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Creating balance in dynamic competitions[☆]

Derek J. Clark[a,*], Tore Nilssen[b]

[a] School of Business and Economics, UiT - The Arctic University of Norway, Tromsø NO-9037, Norway
[b] Department of Economics, University of Oslo, P.O. Box 1095, Blindern, Oslo NO-0317, Norway

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ABSTRACT

We consider incentives for organizing competitions in multiple rounds, focusing on situations where there is heterogeneity among the contestants *ex ante*, which discourages effort in a single contest. Heterogeneity evolves across rounds depending upon the outcomes of previous rounds. We present conditions under which balance in such a competition can be created, by determining the number of rounds and dividing the prize fund carefully across them, so that full rent dissipation entails. In the model, each round is an all-pay auction where contestants differ in their abilities to gain a momentum from winning. We also discuss the case when negative prizes are feasible, demonstrating that this strengthens the full dissipation result; and we consider a case where the size of the winner's momentum is related to the size of the prize attained, showing that the stronger this linkage, the less of the prize is awarded early on.

1. Introduction

Many competitions are organized in multiple rounds. A challenge with this practice is the difficulty of keeping up the efforts of the players as the competition advances in situations where an early loss discourages a player from putting in effort in later rounds (Konrad, 2012). When this so-called discouragement effect is strong, there may be cause to organize the competition as one grand single-stage contest. When multiple-round competitions are so prevalent, this may be for reasons such as convexity of per-round effort costs; budget constraints on the part of contestants, organizer, or both; or the need for the organizer to keep the suspense, which is likely a crucial argument for organizers of sport competitions living off of the audience that the competitions attract.[1]

In this paper, we point to still another reason for organizing a multi-round competition: When contestants come to a competition with *ex-ante* differences in abilities, these differences in themselves may be discouraging.[2] We point to circumstances where, by organizing a competition in multiple rounds, the organizer can not only attenuate the discouragement

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* Corresponding author.
E-mail addresses: derek.clark@uit.no (D.J. Clark), tore.nilssen@econ.uio.no (T. Nilssen).

[1] On convexity of costs in contests see Moldovanu and Sela (2006); for budget constraints in sequential contests refer to Megidish and Sela (2014); for the design of sports contests, see Szymanski (2003).

[2] See the survey by Chowdhury et al. (2019) on the effect of contestant heterogeneity on competition effort.

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effect stemming from such differences but even lift overall efforts so much that full rent dissipation entails. This may happen when the *ex-ante* laggard would get a higher boost in his skills from an early win than the *ex-ante* leader would. This second asymmetry between the contestants creates a scope for the organizer to make use of two or more rounds and split the prize fund across these rounds in such a way that the initial round becomes a balanced contest.

In order to discuss this, we present a model of several rounds of all-pay auctions between two contestants who are different along two dimensions: the productivity of their efforts at the outset, and the ability to obtain a boost in this productivity at later stages from an early win. The principal, who organizes the competition, has available a fixed prize fund to distribute across the sequence of contests with the aim of maximizing total overall expected efforts, subject to an overall budget constraint and (initially) a non-negativity constraint on each round's prize.

We find that the principal is able to obtain full rent dissipation – with total expected efforts among the players equal to the prize fund – when the *ex-ante* heterogeneity in productivity is small enough and the *ex-ante* disadvantaged contestant is sufficiently better at gaining a boost from winning. When this is the case, the principal is able to distribute the prize fund such that the various skills of the two contestants are exactly balanced in the first contest in the sequence and there will be full rent dissipation, with total expected efforts among the players equal to the prize fund. Moreover, we find that the extent to which full rent dissipation is achievable is increased when it is possible to organize the competition over a longer sequence of contests.

The central notion for the possibility of full rent dissipation is that of a balanced contest. Of course, a symmetric contest is balanced. In order to obtain balance in the first contest when there are *ex-ante* asymmetries that are not easily corrected, the underdog must have a greater gain from winning that contest, so that he can catch up with and leapfrog the better rival. When this is the case, there is a scope for splitting the prize fund across the contests such that a balanced first contest is created.

For the two-contest case, we offer in addition a complete characterization of the optimal organization of the contest. For an intermediate range of the *ex-ante* heterogeneity where full rent dissipation is not possible, we find that the principal should put all the prize fund in the second contest, while for large such heterogeneity, the best is to have all the prize fund in the first contest, essentially closing down the second one and run a single contest. We also discuss extensions of our model to the case where the boost gained from winning depends on the size of the prize won, as well as the case where a prize may be negative.

Our analysis is of relevance to many kinds of competition where designers face heterogeneous contestants. Consider, in particular, competitions for research grants, commonly organized by research councils and similar entities. It has long been recognized, at least since Merton (1968) coined the concept of the Matthew effect, that there are win effects in the competition for research grants.[3] As noted by Gallini and Scotchmer (2002, p. 54), "future grants are contingent upon previous success. The linkage between previous success and future funding seems even more specific in the case of the National Science Council". Those who succeed in obtaining research grants may experience an increase in status, winning a grant to fund current work and build up a competent research team, which again improves their chance of winning further grants. Losing teams must use time and resources in seeking presumably inferior forms of funding; in sum, this gives an advantage in future rounds of competition for scarce research funding. Our analysis points to a scope for attenuating the Matthew effect, and even obtain full rent dissipation, by carrying out a research program across several rounds of contests for funds and carefully spreading out the total research budget across these rounds. Taking this view might lead to a distribution of research funds that is less susceptible to discouragement among initial laggards, both for research councils and for organizers of related competitions, such as innovation contests.[4]

Another area where our analysis has a potential to contribute is that of internal labour markets, in particular sales-force management.[5] There, it is not uncommon for the more successful agents to be given less administrative duties, better access to back-office resources, more training than the less successful, and better territories; see, *e.g.*, Skiera and Albers (1998), Farrell and Hakstian (2001), and Krishnamoorthy et al. (2005). This is, again, a mechanism in which winning creates winners, and using several rounds of contests among the sales force may be what is needed in order to create a balanced contest in order to keep total efforts high.[6] Our model may also be applied to franchising, which is suggested by Gillis et al. (2011) to resemble a dynamic tournament setting. Franchisees compete with each other in order to gain more units in the franchise, as a prize for good relative performance over time. Initially, one franchisee may have an advantage over rivals due to location or other factors, and the franchisor can reward the high performer with another franchise unit. This allows the winner to build a business and enjoy economies of scale, scope and/or management in future contests. The franchisor must determine when, and how large a franchise to award. In our analysis, we show that several reward schemes can induce maximal

[3] Win effects are also well documented in biology, where the male of a species experiences an increase in testosterone level following a win in a contest, while the loser has a reduced level of testosterone (Chase et al., 1994). Hence, the winner is in better physiological shape to compete in the next contest. A similar winner effect has been demonstrated in male judo competitions by Cohen-Zada et al. (2017) and in male tennis competitions by Gauriot and Page (2018), while batters in baseball are found to perceive the ball as bigger when they have had recent success in hitting (Witt and Proffitt, 2005). See also Iso-Ahola and Dotson (2014) for a discussion of the occurrence of psychological momentum in sports and other contexts.

[4] See Adamczyk et al. (2012) for a discussion of innovation contests.

[5] But also, and related to the previous paragraph, our analysis may be relevant to the organization of internal innovation contests; see (Höber, 2017).

[6] Our analysis can also find application in political competition, where the winner of a first election may gain additional media attention and funding from campaign contributors which helps to build future momentum (see Strumpf, 2002).

expected effort; some of these involve periods of effort in which no direct prize (franchise) is awarded, but where rivals seek to build their claims for the next prize.

The paper closest to ours is Clark and Nilssen (2019), which analyzes a similar setting to the present one, except that the heterogeneities there are with respect to additive head starts rather than multiplicative productivity biases. In particular, one player has an *ex-ante* head start, while the players also differ in their gains from an early win. Also there, full rent dissipation can be obtained through an optimum distribution of the prize budget if the *ex-ante* heterogeneity is small enough. Although results are similar for the two-contest case, the mechanisms differ. With head starts, cases may occur where the leading player has such a big lead that he can win without exerting effort, simply trusting his head start, and the principal must take such cases, that would entail low total expected efforts, into account when designing the competition. This issue does not show up in the present case of productivity biases. One effect of this difference is that we here are able to get out some results for longer competitions, that is, sequences of three or more contests, thus being able to discuss the merits of such long competitions, while the other paper is limited to the analysis of the two-contest case.

A precursor to both these papers is Clark and Nilssen (2018), which analyzes a sequence of all-pay auctions with win advantages, but where prizes are constant across time and players are identical. In particular, players obtain the same head start and/or productivity boost from winning a contest. In that paper, the design issue that we focus on presently does not appear, since identical players imply that the first contest is always balanced and therefore that there is always full rent dissipation in equilibrium. Instead, the focus is on the extent to which an initial laggard will stay in the game and eventually have higher expected efforts than the leader.[7] Clark and Nilssen (2018) find that heterogeneity between the rivals occurring throughout the competition as wins and losses are recorded can reduce their efforts since the weaker player reduces efforts due to a perceived increase in the probability of losing, and the stronger player reduces effort as a response.[8] Counteracting this mechanism through the contest design is a key issue in the current paper.

Other papers that discuss design in heterogeneous all-pay auctions have focused on how the principal can optimally set, or reset, biases in order to maximize total expected efforts; see, *e.g.*, Epstein et al. (2011), Li and Yu (2012), and Franke et al. (2018).[9] This approach differs from ours, in that we take the view that biases are fixed and not possible to adjust directly and rather explore how, in dynamic competitions, distributing the prize fund across time can affect total expected efforts. A complicating – but realistic – factor in our analysis is that rivals have different productivities of effort initially, and that these evolve at different rates as the series of contests progresses. Contestants improve their effort productivity over time in relation to the pattern of losses and wins, and this opens up for the possibility that a productivity advantage may be enhanced, neutralized or overturned in the course of the play. As Rigney (2010, p. 1) puts it, "i]nitial advantage does not always lead to further advantage, and initial disadvantage does not always lead to further disadvantage".[10]

Our work also relates to Fudenberg et al. (1983). In one version of their patent race, competitors must progress through discrete stages in order to secure the final invention. They assume that a laggard may probabilistically complete a necessary stage in the race before the firm with *ex-ante* higher expected value, giving it an advantage since it can start work on the next stage in the process. In some competitions, successful players may gain access to material goods that make competing easier. Similarly, Konrad and Kovenock (2010) show that the discouragement effect in sequential contests can be mitigated if contestants' abilities are not constant, and rather the result of a stochastic process. This ensures that there are situations in which an underdog may be more able than the favorite on a given day, leading to less pronounced discouragement.

For most of our analysis, we assume that the size of the momentum gained by a contest winner is independent of the size of the prize actually won. Möller (2012), Beviá and Corchón (2013), and Luo and Xie (2018) present two interlinked Tullock contests in which the size of the prize attained in the first affects the probability of winning in the second. We show that our result of attaining contest balance and full rent dissipation also holds for a series of all-pay auctions when the size of the winner's momentum is directly related to the early prize won.

A further extension that we explore is that of a negative prize in the first contest with a fixed budget constraint, a notion that is similar to the idea pursued by Mealem and Nitzan (2016) involving taxation of contest prizes. Clearly, there are many circumstances in which negative prizes are not feasible, including the applications mentioned above of research competitions and internal labour markets. Still, it is of value to note our finding that allowing negative prizes may greatly expand the range of parameters for which full rent dissipation occurs.

Our paper also joins a growing literature that discusses sequential competition, and stands out since it tackles the design issue of how to distribute a prize mass over a sequence of contests, where also the number of contests is a design variable. Several papers assume a structure in which a certain number of rounds (often termed battles) must be won in order to achieve an overall prize (Klumpp and Polborn, 2006; Konrad and Kovenock, 2009; Sela, 2011). Taking a different approach, Feng and Lu (2018) allow the prize achieved to be a non-decreasing function of the number of component battles won in

[7] Clark et al. (2019) discuss a sequence of two Tullock contests in which the winner of the first has a lower cost of exerting effort in the second or a higher win probability compared to ex ante symmetric rivals. In that model, with no discounting of future payoffs, the optimum for the principal is to put the prize fund into the second contest.

[8] A similar mechanism is noted in the innovation tournament of Terwiesch and Xu (2008).

[9] See also the surveys by Mealem and Nitzan (2016) and Chowdhury et al. (2019).

[10] When the direction and magnitude of the contestants' heterogenity evolves, the principal must continually rebias contest efforts if this is the instrument being used. This may be seen as a rather erratic policy in which the favoured contestant constantly changes. Our approach in this setting is simple, involving only a division of the prize mass over contest rounds.

a three-stage contest. They find that intermediate prizes in component battles may be awarded as a way of mitigating the discouragement effect; see also Konrad and Kovenock (2009).

Similarly to us, Fu and Lu (2012) allow a principal to choose the number of contests and how to divide the prize mass between them; since they consider an elimination tournament, the principal can also decide the number of participants remaining at each stage. The component battle is a Tullock contest, and the results echo those of Feng and Lu (2018): low discriminatory power in the contest success function tends to lead to a single contest being optimal, whereas higher levels of discriminatory power make the multi-contest environment more efficient at eliciting effort.

The paper is organized as follows. In the next section, we offer a preliminary analysis of the stage game. In Section 3, we present our model of the two-contest competition and provide a complete solution for the principal's optimum distribution of the prize fund. Section 4 extends the analysis to more than two contests, limiting the discussion to finding conditions such that full rent dissipation is feasible. Section 5 discusses two extensions of our analysis; one allowing negative stage prizes, and one allowing the momentum that the first-contest winner achieves to depend on the size of the prize at that stage. Section 6 offers some concluding remarks, while proofs are relegated to an Appendix.

2. Preliminaries

There are two risk-neutral players, s and w, who compete for a prize that they value at $v_s > 0$ and $v_w > 0$, respectively, by making irreversible efforts $x_s \geq 0$ and $x_w \geq 0$. The probability that player s wins the prize is

$$p_s(x_s, x_w) = \begin{cases} 1 \text{ if } \alpha_s x_s > \alpha_w x_w; \\ \frac{1}{2} \text{ if } \alpha_s x_s = \alpha_w x_w; \\ 0 \text{ if } \alpha_s x_s < \alpha_w x_w; \end{cases}$$

where $\alpha_i > 0$ is a bias parameter in favour of player $i \in \{s, w\}$, and the probability that w wins is $p_w = 1 - p_s$. We assume that $\alpha_s v_s \geq \alpha_w v_w$, implying that player s is the *stronger* one. The expected payoffs of the two players are given by

$$\pi_s(x_s, x_w) = p_s v_s - x_s;$$
$$\pi_w(x_s, x_w) = p_w v_w - x_w.$$

This game has a unique equilibrium, which is described in Lemma 1 in the Appendix. In this equilibrium, the expected efforts of the players are

$$x_s^* = \frac{\alpha_w v_w}{2\alpha_s}, \text{ and } x_w^* = \frac{\alpha_w v_w^2}{2\alpha_s v_s}; \tag{1}$$

expected payoffs are

$$\pi_s^* = v_s - \frac{\alpha_w}{\alpha_s} v_w, \text{ and } \pi_w^* = 0; \tag{2}$$

and probabilities of winning are

$$p_s^* = 1 - \frac{\alpha_w v_w}{2\alpha_s v_s}, \text{ and } p_w^* = \frac{\alpha_w v_w}{2\alpha_s v_s}. \tag{3}$$

From (1), we have that the total expected effort is

$$x_s^* + x_w^* = \frac{\alpha_w v_w}{\alpha_s v_s} \frac{(v_s + v_w)}{2}. \tag{4}$$

We say the contest is *balanced* when

$$\alpha_s v_s = \alpha_w v_w. \tag{5}$$

It follows from the above that, in a balanced contest,

$$x_s^* = \frac{v_s}{2}; \; x_w^* = \frac{v_w}{2}; \; x_s^* + x_w^* = \frac{v_s + v_w}{2}; \tag{6}$$
$$\pi_s^* = \pi_w^* = 0; \text{ and } p_s^* = p_w^* = \frac{1}{2}.$$

As discussed in the Introduction, the notion of a balanced contest is crucial for the principal's search for full rent dissipation. Note that, in a biased contest, total expected effort (4) is a fraction of the average valuation of the prize. Balancing the contest yields expected efforts equal to the players' average valuation. We will return to this below.

The above simple all-pay auction with biases comprises the stage game of our analysis in the next sections.

3. The two-contest model

We start our analysis by completely solving the model for the case of two contests, identifying conditions under which the expected total efforts are equal to the value of the total prize mass. There are two risk-neutral players, $i \in \{1, 2\}$, who compete in two successive contests, $t \in \{1, 2\}$, by making irreversible efforts, $x_{i,t}$. The two players differ in two respects: in the biases they have before the game starts, and in the bias they can obtain before contest two by winning contest one. A principal has a prize mass of size one to divide between the two contests, making $(1-v)$ available in the first, and v in the second; for now we assume non-negative prizes, $v \in [0, 1]$.

Only efforts in the current contest affect the probability of winning, but do so according to a biased version of the all-pay auction. One of the players – player 1, without loss of generality – has a positive bias in contest one, so that the contest success function of player 1 is

$$\rho_{1,1}(x_{1,1}, x_{2,1}) = \begin{cases} 1 \text{ if } bx_{1,1} > x_{2,1}; \\ \frac{1}{2} \text{ if } bx_{1,1} = x_{2,1}; \\ 0 \text{ if } bx_{1,1} < x_{2,1}; \end{cases} \tag{7}$$

where $b > 1$ is the bias in favour of player 1 in contest one; the contest success function of player 2, here and throughout, is $\rho_{2,1}(x_{1,1}, x_{2,1}) = 1 - \rho_{1,1}(x_{1,1}, x_{2,1})$. In contest two, the bias develops according to who has won the first contest. Should the already advantaged player 1 win the first contest, then his bias parameter is increased by a factor of $a_1 > 1$ to $a_1 b$. Should the initial laggard, player 2, win the first contest, then he has a bias parameter of $a_2 > 1$ in contest two, and player 1 retains his bias of b. Hence the probability that player 1 wins the second contest, having won the first, is

$$\rho_{1,2}(x_{1,2}, x_{2,2}; 1) = \begin{cases} 1 \text{ if } a_1 bx_{1,2} > x_{2,2}; \\ \frac{1}{2} \text{ if } a_1 bx_{1,2} = x_{2,2}; \\ 0 \text{ if } a_1 bx_{1,2} < x_{2,2}; \end{cases}$$

whilst the probability that player 1 wins the second contest after the opponent has won the first is

$$\rho_{1,2}(x_{1,2}, x_{2,2}; 2) = \begin{cases} 1 \text{ if } bx_{1,2} > a_2 x_{2,2}; \\ \frac{1}{2} \text{ if } bx_{1,2} = a_2 x_{2,2}; \\ 0 \text{ if } bx_{1,2} < a_2 x_{2,2}. \end{cases}$$

Denote by $\pi^*_{i,2}(i)$ the payoff of player i in the second contest having won the first, and $\pi^*_{i,2}(j)$ the corresponding payoff if i lost the first contest.[11] Seen from contest one, the expected payoff functions of player i can be written as

$$\begin{aligned} \pi_{i,1} &= \rho_{i,1}\left(1 - v + \pi^*_{i,2}(i)\right) + (1 - \rho_{i,1})\pi^*_{i,2}(j) - x_{i,1} \\ &= \pi^*_{i,2}(j) + \rho_{i,1}\left(1 - v + \pi^*_{i,2}(i) - \pi^*_{i,2}(j)\right) - x_{i,1} \\ &= \pi^*_{i,2}(j) + \rho_{i,1} V_{i,1} - x_{i,1}. \end{aligned}$$

Winning the first contest gives the current prize $1-v$ and the expected payoff in the second contest having won the first; losing the first contest gives only the continuation value of proceeding to the second contest as the loser of the first. The value $V_{i,1}$ is the total value that player i fights for in the first contest, consisting of the first contest prize, and the payoff difference in the second contest between winning and losing the first one. The model is solved by backwards induction, starting in the second contest. Proposition 1 characterizes the optimal prize split between the two contests as well as the total expected efforts.

Proposition 1. *In the two-contest model, the optimal setting of v, and the corresponding realized total expected efforts, are as follows:*

(i) If $\frac{a_2(1+a_1)}{a_1(1+a_2)} \geq b > 1$, then $v^* = \widehat{v}$, with total expected efforts 1, where

$$\widehat{v} := \frac{a_1 a_2 (b-1)}{a_2 - a_1 b}. \tag{8}$$

(ii) If $\frac{a_2}{a_1} > b > \frac{a_2(1+a_1)}{a_1(1+a_2)}$, then $v^* = 1$, with total expected efforts $\frac{1}{b} + \frac{a_2 - a_1 b}{a_1 a_2 b} \in \left(\frac{1}{b}, 1\right)$.

(iii) If $b = \frac{a_2}{a_1}$, then $v^* \in [0, 1]$, with total expected efforts $\frac{1}{b} = \frac{a_1}{a_2}$.

(iv) If $b > \frac{a_2}{a_1}$, then $v^* = 0$, with total expected efforts $\frac{1}{b}$.

Fig. 1 gives an illustration of the results, depending on the value of b. When the initial bias in favour of player 1 is sufficiently small, it is possible to use the division of the prize mass to ensure balance in the first contest so that the principal can achieve expected effort equal to the total prize mass. For intermediate values of b, this is not attainable, but

[11] Closed expressions for these payoffs are provided in the proof of Proposition 1 in the Appendix.

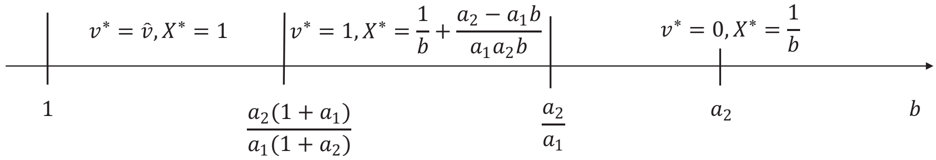

Fig. 1. Illustration of Proposition 1.

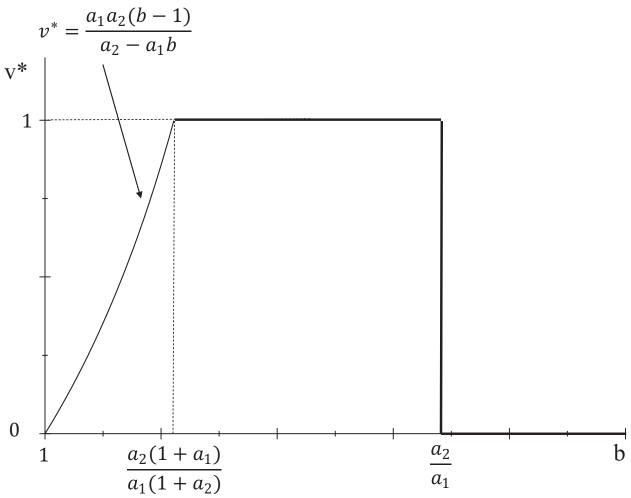

Fig. 2. Optimal second contest prize.

saving the whole prize for contest two yields the most expected effort. When the lead of player 1 at the outset is too large, the principal can do no better than to run a single (biased) contest.

Key to achieving full dissipation of the prize is making the first contest balanced. This requires dividing up the prize mass so that $bV_{1,1} = V_{2,1}$, giving $\widehat{v}$ as the second contest prize. The parameter restriction in part (i) of Proposition 1 derives from the fact that $\widehat{v} \in [0, 1]$.[12] Fig. 2 indicates how the second-contest prize depends on the initial bias in favour of player 1, increasing in this bias until $v^* = 1$ is reached, and then falling to zero when the bias is too large.

When the first contest is balanced, each player has an equal probability of being the victor there. In the second contest, each player has valuation v of winning, and from (1), one can see that each player has the same expected effort in that contest.[13] That second-contest effort depends on who is the winner of the first, being $\frac{v}{a_1 b}$ if player 1 wins, and $\frac{bv}{a_2}$ if 2 wins. In the Appendix, we show that expected effort in contest one is $1 - \frac{\widehat{v}}{2}\left(\frac{1}{a_1 b} + \frac{b}{a_2}\right)$, which is decreasing in the second-period balancing prize. Expected effort in contest two is $\frac{\widehat{v}}{2}\left(\frac{1}{a_1 b} + \frac{b}{a_2}\right)$, which exactly neutralizes the effect that $\widehat{v}$ has on first-contest effort, leaving an expected effort of 1.

With the first contest balanced, the players compete away the full value of the total prize, each ending up with an overall payoff of zero. To see this, note that, in the balanced contest, $\rho^*_{i,1} = \frac{1}{2}$ and $x^*_{i,1} = \frac{V_{i,1}}{2}$, so that $\pi^*_{i,1} = \frac{V_{i,1}}{2} - \frac{V_{i,1}}{2} = 0$. This occurs for sufficiently small values of the *ex-ante* heterogeneity b. When $b > \frac{a_2(1+a_1)}{a_1(1+a_2)}$, it is no longer possible to balance the first contest, since doing so would require $\widehat{v} > 1$, which is ruled out by assumption. It is well known that distributing the whole prize in contest one will give an expected effort of $\frac{1}{b}$. For intermediate values of b, part (ii) of Proposition shows that it is possible to get some benefit from the possible catching-up by player 2 by awarding the whole prize mass in contest two. This works as long as the catching-up parameter of player 2 is sufficiently large ($a_2 > a_1 b$). We show in the Appendix that, when the first contest is not balanced, the total expected effort in the two contests is

$$\frac{v(a_2 - a_1 b)}{a_1 a_2 b} + \frac{1}{b},$$

[12] In Section 5.1 we consider the case when this restriction is not imposed and the first-contest prize may be negative.

[13] From (1), we have that $\frac{x_s^*}{x_w^*} = \frac{v_s}{v_w}$.

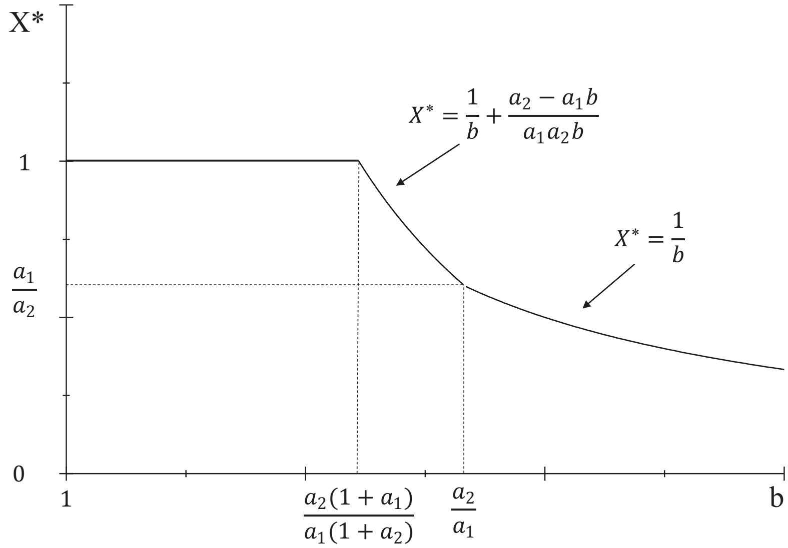

Fig. 3. Maximum expected effort.

which is linear in v. Thus, the optimum decision for the principal depends on the sign of $a_2 - a_1 b$. When $a_1 b > a_2$, the principal can do no better than setting $v = 0$, and getting the expected effort from a single contest, $\frac{1}{b}$. When $a_2 > a_1 b$, $v = 1$ gives more expected effort than $\frac{1}{b}$.

Fig. 3 indicates the total expected efforts achieved, which are $X^* = 1$ for sufficiently low values of b, and falling in b after this.

4. More than two contests

While the previous Section offers a complete solution of the two-contest case, the question remains whether it would be in the interest of the principal to split the competition into even more rounds than two. In the present Section, we provide an answer to this. In particular, we delineate cases where it is in the interest of the principal to have more than two rounds of contests in order to enlarge the scope for full rent dissipation.

In constructing a series of more than two contests, we assume that the momentum/win effect is multiplicative; when contest t is about to be played, and player 1 has won m of the previous $t-1$, the bias for each competitor is $a_1^m b$ and a_2^{t-1-m}, respectively.[14] In our extended model, we retain the assumption from the two-contest case that the initial laggard can catch up and surpass the leader. The key to full rent dissipation is again balancing the first contest. As in the two-contest case, when the first contest is balanced, the value of the game to each player is zero, since they compete away the full value of the prize mass in expectation. We present below a condition such that, in the general case, the first contest is balanced and there is full rent dissipation across the sequence of contests.

Proposition 2. *Suppose that $b \le \frac{a_2}{a_1}$, that the sequence consists of $T \ge 2$ contests, and that the principal allocates her total prize fund of 1 over the T contests such that the prize in contest $t \in \{1, \dots, T\}$ is $v_t \ge 0$, and $\sum_{t=1}^{T} v_t = 1$. The first contest in the series is balanced, and the total expected effort equals the total prize fund, when*

$$v_1(b-1) = \sum_{t=2}^{T} \phi_t v_t, \text{ where} \tag{9}$$

$$\phi_t = \frac{a_1^t - 1}{a_1^{t-1}(a_1 - 1)} - b\frac{\left(a_2^t - 1\right)}{a_2^{t-1}(a_2 - 1)}, t = 2, \dots, T. \tag{10}$$

Since the left-hand-side of (9) is at least zero, balance requires that the right-hand-side be non-negative. When $T = 2$, we see, from (10), that $\phi_2 > 0$ for $\frac{a_2(1+a_1)}{a_1(1+a_2)} \ge b$, consistent with part (i) of Proposition 1. If, on the other hand, $T = 2$, and

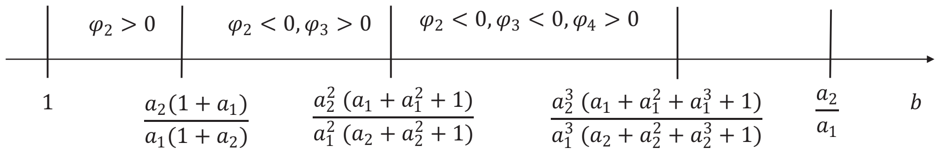

Fig. 4. Balance with T contests.

$\frac{a_2(1+a_1)}{a_1(1+a_2)} < b$, then $\phi_2 < 0$, and a series of two contests cannot be balanced. Fig. 4 depicts how the sign of the ϕ_t depends upon the initial bias parameter b.

This figure shows that, for

$$\frac{a_2(a_1+1)}{a_1(a_2+1)} < b \leq \frac{a_2^2\left(a_1+a_1^2+1\right)}{a_1^2\left(a_2+a_2^2+1\right)},$$

the principal can obtain a balanced first contest by using three contests, $T = 3$. Increasing b outside of this interval means that ϕ_2, $\phi_3 < 0$, and a fourth contest must be added to achieve balance, and so on for further increases in b. Note that the critical value of b making ϕ_t positive can be expressed as the ratio between two geometrical series. As we increase the number of contests, the restriction gets weaker and weaker, modulo the other restriction of $b < \frac{a_2}{a_1}$. From (10), we have that $\phi_t > 0$ for

$$b < \frac{\left(a_1 - \frac{1}{a_1^{t-1}}\right)(a_2-1)}{\left(a_2 - \frac{1}{a_2^{t-1}}\right)(a_1-1)} := \theta_t.$$

Hence, the series of critical values for the initial bias parameter θ_t that makes ϕ_t positive increases as each new contest is added.[15] However, it may not be possible to achieve full rent dissipation for all $b \in \left(1, \frac{a_2}{a_1}\right)$, as detailed in the following corollary.

Corollary 1. *Consider a prize structure $v_t \geq 0$, $t = 1, 2, \ldots, T$. Full rent dissipation can be achieved for $b \in \left(1, \frac{a_2}{a_1}\right)$ iff $\frac{a_1}{a_1-1} \geq a_2$. If $a_2 > \frac{a_1}{a_1-1}$, then full rent dissipation can be achieved for $b \in \left(1, \frac{a_1(a_2-1)}{a_2(a_1-1)}\right]$, but not for $b \in \left(\frac{a_1(a_2-1)}{a_2(a_1-1)}, \frac{a_2}{a_1}\right)$.*

There are three types of asymmetry captured by the model: the initial bias in favour of player 1, and the rates of momentum following a contest win. Achieving full rent dissipation involves adding contests until the first one is balanced. In order to do this, the asymmetry cannot be too large, and this is the essence of Corollary 1. If the initial asymmetry is too large ($b > \frac{a_2}{a_1}$), then the first contest cannot be balanced by adding more since player 2 cannot catch up the lead; if the momentum of the initially disadvantaged player 2 is too large ($a_2 > \frac{a_1}{a_1-1}$), then player 1 cannot catch up if the rival pulls ahead.

A reward scheme for achieving full rent dissipation consists of any combination of non-negative prizes that sum to 1 and solve (9); when there are more than two prizes on offer, there will hence not be a unique solution. As an example, consider $a_1 = 1.2$, $a_2 = 2$, $b = 1.3$. This implies $\phi_2 = -0.11667$, so that two contests cannot be used to create balance, and a third one must be added. We can calculate $\phi_3 = 0.25278$, so that balance in the first contest requires

$$0.3v_1 = -0.11667v_2 + 0.25278v_3; \text{ and} \tag{11}$$

$$v_1 + v_2 + v_3 = 1, \tag{12}$$

where (11) follows from insertions into (9).

Fig. 5 depicts combinations of first-contest and third-contest prizes in (v_3, v_1) space for this example. The $v_1(v_3)$ line in the figure is the locus of combinations of the two prizes that satisfy the two Eqs. (11) and (12), in addition to the non-negativity constraints $v_t \geq 0$, $t \in \{1, 2, 3\}$. At point y in the figure, the first-contest prize is at zero and we have the prize vector $(v_1 = 0, v_2 = 0.68, v_3 = 0.32)$. At point z, it is the second-contest prize that is at zero, and the prize vector is $(v_1 = 0.46, v_2 = 0, v_3 = 0.54)$. In between the two extremes, the third-contest prize moves in the range [0.32, 0.54].

[14] This is not the only way of modelling this parameter over time but is in line with the approach taken by de Roos and Sarafidis (2018) for modelling momentum in continuous-time dynamic contests.

[15] Also, $\frac{\partial\theta_t}{\partial a_1} < 0$, and $\frac{\partial\theta_t}{\partial a_2} > 0$.

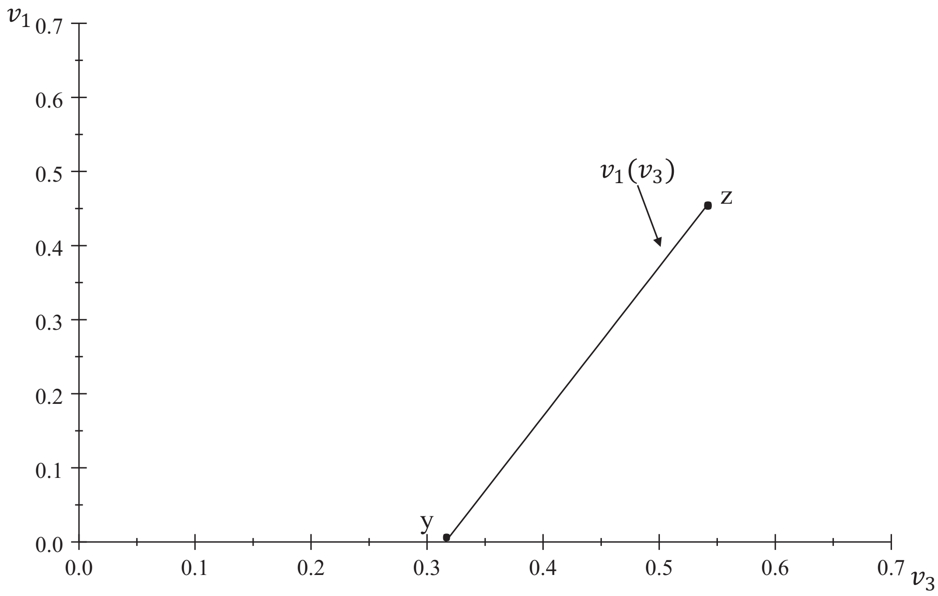

Fig. 5. Reward Schemes for $T = 3, b = 1.3, a_1 = 1.2, a_2 = 2$.

4.1. First-and-last prize schemes

With only two equations to tie down the profile of T rewards, the exact prize structure cannot be determined. It is possible to determine the system by specifying that the principal will just use two prizes, one in the first contest and one in the last; this is represented in Fig. 5 by the point z. We have the following Corollary to Proposition 2 for this "first-and-last" prize scheme:

Corollary 2. *With a series of T contests, the principal obtains full rent dissipation by distributing the prize fund strictly between the two contests 1 and T, keeping v_2 through v_{T-1} at zero, as long as*

$$b \leq \min\left\{\frac{a_2}{a_1}, \frac{\left(a_1 - \frac{1}{a_1^{T-1}}\right)(a_2 - 1)}{\left(a_2 - \frac{1}{a_2^{T-1}}\right)(a_1 - 1)}\right\}. \tag{13}$$

with

$$\begin{aligned} v_T &= \frac{b-1}{b-1+\phi_T}; \\ v_1 &= \frac{\phi_T}{b-1+\phi_T}; \\ v_2, \ldots, v_{T-1} &= 0; \end{aligned} \tag{14}$$

where ϕ_T is defined in (10).

Note, from (10), that $\frac{\partial \phi_t}{\partial t} > 0$. By inserting from (10) into (14), we can derive the following comparative-statics properties of this final-contest prize:

$$\frac{\partial v_T}{\partial b} > 0; \ \frac{\partial v_T}{\partial a_1} > 0; \ \frac{\partial v_T}{\partial a_2} < 0; \ \frac{\partial v_T}{\partial T} < 0.$$

If the initial bias in favor of player 1 increases, and/or if the gain to this player from winning increases, then the final contest prize should be increased, as long as (13) is still satisfied. Saving the prize mass until later encourages the laggard to stay in the game, fighting for the possibility of winning a large reward at the end of the series. The more the disadvantaged player gains from winning a contest, the more of the prize mass it is optimal to have early. This gives the initial laggard a large incentive to win early, catching up and surpassing the initial leader. As the number of contests in the series becomes larger, the principal should give a larger share of the spoils early to balance the contest. This is easily seen from (9), since ϕ_T is increasing in T and all prizes from v_2 through v_{T-1} are zero in this particular reward scheme. The principal induces most effort in the first contest, since the following contests are simply for position, with a modest prize in the end. Note, however, that the final prize is always positive.

There is an interesting interplay between the comparative-statics effects noted above. Ceteris paribus, increasing the initial bias b makes it optimal to shift prize mass to late in the series. However, this also increases the number of contests that must be used in order to achieve full rent dissipation, which lowers the optimal final prize. This can be illustrated by recalling the numerical example in which $a_1 = 1.2,$ and $a_2 = 2,$ and where we vary the initial bias b and the number of contests T; at $b = 1.3$ and $T = 3,$ this example is identical to the one used in conjunction with Fig. 5 above. In the following table, for each row, a "+" indicates the contest in which ϕ_t turns positive.

b	ϕ_2	ϕ_3	ϕ_4	ϕ_5	v_T
1.1	+				0.35
1.3	−	+			0.54
1.5	−	−	+		0.63
1.6	−	−	+		0.85
1.66	−	−	−	+	0.64

Two contests can be used for a low value of the initial bias ($b = 1.1$), with most of the prize mass given in the first contest. With a bias of $b = 1.3,$ three contests are utilized and more than half of the prize is distributed in the final contest. Four contests are used for biases of 1.5 and 1.6, with the final prize increasing in b. When the initial bias is at 1.66, which is close to its top level at $\frac{a_2}{a_1} = \frac{2}{1.2} = \frac{5}{3},$ five contests are necessary, and here we see that the amount of prize that is awarded late falls from 0.85 with $b = 1.6$ to 0.64 with the higher bias parameter. As discussed above, the increase in the bias parameter tends to increase the late prize, whereas the fact that it is awarded one contest later reduces the late prize.

As noted, the simple reward scheme in Corollary 2 is not the unique one that balances the first contest. But there is no other reward scheme that achieves this goal by using fewer contests in the series. To see this, consider (9). If the left-hand side is positive (*i.e.* $v_1 > 0$), then the right-hand side must also be. Hence contests must be added until we find T such that $\phi_T > 0$, just as in the simple scheme above. If $v_1 = 0,$ then the right-hand-side must sum to zero, so that the early negative values of ϕ_t must be canceled out by the first positive one, just as above. Hence, we can state the following:

Corollary 3. *A reward scheme in which $v_1 > 0$, $v_T > 0$, $v_1 + v_T = 1$, and $v_2, \ldots, v_{T-1} = 0$, and where (9) is satisfied, achieves balance in contest one with the fewest number of contests possible, which is the lowest T such that $\phi_T > 0$.*

It is not difficult to construct examples in which a reward scheme uses more contests than that in Corollary 3. Suppose for instance that the principal wants to divide the prize mass as equally as possible across contests, whilst still maintaining balance, so that all contest prizes after the first are of equal value v, whilst the prize in contest one is $v_1 > 0$, and $v_1 + (T-1)v = 1$. The condition for balance in this case is

$$v_1(b-1) = v\sum_{t=2}^{T}\phi_t,$$

requiring that $\sum_{t=2}^{T}\phi_t > 0$. As an illustration, return to the numerical example above, and put $b = 1.5$. The condition $\sum_{t=2}^{T}\phi_t > 0$ is not fulfilled for four contests, so the principal must use five in this instance, one more than discussed above. Contest one is balanced in this case for $T = 5,$ $v_1 = 0.188,$ and $v_2 = v_3 = v_4 = v_5 = 0.203$. A more even distribution of the prize mass thus requires more contests in order to preserve balance.

5. Extensions

Heterogeneity tends to reduce expected effort in a single contest, and the previous sections have highlighted the circumstances under which a principal can use a sequence of contests in order to elicit full prize dissipation. Adding contests has been shown to facilitate this for larger and larger values of the initial degree of heterogeneity. In this Section, we return to a sequence of two contests, relaxing some core assumptions. First, we consider how the scope for balance is changed when the principal can offer a negative first-contest prize;[16] then we relax the assumption that the momentum from winning is constant, allowing it to depend on the size of the prize won in the first contest.

5.1. Negative prizes

In the analysis above, we restrict prizes to be non-negative. This is a natural restriction to impose in many circumstances, which is why we have maintained it in our main analysis. But the same set of contestants competing several times does open for up the possibility of making some prizes negative. With two contests, the second contest prize would have to be positive in order to give the players an incentive to compete at that stage; the first contest prize can be negative, but the

[16] We would like to thank a referee for suggesting this.

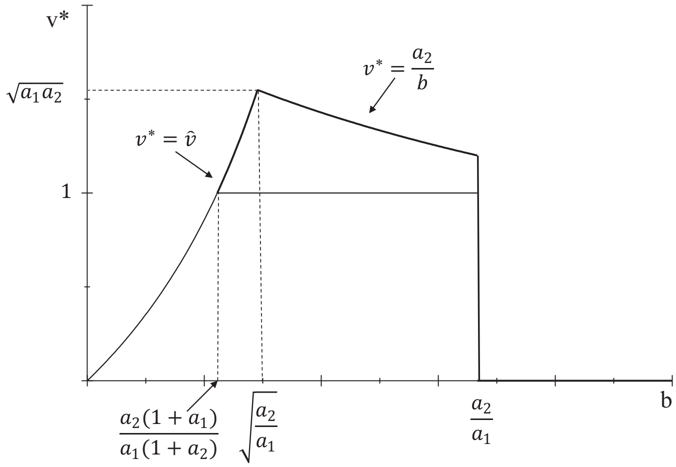

Fig. 6. Optimal second contest prize greater than 1.

contestants will still be interested in winning in order to build up a momentum which is then used to secure the large prize in the following contest. It is conceivable that a negative prize is linked to the momentum achieved by the contest winner; a first contest winner may have the possibility of undertaking a costly activity (training or taking a course) that increases his ability to compete at the next opportunity.

To see the effect of allowing negative prizes, we consider first the case of $T = 2$. The crucial effect of allowing negative prizes is that now v can be above 1, since the budget can be balanced by putting the first-contest prize $1 - v$ below zero. The restrictions are rather that, for both players, the value of taking part in the two-contest competition is at least as great as not taking part, *i.e.*, it is the case that $V_{i,1}$, the value of winning the first contest, is non-negative. This limits the size of the negative first contest prize. Recalling the definition of $\widehat{v}$ in (8), we have the following optimal prize structure and expected efforts when a negative first-contest prize is allowed:

Proposition 3. *Let $T = 2$, with the prize in contest one equal to $(1 - v)$ and that in contest two equal to v. When negative prizes are allowed, the optimal setting of v, and the corresponding realized total expected efforts, are as follows:*

(i) If $\sqrt{\frac{a_2}{a_1}} \geq b > 1$, then $v^* = \widehat{v} \in (0, \sqrt{a_1 a_2}]$, with total expected efforts 1.

(ii) If $\frac{a_2}{a_1} > b > \sqrt{\frac{a_2}{a_1}}$, then $v^* = \frac{a_2}{b}$, with total expected efforts $\frac{a_2}{a_1 b^2} = \frac{1}{b} + \frac{a_2 - a_1 b}{a_1 b^2} \in \left(\frac{a_1}{a_2}, 1\right)$.

(iii) If $b = \frac{a_2}{a_1}$, then $v^* \in [0, a_1]$, with total expected efforts $\frac{1}{b} = \frac{a_1}{a_2}$.

(iv) If $b > \frac{a_2}{a_1}$, then $v^* = 0$, with total expected efforts $\frac{1}{b}$.

Note that, according to Proposition 3, part (i), the principal uses the same prize formula as before, but this now extends beyond $v = 1$, hence implying a negative prize in contest one, for $b \in \left(\frac{a_2(1+a_1)}{a_1(1+a_2)}, \frac{a_2}{a_1}\right)$. Also, as b approaches $\frac{a_2}{a_1}$ from below, v^* approaches $a_1 > 1$. The optimal second contest prize is illustrated in Fig. 6, in which the thick line represents the case in which a negative first contest prize is allowed, whilst the thin line replicates the depiction in Fig. 2 where this is not possible.

The effect of allowing a negative first-contest prize on total expected efforts is depicted in Fig. 7, in which the thicker lines are the efforts from Proposition 3. It is apparent that the principal now can use the prize split to extend the range of b for which $X^* = 1$ to include $b \in \left(\frac{a_2(1+a_1)}{a_1(1+a_2)}, \sqrt{\frac{a_2}{a_1}}\right)$, compared to the case of non-negative prizes. Moreover, the optimal prize split now gives greater efforts also in the interval $b \in \left(\sqrt{\frac{a_2}{a_1}}, \frac{a_2}{a_1}\right)$; as in the analysis of non-negative prizes above, total expected efforts are continuous at $b = \frac{a_2}{a_1}$ (and equal to $\frac{1}{b}$) even though the optimum prize split is not. The negative prize in the first contest would tend to dampen competition there, but the fact that the winner gains momentum going into contest two pulls effort in the other direction. Additionally, the final prize is large due to the budget constraint of the principal, and this incites extra effort in contest two.

It is natural to ask the question as to what can be achieved with the possibility of negative prizes in longer series of contests. The next proposition gives a simple answer:

Proposition 4. *Suppose $T = 3, v_1 < 0, v_2 = 0, v_3 > 0, v_1 + v_3 = 1$. Then full rent dissipation is achieved for all $b < \frac{a_2}{a_1}$.*

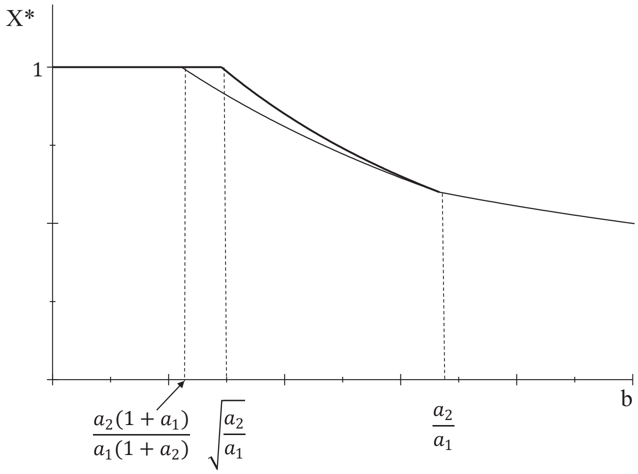

Fig. 7. Maximum expected effort with a negative first contest prize.

Here we consider a first-and-last reward scheme in which the first contest prize is negative, and the third and final one is positive such that the whole prize mass is used up. In the case of two contests, it is possible to increase the range of the initial heterogeneity for which full dissipation occurs, but not to cover the whole interval considered. Introducing a third contest – in which the second one is purely for position – potentially allows the initial laggard to catch up a larger lead by the time the third-contest prize is awarded. Both contestants fight hard to gain an advantageous position to win this prize, and full rent dissipation ensues.

5.2. Prize-dependent momentum

In the analysis so far, the amount of momentum gained by a winner has been independent of the size of the prize attained. In some contexts, there may be a natural connection between the prize and the momentum; a research group that wins a project will be able to build up a bigger team when a grant is large, compared to another who achieves seed funding, for example. Beviá and Corchón (2013) and Möller (2012) have investigated this possibility in the case of Tullock contests. We show here that balance and full rent dissipation can still be achieved in our model with all-pay auctions. We return to the case of two contests, in which player 1 has an initial bias of $b > 1$. The principal divides the prize into $v_1 + v_2 = 1$ over the two contests, and the size of the bias in the second contest is $bf_1(v_1)$ if 1 wins contest one, and $f_2(v_1)$ if 2 wins it. We assume further that $f_i(v_1) > 1, f_i(0) = a_i > 1, f_i^{'}(v_1) \geq 0,$ so that this formulation encompasses that in Section 3. In Section 3, we had $v_1 = 1 - v,\ v_2 = v,$ and $f_i(v_1) = a_i, \forall v_1, i \in \{1, 2\}$. We have the following result.

Proposition 5. *If the first-contest prize* $v_1 \in [0, 1]$ *satisfies* $f_2(v_1) > b$ *and*

$$\frac{bf_1(v_1) - (1 - v_1)}{f_1(v_1)} = \frac{f_2(v_1) - b(1 - v_1)}{f_2(v_1)}, \tag{15}$$

then the total amount of effort expected over the two contests is equal to 1.

When $f_2(v_1) > b$, the initial laggard overtakes the bias of player 1 if he wins the first contest; that this degree of catching up is necessary for the full dissipation result is also a feature of our previous analysis. Balancing contest one requires (15) to be fulfilled, and in this case the full value of the prize is dissipated.

As an example, consider

$$f_i(v_1) = a_i + \delta v_1, \tag{16}$$

where $\delta \geq 0$ captures the momentum effect associated with the first-contest prize, and $\delta = 0$ is the case examined in Section 3. Solving (15) gives the following result.

Corollary 4. *Suppose that* $b < \frac{a_2}{a_1}$ *and that the win advantage of player i is given by (16). The optimal first-contest prize is*

$$v_1^*(\delta) = \begin{cases} \frac{a_2(1+a_1)-a_1b(1+a_2)}{a_2-a_1b} & \text{for } \delta = 0, \\ \frac{a_2(1+a_1)-a_1b(1+a_2)}{b(1+a_2)-(1+a_1)} & \text{for } \delta = 1, \\ \frac{[(a_1+a_2+1)(b-1)\delta+(a_2-a_1b)]-\sqrt{\Psi}}{2\delta(1-\delta)(b-1)} & \text{for } \delta \notin \{0, 1\}, \end{cases}$$

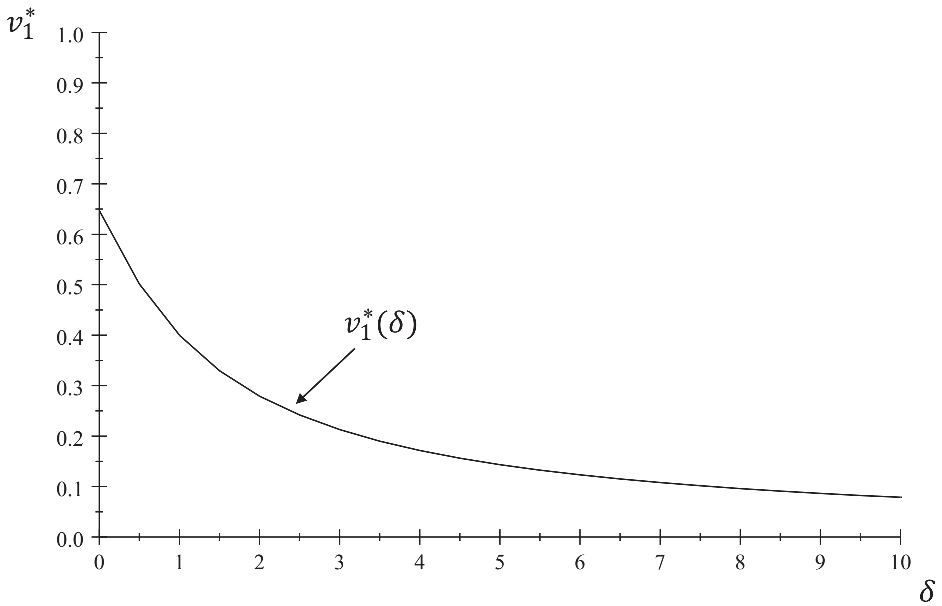

Fig. 8. Optimal first contest prize with prize-dependent momentum ($a_1 = 1.2, a_2 = 2, b = 1.1$).

where

$$\Psi := [(a_1 + a_2 + 1)(b - 1)\delta + (a_2 - a_1 b)]^2 - 4\delta(\delta - 1)(b - 1)[b(a_1 + a_1 a_2) - (a_2 + a_1 a_2)].$$

When $\delta = 0$, we have that $v_1^*(0) = 1 - \widehat{v}$, where $\widehat{v}$ is given in (8). Hence, the same condition as before ($\frac{a_2(1+a_1)}{a_1(1+a_2)} \geq b > 1$) ensures that $1 > v_1^*(0) \geq 0$. Moreover, $v_1^*(\delta)$ is monotonically decreasing in δ, converging to 0 as δ gets large. Hence, $1 > v_1^*(\delta) \geq 0$ for all values of δ, given that it holds for $\delta = 0$. Furthermore, $\frac{\partial v_1^*(\delta)}{\partial b} < 0$, whenever b is such that the optimal solution exists; this is in line with the finding from Section 3 and Fig. 2 (which depicts the second round prize as increasing in b).[17] As an illustration, consider the numerical example used previously, with $a_1 = 1.2$, $a_2 = 2$, $b = 1.1$, which gives the depiction of $v_1^*(\delta)$ in Fig. 8.

The stronger is the connection between the momentum achieved from winning the first contest, and the size of the first-contest prize, *i.e.*, the higher is δ, the lower the principal sets that prize. A larger bias makes it more difficult to balance the series of contests, and hence the principal neutralizes the effect of the prize-dependent momentum by moving prize mass later on. When an early winner can use the prize to boost future winning chances, the prize distribution gets skewed in favour of the late prize.

6. Conclusion

Rivals often face each other in competition repeatedly, and the experience of winning or losing is suggested to affect future competitions. A winner may gain some physiological or psychological momentum, or some material gains that makes competing relatively easier, in line with the saying that “success breeds success”, or “winning makes winners”. In such a situation, it is easy to think that a laggard may simply give up, rendering the competition as a futile method for inducing effort.

We have looked at conditions under which a principal may divide a prize mass among a sequence of several contests in order to induce effort, when strength or ability evolves according to previous wins and losses. The synergy created between the contests in the series can be exploited in order to achieve full rent dissipation even if the rivals are not equally strong at the outset.

In order to derive this result we have shown that the initial laggard must have the possibility of catching up and surpassing his rival’s strength in the course of the play. This echoes the leapfrogging requirement in the patent race of Fudenberg et al. (1983). We have characterized the result completely for the case of two interlinked contests. We have also presented conditions under which the principal can expect the whole value of the prize fund to be dissipated in longer series. To achieve this, the principal must set the optimal number of contests, and set an appropriate prize division across these contests. We have shown that a particularly simple scheme can be used, and that this minimizes the number of contests needed in the series. The principal should just place weight on the first and the last prize in the series, rendering

[17] The properties of $v_1^*(\delta)$ reported here are proved in the Appendix.

intermediate rounds as contests for position to win the final prize. In some applications, such awarding of prizes may not be possible; researchers compete yearly for grants, for example, and not one every few years. If the principal has access to a more or less even stream of financing for the prize (a research council with a yearly budget for example), we have demonstrated that balance can still be achieved – with ensuing full rent dissipation in expectation – albeit at the cost of having to use more contests. Allowing for negative prizes – if these are feasible for the application considered – reinforces the full rent dissipation result, since this can be achieved in as little as three contests for all feasible levels of the initial heterogeneity.

In our model, the effort-maximizing principal seeks to design a series of contests that balances the boost from winning and the negative effect of losing; the instruments at her disposal are the number of contests used and the division of the prize mass between them. Our model is highly stylized and can be seen as a first attempt to model the heterogeneity than can arise in dynamic contests due to previous contest results. In particular, future work should focus on the modelling of the momentum emanating from wins and losses. We have mainly assumed a fixed exponential effect, and this may be easily defended for short series of contests, but is more problematic for longer ones. The amount that a competitor can gain from winning many times is likely to fall as the number of wins increases, so that the winner advantage does not increase indefinitely. We will pursue these avenues in future research.

Appendix A

Equilibrium in the stage game of Section 2

Let $F_i(x)$ be the cumulative distribution function of player i's mixed strategy, $i \in \{s, w\}$. The expected payoffs of the two players are given by

$$\pi_s(x_s) = F_w\left(\frac{\alpha_s}{\alpha_w}x_s\right)v_s - x_s;$$
$$\pi_w(x_w) = F_s\left(\frac{\alpha_w}{\alpha_s}x_w\right)v_w - x_w.$$

The equilibrium of this game is well known, and the following Lemma is stated without proof.

Lemma 1. *The game has a unique equilibrium given by the mixed strategies*

$$F_s(x_s) = \frac{\alpha_s}{\alpha_w v_w}x_s, \quad x_s \in \left[0, \frac{\alpha_w v_w}{\alpha_s}\right];$$
$$F_w(x_w) = 1 - \frac{\alpha_s(v_w - x_w)}{\alpha_s v_s}, \quad x_w \in [0, v_w].$$

Proof of Proposition 1. Consider the second contest, where the prize is v and there is no future contest. We use the results from Section 2 to calculate expected efforts and payoffs. Suppose first that player 1 has won the first contest, in which case the bias parameter of this player is $a_1 b$. In terms of Section 2, we have player 1 as the stronger one, with $\alpha_s = a_1 b$, $\alpha_w = 1$, and $v_s = v_w = v$, and the following equilibrium values:

$$\begin{aligned}
x^*_{1,2}(1) = x^*_{2,2}(1) &= \frac{v}{2a_1 b} \\
x^*_{1,2}(1) + x^*_{2,2}(1) &= \frac{v}{a_1 b} \\
\pi^*_{1,2}(1) &= \frac{(a_1 b - 1)v}{a_1 b} \\
\pi^*_{2,2}(1) &= 0
\end{aligned} \tag{A.1}$$

where the number in brackets identifies the winner of the first contest.

When player 2 wins the first contest, we have to consider two different cases, A and B, depending on who is the stronger player in contest two.[18]

A) $a_2 \geq b$. In this case, player 2 is the stronger player, and we use (1) and (2) to get

$$\begin{aligned}
x^{A*}_{1,2}(2) = x^{A*}_{2,2}(2) &= \frac{bv}{2a_2} \\
x^{A*}_{1,2}(2) + x^{A*}_{2,2}(2) &= \frac{bv}{a_2} \\
\pi^{A*}_{1,2}(2) &= 0 \\
\pi^{A*}_{2,2}(2) &= \frac{(a_2 - b)v}{a_2}
\end{aligned} \tag{A.2}$$

[18] Note that, when $a_2 = b$, the two cases coincide and lead to the same outcome, such that $\pi^*_{1,2} = \pi^*_{2,2} = 0$.

B) $b \geq a_2$. Here player 1 is still the stronger player, despite losing contest one, but with less of a bias in his favour than he had earlier. Equilibrium values are

$$\begin{aligned} x_{1,2}^{B*}(2) &= x_{2,2}^{B*}(2) = \frac{a_2 v}{2b} \\ x_{1,2}^{B*}(2) + x_{2,2}^{B*}(2) &= \frac{a_2 v}{b} \\ \pi_{1,2}^{B*}(2) &= \frac{(b-a_2)v}{b} \\ \pi_{2,2}^{B*}(2) &= 0 \end{aligned}$$

Turning to contest one, we set up the expected payoff functions for the players, where $k \in \{A, B\}$ denotes the case, as discussed above following a win for player 2 in contest one:

$$\begin{aligned} \pi_{1,1}^k &= \rho_{1,1}\left(1 - v + \pi_{1,2}^*(1)\right) + (1-\rho_{1,1})\pi_{1,2}^{k*}(2) - x_{1,1} \\ &= \pi_{1,2}^{k*}(2) + \rho_{1,1}\left(1 - v + \pi_{1,2}^*(1) - \pi_{1,2}^{k*}(2)\right) - x_{1,1} \\ &= \pi_{1,2}^{k*}(2) + \rho_{1,1}V_{1,1}^k - x_{1,1}; \\ \pi_{2,1}^k &= \rho_{2,1}\left(1 - v + \pi_{2,2}^{k*}(2)\right) - x_{2,1} \\ &= \rho_{2,1}V_{2,1}^k - x_{2,1}. \end{aligned}$$

Winning the first contest gives the stage prize of $1-v$ and the continuation value after having won: $\pi_{1,2}^{k*}(1)$ for player 1 and $\pi_{2,2}^{k*}(2)$ for player 2, $k \in \{A, B\}$. Losing gives the promise of $\pi_{i,2}^{k*}(j)$ in the next contest, for $i \neq j$ and $k \in \{A, B\}$. The contest success function is biased in favour of player 1, who has a bias of b. If player 2 wins contest one, then he expects a positive payoff in contest two in case A, and zero in case B; he expects zero if he loses the first contest. Player 1 expects a positive losing payoff only in case B.

Player i is guaranteed $\pi_{i,2}^{k*}(j) \geq 0$ in the first contest in case $k = \{A, B\}$; this is his expected payoff in contest two if he loses contest one. Player 2 has no positive guaranteed payoff, though; *i.e.*, $\pi_{2,2}^{k*}(1) = 0$. Each player can win a further value of

$$V_{i,1}^k = 1 - v + \pi_{i,2}^{k*}(i) - \pi_{i,2}^{k*}(j),$$

which is the stage prize in contest one, $(1-v)$, plus the difference in expected contest-two payoff between winning and losing contest one. This value of winning the first contest for each player for each case is

$$V_{1,1}^A = 1 - v + \frac{(a_1 b - 1)v}{a_1 b} = 1 - \frac{1}{a_1 b}v; \tag{A.3}$$

$$\begin{aligned} V_{2,1}^A &= 1 - v + \frac{(a_2 - b)v}{a_2} = 1 - \frac{b}{a_2}v; \\ V_{1,1}^B &= 1 - v + \frac{(a_1 b - 1)v}{a_1 b} - \frac{(b-a_2)v}{b} = 1 - \frac{a_1 a_2 - a_1 b - 1}{a_1 b}v; \\ V_{2,1}^B &= 1 - v. \end{aligned} \tag{A.4}$$

Consider first case B, where clearly player 1 is stronger, since $bV_{1,1}^B > V_{2,1}^B$.[19] We have, from Eqs. (3) and (4),

$$\begin{aligned} x_{1,1}^{B*} + x_{2,1}^{B*} &= \frac{1-v}{2b}\frac{v - 2a_1 b + 2a_1 bv - a_1 a_2 v}{v - a_1 b + a_1 bv - a_1 a_2 v} \\ \rho_{1,1}^{B*} &= 1 - \frac{1}{2}\frac{a_1(1-v)}{a_1 b - a_1 bv + a_1 a_2 v - v} \\ \rho_{2,1}^{B*} &= \frac{1}{2}\frac{a_1(1-v)}{a_1 b - a_1 bv + a_1 a_2 v - v} \end{aligned}$$

Given this, we can calculate the total expected effort over the two contests for case B as

$$\begin{aligned} X^{B*} &= \frac{1-v}{2b}\frac{v - 2a_1 b + 2a_1 bv - a_1 a_2 v}{v - a_1 b + a_1 bv - a_1 a_2 v} + \left(1 - \frac{1}{2}\frac{a_1(1-v)}{a_1 b - a_1 bv + a_1 a_2 v - v}\right)\frac{v}{a_1 b} + \left(\frac{1}{2}\frac{a_1(1-v)}{a_1 b - a_1 bv + a_1 a_2 v - v}\right)\frac{a_2 v}{b} \\ &= \frac{a_1 - v(a_1 - 1)}{a_1 b}, \end{aligned}$$

which is decreasing in v, since $a_1 > 1$, so that $v = 0$ is the optimal contest-two prize for a principal who wants to maximize total expected effort. This gives $X^{B*}(v=0) = \frac{1}{b}$.

[19] This can be verified since the condition amounts to $\frac{v(a_1 a_2 - 1)}{a_1} + (b-1)(1-v) > 0$, where each component is at least zero, and one component is positive if the other is exactly zero.

Consider next case A. When

$$bV^A_{1,1} = V^A_{2,1}, \tag{A.5}$$

contest one is balanced, and each player has an equal chance of winning. From Eqs. (A.3), (A.4), and (A.5), we see that we obtain balance in contest one by putting $v = \widehat{v}$. Because of the restriction $v \in [0, 1]$, we can only have balance in contest one if $\widehat{v} \in [0, 1]$, *i.e.*, if $\frac{a_2(1+a_1)}{a_1(1+a_2)} \geq b > 1$.

With balance in contest one, total expected effort in that contest, by Eqs. (6), (A.3), and (A.4), is

$$\frac{V^A_{1,1} + V^A_{2,1}}{2} = \frac{1}{2}\left(\frac{a_1 b - v}{a_1 b} + \frac{a_2 - bv}{a_2}\right),$$

and win probabilities are $\frac{1}{2}$ for each player. If player 1 wins contest one, then total expected effort in contest two is $\frac{v}{a_1 b}$, by Eq. (A.1). If player 2 wins contest one, then total expected effort in contest two is $\frac{bv}{a_2}$, by Eq. (A.2). Thus, total expected efforts across the two contests when contest one is balanced is

$$\frac{1}{2}\left(\frac{a_1 b - v}{a_1 b} + \frac{a_2 - bv}{a_2} + \frac{v}{a_1 b} + \frac{bv}{a_2}\right) = 1.$$

This is obviously the best the principal can obtain, so the principal's choice is to get contest one balanced by putting $v^* = \widehat{v}$; this proves part (i) of the Proposition.

Consider next part (ii). Now, $\widehat{v} > 1$. This means that it can never be the case that player 2 is the stronger player in contest 1, since this would require $v \geq \widehat{v}$. Thus, in the present case, we have $0 \leq v < \widehat{v}$, so that $bV^A_{1,1} > V^A_{2,1}$, with player 1 being the strong player in contest 1. By Eqs. (4), (A.3), and (A.4), total expected effort in contest 1 is now

$$\frac{V^A_{2,1}\left(V^A_{1,1} + V^A_{2,1}\right)}{2bV^A_{1,1}} = \frac{1}{2}\left(1 - \frac{bv}{a_2}\right)\left(\frac{1}{b} + \frac{a_1(a_2 - bv)}{a_2(a_1 b - v)}\right),$$

while the win probability of the weak player 2 in contest 1, by equation (3), is

$$\frac{V^A_{2,1}}{2bV^A_{1,1}} = \frac{a_1(a_2 - bv)}{2a_2(a_1 b - v)}.$$

By again using Eqs. (A.1) and (A.2), we have that total expected effort over the two contests is

$$\begin{aligned} &\frac{1}{2}\left(1 - \frac{bv}{a_2}\right)\left(\frac{1}{b} + \frac{a_1(a_2 - bv)}{a_2(a_1 b - v)}\right) + \left(1 - \frac{a_1(a_2 - bv)}{2a_2(a_1 b - v)}\right)\frac{v}{a_1 b} + \frac{a_1(a_2 - bv)}{2a_2(a_1 b - v)}\frac{bv}{a_2} \\ &= \frac{v(a_2 - a_1 b) + a_1 a_2}{a_1 a_2 b}. \end{aligned} \tag{A.6}$$

This is linear in v and depends upon the sign of $a_2 - a_1 b$. When $b > \frac{a_2}{a_1}$,it is optimal to set $v = 0$, with a total expected effort of $\frac{1}{b}$; together with the analysis of Case B above, this proves part (iv) of the Proposition. Here in part (ii), $b < \frac{a_2}{a_1}$, and the expression is increasing in v, which should be set at the top of its range, *i.e.*, at $v^* = 1$, for a total expected effort of $\frac{1}{b} + \frac{a_2 - ba_1}{a_1 a_2 b} > \frac{1}{b}$. In the knife-edge case of $b = \frac{a_2}{a_1}$ in part (iii), the principal is indifferent, since any $v \in [0, 1]$ makes contest one balanced. □

Proof of Proposition 2. Let the history of the game up until contest $t \in \{2, \ldots, T\}$ be summarized by (m_t, n_t), where m_t (n_t) is the number of wins so far by player 1 (2), and $m_t + n_t = t - 1$. We extend the notation from Section 3 to let the payoff to player i in contest t be given by $\pi^*_{i,t}(m_t, n_t)$.

With this notation, we can express the requirement of a balanced first contest. From Section 2, we have that a contest is balanced when $\alpha_s v_s = \alpha_w v_w$. Player 1 is the strong player in contest 1, since he has an *ex-ante* bias. We have $\alpha_s = b$, $\alpha_w = 1$, $v_s = v_1 + \pi^*_{1,2}(1, 0)$, and $v_w = v_1 + \pi^*_{2,2}(0, 1)$. Thus, contest 1 is balanced when $b\big(v_1 + \pi^*_{1,2}(1, 0)\big) = v_1 + \pi^*_{2,2}(0, 1)$, or:

$$v_1(b - 1) = \pi^*_{2,2}(0, 1) - b\pi^*_{1,2}(1, 0). \tag{A.7}$$

Consider a contest $t \in \{2, \ldots, T\}$ where player 1 is strong whenever $m_t > n_t$, whilst player 2 is strong whenever $m_t \leq n_t$.[20] If $m_t > n_t + 1$, then player 2 only fights for the stage prize at t, and not for the continuation value of being the leader after

[20] Analysis of the two-contest case has shown that full dissipation is achievable only if the initial laggard can catch up to a sufficient degree. The assumption made here ensures this. Restricting attention to this case is not limiting. To see this, note that we could have considered the case where player 1 is strong in contest t if and only if $m_t > n_t + 1$, the case where player 1 is strong in contest t if and only if $m_t > n_t + 2$, and so on, as far as it is appropriate. All these cases would involve the calculations we present below. The important point for the result is that player 2 is strong if each have won equally many times; such catching up is necessary for the principal to be able to balance the first contest.

contest t, since player 1 will be strong in contest $t+1$ whatever the outcome in contest t. Hence, following the analysis in Section 2, we have

$$\begin{aligned}\pi^*_{1,t}(m_t,n_t)|_{m_t>n_t+1} &= v_t + \pi^*_{1,t+1}(m_t+1,n_t)|_{m_t>n_t+1} - \frac{a_2^{n_t}}{a_1^{m_t}b}v_t \\ &= v_t\left(\frac{a_1^{m_t}b - a_2^{n_t}}{a_1^{m_t}b}\right) + \pi^*_{1,t+1}(m_t+1,n_t)|_{m_t>n_t+1}.\end{aligned}$$

In turn, working out $\pi^*_{i,t+1}(m_t+1,n_t)|_{m_t>n_t+1}$ reveals that player 1 is strong at each continuation, and we can by backward recursion find

$$\pi^*_{1,t}(m_t,n_t)|_{m_t>n_t+1} = \sum_{k=t}^{T} v_k\left(\frac{a_1^{m_t+k-t}b - a_2^{n_t}}{a_1^{m_t+k-t}b}\right). \tag{A.8}$$

Similarly, if $m_t < n_t$, then player 2 is strong in contest $t+1$, whatever the outcome of contest t. By the same reasoning as above,

$$\pi^*_{2,t}(m_t,n_t)|_{m_t<n_t} = \sum_{k=t}^{T} v_k\left(\frac{a_2^{n_t+k-t} - a_1^{m_t}b}{a_2^{n_t+k-t}}\right). \tag{A.9}$$

Hence, we can write the value for player 2 of winning contest 1, with $t=2$, $n_2=1$, and $m_2=0$, as

$$\pi^*_{2,2}(0,1) = \sum_{k=2}^{T} v_k\left(\frac{a_2^{k-1} - b}{a_2^{k-1}}\right). \tag{A.10}$$

Consider next the value for player 1 of winning contest 1, $\pi^*_{1,2}(1,0)$, the determination of which follows

$$\pi^*_{1,2}(1,0) = v_2 + \pi^*_{1,3}(2,0) - \frac{1}{a_1 b}\big(v_2 + \pi^*_{2,3}(1,1)\big),$$

where, by (A.8), we can write

$$\pi^*_{1,3}(2,0) = \sum_{k=3}^{T} v_k\left(\frac{a_1^{k-1}b - 1}{a_1^{k-1}b}\right).$$

Thus,

$$\pi^*_{1,2}(1,0) = \sum_{k=2}^{T} v_k\left(\frac{a_1^{k-1}b - 1}{a_1^{k-1}b}\right) - \frac{1}{a_1 b}\pi^*_{2,3}(1,1). \tag{A.11}$$

The challenge is now to find an expression for $\pi^*_{2,3}(1,1)$, since the continuation value here is positive for the player who wins contest 2. We write

$$\pi^*_{2,3}(1,1) = v_3 + \pi^*_{2,4}(1,2) - \frac{a_1 b}{a_2}\big(v_3 + \pi^*_{1,4}(2,1)\big),$$

where, by (A.9),

$$\pi^*_{2,4}(1,2) = \sum_{k=4}^{T} v_k\left(\frac{a_2^{k-2} - a_1 b}{a_2^{k-2}}\right).$$

Hence,

$$\pi^*_{2,3}(1,1) = \sum_{k=3}^{T} v_k\left(\frac{a_2^{k-2} - a_1 b}{a_2^{k-2}}\right) - \frac{a_1 b}{a_2}\pi^*_{1,4}(2,1).$$

Continuing, we have

$$\begin{aligned}\pi^*_{1,4}(2,1) &= v_4 + \pi^*_{1,5}(3,1) - \frac{a_2}{a_1^2 b}\big(v_4 + \pi^*_{2,5}(2,2)\big) \\ &= \sum_{k=4}^{T} v_k\left(\frac{a_1^{k-2}b - a_2}{a_1^{k-2}b}\right) - \frac{a_2}{a_1^2 b}\pi^*_{2,5}(2,2),\end{aligned}$$

where

$$\begin{aligned}\pi^*_{2,5}(2,2) &= v_5 + \pi^*_{2,6}(2,3) - \frac{a_1^2 b}{a_2^2}\big(v_5 + \pi^*_{1,6}(3,2)\big)\\ &= \sum_{k=5}^{T} v_k\left(\frac{a_2^{k-3} - a_1^2 b}{a_2^{k-3}}\right) - \frac{a_1^2 b}{a_2^2}\pi^*_{1,6}(3,2),\end{aligned}$$

$$\begin{aligned}\pi^*_{1,6}(3,2) &= v_6 + \pi^*_{1,7}(4,2) - \frac{a_2^2}{a_1^3 b}\big(v_6 + \pi^*_{2,7}(3,3)\big)\\ &= \sum_{k=6}^{T} v_k\left(\frac{a_1^{k-3} b - a_2^2}{a_1^{k-3} b}\right) - \frac{a_2^2}{a_1^3 b}\pi^*_{2,7}(3,3),\end{aligned}$$

and

$$\begin{aligned}\pi^*_{2,7}(3,3) &= v_7 + \pi^*_{2,8}(3,4) - \frac{a_1^3 b}{a_2^3}\big(v_7 + \pi^*_{1,8}(4,3)\big)\\ &= \sum_{k=7}^{T} v_k\left(\frac{a_2^{k-4} - a_1^3 b}{a_2^{k-4}}\right) - \frac{a_1^3 b}{a_2^3}\pi^*_{1,8}(4,3).\end{aligned}$$

Substituting in these expressions, we have

$$\begin{aligned}\pi^*_{2,3}(1,1) &= \sum_{k=3}^{T} v_k\left(\frac{a_2^{k-2} - a_1 b}{a_2^{k-2}}\right) - \frac{a_1 b}{a_2}\pi^*_{1,4}(2,1)\\
&= \sum_{k=3}^{T} v_k\left(\frac{a_2^{k-2} - a_1 b}{a_2^{k-2}}\right) - \frac{a_1 b}{a_2}\left(\sum_{k=4}^{T} v_k\left(\frac{a_1^{k-2} b - a_2}{a_1^{k-2} b}\right) - \frac{a_2}{a_1^2 b}\pi^*_{2,5}(2,2)\right)\\
&= \sum_{k=3}^{T} v_k\left(\frac{a_2^{k-2} - a_1 b}{a_2^{k-2}}\right) - \frac{a_1 b}{a_2}\sum_{k=4}^{T} v_k\left(\frac{a_1^{k-2} b - a_2}{a_1^{k-2} b}\right) + \frac{a_1 b}{a_2}\frac{a_2}{a_1^2 b}\pi^*_{2,5}(2,2)\\
&= \sum_{k=3}^{T} v_k\left(\frac{a_2^{k-2} - a_1 b}{a_2^{k-2}}\right) - \frac{a_1 b}{a_2}\sum_{k=4}^{T} v_k\left(\frac{a_1^{k-2} b - a_2}{a_1^{k-2} b}\right)\\
&\quad + \frac{a_1 b}{a_2}\frac{a_2}{a_1^2 b}\left(\sum_{k=5}^{T} v_k\left(\frac{a_2^{k-3} - a_1^2 b}{a_2^{k-3}}\right) - \frac{a_1^2 b}{a_2^2}\pi^*_{1,6}(3,2)\right)\\
&= \sum_{k=3}^{T} v_k\left(\frac{a_2^{k-2} - a_1 b}{a_2^{k-2}}\right) - \frac{a_1 b}{a_2}\sum_{k=4}^{T} v_k\left(\frac{a_1^{k-2} b - a_2}{a_1^{k-2} b}\right)\\
&\quad + \frac{a_1 b}{a_2}\frac{a_2}{a_1^2 b}\sum_{k=5}^{T} v_k\left(\frac{a_2^{k-3} - a_1^2 b}{a_2^{k-3}}\right) - \frac{a_1 b}{a_2}\frac{a_2}{a_1^2 b}\frac{a_1^2 b}{a_2^2}\pi^*_{1,6}(3,2)\\
&= \sum_{k=3}^{T} v_k\left(\frac{a_2^{k-2} - a_1 b}{a_2^{k-2}}\right) - \frac{a_1 b}{a_2}\sum_{k=4}^{T} v_k\left(\frac{a_1^{k-2} b - a_2}{a_1^{k-2} b}\right)\\
&\quad + \frac{a_1 b}{a_2}\frac{a_2}{a_1^2 b}\sum_{k=5}^{T} v_k\left(\frac{a_2^{k-3} - a_1^2 b}{a_2^{k-3}}\right) - \frac{a_1 b}{a_2}\frac{a_2}{a_1^2 b}\frac{a_1^2 b}{a_2^2}\sum_{k=6}^{T} v_k\left(\frac{a_1^{k-3} b - a_2^2}{a_1^{k-3} b}\right)\\
&\quad + \frac{a_1 b}{a_2}\frac{a_2}{a_1^2 b}\frac{a_1^2 b}{a_2^2}\frac{a_2^2}{a_1^3 b}\pi^*_{2,7}(3,3)\\
&= \sum_{k=3}^{T} v_k\left(\frac{a_2^{k-2} - a_1 b}{a_2^{k-2}}\right) - \frac{a_1 b}{a_2}\sum_{k=4}^{T} v_k\left(\frac{a_1^{k-2} b - a_2}{a_1^{k-2} b}\right)\\
&\quad + \frac{1}{a_1}\sum_{k=5}^{T} v_k\left(\frac{a_2^{k-3} - a_1^2 b}{a_2^{k-3}}\right) - \frac{a_1 b}{a_2^2}\sum_{k=6}^{T} v_k\left(\frac{a_1^{k-3} b - a_2^2}{a_1^{k-3} b}\right)\\
&\quad + \frac{1}{a_1^2}\left(\sum_{k=7}^{T} v_k\left(\frac{a_2^{k-4} - a_1^3 b}{a_2^{k-4}}\right) - \frac{a_1^3 b}{a_2^3}\pi^*_{1,8}(4,3)\right),\end{aligned}$$

and, eventually,

$$\pi^*_{2,3}(1,1) = \sum_{k=3}^{T} v_k\left(\frac{a_2^{k-2} - a_1 b}{a_2^{k-2}}\right) - \frac{a_1 b}{a_2}\sum_{k=4}^{T} v_k\left(\frac{a_1^{k-2}b - a_2}{a_1^{k-2}b}\right) + \frac{1}{a_1}\sum_{k=5}^{T} v_k\left(\frac{a_2^{k-3} - a_1^2 b}{a_2^{k-3}}\right)$$
$$- \frac{a_1 b}{a_2^2}\sum_{k=6}^{T} v_k\left(\frac{a_1^{k-3}b - a_2^2}{a_1^{k-3}b}\right) + \frac{1}{a_1^2}\sum_{k=7}^{T} v_k\left(\frac{a_2^{k-4} - a_1^3 b}{a_2^{k-4}}\right) - \frac{a_1 b}{a_2^3}\pi^*_{1,8}(4,3). \tag{A.12}$$

The process continues until we reach T.

Inserting from Eqs. (A.10), (A.11), and (A.12), we can write the right-hand side of (A.7) as

$$\pi^*_{2,2}(0,1) - b\pi^*_{1,2}(1,0) = \sum_{k=2}^{T} v_k\left(\frac{a_2^{k-1} - b}{a_2^{k-1}}\right)$$
$$- b\left(\sum_{k=2}^{T} v_k\left(\frac{a_1^{k-1}b - 1}{a_1^{k-1}b}\right) - \frac{1}{a_1 b}\sum_{k=3}^{T} v_k\left(\frac{a_2^{k-2} - a_1 b}{a_2^{k-2}}\right) + \frac{1}{a_2}\sum_{k=4}^{T} v_k\left(\frac{a_1^{k-2}b - a_2}{a_1^{k-2}b}\right)\right.$$
$$\left. - \frac{1}{a_1^2 b}\sum_{k=5}^{T} v_k\left(\frac{a_2^{k-3} - a_1^2 b}{a_2^{k-3}}\right) + \frac{1}{a_2^2}\sum_{k=6}^{T} v_k\left(\frac{a_1^{k-3}b - a_2^2}{a_1^{k-3}b}\right) - \frac{1}{a_1^3 b}\sum_{k=7}^{T} v_k\left(\frac{a_2^{k-4} - a_1^3 b}{a_2^{k-4}}\right) + \dots\right),$$

or, collecting terms,

$$\pi^*_{2,2}(0,1) - b\pi^*_{1,2}(1,0) =$$
$$v_2\left(\frac{a_2 - b}{a_2} - \frac{a_1 b - 1}{a_1}\right) + v_3\left(\frac{a_2^2 - b}{a_2^2} - \frac{a_1^2 b - 1}{a_1^2} + \frac{1}{a_1}\left(\frac{a_2 - a_1 b}{a_2}\right)\right)$$
$$+ v_4\left(\frac{a_2^3 - b}{a_2^3} - \frac{a_1^3 b - 1}{a_1^3} + \frac{1}{a_1}\left(\frac{a_2^2 - a_1 b}{a_2^2}\right) - \frac{1}{a_2}\frac{a_1^2 b - a_2}{a_1^2}\right) +$$
$$v_5\left(\frac{a_2^4 - b}{a_2^4} - \frac{a_1^4 b - 1}{a_1^4} + \frac{1}{a_1}\left(\frac{a_2^3 - a_1 b}{a_2^3}\right) - \frac{1}{a_2}\frac{a_1^3 b - a_2}{a_1^3} + \frac{1}{a_1^2}\left(\frac{a_2^2 - a_1^2 b}{a_2^2}\right)\right) + \dots$$

This expression can be further simplified so that it becomes

$$\pi^*_{2,2}(0,1) - b\pi^*_{1,2}(1,0) = \sum_{t=2}^{T}\phi_t v_t, \text{ where}$$
$$\phi_t = \sum_{k=1}^{t}\left(\frac{1}{a_1^{k-1}} - \frac{b}{a_2^{k-1}}\right) = \frac{a_1^t - 1}{a_1^{t-1}(a_1 - 1)} - b\frac{\left(a_2^t - 1\right)}{a_2^{t-1}(a_2 - 1)}, t = 2, \dots, T$$

Combining this with (A.7) gives the result.

To show that a balanced contest yields full rent dissipation involves computing the expected efforts at each node in the game tree, and multiplying by the probability that the node is reached. Denote by $X_2(1, 0)$ and $X_2(0, 1)$ the total expected efforts from nodes (1, 0) and (0, 1). Rounds of recursion from the final contest and backwards yields the following pattern:

$$\pi^*_{1,2}(1,0) = \sum_{t=2}^{T} v_t - X_2(1,0)$$
$$\pi^*_{2,2}(0,1) = \sum_{t=2}^{T} v_t - X_2(0,1)$$

The value for each player of winning the first contest is $V_{1,1}(0,0) = v_1 + \pi^*_{1,2}(1,0)$ and $V_{2,1}(0,0) = v_1 + \pi^*_{2,2}(0,1)$. From (4) this gives a total expected effort in contest one of

$$X_1 = \frac{v_1 + \pi^*_{2,2}(0,1)}{b\left(v_1 + \pi^*_{1,2}(1,0)\right)}\left(\frac{2v_1 + \pi^*_{1,2}(1,0) + \pi^*_{2,2}(0,1)}{2}\right).$$

When the first contest is balanced, $v_1 + \pi^*_{2,2}(0,1) = b\left(v_1 + \pi^*_{1,2}(1,0)\right)$, and the probability of each player winning is $\frac{1}{2}$ (*i.e.* nodes (1, 0) and (0, 1) are reached with equal probability). Using balance, the total expected effort X is

$$X = X_1 + \frac{1}{2}(X_2(1,0) + X_2(0,1))$$
$$= \left(\frac{2v_1 + \pi^*_{1,2}(1,0) + \pi^*_{2,2}(0,1)}{2}\right) + \frac{1}{2}\left(\sum_{t=2}^{T} v_t - \pi^*_{1,2}(1,0) + \sum_{t=2}^{T} v_t - \pi^*_{2,2}(0,1)\right)$$
$$= \sum_{t=1}^{T} v_t = 1.$$

□

Proof of Corollary 1. Achieving full prize dissipation is critically dependent on being able to find a $\phi_T > 0$ for all $b \in \left(0, \frac{a_2}{a_1}\right)$. This requires $\theta_T > b$. Note, from (10), that $\frac{\partial \phi_t}{\partial t} > 0$, so that the value of θ_T increases in T, reaching in the limit $\theta_{T\to\infty} = \frac{a_1(a_2-1)}{a_2(a_1-1)}$. If $\frac{a_1(a_2-1)}{a_2(a_1-1)} \geq \frac{a_2}{a_1}$ (i.e. $\frac{a_1}{a_1-1} > a_2$), then T can be found such that $\theta_T > b \in \left(0, \frac{a_2}{a_1}\right)$, and $\phi_T > 0$, ensuring full dissipation. If $\frac{a_1(a_2-1)}{a_2(a_1-1)} < \frac{a_2}{a_1}$, then there is a range of $b \in \left(\frac{a_2(1+a_1)}{a_1(1+a_2)}, \frac{a_2}{a_1}\right)$ for which there is no T such that $\theta_T > b$, and no possibility of achieving $\phi_T > 0$ which is required for balance and full rent dissipation. □

Proof of Proposition 3. The participation constraints for the contestants in contest one are $V_{i,1} \geq 0, i = 1, 2,$ where $V_{1,1}$ and $V_{2,1}$ are given in Eqs. (A.3) and (A.4), respectively.

Part (i) follows from Proposition 1 and the observation that, in this case, balance requires $bV_{1,1} = V_{2,1}$, implying that $V_{1,1} \geq 0$ is the binding constraint; this is satisfied if and only if $b \leq \sqrt{\frac{a_2}{a_1}}$.

For parts (ii)-(iv), note that $b > \sqrt{\frac{a_2}{a_1}}$ if and only if $a_1 b > \frac{a_2}{b}$, so that $V_{2,1} \geq 0$ is the only binding constraint of the two. Total effort if $v \leq \frac{a_2}{b}$ is given by Eq. (A.6) and is increasing in v; hence v should be set as high as possible, i.e., at $\frac{a_2}{b}$. At $v = \frac{a_2}{b}$, both players take part in contest one without exerting effort and player 1 wins the negative first-contest prize and obtains the boost ba_1 before contest two.[21] Total effort in contest two, and therefore total effort overall, is $\frac{v}{a_1 b} = \frac{a_2}{a_1 b^2}$. This is valid as b increases until we reach the point at which $\frac{a_2}{b^2 a_1} = \frac{1}{b}$, where the latter is total expected effort from a single contest with a prize of size 1. This occurs for $b = \frac{a_2}{a_1}$; this is part (ii). For this value, $b = \frac{a_2}{a_1}$, total effort in (A.6) is independent of v, and hence a range of prizes achieves the maximum effort; this is part (iii). As b increases further, the principal can do no better than run a single contest, which is part (iv). □

Proof of Proposition 4. Given the suggested prize structure, balance in contest one requires $v_1(b-1) = \phi_3 v_3$ as expressed in Proposition 2. Balance also implies that player 1 has a lower expected payoff as seen from contest one, so that the participation constraint of this player is

$$v_1 + \pi^*_{1,2}(1,0) \geq 0, \quad \text{where} \tag{A.13}$$

$$\pi^*_{1,2}(1,0) = \frac{v_3}{b}\left(1 - \frac{b}{a_2^2} - \phi_3\right)$$

The expression for the continuation payoff for 1 having won the first contest is recovered from the proof of Proposition 2. Inserting the condition for balance and the definition of ϕ_3 into (A.13) gives the condition

$$v_3(a_2 - a_1 b)\frac{a_2 + a_1 a_2 + a_1 b}{a_1^2 a_2^2 b(b-1)} > 0,$$

which is satisfied for the region that we are considering: $\frac{a_2}{a_1} > b$. The equations for competitive balance and budget balance then give the prizes as

$$v_1 = \frac{a_2^2\left(a_1 + 1 + a_1^2\right) - a_1^2\left(a_2 + 1 + a_2^2\right)b}{a_2^2(a_1+1) - a_1^2(a_2+1)b}$$

$$v_3 = \frac{a_1^2 a_2^2(b-1)}{a_2^2(a_1+1) - a_1^2(a_2+1)b}$$

We have that $v_1 < 0$ and $v_3 > 0$ for $\frac{a_2^2(a_1+1)}{a_1^2(a_2+1)} > b > \frac{a_2^2(a_1+1+a_1^2)}{a_1^2(a_2+1+a_2^2)}$, where the right-hand-side represents the maximum level of b for which balance can be achieved with non-negative prizes. This new prize structure with a negative first-contest prize and a positive third-contest one achieves balance for b up to $\min\left(\frac{a_2}{a_1}, \frac{a_2^2(a_1+1)}{a_1^2(a_2+1)}\right)$. The result in the proposition follows since $\frac{a_2^2(a_1+1)}{a_1^2(a_2+1)} > \frac{a_2}{a_1}$ for $a_2 > a_1$. □

Proof of Proposition 5. Expected efforts and payoffs in contest two, following a win by player 1, are, by the discussion in Section 2,

$$x^*_{1,2}(1) + x^*_{2,2}(1) = \frac{1-v_1}{bf_1(v_1)};$$

$$\pi^*_{1,2}(1) = \frac{bf_1(v_1) - 1}{bf_1(v_1)}(1 - v_1);$$

$$\pi^*_{2,2}(1) = 0.$$

[21] This statement requires resolution of a technicality with respect to maximization over an open set, for which we make the following assumption: When the weaker player is indifferent between (i) taking part in a contest and earning zero in expectation and (ii) not taking part and earning zero for sure, he always takes part.

Following a win by player 2 there are two cases, but we focus on the one in which player 2 is strong in contest two having won the first, $b < f_2(v_1)$. Expected effort and payoffs are then

$$\begin{aligned} x^*_{1,2}(2) + x^*_{2,2}(2) &= \frac{b(1-v_1)}{f_2(v_1)}; \\ \pi^*_{1,2}(2) &= 0; \\ \pi^*_{2,2}(2) &= \frac{f_2(v_1) - b}{f_2(v_1)}(1-v_1). \end{aligned}$$

The payoff functions in contest one can be written as

$$\begin{aligned} \pi_{1,1} &= \rho_{1,1}\left(v_1 + \pi^*_{1,2}(1)\right) - x_{1,1}; \\ \pi_{2,1} &= \rho_{2,1}\left(v_1 + \pi^*_{2,2}(2)\right) - x_{2,1}. \end{aligned}$$

The key to balancing contest one is then $b(v_1 + \pi^*_{1,2}(1)) = v_1 + \pi^*_{2,2}(2)$, which is (15). The probability of each player winning contest one is now $\frac{1}{2}$, so that total expected effort is

$$\begin{aligned} X^* = \frac{1}{2}\left[v_1 + \pi^*_{1,2}(1)\right] + \frac{1}{2}\left[v_1 + \pi^*_{2,2}(2)\right] + \frac{1}{2}\left[x^*_{1,2}(1) + x^*_{2,2}(1)\right] + \frac{1}{2}\left[x^*_{1,2}(2) + x^*_{2,2}(2)\right] \\ = \frac{1}{2}\left[v_1 + \frac{bf_1(v_1) - 1}{bf_1(v_1)}(1-v_1) + v_1 + \frac{f_2(v_1) - b}{f_2(v_1)}(1-v_1)\right] \\ + \frac{1}{2}\left[\frac{1-v_1}{bf_1(v_1)} + \frac{b(1-v_1)}{f_2(v_1)}\right] = 1. \end{aligned}$$

□

Proof of Corollary 4. With the suggested functional form for $f_i(v_1)$, (15) becomes

$$\frac{b(a_1 + \delta v_1) - (1-v_1)}{a_1 + \delta v_1} = \frac{a_2 + \delta v_1 - b(1-v_1)}{a_2 + \delta v_1},$$

which is solved for a $v_1 \in [0, 1]$ that satisfies

$$\begin{aligned} 0 = \omega &:= \delta(\delta - 1)(b-1)v_1^2 \\ &+ [(a_1 + a_2 + 1)(b-1)\delta + (a_2 - a_1 b)]v_1 \\ &+ b(a_1 + a_1 a_2) - a_2 - a_1 a_2. \end{aligned}$$

Retaining the assumption that $a_2 > a_1 b$, we see that ω is a convex function of v_1 for $\delta > 1$, concave for $\delta < 1$, and linear for $\delta = 1$. The graph of ω has positive slope and negative value at $v_1 = 0$; the value at $v_1 = 0$ is independent of δ. It can readily be determined that $\frac{\partial \omega}{\partial \delta} > 0$ for $v_1 > 0$; this implies also that the root of ω is decreasing in δ, *i.e.*, $\frac{\partial v_1}{\partial \delta} < 0$. It is straightforward to determine the solutions for v_1 when $\delta = 0$ and $\delta = 1$. ω has a single positive root for $\delta > 1$, since it now is convex. Consider next the case of $\delta < 1$. Only roots below 1 are valid solutions. The value of ω at $v_1 = 1$ is $(\delta + a_1)(\delta + a_2)(b-1) > 0$. Since $\omega < 0$ at $v_1 = 0$, and positive at $v_1 = 1$, the roots are real. To see that only one of the roots is below 1, consider the slope of ω measured at $v_1 = 1$: $\frac{\partial \omega}{\partial v_1}\big|_{v_1 = 1} = 2(b-1)\delta^2 + (a_1 + a_2 - 1)(b-1)\delta + (a_2 - a_1 b) > 0$, for all δ. It follows that ω is positive-valued and increasing at $v_1 = 1$, so that the larger root must be above 1. The lower root, which coincides with the single positive root for $\delta > 1$, is given in the Corollary. □

Properties of $v_1^*(\delta)$

In the text after Corollary 4, we make two claims about the comparative statics of $v_1^*(\delta)$. Both claims follow from the above proof of the Corollary. (i) That $v_1^*(\delta)$ is falling in δ follows from $\frac{\partial \omega}{\partial \delta} > 0$, where ω is defined in the proof. (ii) To show that $\frac{\partial v_1^*(\delta)}{\partial b} < 0$, note that $\frac{\partial^2 \omega}{\partial \delta \partial b} > 0$ for all values of δ. Since $\frac{\partial \omega}{\partial b} > 0$ at $\delta = 0$, it is positive for all δ. This implies that the root of ω falls as b increases (up until the point at which b is so large that $v_1^*(\delta) < 0$ in which case balance can no longer be achieved in the first contest).

Supplementary material

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.ijindorg.2019.102578

References

Adamczyk, S., Bullinger, A.C., Möslein, K.M., 2012. Innovation contests: a review, classification and outlook. Creativ. Innovat. Manag. 21, 335–360.
Beviá, C., Corchón, L.C., 2013. Endogenous strength in conflicts. Int. J. Ind. Org. 31, 297–306.
Chase, I.D., Bartolomeo, C., Dugatkin, L.A., 1994. Aggressive interactions and inter-contest interval: how long do winners keep winning? Anim. Behav. 48, 393–400.

Chowdhury, S.M., Esteve-González, P., Mukherjee, A., 2019. Heterogeneity, leveling the playing field, and affirmative action in contests. Unpublished manuscript.
Clark, D.J., Nilssen, T., 2018. Keep on fighting: the dynamics of head starts in all-pay auctions. Game Econ. Behav. 110, 258–272.
Clark, D.J., Nilssen, T., 2019. Competitive balance when winning breeds winners. Unpublished manuscript.
Clark, D.J., Nilssen, T., Sand, J.Y., 2019. Gaining advantage by winning contests. Unpublished manuscript.
Cohen-Zada, D., Krumer, A., Shtudiner, Z., 2017. Psychological momentum and gender. J. Econ. Behav. Org. 135, 66–81.
Epstein, G.S., Mealem, Y., Nitzan, S., 2011. Political culture and discrimination in contests. J. Publ. Econ. 95, 88–93.
Farrell, S., Hakstian, A.R., 2001. Improving salesforce performance: a meta-analytic investigation of the effectiveness and utility of personnel selection procedures and training interventions. Psychol. Market. 18, 281–316.
Feng, X., Lu, J., 2018. How to split the pie: optimal rewards in dynamic multi-battle competitions. J. Publ. Econ. 160, 82–95.
Franke, J., Leininger, W., Wasser, C., 2018. Optimal favoritism in all-pay auctions and lottery contests. Eur. Econ. Rev. 104, 22–37.
Fu, Q., Lu, J., 2012. The optimal multi-stage contest. Econ. Theory 51, 351–382.
Fudenberg, D., Gilbert, R., Stiglitz, J., Tirole, J., 1983. Preemption, leapfrogging and competition in patent races. Eur. Econ. Rev. 22, 3–31.
Gallini, N., Scotchmer, S., 2002. Intellectual property: when is it the best incentive system? Innovat. Policy Econ. 2, 51–77.
Gauriot, R., Page, L., 2018. Does success breed success? A quasi-experiment on strategic momentum in dynamic contests. Econ. J. Forthcoming.
Gillis, W.E., McEwan, E., Cook, T.R., Michael, S.C., 2011. Using tournaments to reduce agency problems: the case of franchising. Entrepreneur. Theory Pract. 35, 427–447.
Höber, B., 2017. Firm Internal Innovation Contests: Work Environment Perceptions and Employees' Participation. Springer.
Iso-Ahola, S.E., Dotson, C.O., 2014. Psychological momentum: why success breeds success. Rev. Gen. Psychol. 18, 19–33.
Klumpp, T., Polborn, M.K., 2006. Primaries and the New Hampshire effect. J. Publ. Econ. 90, 1073–1114.
Konrad, K.A., 2012. Dynamic contests and the discouragement effect. Revue d'Économie Politique 122, 233–256.
Konrad, K.A., Kovenock, D., 2009. Multi-battle contests. Game Econ. Behav. 66, 256–274.
Konrad, K.A., Kovenock, D., 2010. Contests with stochastic abilities. Econ. Inquir. 48, 89–103.
Krishnamoorthy, A., Misra, S., Prasad, A., 2005. Scheduling sales force training: theory and evidence. Int. J. Res. Market. 22, 427–440.
Li, S., Yu, J., 2012. Contests with endogenous discrimination. Econ. Lett. 117, 834–836.
Luo, Z., Xie, X., 2018. A model of rivalries with endogenous prize and strength. J. Econ. Behav. Org. 152, 215–223.
Mealem, Y., Nitzan, S., 2016. Discrimination in contests: a survey. Rev. Econ. Des. 20, 145–172.
Megidish, R., Sela, A., 2014. Sequential contests with synergy and budget constraints. Soc.Choice Welfare 42, 215–243.
Merton, R.K., 1968. The Matthew effect in science. Science 159, 56–63.
Moldovanu, B., Sela, A., 2006. Contest architecture. J. Econ. Theory 126, 70–96.
Möller, M., 2012. Incentives *versus* competitive balance. Econ. Lett. 117, 505–508.
Rigney, D., 2010. The Matthew Effect: How Advantage Begets Further Advantage. Columbia University Press.
de Roos, N., Sarafidis, Y., 2018. Momentum in dynamic contests. Econ. Modell. 70, 403–416.
Sela, A., 2011. Best-of-three all-pay auctions. Econ. Lett. 112, 67–70.
Skiera, B., Albers, S., 1998. COSTA: Contribution optimizing sales territory alignment. Market. Sci. 17, 196–213.
Strumpf, K.S., 2002. Strategic competition in sequential election contests. Publ. Choice 111, 377–397.
Szymanski, S., 2003. The economic design of sporting contests. J. Econ. Literat. 41, 1137–1187.
Terwiesch, C., Xu, Y., 2008. Innovation contests, open innovation, and multiagent problem solving. Manag. Sci. 54, 1529–1543.
Witt, J.K., Proffitt, D.R., 2005. See the ball, hit the ball. Psychol. Sci. 16, 937–938.