ANOMALY OF LINEARIZATION AND AUXILIARY INTEGRALS.

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Abstract. In this note we discuss some formal properties of universal linearization operator, relate this to brackets of non-linear differential operators and discuss application to the calculus of auxiliary integrals, used in compatibility reductions of PDEs.

Introduction

Commutator $[\Delta,\nabla]$ of linear differential operators $\Delta,\nabla\in\operatorname{Diff}(\pi,\pi)$ in the context of non-linear operators $F,G\in\operatorname{diff}(\pi,\pi)$ is up-graded to the higher Jacobi bracket $\{F,G\}$, which plays the same role in compatibility investigations and symmetry calculus.[1]

The linearization operator relates non-linear operators on a bundle π with linear operators on the same bundle, whose coefficients should be however smooth functions on the space of infinite jets. The latter space is the algebra of $\mathscr{C}$-differential operators and we get the map

$$\ell:\operatorname{diff}(\pi,\pi)\to\mathscr{C}\operatorname{Diff}(\pi,\pi)=C^\infty(J^\infty\pi)\otimes_{C^\infty(M)}\operatorname{Diff}(\pi,\pi),$$

defined by the formula [KLV]

$$\ell_F(s)h=\tfrac{d}{dt}F(s+th)|_{t=0},\qquad F\in\operatorname{diff}(\pi,\pi),\quad s,h\in C^\infty(\pi).$$

However it does not respect the commutator:

$$[\ell_F,\ell_G]\neq\ell_{\{F,G\}}.$$

Example: Consider the scalar differential operators on $\mathbb{R}$, so that $\pi=\mathbf{1}$ and $J^\infty(\pi)=\mathbb{R}^\infty(x,u,p=p_1,p_2,\dots)$. Choose

$$F=p^2, G=p+c\cdot x;\quad \{F,G\}=2c\,p\implies \ell_{\{F,G\}}=2c\,\mathcal{D}_x.$$

If we commute $\ell_F=2p\,\mathcal{D}_x$ and $\ell_G=\mathcal{D}_x$, we get: $[\ell_F,\ell_G]=-2p_2\,\mathcal{D}_x$, so that we observe an anomaly.

There are two reasons for this. The first is that the operator of linearization disregards non-homogeneous linear terms, which are important for the Jacobi bracket. The second is the non-linearity itself.

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The goal of this note is to discuss reasons and consequences of this anomaly (this also plays a significant role in investigation of coverings and non-local calculus [KKV]).

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1. Anomaly via Hessian

The Jacobi bracket of non-linear operators $F, G \in \operatorname{diff}(\pi,\pi)$ is expressed via linearization as follows:

$$\{F, G\} = \ell_F G - \ell_G F.$$

We also consider the evolutionary operators defined by duality:

$$\mathit{Э}_F G = \ell_G F.$$

Since ℓ_G is a derivation in G, $\mathit{Э}_F$ is a derivation (satisfies the Leibniz rule) and their union can be treated as the module of vector fields. These operators have no anomaly, i.e. the map $\mathit{Э} : C^\infty(J^\infty\pi) \to \operatorname{Vect}(J^\infty\pi)$ is an anti-homomorphism:

$$[\mathit{Э}_F, \mathit{Э}_G] = -\mathit{Э}_{\{F,G\}}.$$

This instantly implies Jacobi identity for the bracket $\{F, G\}$, so that $\big(\operatorname{diff}(\pi,\pi), \{,\}\big)$ is a Lie algebra [KLV].

The operators of universal linearization and evolutionary differentiation do not commute and this leads to the following

Definition. *The Hessian operator* $\operatorname{diff}(\pi,\pi)\times\operatorname{diff}(\pi,\pi) \to \mathscr{C}\operatorname{Diff}(\pi,\pi)$ *is defined by the formula*

$$\operatorname{Hess}_F G = [\mathit{Э}_G, \ell_F].$$

We will also write $\operatorname{Hess}_F(G,H) = \operatorname{Hess}_F G(H)$ for $F, G, H \in \operatorname{diff}(\pi,\pi)$ and note that $\operatorname{Hess}_F \equiv 0$ for linear operators F, because in this case $\ell_F = F$, which reduces the claim to the commutation of left and right multiplications.

Next we note that the Hessian Hess_F is symmetric:

Lemma 1. $\operatorname{Hess}_F(G,H) = \operatorname{Hess}_F(H,G)$.

Indeed:

$$\operatorname{Hess}_F(G,H) = \mathit{Э}_G \ell_F H - \ell_F \mathit{Э}_G H = \mathit{Э}_G \mathit{Э}_H F - \ell_F \ell_H G,$$

so that

$$\begin{aligned}\operatorname{Hess}_F(G,H) - \operatorname{Hess}_F(H,G) &= [\mathit{Э}_G, \mathit{Э}_H]F - \ell_F\{H,G\} \\ &= -\mathit{Э}_{\{G,H\}}F - \ell_F\{H,G\} = 0.\end{aligned}$$

Now we can express the anomaly of linearization via the Hessian:

Proposition 2. $[\ell_F, \ell_G] - \ell_{\{F,G\}} = \mathrm{Hess}_G F - \mathrm{Hess}_F G.$

Indeed we have:

$$
\begin{aligned}
[\ell_F, \ell_G]H &= \ell_F \mathcal{Э}_H G - \ell_G \mathcal{Э}_H F \\
&= \mathcal{Э}_H(\ell_F G - \ell_G F) - \mathrm{Hess}_F(H, G) + \mathrm{Hess}_G(H, F) \\
&= \mathcal{Э}_H\{F, G\} - \mathrm{Hess}_F(G, H) + \mathrm{Hess}_G(F, H) \\
&= \ell_{\{F,G\}} H + (\mathrm{Hess}_G F - \mathrm{Hess}_F G)H.
\end{aligned}
$$

Finally let us express the Leibniz identity for non-linear operators and the Jacobi bracket. For linear operators it is well-known, but for non-linear ones there's an anomaly:

Proposition 3. $\{F, \ell_G H\} = \ell_{\{F,G\}} H + \ell_G\{F, H\} - \mathrm{Hess}_F(G, H).$

This is obtained as follows:

$$
\begin{aligned}
\{F, \ell_G H\} &= \ell_F \ell_G H - \mathcal{Э}_F \ell_G H \\
&= [\ell_F, \ell_G]H + \ell_G(\ell_F - \mathcal{Э}_F)H - \mathrm{Hess}_G(F, H) \\
&= \ell_{\{F,G\}} H + \ell_G\{F, H\} - \mathrm{Hess}_F(G, H).
\end{aligned}
$$

2. Coordinate expressions

A local coordinate system (x^i, u^j) on π induces the canonical coordinates (x^i, p^j_σ) on the space $J^\infty\pi$, where $\sigma = (i_1, \dots, i_n)$ is a multi-index of length $|\sigma| = i_1 + \dots + i_n$. The operator of total derivative of multi-order σ (and order $|\sigma|$) is $\mathcal{D}_\sigma = \mathcal{D}_1^{i_1} \cdots \mathcal{D}_n^{i_n}$, where $\mathcal{D}_i = \partial_{x^i} + \sum p^j_{\tau+1_i} \partial_{p^j_\tau}$.

The linearization of $F = (F_1, \dots, F_r)$ is $\ell_F = (\ell(F_1), \dots, \ell(F_r))$ with

$$\ell(F_i) = \sum (\partial_{p^j_\sigma} F_i) \cdot \mathcal{D}_\sigma^{[j]},$$

where $\mathcal{D}_\sigma^{[j]}$ denotes the operator $\mathcal{D}_\sigma$ applied to the j-th component of the section from $C^\infty(\pi)$.

The i-th component of the evolutionary differentiation $\mathcal{Э}_G$ corresponding to $G = (G_1, \dots, G_n)$ equals

$$\mathcal{Э}_G^i = \sum (\mathcal{D}_\sigma G_j) \cdot \partial_{p^j_\sigma}{}^{[i]},$$

where $\partial_{p^j_\sigma}{}^{[i]}$ denotes the operator $\partial_{p^j_\sigma}$ applied to the i-th component of the section from $C^\infty(\pi)$.

Then i-th components of the Jacobi bracket is given by

$$\{F,G\}_i = \sum \big(\mathcal{D}_\sigma(G_j)\cdot\partial_{p^j_\sigma}F_i - \mathcal{D}_\sigma(F_j)\cdot\partial_{p^j_\sigma}G_i\big).$$

These formulas are known [KLV]. It is instructive to demonstrate the Jacobi identity in coordinates. For this we need the following assertion.

Lemma 4. *In canonical coordinates on $J^\infty\pi$:*

$$\partial_{p^i_\sigma}\mathcal{D}_\tau = \sum \mathcal{D}_{\tau-\varkappa}\partial_{p^i_{\sigma-\varkappa}}$$

(the difference of multi-indices $\sigma-\varkappa$ is defined whenever $\varkappa\subset\sigma$), the summation is by $\varkappa$ counted with multiplicity. More generally for vector differential operators if $\mathcal{D}^{[j]}_\sigma$ is the operator $\mathcal{D}_\sigma$ acting on the j-th component, then the above formula holds true for such specification.

This follows from iteration of the formula $[\partial_{p^j_\sigma},\mathcal{D}_i]=\partial_{p^j_{\sigma-1_i}}$. Thus

$$\begin{aligned}\{F,\{G,H\}\} = \sum F_{p_\sigma}\mathcal{D}_{\sigma-\varkappa}(G_{p_\tau})\,\mathcal{D}_{\tau+\varkappa}(H) - F_{p_\sigma}\mathcal{D}_{\sigma-\varkappa}(H_{p_\tau})\,\mathcal{D}_{\tau+\varkappa}(G)\\ - G_{p_\sigma p_\tau}\mathcal{D}_\tau(H)\mathcal{D}_\sigma(F) + H_{p_\sigma p_\tau}\mathcal{D}_\tau(G)\mathcal{D}_\sigma(F)\\ - (G_{p_\sigma}\mathcal{D}_{\sigma-\varkappa}(H_{p_{\tau-\varkappa}}) - H_{p_\sigma}\mathcal{D}_{\sigma-\varkappa}(G_{p_{\tau-\varkappa}}))\mathcal{D}_\tau(F),\end{aligned}$$

which yields $\sum_{\text{cyclic}}\{F,\{G,H\}\}=0$.

Now we write the Hessian:

$$\mathrm{Hess}_F(G,H) = \sum F_{p_\sigma p_\tau}\mathcal{D}_\sigma G\cdot\mathcal{D}_\tau H,$$

and its symmetry in G,H and vanishing for linear F is obvious.

The compensated Leibniz formula can be written as follows:

$$\begin{aligned}&\{F,\ell_G H\} - \ell_{\{F,G\}}H - \ell_G\{F,H\} =\\ &\sum F_{p_\sigma}\mathcal{D}_{\sigma-\varkappa}(G_{p_\tau})\,\mathcal{D}_{\tau+\varkappa}(H) - (G_{p_\sigma p_\tau}\mathcal{D}_\tau(H)\mathcal{D}_\sigma(F) + G_{p_\tau}\partial_{p_\sigma}\mathcal{D}_\tau(H))\mathcal{D}_\sigma(F)\\ &-(F_{p_\sigma p_\tau}\mathcal{D}_\sigma(G) + F_{p_\sigma}\partial_{p_\tau}\mathcal{D}_\sigma(G))\mathcal{D}_\tau(H) + (G_{p_\sigma p_\tau}\mathcal{D}_\sigma(F) + G_{p_\sigma}\partial_{p_\tau}\mathcal{D}_\sigma(F))\mathcal{D}_\tau(H)\\ &- G_{p_\sigma}\,(\mathcal{D}_{\sigma-\varkappa}(F_{p_\tau})\mathcal{D}_{\tau+\varkappa}(H) - \mathcal{D}_{\sigma-\varkappa}(H_{p_\tau})\mathcal{D}_{\tau+\varkappa}(F)) = -\,\mathrm{Hess}_F(G,H)\end{aligned}$$

and the anomaly in commuting linearizations is:

$$\begin{aligned}&[\ell_F,\ell_G] - \ell_{\{F,G\}} =\\ &\qquad\sum F_{p_\sigma}\mathcal{D}_{\sigma-\varkappa}(G_{p_\tau})\,\mathcal{D}_{\tau+\varkappa}(H) - G_{p_\sigma}\mathcal{D}_{\sigma-\varkappa}(F_{p_\tau})\,\mathcal{D}_{\tau+\varkappa}(H)\\ &-(F_{p_\sigma p_\tau}\mathcal{D}_\sigma(G) + F_{p_\sigma}\partial_{p_\tau}\mathcal{D}_\sigma(G))\mathcal{D}_\tau(H) + (G_{p_\sigma p_\tau}\mathcal{D}_\sigma(F) + G_{p_\sigma}\partial_{p_\tau}\mathcal{D}_\sigma(F))\mathcal{D}_\tau(H)\\ &\qquad\qquad = \mathrm{Hess}_G(F,H) - \mathrm{Hess}_F(G,H).\end{aligned}$$

This gives an alternative proof of Propositions 3 and 2.

3. Auxiliary integrals

Definition. *An operator $G \in \operatorname{diff}(\pi,\pi)$ is called an auxiliary integral for $F \in \operatorname{diff}(\pi,\pi)$ if*

$$\{F,G\} = \ell_\lambda F + \ell_\mu G$$

for some operators $\lambda \in \operatorname{diff}(\pi,\pi)$ and $\mu \notin \mathscr{C}\operatorname{Diff}(\pi,\pi)\cdot F \setminus \{0\}$. The set of such G is denoted by $\operatorname{Aux}(F)$.

It is better to denote $\operatorname{Aux}_\mu(F)$ the space of G satisfying the above formula with some fixed $\mu \in \operatorname{diff}(\pi,\pi)$, because it is a vector space. Then $\operatorname{Aux}(F) = \cup_\mu \operatorname{Aux}_\mu(F)$. We can assume $\operatorname{ord}(\mu) < \operatorname{ord}(F)$ for scalar operators, i.e. $\operatorname{rank}\pi = 1$.

With certain non-degeneracy condition for the symbols of F, G the following statement holds:

Theorem 5. *A non-linear differential operator G is an auxiliary integral for another operator F iff the system $F = 0, G = 0$ is compatible (formally integrable).*

The generic position condition for the symbols of F, G is essential. If $\pi = \mathbf{1}$ is the trivial one-dimensional bundle, this condition is just the transversality of the characteristic varieties $\operatorname{Char}^{\mathbb{C}}(F)$ and $\operatorname{Char}^{\mathbb{C}}(G)$ in the bundle $\mathbb{P}^{\mathbb{C}}T^*M$ (after pull-back to the joint system $F = G = 0$ in jets); in this form it is a particular form of the statement proved in [KL$_2$]. For $\operatorname{rank}\pi > 1$ the condition is more delicate and will be presented elsewhere.

Notice that $\operatorname{Aux}_0(F) = \operatorname{Sym}(F)$ is the space of symmetries of F. This is a Lie algebra with respect to the Jacobi bracket. It can be represented as a union of spaces

$$\operatorname{Sym}_\theta(F) = \{H \,:\, \ell_F H = \ell_{\theta+H} F\}, \qquad \theta \in \operatorname{diff}(\pi,\pi),$$

which are modules over $\operatorname{Sym}_0(F)$. More generally we have the graded group: $\operatorname{Sym}_{\theta'}(F) + \operatorname{Sym}_{\theta''}(F) \subset \operatorname{Sym}_{\theta'+\theta''}(F)$

Let us assume $G \in \operatorname{Aux}_\mu(F)$, $H \in \operatorname{Sym}_\theta(F)$, i.e.

$$\{F,G\} = \ell_\lambda F + \ell_\mu G, \qquad \{F,H\} = \ell_\theta F.$$

Then denoting $\operatorname{ad}_H = \{H,\cdot\} = \ell_H - \mathcal{D}_H$ we get:

$$\begin{aligned}
\mathrm{ad}_F\{G,H\} &= \{\mathrm{ad}_F G, H\} + \{G, \mathrm{ad}_F H\} \\
&= -\{H, \ell_\lambda F + \ell_\mu G\} + \{G, \ell_\theta F\} \\
= \ell_{\{\lambda,H\}}F + \ell_\lambda\{F,H\} &+ \mathrm{Hess}_H(\lambda,F) + \ell_{\{\mu,H\}}G + \ell_\mu\{G,H\} + \mathrm{Hess}_H(\mu,G) \\
&- \ell_{\{\theta,G\}}F - \ell_\theta\{F,G\} - \mathrm{Hess}_G(\theta,F) \\
&= (\ell_{\{\lambda,H\}} + [\ell_\lambda,\ell_\theta] - \ell_{\{\theta,G\}} + \mathrm{Hess}_H \lambda - \mathrm{Hess}_G \theta)F + \ell_\mu\{G,H\} \\
&\qquad + (\ell_{\{\mu,H\}} - \ell_\theta\ell_\mu + \mathrm{Hess}_H \mu)G.
\end{aligned}$$

Thus $\{G,H\}$ is an auxiliary integral for F if $\ell_\theta\ell_\mu = \ell_{\{\mu,H\}} + \mathrm{Hess}_H \mu$ (the "iff" condition means the difference annihilates G), which can be written as

$$\mu \in \mathrm{Ker}[(\ell_\theta + \ell_{\mathrm{ad}_H} - \mathrm{Hess}_H) \circ \ell\,].$$

Such a pair $\theta \in \mathrm{sym}^*(F) = \mathrm{Sym}(F)/\,\mathrm{Sym}_0(F)$, $H \in \mathrm{Sym}_\theta(F)$ determines the action of the second component

$$\mathrm{ad}_H : \mathrm{Aux}_\mu(F) \to \mathrm{Aux}_\mu(F).$$

Also since

$$\begin{aligned}
\ell_{\{\mu,H\}}G = \ni_G\{\mu,H\} = \ni_G\ni_H(\mu) - \ni_G\ell_H(\mu) = (\ni_H - \ell_H)\ni_G(\mu) \\
- \ni_{\{G,H\}}\mu - \mathrm{Hess}_H(G,\mu) = -\,\mathrm{ad}_H \ni_G(\mu) - \mathrm{Hess}_H(\mu,G) - \ell_\mu\{G,H\},
\end{aligned}$$

we have:

$$\ell_\mu\{G,H\} + (\ell_{\{\mu,H\}} - \ell_\theta\ell_\mu + \mathrm{Hess}_H \mu)G = -(\mathrm{ad}_H + \ell_\theta)\ell_\mu G.$$

Thus if $H \in \mathrm{Sym}_\theta(F)$, i.e. $(\mathrm{ad}_H + \ell_\theta)F = 0$, and $\mu \in \mathrm{Ker}[(\mathrm{ad}_H + \ell_\theta)\circ\ell\,]$, i.e. $(\mathrm{ad}_H + \ell_\theta)\ell_\mu = 0$, then

$$\mathrm{ad}_H : \mathrm{Aux}_\mu(F) \to \mathrm{Sym}(F).$$

4. Symmetries and compatibility

It has been a common belief that if $G \in \mathrm{Sym}(F)$, then the system $F = 0, G = 0$ is compatible, which forms the base of investigation for auto-model solutions. This is however not always true.

Example: Let F, G be two linear diagonal operators with constant coefficients. Then $\{F, G\} = 0$ (in this case the Jacobi bracket is the standard commutator), so that G is a symmetry of F. However the system $F = 0, G = 0$ is usually incompatible: for generic F, G of the considered type the only solution will be the trivial zero vector-function.

More complicated non-diagonal operators are possible, but it would be better to consider non-homogeneous linear operators. Then if the coefficients are constant and generic, the linear matrix part commute, but the system $F = 0, G = 0$ may have no solutions at all.

For instance if we take

$$F = \begin{bmatrix} (\mathcal{D}_x^2 - \mathcal{D}_y) & 0 \\ 0 & (\mathcal{D}_x\mathcal{D}_y + 1) \end{bmatrix} \cdot \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$G = \begin{bmatrix} (\mathcal{D}_x\mathcal{D}_y - 1) & 0 \\ 0 & (\mathcal{D}_y^2 - \mathcal{D}_x) \end{bmatrix} \cdot \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

then $\{F, G\} = 0$, so that $G \in \mathrm{Sym}(F)$, while the system $F = 0, G = 0$ is not compatible, and moreover its solutions space is empty.

Thus the flow $u_t = G(u)$ on the equation $F = 0$ has no fixed points (no auto-model solutions). Here t is an additional variable (x is the base multi-variable for PDEs $F = 0$ and $G = 0$), so that $G \in \mathrm{Sym}(F)$ can be expressed as compatibility of the system

$$F(u) = 0, \quad u_t = G(u),$$

while symmetric solutions correspond to the stationary case $u_t = 0$, i.e. compatibility of the system $F(u) = 0, G(u) = 0$[2].

However if the non-degeneracy condition assumed in Theorem 5 is satisfied, then auto-model (or invariant) solutions exist in abundance, namely they have the required functional dimension and rank as Hilbert polynomial (or Cartan test [C]) predicts, see [KL_4].

Remark. *Symmetric solutions are the stationary points of the evolutionary fields and they are similar to the fixed points of smooth vector fields on $\mathbb{R}^n$, which must exist provided the vector field is Morse at infinity. The non-degeneracy condition plays a similar role.*

Many examples of auto-model solutions and their generalizations can be found in [BK, Ol, Ov], non-local analogs use the same technique and similar theory [KLV, KK, KKV].

Compatible systems correspond to reductions of PDEs and are sometimes called conditional symmetries by analogy with finite-dimensional integrable systems on one isoenergetic surface [FZ]. But the rigorous result must rely on certain general position property for the symbol of differential operators, otherwise it can turn wrong [KL_2, KL_3]. The method based on this approach makes specification of the general idea of differential constraint and is described in [KL_1].

[2] I am grateful to S.Igonin and A.Verbovetsky for an enlightening discussion about the results of [KL_2, KL_3] and the symmetry condition.

5. Conclusion

In this note we described the higher-jets calculus corresponding to symmetries and compatible constraints, basing on the Jacobi brackets. Another approach to integrability of vector systems is given by minimal overdetermination and it uses multi-brackets of differential operators

$$\{\cdots\} : \Lambda^{m+1} \operatorname{diff}(m \cdot \mathbf{1}, \mathbf{1}) \to \operatorname{diff}(m \cdot \mathbf{1}, \mathbf{1})$$

introduced in [KL_3], which are governed by the non-commutative Plücker identity.

Following this approach a minimal generalization of symmetry for $F = (F_1, \dots, F_m) \in \operatorname{diff}(\pi, \pi)$ with $\pi = m \cdot \mathbf{1}$ is such $G \in \operatorname{diff}(\pi, \mathbf{1})$ that

$$\{F_1, \dots, F_m, G\} = \ell_{\theta_1} F_1 + \cdots + \ell_{\theta_m} F_m.$$

With certain non-degeneracy assumption [KL_3] this implies that the overdetermined system $F = 0, G = 0$ is compatible (formally integrable).

A more advanced algebraic technique would yield another higher-jets calculus producing anomaly that manifests in non-vanishing of the expression

$$\{\ell_{F_1}, \cdots, \ell_{F_{m+1}}\} - \ell_{\{F_1, \cdots, F_{m+1}\}}.$$

Implications for vector auxiliary integrals and generalized Lagrange-Charpit method follow the same scheme.

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