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Joint Invariants of Symplectic and Contact Lie Algebra Actions

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Abstract

By restricting generating functions of infinitesimal symmetries of symplectic and contact vector spaces to quadratic forms, we obtain a finite-dimensional Lie subalgebra $\mathfrak{g}$, consisting of vector fields isomorphic to the linear symplectic or conformal symplectic algebra. This allows us to look for joint invariants of the diagonal action of $\mathfrak{g}$ on product manifolds $M^{\times m}$. We find an explicit recipe for creating a transcendence basis for the field of m-fold rational joint invariants over $\mathbb{R}$, starting from a base space M of any dimension $n \geq 2$.

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1 Introduction

Our main objective is to obtain a complete description of the field of joint rational invariants of an extended Lie algebra action defined on symplectic and contact manifolds of varying dimensions. Along the way, we aim to illustrate how to describe the algebra of joint polynomial invariants using minimal free resolutions. We will also briefly consider how our results can be used to generate symmetric invariants. Before diving into the computations, we introduce the necessary tools and definitions needed to make sense of them.

Chapter 2 opens with a section on classical invariant theory, using binary forms to illustrate the key ideas. Those ideas will be used to motivate a more general notion of an invariant, to be described in the following section. We proceed to describe a small variety of computational strategies for finding invariants of a given group action, all of which will be put to use in the central part of the thesis. Finally, we consider spaces of invariants consisting of polynomials as well as of rational functions.

In Chapter 3 we start looking for joint invariants on symplectic manifolds. Most of the groundwork will be laid in the 2-dimensional case, where the limited complexity of the problem admits a direct approach using infinitesimal methods. As we obtain polynomial generators for our space of invariants, we start by describing it as an algebra for the first few product spaces. We then turn to the task of describing the field of invariants for arbitrarily large products. Our results will be of great use in the equivalent endeavor in 4 dimensions and beyond.

We will compute joint invariants on contact manifolds in Chapter 4. Here the infinitesimal approach fails, but we are able to find invariants by obtaining an explicit description of the group action. On contact manifolds, generators for the space of invariants are found to be rational functions. After finding a description for the field of joint invariants in 3 and 5 dimensions, we proceed to consider spaces of higher dimensions.

The thesis is concluded in chapter 5 with a few computations illustrating how to find symmetric joint invariants on products of manifolds of low dimensions. By averaging the action of the symmetric group on the space of invariants, we can find generators for the field and algebra of symmetric invariants. However this time the structure of the algebra of invariants, even the generating set, is quite complicated, and so we describe only some particular examples.

Having computed the joint invariants we can solve the equivalence problem for the finite collection of points (ordered or symmetric) with respect to our group $Sp(2n, \mathbb{R})$ or $CSp(2n, \mathbb{R})$. For this the method of joint invariant signatures can be applied. We refer to [Olv01] for details.

2 A brief overview of Invariant Theory

This chapter will provide an introduction to the main elements of invariant theory.

2.1 The classical perspective

Classical invariant theory is concerned with those properties of mathematical objects which are intrinsic to the objects themselves (i.e. not an artifact of the underlying coordinates in which they are represented). This section will outline this theory using homogeneous polynomials of two variables, also known as "binary forms" in the literature, as well as their inhomogeneous counterparts. For more on classical invariant theory, see [Olv99].

2.1.1 Homogeneous polynomials

Formally, a **homogeneous polynomial** $Q(x, y)$ in two variables is an expression of the following form:

$$Q(x,y) = \sum_{i=0}^{n} a_i x^i y^{n-i} = \sum_{i=0}^{n} \binom{n}{i} \tilde{a}_i x^i y^{n-i}, a_i \in \mathbb{R} \text{ (or } \mathbb{C})$$

It is natural to interpret the above as the local coordinate expression of some function $\mathcal{Q}$ on a 2-dimensional manifold M, given in the local coordinates (x, y). In principle, M could be any such manifold. However, we will require M to be either $\mathbb{R}^2$ or $\mathbb{C}^2$, and our coordinates (x, y) to be *global*.

Since we are interested in finding properties of homogeneous polynomials which do not depend on the coordinates of the space they are defined on, a good place to start is by examining how the expression for $\mathcal{Q}$ changes under a linear change of coordinates.

If we let $M = \mathbb{R}^2$, we can identify the points $(1, 0), (0, 1)$ with the vectors e_1, e_2, and in this sense regard our space as a *vector space* with basis (e_1, e_2). Under this identification, a point $(v^1, v^2) \in \mathbb{R}^2$ can be regarded as the vector $v = v^1 e_1 + v^2 e_2$. From this point of view, changing the coordinates of our space corresponds to changing to some new basis $(\bar{e}_1, \bar{e}_2)$.

Denoting the same point after a change of coordinates by $\bar{v} = \bar{v}^1 \bar{e}_1 + \bar{v}^2 \bar{e}_2$, we have the following relationship between the old and the new coordinates:

$$\begin{pmatrix} \bar{v}^1 \\ \bar{v}^2 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$$

This can be written more concisely as $\bar{v} = Av$, where A is the invertible matrix above. There is a known isomorphism between the space of homogeneous polynomials and the space of symmetric tensors:

$$\mathcal{P}^{(n)}(\mathbb{R}^2) \simeq \mathcal{S}^{(n)}(\mathbb{R}^2)^*$$

For the case where $n = 1$, we can regard $\mathcal{Q}$ as a member of $\mathcal{S}^{(1)}(\mathbb{R}^2)^* = (\mathbb{R}^2)^*$ with (x, y) being the dual basis of (e_1, e_2). Expanding in this basis, $Q(x, y) = a_0 x + a_1 y$ with the action on v given by $Q(v) := \langle Q, v \rangle = v^1 a_0 + v^2 a_1$.

Let $(\bar{x}, \bar{y})$ be the dual basis of $(\bar{e}_1, \bar{e}_2)$. Then we can also express $\mathcal{Q}$ in this basis as $\bar{Q}(\bar{x}, \bar{y}) = \bar{a}_0 \bar{x} + \bar{a}_1 \bar{y}$. We would like to find a relation between the old and new coefficients. Using the well known relationship between the components of a covector under a change of coordinates, we can quickly conclude that: $\bar{a} = A^{-T} a$. In matrix notation:

$$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \begin{pmatrix} \bar{a}_0 \\ \bar{a}_1 \end{pmatrix}$$

Generalizing to the case where $n = k$, we can regard $\mathcal{Q}$ as a member of $\mathcal{S}^k(\mathbb{R}^2)^*$, with the action on v given by $Q(v) := \langle Q, \underbrace{v \otimes ... \otimes v}_{k \text{ entries}} \rangle$

The relationship between the old and new coefficients then becomes:
$\bar{a} = \underbrace{A^{-T} \odot ... \odot A^{-T}}_{k \text{ entries}}(a)$

Notice that from this point of view, $\mathbb{R}^2$ as a *manifold* stays fixed under a change of coordinates. We are merely changing the labels assigned to each point. It is also possible to interpret the invertible matrix A as an automorphism of M. From this second perspective, the coordinates (x, y) and $(\bar{x}, \bar{y})$ are both the standard coordinate functions on $\mathbb{R}^2$. However, the points they are labeling are differently arranged. Here $\bar{Q}$ can be interpreted as a transformed version of Q, defined by the relation $\bar{Q}(\bar{x}, \bar{y}) = Q(x, y)$. Of course, the transformation properties of its coefficients will be exactly the same in either interpretation.

2.1.2 Inhomogeneous polynomials

We define an **inhomogeneous polynomial** $Q(p)$ of one variable as a formal expression of the the following form:

$$Q(p) = \sum_{i=0}^{n} b_i p^i$$

As was the case with homogeneous polynomials of two variables, we can interpret such an expressions as representing some function $\mathcal{Q}$ defined on a 1-dimensional manifold N in the coordinate p.

Here, we will take N to be the **projective line** $\mathbb{P}(\mathbb{R}^2)$, where points corresponds to linear subspaces of $\mathbb{R}^2$. To every homogeneous polynomial $Q(x, y) : \mathbb{R}^2 \to \mathbb{R}$, there is a corresponding function $\mathcal{Q} : \mathbb{P}(\mathbb{R}^2) \to \mathbb{R}$. In affine coordinates on $\mathbb{P}(\mathbb{R}^2)$, this function will be represented as an inhomogeneous polynomial.

Ex: $Q(x, y) = x^2 + 3xy + 2y^2$, let $x = p, y = 1$
$Q(p) = p^2 + 3p + 2$

As before, we are interested in how this expression for $Q(p)$ changes as we change to different coordinates $\bar{p}$.

A linear transformation: $\begin{cases} \bar{x} = \alpha x + \beta y \\ \bar{y} = \gamma x + \delta y \end{cases}$ of $\mathbb{R}^2$,

induces a *Möbius transformation* $\bar{p} = \dfrac{\alpha p + \beta}{\gamma p + \delta}$ of $\mathbb{P}(\mathbb{R}^2)$.

In the special case where $\bar{y} = y$, the linear transformation induces an *affine transformation.* Starting with the relationship between the homogeneous polynomials Q and $\bar{Q}$, we can deduce the relationship between the inhomogeneous polynomials after a change of coordinates:

$$\begin{aligned} Q(x, y) &= \bar{Q}(\bar{x}, \bar{y}) \\ y^n Q\left(\frac{x}{y}\right) &= \bar{y}^n \bar{Q}\left(\frac{\bar{x}}{\bar{y}}\right), \text{ let } x = p, y = 1 \\ Q(p) &= \bar{y}^n \bar{Q}\left(\frac{\bar{x}}{\bar{y}}\right) \\ Q(p) &= (\gamma p + \delta)^n \bar{Q}\left(\frac{\alpha p + \beta}{\gamma p + \delta}\right) \end{aligned}$$

This relationship warrants a couple of remarks:

(i) The "naive degree" of $Q(p)$ is not always preserved under Möbius transformations.

Ex: $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \implies \bar{p} = \dfrac{1}{p+1}$

$$Q(p) = p^2 - 1 \implies \bar{Q}(\bar{p}) = -2\bar{p} + 1$$

In fact, we will *define* the degree of an inhomogeneous polynomial to be the degree of its homogeneous counterpart.

(ii) Roots of $Q(p)$ are mapped to roots of $\bar{Q}(\bar{p})$.

That is, $Q(p_0) = 0$ iff $\bar{Q}(\bar{p}_0) = 0$. This implies that the number of distinct roots of a polynomial does not depend on the coordinates in which it is being represented. It is well known that p_0 is a root of $Q(p)$ of multiplicity k *iff*:

$$Q(p_0) = Q'(p_0) = ... = Q^{(k-1)}(p_0) = 0$$

The resultant gives us a way to determine whether or not two polynomials have any common roots:

Let $P(x) = a_m x^m + a_{m-1} x^{m-1} y + ... + a_0 y^m$
$Q(x) = b_n x^n + b_{n-1} x^{n-1} y + ... + b_0 y^n$

Then the **resultant** of P and Q is defined as the determiant of the $(m+n)\times(m+n)$ Sylvester matrix:

$$Res[P,Q] = \begin{vmatrix} a_m & 0 & \cdots & 0 & b_n & 0 & \cdots & \cdots & 0 \\ a_{m-1} & a_m & \cdots & \vdots & b_{n-1} & b_n & \cdots & \cdots & \vdots \\ \vdots & a_{m-1} & \ddots & \vdots & \vdots & b_{n-1} & \ddots & \cdots & \vdots \\ \vdots & \vdots & \cdots & a_m & \vdots & \vdots & \cdots & \ddots & \vdots \\ \vdots & \vdots & \cdots & a_{m-1} & b_0 & \vdots & \cdots & \cdots & b_n \\ a_0 & \vdots & \cdots & \vdots & 0 & b_0 & \cdots & \cdots & b_{n-1} \\ 0 & a_0 & \cdots & \vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_0 & 0 & 0 & \cdots & \cdots & b_0 \end{vmatrix}$$
$$\underbrace{\qquad\qquad\qquad}_{n \text{ columns}} \quad \underbrace{\qquad\qquad\qquad}_{m \text{ columns}}$$

$P(x)$ and $Q(x)$ have a common root *iff* $Res[P,Q] = 0$.

We identify $Q'(p)$ with $Q_x(x, y)$ and define the **discriminant** of Q as:

$$\Delta[Q] = \frac{Res[Q, Q_x]}{n^n \tilde{a}_n}$$

2.1.3 Classical invariants

Now we come to the main definitions of this section. A **classical invariant** of weight k of a binary form $Q(x, y)$ of degree n, is a function $I(a_0, \ldots, a_n)$ satisfying the following equation under linear transformations:

$$I(a_0, \ldots, a_n) = (\alpha\delta - \beta\gamma)^k I(\bar{a}_0, \ldots, \bar{a}_n)$$

Ex: Let n =2. $\Delta[Q]$ is an invariant of weight 2:

$$\begin{aligned} \Delta &= (\alpha\delta - \beta\gamma)^2 \bar{\Delta} \\ ac - b^2 &= (\alpha\delta - \beta\gamma)^2 (\bar{a}\bar{c} - \bar{b}^2) \end{aligned}$$

A **classical covariant** of weight k of a binary form $Q(x, y)$ of degree n is a function $J(a_0, \ldots, a_n, x, y)$ satisfying the following equation under linear transformations:

$$J(a_0, \ldots, a_n, x, y) = (\alpha\delta - \beta\gamma)^k J(\bar{a}_0, \ldots, \bar{a}_n, \bar{x}, \bar{y})$$

Ex1: Q itself is a covariant of weight 0 under the relation $Q(x, y) = \bar{Q}(\bar{x}, \bar{y})$

Ex2: The Hessian $H = Q_{xx}Q_{yy} - Q_{xy}^2$ is a covariant of weight 2.

A **classical joint covariant** of weight k of a system $Q_1(x, y), \ldots, Q_m(x, y)$ is a function $J(a_0^1, \ldots, a_n^1, \ldots, a_0^m, \ldots, a_n^m, x, y)$ satisfying the following equation under linear transformations:

$$J(a_0^1, \ldots, a_n^1, \ldots, a_0^m, \ldots, a_n^m, x, y) = (\alpha\delta - \beta\gamma)^k J(\bar{a}_0^1, \ldots, \bar{a}_n^1, \ldots, \bar{a}_0^m, \ldots, \bar{a}_n^m, \bar{x}, \bar{y})$$

A joint covariant not depending on (x, y) is called a (classical) **joint invariant**.

Ex: The resultant $Res[Q, P]$ is a joint invariant of weight $mn + mk + nj$, where $m = deg[Q]$, $n = deg[P]$, $k = weight[Q]$, $j = weight[P]$

Using covariants and invariants we can completely classify all quadratic and cubic polynomials under Möbius transformations:

Canonical forms for complex quadratics:			$Q(p) = ap^2 + 2bp + c$
distinct roots	I	p	$\Delta \neq 0$
double root	II	1	$\Delta = 0,\ Q \not\equiv 0$
identically zero	III	0	$Q \equiv 0$

Canonical forms for complex cubics:			$Q(p) = ap^3 + 3bp^2 + 3cp + d$
distinct roots	I	$p^2 - 1$	$\Delta \neq 0$
double root	II	p	$\Delta = 0,\ H \not\equiv 0$
triple root	III	1	$H \equiv 0,\ Q \not\equiv 0$
identically zero	IV	0	$Q \equiv 0$

For quadratics, Δ and Q are the only covariants. Cubics have one more (T, the Jacobian of Q and H). A quartic has 5 covariants (2 invariants). We can construct new covariants by multiplying any two covariants, whose weight will equal the product of their respective weights, or by adding two of the same weight. We are however, mostly interested in independent covariants.

2.2 A broader picture

In the previous section we considered coordinate changes on $\mathbb{R}^2$ and $\mathbb{C}^2$ from two different perspectives. From the first perspective, the spaces stayed fixed, as we were merely transforming the coordinates used to label their points. From the second perspective, a coordinate change is associated with an *automorphism* of the underlying space. As it turns out, adopting the second point of view allows us to significantly broaden our notion of an invariant and the spaces they inhabit. The key observation will be that the set of automorphisms of any space forms a *group*. We can thus consider a change of coordinates to be a particular instance of an action by some element of a transformation group. In this broader picture, we will consider any function defined on the space in question that is unaffected by the action an *invariant*. The goal of this section is to make all of this precise.

2.2.1 Lie group actions and representations

To start off with, we would like to restrict ourselves to those automorphisms that preserve the smooth structure of a smooth manifold M, i.e. diffeomorphisms. The set of all diffeomorphisms of M is denoted by $\mathcal{D}\mathit{iff}(M)$ and forms a group. All transformation groups we consider will be Lie groups.

A **(left) smooth Lie group action** is a smooth map $\Phi : G \times M \to M$ where $(g, p) \mapsto g \cdot p$ satisfies:

$$g_1 \cdot (g_2 \cdot p) = (g_1 g_2) \cdot p, \forall g_i \in G, p \in M$$
$$e \cdot p = p$$

An action of a Lie group G induces a homomorphism $\sigma : G \to \mathcal{D}iff(M)$. A group action is *faithful* if $\sigma(G) \simeq G$. A group action is *regular* if all orbits have the same dimension, and there exists neighborhoods of every point on M such that the intersection with each orbit is connected (see [Olv95], p.41). Unless otherwise stated, all group actions considered are assumed to be regular and faithful.

Ex1: Let $M = \mathbb{R}^2$. The group of rotations $SO(2)$ is a one-dimensional group which rotates the plane around a fixed point. Group elements correspond to the angle of rotation, and orbits are circles of constant radii around the fixed point, as well as the point itself.

Ex2: Let $M = \mathbb{R}^n$. The Euclidean group $E(n)$ is the group consisting of translations, rotations and reflections preserving the euclidean metric on $\mathbb{R}^n$. This group acts transitively, which means that there is only one orbit consisting of the entire space.

There is one particular kind of group action which has especially desirable properties: a **Lie Group representation** is a smooth group homomorphism $\Pi : G \to GL(V)$, where $GL(V)$ is the general linear group consisting of all linear automorphisms of some vector space V.

In general, group actions will be non-linear. However, given any group action Φ, there is a way to induce a linear action via a representation on an associated space. Given a manifold M, its function space $\mathcal{F}(M)$, consisting of all functions $F : M \to \mathbb{R}$ is a vector space.
We define the **induced representation** $\Pi_\Phi : G \to GL(\mathcal{F}(M))$ with the action given by $g \cdot F = \bar{F}$, where we define $\bar{F}(\bar{x}) := F(g^{-1} \cdot \bar{x})$.

It is often the case that we are only interested in subrepresentations of Π_Φ. For instance, in the setting of smooth manifolds it is natural to consider only the induced representations of smooth functions, in the case of algebraic manifolds of rational functions, and in the case of projective varieties of homogeneous polynomials. If our manifold M is a vector space, with the group action being the standard action of $GL(n, \mathbb{R})$, we will find that in all the above mentioned cases, restricting to our desired subspace of $\mathcal{F}(M)$ yields a subrepresentation. However, this will not always be the case. Let $M = P(\mathbb{R}^2)$, and let the action on M be the linear fractional action of $GL(2, \mathbb{R})$. Here the space of polynomials is not a subrepresentation of Π_Φ. To see why, consider the effect of the induced action on the coordinate representations of $Q(p)$ as defined in section 2.1:

$$\bar{Q}(\bar{p}) = Q(g^{-1} \cdot \bar{p})$$
$$\bar{Q}\left(\frac{\alpha p + \beta}{\gamma p + \delta}\right) = Q(p)$$

The left hand side of the above is in general not a polynomial expression. If we generalize our notion of an induced representation slightly, we can fix this problem. We start with a preliminary definition:

A **multiplier** for a Lie group action defined on the space M is a map $\mu : G \times M \to \mathbb{C} \setminus \{0\}$ satisfying:

$$\mu(g_1 \cdot g_2, x) = \mu(g_1, g_2 \cdot x)\mu(g_2, x),\ \forall g_i \in G, x \in M$$
$$\mu(e, x) = 1$$

A **multiplier representation** is a homomorphism $\Pi_{\Phi,\mu} : G \to GL(\mathcal{F}(M))$ with an action given by $g \cdot F = \bar{F}$, where we define $\bar{F}(\bar{x}) := F(g^{-1} \cdot \bar{x}) = \mu(g, x)F(x)$.

For the $GL(2, \mathbb{R})$-action on the functions of two variables, if we let our multiplier be $\mu(g, x) = (\gamma p + \delta)^{-n}$, the space of polynomials corresponding to our previous example above will now be a subrepresentation of $\Pi_{\Phi,\mu}$:

$$Q(p) = (\gamma p + \delta)^n \bar{Q}\left(\frac{\alpha p + \beta}{\gamma p + \delta}\right)$$

2.2.2 Invariants

Let G be a Lie Group with an action defined on the space M. An **invariant** is a function $I : M \to \mathbb{R}$ satisfying $I(g \cdot x) = I(x),\ \forall g \in G$. Equivalently, an invariant $I(x)$ is a fixed point on the function space under the action of the induced representation Π_Φ.

Proposition: Let I denote a real-valued function on a manifold M. The following conditions are equivalent:
i) I is a G-invariant function
ii) I is constant on the orbits of G.
iii) All level sets $\{I(x) = c\}$ are G-invariant subsets of M.

(See [Olv99], p.73). It immediately follows from iii) that constant functions are always invariant. It follows from ii) that these are the only invariants under a transitive group action.

Ex1: Let $M = \mathbb{R}^2$, $G = SO(2)$. If we let 0 be our fixed point under the action of G, any function of the form $I(x^2 + y^2)$ will be an invariant.

Ex2: Let $M = \mathbb{R}^2$, $G = E(n)$. The Euclidean group here acts transitively on M, thus the only invariants will be constant functions.

Let G be a Lie Group with actions defined on the spaces $M_1, \ldots, M_m$. We can then define an induced *Cartesian Product action* on the space $M_1 \times \ldots \times M_m$ given by $g \cdot (x_1, \ldots, x_m) := (g \cdot x_1, \ldots, g \cdot x_m)$, $\forall g \in G$, $x_i \in M_i$.

A **joint invariant** is a function $J : M_1 \times \ldots \times M_m \to \mathbb{R}$ satisfying
$J(g \cdot x_1, \ldots, g \cdot x_m) = J(x_1, \ldots, x_m)$, $\forall g \in G$.

Usually, we are most interested in the case where the M_i's are all copies of the same space. A Joint invariant on the cartesian product of m copies of M, under the induced Cartesian product action, is referred to as an **m-fold joint invariant**. In this case, we refer to the induced action as the **extended action** of G.

Ex: The Euclidean group acting on $M = \mathbb{R}^2$, failed to yield any non-trivial ordinary invariants. However, consider $M \times M$ with the extended action of $G = E(n)$. Any function of the form $I(d((x_1, y_1), (x_2, y_2))$, where d is the Euclidean distance function, will be a 2-fold joint invariant.

A **symmetric m-fold joint invariant** is an invariant of the *extended group* $G \times S^m$, with an action given by $\sigma : G \times S^m \times M^{\times m} \to M^{\times m}$.

Ex: The joint invariant from the previous example is also a symmetric 2-fold joint invariant.

Let μ be a multiplier for a Lie Group action defined on the space X. A **relative invariant** is a function $R : X \to \mathbb{R}$ satisfying $R(g \cdot x) = \mu(g, x)R(x)$, $\forall g \in G$. If the group acting is $GL(n, \mathbb{R})$, it can be shown that the multiplier will always be a determinantal factor raised to the power k (see [Olv99]). In this case we define the *weight* of a relative invariant to be the value of k. The product of two relative invariants of weight k and m, is a new relative invariant of weight $k + m$. In particular, multiplying an invariant of weight k with an absolute invariant of weight $-k$, yields an absolute invariant of weight 0.

We can fit our classical notions of invariants into this broader picture as follows: Let the $G = GL(2)$, with the induced action on $\mathcal{F}(\mathbb{C}^2)$.

A **classical invariant** is a relative invariant
$I : \mathcal{P}^{(n)} \to \mathbb{C}$

A **classical covariant** is a joint relative invariant
$J : \mathcal{P}^{(n)} \times \mathbb{C}^2 \to \mathbb{C}$

A **classical joint covariant** is a joint relative invariant
$I : \mathcal{P}^{(n)} \times \ldots \times \mathcal{P}^{(m)} \times \mathbb{C}^2 \to \mathbb{C}$

2.3 Computational methods

This section will be discussing different computational strategies to finding invariants of a given Lie group acting on a space.

Recall that an invariant is a function $I(x) \in \mathcal{F}(M)$ satisfying $I(g \cdot x) = I(x)$, $\forall g \in \sigma(G)$. Here every element g is a diffeomorphism of M, which suggests that it might be possible to rephrase our defining equation for an invariant slightly. Consider the following diagram:

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[dr, "(I\circ g)(x)"'] & M \arrow[d, "I(x)"] \\ & \mathbb{R} \end{tikzcd}$$

It should be clear that $I(x)$ is an invariant if and only if the above diagram commutes. Recognizing $(I \circ g)(x)$ as the pullback of $I(x)$ under g, we can write:

$$g^* I(x) = I(x),\ \forall g \in \sigma(G)$$

We can recognize the form of this equation as a symmetry equation. However, when computing symmetries of a geometric object, the unknown element is the group acting. In our case, the group is known, and the unknown elements are the functions its actions preserve.

2.3.1 The method of moving frames

In general there is no one way to solve the system described above, which generically will be highly nonlinear. If, under certain conditions, we have an explicit local expression for the action of G on M, one approach is given by **The method of moving frames**.

Let $(x^1, \dots, x^n)$ be local coordinates around a point p in some n-dimensional manifold M, and let $\Phi : G \times M \to M$ be a Lie group action. A local expression for Φ is given by:

$$\varphi(g, (x^1, \dots, x^n)) = (\varphi_1(g, (x^1, \dots, x^n)), \dots, \varphi_n(g, (x^1, \dots, x^n)))$$

As G is also a smooth manifold, we can pick local coordinates $(y^1, \dots, y^r)$ around the identity $e \in G$. The idea will be to eliminate the r group parameters by equating the first r component functions of φ to a set of constants $\tilde{x}^1, \dots, \tilde{x}^r$. Consider the following system of equations, called the **normalization equations**:

$$
\begin{aligned}
\varphi_1((y^1, \dots, y^r), (x^1, \dots, x^n)) &= \tilde{x}^1 \\
&\vdots \\
\varphi_r((y^1, \dots, y^r), (x^1, \dots, x^n)) &= \tilde{x}^r
\end{aligned}
$$

After solving this system for the r group parameters $(y^1, \dots, y^r)$, we define a **moving frame** as a map $\gamma : M \to G$ given by substituting its solution back into $\varphi_1, \cdots, \varphi_r$.

Proposition: Assume that the action is free and regular. Then the functions $I_1, \dots, I_{n-r}$ defined below form a complete system of $n-r$ functionally independent invariants for the action of G:

$$
\begin{aligned}
I_1(x^1, \dots, x^n) &= \varphi_{r+1}(\gamma(x^1, \dots, x^n), (x^1, \dots, x^n)) \\
&\vdots \\
I_{n-r}(x^1, \dots, x^n) &= \varphi_n(\gamma(x^1, \dots, x^n), (x^1, \dots, x^n))
\end{aligned}
$$

(See [Olv99], thm 8.25, p.164)

2.3.2 Infinitesimal methods

Another, often more computationally straightforward approach to finding invariants of a Lie group action, can be taken by making use of the corresponding *Lie algebra*. Given a Lie group G, there is a natural action of G on itself given by left translations. Let $l_{g1} : G \to G$ denote a left translation by g_1, which we define by $l_{g1}(g_2) = g_1 g_2$. A vector field $X \in \mathcal{D}(G)$ is said to be *left invariant* if it satisfies $(l_g)_* X = X$, $\forall g \in G$. We denote the space of all left invariant vector fields of G by Lie(G), and call it the **Lie algebra** of G. It has some very useful properties:

- Lie(G) is closed under the Lie bracket operation (hence the name).
- Every vector field in Lie(G) is complete.
- It is isomorphic to the tangent space at the identity of G.

There is a natural map Exp: Lie$(G) \to G$ from the algebra to the group known as the **Exponential map**. It is given by $\mathrm{Exp}(X) = \gamma(1)$, where γ is the unique integral curve of X starting at the identity. Taking the linear span of X yields a curve $\gamma(t) = \mathrm{Exp}(Xt)$, corresponding to a one-parameter subgroup in G.

In general, not every element of G can be reached via the exponential map of some $X \in \text{Lie}(G)$. The image of $\text{Lie}(G)$ under Exp is some neighborhood around the identity of G. Generically, the map also fails to be injective. Only in some cases, like when G is compact or nilpotent and simply connected, do we have that $\text{Exp}(\text{Lie}(G)) \simeq G$. For more information on this topic, see [Hal15].

A Lie group action $\Phi : G \times M \to M$, will induce a Lie algebra homomorphism $\hat{\Phi} : \text{Lie}(G) \to \mathcal{D}(M)$. We can associate a global flow $\varphi_X : \mathbb{R} \times M \to M$, where $\varphi_X(t,p) = \gamma_X(t) \cdot p$ to each element $X \in \text{Lie}(G)$. Then we can define $\hat{\Phi}(X) = \hat{X}$, such that $\hat{X}_p = \left.\dfrac{d\varphi_X(t,p)}{dt}\right|_{t=0}$.

$$\begin{array}{ccc} \text{Lie}(G) & \xrightarrow{\hat{\Phi}} & \mathfrak{g} \\ {\scriptstyle \text{Exp}}\downarrow & & \downarrow{\scriptstyle \varphi_{\hat{X}}} \\ G & \xrightarrow{\sigma} & \mathfrak{G} \end{array}$$

We denote the image of G under σ by $\mathfrak{G} \subset \mathcal{D}\mathit{iff}(M)$. Similarly, let $\mathfrak{g} \subset \mathcal{D}(M)$ be the image of $\text{Lie}(G)$ under $\hat{\Phi}$.

The map $\hat{\Phi}$ is sometimes referred to as an **infinitesimal generator of group actions** (see [Lee13], p.526). Our strategy will be to reach for the connected component of $\mathfrak{G}$ via the flow of vector fields in $\mathfrak{g}$.

Recall our defining equation for an invariant function: $g^*I(x) = I(x), \forall g \in G$. Given that this holds for every $g \in G$, it will in particular hold for any $\varphi_{\hat{X}}(t,p) \in \mathfrak{G}$, where $\hat{X} \in \mathfrak{g}$. Thus, we can write:

$$\begin{aligned} \varphi_{\hat{X}}(t,p)^* I(p) &= I(p) \\ \left.\frac{d\varphi_{\hat{X}}(t,p)^*}{dt}\right|_{t=0} I(p) &= \left.\frac{dI(p)}{dt}\right|_{t=0} \\ L_{\hat{X}} I(p) &= 0, \, \forall \hat{X} \in \mathfrak{g} \end{aligned}$$

We recognize the last expression as the **Lie equation**. As before, when the vector fields are the unknowns, this is an (infinitesimal) symmetry equation. Even though there is some loss of information in going from the full picture to the infinitesimal one, the Lie equation has the advantage of always being a *linear* system of PDE's. In practice, solutions to this system will often yield all invariants of a given group action. Another useful fact is that orbits of the action of G are integral submanifolds of the flow of the Lie algebra (see [Lee13]).

It is not a given that we have a full group acting on a space to begin with. Any Lie algebra homomorphism, defines a **Lie algebra action**. Finding invariants of a Lie algebra action is the task we will be doing in the latter half of this thesis. Generically there is a slight complication when considering Lie algebra actions due to the fact that there is no guarantee that every $\hat{X} \in \mathfrak{g}$ will be a complete vector field. All we can say is that *locally*, the flow $\varphi_{\hat{X}}(t,p)$ will correspond to a family of diffeomorphisms which we identify with elements in some $\mathfrak{G}$. For our purposes, this turns out to be good enough.

When we are looking for m-fold joint invariants of a group action, the general way to go about it remains very much the same. Let G be a Lie group with an action Φ defined on M. Recall that we defined the *extended group action* $\Phi^{\times k}$ on $M^{\times k}$ to be the map given by $\Phi^{\times k} : G \times M^{\times k} \to M^{\times k}$ such that:
$g \cdot (p_1, ..., p_k) = (\Phi(g,p_1), ..., \Phi(g,p_k))$. We can consider a joint invariant of Φ an ordinary invariant under the action of $\Phi^{\times k}$ on $M^{\times k}$, and proceed as before. This action induces a new homomorphsm $\sigma^{\times k} : G \to \mathcal{D}\mathit{iff}(M^{\times k})$. We label the image of G under $\sigma^{\times k}$ by $\mathfrak{G}^{\times k}$.

Similarly, given a Lie algebra $\mathfrak{g} \in \mathcal{D}(M)$, we define the **extended Lie algebra** $\mathfrak{g}^{\times k} \in \mathcal{D}(M^{\times k})$ as the image of the induced Lie algebra homomorphism $\hat{\Phi}^{\times k}$. In local coordinates: If $\hat{X} = f(x^1, ..., x^n)^i \partial_{x^i}$ is an element of $\mathfrak{g}$,
$\hat{X}^{\times k} = f(x_1^1, ..., x_1^n)^i \partial_{x_1^i} + ... + f(x_k^1, ..., x_k^n)^i \partial_{x_k^i}$ is the extended vector field in $\mathfrak{g}^{\times k}$.

2.4 The space of invariants

Having spent some time defining invariants and looking at different ways to find them, we now turn our attention to the space of invariants itself. Given a Lie group G acting on a manifold M, we will denote the space of all functional invariants by $\mathcal{I}^G$.

Let $I_1, I_2 \in \mathcal{I}^G$. Then we have allready seen that the sum $I_1 + I_2$ as well as the product $I_1 I_2$ yields another invariant. This makes the space I^G into a *ring*. If our invariants are real functions, then multiplication of invariants by elements $r \in \mathbb{R}$ turns out to yield another invariant as well. We can thus consider the space $\mathcal{I}^G \subset \mathcal{F}(M)$ as a sub-*algebra*.

2.4.1 Generating sets

The first question we would like to answer, is whether or not a subset $\mathbf{a} \subset \mathcal{I}^G$ is a generating set for the space of invariants as an $\mathbb{R}$-algebra.

Recall that orbits of a regular group action are immersed submanifolds of M, which are integral manifolds of Π, the distribution defined by the Lie algebra $\mathfrak{g} \subset \mathcal{D}(M)$. The dimension of the orbits is the dimension of these submanifolds, which is equal to the rank of Π (proposition 9.26 in [Olv99], p.209). Given that our group acts regularly on M, the following theorem completely determines the number of functionally independent invariants of G:

Theorem

Let G be a Lie group with a regular action Φ defined on an n-dimensional manifold M. If the orbits of Φ are of dimension s, then there exists $m - s$ functionally independent local invariants $I_1, ..., I_s \in \mathcal{F}(\mathcal{M})$.

(See [Olv95], p.46). We conclude that $\mathbf{a}$ is a generating set of $\mathcal{I}^G$ iff it contains s functionally independent elements. From this we can also infer that when Π has reached maximal rank, $s = \dim(M) - \dim(G)$.

2.4.2 The algebra of polynomial invariants

In the special case where the group G is semi-simple, and the manifold M is any affine space, there exists a generating set $\mathbf{a} \subset \mathcal{I}^G$ consisting of *polynomials*. This is result is known as **Hilbert's theorem**. For proof and further reading, see [Hil93] and [MFK94].

In particular, if $M = \mathbb{R}^n$, we can consider our generating set $\mathbf{a}$ as a subset of $\mathbb{R}[\mathbf{x}]$, where $\mathbf{x} = (x^1, ..., x^n)$. We define the **algebra of polynomial invariants** as the subalgebra generated by $\mathbf{a} \subset \mathbb{R}[\mathbf{x}]$ and label it I^G.

We can learn more about I^G by a deeper examination of its set of generators. How do we know that a given generating set is a minimal one? Are the generators independent or is there some relation between them? The answers to these questions are to be found within the framework of *syzygy-modules*.

Given a generating set of cardinality m, $\mathbf{a} = \{a_1, a_2, \dots, a_m \mid a_i \in I^G\}$, we denote by F the free commutative $\mathbb{R}$-algebra generated by $\mathbf{a}$. It should be clear that $F \simeq \mathbb{R}[a_1, \dots, a_m]$. We can also consider F a free module over itself. In fact, we can consider both F and I^G to be $\mathbb{R}[a_1, \dots, a_m]$-modules. There is a natural map $\phi : F \to I^G$ given by $\phi(\tilde{a_i}) = a_i$. Denoting the kernel of ϕ by S_1, we get the following exact sequence:

$$0 \to S_1 \to F \to I^G \to 0$$

The $\mathbb{R}[a_1, \dots, a_m]$-module S_1 is called **the 1st module of syzygies** of I^G. By a **syzygy** we mean an element of S_1. In other words, a syzygy is a relation between the generators of I^G, of the form:

$$r_{i1}a_{j1} + ... + ra_{ik}a_{jk} = 0,$$

where $r_{i1}, ..., r_{ik} \in F, a_{j1}, ..., a_{jk} \in I^G, k \leq m$. If S_1 is a free module, there is nothing more to be done. However, there might be relations between its generators as well. In that case, we can proceed as before. Let $\mathbf{b} = \{b_1, b_2, \dots, b_l \mid b_i \in S_1\}$ be a generating set of S_1. Then $F_1 \simeq \mathbb{R}[b_1, \dots, b_l]$ is the free algebra generated by $\mathbf{b}$. We get another exact seqence:

$$0 \to S_2 \to F_1 \to S_1 \to 0,$$

where S_2 is **the 2nd module of syzygies** of I^G. Equivalently, we can define a map $\phi_1 : F_1 \to F$ such that $\phi_1(F_1) = S_1$. Continuing this way we get a long exact sequence, called a **free resolution** of I^G:

$$... \xrightarrow{\phi_3} F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F \xrightarrow{\phi} I^G \to 0$$

All modules F_i are free and each map is a surjection onto the kernel of the next. It is natural to ask whether or not the free resolution of I^G is a finite sequence. The following theorem answers the question:

Hilbert's syzygy theorem

If M is a finitely generated $\mathbb{R}[a_1, ..., a_m]$-module, then the m-th module of syzygies S_m of M is free.

(For proof and more information on free resolutions, see [Eis06]). It follows immediately that $F_i = 0$ for $i > m$. By Hilbert's basis theorem (see [Hil93]), each S_i is finitely generated. If we at each point choose a minimal generating set, we get a **minimal free resolution** of I^G, which is unique up to isomorphism and finite with length at most m.

By homological methods, it is possible to show that the alternating sum of the dimensions of the free modules F_i equals 0 for a minimal free resolution. These considerations are outside the scope of this text, but for further information on this topic, see [Eis13].

The notation F_i emphasizes that the modules in question are *free*. To emphasize their generating sets $\mathbf{a}, \mathbf{b}, \mathbf{c}, ..$, we will adopt the following convention for depicting free resolutions:

$$\mathbb{R}[\mathbf{x}] \supset I^G \leftarrow \mathbb{R}[\mathbf{a}] \leftarrow \mathbb{R}[\mathbf{b}] \leftarrow \mathbb{R}[\mathbf{c}] \leftarrow ... \leftarrow 0$$

2.4.3 The field of rational invariants

In general, the Lie group G acting on our manifold M will not be semi-simple, and so we can't consider our space of invariants as a polynomial subalgebra even if $M = \mathbb{R}^n$. However, if the *center* of the group acts by semi-simple elements, there exists a generating set $\mathbf{a} \subset \mathcal{I}^G$ consisting of *rational* functions. This is a consequence of the following more general theorem:

Theorem (Rosenlicht)

If the action of G is *algebraic*, and M is any affine or projective space, a finite set of rational invariants separates the orbits.

(For proof and further reading, see [Ros56] and [KL16]). In light of the above, we can define the **field of rational invariants** as the subfield generated by $\mathbf{a} \subset \mathbb{R}(\mathbf{x})$, labeled J^G. As the kernel of any field homomorphism is either 0 or the the whole field, the notions of syzygy-modules and free resolutions are inapplicable in this context. Instead, we will make use of concepts from field theory to further describe J^G.

Given a field extension $L|K$, we define its **Transcendence degree** to be the largest cardinality of an algebraically independent subset S of L over K. If in addition, L is an algebraic extension of the field $K(S)$, we refer to S as a **Transcendence basis**. For more on field extensions, see [DF04].

In particular, the field $\mathbb{R}(\mathbf{x})$ is a field extension of transcendence degree n over $\mathbb{R}$. If a generating set $\bar{\mathbf{a}} \subseteq \mathbf{a}$ of J^G of cardinality d is a transcendence basis, then d is also the transcendence degree of J^G over $\mathbb{R}$. We denote this as follows:

$$\mathbb{R}(\mathbf{x}) \supset J^G \simeq \mathbb{R}(\bar{\mathbf{a}}) \overset{d}{\supset} \mathbb{R}$$

Of course, even in the cases when our space of invariants has a polynomial generating set, we can still consider the space as a subspace of the field of rational functions. From this point of view, we look for a rationally independent generating set to describe the space J^G as a field extension over $\mathbb{R}$.

3 Computations on even-dimensional symplectic manifolds

In this chapter we will compute ordered joint invariants on symplectic manifolds of dimensions 2 and 4, concluding with a discussion on how our results generalize to higher dimensions. We begin with a preliminary exposition of symplectic geometry.

A **symplectic manifold** (M, ω) is an even-dimensional manifold M equipped with a closed, alternating, non-degenerate 2-form ω, called a *symplectic form*. In contrast to Riemannian geometry, there are no notions of lengths or angles in symplectic geometry. However, ω does provide a canonical volume form on M, given by $\omega^n = \omega \wedge ... \wedge \omega$ (n entries, where $\dim M = 2n$), which means there is still a notion of $2n$-dimensional hypervolume. Restricting to subspaces S on which $\omega|_S$ is non-degenerate, we can define even-dimensional volumes of lower dimension as well. Such a subspace is called a *symplectic subspace*. A volume form provides an orientation, which means that a symplectic manifold is *oriented*. Unlike in the Riemannian case, symplectic manifolds are locally similar, by the following theorem:

Theorem(Darboux)

Let (M, ω) be a $2n$-dimensional symplectic manifold. For any point $q \in M$, there exists local coordinates $(x^1, ..., x^n, p^1, ..., p^n)$ centered at q, in which ω has the following form:

$$\omega = \sum_{i=1}^{n} dx^i \wedge dp^i$$

(For a proof, see [Lee13]). We can interpret the action of ω_q on a pair of tangent vectors $X_q, Y_q \in T_qM$ as a sum of areas of parallelograms defined by the projections of X_q, Y_q to the symplectic subspaces $\mathbb{R}^2_{(x^i, p^i)}$ of T_qM.

The symplectic form establishes a canonical isomorphism between TM and T^*M given by $\omega : TM \to T^*M$, where $X \mapsto \iota_X \omega$. Given any $H \in \mathcal{C}^\infty(M)$, we can use this isomorphism to associate a **Hamiltonian vector field** X_H to H, defined by the relation $\iota_{X_H}\omega = dH$.

In local Darboux coordinates, the Hamiltonian vector field corresponding to the function H is given by:

$$X_H = \sum_{i=1}^{n} \frac{\partial H}{\partial p^i}\frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i}\frac{\partial}{\partial p^i}$$

It is straightforward to deduce that Hamiltonian vector fields correspond to the infinitesimal symmetries of ω, i.e. $L_X\omega = 0$, *iff* $X = X_H$ for some $H \in \mathcal{C}^\infty(M)$. By the *infinitesimal Stoke's theorem* $L_{X_H}\omega = d(\iota_{X_H}\omega) + \iota_{X_H}d\omega$. Since ω is closed, the second term vanishes. Using the definition of X_H, we see that $d(\iota_{X_H}\omega) = d(dH) = 0$ as well, which verifies the claim.

Thus, there is a correspondence between elements of $\mathcal{C}^\infty(M)$ and $sym(\omega)$. We can use the Hamiltonian vector fields to induce an operation on $\mathcal{C}^\infty(M)$, turning the space into a Lie algebra. Let $F, H \in \mathcal{C}^\infty(M)$. Then we define the **Poisson bracket** $\{\ ,\ \} : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ by $\{F, H\} := X_H(F)$.
In local Darboux coordinates:

$$\{F, H\} = \sum_{i=1}^{n} \frac{\partial H}{\partial p^i}\frac{\partial F}{\partial x^i} - \frac{\partial H}{\partial x^i}\frac{\partial F}{\partial p^i}$$

It is follows that $X_{\{F,H\}} = [X_F, X_H]$, and so the map $\Phi : \mathcal{C}^\infty(M) \to sym(\omega)$ defined by $\Phi(H) = X_H$ is a Lie algebra homomorphism, where the kernel of Φ consists of constant functions. Hence, $sym(\omega)$ is an infinite-dimensional Lie algebra. We would like to consider a finite-dimensional subalgebra $\mathfrak{g}$. Let M be a linear symplectic manifold. It should be clear that the subspace $\mathcal{P}^{(2)}(M) \subset \mathcal{C}^\infty(M)$ is closed under the Poisson bracket.

This allows us to define $\mathfrak{g} = Im(\Phi|_{\mathcal{P}^{(2)}(M)}(\mathcal{C}^\infty(M))) \subset sym(\omega)$.

3.1 2-dimensional M

We start with the case where $M = \mathbb{R}^2(x, p)$ is our base space endowed with the standard symplectic form $\omega = dx \wedge dp$. Its infinitesimal symmetries are given by:

$$sym(\omega) = \{X_f = f_p\partial_x - f_x\partial_p \mid f \in \mathcal{C}^\infty(M)\}$$

By restricting the generating functions f to consist of quadratic functions of the form $f = a_0x^2 + a_1xy + a_2y^2$, we get a 3-dimensional subalgebra:

$$\mathfrak{g} = \langle -x\partial_p, x\partial_x - p\partial_p, p\partial_x\rangle$$

We observe that $\mathfrak{g} \simeq sp(2) = sl(2)$.

The action of $\mathfrak{g}$ on M has two orbits: the fixed point $\{0\}$ and the open orbit $\mathbb{R}^2 \setminus \{0\}$. We regard a generic point as a point contained in the latter orbit, and say that the algebra acts transitively on generic points of M. Thus, there will be no invariants on the base space. Solving the system $L_X f = 0, \forall X \in \mathfrak{g}$ gives us the trivial solution where $f = const$, which confirms that this is the case. We conclude that $I^G = \mathbb{R}$.

3.1.1 $M \times M$

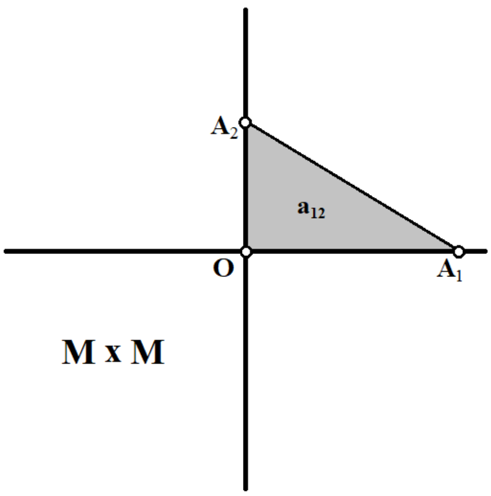
We now move onto the space $M \times M = \mathbb{R}^4(x_1, x_2, p_1, p_2)$. To look for joint invariants, we start by extending the algebra by applying the recipe from section 2.3. It gives us the following:

$$\mathfrak{g}^{\times 2} = \langle -x_1\partial_{p_1} - x_2\partial_{p_2}, x_1\partial_{x_1} - p_1\partial_{p_1} + x_2\partial_{x_2} - p_2\partial_{p_2}, p_1\partial_{x_1} + p_2\partial_{x_2}\rangle$$

The rank of the distribution defined by the vector fields in $\mathfrak{g}^{\times 2}$ is 3, which is the maximal rank. This implies that there will be one independent invariant. Moreover, as the algebra acting is semi-simple, we know that it will be a polynomial. Solving the Lie equation gives us:

$$a_{12} = x_1 p_2 - x_2 p_1$$

The chosen label for the invariant will become clear as we proceed. This expression can be recognized as a signed area. If we regard $A_1 = (x_1, p_1)$ and $A_2 = (x_2, p_2)$ as being two points on the base space, then $|a_{12}|$ is proportional the area shown in the figure below:



As our base space is $\mathbb{R}^2$, we can identify our manifold with the tangent space at the origin. Since the origin is fixed by the action of $\mathfrak{g}^{\times 2}$, we can interpret a_{12} the following way: Denote the position vectors from the origin to two points $(x_1, p_1), (x_2, p_2)$ by OA_1, OA_2. Then $a_{12} = \omega(OA_1, OA_2)$. We denote the algebra of 2-fold joint invariants by $I^{G\times 2}$. It is generated by one element, with the following minimal free resolution:

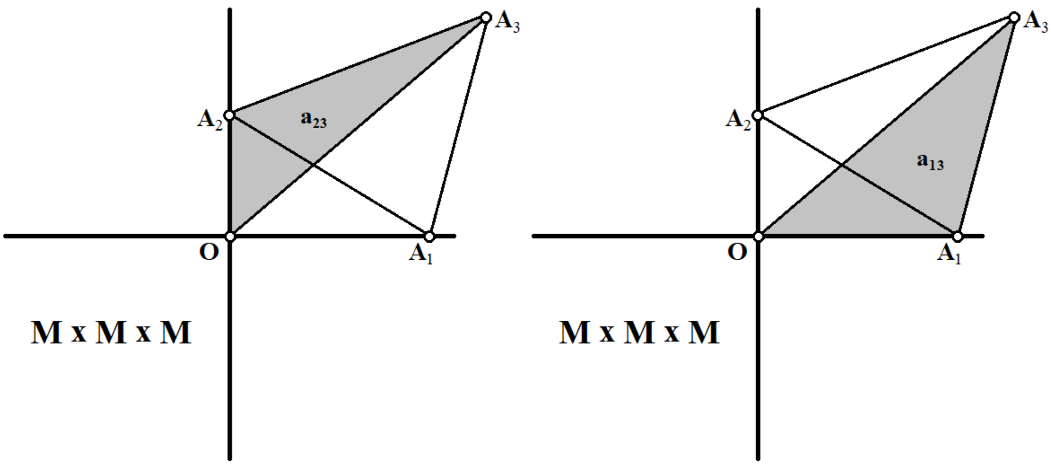
$$\mathbb{R}[x_1, x_2, p_1, p_2] \supset I^{G\times 2} \leftarrow \mathbb{R}[a_{12}] \leftarrow 0$$

This implies that $I^{G\times 2} \simeq \mathbb{R}[a_{12}]$

3.1.2 $M \times M \times M$

Prolonging the algebra further, we find that a_{12} is still an invariant when we solve the Lie equation. In addition, we get two more:

$$a_{13} = x_1 p_3 - x_3 p_1$$
$$a_{23} = x_2 p_3 - x_3 p_2$$



As before, these can be regarded as signed areas, or equivalently:

$$a_{13} = \omega(OA_1, OA_2)$$
$$a_{23} = \omega(OA_2, OA_3)$$

As the distribution determined by the vector fields reached maximal rank on $M \times M$, from this point on we can simply use the formula $s = \dim(M^k) - \dim(G)$, to find the number of independent invariants. We have a 3-dimensional algebra acting on a 6-dimensional space, and the total number of invariants we got is 3 as expected. Let us confirm that there are no relations between our invariants. Using computer elimination algorithms, we can take our 3 defining equations for a_{12}, a_{13}, a_{23} and try to eliminate the variables $x_1, x_2, x_3, p_1, p_2, p_3$. The ouptut is empty as expected, and we get the following minimal free resolution for $I^{G\times 3}$:

$$\mathbb{R}[x_1, x_2, x_3, p_1, p_2, p_3] \supset I^{G\times 3} \leftarrow \mathbb{R}[a_{12}, a_{13}, a_{23}] \leftarrow 0$$

Once again, $I^{G\times 3} \simeq \mathbb{R}[a_{12}, a_{13}, a_{23}]$.

3.1.3 $M^{\times 4}$

At this point there seems to have emerged a pattern:
We consider $a_{ij} = \omega(OA_i, OA_j) = x_i p_j - x_j p_i,$ s.t. $1 \leq i < j \leq 3$. Substituting for f in the Lie equation, we find that all 6 a_{ij} are indeed invariant. However, dimensional analysis tells us that there can be at most 5 independent invariants at this point. There has to be a relation between the terms.

Using the same method as before, we take our set of 6 defining equations for a_{ij} and try to eliminate the variables $x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4$. At first, the elimination-algorithm fails to find any relation between the a_{ij}'s. However, by the transitivity of the action on the open orbit in M, we can fix a generic point contained in it in order to simplify the system. We declare that $x_1 = 1, p_1 = 0$, and run the algorithm again. This time we get the following relation between the generators of $I^{G\times 4}$:

$$b_{1234} = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} = 0$$

In general we can eliminate a number of variables equal to the dimension of the orbit, when the algebra has reached maximal rank. At this point, the action is free. When a point $q = (x_i, p_i)$ is fixed only the stabilizer G_q of this point acts on the remaining points. Thus the dimension of the orbit becomes smaller and the elimination process is simplified. We will employ this strategy in all subsequent computations.

As an aside, we can recognize the expression $b_{1234} = 0$ as the *Plücker relation.*

We label the 1st module of syzygies by S_1, which is the kernel of the map $\Phi : \mathbb{R}[a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34}] \to I^{G\times 4}$ containing the generator b_{1234}.

Hence, this is our minimal free resolution of $I^{G\times 4}$:

$$\mathbb{R}[\mathbf{x}, \mathbf{p}] \supset I^{G\times 4} \leftarrow \mathbb{R}[a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34}] \leftarrow \mathbb{R}[b_{1234}] \leftarrow 0$$

So far we have considered our generators for the algebra of joint invariants as polynomials. However, it is also possible to consider them rational functions. From this perspective, our 6 generators are no longer independent, as we can solve for any one of them from the relation $b_{1234} = 0$ by dividing out their coefficient. For instance:

$$a_{34} = \frac{a_{13}a_{24} - a_{14}a_{23}}{a_{12}}$$

Hence, we can drop a_{34} from the set of generators of the field of rational joint invariants, which we will denote $J^{G\times 4}$. From this perspective, $J^{G\times 4}$ is a field extension over $\mathbb{R}$ of transcendence degree 5:

$$\mathbb{R}(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) \supset J^{G\times 4} \simeq \mathbb{R}(a_{12}, a_{13}, a_{14}, a_{23}, a_{24}) \overset{5}{\supset} \mathbb{R}$$

3.1.4 $M^{\times 5}$

Our previous formula still produces invariants as we extend our space to $M^{\times 5}$. We have $a_{ij} = x_i y_j - x_j y_i$, s.t. $1 \leq i < j \leq 5$. This time, the number of generators is 10. We have a 3-dimensional group acting on a 10-dimensional space, and so only 7 of them can be independent. The kernel of Φ now contains 5 elements of the form:

$$b_{ijkl} = a_{ij}a_{kl} - a_{ik}a_{jl} + a_{il}a_{jk}, \text{ where } 1 \leq i < j < k < l \leq 5.$$

To shorten our expressions, we hereby adopt a shorthand notation for the set of generators for the free algebras we will be constructing. Let $\mathbf{a}$ denote the set of all a_{ij}'s. Likewise, let $\mathbf{b}$ denote the set of all b_{ijkl}'s.

Let us find out if S_1 is a free module, or if there are relations between its generators. We proceed as before, by constructing the free algebra $F_1 \simeq \mathbb{R}[\mathbf{b}]$, which we consider an $\mathbb{R}[\mathbf{a}]$-module. Relations between elements of S_1 will linear combinations of elements in $\mathbb{R}[\mathbf{b}]$ with coefficients in $\mathbb{R}[\mathbf{a}]$ which equals zero.

We get 5 elements of S_2:

$$\begin{aligned}
c_1 &= a_{12}b_{1345} - a_{13}b_{1245} + a_{14}b_{1235} - a_{15}b_{1234} = 0\\
c_2 &= a_{12}b_{2345} - a_{23}b_{1245} + a_{24}b_{1235} - a_{25}b_{1234} = 0\\
c_3 &= a_{13}b_{2345} - a_{23}b_{1345} + a_{34}b_{1235} - a_{35}b_{1234} = 0\\
c_4 &= a_{14}b_{2345} - a_{24}b_{1345} + a_{34}b_{1245} - a_{45}b_{1234} = 0\\
c_5 &= a_{15}b_{2345} - a_{25}b_{1345} + a_{35}b_{1245} - a_{45}b_{1235} = 0
\end{aligned}$$

Then, we can look for generators between c_i's. These will be linear combinations of elements in $\mathbb{R}[\mathbf{c}]$ with coefficients in $\mathbb{R}[\mathbf{a}]$. We find one, which is an element of S_3:

$$\begin{aligned}
d = {} & (a_{23}a_{45} - a_{24}a_{35} + a_{25}a_{34})c_1 + (-a_{13}a_{45} + a_{14}a_{35} - a_{15}a_{34})c_2\\
& + (a_{12}a_{45} - a_{14}a_{25} + a_{15}a_{24})c_3 + (-a_{12}a_{35} + a_{13}a_{25} - a_{15}a_{23})c_4\\
& + (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})c_5 = 0
\end{aligned}$$

This is our minimal free resolution of $I^{G\times 5}$:

$$\mathbb{R}[\mathbf{x},\mathbf{p}] \supset I^{G\times 5} \leftarrow \mathbb{R}[\mathbf{a}] \leftarrow \mathbb{R}[\mathbf{b}] \leftarrow \mathbb{R}[\mathbf{c}] \leftarrow \mathbb{R}[\mathbf{d}] \leftarrow 0$$

As was the case in the previous section, we can also here consider our generating set to consist of rational functions. In the previous section we made use of the relation $b_{1234} = 0$ to solve for the generators a_{34} in terms of the other 5. We now have 5 such relations $b_{ijkl} = 0$ constituting a system of equations, and we can solve for 3 of the generators simultaneously, expressing them in terms of the other 7:

$$\begin{aligned}
a_{34} &= \frac{a_{13}a_{24} - a_{14}a_{23}}{a_{12}}\\
a_{35} &= \frac{a_{13}a_{25} - a_{15}a_{23}}{a_{12}}\\
a_{45} &= \frac{a_{14}a_{25} - a_{15}a_{24}}{a_{12}}
\end{aligned}$$

Thus, we can remove these from our set of generators for $J^{G\times 5}$. Let us denote our set of 7 independent generators by $\bar{\mathbf{a}} = \mathbf{a} \setminus \{a_{34}, a_{35}, a_{45}\}$. From this point of view, the field of rational joint invariants is the field $\mathbb{R}(\bar{\mathbf{a}})$ which has transcendence degree 7 over $\mathbb{R}$:

$$\mathbb{R}(\mathbf{x},\mathbf{p}) \supset J^{G\times 5} \simeq \mathbb{R}(\bar{\mathbf{a}}) \overset{7}{\supset} \mathbb{R}$$

3.1.5 $M^{\times 6}$

On this space we have 15 generators of the form:
$a_{ij} = x_i y_j - x_j y_i$, s.t. $1 \leq i < j \leq 6$.
We also get 15 relations of the form:
$b_{ijkl} = a_{ij}a_{kl} - a_{ik}a_{jl} + a_{il}a_{jk} = 0$, s.t. $1 \leq i < j < k < l \leq 6$.

In principle, we now know how to find relations between the b_{ijkl}'s. By Hilbert's Syzygy Theorem, we also know that there will be a free resolution of $I^{G\times 6}$ of length at most 12. This is good enough for our purposes. As far as illustrating the evolution of the algebra of polynomial joint invariants as we extend our base space, we are done. Now we turn to the task of completely describing the field of rational invariants as we extend our space to arbitrarily large products.

By a dimension count, we know that only 9 of the generators will be functionally independent. As before, our goal will be to express the 6 that are superfluous in terms of the other 9. Using the exact same approach we get the 3 expressions we already had for a_{34}, a_{35}, a_{45}, as well as 3 more:

$$a_{36} = \frac{a_{13}a_{26} - a_{16}a_{23}}{a_{12}}$$
$$a_{46} = \frac{a_{14}a_{26} - a_{16}a_{24}}{a_{12}}$$
$$a_{56} = \frac{a_{15}a_{26} - a_{16}a_{25}}{a_{12}}$$

We remove the elements $a_{34}, a_{35}, a_{36}, a_{45}, a_{46}, a_{56}$ from our generating set, and label the ordered set of independent generators $\bar{\mathbf{a}}$ as before. The field of invariants can be identified with the following field extension of transcendence degree 9 over $\mathbb{R}$:

$$\mathbb{R}(\mathbf{x}, \mathbf{p}) \supset J^{G\times 6} \simeq \mathbb{R}(\bar{\mathbf{a}}) \overset{9}{\supset} \mathbb{R}$$

3.1.6 Larger products

Finally, we would like to consider spaces of the form $M^{\times m}$. The number of generators a_{ij} will be $\binom{m}{2} = \frac{m(m-1)}{2}$, where $1 \leq k < l \leq m$. The number of *independent* generators is given by $2m-3$. We can generalize the expressions we've found so far for the dependent generators in terms of the independent ones. Consider expressions of the form:

$$a_{kl} = \frac{a_{1k}a_{2l} - a_{1l}a_{2k}}{a_{12}}$$

where $3 \leq k < l \leq m$. Removing those from the set of generators yields us precisely an independent generating set of desired cardinality. We can summarize this entire section by the following diagram:

$$\mathbb{R}(\mathbf{x}, \mathbf{p}) \supset J^{G\times m} \simeq \mathbb{R}(\bar{\mathbf{a}}) \overset{2m-3}{\supset} \mathbb{R}$$

3.2 4-dimensional M

Now let $M = \mathbb{R}^4(x, y, p, q)$ serve as the base space. The standard symplectic form in this case will be $\omega = dx \wedge dp + dy \wedge dq$. As before, the infinitesimal symmetries of ω constitutes an infinite dimensional Lie algebra:

$$sym(\omega) = \{X_f = f_p\partial_x - f_x\partial_p + f_q\partial_y - f_y\partial_q \mid f \in \mathcal{C}^\infty(M)\}$$

We restrict our generating functions to be of the form:
$f = a_1x^2 + a_2xy + a_3xp + a_4xq + a_5y^2 + a_6yp + a_7yq + a_8p^2 + a_9pq + a_{10}q^2$, which again gives us a finite-dimensional Lie algebra:

$$\begin{aligned}\mathfrak{g} = \ &\langle -x\partial_p, -y\partial_p - x\partial_q, x\partial_x - p\partial_p, x\partial_y - q\partial_p, -y\partial_q,\\ &y\partial_x - p\partial_q, y\partial_y - q\partial_q, p\partial_x, q\partial_x + p\partial_y, q\partial_y\rangle\end{aligned}$$

Here $\mathfrak{g} \simeq sp(4)$. This algebra is 10-dimensional and its action has two orbits on M, just like in the 2-dimensional case. $\mathfrak{g}$ acts transitively on generic points, contained in the open orbit $\mathbb{R}^4\setminus\{0\}$, while it fixes the origin. We solve the system $L_X f = 0, \forall X \in \mathfrak{g}$, which again only has the trivial solution where $f = const$. This confirms that there are no functional invariants on M. Thus, $I^G = \mathbb{R}$.

3.2.1 $M \times M$

As we increase dimensions, the systems of PDEs corresponding to the Lie equation will eventually reach a size where finding invariants by solving the system directly becomes impractical. However, it might be possible to simplify the search for invariants a great deal by making good use of the geometric intuition we obtained from working on the 2-dimensional case. On $M \times M$, the rank of distribution determined by the vector fields in $\mathfrak{g}^{\times 2}$ is 7, so we expect one invariant.

Recall that in 2 dimensions, we could interpret the invariant we got on $M \times M$ by solving the Lie equation directly as the output of a symplectic form acting on two position vectors on M. The corresponding expression where M is 4-dimensional is:

$$a_{12} = \omega(OA_1, OA_2) = x_1p_2 - x_2p_1 + y_1q_2 - y_2q_1,$$

where $OA_i = (x_i, y_i, p_i, q_i)$. Substituting back into the Lie equation, we discover that the above expression is indeed an invariant, which we will label a_{12} as before.

The minimal free resolution is given by:

$$\mathbb{R}[x_1, y_1, p_1, q_1, x_2, y_2, p_2, q_2] \supset I^{G\times 2} \leftarrow \mathbb{R}[a_{12}] \leftarrow 0$$

Which implies that $I^{G\times 2} \simeq \mathbb{R}[a_{12}]$.

3.2.2 $M \times M \times M$

Perhaps not surprisingly, we again look to expressions of the form:

$$a_{ij} = \omega(OA_i, OA_j) = x_ip_j - x_jp_i + y_iq_j - y_jq_i, \text{ s.t. } 1 \leq i < j \leq 3$$

And again, they are all easily verified to be invariant functions. The rank of the distribution defined by the extended vector fields is now 9, and we got 3 generators for $I^{G\times 3}$ as expected.

$$\mathbb{R}[\mathbf{x}, \mathbf{y}, \mathbf{p}, \mathbf{q}] \supset I^{G\times 3} \leftarrow \mathbb{R}[\mathbf{a}] \leftarrow 0$$

3.2.3 $M^{\times 4}$

Here our distribution have reached rank 10, which is the maximal rank. We have a 10-dimensional algebra acting on a 16-dimensional space, and so we expect to find 6 generators for $I^{G\times 4}$. This is exactly what our recipe outputs:

$$a_{ij} = x_i p_j - x_j p_i + y_i q_j - y_j q_i, \text{ s.t. } 1 \leq i < j \leq 4$$

We produced 6 generators on $\mathbb{R}^{2\times 4}$ as well, but in that case there was a relation between them. This time, they are all independent.

$$\mathbb{R}[\mathbf{x}, \mathbf{y}, \mathbf{p}, \mathbf{q}] \supset I^{G\times 4} \leftarrow \mathbb{R}[\mathbf{a}] \leftarrow 0$$

3.2.4 $M^{\times 5}$

Now we have a 10-dimensional algebra acting on a 20-dimensional space, and thus we need 10 generators for $I^{G\times 4}$. Again, this is precisely the number we get:

$$a_{ij} = x_i p_j - x_j p_i + y_i q_j - y_j q_i, \text{ s.t. } 1 \leq i < j \leq 5$$

It is quick to verify that they are still independent.

$$\mathbb{R}[\mathbf{x}, \mathbf{y}, \mathbf{p}, \mathbf{q}] \supset I^{G\times 5} \leftarrow \mathbb{R}[\mathbf{a}] \leftarrow 0$$

3.2.5 $M^{\times 6}$

Our formula generates 15 invariants on this space. We have a 10-dimensional algebra acting on a 24-dimensional space, which means that only 14 of our generators can be independent. Hence, there has to be a relation between them. In this case, we have to eliminate 24 variables from a set of 15 equations. By carefully fixing points in accordance with our previously described method, we can simplify the problem enough for the elimination-algorithm to output the following relation between our 16 generators:

$$\begin{aligned} b_{123456} = & a_{12}a_{34}a_{56} - a_{12}a_{35}a_{46} + a_{12}a_{36}a_{45} - a_{13}a_{24}a_{56} + a_{13}a_{25}a_{46} - a_{13}a_{26}a_{45} + \\ & a_{14}a_{23}a_{56} - a_{14}a_{25}a_{36} + a_{14}a_{26}a_{35} - a_{15}a_{23}a_{46} + a_{15}a_{24}a_{36} - a_{15}a_{26}a_{34} + \\ & a_{16}a_{23}a_{45} - a_{16}a_{24}a_{35} + a_{16}a_{25}a_{34} = 0 \end{aligned}$$

Recall that the first relation that appeared between our generators as we extended the 2-dimensional base space was a quadratic relation. Here, the relation we find is a *cubic* relation. We get the following free resolution of $I^{G\times 6}$:

$$\mathbb{R}[\mathbf{x},\mathbf{y},\mathbf{p},\mathbf{q}] \supset I^{G\times 6} \leftarrow \mathbb{R}[\mathbf{a}] \leftarrow \mathbb{R}[\mathbf{b}] \leftarrow 0$$

Up until now, we have considered the space of joint invariants as a subalgebra of $\mathbb{R}[\mathbf{x},\mathbf{y},\mathbf{p},\mathbf{q}]$. From this point on, we would like to consider our generators as rational functions, which makes the space of joint invariants a subfield of $\mathbb{R}(\mathbf{x},\mathbf{y},\mathbf{p},\mathbf{q})$.

On the 2-dimensional base space we were able to use the relation $b_{1234} = 0$ to solve for one of the generators of $J^{G\times 4}$ in terms of the others. Using the relation $b_{123456} = 0$ we are able to do the same thing here:

$$\begin{aligned} a_{56} =&(a_{12}a_{35}a_{46} - a_{12}a_{36}a_{45} - a_{13}a_{25}a_{46} + a_{13}a_{26}a_{45} + a_{14}a_{25}a_{36} - a_{14}a_{26}a_{35} \\ &+ a_{15}a_{23}a_{46} - a_{15}a_{24}a_{36} + a_{15}a_{26}a_{34} - a_{16}a_{23}a_{45} + a_{16}a_{24}a_{35} - a_{16}a_{25}a_{34}) \\ &/(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}) \end{aligned}$$

We can thus remove a_{56} from our generating set, and consider $J^{G\times 4}$ as a field extension over $\mathbb{R}$ of transcendence degree 14:

$$\mathbb{R}(\mathbf{x},\mathbf{y},\mathbf{p},\mathbf{q}) \supset J^{G\times 6} \simeq \mathbb{R}(\bar{\mathbf{a}}) \overset{14}{\supset} \mathbb{R}$$

3.2.6 $M^{\times 7}$

At this space, our formula yields 21 generators of the form a_{ij}. We get 6 relations between them of the following form:

$$\begin{aligned} b_{ijklmn} =& a_{ij}a_{kl}a_{mn} - a_{ij}a_{km}a_{ln} + a_{ij}a_{kn}a_{lm} - a_{ik}a_{jl}a_{mn} + a_{ik}a_{jm}a_{ln} - a_{ik}a_{jn}a_{lm} + \\ & a_{il}a_{jk}a_{mn} - a_{il}a_{jm}a_{kn} + a_{il}a_{jn}a_{km} - a_{im}a_{jk}a_{ln} + a_{im}a_{jl}a_{kn} - a_{im}a_{jn}a_{kl} + \\ & a_{in}a_{jk}a_{lm} - a_{in}a_{jl}a_{km} + a_{in}a_{jm}a_{kl} = 0 \end{aligned}$$

where $1 \leq i < j < k < l < m < n \leq 7$. By counting dimensions, we know that 18 of our 21 generators are independent. Our formula expressing a_{56} in terms of 14 other generators is still valid.

Using the above relations, we can solve for 2 more:

$$\begin{aligned} a_{57} =& (a_{12}a_{35}a_{47} - a_{12}a_{37}a_{45} - a_{13}a_{25}a_{47} + a_{13}a_{27}a_{45} + a_{14}a_{25}a_{37} - a_{14}a_{27}a_{35} \\ & + a_{15}a_{23}a_{47} - a_{15}a_{24}a_{37} + a_{15}a_{27}a_{34} - a_{17}a_{23}a_{45} + a_{17}a_{24}a_{35} - a_{17}a_{25}a_{34}) \\ & /(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}) \end{aligned}$$

$$\begin{aligned} a_{67} =& (a_{12}a_{36}a_{47} - a_{12}a_{37}a_{46} - a_{13}a_{26}a_{47} + a_{13}a_{27}a_{46} + a_{14}a_{26}a_{37} - a_{14}a_{27}a_{36} \\ & + a_{16}a_{23}a_{47} - a_{16}a_{24}a_{37} + a_{16}a_{27}a_{34} - a_{17}a_{23}a_{46} + a_{17}a_{24}a_{36} - a_{17}a_{26}a_{34}) \\ & /(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}) \end{aligned}$$

Removing a_{56}, a_{57}, a_{67} from our set of generators, allows us to consider $J^{G\times 7}$ as the following field extension over $\mathbb{R}$:

$$\mathbb{R}(\mathbf{x}, \mathbf{y}, \mathbf{p}, \mathbf{q}) \supset J^{G\times 7} \simeq \mathbb{R}(\bar{\mathbf{a}}) \overset{18}{\supset} \mathbb{R}$$

3.2.7 Larger products

We can now generalize what we've found to describe any $M^{\times m}$ with a 4-dimensional base space. The number of generators a_{ij} produced at a given m, is exactly the same as when we considered products of 2-dimensional spaces, i.e. $\dfrac{m(m-1)}{2}$.

This time our algebra is 10-dimensional, and so the number of independent generators is given by $4m - 10$. As before, we can generalize the expressions for the dependent generators in terms of the independent generators:

$$\begin{aligned} a_{kl} =& (a_{12}a_{3k}a_{4l} - a_{12}a_{3l}a_{4k} - a_{13}a_{2k}a_{4l} + a_{13}a_{2l}a_{4k} + a_{14}a_{2k}a_{3l} - a_{14}a_{2l}a_{3k} \\ & + a_{1k}a_{23}a_{4l} - a_{1k}a_{24}a_{3l} + a_{1k}a_{2l}a_{34} - a_{1l}a_{23}a_{4k} + a_{1l}a_{24}a_{3k} - a_{1l}a_{2k}a_{34}) \\ & /(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}) \end{aligned}$$

where $5 \leq k < l \leq m$. Removing those from the set of generators for $J^{\times m}$ gives us an independent set of generators consisting of $4m - 10$ elements as desired. Hence:

$$\mathbb{R}(\mathbf{x}, \mathbf{y}, \mathbf{p}, \mathbf{q}) \supset J^{G\times m} \simeq \mathbb{R}(\bar{\mathbf{a}}) \overset{4m-10}{\supset} \mathbb{R}$$

3.3 Higher dimensions

Noticing the pattern for generating invariants, we can consider spaces where $M = \mathbb{R}^{2n}(x^1, ..., x^n, p^1, ..., p^n)$ is the base space for $n > 2$. The expression for the standard symplectic form becomes $\omega = dx^1 \wedge dp^1 + ... + dx^n \wedge dp^n$.

The number of vector fields constituting the finite subalgebra $\mathfrak{g}$ will equal the dimension of the space of homogeneous polynomials in $2n$ variables, which is given by $\begin{pmatrix} 2n+2-1 \\ 2 \end{pmatrix} = n(2n+1)$. We have that $\mathfrak{g} \simeq sp(2n)$.

Our formula for producing invariants on $M^{\times m}$ now becomes:

$$a_{ij} = \omega(OA_i, OA_j) = \sum_{k=1}^{n} x_i^k p_j^k - x_j^k p_i^k, \text{ s.t. } 1 \leq i < j \leq m$$

Going from a 2-dimensional base space to a 4-dimensional base space, we found that even though we could generate the same number of invariants for each extension, the difference between the number of vector fields defining $\mathfrak{g}$ and the dimension of the extended manifold grew. A natural question to ask is whether or not this difference will increase to a point where we will obtain insufficient generators for the algebra of invariants by only using our previous method.

To answer this question, let us examine how the rank of the distribution defined by vector fields in $\mathfrak{g}$ evolves as we extend base spaces of higher dimensions. It should be clear that the action is transitive on the open orbit $\mathbb{R}^{2n} \setminus \{0\}$ for any any $n > 2$. Thus the distribution always starts out with a rank equal to the dimension of $\mathbb{R}^{2n}$, i.e. $2n$. On $\mathbb{R}^{2n \times 2}$, the group G acts on pairs (p_1, p_2). To obtain the dimension of the orbits, we can picture the group acting on each copy of $\mathbb{R}^{2n}$ in succession.

We have $2n$ degrees of freedom in fixing the point p_1. The remaining degrees of freedom is given by $\dim(G_{p_1})$, where G_{p_1} is the stabilizer of p_1. As the expression a_{12} is always an invariant on $\mathbb{R}^{2n \times 2}$ for generic points, we can conclude that $\dim(G_{p_1}) = 2n - 1$. Hence the rank of the distribution is always $4n - 1$ on $\mathbb{R}^{2n \times 2}$.

Similarly, on $\mathbb{R}^{2n \times 3}$ we have an action on triplets (p_1, p_2, p_3). After fixing the first two points, the remaining degrees of freedom is given by $\dim G_{p_1, p_2}$. But here we have two new invariants a_{13}, a_{23}, and so $\dim(G_{p_1, p_2}) = 2n - 2$.

Proceeding in this fashion, it should be clear that we obtain maximal rank at $\mathbb{R}^{2n\times 2n}$. In particular, $G_{p_1,\dots,p_{2n}} = \{0\}$. Of course, this also follows from the fact that a generic $2n$-tuple is a basis for $\mathbb{R}^{2n}$, and that a linear transformation which preserves it must be the identity.

By comparing dimensions of our spaces to the number of generators, we can deduce that the first syzygy appears at $\mathbb{R}^{2n\times(2n+2)}$. By inspecting the syzygies appearing on $\mathbb{R}^{2\times 4}$ and $\mathbb{R}^{4\times 6}$, we can understand them as nested Plücker relations. Denoting their generators by b^2_{1234} and b^4_{123456} respectively, we see that:

$$
\begin{aligned}
b^2_{1234} &= a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} \\
b^4_{123456} &= \sum_{j=2}^{6} (-1)^j a_{1j} b^2_{klst}
\end{aligned}
$$

where the letters $klst$ are taken from $\{2,3,4,5,6\}\setminus\{j\}$ and arranged in ascending order. It is easy to verify that the following constitutes a generator for the first module of syzygies for $\mathbb{R}^{6\times 8}$:

$$
b^6_{12345678} = \sum_{j=2}^{8} (-1)^j a_{1j} b^4_{klstuv}
$$

Generalizing this result, we can define the generator for the first module of syzygies for $\mathbb{R}^{2n\times(2n+2)}$ inductively:

$$
b^{2n}_{1\dots 2n} = \sum_{j=2}^{2n+2} (-1)^j a_{1j} b^{2n-2}_{i1\dots i(2n-2)}
$$

Hence, once we reach this point we can always use these relations to express excess generators in terms of the others, generalizing our previous procedure. Denote the set of independent generators obtained by removing those that are not needed by $\bar{\mathbf{a}}$. It is quick to verify that this set will contain $2nm - n(2n+1)$ elements.

Summarizing:

$$
\mathbb{R}(\mathbf{x},\mathbf{p}) \supset J^{G\times m} \simeq \mathbb{R}(\bar{\mathbf{a}}) \overset{2nm-n(2n+1)}{\supset} \mathbb{R}
$$

4 Computations on odd-dimensional contact manifolds

In this chapter we will compute ordered joint invariants on contact manifolds of dimensions 3 and 5, concluding with some remarks on how to extend our results to higher dimensions. We start with a brief introduction to contact geometry.

A **contact manifold** (M, τ) is an odd-dimensional manifold M equipped with a completely non-integrable smooth distribution $\Pi \subset TM$ of co-dimension 1, known as a a *contact structure*. Elements $\tau \in Ann(\Pi)$ are non-vanishing one-forms called *contact forms*, and completely characterize Π. Contact geometry is in many ways an odd-dimensional counterpart to symplectic geometry. Any contact form τ has the property that $d\tau_p|_{\Pi_p}$ is a symplectic tensor for any $p \in M$. The following theorem illustrates another similarity:

Theorem(Contact Darboux)

Let (M, τ) be a $2n+1$-dimensional contact manifold. For any point $q \in M$ and any $\tau \in Ann(\Pi)$, there exists local coordinates $(x^1, ..., x^n, u, p^1, ..., p^n)$ centered at q, in which τ has the following form:

$$\tau = du - \sum_{i=1}^{n} p_i dx^i$$

(For a proof, see [Lee13]). An important difference from symplectic geometry, is that while ω is *unique*, $f\tau$ is still a contact form if $f \in \mathcal{C}^\infty(M)$ is non-vanishing. Hence it makes sense in defining infinitesimal symmetries of contact forms, to require only that τ is preserved up to scale. That is:

$$sym(\tau) := \{X \mid L_X \tau = \lambda \tau\}$$

Solutions to the above equation are called **contact vector fields**.
Let $h \in \mathcal{C}^\infty(M)$. Then in local contact Darboux coordinates, any contact vector field has the following form (see [Arn13]):

$$X_h = -\sum_{i=1}^{n} \frac{\partial h}{\partial p_i} \left(\frac{\partial}{\partial x^i} + p_i \frac{\partial}{\partial u} \right) + h \frac{\partial}{\partial u} + \sum_{i=1}^{n} \left(\frac{\partial h}{\partial x^i} + p_i \frac{\partial h}{\partial u} \right) \frac{\partial}{\partial p_i}$$

Just as was the case on symplectic manifolds, there is a correspondence between $sym(\tau)$ and $\mathcal{C}^\infty(M)$. Also here we can induce an operation on the latter space,

providing a Lie algebra structure. Let $f, h \in \mathcal{C}^\infty(M)$. Define the **Lagrange bracket** $[\ ,\ ] : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ by $[f, h] := X_h(f)$.
In local contact Darboux coordinates:

$$[f, h] = -\sum_{i=1}^{n} \frac{\partial h}{\partial p_i}\left(\frac{\partial f}{\partial x^i} + p_i \frac{\partial f}{\partial u}\right) + h\frac{\partial f}{\partial u} + \sum_{i=1}^{n}\left(\frac{\partial h}{\partial x^i} + p_i\frac{\partial h}{\partial u}\right)\frac{\partial f}{\partial p_i}$$

As before, the property $X_{[f,h]} = [X_f, X_h]$ allows us to construct a Lie algebra homomorphism. Define $\Phi : \mathcal{C}^\infty(M) \to sym(\tau)$ by $\Phi(h) = X_h$. Here the kernel is trivial, and so the map Φ is an *isomorphism*. Also in this case, we would like to define a finite-dimensional subalgebra $\mathfrak{g}$. Let M be a linear contact manifold. This time, $\mathcal{P}^{(2)}(M)$ is not closed under the Lagrange bracket. To remedy this situation, we consider the space $\mathcal{Q}^{(2)}(M)$, consisting of quasi-homogeneous polynomials of degree 2. In this space, the variables x^i, p_i is given weight 1, while the variable u is given weight 2. This construction is known as the *Heisenberg algebra*, which is the Tanaka algebra (see [Kru14]) of the contact structure.

Now we can define $\mathfrak{g} = Im(\Phi|_{\mathcal{Q}^{(2)}(M)}(\mathcal{C}^\infty(M))) \subset sym(\tau)$.

4.1 3-dimensional M

Let $M = \mathbb{R}^3(x, u, p)$ be our base space endowed with the standard contact structure, whose annihilator is generated by the form $\tau = du - pdx$. We have the following infinite-dimensional Lie algebra:

$$sym(\tau) = \{X_f = -f_p(\partial_x + p\partial_u) + f\partial_u + (f_x + pf_u)\partial_p \mid f \in \mathcal{C}^\infty(M)\}$$

We restrict our generating functions to those of the form:
$f = a_0x^2 + a_1xp + a_2p^2 + a_3u$.
This yields the following 4-dimensional subalgebra:

$$\mathfrak{g} = \langle x^2\partial_u + 2x\partial_p, -x\partial_x + p\partial_p, -2p\partial_x - p^2\partial_u, x\partial_x + 2u\partial_u + p\partial_p\rangle$$

To look for functional invariants on the base space, we again solve the system $L_X f = 0, \forall X \in \mathfrak{g}$. The only solution to this system is the trivial one, $f = const$. Thus, there are no non-trivial invariants on the base space.

4.1.1 $M \times M$

Moving on to the space $M \times M = \mathbb{R}^6(x_1, u_1, p_1, x_2, u_2, p_2)$, we start by computing the rank of the distribution defined by the new vector fields, which is found to be the maximal rank of 4. This implies that there are 2 generators for our space of joint invariants. Next, we attempt to solve the Lie equation as before. However, unlike in the previous even-dimensional cases, the PDE-solve program is unable to obtain a solution for this system. We need to try another approach.

One such approach is to simplify the system of equations we need to solve by eliminating group parameters, making use of the method of moving frames. In order to do so, we have to be able to determine the explicit action of the group obtained by exponentiating the Lie algebra.

We start by looking closer at our initial Lie algebra on the base space. Its Levi decomposition turns out to be $\mathfrak{g} = \mathbb{R} \oplus sp(2)$, which implies that:
$\mathfrak{g} \simeq csp(2) \simeq gl(2)$. As its center acts by semi-simple elements, we expect a *rational* generator for the space of invariants.

Exponentiating yields the connected component of $GL(2)$.
We denote our group $\mathfrak{G}$.

Computing the flow of each vector field in $\mathfrak{g}$ might tell us something about how the group acts. We get the following families of diffeomorphisms:

$$\begin{aligned}
\Phi_0(t, x, u, p) &= (x, tx^2 + u, 2tx + p) \\
\Phi_1(t, x, u, p) &= (xe^{-t}, u, pe^t) \\
\Phi_2(t, x, u, p) &= (-2pt + x, -p^2t + u, p) \\
\Phi_3(t, x, u, p) &= (xe^t, ue^{2t}, pe^t)
\end{aligned}$$

From this we deduce that the action of $\mathfrak{G}$ projects to the standard linear action on $\mathbb{R}^2(x, p)$. Let $\varphi \in \mathfrak{G}$. Then we know it is an automorpism of the form:
$\varphi(x, u, p) = (\alpha x + \beta p, f(x, u, p), \gamma x + \delta p)$ such that $\varphi^*\tau = \lambda\tau$,
where $\tau = du - pdx$ is a contact form.

Substituting:

$$\begin{aligned}
\varphi^*\tau &= d(\varphi^*(u)) - \varphi^*(p)d(\varphi^*(x)) \\
&= d(f(x, u, p) - (\gamma x + \delta p)d(\alpha x + \beta p) \\
&= f_x dx + f_u du + f_p dp - \alpha(\gamma x + \delta p)dx - \beta(\gamma x + \delta p)dp \\
&= \lambda du - \lambda pdx
\end{aligned}$$

Collecting terms we get a system of PDEs:

$$\begin{aligned}
f_u &= \lambda \\
f_x &= \alpha\gamma\delta + \alpha\delta p - f_u p \\
f_p &= \beta\gamma x + \beta\delta p
\end{aligned}$$

Solving this system for f gives us:

$$f(x,u,p) = (\alpha\delta - \beta\gamma)\left(u - \frac{px}{2}\right) + \frac{(\alpha x + \beta p)(\gamma x + \delta p)}{2}$$

Now we have an explicit expression for the action of $\mathfrak{G}$ on M. Notice that even though $\mathfrak{G}$ is in principle only the connected component of $GL(2)$ (i.e. elements with positive determinant), our formula is valid for all real values of $\alpha, \beta, \gamma, \delta$, as long as $\alpha\delta - \beta\gamma \neq 0$. We can understand our expression as the action of the Zarisky closure of $\mathfrak{G}$. By extending, we also get an expression for the action on $M \times M$:

$$\begin{aligned}
\varphi^{\times 2}(x_1, x_2, u_1, u_2, p_1, p_2) =& (\alpha x_1 + \beta p_1, \alpha x_2 + \beta p_2, f(x_1, u_1, p_1), \\
& f(x_2, u_2, p_2), \gamma x_1 + \delta p_1, \gamma x_2 + \delta p_2)
\end{aligned}$$

At this point we can eliminate the group parameters. Consider the system of equations:

$$\begin{aligned}
\varphi^{\times 2*} x_1 &= \tilde{x}_1 \\
\varphi^{\times 2*} p_1 &= \tilde{p}_1 \\
\varphi^{\times 2*} x_2 &= \tilde{x}_2 \\
\varphi^{\times 2*} p_2 &= \tilde{p}_2
\end{aligned}$$

We solve for the group parameters $\alpha, \beta, \gamma, \delta$ that satisfy the above, and substitute the output into the expression for $\varphi^{\times 2}$. Next, we compute the pullback for the remaining two coordinates:

$$\begin{aligned}
\varphi^{\times 2*} u_1 &= \frac{x_1 p_1 - 2u_1}{2(p_1 x_2 - p_2 x_1)} \\
\varphi^{\times 2*} u_2 &= \frac{x_2 p_2 - 2u_2}{2(p_1 x_2 - p_2 x_1)}
\end{aligned}$$

Substituting the results for f into our original Lie equation, we confirm that both functions are invariant. We have a 4-dimensional group acting on a 6-dimensional space, so we don't expect any more. Looking closer at the invariant functions obtained, we notice another difference from the previous even-dimensional cases. This time the invariants are rational functions, rather than polynomial. These can be understood as a ratio of relative invariants. Collecting all 3 terms appearing in either the numerator or denominator of our invariant functions, ignoring the factor of 2:

$$\begin{aligned} R_{11} &= x_1 p_1 - 2u_1 \\ R_{22} &= x_2 p_2 - 2u_2 \\ R_{12} &= x_1 p_2 - x_1 p_2 \end{aligned}$$

The terms R_{11}, R_{22} can be understood in the following sense: Consider the vector field $X_3 = x\partial_x + 2u\partial_u + p\partial_p$, corresponding to the action of the center of $\mathfrak{g}$. We saw that its flow lines are curves radiating away from the origin. The extended vector field $X_3^{\times 2} = x_1\partial_{x_1} + 2u_1\partial_{u_1} + p_1\partial_{p_1} + x_2\partial_{x_2} + 2u_2\partial_{u_2} + p_2\partial_{p_2}$ is a sum of two copies of X_3.

By labeling the copies X_3^1, X_3^2, we have that $X_3^{\times 2} = X_3^1 + X_3^2$. As was the case on the symplectic manifolds, we can identify M with the tangent space at the origin. Since the origin is fixed by the action, we can interpret the relative invariants as:

$$R_{11} = -\tau_0(X_3^1), R_{22} = -\tau_0(X_3^2)$$

We recognize the expression R_{12} as a projected symplectic area. Computing the pullback, we find that $\varphi^{\times 2*}R_{ij} = (\alpha\delta - \beta\gamma)R_{ij}$ in all 3 cases, which means that our 3 R_{ij}'s are all relative invariants of weight 1. Recall that the product of two relative invariants of weight m and n respectively, equals a new relative invariant of weight $m+n$. As we multiply a relative invariant with the reciprocal of a different relative invariant of the same weight, the result is an absolute invariant of weight 0. We can choose any two such products as our generators of $J^{G\times 2}$, as long as they are independent. Let us choose the following:

$$\begin{aligned} I_{11} &= \frac{R_{11}}{R_{22}} \\ I_{12} &= \frac{R_{12}}{R_{22}} \end{aligned}$$

The following describes our field of rational joint invariants:

$$\mathbb{R}(x_1, x_2, u_1, u_2, p_1, p_2) \supset J^{G\times 2} \simeq \mathbb{R}(I_{11}, I_{12}) \overset{2}{\supset} \mathbb{R}$$

4.1.2 $M \times M \times M$

Again, we have discovered a pattern. Define the following functions:

$$R_{ij} = \begin{cases} x_i p_i - 2u_i, & \text{if } i = j \\ x_i p_j - x_j p_i, & \text{if } i \neq j \end{cases}, \text{ where } 1 \leq i \leq j \leq 3.$$

All 6 turn out to be relative invariants of weight 1, as before. Dividing all of them by the last one, gives us 5 absolute invariants:

$$I_{ij} = \frac{R_{ij}}{R_{33}}$$

We will adopt the convention that indices in expressions I_{ij} are the usual symmetric ones, with the largest value of ij removed, in this case 33. Here we have a 4-dimensional group acting on a 9-dimensional space, so we expect no more invariants. As we did before, we adopt the notation $\bar{\mathbf{I}}$ for our rationally independent set of generators.

$$\mathbb{R}(\mathbf{x}, \mathbf{u}, \mathbf{p}) \supset J^{G \times 3} \simeq \mathbb{R}(\bar{\mathbf{I}}) \overset{5}{\supset} \mathbb{R}$$

4.1.3 $M^{\times 4}$

Using the same formula for the R_{ij}'s as before, where $1 \leq i \leq j \leq 4$, we now generate 10 relative invariants. Dividing every expression by R_{44}, gives us 9 absolute invariants I_{ij}. Now we expect to find a relation between them.

Using elemination methods, we find one relation between the absolute invariants with skew-symmetric indices:

$$b_{1234} = I_{12}I_{34} - I_{13}I_{24} + I_{14}I_{23} = 0$$

The invariants I_{11}, I_{22} and I_{33} are all independent. As our invariants are already rational functions, we can use the relation $b_{1234} = 0$ to solve for one of them in terms of the other:

$$I_{34} = \frac{I_{13}I_{24} - I_{14}I_{23}}{I_{12}}$$

Thus, we can remove I_{34} from $\bar{\mathbf{I}}$. It follows that our field of invariants $J^{G \times 4}$ has transcendence degree 8 over $\mathbb{R}$:

$$\mathbb{R}(\mathbf{x}, \mathbf{u}, \mathbf{p}) \supset J^{G \times 4} \simeq \mathbb{R}(\bar{\mathbf{I}}) \overset{8}{\supset} \mathbb{R}$$

4.1.4 $M^{\times 5}$

Continuing, we get 14 invariants $I_{ij} = \dfrac{R_{ij}}{R_{33}}$

As we did on $\mathbb{R}^{2\times 5}$, we get 5 relations between our generators with skew-symmetric indices:

$$b_{ijkl} = I_{ij}I_{kl} - I_{ik}I_{jl} + I_{il}I_{jk}, \text{ where } 1 \leq i < j < k < l \leq 5.$$

The 4 generators with symmetric indices are independent. Completely analogously to what we did on $\mathbb{R}^2 \times 5$ we can use our relations to express two more generators in terms of the others:

$$I_{35} = \frac{I_{13}I_{25} - I_{15}I_{23}}{I_{12}}$$
$$I_{45} = \frac{I_{14}I_{25} - I_{15}I_{24}}{I_{12}}$$

Now we can describe our field of invariants as follows:

$$\mathbb{R}(\mathbf{x}, \mathbf{u}, \mathbf{p}) \supset J^{G\times 5} \simeq \mathbb{R}(\bar{\mathbf{I}}) \overset{11}{\supset} \mathbb{R}$$

4.1.5 Larger products

Our results on $M^{\times 4}$ and $M^{\times 5}$ can be generalized to larger products. On the space $M^{\times m}$, the number of generators I_{ij} is given by:

$$\binom{m+1}{2} - 1 = \frac{m(m+1)-2}{2}$$

The number of *independent* generators is $3m-4$. Comparing our number to the number of independent generators for $\mathbb{R}^{2\times 2}$, which was $2m-3$, we can understand the difference to consist of generators with equal indices.

Our previous formula for expressing the dependent generetors in terms of those that are independent, can be applied here as well:

$$I_{kl} = \frac{I_{1k}I_{2l} - I_{1l}I_{2k}}{I_{12}}$$

where $3 \leq k < l \leq m$. We can remove the dependent genereators from our generating set as before. The following diagram summarizes this section:

$$\mathbb{R}(\mathbf{x}, \mathbf{u}, \mathbf{p}) \supset J^{G\times m} \simeq \mathbb{R}(\bar{\mathbf{I}}) \overset{3m-4}{\supset} \mathbb{R}$$

4.2 5-dimensional M

Here $M = \mathbb{R}^5(x, y, u, p, q)$ is our base space, again with the standard contact structure. In this case the annihilator is generated by the form:
$\tau = du - pdx - qdy$. We have:

$$sym(\tau) = \{X_f = -f_p(\partial_x+p\partial_u)-f_q(\partial_y+q\partial_u)+f\partial_u+(f_x+pf_u)\partial_p+(f_y+qf_u)\partial_q\}$$

where $f \in \mathcal{C}^\infty(M)$. As before, we need to use weighted polynomials as our generators to produce a finite dimensional subalgebra. They will be of the form:
$f = a_1x^2+a_2xy+a_3xp+a_4xq+a_5y^2+a_6yp+a_7yq+a_8p^2+a_9pq+a_{10}q^2+a_{11}u$.
We now get an 11-dimensional Lie-algebra:

$$\begin{aligned}\mathfrak{g} = \ &\langle x^2\partial_u + 2x\partial_p, xy\partial_u + y\partial_p + x\partial_q, -x\partial_x + p\partial_p, -x\partial_y + q\partial_p, y^2\partial_u + 2y\partial_q,\\ &- y\partial_x + p\partial_q - y\partial_y + q\partial_q, -2p\partial_x - p^2\partial_u, -q\partial_x - p\partial_y - pq\partial_u,\\ &- 2q\partial_y - q^2\partial_u, x\partial_x + y\partial_y + 2u\partial_u + p\partial_p + q\partial_q\rangle\end{aligned}$$

Here we find that $\mathfrak{g} \simeq \mathbb{R} \oplus sp(4) = csp(4)$

We quickly confirm that there are no functional invariants on the base space by solving the Lie equation, which only has constant solutions for this system. As in all previous cases, the algebra of invariants on the base space $I^G = \mathbb{R}$.

4.2.1 $M \times M$

Like before, increasing dimensions increases the computational complexity of the problem of finding invariants. In 3 dimensions, we were unable to solve the Lie equation on $M \times M$, so we can't expect to be able to do so here. It is possible to find an explicit expression for the group action on the base space again, but maybe we can do without it. When finding generators for the algebra of invariants in 4 dimensions (in fact, in all even dimensions), we could make use of our previous discoveries in 2 dimensions. We will analogously make use of the results from the 3-dimensional case here.

In 3 dimensions, we could interpret all generators for the algebras of joint invariants as a ratio of relative invariants. These relative invariants came in two basic forms, one of which could be recognized as a projection of a symplectic area to a submanifold isomorphic to the tangent bundle on the base space. The corresponding submanifold this time is going to be $\mathbb{R}^4(x, y, p, q)$, so we naturally consider the expression:

$$R_{12} = x_1 p_2 - x_2 p_1 + y_1 q_2 - y_2 q_1$$

We verify that it is a relative invariant of weight 1. The other relative invariants could be understood as the standard contact form τ acting on components of the extended vector fields corresponding to the center of the action of $\mathfrak{g}$, evaluated at the origin. On the base space the center is given by the vector field $X_{11} = x\partial_x + y\partial_y + 2u\partial_u + p\partial_p + q\partial_q$. This leads us to consider the following two expressions:

$$\begin{aligned} R_{11} &= x_1 p_1 + y_1 q_1 - 2u_1 \\ R_{22} &= x_2 p_2 + y_2 q_2 - 2u_2 \end{aligned}$$

Both are verified to be relative invariants of weight 1. Dividing all 3 expressions by R_{22} gives us two absolute invariants, like before:

$$\begin{aligned} I_{11} &= \frac{R_{11}}{R_{22}} \\ I_{12} &= \frac{R_{12}}{R_{22}} \end{aligned}$$

The rank of the distribution defined by the vector fields is confirmed to be 8 at this point. As our space is 10-dimensional these are the only invariants.

$$\mathbb{R}(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{p}, \mathbf{q}) \supset J^{G\times 2} \simeq \mathbb{R}(\bar{\mathbf{I}}) \overset{2}{\supset} \mathbb{R}$$

4.2.2 $M \times M \times M$

The rank of our vector fields is now 10, while our space is 15-dimensional. We will have 5 independent generators for our field of invariants. Our formula provides us with 6 relative invariants R_{ij} where $1 \leq i \leq j \leq 3$. We divide every one of them by R_{33} and get 5 absolute invariants I_{ij}.

$$\mathbb{R}(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{p}, \mathbf{q}) \supset J^{G\times 3} \simeq \mathbb{R}(\bar{\mathbf{I}}) \overset{5}{\supset} \mathbb{R}$$

4.2.3 $M^{\times 4}$

Our distribution have now reached rank 11, which is the maximal rank. We are able to produce 9 independent generators for $J^{G\times 4}$, which is what we need.

$$\mathbb{R}(\mathbf{x},\mathbf{y},\mathbf{u},\mathbf{p},\mathbf{q}) \supset J^{G\times 4} \simeq \mathbb{R}(\bar{\mathbf{I}}) \overset{9}{\supset} \mathbb{R}$$

4.2.4 $M^{\times 5}$

Here we get an 11-dimensional algebra acting on a 25-dimensional space. Our formula produces 14 independent generators.

$$\mathbb{R}(\mathbf{x},\mathbf{y},\mathbf{u},\mathbf{p},\mathbf{q}) \supset J^{G\times 5} \simeq \mathbb{R}(\bar{\mathbf{I}}) \overset{14}{\supset} \mathbb{R}$$

4.2.5 $M^{\times 6}$

Analogously to the situation where we had a 4-dimensional symplectic base space, we now reach a point were our formula produces more generators than we need. We have an 11-dimensional algebra acting on a 30-dimensional space. There is 20 generators I_{ij}, which means there must be some relation between them. Recall that on the space $\mathbb{R}^{3\times 4}$ we found a relation between generators with skew-symmetric indices. Moreover, the relation turned out to be the same one as the relation $b_{1234} = 0$ we found on $\mathbb{R}^{2\times 4}$. From this we are able to predict that we will find the same relation between generators with skew-symmetric indices on $\mathbb{R}^5 \times 6$, as the relation $b_{123456} = 0$ we found on $\mathbb{R}^{4\times 6}$. We verify that this is the case:

$$\begin{aligned} b_{123456} =& I_{12}I_{34}I_{56} - I_{12}I_{35}I_{46} + I_{12}I_{36}I_{45} - I_{13}I_{24}I_{56} + I_{13}I_{25}I_{46} - I_{13}I_{26}I_{45} + \\ & I_{14}I_{23}I_{56} - I_{14}I_{25}I_{36} + I_{14}I_{26}I_{35} - I_{15}I_{23}I_{46} + I_{15}I_{24}I_{36} - I_{15}I_{26}I_{34} + \\ & I_{16}I_{23}I_{45} - I_{16}I_{24}I_{35} + I_{16}I_{25}I_{34} = 0 \end{aligned}$$

We find that the generators $I_{11}, I_{22}, I_{33}, I_{44}, I_{55}$ are all independent. Like before, we have no problem expressing one generator in terms of the others:

$$\begin{aligned} I_{56} =& (I_{12}I_{35}I_{46} - I_{12}I_{36}I_{45} - I_{13}I_{25}I_{46} + I_{13}I_{26}I_{45} + I_{14}I_{25}I_{36} - I_{14}I_{26}I_{35} \\ & + I_{15}I_{23}I_{46} - I_{15}I_{24}I_{36} + I_{15}I_{26}I_{34} - I_{16}I_{23}I_{45} + I_{16}I_{24}I_{35} - I_{16}I_{25}I_{34}) \\ & /(I_{12}I_{34} - I_{13}I_{24} + I_{14}I_{23}) \end{aligned}$$

Having removed I_{56} from our generating set $\bar{\mathbf{I}}$, we can describe $J^{G\times 6}$:

$$\mathbb{R}(\mathbf{x},\mathbf{y},\mathbf{u},\mathbf{p},\mathbf{q}) \supset J^{G\times 6} \simeq \mathbb{R}(\bar{\mathbf{I}}) \overset{19}{\supset} \mathbb{R}$$

4.2.6 Larger products

It is now possible to describe the field of joint invariants on larger products $M^{\times m}$. We will produce the same number of generators I_{ij} as we did on $\mathbb{R}^{3\times m}$, that is: $\dfrac{m(m+1)-2}{2}$. Of those, $5m-11$ will be independent.

Generators of the form:

$$\begin{aligned} I_{kl} =&(I_{12}I_{3k}I_{4l} - I_{12}I_{3l}I_{4k} - I_{13}I_{2k}I_{4l} + I_{13}I_{2l}I_{4k} + I_{14}I_{2k}I_{3l} - I_{14}I_{2l}I_{3k} \\ &+ I_{1k}I_{23}I_{4l} - I_{1k}I_{24}I_{3l} + I_{1k}I_{2l}I_{34} - I_{1l}I_{23}I_{4k} + I_{1l}I_{24}I_{3k} - I_{1l}I_{2k}I_{34}) \\ &/(I_{12}I_{34} - I_{13}I_{24} + I_{14}I_{23}) \end{aligned}$$

where $5 \le k < l \le m$, can be removed from our generating set. Thus:

$$\mathbb{R}(\mathbf{x},\mathbf{y},\mathbf{u},\mathbf{p},\mathbf{q}) \supset J^{G\times m} \simeq \mathbb{R}(\bar{\mathbf{a}}) \overset{5m-11}{\supset} \mathbb{R}$$

4.3 Higher dimensions

As in the even-dimensional case, we observe a pattern for generating invariants on our odd-simensional contact manifolds. We can consider the cases when $M = \mathbb{R}^{2n+1}(x^1,...,x^n,u,p_1,...,p_n)$ is our base space for $n > 2$. Now the annihilator for the standard contact structure is given by $\tau = du - p_1dx^1 - ... - p_ndx^n$.

The finite subalgebra $\mathfrak{g}$ will contain $n(2n+1)+1$ vector fields, and we can produce relative invariants on $M^{\times m}$ of the following form:

$$R_{ij} = \begin{cases} -2u_i + \sum_{k=1}^n x_i^k p_i^k, & \text{if } i = j \\ \sum_{k=1}^n x_i^k p_j^k - x_j^k p_i^k, & \text{if } i \ne j \end{cases} \quad \text{, where } 1 \le i \le j \le m.$$

Dividing every relative invariant by R_{mm} gives us absolute invariants:

$$I_{ij} = \frac{R_{ij}}{R_{mm}}$$

We showed that on linear symplectic manifolds of higher dimensions, we were always able to generate enough invariants. By similar arguments, the same holds true for linear contact manifolds of increasing dimensions.

Like on our symplectic manifolds, we get our first relation between the generators of $J^{\times m}$ on $\mathbb{R}^{(2n+1)\times(2n+2)}$. We learned that all generators of the form I_{ii} will be completely independent. Generators of the form I_{ij} where $i \neq j$ will have relations of the same type as the generators on symplectic manifolds. If we let $b^2_{1234} = I_{12}I_{34} - I_{13}I_{24} + I_{14}I_{23}$, we find that on $\mathbb{R}^{(2n+1)\times(2n+2)}$:

$$b^{2n}_{1\ldots 2n} = \sum_{j=2}^{2n+2} (-1)^j I_{1j} b^{2n-2}_{i1\ldots i(2n-2)} = 0$$

This implies that we can always express excess generators in terms of a minimal generating set $\bar{\mathbf{I}}$, which is obtained by removing elements that are unneeded the same way as before.

To Summarize:

$$\mathbb{R}(\mathbf{x}, \mathbf{u}, \mathbf{p}) \supset J^{G\times m} \simeq \mathbb{R}(\bar{\mathbf{I}}) \overset{(2n+1)m-n(2n+1)-1}{\supset} \mathbb{R}$$

5 Computations of symmetric joint invariants

In this chapter we will use our previous results to illustrate how to compute symmetric joint invariants on the symplectic manifold of dimension 2, as well as the contact manifold of dimension 3. The approach used is applicable to both larger products and base spaces of higher dimension.

Recall that we defined an unordered joint invariant as an invariant of the extended group $G \times S^m$, with an action given by $\sigma : G \times S^m \times M^{\times m} \to M^{\times m}$.

Alternatively, we can view an unordered joint invariant as an invariant of the symmetric group S^m acting on the space of invariants, with an action given by $\sigma_S : S^m \times I^{G\times m} \to I^{G\times m}$.

5.1 2-dimensional M

We return to the 2-dimensional symplectic manifold $M = \mathbb{R}^2(x, p)$, with an action given by $\mathfrak{g} = \langle x\partial_p, x\partial_x - p\partial_p, p\partial_x\rangle$. We will denote our space of symmetric joint invariants by I_S^G. Recall that the only invariants on the base space was constant functions. These are trivially symmetric joint invariants as well, and so we can conclude that $I_S^G = I^G = \mathbb{R}$.

5.1.1 $M \times M$

On this space, we found one generator for $I^{G\times 2}$, namely: $a_{12} = x_1p_2 - x_2p_1$. Let us denote the result of permuting the indices 1 and 2 by a_{21}. Observe that $a_{21} = x_2p_1 - x_1p_2 = -a_{12}$, which means that a_{12} is not a symmetric joint invariant. There exists a natural projection ρ from the space of functions to the space of invariants under a group action, given by:

$$\rho(f) = \sum_{g\in G} g \cdot f$$

We would like to find an element $f \in I^{G\times 2}$, (a polynomial function of the variable a_{12}), such that $\rho(f) = f$, under the action of S^2 . Applying this projection to $a_{12} \in I^{G\times 2}$ gives us: $\rho(a_{12}) = \sum_{\alpha\in S^2} \alpha \cdot a_{12} = a_{12} + a_{21} = 0$.
Even though ρ is surjective, it is clear that it does not map generators of $I^{G\times 2}$ to generators of $I_S^{G\times 2}$.

However, by linearity of ρ, we can see that for f to satisfy $\rho(f) = f$, it necessarily satisfies the condition $f(a_{12}) = f(-a_{12})$. Hence letting $f = (a_{12})^2$, we get: $\rho((a_{12})^2) = (a_{12})^2$.

As any linear polynomial in a_{12} is contained in the kernel of ρ, $(a_{12})^2$ is in fact a generator for $I_S^{G\times 2}$. We know that the number of generators for the symmetric algebra will be the same as for the ordinary one by dimensional reasons, as $I_S^{G\times 2}$ is a subfield of $I^{G\times 2}$. It is thus the only generator, which we will denote by s, where $s = (a_{12})^2$. Summarizing:

$$\mathbb{R}[s] \simeq I_S^{G\times 2} \subset I^{G\times 2} \simeq \mathbb{R}[a_{12}]$$

5.1.2 $M \times M \times M$

We found 3 generators for $I^{G\times 3}$ of the following form:
$a_{ij} = x_i y_j - x_j y_i$, s.t. $1 \leq i < j \leq 3$.

Again, we note that all linear polynomials in the variables a_{ij} maps to 0 under ρ. We could hope that squaring the generators of $I^{G\times 3}$ would yield all generators of $I_S^{G\times 3}$, but as it turns out, ρ maps all $(a_{ij})^2$ to the same element:
$\rho((a_{12})^2) = \rho((a_{13})^2) = \rho((a_{23})^2) = (a_{13})^2 + (a_{13})^2 + (a_{23})^2$

Of course, other quadratic expressions can also be mapped under ρ:

$$\rho(a_{12}a_{13}) = \rho(a_{12}a_{23}) = a_{12}a_{13} - a_{12}a_{23} + a_{13}a_{23}$$

We can also consider inputting cubic and quartic expressions, like for instance:

$$\begin{aligned} \rho((a_{12})^2 a_{13}) &= (a_{12})^2 a_{23} - (a_{23})^2 a_{13} + (a_{13})^2 a_{12} \\ \rho((a_{12})^4) &= (a_{13})^4 + (a_{13})^4 + (a_{23})^4 \end{aligned}$$

All we can say without further investigations, is that the algebra $I_S^{G\times 3}$ is generated by a finte number of images of monomials in $I^{G\times 3}$ under ρ. Finding a minimal generating set and all possible syzygies can be done in principle, but could be very difficult to compute in practice and is outside the scope of this thesis.

5.2 3-dimensional M

Let $M = \mathbb{R}^3(x,u,p)$ equipped with the standard contact structure, with the action given by:

$$\mathfrak{g} = \langle x^2\partial_u + 2x\partial_p, -x\partial_x + p\partial_p, -2p\partial_x - p^2\partial_u, 2u\partial_u + x\partial_x + p\partial_p\rangle$$

There are no nontrivial invariants on the base space, and so:
$I_S^G = I^G = \mathbb{R}$.

5.2.1 $M \times M$

Recall that we found 3 relative invariants on this space of the following form:

$$R_{ij} = \begin{cases} x_i p_i - 2u_i, & \text{if } i = j \\ x_i p_j - x_j p_i, & \text{if } i \neq j \end{cases} \quad \text{, where } 1 \leq i \leq j \leq 2.$$

Dividing all of them by the last one, gave us two rational absolute invariants:

$$I_{ij} = \frac{R_{ij}}{R_{22}}$$

In the symplectic case, the natural projection ρ to the space of symmetric invariant mapped generators of I^G to 0. On this space however, we find that:

$$\begin{aligned} t_1 = \rho(I_{11}) &= \frac{R_{11}}{R_{22}} + \frac{R_{22}}{R_{11}} \\ t_2 = \rho(I_{12}) &= \frac{R_{12}}{R_{22}} + \frac{R_{21}}{R_{11}} \end{aligned}$$

We can take those two elements to be a generating set for $J_S^{G\times 2}$ and conclude that:

$$\mathbb{R}(t_1,t_2) \simeq J_S^{G\times 2} \subset J^{G\times 2} \simeq \mathbb{R}(I_{11}, I_{12})$$

5.2.2 $M \times M \times M$

Here we have 5 absolute invariants:

$$I_{ij} = \frac{R_{ij}}{R_{33}}$$

There are many possible linear expressions to map under ρ:

$$\begin{aligned}
\rho(I_{11}) = \rho(I_{22}) &= \frac{R_{11}}{R_{33}} + \frac{R_{11}}{R_{22}} + \frac{R_{22}}{R_{11}} + \frac{R_{22}}{R_{33}} + \frac{R_{33}}{R_{11}} + \frac{R_{33}}{R_{22}} \\
\rho(I_{12}) &= \frac{R_{12}}{R_{33}} + \frac{R_{13}}{R_{22}} + \frac{R_{21}}{R_{33}} + \frac{R_{23}}{R_{11}} + \frac{R_{31}}{R_{22}} + \frac{R_{32}}{R_{11}} \\
\rho(I_{13}) &= \frac{R_{13}}{R_{33}} + \frac{R_{12}}{R_{22}} + \frac{R_{23}}{R_{33}} + \frac{R_{21}}{R_{11}} + \frac{R_{32}}{R_{22}} + \frac{R_{31}}{R_{11}} \\
\rho(I_{23}) &= \frac{R_{23}}{R_{33}} + \frac{R_{32}}{R_{22}} + \frac{R_{13}}{R_{33}} + \frac{R_{31}}{R_{11}} + \frac{R_{12}}{R_{22}} + \frac{R_{21}}{R_{11}}
\end{aligned}$$

We have even more quadratic expressions:

$$\begin{aligned}
\rho(I_{11}I_{12}) &= \frac{R_{11}}{R_{33}}\frac{R_{12}}{R_{33}} + \frac{R_{11}}{R_{22}}\frac{R_{13}}{R_{22}} + \frac{R_{22}}{R_{33}}\frac{R_{21}}{R_{33}} + \frac{R_{22}}{R_{11}}\frac{R_{23}}{R_{11}} + \frac{R_{33}}{R_{22}}\frac{R_{31}}{R_{22}} + \frac{R_{33}}{R_{11}}\frac{R_{32}}{R_{11}} \\
\rho(I_{11}I_{13}) &= \frac{R_{11}}{R_{33}}\frac{R_{13}}{R_{33}} + \frac{R_{11}}{R_{22}}\frac{R_{12}}{R_{22}} + \frac{R_{22}}{R_{33}}\frac{R_{23}}{R_{33}} + \frac{R_{22}}{R_{11}}\frac{R_{21}}{R_{11}} + \frac{R_{33}}{R_{22}}\frac{R_{32}}{R_{22}} + \frac{R_{33}}{R_{11}}\frac{R_{31}}{R_{11}} \\
\rho(I_{11}I_{22}) &= \frac{R_{11}}{R_{33}}\frac{R_{22}}{R_{33}} + \frac{R_{11}}{R_{22}}\frac{R_{33}}{R_{22}} + \frac{R_{22}}{R_{33}}\frac{R_{11}}{R_{33}} + \frac{R_{22}}{R_{11}}\frac{R_{33}}{R_{11}} + \frac{R_{33}}{R_{22}}\frac{R_{11}}{R_{22}} + \frac{R_{33}}{R_{11}}\frac{R_{22}}{R_{11}} \\
\rho(I_{11}I_{23}) &= \frac{R_{11}}{R_{33}}\frac{R_{23}}{R_{33}} + \frac{R_{11}}{R_{22}}\frac{R_{32}}{R_{22}} + \frac{R_{22}}{R_{33}}\frac{R_{13}}{R_{33}} + \frac{R_{22}}{R_{11}}\frac{R_{31}}{R_{11}} + \frac{R_{33}}{R_{22}}\frac{R_{12}}{R_{22}} + \frac{R_{33}}{R_{11}}\frac{R_{21}}{R_{11}} \\
\rho(I_{12}I_{13}) &= \frac{R_{12}}{R_{33}}\frac{R_{13}}{R_{33}} + \frac{R_{13}}{R_{22}}\frac{R_{12}}{R_{22}} + \frac{R_{21}}{R_{33}}\frac{R_{13}}{R_{33}} + \frac{R_{23}}{R_{11}}\frac{R_{21}}{R_{11}} + \frac{R_{31}}{R_{22}}\frac{R_{32}}{R_{22}} + \frac{R_{32}}{R_{11}}\frac{R_{31}}{R_{11}} \\
\rho(I_{12}I_{22}) &= \frac{R_{12}}{R_{33}}\frac{R_{22}}{R_{33}} + \frac{R_{13}}{R_{22}}\frac{R_{33}}{R_{22}} + \frac{R_{21}}{R_{33}}\frac{R_{11}}{R_{33}} + \frac{R_{23}}{R_{11}}\frac{R_{33}}{R_{11}} + \frac{R_{31}}{R_{22}}\frac{R_{11}}{R_{22}} + \frac{R_{32}}{R_{11}}\frac{R_{22}}{R_{11}} \\
\rho(I_{12}I_{23}) &= \frac{R_{12}}{R_{33}}\frac{R_{23}}{R_{33}} + \frac{R_{13}}{R_{22}}\frac{R_{32}}{R_{22}} + \frac{R_{21}}{R_{33}}\frac{R_{13}}{R_{33}} + \frac{R_{23}}{R_{11}}\frac{R_{31}}{R_{11}} + \frac{R_{31}}{R_{22}}\frac{R_{12}}{R_{22}} + \frac{R_{32}}{R_{11}}\frac{R_{21}}{R_{11}} \\
\rho(I_{13}I_{22}) &= \frac{R_{13}}{R_{33}}\frac{R_{22}}{R_{33}} + \frac{R_{12}}{R_{22}}\frac{R_{33}}{R_{22}} + \frac{R_{23}}{R_{33}}\frac{R_{11}}{R_{33}} + \frac{R_{21}}{R_{11}}\frac{R_{33}}{R_{11}} + \frac{R_{32}}{R_{22}}\frac{R_{11}}{R_{22}} + \frac{R_{31}}{R_{11}}\frac{R_{22}}{R_{11}} \\
\rho(I_{13}I_{23}) &= \frac{R_{13}}{R_{33}}\frac{R_{23}}{R_{33}} + \frac{R_{12}}{R_{22}}\frac{R_{32}}{R_{22}} + \frac{R_{23}}{R_{33}}\frac{R_{13}}{R_{33}} + \frac{R_{21}}{R_{11}}\frac{R_{31}}{R_{11}} + \frac{R_{32}}{R_{22}}\frac{R_{12}}{R_{22}} + \frac{R_{31}}{R_{11}}\frac{R_{21}}{R_{11}} \\
\rho(I_{22}I_{23}) &= \frac{R_{22}}{R_{33}}\frac{R_{23}}{R_{33}} + \frac{R_{33}}{R_{22}}\frac{R_{32}}{R_{22}} + \frac{R_{11}}{R_{33}}\frac{R_{13}}{R_{33}} + \frac{R_{33}}{R_{11}}\frac{R_{31}}{R_{11}} + \frac{R_{11}}{R_{22}}\frac{R_{12}}{R_{22}} + \frac{R_{22}}{R_{11}}\frac{R_{21}}{R_{11}}
\end{aligned}$$

We also get 5 more, of the form $\rho((I_{ij})^2)$. .

Note that some of these expressions are identically 0, like for instance $\rho(I_{12})$, due to the fact that $R_{ij} = -R_{ji}$. Like in the 2-dimensional case, on $M \times M \times M$, all we can say a priori is that the field $J_S^{G\times 3}$ is generated by a finte number of images of monomials in $J^{G\times 3}$ under ρ. Finding relations and a minimal generating set for $J^{G\times 3}$ is outside the scope of this thesis, and the above expressions illustrate the complexity of the problem.

References

[Arn13] Vladimir Igorevich Arnol'd. *Mathematical methods of classical mechanics*, volume 60. Springer Science & Business Media, 2013.

[DF04] David Steven Dummit and Richard M Foote. *Abstract algebra*, volume 3. Wiley Hoboken, 2004.

[Eis06] David Eisenbud. *The geometry of syzygies: a second course in algebraic geometry and commutative algebra*, volume 229. Springer Science & Business Media, 2006.

[Eis13] David Eisenbud. *Commutative Algebra: with a view toward algebraic geometry*, volume 150. Springer Science & Business Media, 2013.

[Hal15] Brian Hall. *Lie groups, Lie algebras, and representations: an elementary introduction*, volume 222. Springer, 2015.

[Hil93] David Hilbert. *Theory of algebraic invariants*. Cambridge university press, 1993.

[KL16] Boris Kruglikov and Valentin Lychagin. Global Lie-Tresse Theorem. *Selecta Mathematica*, 22(3):1357–1411, 2016.

[Kru14] Boris Kruglikov. Symmetries of filtered structures via filtered lie equations. *Journal of Geometry and Physics*, 85:164–170, 2014.

[Lee13] John M Lee. Smooth manifolds. Springer, 2013.

[MFK94] David Mumford, John Fogarty, and Frances Kirwan. *Geometric invariant theory*, volume 34. Springer Science & Business Media, 1994.

[Olv95] Peter J Olver. *Equivalence, invariants and symmetry*. Cambridge University Press, 1995.

[Olv99] Peter J Olver. *Classical invariant theory*, volume 44. Cambridge University Press, 1999.

[Olv01] Peter J Olver. Joint invariant signatures. *Foundations of computational mathematics*, 1(1):3–68, 2001.

[Ros56] Maxwell Rosenlicht. Some basic theorems on algebraic groups. *American Journal of Mathematics*, 78(2):401–443, 1956.