Differential invariants
of the motion group actions.

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Abstract

Differential invariants of a (pseudo)group action can vary when restricted to invariant submanifolds (differential equations). The algebra is still governed by the Lie-Tresse theorem, but may change a lot. We describe in details the case of the motion group $O(n) \ltimes \mathbb{R}^n$ acting on the full (unconstraint) jet-space as well as on some invariant equations.[1]

Introduction

Let G be a pseudogroup acting on a manifold M or a bundle $\pi : E \to M$. This action can be prolonged to the higher jet-spaces $J^k(\pi)$ (one can also start with an action in some PDE system $\mathcal{E} \subset J^k(\pi)$ and prolong it).

The natural projection $\pi_{k,k-1} : J^k(\pi) \to J^{k-1}(\pi)$ maps the orbits in the former space to the orbits in the latter. If the pseudogroup is of finite type (i.e. a Lie group), this bundle (restricted to orbits) is occasionally a covering outside the singularity set. Otherwise it will become a sequence of bundles for $k \gg 1$. Ranks of these bundles varies but it is occasionally given by the Hilbert-Poincaré polynomial of the pseudogroup action.

The orbits can be described via differential invariants, i.e. invariants of the action on some jet level k. Existence and stability of the above mentioned Hilbert-Poincaré polynomial is a consequence of the Lie-Tresse theorem, which claims that the algebra of differential invariants is finitely generated via the algebraic-functional operations and invariant derivations.

This theorem in the ascending degree of generality was proved in different sources [Lie_1, Tr, O, Ku, KL$_1$]. In particular, the latter reference contains the full generality statement, when the pseudogroup acts on a system of differential equations $\mathcal{E} \subset J^l(\pi)$ (the standard regularity assumption is imposed, which is an open condition in finite jets).

In the case the pseudogroup G acts on the jet space, $\mathcal{E}$ must be invariant and so consist of the orbits, or equivalently it has an invariant representation $\mathcal{E} = \{J_1 = 0, \ldots, J_r = 0\}$, where J_l are (relative) differential invariants. Now the following dichotomy is possible.

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If the orbits forming $\mathcal{E}$ are regular, the structure of the algebra of differential invariants on $\mathcal{E}$ can be read off from that one of the pure jet-space.

On the other hand if $\mathcal{E}$ consists of singular orbits[2] (which is often the case when the system is overdetermined, so that differential syzygy should be calculated, which is an invariant count of compatibility conditions), then the structure of the algebra of differential invariants is essentially invisible from the corresponding algebra $\mathcal{I}$ of the pure jet-space, because $\mathcal{E}$ is the singular locus for differential invariants $I \in \mathcal{I}$ (if these exist, cf. just remarked).

In this note we demonstrate this effect on the example of motion group G acting naturally on the Euclidean space $\mathbb{R}^n$. The group is finite dimensional, but even in this case the described effect is visible. For infinite pseudogroups this follow the same route (see, for instance, the pseudogroup of all local diffeomorphisms acting on the bundle of Riemannian metrics in [K]).

We lift the action of G to the jets of functions on $\mathbb{R}^n$ and describe in details the structure of algebra of scalar differential invariants in the unconstrained ($J^\infty \mathbb{R}^n$) and constrained (system of PDEs) cases. This motion group was a classical object of investigations (see e.g. the foundational work [Lie$_2$]), but we have never seen the complete description of the differential invariants algebra.

1 Differential invariants and Lie-Tresse theorem

We refer to the basics on pseudogroup actions to [Ku, KL$_2$], but recall the relevant theory about differential invariants (see also [Tr, O, KJ]). Since we'll be concerned with a Lie group in this paper, it will be denoted by one symbol G (in infinite case G should be co-filtered as the equations in formal theory).

A function $I \in C^\infty(J^\infty \pi)$ (this means that I is a function on a finite jet space $J^k \pi$ for some $k > 1$) is called a differential invariant if it is constant along the orbits of the lift of the action of G to $J^k \pi$. For connected groups G we have an equivalent formulation: The Lie derivative vanishes $L_{\hat{X}}(I) = 0$ for all vector fields X from the lifted action of the Lie algebra.

Note that often functions I are defined only locally near families of orbits. Alternatively we should allow I to have meromorphic behavior over smooth functions (but we'll be writing though about local functions in what follows, which is a kind of micro-locality, i.e. locality in finite jet-spaces).

The space $\mathcal{I} = \{I\}$ forms an algebra with respect to usual algebraic operations of linear combinations over $\mathbb{R}$ and multiplication and also the composition $I_1, \ldots, I_s \mapsto I = F(I_1, \ldots, I_s)$ for any $F \in C^\infty_{\mathrm{loc}}(\mathbb{R}^s, \mathbb{R})$, $s = 1, 2, \ldots$ any finite number. However even with these operations the algebra $\mathcal{I}$ is usually not locally finitely generated. Indeed, the subalgebras $\mathcal{I}_k \subset \mathcal{I}$ of order k differential invariants are finitely generated on non-singular strata with respect to the above operations, but their injective limit $\mathcal{I}$ is not.

To cure this difficulty S.Lie and later his French student A.Tresse introduced invariant derivatives, i.e. such differentiations ϑ that belong to the centralizer

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[2]In this case $\mathcal{E}$ can be defined via vanishing of an invariant tensor **J**, with components J_i, though in general the latter cannot be chosen as scalar differential invariants.

of the Lie algebra $\mathfrak{g} = \text{Lie}(G)$ lifted as the space of vector fields on $J^\infty(\pi)$. To be more precise we consider the derivations $\vartheta \in C^\infty(J^\infty \pi) \otimes_{C^\infty(M)} \mathcal{D}(M)$ ($\mathscr{C}$-vector fields on π), which commute with the G-action. These operators map differential invariants to differential invariants $\vartheta : \mathcal{I}_k \to \mathcal{I}_{k+1}$.

We can associate invariant differentiations to a collection of differential invariants $I_1, \ldots, I_n$ ($n = \dim M$) in general position, meaning $\hat{d}I_1 \wedge \ldots \wedge \hat{d}I_n \neq 0$. Moreover the whole theory discussed above transforms to the action on equations[3] $\mathcal{E} \subset J^\infty(\pi)$.

Namely, given n functionally independent invariants $I^1, \ldots, I^n$ we assume their restrictions $I^1_\mathcal{E}, \ldots, I^n_\mathcal{E}$ are functionally independent[4] (in fact we can have the latter invariants only without the former), so that they can be considered as local coordinates.

Then one can introduce the horizontal basic forms (coframe) $\omega^i = \hat{d}I^i_\mathcal{E}$. Its dual frame consists of invariant differentiations $\hat{\partial}/\hat{\partial}I^i_\mathcal{E} = \sum_j [\mathcal{D}_a(I^b_\mathcal{E})]^{-1}_{ij} \mathcal{D}_j$. The invariant derivative of a differential invariant I are just the coefficients of the decomposition of the horizontal differential by the coframe:

$$\hat{d}I = \sum_{i=1}^n \frac{\hat{\partial}I}{\hat{\partial}I^i_\mathcal{E}} \, \omega^i$$

and they are called Tresse derivatives.

All invariant tensors and operators can be expressed through the given frame and coframe and this is the base for the solution of the equivalence problem.

Lie-Tresse theorem claims that the algebra of differential invariants $\mathcal{I}$ is finitely generated with respect to algebraic-functional operations and invariant derivatives.

2   Motion group action

Consider the motion group $\text{O}(n) \ltimes \mathbb{R}^n$. It is disconnected and for the purposes of further study of differential invariants we restrict to the component of unity $G = \text{SO}(n) \ltimes \mathbb{R}^n$. The two Lie groups have the same Lie algebra $\mathfrak{g} = o(n) \ltimes \mathbb{R}^n$ and the differential invariants of the latter become the differential invariants of the second via squaring.

Since the latter is inevitable even for the group G, the difference between two algebras of invariants is by an extension via finite group and will be ignored.

Below we will make use of the action of G on the space of codimension m affine subspaces of $\mathbb{R}^n$:

$$\text{AGr}(m,n) \equiv \{\Pi + c\} \simeq \{(\Pi, c) : \Pi \in \text{Gr}(n-m, n), c \in \Pi^\perp\}.$$

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[3]At this point we do not need to require even formal integrability of the system $\mathcal{E}$ [KL$_1$], but this as well as regularity issues will not be discussed here.

[4]Here and in what follows one can assume (higher micro-)local treatment.

The action of G is $x \mapsto Ax + b$, $x \in \mathbb{R}^n$, it is transitive on $\mathrm{AGr}(m,n)$ and the stabilizer equals

$$\mathrm{St}(\Pi + c) = \{(A,b) \in G : A\Pi = \Pi, b \in (1-A)c + \Pi\} \simeq \mathrm{SO}(\Pi) \times \mathrm{SO}(\Pi^\perp) \ltimes \Pi.$$

We have $\dim G = \dfrac{n(n+1)}{2}$, $\dim \mathrm{AGr}(m,n) = m(n-m+1)$ and

$$\mathrm{AGr}(k,n) \simeq G/(\mathrm{SO}(m) \times \mathrm{SO}(n-m) \ltimes \mathbb{R}^{n-m})$$

(note that this implies $\mathrm{AGr}(m,n) \neq \mathrm{AGr}(n-m,n)$ except for $n = 2m$ contrary to the space $\mathrm{Gr}(m,n)$).

We can extend the action of G on $\mathbb{R}^n$ to the space $\mathbb{R}^n \times \mathbb{R}^m$ by letting $g \in G$ act

$$g \cdot (x,u) = (g \cdot x, u).$$

We can prolong the action to the space $J^k(n,m)$.

For $k = 1$ the action commutes with the natural $\mathrm{Gl}(m)$-action in fibers of the bundle $\pi_{10} : J^1(n,m) \to J^0(n,m)$ and the action descends on the projectivization, which can be identified with the open subset in $\mathbb{R}^n \times \mathrm{AGr}(k,n)$ by associating the space $\mathrm{Ker}(d_x f)$ to a (surjective at x if we assume $n > m$) function $f : \mathbb{R}^n \to \mathbb{R}^m$.

Thus u is indeed an invariant of the G-action (scalar invariants are its components u^i, so that we can assume the fiber $\mathbb{R}^m$ being equipped with coordinates), and the scalar differential invariants of order 1 are[5] $\langle \nabla u^i, \nabla u^j \rangle = \sum u^i_{x^s} u^j_{x^s}$.

These form the generators of scalar differential invariants of order[6] ≤ 1.

Remark 1 *Sophus Lie investigated the vertical actions of G in $J^0(m,n) = \mathbb{R}^m \times \mathbb{R}^n$ and the invariants of its lift to $J^\infty(m,n)$ [Lie$_2$] (actually in this paper for $m = 1, n = 3$). This case is easier since the total derivatives $\mathcal{D}_1, \ldots, \mathcal{D}_m$ are obvious invariant derivations.*

In what follows we restrict to the case $m = 1$ and investigate invariants of the G-action in $J^\infty(n,1) = J^\infty(\mathbb{R}^n)$. Partially the results extend to the case of general m, though the theory of vector-valued symmetric forms $S^k(\mathbb{R}^n)^* \otimes \mathbb{R}^m$ is more complicated.

3 Differential invariants: Space $J^\infty(\mathbb{R}^n)$

Denote $V = T_0 \mathbb{R}^n$. Our affine space $\mathbb{R}^n$ (as well as the vector space V) is equipped with the Euclidean scalar product $\langle,\rangle$ and G is the symmetry group of it. In what follows we will identify the tangent space $T_x \mathbb{R}^n$ with V via translations (using the affine structure on $\mathbb{R}^n$).

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[5]Recall that the base space $\mathbb{R}^n$ is equipped with the Euclidean metric preserved by G.

[6]This claim holds at an open dense subset of $J^1(n,m)$. However if we restrict to the set of singular orbits with $\mathrm{rank}(d_x u) = r < m$, the basic set of invariants will be quite different.

The space $J^\infty(\mathbb{R}^n)$, which is the projective limit of the finite-dimensional manifolds $J^k(\mathbb{R}^n)$, has coordinates (x^i, u, p_σ), where $\sigma = (i_1, \ldots, i_n) \in \mathbb{Z}_{\geq 0}^n$ is a multiindex with length $|\sigma| = i_1 + \cdots + i_n$.

The only scalar differential invariants[7] of order ≤ 1 are

$$I_0 = u \text{ and } I_1 = |\nabla u|^2.$$

For each $x_1 \in J^1(\mathbb{R}^n)$ the group G has a large stabilizer. Provided x_1 is non-singular the dimension of the stabilizer St_1 is $\dim G - 2n + 1 = \frac{1}{2}(n-1)(n-2)$.

However the stabilizer completely evolves upon the next prolongation: the action of G on an open dense subset of $J^k(\mathbb{R}^n)$ for any $k \geq 2$ is free. Note that due to the trivial connection in $J^0(\mathbb{R}^n) = \mathbb{R}^n \times \mathbb{R}$ we can decompose

$$J^k(\mathbb{R}^n) = \mathbb{R}^n \times \mathbb{R} \times V^* \times S^2 V^* \times \cdots \times S^k V^*. \tag{1}$$

Thus we can represent a point $x_k \in J^k(\mathbb{R}^n)$ as the base projection $x \in \mathbb{R}^n$ and a sequence of "pure jets" $Q_t = d^t u \in S^t V^*$, $t = 0, \ldots, k$.

Covector Q_1 can be identified with the vector $\text{v} = \nabla u$.

Consider the quadric $Q_2 \in S^2 V^*$. Due to the metric we can identify it with a linear operator $A \in V^* \otimes V$, which has spectrum

$$\text{Sp}(A) = \{\lambda_1, \ldots, \lambda_n\}$$

and the normalized eigenbasis $e_1, \ldots, e_n$ (each element defined up to a sign!), provided Q_2 is semi-simple. Since Q_2 is symmetric, the basis is orthonormal.

In what follows we assume to work over the open dense subset $U \subset J^2(\mathbb{R}^n)$, where A is simple, so that the basis is defined (almost) uniquely (this can be relaxed to semi-simplicity, but then the stabilizer is non-trivial and the number of scalar invariants drops a bit).

There are precisely $(2n - 1) = \dim J^2(\mathbb{R}^n) - \dim \text{St}_1$ differential invariants of order 2. One choice is to take $\bar{I}_{2,i} = \lambda_i$ and $\bar{I}_{2,(i)} = \langle e_i, \text{v} \rangle$, $i = 1, \ldots, n$. There is an obvious relation $\sum_{i=1}^n \bar{I}_{2,(i)}^2 = 1$, so that we can restrict to the first $(n - 1)$ invariants in this group, but beside this the invariants are functionally independent.

Another choice of invariants is provided by the restriction Q_Π of Q_2 to $\Pi = \text{v}^\perp$, which has spectrum (again by converting quadric to an operator) $\text{Sp}(Q_\Pi) = \{\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{n-1}\}$ and normalized eigenvectors $\tilde{e}_i$. So the following invariants can be chosen: $\tilde{I}_{2,i} = \tilde{\lambda}_i$, $\tilde{I}_{2,n} = Q_2(\text{v}, \text{v})$ and $\tilde{I}_{2,(i)} = Q_2(\text{v}, \tilde{e}_i)$.

Both choices have disadvantages of using transcendental functions (solutions to algebraic equations), but we can overcome this with the following choice:

$$I_{2,i} = \text{Tr}(A^i), \ I_{2,(i)} = \langle A^i \text{v}, \text{v} \rangle, \quad i = 1, \ldots, n.$$

Here the number of invariants is $2n$, but they are dependent[8] due to Newton-Girard formulas, which relate the elementary symmetric polynomials $E_k(A) =$

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[7]From now on by this we mean the minimal set of generators.
[8]The first $(2n - 1)$ invariants are however independent and algebraic in the jets.

$\sum_{1_1 < \cdots < 1_k} \lambda_{i_1} \cdots \lambda_{i_k}$ and power sums $S_k(A) = \mathrm{Tr}(A^k) = \sum \lambda_i^k$ (these are $I_{2,k}$):

$$kE_k(A) = \sum_{i=1}^{k} (-1)^{i-1} S_i(A) E_{k-i}(A),$$

which together with $E_0 = 1$ gives an infinite chain of formulas

$$E_1 = S_1, \quad 2E_2 = S_1^2 - S_2, \quad 6E_3 = S_1^3 - 3S_1 S_2 + 2S_3, \quad \ldots$$

Now with the help of Cayley-Hamilton formula

$$A^n = E_1(A)A^{n-1} - E_2(A)A^{n-2} + \cdots + (-1)^n E_{n-1}(A)A - (-1)^n E_n(A)$$

we can express

$$I_{2,(n)} = E_1(A)I_{2,(n-1)} - E_2(A)I_{2,(n-2)} + \cdots - (-1)^n \det A$$

through our invariants since $E_i(A)$ are functions of $I_{2,i}$.

Remark 2 *We could restrict only to invariants $I_{2,(i)}$, $i = 1, \ldots, 2n-1$. This is helpful as we shall see. But when we restrict to singular (from the orbits point of view) PDEs these differential invariants may turn to be non-optimal, and this will be precisely the case in the example we investigate.*

Now there are precisely $\binom{n+2}{3} = \dim S^3 V^*$ differential invariants of order 3, $\binom{n+3}{4} = \dim S^4 V^*$ differential invariants of order 4, ..., $\binom{n+k-1}{k} = \dim S^k V^*$ differential invariants of order k.

The third order invariants are the following:

$$\bar{I}_{3,\sigma} = Q_3(e_i, e_j, e_l), \quad \text{where } \sigma = (ijl) \in S^3\{1, \ldots, n\}.$$

Generating invariants of orders 4 and higher are obtained from the similar formulae, namely as the coefficients q_σ of the decomposition

$$Q_k = \sum_{\sigma = (i_1, \ldots, i_k)} q_\sigma \omega^\sigma, \quad \text{where } \omega^\sigma = \omega^{i_1} \cdots \omega^{i_k}, \ 1 \le i_1 \le \cdots \le i_k \le n.$$

They are again transcendental functions. To get algebraic expressions one can use the third order functions

$$I_{3,\sigma} = Q_3(A^i \mathrm{v}, A^j \mathrm{v}, A^l \mathrm{v}), \quad \sigma = (ijk) \text{ with } 1 \le i \le j \le l \le n$$

and similar expressions for the higher order.

Theorem 1 *The invariants $I_{i,\sigma}$ with $i \le 3$ is the base of differential invariants for the Lie group G action in $J^\infty(\mathbb{R}^n)$ via algebraic-functional operations and Tresse derivatives.*

This statement is an easy dimensional count[9] together with examination of independency condition. To get Tresse derivatives n invariants (for instance of order ≤ 2) should be chosen.

However this is not necessary, if one does not care about transcendental functions. Indeed, the vector fields $e_1, \ldots, e_n$ are invariant differentiations (they can be expressed through the total derivatives $\mathcal{D}_1, \ldots, \mathcal{D}_n$ with coefficients of the second order).

Remark 3 *Notice that the moving frame*

$$e_1, \ldots, e_n \in C^\infty(U, \pi_2^* T\mathbb{R}^n)$$

uniquely fixes an element $g \in G$, which transforms it to the standard orthonormal frame at $0 \in \mathbb{R}^n$. This leads to the equivariant map defined on the open dense set $\pi_{\infty,2}^{-1}(U)$:

$$J^\infty(\mathbb{R}^n) \to J^2(\mathbb{R}^n) \supset U \to G.$$

Such map is called the moving frame in the approach of Fells and Olver [FO].

4 Relations in the algebra $\mathcal{I}$

Since the commutator of invariant differentiations is an invariant differentiation, decomposition $[e_i, e_j] = \sum c_{ij}^k e_k$ yields $\leq \frac{1}{2}n^2(n-1)$ (in general precisely this number) 3rd order differential invariants c_{ij}^k. The number of pure 3rd order invariants obtained via invariant differentiations of the 2nd order invariants is $n(2n-1)$. So since

$$\frac{n(n+1)(n+2)}{6} - n(2n-1) - \frac{n^2(n-1)}{2} = \frac{n(n+4)(1-n)}{3} \leq 0$$

we can conclude that differential invariants $I_{i,\sigma}$ with $i \leq 2$ and invariant differentiations $\{e_i\}_{i=1}^n$ generate the whole algebra $\mathcal{I}$ on an open set $\hat{U} \subset J^\infty(\mathbb{R}^n)$.

Thus we are lead to the question on relations in this algebra. They can be all deduced from the expressions for pure jets of u

$$Q_3 = \hat{\nabla} Q_2, \ Q_4 = \hat{\nabla} Q_3 \ \text{ etc}$$

using the structural equations. Here

$$\hat{\nabla} : C^\infty(\pi_i^* S^i V^*) \to C^\infty(\pi_{i+1}^* S^{i+1} V^*)$$

is the symmetric covariant derivative induced by the flat connection ∇ in the trivial bundle $J^0(\mathbb{R}^n) = \mathbb{R}^n \times \mathbb{R}$, $V = T\mathbb{R}^n$ (the map is the composition of the horizontal differential $\hat{d}$ and symmetrization).

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[9]In fact for $n \leq 4$ the same arguments imply that the base can formed only by the invariants $I_{i,\sigma}$ with $i \leq 2$.

However for the sake of algebraic formulations we change invariant differentiations e_i to the following ones:

$$v_1 = \hat{v} = v \cdot \mathcal{D}_x = \sum u_i \mathcal{D}_i$$

$$v_2 = \hat{A}v = Av \cdot \mathcal{D}_x = \sum u_i u_{ij} \mathcal{D}_j$$

$$v_3 = \widehat{A^2 v} = A^2 v \cdot \mathcal{D}_x = \sum u_i u_{ij} u_{jk} \mathcal{D}_k$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$v_n = \widehat{A^{n-1} v} = A^{n-1} v \cdot \mathcal{D}_x = \sum u_{i_1} u_{i_1 i_2} \dots u_{i_{n-1} i_n} \mathcal{D}_{i_n}.$$

Now we are going to change the basis of differential invariants in $\mathcal{I}_k$ to describe the relations in the simplest way.

Namely for the basis of invariants of order 2 we can take $I_{2,(ij)} = Q_2(A^i v, A^j v)$, $0 \le i \le j < n$. However since $Q_2(v, w) = \langle Av, w \rangle$ and A is self-adjoint we get

$$I_{2,(ij)} = \langle A^{i+1} v, A^j v \rangle = \langle A^{i+j+1} v, v \rangle = I_{2,(i+j+1)},$$

so that the new invariants are precisely the old ones $I_{2,(i)}$, just with the larger index range $i = 1, \dots, 2n - 1$ (we can allow arbitrary index i, but the corresponding invariants are expressed via these ones, see Remark 2 and before).

Basic higher order invariants are introduced in the same fashion:

$$I_{s,(i_1 \dots i_s)} = Q_s(A^{i_1} v, \dots, A^{i_s} v), \quad 0 \le i_1 \le \dots \le i_s < n.$$

Suppose now that our set of generic (regular) points $U \subset J^2(\mathbb{R}^n)$ is given by not only the constraint that $\mathrm{Sp}(A)$ is simple, but also the claim that the $n \times n$ matrix $\|\gamma_{ij}\|_{0 \le i,j < n}$ with entries $\gamma_{ij} = \langle A^i v, A^j v \rangle = I_{2,(i+j)}$ is non-degenerate. Let

$$[\gamma^{ij}] = \begin{pmatrix} 1 & I_{2,(1)} & \cdots & I_{2,(n-1)} \\ I_{2,(1)} & I_{2,(2)} & \cdots & I_{2,(n)} \\ \vdots & \vdots & \ddots & \vdots \\ I_{2,(n-1)} & I_{2,(n)} & \cdots & I_{2,(2n-2)} \end{pmatrix}^{-1}$$

be the inverse matrix. Note that all its entries are invariants. Now

$$(A^{i_0} v \cdot \mathcal{D}_x) Q_s(A^{i_1} v, \dots, A^{i_s} v) = Q_{s+1}(A^{i_0} v, A^{i_1} v, \dots, A^{i_s} v)$$

$$+ \sum_{j=1}^{s} Q_s(A^{i_1} v, \dots, A^{i_j - 1} v, \theta_{i_0 i_j}, A^{i_j + 1} v, \dots, A^{i_s} v),$$

where $\theta_{i_0 i_j} = \nabla_{A^{i_0} \hat{v}}(A^{i_j} v)$ is the vector which, due to metric duality, is dual to the covector $\sum_{\alpha+\beta=i_j-1} Q_3(A^{i_0} v, A^\alpha v, A^\beta \cdot)$. Thus we obtain

Theorem 2 *The algebra $\mathcal{I}$ is generated by the invariants $I_{s,\sigma}$ and invariants derivatives $v_1, \dots, v_n$, which are related by the formulae ($s \ge 2$):*

$$v_{i_0} \cdot I_{s,(i_1 \dots i_s)} = I_{s+1,(i_0 i_1 \dots i_s)} + \sum_{j=1}^{s} \sum_{a,b=0}^{n-1} \sum_{\alpha+\beta=i_j-1} I_{s,(i_1 \dots i_{j-1}, a, i_{j+1} \dots i_s)} \gamma^{ab} I_{3,(i_0,\alpha,b+\beta)}.$$

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In this case we can choose $I_{s,\sigma}$, $s \le 3$ and v_i as the generators.

This representation for $\mathcal{I}$ via generators and relations is not minimal, as clear from the first part of the section. However the relations are algebraic, explicit and quite simple.

To explain how to achieve minimality let us again change the set of generators (basic differential invariants). For the second order we return to $I_{2,i}, I_{2,(i)}$, $1 \le i \le n$. For the third order we add the invariants

$$I_{3,[ij]l} = \mathrm{Tr}(Q_3(A^i\cdot, A^j\cdot, A^l\mathrm{v})).$$

They can indeed be expressed algebraically through the invariants $I_{3,(ijk)}$ together with the lower order invariants.

For higher order we have more possibilities of inventing new invariants (which can be described via graphs of the type $(k,1)$-tree), but they are again algebraically dependent with already known differential invariants.

The relations are as follows ($0 \le k < n$ and we show only top of the list):

$$v_1 \cdot I_0 = I_1, \ v_2 \cdot I_0 = I_{2,(1)}, \ \ldots, \ v_n \cdot I_0 = I_{2,(n-1)},$$
$$v_1 \cdot I_1 = 2I_{2,(1)}, \ v_2 \cdot I_1 = 2I_{2,(2)}, \ \ldots, \ v_n \cdot I_1 = 2I_{2,(n)},$$
$$v_{k+1} \cdot I_{2,l} = \sum_{\alpha+\beta=l-1} I_{3,[\alpha\beta]k}, \ v_{k+1} \cdot I_{2,(l)} = \sum_{\alpha+\beta=l-1} I_{3,(\alpha\beta k)} + 2I_{2,(k+l+1)} \ \ \text{etc.}$$

Elaborate work with these shows that all the invariants can be obtained from I_0 and structural constants $\bar{c}_{ij}^k$ of the frame $[v_i, v_j] = \sum \bar{c}_{ij}^k v_k$.

Corollary 1 *By shrinking $\hat{U} \subset J^\infty(\mathbb{R}^n)$ further (but leaving it open dense) we can arrange that the algebra $\mathcal{I}$ of differential invariants is generated only by I_0 and the derivations $v_1, \ldots, v_n$.*

5 Algebra of differential invariants: Equation $\mathcal{E}$

Consider the PDE $\mathcal{E} = \{\|\nabla u\| = 1\}$. By the standard arguments it determines a cofiltered manifold in $J^\infty(\mathbb{R}^n)$ and we identify $\mathcal{E}$ with it, so that it consists of the sequence of prolongations $\mathcal{E}_k \subset J^k(\mathbb{R}^n)$ and projections $\bar{\pi}_{k,k-1} : \mathcal{E}_k \to \mathcal{E}_{k-1}$.

Since the prolongation of the defining equation for $\mathcal{E}$ to the second jets is $Q_2(\mathrm{v}, \cdot) = 0$ or $\mathrm{v} \in \mathrm{Ker}(A)$ we conclude that most of the invariants, introduced on the previously defined subset $\hat{U}$, vanish: the equation is singular. Indeed, $0 \in \mathrm{Sp}(A)$, so that $\det A = 0$, the matrix $[\gamma_{ij}]$ is not invertible etc.

In particular, $I_{2,(i)} = 0$, $I_{s,(i_1\ldots i_s)} = 0$ if at least one $i_t \ne 0$, $v_2 = \cdots = v_n = 0$. Thus the algebra $\mathcal{I}$ description from the previous section does not induce any description of the algebra $\mathcal{I}_\mathcal{E}$ of differential invariants of the group G action on $\mathcal{E}$: the notion of regularity and basic invariants are changed completely!

Again the group acts freely on the second jets. So there is 1 invariant of order 0

$$I_0 = u,$$

no invariants of order 1 and $(n-1)$ invariants of order 2:

$$I_{2,1}, \ldots, I_{2,n-1} \qquad \text{or equivalently} \quad E_1(A), \ldots, E_{n-1}(A).$$

The number of invariants of pure order $k > 2$ coincides with the ranks of the projections:

$$\dim \bar{\pi}_{k,k-1}^{-1}(*) = \binom{n+k-2}{k}.$$

The principal axes of Q_2 (or normalized eigenbasis of A) are now $e_1 = v, e_2, \ldots, e_n$. These are still the invariant derivations and the invariants of order $k > 2$ are the coefficients[10] of the decomposition by basis in $S^k \operatorname{Ann}(v) \subset S^k V^*$:

$$Q_k|_{\mathcal{E}} = \sum_{\sigma=(i_1 \ldots i_k):i_t>1} q_\sigma \, \omega^\sigma, \quad q_\sigma = Q_k(v_{i_1}, \ldots, v_{i_k}).$$

Theorem 3 *The invariants I_0, $I_{2,i}$ and $I_{3,\sigma}$ $\left(1 \leq i < n, \ \sigma = (i_1, i_2, i_3), \ i_t \neq 1\right)$ form a base of differential invariants of the algebra $\mathcal{I}_{\mathcal{E}}$ via algebraic-functional operations and Tresse derivatives.*

Algebra of differential invariants can again be represented in a simpler form via differential invariants and invariant derivatives. If we choose e_i for the latter the relations can be read off from the algebra $\mathcal{I}$, though this again involves transcendental functions.

Denote the Christoffel symbols of $\hat{\nabla}$ in the basis e_α by Γ_{ij}^k (these are differential invariants of order 3):

$$\hat{\nabla}_{e_i} e_j = \sum \Gamma_{ij}^k e_k \quad \Longleftrightarrow \quad \hat{\nabla}_{e_i} \omega^j = -\sum \Gamma_{ik}^j \omega^k.$$

Notice that since the connection is torsionless, $T_\nabla = 0$, these invariants determine the structure functions $c_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k$.

Let us now substitute the formulas (eigenvalues λ_i can be expressed through the invariants $I_{2,i}$, however in a transcendental way; $\lambda_1 = 0$ corresponds to e_1)

$$Q_2 = \sum_{1<i\leq n} \lambda_i (\omega^i)^2, \quad Q_3 = \sum_{1<i\leq j\leq k\leq n} q_{ijk} \omega^i \omega^j \omega^k$$

into the identity $\hat{\nabla} Q_2 = Q_3$:

$$\hat{\nabla} \sum \lambda_i (\omega^i)^2 = \sum (\hat{\nabla}\lambda_i)(\omega^i)^2 + 2\sum \lambda_i \omega^i \cdot \hat{\nabla}\omega^i$$
$$= \sum \partial_{e_k}(\lambda_i)\omega^i \omega^i \omega^k - 2\sum \lambda_i \Gamma_{jk}^i \omega^i \omega^j \omega^k.$$

We get for $1 < i \leq j \leq k \leq n$:

$$q_{ijk} = \left(\partial_{e_k}(\lambda_i)\delta_{ij} + \partial_{e_i}(\lambda_k)\delta_{jk} - \partial_{e_k}(\lambda_i)\delta_{ik}\right) - 2\sum_{\tau\in S_3} \lambda_{\tau(i)}\Gamma_{\tau(j)\tau(k)}^{\tau(i)}$$

---

[10]Note that these invariants are defined up to $\pm$ and so should be squared to become genuine invariants; alternatively certain products/ratios of them define absolute invariants.

Since in addition, in general position the invariants λ_i can be expressed through the invariants $e_i \cdot I_0$ $(1 < i \leq n)$[11], then by adding decomposition of the covariant derivatives by the frame into the set of operations, we obtain the following

Corollary 2 *By shrinking $\hat{U} \subset \mathcal{E}$ further (but leaving it open dense) we can arrange that the algebra $\mathcal{I}_\mathcal{E}$ of differential invariants is generated only by I_0 and the derivations $e_1, \ldots, e_n$.*

6   Algebra of differential invariants: Equation $\tilde{\mathcal{E}}$

Completely new picture for the algebra of differential invariants emerges, when we add one more invariant PDE: the system becomes overdetermined and compatibility conditions (or differential syzygies) come into the play.

We will study the following system[12], which comes from application to relativity [C] (when Laplacian Δ is changed to Dalambertian $\square$):

$$\{\|u\| = 1, \Delta u = f(u)\} \subset \mathcal{E}.$$

This equation is a non-empty submanifold in $J^2(\mathbb{R}^n)$, but when we carry the prolongation-projection scheme, it becomes much smaller.

It turns out that for most functions $f(u)$ the resulting submanifold $\tilde{\mathcal{E}}$ is just empty. We are going to decompose it into the strata

$$\tilde{\mathcal{E}} = \Sigma_1(\tilde{\mathcal{E}}) \cup \cdots \cup \Sigma_n(\tilde{\mathcal{E}}),$$

where $\Sigma_i(\tilde{\mathcal{E}}) = \{x \in \tilde{\mathcal{E}} : \ \#[\mathrm{Sp}(A_{\tilde{\mathcal{E}}})] = i\}$ for the operator $A_{\tilde{\mathcal{E}}}$ corresponding to the 2-jet $Q_2|_{\tilde{\mathcal{E}}}$.

It is possible to show that the spectrum of A on $\tilde{\mathcal{E}}$ depends on u (and some constants) only. This was done in [FZY] via the Cayley-Hamilton theorem, though they used the Dalambertian instead of the Laplace operator. In the next section we prove it for the Laplace operator via a different approach.

More detailed investigation leads to the following claim:

> **Conjecture:** The strata $\Sigma_n(\tilde{\mathcal{E}}), \ldots, \Sigma_3(\tilde{\mathcal{E}})$ are empty, while $\Sigma_2(\tilde{\mathcal{E}}), \Sigma_1(\tilde{\mathcal{E}})$ are not and they are finite-dimensional manifolds.

Let us indicate the idea of the proof for the stratum $\Sigma_n(\tilde{\mathcal{E}})$ because on other strata the eigenbasis e_i is not defined (but the arguments can be modified). It turns out that the compatibility is related to dramatic collapse of the algebra $\mathcal{I}_{\tilde{\mathcal{E}}}$ of differential invariants.

Indeed, as follows from the discussion above and the next section, there is only one invariant u of order ≤ 2 for the G-action on $\tilde{\mathcal{E}}$. Since the coefficients of the invariant derivations have the second order, we obtain the following statement:

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[11]We have $e_1 \cdot I_0 = 1$ on $\mathcal{E}$.

[12]This interesting system was communicated to the first author by Elizabeth Mansfield.

Theorem 4 *All differential invariants of the Lie group G-action on the PDE system $\tilde{\mathcal{E}}$ can be obtained from the function $I_0 = u$ and invariant derivations.*

Now relations in the algebra $\mathcal{I}_{\tilde{\mathcal{E}}}$ are differential syzygies for $\tilde{\mathcal{E}}$ and they boil down to a system of ODEs on $f(u)$, which completely determines it.

The details of this program will be however realized elsewhere.

7 Geometry of the system

In this section we justify the claim from §6 and prove that the spectrum of the operator $A = A_{\tilde{\mathcal{E}}}$, obtained from the pure 2-jet $Q_2|_{\tilde{\mathcal{E}}}$ via the metric, depends on u only. To do this we reformulate the problem with nonlinear differential equations in the geometric language from contact geometry [Ly].

The first equation $\mathcal{E}$ we represent as a level surface $H = \frac{1}{2}(1 - \sum_{i=1}^{n} p_i^2) = 0$ in the jet-space $J^1(\mathbb{R}^n)$. The second equation from $\tilde{\mathcal{E}}$ can be represented as Monge-Ampère type via n-form

$$\Omega_1 = \sum_{i=1}^{n} dx_1 \wedge \ldots \wedge dx_{i-1} \wedge dp_i \wedge \ldots \wedge dx_n - f(u)dx_1 \wedge \ldots \wedge dx_n.$$

Namely a solution to the system is a Lagrangian submanifold $L^n \subset \{H = 0\}$ such that $\Omega_1|_{L^n} = 0$. Representing $L^n = \text{graph}[j^1(u)]$ we obtain the standard description.

The contact Hamiltonian vector field X_H preserves the contact structure and being restricted to the surface $H = 0$ it coincides with the field of Cauchy characteristic $Y_H = X_H|_{H=0} = \sum p_i \mathcal{D}_{x_i} = \sum p_i \partial_{x^i} + \partial_u$.

Since Cauchy characteristics are always tangent to any solution, the forms $\Omega_{1+i} = (L_{X_H})^i \Omega_1$ also vanish on any solution of the system $\mathcal{E}$. We simplify them modulo the form Ω_1 and get:

$$\Omega_2 = L_{X_H}\Omega_1 + f(u)\Omega_1$$

$$= 2\sum dx_1 \wedge \ldots dp_i \wedge dx_{i+1} \ldots dp_j \wedge dx_{j+1} \ldots \wedge dx_n - (f' + f^2)\, dx_1 \wedge \ldots dx_n,$$

$$\Omega_3 = L_{X_H}\Omega_2 + (f'(u) + f^2(u))\Omega_1$$

$$= 3!\sum dx_1 \wedge \ldots dp_i \wedge dx_{i+1} \ldots dp_j \wedge dx_{j+1} \ldots dp_k \wedge dx_{k+1} \ldots \wedge dx_n$$
$$- (D + f)^2(f)\, dx_1 \wedge \ldots \wedge dx_n,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\Omega_n = n!\, dp_1 \wedge \ldots dp_n - (D + f)^{n-1}(f)\, dx_1 \wedge \ldots \wedge dx_n,$$

$$\Omega_{n+1} = -(D + f)^n(f)\, dx_1 \wedge \ldots \wedge dx_n,$$

where D is the operator of differentiation by u and f is the operator of multiplication by $f(u)$. Thus a necessary condition for solvability is the following non-linear ODE:

$$(D + f)^{n+1}(1) = 0. \tag{2}$$

This equation can be solved via conjugation $D + f = e^{-g}De^g$ with $g(u) = \int f(u)\,du$ [Ko], which reduces the ODE to the form $D^{n+1}e^g = 0$, so that $g = \mathrm{Log}\,P_n(u)$, where $P_n(u)$ is a polynomial of degree n, whence[13]

$$f(u) = \sum_{i=1}^{n} \frac{1}{u - \alpha_i}, \qquad \alpha_i = \mathrm{const}. \tag{3}$$

However there are more compatibility conditions, which produce further constraints on numbers α_i. The above relations $\Omega_i = 0$ can be used to find $\mathrm{Sp}(A)$. Namely let us rewrite them as follows:

$$E_1(A) = \sum \lambda_i = f, \quad E_2(A) = \sum_{i<j} \lambda_i \lambda_j = \tfrac{1}{2}(D+f)^2(1),$$

$$E_3(A) = \sum_{i<j<k} \lambda_i \lambda_j \lambda_k = \tfrac{1}{3!}(D+f)^3(1), \ \ldots, \ E_n(A) = \lambda_1 \cdots \lambda_n = \tfrac{1}{n!}(D+1)^n(1).$$

These, due to Newton-Girard formulas, imply the equivalent identities:

$$I_{2,1} = \sum \lambda_i = f(u), \quad I_{2,2} = \sum \lambda_i^2 = -f'(u),$$

$$I_{2,3} = \sum \lambda_i^3 = \tfrac{1}{2}f''(u), \quad I_{2,4} = \sum \lambda_i^4 = -\tfrac{1}{3!}f'''(u), \quad \ldots$$

In particular we get $\lambda_i = (u - \alpha_i)^{-1}$ and so

$$A \sim \mathrm{Diag}\left(\frac{1}{u - \alpha_1}, \ldots, \frac{1}{u - \alpha_n}\right).$$

The fact that $\det(A) = 0$ on $\tilde{\mathcal{E}}$ implies that $\alpha_n = \infty$ and using symmetry ∂_u (shift along u) we can arrange $\alpha_1 = 0$ (we use freedom of renumbering the spectral values).

The conjecture from the previous section is equivalent to the claim that other α_i equal either 0 or ∞. But this will be handled in a separate paper.

8   Integrating the system along characteristics

Let us now consider the quotient of the submanifold $\{H = 0\} \subset J^1(\mathbb{R}^n)$ by the Cauchy characteristics. We can identify it with the transversal section $\Sigma^{2n-1} = \{H = 0, u = \mathrm{const}\}$. The solutions will be $(n-1)$-dimensional manifolds of the induced exterior differential system.

Note that we should augment the system with the contact form $\omega = du - \sum p_i\,dx^i$ and its differential $\Omega_0 = \sum dx^i \wedge dp_i$. Note that if we choose $f(u)$ to be the solution of the ODE (2), then $\frac{1}{n!}\Omega_n = dp_1 \wedge \ldots \wedge dp_n = 0$ on solutions.

Let us start investigation from the case **n = 2**. In this case the induced differential system is given by two 1-forms:

$$\theta = i_{X_H}\Omega_1|_\Sigma = p_1\,dp_2 - p_2\,dp_1 - \frac{1}{u}(p_1\,dx_2 - p_2\,dx_1)$$

---

[13]Here we can assume we are working over $\mathbb{C}$, though this turns out to be inessential.

and $\theta_0 = i_{X_H}\Omega_0|_\Sigma = p_1\,dp_1 + p_2\,dp_2$, but it vanishes on Σ. The form θ is contact: $\theta \wedge d\theta \neq 0$, so solutions of $\mathcal{E}$ are represented by all Legendrian curves on (Σ^3, θ).

Consider now $\mathbf{n = 3}$. In this case we know that $\mathrm{Sp}(A) = \{0, \frac{1}{u-\alpha}, \frac{1}{u+\alpha}\}$ (in fact, $\alpha = 0$, but let us pretend we do not know it yet).

We have: $f = \frac{2u}{u^2-\alpha^2}$, $f' + f^2 = \frac{2}{u^2-\alpha^2}$.

Again $\theta_0 = i_{X_H}\Omega_0$ vanishes on Σ^5, so the exteriour differential system is generated by two 2-forms:

$$\theta_1 = i_{X_H}\Omega_1 = (p_1\,dp_2 - p_2\,dp_1) \wedge dx_3 + (p_2\,dp_3 - p_3\,dp_2) \wedge dx_1$$
$$+(p_3\,dp_1 - p_1\,dp_3) \wedge dx_2 - \tfrac{2u}{u^2-\alpha^2}(p_1\,dx_2 \wedge dx_3 + p_2\,dx_3 \wedge dx_1 + p_3\,dx_1 \wedge dx_2);$$
$$\theta_2 = \tfrac{1}{2}i_{X_H}\Omega_2 = p_1\,dp_2 \wedge dp_3 + p_2\,dp_3 \wedge dp_1 + p_3\,dp_1 \wedge dp_2$$
$$-\tfrac{1}{u^2-\alpha^2}(p_1\,dx_2 \wedge dx_3 + p_2\,dx_3 \wedge dx_1 + p_3\,dx_1 \wedge dx_2).$$

The integral surfaces of this system integrate to solutions of $\mathcal{E}$.

Digression. Let us choose another section for $\Sigma' \subset J^1(\mathbb{R}^3)$: since the Cauchy characteristics are given by the system $\{\dot{x}_i = p_i, \dot{u} = 1\}$, we can take in the domain $p_3 > 0$: $x_3 = \text{const}$, $p_3 = \sqrt{1 - p_1^2 - p_2^2}$. Then the forms giving the differential system are given by (being multiplied by p_3):

$$\theta_1' = \left((1 - p_2^2)\,dp_1 + p_1 p_2\,dp_2\right) \wedge dx_2 + dx_1 \wedge \left(p_1 p_2\,dp_1 + (1 - p_1^2)\,dp_2\right)$$
$$-\tfrac{2u}{u^2-\alpha^2}(1 - p_1^2 - p_2^2)\,dx_1 \wedge dx_2;$$
$$\theta_2' = dp_1 \wedge dp_2 - \frac{1 - p_1^2 - p_2^2}{u^2 - \alpha^2}\,dx_1 \wedge dx_2.$$

If we identify $\Sigma' \simeq J^1(\mathbb{R}^2)$ with the contact form $\omega' = du - p_1 dx^1 - p_2 dx^2$, the above 2-forms become represented by the following Monge-Ampere equations:

$$(1 - u_x^2)u_{xx} + 2u_x u_y \cdot u_{xy} + (1 - u_x^2)u_{yy} = \tfrac{2u}{u^2-\alpha^2}(1 - u_x^2 - u_y^2),$$
$$u_{xx}u_{yy} - u_{xy}^2 = \tfrac{1}{u^2-\alpha^2}(1 - u_x^2 - u_y^2).$$

Compatibility of this pair yields $\alpha = 0$.

Remark 4 *The above system is of the kind investigated in [KL$_3$]: when the surface $\Sigma^2 = \mathrm{graph}\{u : \mathbb{R}^2 \to \mathbb{R}^1\} \subset \mathbb{R}^3$ has prescribed Gaussian and mean curvatures, K and H respectively (this leads to a complicated overdetermined system). In fact the PDEs of the above system can be written in the form $H = F_1(u, \nabla u), K = F_2(u, \nabla u)$.*

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