



# Discovering key interactions. How student interactions relate to progress in mathematical generalization

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## Abstract

This article presents a study of 8th grade students working in groups to solve a task about generalizing patterns. The study aimed to openly explore how progress in mathematical thinking might relate to the discourse. To do this, we first studied both separately. The progress in mathematical thinking was studied by inspecting how the groups progressed through different levels of generalization. The discourse was studied by characterizing each student interaction. When combining these, we realized that some specific types of interactions were related to students progressing to a higher level of generalization. We call these key interactions, and they were mainly of the types of advocating, locating, and reformulating. These seem clearly important for identifying evidence of progress during the discourses, but might also be helpful for understanding how specific types of interactions relates to sharing and growing mathematical thinking.

**Keywords** Key interactions · Student interaction · Generalization · Communication in mathematics

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## Literature

### Describing patterns

Expressing generality is a natural and rewarding part of how we, as people, make sense of the world around us. The ability to see the general in the special is fundamental and is a cornerstone of mathematics (Wilkie 2016). According to Mason et al. (2005), algebra gives us a vocabulary of symbols that we can handle and a language to express and treat the general. Developing algebraic thinking through generalization is a key activity in the work of algebra in school, and is the key to the process of a deep underlying understanding of algebra (Mason 1996). A widely used way to introduce and learn about generalization in elementary schools is to explore and describe figurative patterns as functions or algebraic expressions (Lannin et al. 2006; Mason 1996; Wilkie 2016). Within the algebraic generalization of such patterns, one tries to describe the pattern using a mathematical expression of the general element in the number sequence. Typically, one distinguishes between an explicit and a recursive generalization (Mason 1996). A recursive solution involves finding a local rule, and recognizing and using elements and changes from previous information in the pattern to find the next one in the series (Lannin et al. 2006). An explicit solution involves detecting a pattern that leads to a direct formula that will always work for any number in the series (Mason 1996).

Working with patterns is also about using symbols to think distinctively; this is more or less the essence of algebra (Radford 2010). According to Lannin (2005), figurative patterns will create a context where students may generalize one or more rules that can be used to determine other specific instances of the pattern. The generalization of numerical situations can be an aid in the transition to more formal algebra and can help students see relations in symbolic representations and elevate the students' arithmetic knowledge (Wilkie 2016). At the same time, students often find it difficult to see and express relations in growing figurative patterns (Lee 1996; Warren & Cooper 2007; Wilkie 2016). Many students demonstrate a lack of ability to visualize figurative patterns, express generalizations with natural language, and to convert patterns into tables of values and look for relations (Warren & Cooper 2007). Algebraic reasoning involves an understanding of mathematical and arithmetic contexts. Languages, actions, and the use of different representations and concrete materials express these contexts (Warren & Cooper 2007). Also, moving between different forms of representation can help students understand mathematical ideas and is fundamental to students' understanding of the interaction between the different representations (Wilkie 2016). Being able to express themselves visually can help reveal relations and solve problems, while at the same time, help students think about a higher and more abstract level (Tripathi 2008).

Kieran (2004) argues that the main elements of school algebra involve problem-solving, modeling, and the generalization of numerical and figurative patterns. Based on this, Kieran (2007) distinguishes between three different algebraic activities in school, i.e., *generational*, *transformational*, and *global/meta-level* activities. Generational activities refer to the actual objects of the algebra and are designed to create expressions and equations that represent a problem, figurative patterns, or numerical relations. Transformational activities deal with the computational processes in the

algebra, such as extension, factorization, and solving equations and inequalities. Global/meta-level activities are activities where algebra is used as a tool, typically without using formal algebraic symbols. This includes problem-solving, argumentation, making assumptions, and looking for relations and structures (Kieran 2004). Global/meta-level activities also relate to early algebra and the development of algebraic thinking without formal symbols. Carraher and Schliemann (2007) describe three entry points into algebra, namely, *arithmetic and numerical reasoning*, *arithmetic and quantitative reasoning*, and *arithmetic and functions*. All these seem to start with global/meta-level activities but aim to move the students into using formal generalization using symbols gradually.

While there might be different types of algebraic activities and entry points into algebra, it is also interesting to look into what algebraic thinking is. Based on prior research, Radford (2013) suggests three conditions that characterize algebraic thinking. The first condition, *indeterminacy*, is that the problem involves not-known numbers such as unknowns, variables, or parameters. The second condition, *denotation*, is that the indeterminate number(s) needs to be named or symbolized. This naming does not limit itself to using letters like  $x$  and  $y$ , but might also be symbolized through natural language, gestures, or unconventional signs. The third condition, *analyticity*, is that the indeterminate number(s) are treated as if they were known so that one can operate on them (i.e., adding, subtracting, multiplying, dividing). Then one can find solutions by deducing, not only by guessing or trial-and-error.

Lee (1996) further details the concept of generalization by suggesting three levels of students' work with generalization through figurative patterns. The *level of perception* provides a picture of how students interpret and perceive the pattern, which can guide the further work of generalization. The ability to verbally express the pattern falls within the *level of verbalization*. The transition between these two levels can often be a sliding one. The *level of simulation* is where students can express a formal solution with one or more variables, recursively or explicitly (Lee 1996). In this way, Lee (1996) details the process of generalization. At the same time, both the perception level and the verbalization level of generalization might be seen as global/meta-level types of activities, where formal symbols are not used or at least not the main point of the work. Looking at it this way illustrates that the separation of generalization and global/meta-level activities might not always be meaningful, as generalization might be a vital part of many global/meta-level activities. After all, Radford (2006) argues that letters do not amount to doing algebra because, fundamentally, algebraic thinking is a way of reflecting in mathematics, and that other semiotic representation might work as well as any letter. The three levels suggested by Lee (1996) illustrates just this, by describing all three levels as algebra while just the last one includes formal algebraic symbols.

Like Lee (1996), Radford (2010) suggests different levels of algebraic generalization. Radford (2010) divides the students' solution strategies into two main levels, *arithmetic generalization* and *algebraic generalization*, and further divides the last one into three sub-levels (see Table 1). Arithmetic generalization consists of practice-based solution strategies, and students base their solutions on trial and error and other guess strategies. They are unable to use algebraic reasoning to give an explicit expression of any element. The algebraic generalization level is where students can develop general rules for any number, such as identifying general objects in a figurative pattern. The solution is explicit, with rules that give an expression for any number in the pattern.

Radford (2010) divides the algebraic generalization into three sub-levels. *Factual generalization* means that the general objects are expressed implicitly through actions, such as gesticulation, words, or gestures. Radford (2010) calls this a generated solution for emerging algebraic reasoning, and it contributes to the development that occurs later in a higher level of algebraic generalization. The second level, *contextual generalization*, implies contextual references to the variables in the pattern, for example, by a mixture of mathematical symbols and concepts through natural language. *Symbolic generalization* is the highest level. The students are here, able to describe the rule and express it with symbols or to describe the rule for any number with an entirely symbolic equation (Radford 2010).

Compared with Lee (1996), arithmetic generalization is new. While it is possible to argue against that this can be called generalization, it is also possible to argue that this type of practice-based trial and error might be an important part of initial thinking or searching for generalizable ideas. The three sub-levels of algebraic generalization has similarities with the last two of Lee's (1996) levels. Factual and contextual generalization seems to be two levels of verbalization, while symbolic generalization is quite similar to simulation. In this way, Radford (2010) further details the students' generalization process while leaving out the unobservable level of perception.

In a further development and operationalization, Wilkie (2016) first adapted a rubric for analyzing levels of generalization from Markworth (2010) and later incorporated Radford's (2010) levels into this rubric. The result was a rubric with six levels, from pre-1 to 5, as illustrated in Fig. 1.

In addition to developing this rubric, Wilkie also used it for analyzing students' work with pattern generalization. In a study of 102 Australian students aged 12–13 years, Wilkie (2016) used this rubric to find each student's highest level of generalization and then linking this to different aspects of their representation and reasoning. In another article based on the same data, Wilkie (2019) used this rubric to study how the students' pattern generalization might relate to their attention to covariation with their graphs and rules.

## Communication

The concepts above, describing students' work with generalization, are developed from students sharing their ideas. There are, of course, different ways to access student ideas, from constructed situations like interviews to observing them work in more natural

**Table 1** Radford's (2010) levels of algebraic generalization

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Arithmetic generalizations are practice-based solutions strategies where students base their solutions on trial and error and other guessing strategies.
Algebraic generalizations with three sub-levels
Factual generalization means that the general objects are expressed implicitly through actions, such as gesticulation, words, or gestures.
Contextual generalization implies contextual references to the variables in the pattern, for example, by a mixture of mathematical symbols and concepts through natural language.
Symbolic generalization is when students can describe the rule and express it with symbols and/or to describe the rule for any number with an entirely symbolic equation.

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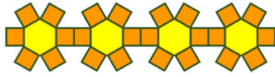
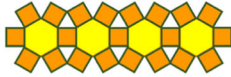
Score	Highest level of generalisation evidenced	Illustrative examples (post-task)
Pre-1	Extend figural growing pattern incorrectly	
1	Extend a growing (figural) pattern by identifying its physical structure, features that change, and features that remain the same (evidence of arithmetic generalisation)	
2	Identify quantifiable aspects of items that vary in a growing pattern (evidence of emerging factual generalisation)	Num. blocks for 7 daisies in a chain: '43 because it has been added by six blocks every time so $6 \times 6 = 36 + 7 = 43$ '
3	Articulate functional relationship between quantifiable aspects of a growing pattern by identifying the change between successive items in the sequence (evidence of factual generalisation)	Num. blocks for 17 daisies: ' $7 + (16 \times 6) = 103$ ' 'squares = you start with 6 and add 5 on each time hexagons = whatever the day is that's what the number of hexagons is'
4.1	Describe the relationship between a quantifiable aspect of an item and its position in the sequence (evidence of contextual generalisation)	'You times the number of flowers by 6 blocks and add 1' 'Take one away from the daisy chain number and multiply that by 6 then add 7'
4.2	Use symbols or letters to represent variables in explicit functional rule (evidence of emerging symbolic generalisation)	' $dn \times 6 + 1$ dn=day number' ' $7f - (f - 1)$ '
4.3	Represent generalization in a full, symbolic equation (evidence of symbolic generalisation)	'n = number of flowers b = number of blocks $b = 1 + (5n) + n$ ' 'f = number of flowers b = number of blocks $b = 6f + 1$ '
5	Apply an understanding of linear functional relationships between variables to further pattern analysis and multiple representations	Correct construction of Cartesian graph with discrete points; Reverse functional thinking: 'It's not possible to make a flower with 100 blocks because 6 doesn't go into 99 and you have to plus 1'

Fig. 1 Rubric for analyzing pattern generalization (Wilkie 2019)

situations. In order to access ideas in natural situations, there needs to be some discourse between students or between students and teachers, and scholars have described several types of such discourses. One such is exploratory talk that, according to Barnes (2008), often occurs early in a process and can often be incomplete and hesitant while the participants try out ideas and thoughts. More generally, Cobb (1995) distinguishes between two types of cooperation between students. The first type is direct, when the task can be solved by routine operations and by students coordinating their efforts to solve the task. According to Cobb (1995), such direct cooperation creates few opportunities to learn. The second type is indirect, where students solve tasks by thinking aloud, building on each other's suggestions, and solving the tasks independently of each other. The learning during indirect cooperation occurs when a student says something significant and clarifying for the other student's thinking at that point. Indirect communication is similar to exploratory talk in the sharing of ideas and suggestions. Also, Webb (1991) emphasizes that students working in small groups might develop an understanding as they share a common daily language that makes it easier to explain difficult concepts and words to each other. Another effect is that students need and get confirmation from each other.

While such general descriptions of discourses might be helpful, the quality of the discourse clearly depends on what the participants say. There are several frameworks

offering concepts to describe interactions turn-by-turn. One such, the *inquiry-cooperation model* by Alrø and Skovsmose (2002, 2004), describes eight different types of interactions used by both teachers and students (see Table 2).

The eight types of interactions in the inquiry-cooperation model (Alrø & Skovsmose 2004) can be seen as detailing more general descriptions of discourses. For example, getting in contact, locating, and identifying are all related to exploratory talk (Barnes 2008), while thinking aloud and reformulating can be seen as indirect cooperation as described by Cobb (1995). Also, the eight types of interaction can be used to describe communication on a turn-by-turn basis, which might be vital to understanding how individual or common understanding develops during a solution process. The inquiry-cooperation model were developed from classrooms where the teachers and students worked tightly together, in classrooms that might be described as *inquiry/argument classroom culture* (Wood et al. 2006), where the students share ideas in order to create discussions with fellow students, often including challenges, disagreements, argumentation, and justification. It also seems to be similar to what Brendefur and Frykholm (2000) call *instructive communication*, where the students and teachers participate jointly together to build upon and deepen students' present understanding. Other has developed frameworks based on more ordinary classrooms. For example, in studies of classrooms characterized by the IRF pattern where the teacher *initiates*, the student *responds*, and the teacher gives *feedback*, as described by Sinclair and Coulthard (1975), Drageset (2014, 2015a, b) describes qualitatively different types of interactions. Typically, the teachers tended to simplify the tasks by giving hints that changed the tasks to easier ones, they frequently asked one question for each step until the student arrived at a solution, and the primary type of student response was answers to very easy questions (teacher-led responses). Such interactions cannot be found within the inquiry-cooperation model. This does not limit the value of any of the frameworks as long as one is aware that the types of student and teacher interactions that are used probably depend on the classroom culture. Based on the description of how the inquiry-cooperation model was developed, one can assume that this will work best when describing problem-solving activities.

**Table 2** The inquiry cooperation model (Alrø & Skovsmose 2004)

Getting in contact	Means tuning in to each other, being present and aware of each other's contributions
Locating	Means zooming in on a topic and finding out something new or something of which one was unaware
Identifying	Means identifying the subject matter and making it accessible to the others
Advocating	Is constituted by experimental or hypothetical argumentation in which one examines one's own or others' ideas or proposals
Thinking aloud	Means expressing one's thoughts, ideas, and emotions
Reformulating	Means repeating what has been said in a slightly different tone of voice or adding something to it
Challenging	Means questioning already established knowledge
Evaluating	Means correcting mistakes, giving positive or negative feedback, providing advice, supporting, giving confirmation or credit

As described above, there are different types of algebraic activities (such as generalization, transformation, global/meta), and there are different levels of generalization. Little is known about how different types of interactions might relate to different types of algebraic activity, or if the interactions would look different at different levels of generalization. Moreover, it might be that some of the categories will reveal more about thinking than others.

## Method

This project aimed to study how students work through collaboration and discussion to find generalizations. We particularly wanted to study students' interactions and the development of their mathematical thinking. The research question was formulated as How do students cooperate and contribute to finding solutions for mathematical generalization tasks, and how are interactions and progress related?

To achieve this, we chose to study students working on mathematical problems in groups. We selected twelve students from an eighth grade (ages 13–14) classroom in Norway. Their teacher helped us form three groups of students who were on similar levels in mathematics and were clever at formulating their thinking. This means that most students were at or above the classroom's average mathematical level. We divided them into three groups of four students each to allow every student to participate actively.

The task chosen was related to algebraic generalization. The students were presented with illustrations of a table formed as a trapezium and how the tables can be put together (see Fig. 2). Their tasks were to find the number of chairs with two, five, ten, one hundred, and finally  $n$  tables. In Norway, students in the eighth grade have some experience with early algebra but have just started with more formal algebra and the use of symbols. Most students have not worked on formulating a symbolic formula describing a pattern.

Each group used approximately 25 min on the task (about the same time for all groups), and they were not stopped or rushed before finishing the task. During the work, the researcher was an observer-as-participant (Cohen et al. 2011). The main intention was to be an observer, but questions and hints were prepared in case the group could not find a way forward. Also, the researcher sometimes asked the students to explain their thinking to each other (not to the researcher). The groups came into a small room where the researcher had prepared a table to sit around with the tasks written on handouts (Fig. 2 and the tasks of finding the number of chairs for two, five, ten, hundred, and  $x$  tables). The researcher's video recorded the activity by letting the camera standstill all the time, and no other was in the room.

All video recordings were then transcribed and mainly analyzed by the first author. The second author read through vital parts of the analysis regularly. Any disagreements



Fig. 2 Illustration of the diagrams presented with the task



or uncertainty were solved through discussion between the first and second authors. Colleagues of the first author were also discussion partners. The actual analysis was separated into three steps. The first step was related to the mathematics and was a turn-by-turn conversation analysis (Hutchby & Wooffitt 1998; Sidnell 2010) of each group's solution process, trying to identify interactions that related to the four levels of generality suggested by Radford (2010). The choice of Radford's (2010) levels were based on a thorough search for useful frameworks for describing students' development of their generalizations during their task work. Not many such frameworks were found, and Radford (2010) were preferred to Lee (1996) as the first level; perception were not possible to observe. Also, Radford (2010) were preferred to Wilkie's (2016, 2019) rubric as it was easier to separate between the levels, and hence more precise, with our data. In the second step of the analysis, we also used conversation analysis, where each turn was categorized in terms of the eight communicative features of the inquiry-cooperation model (Alrø & Skovsmose 2002). Other possible frameworks exist, but since the discourse was characterized by problem-solving, we chose the inquiry-cooperation model as this model is developed from inquiry-based activities. It turned out that the inquiry-cooperation framework was useful as it had diversity with its eight categories; all eight categories were found represented in all group discourses, all interactions fitted into the categories, and the framework revealed differences between the groups that we could not see without it. The third step of the analyses sought to find connections between the two first parts, particularly related to which types of interactions (from the inquiry-cooperation model) gave information about the level of generalization at a given time. More specifically, we ended up looking for which types of interactions gave evidence for progress from one level of generalization to the next.

In the findings below, all the names of the students are anonymized. To separate the groups from each other, all students in the first group have names starting with "A," all students in the second group have names starting with "M," and all students in the third group have names starting with "J."

## Findings, part one: The mathematical solution process

This part of the analysis addresses the mathematical solution process that emerged in the conversation and the different solutions that the students generated. These solution strategies are seen in the light of Radford's (2010) four levels of generalization. The purpose of the analysis is to show the diversity of solutions and progress related to generalization, regardless of which group generated them.

### Arithmetic and non-algebraic generalization

In the following example, the students are trying to figure out how many chairs could be placed around two merged tables.

*Ada:(reads the task) If the tables are placed like this. How many are there then?  
Okay, so then it is two tables?*  
*Anders:Yes.*



Alex: *So, at one table there are five chairs?*

Anna: *When one place two tables together, you have to subtract two, because there cannot be anyone sitting in between.*

Ada: *Yes, you cannot sit where the table merged, so then it is only four at each table then. It will be eight then (the rest of the group confirms by nodding.)*

Anders: *Shall we write eight then, or?*

Researcher: *Yes, you can write down the answer (the student starts to write).*

The students used the table on the assignment sheet, then added the next table by placing two tables together and subtracting two, as two chairs disappear between the tables. The students recognized and used the elements from earlier in the pattern to find the next one in the series. In this process, the students use the knowledge at one step to find the next step, which is typical for recursive methods as described by Mason (1996). The answer is given verbally and not by an algebraic expression, which is an example of arithmetic generalization, which is the lowest generalization-level according to Radford (2010).

Later, the students were to find the number of chairs around five tables.

As Fig. 3 illustrates, several groups chose to draw the tables, and then count the number of chairs. Although there were suggestions from a student to use multiplicative thinking where they multiplied four chairs by the total number of tables, most chose to make the drawing as a visual aid to ensure a correct answer. This method could be called “drawing and counting.” The students created a larger structure and moved a step further in the process from the previous situation. It is a visual representation that is more advanced and systematic than the solution used for two tables. According to Weingarden et al. (2019), such solutions are at a low level of algebraic thinking because the students reproduced facts already given.

The drawing of five tables is more advanced than the discussion about two tables using a systematic drawing approach. However, both of these solutions are what Cobb (1995) calls practice-based solutions (drawing and counting) and thus clearly arithmetic solutions (practical) with no apparent signs of algebraic generalization. Also, the drawings in Fig. 3 might be seen as somewhat surprising as all pairs of tables are drawn on one side (which can also be seen in Fig. 4). The students did not discuss if this way of drawing, or with the pairs of chairs alternating, was correct. One interpretation is that this is only a simplification, as it seems to be faster to draw it this way, and the counting is easier and not any kind of misunderstanding. However, as we never illustrated where the chairs should be placed when two tables were put together, it might also be that some of the students thought it should be done this way while others thought it should alternate. Independent of reason, and since this was not discussed, choose of placement for the pairs of chairs did not seem to disturb or affect the

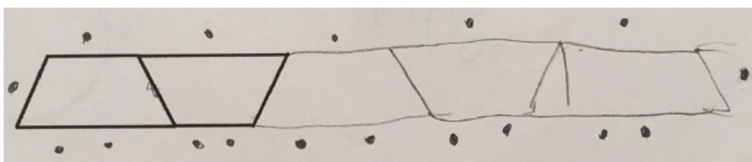


Fig. 3 Student illustration from group 1

algebraic generalization. Figures 3 and 4 might also illustrate flexibility in thinking as they can adjust the figures but keep the main features.

### Factual generalization

When the students discussed solutions for the task of ten tables, one student suggested this:

Mons: *But 10 tables, isn't that twice as much?*

Mie: *No (shaking her head).*

The discussion that follows shows that the previous solution strategies are not an optimal strategy when the number of tables increases to ten. The students, therefore, started to reason based on what they already knew. Expressing a connection between a larger number of tables proved to be problematic for the students. Mie and Mons decided that the solution to this task had to be 34 chairs, but they could not argue for the solution. They decided to draw the tables and chairs.

Mie: *So,  $17 + 17$  is 32?*

It is an interesting finding that Mie now seems to doubt her computing skills when she asked the question above. When the students chose to return to the drawing method, they discovered a conflict between the doubling/halving answer and the drawing. When Mie looked closer at the drawing, she decided to draw a line between the tables and manage to find the reason why the answer is 32 and not 34.

Mie: *Yes! If we divide it here. Then it will be a person there because it is like the end of the table. So then it will be minus two. Two people because the table will be re-assembled, and then they will be gone.*

By making use of both the doubling method and the visual draw and count method, she seemed to understand the reason for the answer. When there was a discrepancy between the two different responses, the students had generated that there was an uncertainty in the group as to which solution they should rely on. The students expressed the highest confidence in the practical solution where they could see the answer in front of them.

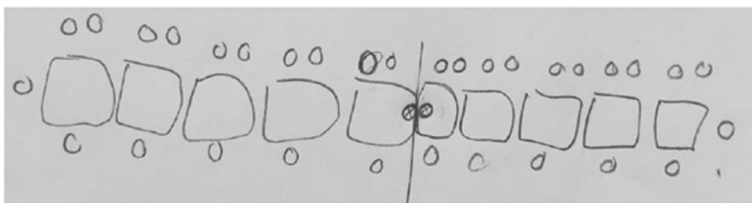


Fig. 4 Student illustration in group 2

This is a sign that they were somewhere between the practice-based solution strategies and factual generalization. *Factual generalization* (Cobb 1995) means that the general objects are expressed implicitly through actions, such as gesticulation, words, or gestures. Also, the thoughts that emerge within factual generalization form the basis for further thinking and development of algebraic reasoning. While this example illustrates some early generalization, the students had not developed an understanding of the continuous part of the pattern and had no basis for expressing a mathematical formula either verbally or symbolically. However, Mie showed an ability to move between different representations, and according to Tripathi (2008), visually expressing something is an essential factor in problem-solving and can help students see relations and solutions.

Another solution strategy based on previous findings appeared when one of the groups was to find the number of chairs at one hundred tables.

*Mons: So now we multiply the 32 chairs by 10, but we have to minus something. So it will be 320 minus something.*

Here, Mons uses the knowledge that at ten tables, there are 32 chairs. The question that appeared was what they could subtract to find the correct answer? How many chairs “disappear” between the tables? In this case, the students are moving away from recursive thinking where they are dependent on the last number in the pattern and moving towards a more explicit way of producing ideas and thoughts. They build on already solved tasks, where they calculated specific cases. The students develop a partial generalization (multiplication) and identify what needs to be generalized (what to subtract).

As the examples illustrate, the students use doubling and multiplication. Doubling can be seen as emerging recursive generalization, while multiplication seems to be emerging explicit generalization. In both cases, the students struggle with what must be taken away. Such emerging algebraic reasoning is a feature of factual generalization (Radford 2010).

### **Context-based generalization**

One of the other groups chose another solution strategy for the task of 100 tables. These students also took the starting point for the solution they found for 10 tables.

*Jens: Then I think we can only remove those on the end and multiply by ten.*

Here, Jens tells us that in order to find the answer for 100 tables, he will remove the chairs at the ends and multiply that number by 10. He already says at the start of the conversation that the tables have a chair on the short side and two on the long side. This thinking made the group to discover the general in the pattern already in the beginning. Expressing the general objects orally in the pattern is one of the characteristics of both contextual and symbolic generalization. However, since the students were only able to attach words to the pattern and not a formula with

numbers and variables, this solution is similar to what Cobb (1995) describes as context-based generalizations. What this group does differently from the above example is to remove the two end chairs before they multiply. This indicates increased insight into the general features of the pattern.

Some students noticed that the tables were shaped like a trapezium, which holds a chair on the short side and two chairs on the long side, and used this as a basis for further reasoning. The characteristics of these situations were that the students explicitly attached words to the general object in the pattern, but without expressing the formula symbolically. Such explicit expressions of the general features of a pattern using natural language are similar to what Radford (2010) describes as a contextual generalization.

### Symbolic generalization

Towards the end of the sequence, examples indicated that all groups were able to generate solutions at the level of symbolic generalization, which is Radford's (2010) highest level of generalization.

*Ada: So yes, then it really will be like multiplying the number of tables by how many chairs it is, and then adding by two. Will this be right?*

*Anna: I wrote three times the number of tables and added two.*

Ada's explanation can be considered to contain some generality in multiplying the number of chairs by the number of tables without expressing specifically the number of chairs. This statement, therefore, fits better into the category of contextual generalization. Anna expresses symbolic generalization by multiplying the number of tables by three, then add two. This solution is a valid explicit formula in relation to the pattern. Some of the students were able to express the generality using the letter  $n$ , as requested in the final assignment:

*Mons: Three multiplied by  $n$ .*

In this statement, Mons takes a fellow student's explicit formula as a starting point, three times the number of tables plus two. His statement shows that he can use the variable  $n$  and has an understanding that  $n$  is a notation for the number of tables. Therefore, the solution process lies within the level of symbolic generalization, as the student uses algebra's alphanumeric semiotic system (Radford, 2010). Jenny also showed the ability to express herself with symbolic generalization towards the end of the sequence:

*Jenny writes: We multiply the chairs that are not on the outside by the number of tables and then add two.*

*$3 \times \text{number of tables} + 2.$*

*$n \text{ tables} + 2 = ?$*

Jenny shows her ability to use the general information that the group has identified in the work on the generalization task to give an explicit expression of each element in the pattern. Also, some managed to write it as  $3 \times n + 2$ .

### Summary findings part one

All three groups moved through all four levels of generalization. Above this is illustrated with examples from all groups, as a full presentation of all groups' progress would take too much place. This also means that all three groups solved the task at a symbolic level. Nevertheless, although all groups were able to generate one or more solutions at the level of symbolic generalization, not all the individual students seemed to reach this level. Based on the group's dialogs, we found that three out of four students reached the symbolic generalization level in two of the groups, while only one student seemed to reach the symbolic generalization level in the third group.

### Findings, part two: group dialog

The turn-by-turn analysis of the dialog identified examples of all of the eight communicative features suggested by Alrø and Skovsmose (2002) in all three groups. Altogether, 730 turns were analyzed, with 209, 228, and 293 in the respective groups. We do not present examples of all categories here but instead focus on two main areas. First, two findings that lead us to suggest adjustments to the inquiry-cooperating model. Second, a comparison between the group discourses based on the categorization of all interactions using the inquiry-cooperation model.

The inquiry-cooperation categories worked fine to separate different types of student interactions. We were able to categorize all interactions within the existing framework. At the same time, all categories involved variations. In two cases, these variations seemed to form distinct sub-categories. These two separate interactions illustrate the first.

*Mons: Because when there is one and one table, then you have all the sides. But when you put them together, then you lose one of the sides of each table. That is how I think.*

*Anna: Yes (insisting)! This solution works!*

Both examples illustrate advocating a position or proposal. However, while Mons argues and explains, Anna just claims. We found these two types of advocating to be consistently and distinctly different, which suggests that one should look at two types of advocating, i.e., *advocating by claims* and *advocating by arguments*. This is an interesting difference because advocating by reasons involves sharing reasons, while advocating by claims does not.

A second category where we found distinct sub-categories is locating. Here, we observed a distinct difference between *locating as questions* (in order to understand or explore) and locating that involved *suggestions*.

In order to make the thoughts public and help the students, the researcher occasionally participated in the discourse. These interactions were also coded using the eight categories of the inquiry-cooperation model (Alrø & Skovsmose 2002). The researcher used three of these categories, namely, getting in contact (to keep the structure in the discourse), identify (to explain meaning), and challenge (to request an explanation or justification and to give hints).

Looking at all three groups combined, the most frequent type of interaction was advocating (152 turns). Three other types were also rather frequent, i.e., evaluating (127), getting in contact (118), and locating (109), while challenging (52) and identifying (30) were the least frequent ones. Somewhere in the lower middle were reformulating (73) and thinking aloud (69).

The frequency of each category varied somewhat between the groups. Perhaps the most interesting observation is the difference between groups 1 and 2. Getting in contact (22 vs 64) and locating (26 vs 52) were clearly more frequent in group 2, while challenging (25 vs 8) were clearly more frequent in group 1 (see Table 3). Our analysis cannot explain these differences between the groups. But it was possible to see that group 2 used more time to openly explore ideas and listen to each other without arguing or disagreeing. At the same time, the members of group 2 rarely challenged each other's ideas, which meant that their process was more linear and with fewer disagreements to be solved and fewer changes in approach. Furthermore, group 2 used far more interactions than groups 1 and 3 to solve the task. This higher frequency was mainly related to getting in contact, locating, advocating, and thinking aloud. This also illustrates how group 2 explored ideas, shared them, and supported them more frequently than the other groups.

### Findings, part three: key interactions

As illustrated above, all groups moved step by step through the four levels of generalization (Radford 2010). Also, all eight categories from the inquiry-cooperation model (Alrø & Skovsmose 2002) were found in all three groups, with some variation. The next step was to explore how interactions and progress interact. To look at this, we

**Table 3** Frequency of each category of the inquiry-cooperation model (Alrø & Skovsmose 2002)

	Total	Group 1	Group 2	Group 3
Getting in contact	118 (16%)	22 (11%)	64 (22%)	32 (14%)
Locating	109 (15%)	26 (12%)	52 (18%)	31 (14%)
Identifying	30 (4%)	11 (5%)	10 (3%)	9 (4%)
Advocating	152 (21%)	37 (18%)	62 (21%)	53 (23%)
Thinking aloud	69 (9%)	23 (11%)	30 (10%)	16 (7%)
Reformulating	73 (10%)	20 (10%)	25 (9%)	28 (12%)
Challenging	52 (7%)	25 (12%)	8 (3%)	19 (8%)
Evaluating	127 (17%)	45 (22%)	42 (14%)	40 (18%)
Totals ( <i>n</i> )	730	209	293	228

combined these two frameworks to explore what types of interactions were associated with a specific level of generalization. After some exploration, we found that the interactions that first showed that a group had moved to a higher level of generalization were of particular interest. We call these *key interactions*. Table 4 presents the frequencies of key interactions for each group.

In an ideal or linear process, each group would have one key interaction for each level, adding up to four for each group and twelve in total. However, as the different numbers reflect, we found a more complicated picture, and sometimes two or three consecutive interactions in combination gave evidence for a new level. By looking at all key interactions for all three groups, we found that those that belonged to the category of advocating were clearly the most frequent. Looking at the two subcategories we developed for advocating, we found that these were almost equally distributed, with 10 arguments and 8 claims. Altogether, 18 of the 34 key interactions were advocating, 7 were locating, 5 were reformulating, 2 were thinking aloud, 1 was identifying, and 1 was evaluating. This clearly illustrates that advocating was the type of interaction that most frequently evidenced a new and higher level of generalization. Advocating was also the most frequent type of interaction in general, but its share of key interactions was 2.5 times higher than its share of all the interactions. It is also worth noting that no key interactions were challenges or getting in contact. Together, the categories of advocating, locating, and reformulating add up to 30 of 34 key interactions. These three types of interactions evidence almost all the progress in mathematical generalization in the three groups.

This analysis of the key interactions indicates that advocating, locating, and reformulating interactions are important in some way. It might be that the use of advocating, locating, and reformulating is the key to progress and that the use of these three types of interactions more often contributes towards reaching higher levels of generalization than do other types of interactions. This study can only indicate this, but the findings warrant further research related to the connection between particular types of interactions (such as advocating, locating, and reformulating) and the development of students' mathematical thinking. However, it is also possible that advocating, locating, and reformulating are only the most important messengers that inform us that the individuals or the group have moved their thinking to a higher level of

**Table 4** Frequency of key interactions

	Total	Group 1	Group 2	Group 3
Getting in contact	0	0	0	0
Locating	7	3	2	2
Identifying	1	0	1	0
Advocating	18	6	6	6
Thinking aloud	2	2	0	0
Reformulating	5	1	0	4
Challenging	0	0	0	0
Evaluating	1	0	0	1
Totals ( <i>n</i> )	34	12	9	13



generalization. Even so, this finding seems important, as looking for these three types of interactions might be the key to grasp progress in mathematical thinking.

## Discussion and conclusion

This project aimed to study students' mathematical reasoning during group work, and in particular, search for connections between their interactions and their progress. We chose to use a task about patterns where the students needed to work on algebraic generalization. Using Radford's (2010) levels of generalizations, we identified all four levels in chronological order in all three groups. All groups started at the lowest level, algebraic generalization, and then moved through factual and contextual generalization. Also, all groups managed to develop solutions at the level of symbolic generalization. This might be surprising, given that students in the eighth grade in Norway typically have limited knowledge about algebraic generalization or generalization at all.

Even though all groups managed to solve the task using symbolic generalization, there were, of course, differences within the groups, with some students seeming not to go through the generalization levels at the same pace as the rest of the group. However, the evidence of this is too weak to go deeper into, as these students typically said less.

After studying the mathematical content, we chose to use the eight categories of the inquiry-cooperation model (Alrø & Skovsmose 2002) in a conversation analysis on a turn-by-turn basis. We found that all eight categories were used in all three groups, with advocating the most frequent and some minor differences between the groups. We also suggested subcategories for two of the categories. Advocating interactions were observed in two distinct forms, either with arguments or as a claim. Locating interactions were also observed in two distinct forms, either locating as questions in order to understand or explore, or locating including suggestions.

By studying how the content develops through increasing levels of generalization and how students participated using different types of interactions, we obtained a basis for the study of how progress in mathematical thinking related to specific types of interactions. Of particular interest were the first interactions that indicated a specific level of generalization, and we called them key interactions. Typically, these were a few interactions for each level and not only a single one. It is also worth noting that the key interactions did not come from a few students, but from many different students, indicating that most students really participated in bringing the process forward, and that they probably achieved more together than they would have managed separately.

It turned out that the dominating type of key interactions was advocating (18 of 34), while locating (7) and reformulating (5) also were relatively frequent. Also, it is worth noting that no key interactions were challenges. Also, combining the categories of advocating, locating, and reformulating accounts for 29 of 34 key interactions. Together, these categories describe interactions that zoom into, argue for, or claim proposals are correct, and change slightly or add something to proposals. This indicates that zooming into interesting details (locating), claiming and arguing for a stance (advocating), and developing ideas through reformulation (reformulating) might be the way that these groups were able to progress towards higher levels of generalization and use this to find solutions to the task. Or at least these categories are the messengers that inform us of progress.

There is a clear limitation in that this study is limited to a small number of students and tasks. But the findings indicate some interesting connections between progress and particular types of student interactions. This should be further studied with more students, other tasks, and in other grades. Furthermore, research is needed as to how these types of student interactions affect progress beyond generalization tasks.

Further research is also needed on the connection between different types of interactions and students' development of mathematical thinking. In particular, there is a need to develop our understanding of how the deliberate use of, or request for, different types of interactions can be crucial to developing understanding. Such knowledge might enable us to request, or even rehearse, specific types of interactions in order to develop student discourses for learning.

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