficient in the numerator of $\partial_3 \partial_2 \partial_1 p - \partial_2 \partial_3 \partial_1 p$ at the monomial $(\varphi^1)^{12}(\varphi^2)^9(\varphi^3)^6(\eta^1)^6(\eta^2)(\eta^3)^3$ has the form τL^2 where τ is a nonzero expression depending on first-order derivatives of f and h only, and

$$\begin{aligned} L = &(f_r h_b - f_q h_c)(f_r^2 f_{qq} - 2 f_q f_r f_{qr} + f_q^2 f_{rr}) \\ &+ (f_c f_q - f_b f_r + f_r h_q - f_q h_r)(f_r^2 h_{qq} - 2 f_q f_r h_{qr} + f_q^2 h_{rr}). \end{aligned}$$

The condition $L = 0$ is linear in the second-order derivatives of f and h. Let us now utilise the fact that conditions of integrability must be invariant under the action of the equivalence group. Acting on the condition $L = 0$ by transformations from the equivalence group $\mathbf{SL}(6)$ we obtain all of the 20 second-order conditions of linear degeneracy (10).

Implications (b)$\Longrightarrow$(c). Let $[g]$ be the conformal structure defined by the characteristic variety of system (1). We shall demonstrate that, with a proper choice of orientation, the condition of conformal half-flatness implies linear degeneracy. Let us note that in the splitting of the Weyl tensor, $W = W_+ + W_-$, we use the Hodge star operator which depends on the square root of

$$\det g = \Big(\tfrac{1}{4}(f_a f_q h_c - f_a f_r h_b - f_b f_p h_c + f_b f_r h_a + f_c f_p h_b \\ - f_c f_q h_a - f_p h_b h_r + f_p h_c h_q + f_q h_a h_r - f_q h_c h_p - f_r h_a h_q + f_r h_b h_p)\Big)^2,$$

in the notation of (16). Choosing $\sqrt{\det g}$ to be the expression in big parentheses, we define the tensor $*W$ and observe the following. The condition of self-duality, $W_- = 0$, consists of 30 equations that are equivalent to those of Proposition 3, and characterise PDEs of Monge-Ampère type given by the system $\tilde{\mathcal{E}}''$ from Section 3.2. The condition of anti-self-duality, $W_+ = 0$, consists of 24 equations that characterise general linearly degenerate PDEs associated with quadratic maps $\mathbb{P}^6 \dashrightarrow \mathbf{Gr}(4, 6)$. These are given by the system $\tilde{\mathcal{E}}'$ from Section 3.2.

Implication (c)$\Longrightarrow$(a). Since all linearly degenerate equations are classified in Tables 1-6, it is straightforward to check that each of them passes the test for hydrodynamic integrability.

Implication (c)$\Longrightarrow$(b). Again, linearly degenerate equations have normal forms represented in Tables 1-6. It can be straightforwardly verified that conformal structures corresponding to them are half-flat ($*W = \pm W$) on every solution.

More conceptually, the result can be seen as follows. Every equation in Tables 1-6 has a Lax pair with a spectral parameter, and according to [3] this implies self-duality (with a proper choice of orientation). Here is a brief explanation. This Lax pair is a 2-distribution on the correspondence space $(x^1, \dots, x^4, \lambda)$. The integral surfaces of this distribution projected to the $\mathbf{x}$-space form a 3-parametric family of null totally geodesic surfaces with respect to the conformal structure on every solution $u = u(\mathbf{x}), v = v(\mathbf{x})$. According to Penrose [23], the existence of such surfaces (known as α-surfaces) is equivalent to self-duality.

Implications (d)$\Longleftrightarrow$(c). This is a direct corollary of Section 3.2.

This finishes the proof of Theorem 1.

Remark. Geometrically, Theorem 1 can be interpreted as follows. Let X be a sixfold in $\mathbf{Gr}(4, 6)$. Taking a point $o \in X$ and projectivising the intersection of the tangent space $\mathrm{T}_o X$ with the Serge cone C in $\mathrm{T}_o\mathbf{Gr}(4, 6)$, which is the cone over a non-singular rational fourfold of degree four in $\mathbb{P}^7 = \mathbb{P}\mathrm{T}_o\mathbf{Gr}(4, 6)$, one obtains a rational surface of degree four. This surface, known as a rational normal scroll, can be interpreted as the set of matrices of rank one in the tangent space $\mathrm{T}_o X$ (recall that $\mathrm{T}_o\mathbf{Gr}(4, 6)$ is identified with the space of 2×4 matrices: here we utilise the duality between $\mathbf{Gr}(4, 6)$ and $\mathbf{Gr}(2, 6)$). Thus, the projectivised tangent bundle of X is equipped with a field of rational normal scrolls of degree four. The integrability conditions can be reformulated as the requirement of the existence in X of infinitely many holonomic trisecant threefolds whose projectivised tangent spaces intersect the rational normal scroll at three distinct points. These threefolds correspond to three-component hydrodynamic reductions (we refer to [7, 26] for a related discussion). Theorem 1 states that this requirement forces X to be algebraic, more precisely, X must be either a codimension 2 linear section of the Plücker embedding $\mathbf{Gr}(4, 6) \hookrightarrow \mathbb{P}^{14}$, or the image of a quadratic map $\mathbb{P}^6 \dashrightarrow \mathbf{Gr}(4, 6)$. It would be interesting to have a purely geometric proof of this result.

4 Concluding remarks

We have obtained a complete description of integrable systems associated with sixfolds in $\mathbf{Gr}(4,6)$. The corresponding sixfolds are either codimension two linear sections of the Plücker embedding $\mathbf{Gr}(4,6) \hookrightarrow \mathbb{P}^{14}$ or images of quadratic maps $\mathbb{P}^6 \dashrightarrow \mathbf{Gr}(4,6)$. Conversely, every sixfold of one of the above types gives rise to an integrable system.

Let us compare the case of PDE systems in 3D associated to fourfolds in $\mathbf{Gr}(3,5)$ studied in [7] to that of PDE systems in 4D studied in this paper. While the main theorems expressing integrability of the systems $\mathcal{E} = \{F = 0, H = 0\}$ via the geometric properties of solutions (Einstein-Weyl property in 3D and self-duality in 4D) are similar in spirit, there are several important differences.

Integrable systems in 3D	**Integrable systems in 4D**
The parameter space $\mathcal{M} = \mathcal{M}^{30}$ is irreducible	The parameter space $\mathcal{M} = \mathcal{M}_1^{26} \cup \mathcal{M}_2^{32}$ is reducible
The generic equation $\mathcal{E} \in \mathcal{M}$ is transcendental	The generic equation $\mathcal{E} \in \mathcal{M}$ is algebraic
The moduli space $\mathcal{M}/G$ is a rational algebraic variety of positive dimension $(= 6)$ Parametrizable by special functions	The moduli space $\mathcal{M}/G$ is 0-dimensional Each component of $\mathcal{M}$ has a unique Zariski open orbit Finite classification of integrable cases
There exists ∂_λ term in the Lax pair (in general)	There is no ∂_λ term in the Lax pair (hyper-complex case)

Finally, it would also be interesting to investigate the integrability problem for Grassmannians of higher dimensions. For instance, let u, v, w be functions of the independent variables $x^1, \dots, x^4$. A first-order 3-component system,

$$F_i(u_1, \dots, u_4,\ v_1, \dots, v_4,\ w_1, \dots, w_4) = 0, \quad i = 1, 2, 3,$$

is naturally associated with a codimension 3 submanifold $X \subset \mathbf{Gr}(4,7)$. We conjecture that the requirement of integrability forces X to be either a codimension 3 linear section of $\mathbf{Gr}(4,7)$ (the case of Monge-Ampère systems), or the image of a cubic map $\mathbb{P}^9 \dashrightarrow \mathbf{Gr}(4,7)$ (general linearly degenerate systems). The results of [6] suggest that even under these restrictions the corresponding systems will not automatically be integrable, and additional geometric restrictions on X will be required.

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