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HARDY-TYPE INEQUALITIES OVER BALLS IN $\mathbb{R}^N$ FOR SOME BILINEAR AND ITERATED OPERATORS

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Abstract. Some new multidimensional Hardy-type inequalites are proved and discussed. The cases with bilinear and iterated operators are considered and some equivalence theorems are proved.

1. Introduction

The one-dimensional weighted Hardy inequality

$$\left(\int_0^\infty \Big(HF(x)\Big)^q W(x)\,dx\right)^{\frac{1}{q}} \le C\left(\int_0^\infty F^p(x)\,V(x)\,dx\right)^{\frac{1}{p}}, \quad F \ge 0 \tag{1.1}$$

where $HF(x) := \displaystyle\int_0^x F(t)\,dt$ is the Hardy operator, is characterized for various choices of indices p and q. A fairly complete description both of the prehistory (until Hardy [4] proved the first result in 1925), the fascinating continued development and current status can be found in the books [9], [11], [12], [14] and the references therein.

In this paper, we shall continue to study a variant of Hardy-type inequalities, which was not discussed in the books above and we do so even in a multidimensioanl setting. First we mention that Cañestro *et al.* [1] considered the weighted bilinear Hardy operator

$$H_2(F,G)(x) := HF(x)\cdot HG(x) \tag{1.2}$$

and characterized the corresponding inequality

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$$\left(\int_0^\infty \Big(H_2(F,G)(x)\Big)^q W(x)\,dx\right)^{\frac{1}{q}} \le C\left(\int_0^\infty F^{p_1}(x)\,V_1(x)\,dx\right)^{\frac{1}{p_1}} \times \left(\int_0^\infty G^{p_2}(x)\,V_2(x)\,dx\right)^{\frac{1}{p_2}}, \quad F,\,G \ge 0 \tag{1.3}$$

for various combinations of the indices p_1, p_2, q. Recently, a simpler proof was given by Krepela [10] who made use of the information about one-dimensional inequality (1.1) iteratively.

The N-dimensional analogue over balls of the operator (1.2) is given by

$$H_2^N(f,g)(x) := H^N f(x)\cdot H^N g(x) = \int_{B(0,|x|)} f(t)\,dt \int_{B(0,|x|)} g(t)\,dt.$$

Very recently in [2], the authors studied the N-dimensional version of the inequality (1.3), i.e.,

$$\left(\int_{\mathbb{R}^N} \Big[H_2^N(f,g)(x)\Big]^q w(x)\,dx\right)^{\frac{1}{q}} \le C\left(\int_{\mathbb{R}^N} f^{p_1}(x)v_1(x)\,dx\right)^{\frac{1}{p_1}} \times \left(\int_{\mathbb{R}^N} g^{p_2}(x)v_2(x)\,dx\right)^{\frac{1}{p_2}} \tag{1.4}$$

and obtained its weight characterization for several choices of indices p_1, p_2 and q. The authors followed the strategy of Krepela [10] by using iteratively the information about the inequality

$$\left(\int_{\mathbb{R}^N} \Big[H^N f(x)\Big]^q w(x)\,dx\right)^{\frac{1}{q}} \le C\left(\int_{\mathbb{R}^N} f^{p}(x)v(x)\,dx\right)^{\frac{1}{p}}, \tag{1.5}$$

which is already well known in the literature, see, e.g., [3] and [16]. In this strategy, depending upon the relationship among the indices p_1, p_2 and q, different proofs are required.

One of the main aims of the present paper is to reinvestigate (1.4) in a more direct way. For a complete description of standard Hardy-type inequalities in this case, see Chapter 3 of the recent book [12] and the references therein. In particular, in Section 2, we show that the N-dimensional inequality (1.4) is equivalent to the one-dimensional inequality (1.3) regardless of the relationship among the indices p_1, p_2, q (see Theorem 2.1). Moreover, in Section 3, we then use the weight characterization of (1.3) and obtain the corresponding characterization of (1.4). We also remark that a similar equivalence between (1.1) and (1.5) was proved in [16].

We will point out that the equivalence of (1.3) and (1.4) also holds if the Hardy operators H_2 and H_2^n are replaced by the corresponding Hardy-Steklov operators. We recall that the standard one-dimensional Hardy-Steklov operator is given by

$$SF(x) := \int_{a(x)}^{b(x)} F(t)\,dt,$$

where a and b are strictly increasing differentiable functions on $[0,\infty]$ satisfying $a(0) = b(0) = 0$; $a(x) < b(x)$ for $0 < x < \infty$ and $a(\infty) = b(\infty)$. The $L^p - L^q$ boundedness of S has been proved in [5] while the corresponding compactness was proved in [6]. Our corresponding main results are presented as Theorem 2.2 and Theorem 3.2.

Moreover, in this paper, certain N-dimensional iterated Hardy type operators are studied and one of them T^N is defined as follows:

$$T^N f(x) := \left(\int_{\mathbb{R}^N \setminus B(0,|x|)} \left(\int_{B(0,|y|)} f(z)\, dz \right)^q w(y)\, dy \right)^{\frac{1}{q}}. \tag{1.6}$$

We show that the inequality

$$\left(\int_{\mathbb{R}^N} \left(T^N f(x) \right)^r u(x)\, dx \right)^{\frac{1}{r}} \leq C \left(\int_{\mathbb{R}^N} f^p(x)\, v(x)\, dx \right)^{\frac{1}{p}} \tag{1.7}$$

can be proved for any $N \in \mathbb{Z}_+$ by just proving the corresponding one-dimensional result for $T \equiv T^1$. More exactly, we prove that the inequalities (1.7) and

$$\left(\int_0^\infty \left(TF(x) \right)^r U(x)\, dx \right)^{\frac{1}{r}} \leq C \left(\int_0^\infty F^p(x)\, V(x)\, dx \right)^{\frac{1}{p}}, \tag{1.8}$$

where

$$TF(x) := \left(\int_x^\infty \left(\int_0^y F(z)\, dz \right)^q W(y)\, dy \right)^{\frac{1}{q}}.$$

are equivalent. We remark that the inequality (1.8) has been investigated in [15]. Moreover, in Section 4, we state this equivalence result not only for the operator T^N but also for three other iterated operators (see Theorem 4.1).

In order to avoid confusion and ambiguity, let us agree on some notations. All the functions in this paper are measurable and non-negative. The symbols F and G are used for one-dimensional functions while f and g are used for functions defined on $\mathbb{R}^N$. One-dimensional weights are denoted by the symbols W, U, V, V_1 and V_2 and the corresponding weights in $\mathbb{R}^N$ are denoted by w, u, v, v_1 and v_2, respectively. We do not use separate symbols for arguments of one-dimensional functions and higher dimensional functions since it will be clear from the context, e.g., in $F(x)$, $x \in (0, \infty)$ and in $f(x)$, $x \in \mathbb{R}^N$.

2. Equivalence theorems concerning Hardy-type inequalities for bilinear operators

A crucial point in the proofs in this paper is to use polar coordinates, i.e., for $x \in \mathbb{R}^N$, we write $x = t\tau$, where $t \in (0, \infty)$ and $\tau \in \Sigma_N$, the surface of the unit ball in $\mathbb{R}^N$.

The first main result of this section is the following:

Theorem 2.1. *Let $0 < q < \infty$, $1 < p_1, p_2 < \infty$ and w, v_1, v_2 are weight functions defined on $\mathbb{R}^N$. The inequality (1.4) holds for all $f, g \geq 0$ if and only if the inequality (1.3) holds for all $F, G \geq 0$ with*

$$W(t) := \int_{\Sigma_N} w(t\tau)\, t^{N-1}\, d\tau, \tag{2.1}$$

$$V_i(t) := \left(\int_{\Sigma_N} v_i^{1-p_i'}(t\tau)\, t^{N-1}\, d\tau \right)^{1-p_i}, \ i = 1, 2, \quad t > 0, \ \tau \in \Sigma_N. \tag{2.2}$$

Moreover, the constant C in (1.3) and (1.4) is the same.

Proof. Let us first assume that the inequality (1.3) holds. For fixed f and g, we define

$$F(t) := \int_{\Sigma_N} f(t\tau)\, t^{N-1}\, d\tau,$$
$$G(t) := \int_{\Sigma_N} g(t\tau)\, t^{N-1}\, d\tau.$$

By using Hölder's inequality, we get that

$$\begin{aligned} F(t) &= \left(\int_{\Sigma_N} f(t\tau)\, v_1^{\frac{1}{p_1}+\frac{1-p_1'}{p_1'}}(t\tau)\, t^{N-1}\, d\tau\right) \\ &\le \left(\int_{\Sigma_N} f^{p_1}(t\tau)\, v_1(t\tau)\, t^{N-1}\, d\tau\right)^{\frac{1}{p_1}} \left(\int_{\Sigma_N} v_1^{1-p_1'}(t\tau)\, t^{N-1}\, d\tau\right)^{\frac{1}{p_1'}} \\ &= \left(\int_{\Sigma_N} f^{p_1}(t\tau)\, v_1(t\tau)\, t^{N-1}\, d\tau\right)^{\frac{1}{p_1}} \Big(V_1(t)\Big)^{\frac{1}{p_1'(1-p_1)}} \\ &= \left(\int_{\Sigma_N} f^{p_1}(t\tau)\, v_1(t\tau)\, t^{N-1}\, d\tau\right)^{\frac{1}{p_1}} \Big(V_1(t)\Big)^{-\frac{1}{p_1}}. \end{aligned} \tag{2.3}$$

Similarly,

$$G(t) \le \left(\int_{\Sigma_N} g^{p_2}(t\tau)\, v_2(t\tau)\, t^{N-1}\, d\tau\right)^{\frac{1}{p_2}} \Big(V_2(t)\Big)^{-\frac{1}{p_2}}. \tag{2.4}$$

By changing to polar coordinates $x = s\tau$, $y = s_1\sigma$, $z = s_2\gamma$, $s, s_1, s_2 > 0$, $\tau, \sigma, \gamma \in \Sigma_N$ and using the inequalities (1.3), (2.3) and (2.4), we obtain that

$$\begin{aligned} &\left(\int_{\mathbb{R}^N} \Big[H_2^N(f,g)(x)\Big]^q w(x)\, dx\right)^{\frac{1}{q}} \\ &= \left(\int_{\mathbb{R}^N} \left(\int_{B(0,|x|)} f(y)\, dy\right)^q \left(\int_{B(0,|x|)} g(z)\, dz\right)^q w(x)\, dx\right)^{\frac{1}{q}} \\ &= \Bigg\{\int_0^\infty \int_{\Sigma_N} \left(\int_0^s \int_{\Sigma_N} f(s_1\sigma)\, s_1^{N-1}\, d\sigma\, ds_1\right)^q \\ &\qquad \times \left(\int_0^s \int_{\Sigma_N} g(s_2\gamma)\, s_2^{N-1}\, d\gamma\, ds_2\right)^q w(s\tau)\, s^{N-1}\, d\tau\, ds\Bigg\}^{\frac{1}{q}} \\ &= \left(\int_0^\infty \left(\int_0^s F(s_1)\, ds_1\right)^q \left(\int_0^s G(s_2)\, ds_2\right)^q W(s)\, ds\right)^{\frac{1}{q}} \\ &= \left(\int_0^\infty \Big[H_2(F,G)(s)\Big]^q W(s)\, ds\right)^{\frac{1}{q}} \\ &\le C\left(\int_0^\infty F^{p_1}(s)V_1(s)\, ds\right)^{\frac{1}{p_1}} \left(\int_0^\infty G^{p_2}(s)V_2(s)\, ds\right)^{\frac{1}{p_2}} \\ &\le C\left(\int_0^\infty \int_{\Sigma_N} f^{p_1}(s\tau)\, v_1(s\tau)\, s^{N-1}\, d\tau\, ds\right)^{\frac{1}{p_1}} \left(\int_0^\infty \int_{\Sigma_N} g^{p_2}(s\tau)\, v_2(s\tau)\, s^{N-1}\, d\tau\, ds\right)^{\frac{1}{p_2}} \end{aligned}$$

$$= C\left(\int_{\mathbb{R}^N} f^{p_1}(x)v_1(x)\,dx\right)^{\frac{1}{p_1}}\left(\int_{\mathbb{R}^N} g^{p_2}(x)v_2(x)\,dx\right)^{\frac{1}{p_2}},$$

which means that (1.4) holds.

Conversely, assume that the inequality (1.4) holds. For fixed F and G, we set

$$f(t\sigma) := F(t)v_1^{1-p_1'}(t\sigma)\Big(V_1(t)\Big)^{\frac{1}{p_1-1}},$$
$$g(t\gamma) := G(t)v_2^{1-p_2'}(t\gamma)\Big(V_2(t)\Big)^{\frac{1}{p_2-1}},$$

where $t>0$, $\sigma,\gamma\in\Sigma_N$. This gives that

$$F(t) = \int_{\Sigma_N} f(t\sigma)\,t^{N-1}\,d\sigma,$$
$$G(t) = \int_{\Sigma_N} g(t\gamma)\,t^{N-1}\,d\gamma.$$

Therefore, by using the inequality (1.4), we get

$$\begin{aligned}
&\left(\int_0^\infty \Big[H_2(F,G)(s)\Big]^q W(s)\,ds\right)^{\frac{1}{q}}\\
&= \left(\int_0^\infty \left(\int_0^s F(s_1)\,ds_1\right)^q \left(\int_0^s G(s_2)\,ds_2\right)^q W(s)\,ds\right)^{\frac{1}{q}}\\
&= \Bigg\{\int_0^\infty \left(\int_0^s \int_{\Sigma_N} f(s_1\sigma)\,s_1^{N-1}\,d\sigma\,ds_1\right)^q\\
&\qquad \times \left(\int_0^s \int_{\Sigma_N} g(s_2\gamma)\,s_2^{N-1}\,d\gamma\,ds_2\right)^q W(s)\,ds\Bigg\}^{\frac{1}{q}}\\
&= \Bigg\{\int_0^\infty \left(\int_0^s \int_{\Sigma_N} f(s_1\sigma)\,s_1^{N-1}\,d\sigma\,ds_1\right)^q\\
&\qquad \times \left(\int_0^s \int_{\Sigma_N} g(s_2\gamma)\,s_2^{N-1}\,d\gamma\,ds_2\right)^q \int_{\Sigma_N} w(s\tau)\,s^{N-1}\,d\tau\,ds\Bigg\}^{\frac{1}{q}}\\
&= \left(\int_{\mathbb{R}^N} \left(\int_{B(0,|x|)} f(y)\,dy\right)^q \left(\int_{B(0,|x|)} g(z)\,dz\right)^q w(x)\,dx\right)^{\frac{1}{q}}\\
&= \left(\int_{\mathbb{R}^N} \Big(H_2^N(f,g)(x)\Big)^q w(x)\,dx\right)^{\frac{1}{q}}\\
&\le C\left(\int_{\mathbb{R}^N} f^{p_1}(x)\,v_1(x)\,dx\right)^{\frac{1}{p_1}}\left(\int_{\mathbb{R}^N} g^{p_2}(x)\,v_2(x)\,dx\right)^{\frac{1}{p_2}}\\
&= C\left(\int_0^\infty \int_{\Sigma_N} f^{p_1}(s\tau)\,v_1(s\tau)\,s^{N-1}\,d\tau\,ds\right)^{\frac{1}{p_1}}\\
&\qquad \times \left(\int_0^\infty \int_{\Sigma_N} g^{p_2}(s\tau)\,v_2(s\tau)\,s^{N-1}\,d\tau\,ds\right)^{\frac{1}{p_2}}\\
&= C\left(\int_0^\infty \int_{\Sigma_N} v_1(s\tau)\,s^{N-1}\left[F^{p_1}(s)v_1^{p_1(1-p_1')}(s\tau)\Big(V_1(s)\Big)^{\frac{p_1}{p_1-1}}\right]d\tau\,ds\right)^{\frac{1}{p_1}}
\end{aligned}$$

$$
\begin{aligned}
&\times \left(\int_0^\infty \int_{\Sigma_N} v_2(s\tau)\, s^{N-1} \left[G^{p_2}(s) v_2^{p_2(1-p_2')}(s\tau)\Big(V_2(s)\Big)^{\frac{p_2}{p_2-1}}\right] d\tau\, ds\right)^{\frac{1}{p_2}} \\
&= C\left(\int_0^\infty F^{p_1}(s)\left(\int_{\Sigma_N} v_1^{1-p_1'}(s\tau)\, s^{N-1}\, d\tau\right)\Big(V_1(s)\Big)^{\frac{p_1}{p_1-1}}\, ds\right)^{\frac{1}{p_1}} \\
&\times \left(\int_0^\infty G^{p_2}(s)\left(\int_{\Sigma_N} v_2^{1-p_2'}(s\tau)\, s^{N-1}\, d\tau\right)\Big(V_2(s)\Big)^{\frac{p_2}{p_2-1}}\, ds\right)^{\frac{1}{p_2}} \\
&= C\left(\int_0^\infty F^{p_1}(s)\Big(V_1(s)\Big)^{\frac{1}{1-p_1}}\Big(V_1(s)\Big)^{\frac{p_1}{p_1-1}}\, ds\right)^{\frac{1}{p_1}} \\
&\times \left(\int_0^\infty G^{p_2}(s)\Big(V_2(s)\Big)^{\frac{1}{1-p_2}}\Big(V_2(s)\Big)^{\frac{p_2}{p_2-1}}\, ds\right)^{\frac{1}{p_2}} \\
&= C\left(\int_0^\infty F^{p_1}(s)\, V_1(s)\, ds\right)^{\frac{1}{p_1}}\left(\int_0^\infty G^{p_2}(s)\, V_2(s)\, ds\right)^{\frac{1}{p_2}},
\end{aligned}
$$

which means that (1.3) holds and so the proof is complete. □

Next, we consider the bilinear Hardy-Steklov operator

$$
S_2(F,G)(x) := \int_{a_1(x)}^{b_1(x)} F(t)\, dt \int_{a_2(x)}^{b_2(x)} G(t)\, dt, \tag{2.5}
$$

where a_i and b_i are the functions as the functions a and b for the operator S defined in Section 1. For the operator S_2, the inequality

$$
\|S_2(F,G)\|_{L^q_W} \le C\, \|F\|_{L^{p_1}_{V_1}}\, \|G\|_{L^{p_2}_{V_2}} \tag{2.6}
$$

has been characterized for various choices of the indices p_1, p_2, q in [7], [8]. Here, we consider the N-dimensional analogue over the balls of the operator (2.5) given by

$$
S_2^N(f,g)(x) := \int_{a_1(|x|)<|y|<b_1(|x|)} f(y)\, dy \int_{a_2(|x|)<|z|<b_2(|x|)} g(z)\, dz, \quad x,y,z \in \mathbb{R}^N
$$

and thereby consider the inequality

$$
\begin{aligned}
\left(\int_{\mathbb{R}^N}\Big(S_2^N(f,g)(x)\Big)^q w(x)\, dx\right)^{\frac{1}{q}} &\le C\left(\int_{\mathbb{R}^N} f^{p_1}(x) v_1(x)\, dx\right)^{\frac{1}{p_1}} \\
&\times \left(\int_{\mathbb{R}^N} g^{p_2}(x) v_2(x)\, dx\right)^{\frac{1}{p_2}}.
\end{aligned} \tag{2.7}
$$

Our equivalence result for this case reads:

Theorem 2.2. *Let $0 < q < \infty$, $1 < p_1, p_2 < \infty$ and w, v_1, v_2 are weight functions defined on $\mathbb{R}^N$. The inequality (2.7) holds for all $f, g \ge 0$ if and only if the inequality (2.6) holds for all $F, G \ge 0$ with W, V_1, V_2 as given by (2.1) and (2.2), respectively. Also the constant C in (2.6) and (2.7) is the same.*

Proof. The proof is completely similar to that of Theorem 2.1. Hence, we leave out the details. □

3. Weight characterizations of some multidimensional Hardy-type inequalities

In this section, we give the precise weight characterizations of the inequalities (1.4) and (2.7) for a great variety of parameters q, p_1 and p_2. Let us recall the following result proved in [1], [10]:

Theorem A. *Let $0 < q < \infty$, $1 < p_1, p_2 < \infty$. The inequality (1.3) holds for all $F, G \geq 0$ if and only if*

(i) for $1 < \max(p_1, p_2) \leq q < \infty$,

$$B_1 := \sup_{0<x<\infty} \left(\int_x^\infty W(y)\, dy \right)^{\frac{1}{q}} \left(\int_0^x V_1^{1-p_1'}(y)\, dy \right)^{\frac{1}{p_1'}} \left(\int_0^x V_2^{1-p_2'}(y)\, dy \right)^{\frac{1}{p_2'}} < \infty,$$

(ii) for $1 < p_1 \leq q < p_2 < \infty$, $\frac{1}{r_2} = \frac{1}{q} - \frac{1}{p_2}$,

$$B_2 := \sup_{0<x<\infty} \left(\int_0^x V_1^{1-p_1'}(y)\, dy \right)^{\frac{1}{p_1'}} \left(\int_x^\infty \left(\int_y^\infty W(z)\, dz \right)^{\frac{r_2}{p_2}} \right.$$
$$\left. \times \left(\int_0^y V_2^{1-p_2'}(z)\, dz \right)^{\frac{r_2}{p_2'}} W(y)\, dy \right)^{\frac{1}{r_2}} < \infty,$$

(iii) for $1 < p_2 \leq q < p_1 < \infty$, $\frac{1}{r_1} = \frac{1}{q} - \frac{1}{p_1}$,

$$B_3 := \sup_{0<x<\infty} \left(\int_0^x V_2^{1-p_2'}(y)\, dy \right)^{\frac{1}{p_2'}} \left(\int_x^\infty \left(\int_y^\infty W(z)\, dz \right)^{\frac{r_1}{p_1}} \right.$$
$$\left. \times \left(\int_0^y V_1^{1-p_1'}(z)\, dz \right)^{\frac{r_1}{p_1'}} W(y)\, dy \right)^{\frac{1}{r_1}} < \infty,$$

(iv) for $0 < q < \min(p_1, p_2) < \infty$, $\min(p_1, p_2) > 1$, $\frac{1}{q} \leq \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{r_i} = \frac{1}{q} - \frac{1}{p_i}$, $i = 1, 2$,

$$B_4 := \sup_{0<x<\infty} \left(\int_0^x V_1^{1-p_1'}(y)\, dy \right)^{\frac{1}{p_1'}} \left(\int_x^\infty \left(\int_y^\infty W(z)\, dz \right)^{\frac{r_2}{q}} \right.$$
$$\left. \times \left(\int_0^y V_2^{1-p_2'}(z)\, dz \right)^{\frac{r_2}{q'}} V_2^{1-p_2'}(y)\, dy \right)^{\frac{1}{r_2}} < \infty,$$

and

$$B_5 := \sup_{0<x<\infty} \left(\int_0^x V_2^{1-p_2'}(y)\, dy \right)^{\frac{1}{p_2'}} \left(\int_x^\infty \left(\int_y^\infty W(z)\, dz \right)^{\frac{r_1}{q}} \right.$$
$$\left. \times \left(\int_0^y V_1^{1-p_1'}(z)\, dz \right)^{\frac{r_1}{q'}} V_1^{1-p_1'}(y)\, dy \right)^{\frac{1}{r_1}} < \infty,$$

(v) for $0<q<\min(p_1,p_2)<\infty$, $\min(p_1,p_2)>1$, $\frac{1}{q}>\frac{1}{p_1}+\frac{1}{p_2}$, $\frac{1}{k}=\frac{1}{q}-\frac{1}{p_1}-\frac{1}{p_2}$ *and* $\frac{1}{r_i}=\frac{1}{q}-\frac{1}{p_i}$, $i=1,2$,

$$B_6:=\left\{\int_0^\infty\left(\int_x^\infty\left(\int_y^\infty W(z)\,dz\right)^{\frac{r_2}{q}}\left(\int_0^y V_2^{1-p_2'}(z)\,dz\right)^{\frac{r_2}{q'}}V_2^{1-p_2'}(y)\,dy\right)^{\frac{k}{r_2}}\right.$$
$$\left.\times\left(\int_0^x V_1^{1-p_1'}(y)\,dy\right)^{\frac{k}{r_2'}}V_1^{1-p_1'}(x)\,dx\right\}^{\frac{1}{k}}<\infty,$$

and

$$B_7:=\left\{\int_0^\infty\left(\int_x^\infty\left(\int_y^\infty W(z)\,dz\right)^{\frac{r_1}{q}}\left(\int_0^y V_1^{1-p_1'}(z)\,dz\right)^{\frac{r_1}{q'}}V_1^{1-p_1'}(y)\,dy\right)^{\frac{k}{r_1}}\right.$$
$$\left.\times\left(\int_0^x V_2^{1-p_2'}(z)\,dz\right)^{\frac{k}{r_1'}}V_2^{1-p_2'}(x)\,dx\right\}^{\frac{1}{k}}<\infty.$$

Concerning the inequality (1.4), our main result reads:

Theorem 3.1. *Let* $0<q<\infty$, $1<p_1,p_2<\infty$ *and* w,v_1,v_2 *are weight functions defined on* $\mathbb{R}^N$, $N\in\mathbb{Z}_+$. *The inequality* (1.4) *holds for all* $f,g\geq 0$ *if and only if*

(i) for $1<\max(p_1,p_2)\leq q<\infty$,

$$B_1^N:=\sup_{0<\alpha<\infty}\left(\int_{|x|\geq\alpha}w(x)\,dx\right)^{\frac{1}{q}}\left(\int_{|x|\leq\alpha}v_1^{1-p_1'}(x)\,dx\right)^{\frac{1}{p_1'}}\left(\int_{|x|\leq\alpha}v_2^{1-p_2'}(x)\,dx\right)^{\frac{1}{p_2'}}<\infty,$$

(ii) for $1<p_1\leq q<p_2<\infty$, $\frac{1}{r_2}=\frac{1}{q}-\frac{1}{p_2}$,

$$B_2^N:=\sup_{0<\alpha<\infty}\left(\int_{|x|\leq\alpha}v_1^{1-p_1'}(x)\,dx\right)^{\frac{1}{p_1'}}\left(\int_{|y|\geq\alpha}\left(\int_{|x|\geq|y|}w(x)\,dx\right)^{\frac{r_2}{p_2}}\right.$$
$$\left.\times\left(\int_{|x|\leq|y|}v_2^{1-p_2'}(x)\,dx\right)^{\frac{r_2}{p_2'}}w(y)\,dy\right)^{\frac{1}{r_2}}<\infty,$$

(iii) for $1<p_2\leq q<p_1<\infty$, $\frac{1}{r_1}=\frac{1}{q}-\frac{1}{p_1}$,

$$B_3^N:=\sup_{0<\alpha<\infty}\left(\int_{|x|\leq\alpha}v_2^{1-p_2'}(x)\,dx\right)^{\frac{1}{p_2'}}\left(\int_{|y|\geq\alpha}\left(\int_{|x|\geq|y|}w(x)\,dx\right)^{\frac{r_1}{p_1}}\right.$$
$$\left.\times\left(\int_{|x|\leq|y|}v_1^{1-p_1'}(x)\,dx\right)^{\frac{r_1}{p_1'}}w(y)\,dy\right)^{\frac{1}{r_1}}<\infty,$$

(iv) for $0<q<\min(p_1,p_2)<\infty$, $\min(p_1,p_2)>1$, $\frac{1}{q}\leq\frac{1}{p_1}+\frac{1}{p_2}$ *and* $\frac{1}{r_i}=\frac{1}{q}-\frac{1}{p_i}$, $i=1,2$,

$$B_4^N:=\sup_{0<\alpha<\infty}\left(\int_{|x|\leq\alpha}v_1^{1-p_1'}(x)\,dx\right)^{\frac{1}{p_1'}}\left(\int_{|y|\geq\alpha}\left(\int_{|x|\geq|y|}w(x)\,dx\right)^{\frac{r_2}{q}}\right.$$

$$\times \left(\int_{|x|\le|y|} v_2^{1-p_2'}(x)\,dx \right)^{\frac{r_2}{q'}} v_2^{1-p_2'}(y)\,dy \Bigg)^{\frac{1}{r_2}} < \infty,$$

and

$$B_5^N := \sup_{0<\alpha<\infty} \left(\int_{|x|\le\alpha} v_2^{1-p_2'}(x)\,dx \right)^{\frac{1}{p_2'}} \Bigg(\int_{|y|\ge\alpha} \left(\int_{|x|\ge|y|} w(x)\,dx \right)^{\frac{r_1}{q}}$$

$$\times \left(\int_{|x|\le|y|} v_1^{1-p_1'}(x)\,dx \right)^{\frac{r_1}{q'}} v_1^{1-p_1'}(y)\,dy \Bigg)^{\frac{1}{r_1}} < \infty.$$

(v) for $0 < q < \min(p_1,p_2) < \infty$, $\min(p_1,p_2) > 1$, $\frac{1}{q} > \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{k} = \frac{1}{q} - \frac{1}{p_1} - \frac{1}{p_2}$ *and* $\frac{1}{r_i} = \frac{1}{q} - \frac{1}{p_i}$, $i = 1,2$,

$$B_6^N := \Bigg(\int_{\mathbb{R}^N} \Bigg(\int_{|y|\ge|x|} \left(\int_{|z|\ge|y|} w(z)\,dz \right)^{\frac{r_2}{q}} \left(\int_{|z|\le|y|} v_2^{1-p_2'}(z)\,dz \right)^{\frac{r_2}{q'}} v_2^{1-p_2'}(y)\,dy \Bigg)^{\frac{k}{r_2}}$$

$$\times \left(\int_{|z|\le|x|} v_1^{1-p_1'}(z)\,dz \right)^{\frac{k}{r_2'}} v_1^{1-p_1'}(x)\,dx \Bigg)^{\frac{1}{k}} < \infty,$$

and

$$B_7^N := \Bigg(\int_{\mathbb{R}^N} \Bigg(\int_{|y|\ge|x|} \left(\int_{|z|\ge|y|} w(z)\,dz \right)^{\frac{r_1}{q}} \left(\int_{|z|\le|y|} v_1^{1-p_1'}(z)\,dz \right)^{\frac{r_1}{q'}} v_1^{1-p_1'}(y)\,dy \Bigg)^{\frac{k}{r_1}}$$

$$\times \left(\int_{|z|\le|x|} v_2^{1-p_2'}(z)\,dz \right)^{\frac{k}{r_1'}} v_2^{1-p_2'}(x)\,dx \Bigg)^{\frac{1}{k}} < \infty.$$

Proof. In view of our equivalence Theorem 2.1, it is sufficient to show that the conditions B_i^N are equivalent to the conditions B_i of Theorem A, $i = 1,2,\cdots,7$. We prove only the equivalence of B_1^N and B_1 since the proofs of all other cases are completely similar. By using polar coordinates $x = s\tau$, $s > 0, \tau \in \Sigma_N$ and using (2.1) and (2.2), we have that

$$\begin{aligned}
B_1^N &= \sup_{0<\alpha<\infty} \left(\int_{|x|\ge\alpha} w(x)\,dx \right)^{\frac{1}{q}} \left(\int_{|x|\le\alpha} v_1^{1-p_1'}(x)\,dx \right)^{\frac{1}{p_1'}} \left(\int_{|x|\le\alpha} v_2^{1-p_2'}(x)\,dx \right)^{\frac{1}{p_2'}} \\
&= \sup_{0<\alpha<\infty} \left(\int_\alpha^\infty \int_{\Sigma_N} w(s\tau)\, s^{N-1}\,d\tau\,ds \right)^{\frac{1}{q}} \left(\int_0^\alpha \int_{\Sigma_N} v_1^{1-p_1'}(s\tau)\, s^{N-1} d\tau\,ds \right)^{\frac{1}{p_1'}} \\
&\qquad \times \left(\int_0^\alpha \int_{\Sigma_N} v_2^{1-p_2'}(s\tau)\, s^{N-1} d\tau\,ds \right)^{\frac{1}{p_2'}} \\
&= \sup_{0<\alpha<\infty} \left(\int_\alpha^\infty W(s)\,ds \right)^{\frac{1}{q}} \left(\int_0^\alpha V_1^{\frac{1}{1-p_1}}(s)\,ds \right)^{\frac{1}{p_1'}} \left(\int_0^\alpha V_2^{\frac{1}{1-p_2}}(s)\,ds \right)^{\frac{1}{p_2'}}
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{0<\alpha<\infty} \left(\int_\alpha^\infty W(s)\,ds\right)^{\frac{1}{q}} \left(\int_0^\alpha V_1^{1-p_1'}(s)\,ds\right)^{\frac{1}{p_1'}} \left(\int_0^\alpha V_2^{1-p_2'}(s)\,ds\right)^{\frac{1}{p_2'}} \\
&= B_1
\end{aligned}
$$

and the assertion follows. □

Let us choose a function σ_i such that $a_i(x) < \sigma_i(x) < b_i(x)$ and

$$
\int_{a_i(x)}^{\sigma_i(x)} v_i^{p_i'} = \int_{\sigma_i(x)}^{b_i(x)} v_i^{p_i'}, \quad x > 0.
$$

Moreover, let a_i^{-1}, b_i^{-1}, σ_i^{-1} be the inverse functions of a_i, b_i, σ_i, respectively. Denote

$$
\begin{aligned}
&\Delta_i(t) := (a_i(t), b_i(t)),\\
&\Delta_i^{-1}(t) := (a_i^{-1}(t), b_i^{-1}(t)),\\
&\delta_i(t) := (b_i^{-1}(\sigma_i(t)), a_i^{-1}(\sigma_i(t))),\\
&\delta_i^{-1}(t) := (a_i^{(}\sigma_i^{-1}(t)), b_i(\sigma_i^{-1}(t))), \quad i = 1, 2.
\end{aligned}
$$

On the similar lines as in the proof of Theorem 3.1, using the information for the bilinear Hardy-Steklov inequality in [7], [8] and applying Theorem 2.2, the following equivalence theorem can be proved:

Theorem 3.2. *Let $0 < q < \infty$, $1 < p_1, p_2 < \infty$ and w, v_1, v_2 are weight functions defined on $\mathbb{R}^N$, $N \in \mathbb{Z}_+$. The inequality (2.7) holds for all $f, g \geq 0$ if and only if*

(i) for $1 < \max(p_1, p_2) \leq q < \infty$,

$$
BS_1^N := \sup_{t,s>0} \left(\int_{\delta_1(|t|)\cap\delta_2(|s|)} w^q\right)^{\frac{1}{q}} \left(\int_{\Delta_1(|t|)} v_1^{1-p_1'}\right)^{\frac{1}{p_1'}} \left(\int_{\Delta_2(|s|)} v_2^{1-p_2'}\right)^{\frac{1}{p_2'}} < \infty,
$$

(ii) for $1 < p_1 \leq q < p_2 < \infty$, $\frac{1}{r_2} = \frac{1}{q} - \frac{1}{p_2}$,

$$
\begin{aligned}
BS_2^N := \sup_{t>0} \left(\int_{\Delta_1(|t|)} v_1^{1-p_1'}\right)^{\frac{1}{p_1'}} &\left(\int_{\delta_1(|t|)} \left(\int_{\delta_1(|t|)\cap\delta_2(|s|)} w^q\right)^{\frac{r_2}{p_2}}\right.\\
&\left.\times \left(\int_{\Delta_2(|s|)} v_2^{1-p_2'}(x)\,dx\right)^{\frac{r_2}{p_2'}} w^q(s)\,ds\right)^{\frac{1}{r_2}} < \infty,
\end{aligned}
$$

(iii) for $1 < p_2 \leq q < p_1 < \infty$, $\frac{1}{r_1} = \frac{1}{q} - \frac{1}{p_1}$,

$$
\begin{aligned}
BS_3^N := \sup_{s>0} \left(\int_{\Delta_1(|s|)} v_2^{1-p_2'}\right)^{\frac{1}{p_2'}} &\left(\int_{\delta_2(|s|)} \left(\int_{\delta_1(|t|)\cap\delta_2(|s|)} w^q\right)^{\frac{r_1}{p_1}}\right.\\
&\left.\times \left(\int_{\Delta_1(|t|)} v_1^{1-p_1'}(x)\,dx\right)^{\frac{r_1}{p_1'}} w^q(t)\,dt\right)^{\frac{1}{r_1}} < \infty.
\end{aligned}
$$

Remark. *In Theorem 3.2, the remaining cases, namely* $0 < q < \min(p_1, p_2) < \infty$, $\min(p_1, p_2) > 1$, $\frac{1}{q} \leq \frac{1}{p_1} + \frac{1}{p_2}$ *and* $0 < q < \min(p_1, p_2) < \infty$, $\min(p_1, p_2) > 1$, $\frac{1}{q} > \frac{1}{p_1} + \frac{1}{p_2}$ *can also be handled as the other cases but since it requires introducing additional and cumbersome notations, we therefore leave out the formulation of these cases.*

4. An equivalence theorem for iterated Hardy-type operators

Here we consider the N-dimensional iterated Hardy type operators T_1^N, T_2^N, T_3^N and T_4^N defined by

$$T_1^N f(x) := \left(\int_{\mathbb{R}^N \setminus B(0,|x|)} \left(\int_{B(0,|y|)} f(z)\, dz \right)^q w(y)\, dy \right)^{\frac{1}{q}},$$

$$T_2^N f(x) := \left(\int_{B(0,|x|)} \left(\int_{\mathbb{R}^N \setminus B(0,|y|)} f(z)\, dz \right)^q w(y)\, dy \right)^{\frac{1}{q}},$$

$$T_3^N f(x) := \left(\int_{\mathbb{R}^N \setminus B(0,|x|)} \left(\int_{\mathbb{R}^N \setminus B(0,|y|)} f(z)\, dz \right)^q w(y)\, dy \right)^{\frac{1}{q}},$$

$$T_4^N f(x) := \left(\int_{B(0,|x|)} \left(\int_{B(0,|y|)} f(z)\, dz \right)^q w(y)\, dy \right)^{\frac{1}{q}}$$

which are the N-dimensional analogues of the corresponding one-dimensional operators T_1, T_2, T_3 and T_4 defined, respectively, by

$$T_1 F(x) := \left(\int_x^\infty \left(\int_0^y F(z)\, dz \right)^q W(y)\, dy \right)^{\frac{1}{q}},$$

$$T_2 F(x) := \left(\int_0^x \left(\int_y^\infty F(z)\, dz \right)^q W(y)\, dy \right)^{\frac{1}{q}},$$

$$T_3 F(x) := \left(\int_x^\infty \left(\int_y^\infty F(z)\, dz \right)^q W(y)\, dy \right)^{\frac{1}{q}},$$

$$T_4 F(x) := \left(\int_0^x \left(\int_0^y F(z)\, dz \right)^q W(y)\, dy \right)^{\frac{1}{q}}.$$

Our main result in this section reads:

Theorem 4.1. *Let* $0 < r < \infty$, $1 < p < \infty$ *and* u, v *be weight functions defined on* $\mathbb{R}^N$. *The Hardy-type inequality*

$$\left(\int_{\mathbb{R}^N} \left((T_1^N f)(x) \right)^r u(x)\, dx \right)^{\frac{1}{r}} \leq C \left(\int_{\mathbb{R}^N} f^p(x)\, v(x)\, dx \right)^{\frac{1}{p}} \tag{4.1}$$

holds for all $f \geq 0$ *if and only if the inequality*

$$\left(\int_0^\infty \left((T_1 F)(s) \right)^r U(s)\, ds \right)^{\frac{1}{r}} \leq C \left(\int_0^\infty F^p(s)\, V(s)\, ds \right)^{\frac{1}{p}} \tag{4.2}$$

holds for $F \geq 0$ *with* W *given by* (2.1) *and* U, V *given by*

$$U(t) := \int_{\Sigma_N} u(t\tau)\, t^{N-1}\, d\tau, \tag{4.3}$$

$$V(t) := \left(\int_{\Sigma_N} v^{1-p'}(t\tau)\, t^{N-1}\, d\tau\right)^{1-p}, \quad t > 0,\ \tau \in \Sigma_N. \tag{4.4}$$

Proof. Let us first assume that the inequality (4.2) holds. Let us fix f and choose

$$F(t) := \int_{\Sigma_N} f(t\tau)\, t^{N-1}\, d\tau.$$

By using Hölder's inequality, we find that

$$\begin{aligned} F(t) &\leq \left(\int_{\Sigma_N} f^p(t\tau)\, v(t\tau)\, t^{N-1}\, d\tau\right)^{\frac{1}{p}} \left(\int_{\Sigma_N} v^{1-p'}(t\tau)\, t^{N-1}\, d\tau\right)^{\frac{1}{p'}} \\ &= \left(\int_{\Sigma_N} f^p(t\tau)\, v(t\tau)\, t^{N-1}\, d\tau\right)^{\frac{1}{p}} \Big(V(t)\Big)^{-\frac{1}{p}} \end{aligned} \tag{4.5}$$

Changing to polar coordinates $x = s\tau$, $y = s_1\sigma$, $z = s_2\gamma$, $s, s_1, s_2 > 0$, $\tau, \sigma, \gamma \in \Sigma_N$ and using the inequalities (4.2) and (4.5), we get

$$\begin{aligned} &\left(\int_{\mathbb{R}^N} \Big((T_1^N f)(x)\Big)^r u(x)\, dx\right)^{\frac{1}{r}} \\ &= \left(\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N \setminus B(0,|x|)} \left(\int_{B(0,|y|)} f(z)\, dz\right)^q w(y)\, dy\right)^{\frac{r}{q}} u(x)\, dx\right)^{\frac{1}{r}} \\ &= \Bigg\{\int_0^\infty \int_{\Sigma_N} \left(\int_s^\infty \int_{\Sigma_N} \left(\int_0^{s_1} \int_{\Sigma_N} f(s_2\gamma)\, s_2^{N-1}\, d\gamma\, ds_2\right)^q w(s_1\sigma)\, s_1^{N-1}\, d\sigma\, ds_1\right)^{\frac{r}{q}} \\ &\qquad\qquad \times u(s\tau)\, s^{N-1}\, d\tau\, ds\Bigg\}^{\frac{1}{r}} \\ &= \left(\int_0^\infty \left(\int_s^\infty \left(\int_0^{s_1} F(s_2)\, ds_2\right)^q W(s_1)\, ds_1\right)^{\frac{r}{q}} U(s)\, ds\right)^{\frac{1}{r}} \\ &= \left(\int_0^\infty \Big((T_1 F)(s)\Big)^r U(s)\, ds\right)^{\frac{1}{r}} \\ &\leq C \left(\int_0^\infty F^p(s)\, V(s)\, ds\right)^{\frac{1}{p}} \\ &\leq C \left(\int_0^\infty \int_{\Sigma_N} f^p(s\tau)\, v(s\tau)\, s^{N-1}\, d\tau\, ds\right)^{\frac{1}{p}} \\ &= C \left(\int_{\mathbb{R}^n} f^p(x)\, v(x)\, dx\right)^{\frac{1}{p}} \end{aligned}$$

which means that (4.1) holds.

Conversely, assume that the inequality (4.1) holds. Let us fix F and choose

$$f(t\gamma) := F(t) v^{1-p'}(t\gamma) \Big(V(t)\Big)^{\frac{1}{p-1}},$$

where $t > 0$, $\gamma \in \Sigma_N$. That gives that

$$F(t) = \int_{\Sigma_N} f(t\gamma)\, t^{N-1}\, d\gamma. \tag{4.6}$$

Now, by using (4.6) and the inequality (4.1), we obtain that

$$\begin{aligned}
&\left(\int_0^\infty \Big((T_1F)(s)\Big)^r U(s)\, ds\right)^{\frac{1}{r}} \\
&= \left(\int_0^\infty \left(\int_s^\infty \left(\int_0^{s_1} F(s_2)\, ds_2\right)^q W(s_1)\, ds_1\right)^{\frac{r}{q}} U(s)\, ds\right)^{\frac{1}{r}} \\
&= \Bigg\{\int_0^\infty \left(\int_s^\infty \left(\int_0^{s_1}\int_{\Sigma_N} f(s_2\gamma)\, s_2^{N-1}\, d\gamma\, ds_2\right)^q \int_{\Sigma_N} w(s_1\sigma)\, s_1^{N-1}\, d\sigma\, ds_1\right)^{\frac{r}{q}} \\
&\qquad\qquad \times \int_{\Sigma_N} u(s\tau)\, s^{N-1}\, d\tau\, ds\Bigg\}^{\frac{1}{r}} \\
&= \left(\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N\setminus B(0,|x|)}\left(\int_{B(0,|y|)} f(z)\, dz\right)^q w(y)\, dy\right)^{\frac{r}{q}} u(x)\, dx\right)^{\frac{1}{r}} \\
&= \left(\int_{\mathbb{R}^N}\Big((T_1^N f)(x)\Big)^r u(x)\, dx\right)^{\frac{1}{r}} \\
&\le C\left(\int_{\mathbb{R}^N} f^p(x)\, v(x)\, dx\right)^{\frac{1}{p}} \\
&= C\left(\int_0^\infty \int_{\Sigma_N} f^p(s\tau)\, v(s\tau)\, s^{N-1}\, d\tau\, ds\right)^{\frac{1}{p}} \\
&= C\left(\int_0^\infty \int_{\Sigma_N} \left[F(s) v^{1-p'}(s\tau)\Big(V(s)\Big)^{\frac{1}{p-1}}\right]^p v(s\tau)\, s^{N-1}\, d\tau\, ds\right)^{\frac{1}{p}} \\
&= C\left(\int_0^\infty F^p(s)\left(\int_{\Sigma_N} v^{1-p'}(s\tau)\, s^{N-1}\, d\tau\right)\Big(V(s)\Big)^{\frac{p}{p-1}}\, ds\right)^{\frac{1}{p}} \\
&= C\left(\int_0^\infty F^p(s)\Big(V(s)\Big)^{\frac{1}{1-p}}\Big(V(s)\Big)^{\frac{p}{p-1}}\, ds\right)^{\frac{1}{p}} \\
&= C\left(\int_0^\infty F^p(s) V(s)\, ds\right)^{\frac{1}{p}},
\end{aligned}$$

so (4.2) holds. The proof is complete. □

Remark. *Theorem 4.1 can also be proved if T_1^N in* (4.1) *is replaced by any of the remaining operators T_2^N, T_3^N, T_4^N and correspondingly in* (4.2), *T_1 is replaced by any of the operators T_2, T_3 and T_4, respectively.*

Remark. *Weight characterization for the inequality* (4.2) *can be obtained on the similar lines as in Theorems 3.1 and 3.2, as soon as the corresponding weight characterization of the one-dimensional case has been derived (c.f.* (4.2)*).*

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