

ON JORDAN CLASSES FOR VINBERG'S  $\theta$ -GROUPS

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*Dedicated to Prof. È. B. Vinberg*

**Abstract.** V. L. Popov has recently introduced an analogue of Jordan classes (packets or decomposition classes) for the action of a  $\theta$ -group  $(G_0, V)$ , showing that they are finitely-many, locally-closed, irreducible unions of  $G_0$ -orbits of constant dimension partitioning  $V$ . We carry out a local study of their closures showing that Jordan classes are smooth and that their closure is a union of Jordan classes. We parametrize Jordan classes and  $G_0$ -orbits in a given class in terms of the action of subgroups of Vinberg's little Weyl group, and include several examples and counterexamples underlying the differences with the symmetric case and the critical issues arising in the  $\theta$ -situation.

### Introduction

Theta groups (or, equivalently, periodically graded reductive Lie algebras) were deeply studied in [31], [32] as a natural generalisation of symmetric spaces, [12], [13]. In all situations,  $\mathfrak{g}$  is a  $\mathbb{Z}_m$ -graded complex reductive Lie algebra, its degree 0 part  $\mathfrak{g}_0$  is again reductive and the focus is on the action of the corresponding

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connected algebraic group  $G_0$  on the other homogeneous components  $\mathfrak{g}_i$  of  $\mathfrak{g}$ . As observed by Vinberg, there is no loss of generality in studying the action on the degree 1 component  $V = \mathfrak{g}_1$  only. Key results in [31] concern invariant theory and include the introduction of a little Weyl group and the analogue of the Steinberg map and Chevalley's restriction theorem and the proof that the little Weyl groups are complex reflection groups. These results were confirmed also in positive characteristic [14], where an alternative description of the little Weyl group in terms of the usual Weyl group is proposed. Many interesting examples in representation theory can be interpreted in terms of graded Lie algebras; for instance, if  $\mathfrak{g}$  is the Lie algebra of a classical group  $G$ , a grading on the defining representation of  $G$  induces a grading on  $\mathfrak{g}$  and the  $G_0$ -action on  $V$  can be seen as a representation of a cyclic quiver with additional structure, [18, Sect. 0.3, Section 9.5].

A structural feature of theta groups is that they are visible groups: that is, connected reductive algebraic groups with a representation for which each fiber of the Steinberg map consists of finitely many orbits. This property almost characterizes the theta groups; more precisely, a connected simple irreducible visible linear group is either a (commutant of a) theta group or it is isomorphic to  $\text{Spin}(11)$  or  $\text{Spin}(13)$  [11]. Various explicit descriptions of the orbits and invariants for theta groups of order  $m = 2$  are known (see [35, Summary Table]) but a number of cases with  $m \geq 3$  have also been considered in the literature [9], [10], [20], [23], [34].

An important application of theta group theory is in the representation theory of reductive groups over a  $p$ -adic field  $F$ . Indeed, the classification of positive rank gradings [14], [15], [25] over the residue field  $k$  of a maximal unramified extension  $L$  of  $F$  leads to the classification of nondegenerate  $K$ -types, and stable  $G_0$ -orbits in  $V^*$  are strictly related to supercuspidal representations of the rational points of  $G$  over  $F$  attached to elliptic  $\mathbb{Z}$ -regular elements of the Weyl group [26]. Also, in the context of a graded version of the generalized Springer correspondence, the block decomposition of the  $G_0$ -equivariant derived category supported on the nilpotent part of each  $\mathfrak{g}_i$  leads to the construction of representations of various graded double affine Hecke algebras with possibly unequal parameters: one for each block [18], [19]. It emerges from these constructions that parabolic induction is no longer the right instrument in the graded setting, leading to the introduction of spirals. This shows that even though many results in the classical symmetric case have an analogue in the graded setting, generalisations to the case of  $m > 2$  are not always straightforward.

This phenomenon is also visible in the study of related  $G_0$ -stable stratifications in  $V$ . In the setting of the ungraded generalized Springer correspondence, one of the relevant stratifications is given by the decomposition into Jordan classes (packets or decomposition classes) in a reductive group  $G$ , or Lie algebra  $\mathfrak{g}$ . In the Lie algebra setting, Jordan classes were introduced in [5] and were crucial in the construction of sheets for the adjoint action of a semisimple group  $G$  on its Lie algebra. These classes are  $G$ -stable, disjoint, finitely-many, locally-closed, smooth and irreducible. The decomposition into Jordan classes in a Lie algebra turns out to coincide with the decomposition into orbit-types: i.e., into the subsets of elements with same stabilizer up to conjugation. Borho and Kraft proved that

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sheets are easily described as regular closures of those Jordan classes which satisfy some maximality property with respect to closure inclusion, and it was shown in [4] that the closure and regular closure of Jordan classes can be described in terms of Lusztig-Spaltenstein's parabolic induction of adjoint orbits. The symmetric analogue of Jordan classes and sheets has been studied by Tauvel and Yu, (see [29] and references in there) and their closures were studied in [7, 8]. In the latter, it is again observed that parabolic induction is no longer efficient, and slice induction is proposed. One of the difficulties in working with parabolic induction is the fact that many homogenous Levi subalgebras do not necessarily lie in a homogeneous parabolic subalgebra: see the Appendix A for an example of this phenomenon. An analogue of Jordan classes for theta groups when  $\mathfrak{g}$  is semisimple has been recently introduced by Popov in [24], generalizing the classical and symmetric ones. As in these cases, Jordan classes form a partition of  $V$  into finitely-many, locally-closed, irreducible unions of  $G_0$ -orbits of constant dimension, and so sheets for the  $G_0$ -action on  $V$  are regular closures of some Jordan class. In this paper, we introduce a local study of such Jordan classes and their closures, leading us to prove that any Jordan class is smooth and that its closure is a union of Jordan classes. In order to characterize the closure relation, we provide an analogue of the results in [8] on slice induction. For our inductive arguments, we needed to extend slightly the notion of Jordan classes to the case of reductive Lie algebras. Our local approach differs from [8] because we rely on Luna's fundamental Lemma and use the Slodowy slice only after reduction to neighbourhoods of nilpotent points; Luna's slice theorem is also used for the proof of smoothness.

It is also worthwhile to notice that a different, coarser, notion of Jordan equivalence relation could have been introduced by using regularity for the  $G_0$ -action rather than for the action of the full group  $G$ . In the symmetric setting, these two notions coincide by virtue of [13, Prop. 5], but they might differ for  $m > 2$ . Popov's choice of Jordan classes in  $V$  ensures that each of them is contained in a usual Jordan class in  $\mathfrak{g}$ . We devote Section 1.4 and Section 2.3 to comparisons of different notions of regularity and refer to [21], [9] for an analysis of various results on regularity in theta groups.

The paper is structured as follows. In Section 1, we recall the basics on periodically graded complex reductive Lie algebras, introduce the relevant notions of regularity, and extend to the reductive case the general treatment in [24] of Jordan classes and sheets in  $V$ . We then focus in Section 2 on the local study of the closures of Jordan classes in  $V$ ; the main results here are Theorem 28, Theorem 29 and Proposition 30. We conclude Section 2 with some regularity questions, including Proposition 31. The last section is devoted to slice induction, leading to Theorem 34, and to the parametrization of the Jordan classes in  $V$  and the  $G_0$ -orbits contained in a class. The paper finishes with Example 4 on trivectors in 9-dimensional space and with Appendix A, dealing with obstructions to the existence of homogeneous parabolic subalgebras in  $\mathfrak{g}$ .

The results of this paper are valid over any algebraically closed field of characteristic zero, but we expect that they extend to suitable positive characteristics. Indeed the results from [16] needed in Sections 2–3 hold also in positive characteristic under mild restrictions, cf. [3].

During completion of this paper, we were informed that Professor È. B. Vinberg had passed away. Without his work in [31] this manuscript would never have been written, so we would like to dedicate it to his memory.

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## 1. Preliminaries on Vinberg's $\theta$ -groups and Jordan classes

### 1.1. Graded Lie algebras

Let  $\mathfrak{g}$  be a complex reductive Lie algebra which is  $\mathbb{Z}_m$ -graded: that is, it admits a direct sum decomposition of vector spaces

$$\mathfrak{g} = \bigoplus_{l \in \mathbb{Z}_m} \mathfrak{g}_l \tag{1}$$

with  $[\mathfrak{g}_i, \mathfrak{g}_l] \subset \mathfrak{g}_{i+l}$  for all  $i, l \in \mathbb{Z}_m$ . We note that the subspaces of (1) can be recovered as the eigenspaces of the automorphism  $\theta$  of  $\mathfrak{g}$  defined by  $\theta(x) = \omega^l x$  for  $x \in \mathfrak{g}_l$ , where  $\omega = e^{2\pi i/m}$ . Conversely, any automorphism  $\theta$  of  $\mathfrak{g}$  of period  $m$  defines a  $\mathbb{Z}_m$ -grading. Due to this, we will denote a Lie algebra  $\mathfrak{g}$  with a  $\mathbb{Z}_m$ -grading by the triple  $\{\mathfrak{g}, \theta, m\}$ , or often simply by  $\{\mathfrak{g}, \theta\}$ . Whenever a subspace  $A \subset \mathfrak{g}$  is homogeneous, i.e., it satisfies  $A = \bigoplus_l (A \cap \mathfrak{g}_l)$ , we will write  $A_l = A \cap \mathfrak{g}_l$  and  $A = \bigoplus_l A_l$ .

The Lie algebra  $\mathfrak{g}$  has a decomposition into homogeneous ideals

$$\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{s}, \quad \text{where } \mathfrak{s} := [\mathfrak{g}, \mathfrak{g}]. \tag{2}$$

We denote by  $\kappa$  a bilinear form on  $\mathfrak{g}$  that is nondegenerate,  $\mathfrak{g}$ -invariant,  $\theta$ -invariant and such that  $\mathfrak{z}(\mathfrak{g})$  and  $\mathfrak{s}$  are orthogonal. We call any such bilinear form *adapted*.

**Lemma 1.** *There exists an adapted bilinear form  $\kappa$  on  $\mathfrak{g}$  if and only if  $\mathfrak{z}(\mathfrak{g})$  is symmetrically graded: i.e.,  $\dim \mathfrak{z}(\mathfrak{g})_l = \dim \mathfrak{z}(\mathfrak{g})_{-l}$  for all  $l \in \mathbb{Z}_m$ . In this case,  $\dim \mathfrak{g}_l = \dim \mathfrak{g}_{-l}$  for all  $l \in \mathbb{Z}_m$ .*

*Proof.* If  $\kappa$  is adapted, then  $\kappa(\mathfrak{g}_l, \mathfrak{g}_i) = 0$  whenever  $i + l \neq 0$ , hence  $\mathfrak{g}_{-l}$  and  $\mathfrak{g}_l$  are dual spaces, and so are  $\mathfrak{s}_{-l}$  and  $\mathfrak{s}_l$  and also  $\mathfrak{z}(\mathfrak{g})_{-l}$  and  $\mathfrak{z}(\mathfrak{g})_l$ . In particular,  $\mathfrak{z}(\mathfrak{g})$  is symmetrically graded. Conversely, it is enough to consider an appropriate extension of the Killing form of  $\mathfrak{s}$ .  $\square$

With the term *reductive  $\mathbb{Z}_m$ -graded Lie algebra  $\{\mathfrak{g}, \theta\}$* , we will always mean a complex reductive Lie algebra  $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{s}$  together with a  $\mathbb{Z}_m$ -grading such that the center  $\mathfrak{z}(\mathfrak{g})$  is symmetrically graded. This is also the class of graded Lie algebras considered in [31], since they allow for adapted bilinear forms. By Lemma 1 we may assume  $\kappa$  to be an extension of the Killing form of  $\mathfrak{s}$ .

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Let  $G$  be any connected algebraic group with Lie algebra  $\text{Lie}(G) = \mathfrak{g}$ , let  $S$  be the connected subgroup of  $G$  with  $\text{Lie}(S) = \mathfrak{s}$ , and let  $^\circ$  denote the identity component of a closed subgroup, so  $G = Z(G)^\circ S$ . Let  $G_0$  be the connected subgroup of  $G$  with  $\text{Lie}(G_0) = \mathfrak{g}_0$ . Unless otherwise stated, for Lie subalgebras of  $\mathfrak{g}$  we will use a gothic letter, the corresponding Roman capital letter will indicate the connected subgroup of  $G$  with that Lie algebra, a lower index 0 its intersection with  $G_0$ . So, the decomposition  $\mathfrak{g}_0 = \mathfrak{z}(\mathfrak{g})_0 \oplus \mathfrak{s}_0$ , gives an almost direct product  $G_0 = Z(G)_0^\circ S_0^\circ$ , where  $S_0^\circ$  is the reductive, connected subgroup of  $S$  with  $\text{Lie}(S_0^\circ) = \mathfrak{s}_0$ . By restricting the adjoint representation,  $G_0$  and  $S_0^\circ$  act on  $\mathfrak{g}_l$ , for any  $l \in \mathbb{Z}_m$ , with trivial action of  $Z(G)_0^\circ$ . The reduction process in [31, p. 467] shows that it is enough to focus on the case of  $l = 1$ ; we set  $V := \mathfrak{g}_1$ . The linear group of transformations of  $V$  associated to  $G_0$  is called the  $\theta$ -group of the graded Lie algebra  $\{\mathfrak{g}, \theta\}$  and it does not depend on the choice of  $G$  in the class of locally isomorphic groups. However, by abuse of notation, we will directly refer to  $G_0$  as the  $\theta$ -group of  $\{\mathfrak{g}, \theta\}$ . The decomposition (2) in degree 1 gives a decomposition of  $V$  into  $G_0$ -stable subspaces  $V = \mathfrak{z}(\mathfrak{g})_1 \oplus \mathfrak{s}_1$  with trivial  $G_0$ -action on  $\mathfrak{z}(\mathfrak{g})_1$ . Observe that  $\mathfrak{z}(\mathfrak{g})_1 \neq 0$  may occur only if  $\theta$  is not inner.

Let  $x \in \mathfrak{g}$  and  $\mathfrak{m}$  be a Lie subalgebra of  $\mathfrak{g}$  with associated subgroup  $M \subset G$ . The orbit of  $x$  for the action of  $M$  is denoted by  $\mathcal{O}_x^M$ , and the stabilizer of  $x$  in  $M$  by  $M^x$ . The centralizer of  $x$  in  $\mathfrak{m}$  is denoted by  $\mathfrak{m}^x$ , with center  $\mathfrak{z}(\mathfrak{m}^x)$ . If  $x \in V$ , then  $\mathfrak{g}^x$ ,  $\mathfrak{z}(\mathfrak{g}^x)$  and  $[\mathfrak{g}^x, \mathfrak{g}^x]$  are  $\theta$ -stable: in other words, homogeneous. We recall that if  $x \in \mathfrak{g}$  is semisimple, then  $G^x$  is a connected subgroup of  $G$ , the Levi subgroup of a parabolic subgroup of  $G$  [28, 7.3.5]. In this case, the restriction of  $\kappa$  to  $\mathfrak{g}^x = \mathfrak{z}(\mathfrak{g}^x) \oplus [\mathfrak{g}^x, \mathfrak{g}^x]$  is an adapted bilinear form, so  $\mathfrak{z}(\mathfrak{g}^x) = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{z}(\mathfrak{s}^x)$  is symmetrically graded. We stress that  $G_0^x = G^x \cap G_0$  is not connected in general.

We recall the following general results on centralizers, that we will later apply when  $x \in V$ .

**Lemma 2** ([29, Prop. 35.3.1, Cor. 35.3.2]). *Let  $x \in \mathfrak{g}$ . Then,*

$$[\mathfrak{g}, x]^\perp = [\mathfrak{g}, \mathfrak{z}(\mathfrak{g}^x)]^\perp = \mathfrak{g}^x \quad \text{and} \quad [\mathfrak{g}, \mathfrak{g}^x]^\perp = \mathfrak{z}(\mathfrak{g}^x). \quad (3)$$

*The following conditions are equivalent for any  $x, y \in \mathfrak{g}$ :*

- (i)  $y \in \mathfrak{z}(\mathfrak{g}^x)$ ;
- (ii)  $\mathfrak{g}^x \subset \mathfrak{g}^y$ ;
- (iii)  $[\mathfrak{g}, y] \subset [\mathfrak{g}, x]$ ;
- (iv)  $\mathfrak{z}(\mathfrak{g}^y) \subset \mathfrak{z}(\mathfrak{g}^x)$ ;

**Corollary 3.** *Let  $g \in G_0$  and  $x, y \in V$ . Then the following conditions are equivalent:*

- (i)  $g \cdot \mathfrak{g}^x = \mathfrak{g}^y$ ;
- (ii)  $g \cdot \mathfrak{z}(\mathfrak{g}^x) = \mathfrak{z}(\mathfrak{g}^y)$ ;
- (iii)  $g \cdot \mathfrak{z}(\mathfrak{g}^x)_1 = \mathfrak{z}(\mathfrak{g}^y)_1$ .

*Proof.* Clearly (i)  $\Leftrightarrow$  (ii) by Lemma 2, since  $g \cdot \mathfrak{g}^x = \mathfrak{g}^{g \cdot x}$  and  $g \cdot \mathfrak{z}(\mathfrak{g}^x) = \mathfrak{z}(\mathfrak{g}^{g \cdot x})$ , and (ii)  $\Rightarrow$  (iii). If (iii) holds, then  $y \in \mathfrak{z}(\mathfrak{g}^y)_1 = \mathfrak{z}(\mathfrak{g}^{g \cdot x})_1$  and  $x \in \mathfrak{z}(\mathfrak{g}^{g^{-1} \cdot y})_1$ , hence  $g \cdot \mathfrak{g}^x = \mathfrak{g}^y$  by Lemma 2.  $\square$

### 1.2. The Jordan decomposition

Let  $\{\mathfrak{g}, \theta\}$  be a reductive  $\mathbb{Z}_m$ -graded Lie algebra. For elements  $x, y, z$  in  $\mathfrak{g}$ , lower indices  $s$  and  $n$  will always indicate semisimple and nilpotent parts in the Jordan decomposition: i.e., they stand for  $x = x_s + x_n$  with  $x_s \in \mathfrak{g}$  semisimple,  $x_n \in \mathfrak{g}$  nilpotent, and  $[x_s, x_n] = 0$ . Elements of  $\mathfrak{z}(\mathfrak{g})$  are always intended to be semisimple.

Let  $\mathcal{S}$  ( $\mathcal{N}$ , respectively) be the set of semisimple (nilpotent, respectively) elements of  $\mathfrak{g}$ . We note that  $\theta$  preserves both  $\mathcal{S}$  and  $\mathcal{N}$ , so semisimple and nilpotent parts of any  $x \in \mathfrak{g}_l$  also belong to  $\mathfrak{g}_l$ . We set  $\mathcal{S}_V = \mathcal{S} \cap V$ ,  $\mathcal{N}_V = \mathcal{N} \cap V$ , and stress that the number of  $G_0$ -orbits in  $\mathcal{N}_V$  is finite [31].

**Lemma 4.** *The action of the reductive Lie algebra  $\mathfrak{g}_0$  on  $\mathfrak{g}$  is completely reducible.*

*Proof.* Since  $\mathfrak{g}_0 = \mathfrak{z}(\mathfrak{g})_0 \oplus \mathfrak{s}_0$ , it is sufficient to prove the claim for  $\mathfrak{s}_0$ . Now  $\kappa$  restricted to  $\mathfrak{s}_0$  is nondegenerate and  $\mathfrak{s}_0$  contains the semisimple and nilpotent parts of any of its elements. The claim then follows from, e.g., [29, Prop. 20.5.12].  $\square$

We emphasize that  $\mathfrak{g}_0$  is not a subalgebra of maximal rank of  $\mathfrak{g}$  in general: that is, it might not contain any Cartan subalgebra of  $\mathfrak{g}$ . Let  $x \in V$ . A direct consequence of Lemma 2 is as follows.

**Lemma 5** ([31, p. 470]). *The tangent space  $T_x \mathcal{O}_x^{G_0}$  to  $\mathcal{O}_x^{G_0}$  at  $x$  is given by the subspace  $[\mathfrak{g}_0, x]$  of  $V$ . Its orthogonal complement in  $\mathfrak{g}_{-1}$  coincides with  $\mathfrak{g}_{-1}^x$ .*

### 1.3. The Cartan subspace

A *Cartan subspace* of  $\{\mathfrak{g}, \theta\}$  is an abelian subspace  $\mathfrak{c}$  of  $V$  which consists of semisimple elements and it is maximal in the class of such subspaces.

**Theorem 6.** [31, p. 472] *Any two Cartan subspaces of  $\{\mathfrak{g}, \theta\}$  are conjugate by the action of an element in  $G_0$ . As a consequence, if  $x \in \mathcal{S}_V$ , then  $\mathcal{O}_x^{G_0}$  meets any Cartan subspace of  $\{\mathfrak{g}, \theta\}$ .*

The dimension of a Cartan subspace of a graded Lie algebra  $\{\mathfrak{g}, \theta\}$  is called the *rank* of  $\{\mathfrak{g}, \theta\}$ . It is clear that  $\{\mathfrak{g}, \theta\}$  has zero rank if and only if  $V \subset \mathcal{N}_V$ . For any set  $R$  of commuting elements of  $\mathcal{S}_V$ , the centralizer  $\mathfrak{c}_{\mathfrak{g}}(R) = \bigcap_{x \in R} \mathfrak{g}^x$  of  $R$  in  $\mathfrak{g}$  is a homogeneous Levi subalgebra of  $\mathfrak{g}$ , so

$$\mathfrak{c}_{\mathfrak{g}}(R) = \mathfrak{z}(\mathfrak{c}_{\mathfrak{g}}(R)) \oplus [\mathfrak{c}_{\mathfrak{g}}(R), \mathfrak{c}_{\mathfrak{g}}(R)], \quad (4)$$

and these summands are also homogeneous. We recall a useful characterization of a Cartan subspace in terms of its centralizer [31, p. 471].

**Proposition 7.** *A subspace  $\mathfrak{c} \subset V$  consisting of commuting semisimple elements is a Cartan subspace if and only if  $\mathfrak{z}(\mathfrak{c}_{\mathfrak{g}}(\mathfrak{c}))_1 = \mathfrak{c}$  and the graded Lie algebra  $\{[\mathfrak{c}_{\mathfrak{g}}(\mathfrak{c}), \mathfrak{c}_{\mathfrak{g}}(\mathfrak{c})], \theta\}$  has zero rank.*

Let  $\mathfrak{c}$  be a Cartan subspace. By the previous result and equation (4) for  $R = \mathfrak{c}$ , we have a decomposition  $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{c})_1 = \mathfrak{c} \oplus [\mathfrak{c}_{\mathfrak{g}}(\mathfrak{c}), \mathfrak{c}_{\mathfrak{g}}(\mathfrak{c})]_1$ , with  $\mathfrak{c} \subset \mathcal{S}_V$  and  $[\mathfrak{c}_{\mathfrak{g}}(\mathfrak{c}), \mathfrak{c}_{\mathfrak{g}}(\mathfrak{c})]_1 \subset \mathcal{N}_V$ . In other words, this decomposition gives the Jordan components of any element of  $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{c})_1$ .

**Corollary 8.** *For any  $x \in \mathfrak{c}$ , we have  $\mathfrak{z}(\mathfrak{g}^x)_1 \subset \mathfrak{c}$ .*

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*Proof.* Since  $\mathfrak{z}(\mathfrak{g}^x)$  consists of semisimple elements, it follows that

$$\mathfrak{z}(\mathfrak{g}^x)_1 \subset \mathfrak{c}_{\mathfrak{g}}(\mathfrak{c})_1 \cap \mathcal{S}_V = \mathfrak{c}. \quad \square$$

Before turning to the next subsection, we recall that the Weyl group in the sense of Vinberg is the group  $W_{\text{Vin}} = W_{\text{Vin}}(\mathfrak{g}, \theta)$  of linear transformations of  $\mathfrak{c}$  given by  $W_{\text{Vin}} \cong N_{G_0}(\mathfrak{c})/Z_{G_0}(\mathfrak{c})$ , where  $N_{G_0}(\mathfrak{c})$  ( $Z_{G_0}(\mathfrak{c})$ , respectively) is the normalizer (centralizer, respectively) of  $\mathfrak{c}$  in  $G_0$ .

**Theorem 9** ([31, p. 473]). *The group  $W_{\text{Vin}}$  is finite and for  $x, y \in \mathfrak{c}$  we have  $y \in \mathcal{O}_x^{G_0}$  if and only if  $y \in W_{\text{Vin}} \cdot x$ .*

There is a geometric counterpart to this result [31, Sect. 4]. The restriction  $\mathbb{C}[V] \rightarrow \mathbb{C}[\mathfrak{c}]$  of polynomial functions from  $V$  to  $\mathfrak{c}$  induces a ‘‘Chevalley-type’’ isomorphism  $\mathbb{C}[V]^{G_0} \cong \mathbb{C}[\mathfrak{c}]^{W_{\text{Vin}}}$  and each fiber of the ‘‘Steinberg quotient map’’  $\varphi: V \rightarrow V//G_0 \cong \mathfrak{c}/W_{\text{Vin}}$  consists of finitely many  $G_0$ -orbits. Here,  $V//G_0$  is the GIT quotient of  $V$ , and two elements of  $V$  fail to be separated by the invariants if and only if their semisimple parts lie in the same  $G_0$ -orbit. Recall that semisimple and nilpotent orbits can also be characterized as the closed orbits and orbits whose closure contains 0, respectively. Hence, each fiber of  $\varphi$  contains exactly one closed orbit.

#### 1.4. Dimensions of centralizers and regularity conditions

This subsection deals with some general observations, which encompass a classical result of Kostant and Rallis (see [13] and [21]), and motivates the introduction of two distinct notions of regularity.

**Proposition 10.** *Let  $\{\mathfrak{g}, \theta\}$  be a reductive  $\mathbb{Z}_m$ -graded Lie algebra (with symmetrically graded center, as usual). Then  $\dim \mathfrak{g}_l - \dim \mathfrak{g}_l^x = \dim \mathfrak{g}_{-l-1} - \dim \mathfrak{g}_{-l-1}^x$  for all  $x \in V$  and  $l \in \mathbb{Z}_m$ .*

*Proof.* Let  $\kappa$  be an adapted bilinear form on  $\mathfrak{g}$ . The bilinear form given by  $\kappa_x(y, z) := \kappa(x, [y, z])$  is skew-symmetric for all  $y, z \in \mathfrak{g}$  and its radical is the centralizer  $\mathfrak{g}^x$ , which is homogeneous. It induces a nondegenerate bilinear form on the quotient  $\mathfrak{g}/\mathfrak{g}^x = \bigoplus_{l \in \mathbb{Z}_m} \mathfrak{g}_l/\mathfrak{g}_l^x$  with the property that  $\mathfrak{g}_i/\mathfrak{g}_i^x \perp \mathfrak{g}_l/\mathfrak{g}_l^x$  if  $i + l + 1 \neq 0$ , in particular  $\mathfrak{g}_l/\mathfrak{g}_l^x \cong (\mathfrak{g}_{-l-1}/\mathfrak{g}_{-l-1}^x)^*$ .  $\square$

**Corollary 11.**

- (i) *For all  $x \in V$  we have  $\dim \mathcal{O}_x^G = 2 \dim \mathcal{O}_x^{G_0} + \sum_{l \neq -1, 0} (\dim \mathfrak{g}_l - \dim \mathfrak{g}_l^x)$ ;*
- (ii) *If  $x \in \mathcal{S}_V$ , then  $\dim \mathfrak{g}_l - \dim \mathfrak{g}_l^x = \dim \mathfrak{g}_{l+1} - \dim \mathfrak{g}_{l+1}^x$  is independent of  $l \in \mathbb{Z}_m$  and we have  $\dim \mathcal{O}_x^G = m \dim \mathcal{O}_x^{G_0}$ ;*
- (iii) *Let  $x \in V$ , then  $\mathfrak{g}_0^x = \mathfrak{g}_0$  if and only if  $x \in \mathfrak{z}(\mathfrak{g})_1$ .*

*Proof.* Claim (i) is immediate from Proposition 10. If  $x \in \mathcal{S}_V$ , then the restriction of  $\kappa$  to  $\mathfrak{g}^x$  is nondegenerate and  $\dim \mathfrak{g}_l^x = \dim \mathfrak{g}_{-l}^x$  for all  $l \in \mathbb{Z}_m$ , so (ii) follows from Proposition 10 and (i). If  $x \in \mathfrak{z}(\mathfrak{g})_1$ , then clearly  $\mathfrak{g}_0^x = \mathfrak{g}_0$ . Conversely, if  $\mathfrak{g}_0^x = \mathfrak{g}_0$  then  $x \in \mathfrak{c}_{\mathfrak{g}}(\mathfrak{h}_0)$ , where  $\mathfrak{h}_0$  is a Cartan subalgebra of  $\mathfrak{g}_0$ , and  $x$  is semisimple by a classical result: see, e.g., [33, p. 116]. Then  $\mathfrak{g}^x = \mathfrak{g}$  by (ii) and  $x \in \mathfrak{z}(\mathfrak{g})_1$ .  $\square$

If  $x, y \in V$  are two elements with  $\dim \mathcal{O}_x^{G_0} = \dim \mathcal{O}_y^{G_0}$ , then  $\dim \mathfrak{g}_l^x = \dim \mathfrak{g}_l^y$  for  $l = 0, -1$ . The following simple example shows that the hypothesis  $x, y \in \mathcal{S}_V$  is indeed necessary for  $\dim \mathfrak{g}_l^x = \dim \mathfrak{g}_l^y$  to hold also for  $l \neq 0, -1$ .

**Example 1.** Let  $\mathfrak{g}$  be of type  $E_8$  and  $\theta$  be the automorphism of  $\mathfrak{g}$  of order 3 extensively studied in [34]. Here  $\mathfrak{g}_1 \cong \wedge^3 \mathbb{C}^9$ ,  $\mathfrak{g}_0 \cong \mathfrak{sl}(9)$  and  $\mathfrak{g}_{-1} \cong \wedge^3(\mathbb{C}^9)^*$ . The orbits of  $SL(9)$  on  $V = \wedge^3 \mathbb{C}^9$  have been classified in loc. cit. Let  $e_i$ , for  $1 \leq i \leq 9$ , be the canonical basis vectors of  $\mathbb{C}^9$  and let  $e_{ijl} := e_i \wedge e_j \wedge e_l$ . The trivector  $x_s = e_{123} + e_{456} + e_{789}$  is semisimple, with centralizer  $\mathfrak{g}^{x_s}$  a reductive Lie algebra with semisimple part  $\mathfrak{r}$  of type  $E_6$ . More precisely,  $\mathfrak{r} = \mathfrak{r}_{-1} \oplus \mathfrak{r}_0 \oplus \mathfrak{r}_1$  with

$$\mathfrak{r}_1 = X \otimes Y \otimes Z, \quad \mathfrak{r}_0 = \mathfrak{sl}(X) \oplus \mathfrak{sl}(Y) \oplus \mathfrak{sl}(Z), \quad \mathfrak{r}_{-1} = X^* \otimes Y^* \otimes Z^*,$$

where  $X = \text{span}\{e_1, e_2, e_3\}$ ,  $Y = \text{span}\{e_4, e_5, e_6\}$ ,  $Z = \text{span}\{e_7, e_8, e_9\}$  and where we identified tensor products with subspaces of  $\mathfrak{g}_{\pm 1}$  by mapping pure tensors to the corresponding antisymmetrizations. Since  $\mathfrak{g}^{x_s}$  has maximal rank, its center is 2-dimensional and it is not difficult to see that it is spanned by  $x_s \in \mathfrak{g}_1$  and  $x_s^* \in \mathfrak{g}_{-1}$ .

Now  $x_s$  is the semisimple part of trivectors  $x = x_s + x_n$  in the VI family, cf. [34, Table 5]. We consider those trivectors for which  $\dim \mathcal{O}_x^{G_0} = 76$ : i.e.,  $x = x_s + x_n$  with nilpotent part:

**Class 7:**  $x_n = e_{149} + e_{158} + e_{167} + e_{248} + e_{357}$ ;

**Class 8:**  $x_n = e_{149} + e_{167} + e_{258} + e_{347}$ ;

**Class 9:**  $x_n = e_{147} + e_{158} + e_{258} + e_{269}$ .

In all the three cases,  $\dim \mathfrak{g}_0^x = 4$  and  $\dim \mathfrak{g}_{-1}^x = 8$  by Proposition 10. However a direct computation tells us that  $\mathfrak{g}_1^x = \{y \in \mathfrak{g}_1^{x_s} \mid y \wedge x_n = 0\}$  has dimension 6, 8 and 10, respectively.

Corollary 11 and Example 1 motivate the following.

**Definition 1.** For any subset  $A \subset V$ , we set

- (i)  $A^{\text{reg}} = \{x \in A \mid \dim \mathfrak{g}^x \leq \dim \mathfrak{g}^y \text{ for all } y \in A\}$ ;
- (ii)  $A^\bullet = \{x \in A \mid \dim \mathfrak{g}_0^x \leq \dim \mathfrak{g}_0^y \text{ for all } y \in A\}$ .

The subset  $A^{\text{reg}}$  ( $A^\bullet$ , respectively) is called the regular part (the  $G_0$ -regular part, respectively) of  $A$ .

Note that  $A^\bullet = \{x \in A \mid \dim \mathfrak{g}_{-1}^x \leq \dim \mathfrak{g}_{-1}^y \text{ for all } y \in A\}$  due to Lemma 5. A simple relation between the two notions is given by the following.

**Lemma 12.** *Let  $A$  be an irreducible subset of  $V$  (with respect to the induced topology). Then*

$$A^{\text{reg}} = \bigcap_{l \in \mathbb{Z}_m} \{x \in A \mid \dim \mathfrak{g}_l^x \leq \dim \mathfrak{g}_l^y \text{ for all } y \in A\} \quad (5)$$

and so  $A^{\text{reg}} \subset A^\bullet$  as a Zariski open subset.

*Proof.* Clearly each subset on the R.H.S of (5) is nonempty and Zariski open in  $A$ . Since  $A$  is irreducible, the (finite) intersection of all such subsets is nonempty, so equal to  $A^{\text{reg}}$ .  $\square$

Let  $\Sigma$  be the set of restricted roots: that is, the nonzero linear functions on  $\mathfrak{c}$  occurring in the weight space decomposition of the semisimple action of the abelian subalgebra  $\mathfrak{c}$  on  $\mathfrak{g}$ . We write  $\mathfrak{g} = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{c}) \oplus \bigoplus_{\sigma \in \Sigma} \mathfrak{g}_\sigma$ .



**Example 2.** A semisimple element  $y \in \mathcal{S}_V$  belongs to  $\mathcal{S}_V^{\text{reg}}$  if and only if it lies in  $\mathcal{S}_V^\bullet$  if and only if  $\dim \mathfrak{g}^y = \dim \mathfrak{c}_{\mathfrak{g}}(\mathfrak{c})$ . The first equivalence follows from Corollary 11(ii). For the second one, observe that  $y$  is  $G_0$ -conjugated to some  $x \in \mathfrak{c}$ , whose centralizer is stabilized by  $\mathfrak{c}$  and decomposes as

$$\mathfrak{g}^x = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{c}) \oplus \bigoplus_{\sigma \in \Sigma(x)} \mathfrak{g}_\sigma \quad (6)$$

with  $\Sigma(x) = \{\sigma \in \Sigma \mid \sigma(x) = 0\}$ . Hence,  $x \in \mathcal{S}_V^{\text{reg}}$  if and only if  $\sigma(x) \neq 0$  for all  $\sigma \in \Sigma$ : i.e.,  $\mathfrak{g}^x = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{c})$ .

**Example 3.** Contrarily to the ungraded case and the  $(m = 2)$ -case, an element  $x_s \in \mathfrak{c} \cap \mathcal{S}_V^{\text{reg}}$  is not necessarily in  $V^\bullet$  (let alone  $V^{\text{reg}}$  or  $\mathfrak{g}^{\text{reg}}$ , since  $\mathfrak{g}^{\text{reg}} \cap V \subset V^{\text{reg}} \subset V^\bullet$ ). In general,  $x_s$  extends to an element  $x = x_s + x_n \in V^\bullet$  where  $x_n$  is an element in general position in  $[\mathfrak{c}_{\mathfrak{g}}(\mathfrak{c}), \mathfrak{c}_{\mathfrak{g}}(\mathfrak{c})]_1$  (recall that  $[\mathfrak{c}_{\mathfrak{g}}(\mathfrak{c}), \mathfrak{c}_{\mathfrak{g}}(\mathfrak{c})]_1$  consists of nilpotent elements). Then, if  $[\mathfrak{c}_{\mathfrak{g}}(\mathfrak{c}), \mathfrak{c}_{\mathfrak{g}}(\mathfrak{c})]_1 \neq 0$ , also  $x_n \neq 0$  and so  $\mathfrak{g}_0^x \subsetneq \mathfrak{g}_0^{x_s}$  due to (iii) of Corollary 11 applied to the reductive Lie algebra  $\mathfrak{g}^{x_s}$ . The  $G_0$ -orbits in  $V^\bullet$  have codimension in  $V$  equal to the rank of  $\{\mathfrak{g}, \theta\}$ , hence  $\dim \mathfrak{g}_0^x = \dim \mathfrak{g}_0 - \dim V + \dim \mathfrak{c}$ , see [31, Thm. 5].

### 1.5. Jordan classes and sheets for $\theta$ -groups

V. L. Popov has recently generalized the notion of a Jordan class to the case of semisimple  $\mathbb{Z}_m$ -graded Lie algebras  $\{\mathfrak{g}, \theta\}$  and studied its main geometric properties in [24]. For  $m = 1, 2$ , the notion coincides with that studied in [8, 29]. We here briefly extend his general treatment to the reductive case, which is more suitable for our inductive and local arguments of Sections 2–3, and directly refer to [24, Sect. 3] for more details. (We warn the reader that the symbol “reg” in [24] is replaced by “ $\bullet$ ” in the present paper.)

Let  $\{\mathfrak{g}, \theta\}$  be a reductive  $\mathbb{Z}_m$ -graded Lie algebra. Two elements  $x = x_s + x_n$  and  $y = y_s + y_n$  of  $V$  are  $G_0$ -Jordan equivalent if there exists  $g \in G_0$  such that

$$\mathfrak{g}^{y_s} = g \cdot \mathfrak{g}^{x_s}, \quad y_n = g \cdot x_n. \quad (7)$$

This is an equivalence relation  $x \overset{G_0}{\sim} y$  on  $V$ , the equivalence class  $J_{G_0}(x)$  of  $x \in V$  is called the  $G_0$ -Jordan class of  $x$  in  $V$ . Evidently the union of all  $G_0$ -Jordan classes in  $V$  is a partition of  $V$ .

*Remark 1.*

(1) By construction, any  $G_0$ -Jordan class is a  $G_0$ -stable set consisting of  $G_0$ -orbits of the same dimension. For example,  $\mathcal{S}_V^{\text{reg}}$  constitutes a  $G_0$ -Jordan class, as it can be easily seen from Theorem 6 and Example 2.

(2) The equality  $\mathfrak{g}^{x_s} = \mathfrak{g}^{z+x_s}$  for any  $z \in \mathfrak{z}(\mathfrak{g})_1$  and  $x \in \mathfrak{g}_1$  implies that  $z+x \overset{G_0}{\sim} x$ , so the additive group underlying  $\mathfrak{z}(\mathfrak{g})_1$  acts on each  $G_0$ -Jordan class  $J_{G_0}(x)$  by translations.

(3) Since  $G_0 = Z(G)_0^\circ S_0^\circ$ , the element  $g$  from (7) can always be chosen in  $S_0^\circ$ . Then, for  $x = z + x' \in \mathfrak{z}(\mathfrak{g})_1 \oplus \mathfrak{s}_1$  and  $y = w + y' \in \mathfrak{z}(\mathfrak{g})_1 \oplus \mathfrak{s}_1$ , the statement  $x \overset{G_0}{\sim} y$

holds if and only if  $x' \stackrel{S_0^\circ}{\sim} y'$  holds and the decomposition of  $V = \mathfrak{z}(\mathfrak{g})_1 \oplus \mathfrak{s}_1$  induces a decomposition

$$J_{G_0}(x) = J_{G_0}(x') = \mathfrak{z}(\mathfrak{g})_1 \times J_{S_0^\circ}(x') \quad (8)$$

where  $J_{S_0^\circ}(x')$  is the  $S_0^\circ$ -Jordan class of  $x' \in \mathfrak{s}_1$  as introduced in [24].

(4) Equality (8) applied to  $x' \in \mathcal{N}_V \subset \mathfrak{s}_1$  gives  $J_{G_0}(x') = \mathfrak{z}(\mathfrak{g})_1 \times \mathcal{O}_{x'}^{G_0} = \mathfrak{z}(\mathfrak{g})_1 \times \mathcal{O}_{x'}^{S_0^\circ}$ . For  $z \in \mathfrak{z}(\mathfrak{g})_1$  we then get  $J_{G_0}(z) = J_{G_0}(0) = \mathfrak{z}(\mathfrak{g})_1$ .

Observe that if  $x = x_s + x_n \in V$ , then  $x_n$  lies in the degree 1 component of the homogeneous semisimple subalgebra  $[\mathfrak{g}^{x_s}, \mathfrak{g}^{x_s}]$ .

**Lemma 13.** *We have  $\mathfrak{z}(\mathfrak{g}^x) = \mathfrak{z}(\mathfrak{g}^{x_s}) \oplus \mathfrak{z}(\mathfrak{g}^{x_n} \cap [\mathfrak{g}^{x_s}, \mathfrak{g}^{x_s}])$  and the components of an element in  $\mathfrak{z}(\mathfrak{g}^x)$  with respect to this decomposition coincide with its semisimple and nilpotent parts, respectively. Thus,  $\mathfrak{z}(\mathfrak{g}^x)_1 = \mathfrak{z}(\mathfrak{g}^{x_s})_1 \oplus \mathfrak{z}(\mathfrak{g}^{x_n} \cap [\mathfrak{g}^{x_s}, \mathfrak{g}^{x_s}])_1$ .*

*Proof.* The first claim is [29, Prop. 39.1.1], the second follows since  $\mathfrak{z}(\mathfrak{g}^x)$  and its summands are homogeneous.  $\square$

Lemma 2 tells us that

$$\begin{aligned} (\mathfrak{z}(\mathfrak{g}^x)_1)^{\text{reg}} &= \{y \in \mathfrak{z}(\mathfrak{g}^x)_1 \mid \mathfrak{g}^y = \mathfrak{g}^x\} \\ &= \{y \in \mathfrak{z}(\mathfrak{g}^x)_1 \mid \mathfrak{z}(\mathfrak{g}^y) = \mathfrak{z}(\mathfrak{g}^x)\} \\ &= \{y \in \mathfrak{z}(\mathfrak{g}^x)_1 \mid \text{rk}(\text{ad}_{\mathfrak{g}}(y)) = \text{rk}(\text{ad}_{\mathfrak{g}}(x))\}, \end{aligned} \quad (9)$$

which is a Zariski open subset of  $\mathfrak{z}(\mathfrak{g}^x)_1$ , hence irreducible. We note that this is also the set of all  $y \in V$  such that  $\mathfrak{g}^y = \mathfrak{g}^x$  and that  $x \in (\mathfrak{z}(\mathfrak{g}^x)^{\text{reg}})_1$ , so  $(\mathfrak{z}(\mathfrak{g}^x)^{\text{reg}})_1 = (\mathfrak{z}(\mathfrak{g}^x)_1)^{\text{reg}}$  and we will omit the parentheses in the sequel.

The proof of the following result is as in [29, Lem. 39.1.2 & Prop. 39.1.5], once the last claim of Lemma 13 is taken into account. See also [24, Prop. 3.10].

**Proposition 14.** *Let  $x = x_s + x_n \in V$ . Then,*

- (i) *the decomposition in Lemma 13 induces a decomposition*

$$\mathfrak{z}(\mathfrak{g}^x)_1^{\text{reg}} = \mathfrak{z}(\mathfrak{g}^{x_s})_1^{\text{reg}} \times \mathfrak{z}(\mathfrak{g}^{x_n} \cap [\mathfrak{g}^{x_s}, \mathfrak{g}^{x_s}])_1^{\text{reg}, G^{x_s}},$$

where “reg,  $G^{x_s}$ ” is the regular part for the action of  $G^{x_s}$ ;

- (ii) *the  $G_0$ -Jordan class of  $x$  is the irreducible subset of  $V$  given by*

$$J_{G_0}(x) = G_0 \cdot (\mathfrak{z}(\mathfrak{g}^{x_s})_1^{\text{reg}} + x_n).$$

We will need the following results from [24] which readily generalize to the reductive case by virtue of (8).

**Proposition 15** ([24, Prop. 3.9 and Prop.3.17]). *Let  $\{\mathfrak{g}, \theta\}$  be a reductive  $\mathbb{Z}_m$ -graded Lie algebra and  $x, y \in V$ . Then the following conditions are equivalent:*

- (i)  $x \stackrel{G_0}{\sim} y$ ;
- (ii) *there exists  $g \in G_0$  such that  $\mathfrak{g}^y = g \cdot \mathfrak{g}^x$ ;*
- (iii) *there exists  $g \in G_0$  such that  $\mathfrak{z}(\mathfrak{g}^y) = g \cdot (\mathfrak{z}(\mathfrak{g}^x))$ .*

*Moreover the number of  $G_0$ -Jordan classes in  $V$  is finite.*

**Corollary 16.** *The  $G_0$ -Jordan class of  $x \in V$  coincides also with  $J_{G_0}(x) = G_0 \cdot \mathfrak{z}(\mathfrak{g}^x)_1^{\text{reg}}$ , it is locally closed in  $V$  (hence a subvariety of  $V$ ) and  $\dim J_{G_0}(x) = \dim \mathfrak{g}_0 - \dim \mathfrak{g}_0^x + \dim \mathfrak{z}(\mathfrak{g}^{x_s})_1$ .*

*Proof.* The first two statements can be proved as in [29, Cor. 39.1.7], for the last one see [24, Prop. 3.13].  $\square$

It follows from Corollary 16 that any  $G_0$ -Jordan class  $J_{G_0}(x) = G_0 \cdot \mathfrak{z}(\mathfrak{g}^x)_1^{\text{reg}}$  is contained in the  $G$ -Jordan class  $J_G(x) = G \cdot \mathfrak{z}(\mathfrak{g}^x)^{\text{reg}}$ . However, it is well known that two elements  $x, y \in V$  in the same  $G$ -Jordan class are not  $G_0$ -Jordan equivalent in general (see [29, 38.7.18] for an example of nilpotent elements in  $V$  that are  $G$ -conjugate but not  $G_0$ -conjugate). We conclude this subsection recalling the relationship between the sheets for the  $G_0$ -action on  $V$  and the  $G_0$ -Jordan classes.

Let  $H$  be a connected algebraic group acting on a variety  $X$  and let  $d \in \mathbb{N}$ . We set  $X_{(d)} = \{x \in X \mid \dim \mathcal{O}_x^H = d\}$  and for any subset  $A \subset X$  we set  $A_{(d)} = A \cap X_{(d)}$ . Each  $X_{(d)}$  is locally closed and its irreducible components are called *sheets* for the  $H$ -action on  $X$ . We observe that  $X_{(\leq d)} := \bigcup_{j \leq d} X_{(j)}$  is closed so  $\overline{X_{(d)}} \subset X_{(\leq d)}$  [29, Prop. 21.4.4].

If  $A \subset V$ ,  $H = G_0$ , and  $p$  is the largest integer with  $A_{(p)} \neq \emptyset$ , then according to Definition 1, we have  $A_{(p)} = A^\bullet$ , which is a Zariski open subset of  $A$ . In particular, the set  $V^\bullet$  is a Zariski open subset of  $V$ , hence it is irreducible and it is called the  $G_0$ -regular sheet of  $V$ .

**Proposition 17** ([24, Prop. 3.19]). *For any sheet  $S$  in  $V$  there exists a unique  $G_0$ -Jordan class  $J \subset S$  such that  $S = \overline{J}^\bullet$ . Moreover we have  $\overline{S} = \overline{J}$ .*

## 2. Closure of a $G_0$ -Jordan class

### 2.1. Closure of $G_0$ -Jordan classes: the semisimple parts

By virtue of Proposition 17, it is important to understand the closure and  $G_0$ -regular closure of a  $G_0$ -Jordan class and to see which classes are dense in a sheet. We start with a preliminary result and then describe which semisimple parts occur in the closure of a  $G_0$ -Jordan class.

We first describe the image of the closure of a Jordan class through the Steinberg map  $\varphi: V \rightarrow V//G_0 \cong \mathfrak{c}/W_{\text{Vin}}$ .

**Lemma 18.** *Let  $J = J_{G_0}(x)$  be a  $G_0$ -Jordan class in  $V$ , with  $x_s \in \mathfrak{c}$ . Then,  $\varphi(\overline{J}) = W_{\text{Vin}} \cdot \mathfrak{z}(\mathfrak{g}^{x_s})_1/W_{\text{Vin}}$ . In particular, it is a closed subset of  $\mathfrak{c}/W_{\text{Vin}}$ : i.e., an affine variety.*

*Proof.* Observe that  $\mathfrak{z}(\mathfrak{g}^{x_s})_1 \subset \mathfrak{c}$ , cf. Corollary 8, so the expression for the image makes sense. Now,  $\varphi(J) = \varphi(\mathfrak{z}(\mathfrak{g}^{x_s})_1^{\text{reg}})$  by Proposition 14,  $G_0$ -equivariance and [31, Thm. 3], hence

$$\varphi(\overline{J}) \subset \overline{\varphi(J)} = \overline{\varphi(\mathfrak{z}(\mathfrak{g}^{x_s})_1^{\text{reg}})} = \overline{W_{\text{Vin}} \cdot \mathfrak{z}(\mathfrak{g}^{x_s})_1^{\text{reg}}/W_{\text{Vin}}} = W_{\text{Vin}} \cdot \mathfrak{z}(\mathfrak{g}^{x_s})_1/W_{\text{Vin}}. \quad (10)$$

On the other hand, if  $y_s \in \mathfrak{z}(\mathfrak{g}^{x_s})_1^{\text{reg}}$ , then  $y = y_s + x_n \in \mathcal{O}_y^{G_0} \subset J$  and so  $y_s \in \overline{\mathcal{O}_y^{G_0}} \subset \overline{J}$ , [31, Prop. 4], giving  $\mathfrak{z}(\mathfrak{g}^{x_s})_1^{\text{reg}} \subset \overline{J}$ . It follows that

$$\mathfrak{z}(\mathfrak{g}^{x_s})_1 = \overline{\mathfrak{z}(\mathfrak{g}^{x_s})_1^{\text{reg}}} \subset \overline{J}, \quad (11)$$

hence  $W_{\text{Vin}} \cdot \mathfrak{z}(\mathfrak{g}^{x_s})_1 \subset \bar{J}$  and

$$W_{\text{Vin}} \cdot \mathfrak{z}(\mathfrak{g}^{x_s})_1 / W_{\text{Vin}} = \varphi(W_{\text{Vin}} \cdot \mathfrak{z}(\mathfrak{g}^{x_s})_1) \subset \varphi(\bar{J}). \quad \square$$

Let  $J = J_{G_0}(x) \subset V_{(d)}$  be a  $G_0$ -Jordan class in  $V$ . Then its closure  $\bar{J}$  is a union of  $G_0$ -orbits and if  $\mathcal{O}_y^{G_0} \subset \bar{J}$ , then  $\overline{\mathcal{O}_y^{G_0}} \subset \bar{J}$ . Let  $\mathcal{M}_{\bar{J}}$  be the set of  $G_0$ -orbits contained in  $\bar{J}$  which are maximal with respect to the partial order given by inclusion of orbit closures. By construction,  $\bar{J} = \bigcup_{\mathcal{O} \in \mathcal{M}_{\bar{J}}} \overline{\mathcal{O}}$ .

The following proposition can be seen as a partial analogue of [4, Sect. 3.1].

**Proposition 19.** *Let  $J = J_{G_0}(x)$  be a  $G_0$ -Jordan class. Then  $\bar{J}^\bullet = \bigcup_{\mathcal{O} \in \mathcal{M}_{\bar{J}}} \mathcal{O}$ .*

*Proof.* We may assume without loss of generality that  $x = x_s + x_n$  with  $x_s \in \mathfrak{c}$ . First of all,  $\bar{J} \subset \overline{V_{(d)}} \subset V_{(\leq d)}$ , so  $\dim \mathcal{O} \leq d$  for any  $\mathcal{O} \in \mathcal{M}_{\bar{J}}$ . We then consider the restriction  $\psi = \varphi|_{\bar{J}} : \bar{J} \rightarrow \varphi(\bar{J})$  to  $\bar{J}$  of the Steinberg map  $\varphi$ .

Let  $z \in \mathcal{O}$  for  $\mathcal{O} \in \mathcal{M}_{\bar{J}}$ . By [31, Thm. 4], the irreducible component of the fiber  $\psi^{-1}\psi(z)$  containing  $z$  is the closure of a  $G_0$ -orbit in  $\bar{J}$ : i.e., it is  $\overline{\mathcal{O}}$ . Since  $\psi$  is a dominant morphism of irreducible affine varieties, we may argue as in [31, Cor. 2] and the fibers of  $\psi$  are all of the same dimension, which is the maximum dimension of an orbit in  $\bar{J}$ , namely  $d$ . Hence  $\dim \mathcal{O} = d$ ,  $\mathcal{O} \subset \bar{J}^\bullet$  and

$$\bigcup_{\mathcal{O} \in \mathcal{M}_{\bar{J}}} \mathcal{O} \subset \bar{J}^\bullet.$$

The other inclusion follows because  $\overline{\mathcal{O}} \setminus \mathcal{O}$  is always a union of  $G_0$ -orbits of dimension  $< d$ .  $\square$

**Lemma 20.** *Let  $J = J_{G_0}(x)$  be a  $G_0$ -Jordan class and  $y = y_s + y_n \in \bar{J}$ . Then,*

- (i)  $y_s \in \bar{J}$ ;
- (ii)  $y_s$  is  $G_0$ -conjugate to an element of  $\mathfrak{z}(\mathfrak{g}^{x_s})_1$ ;
- (iii) for any  $y'_s \in \mathfrak{z}(\mathfrak{g}^{x_s})_1$  there exists a  $y'_n \in \mathfrak{g}^{y'_s} \cap \mathcal{N}_V$  such that  $y'_s + y'_n \in \bar{J}^\bullet$ ;
- (iv) if  $z \in \mathfrak{z}(\mathfrak{g})_1$ , then  $z + y \in \bar{J}$ , and in that case  $z + y \in \bar{J}^\bullet$  if and only if  $y \in \bar{J}^\bullet$ .

*Proof.* Since  $\bar{J}$  is  $G_0$ -stable, claim (i) follows from [31, Prop. 4] because  $y_s \in \overline{\mathcal{O}_y^{G_0}} \subset \bar{J}$ . We now turn to (ii). We may assume  $y_s \in \mathfrak{c}$  by Theorem 6. Claim (ii) is then an immediate consequence of the following identity

$$\bar{J} \cap \mathfrak{c} = W_{\text{Vin}} \cdot \mathfrak{z}(\mathfrak{g}^{x_s})_1, \quad (12)$$

which we now establish.

First of all,  $W_{\text{Vin}} \cdot \mathfrak{z}(\mathfrak{g}^{x_s})_1 \subset \bar{J}$  by (11) and  $W_{\text{Vin}} \cdot \mathfrak{z}(\mathfrak{g}^{x_s})_1 \subset \mathfrak{c}$  by Corollary 8, so one inclusion is clear.

Conversely  $\varphi(\bar{J} \cap \mathfrak{c}) \subset \varphi(\bar{J}) = \varphi(W_{\text{Vin}} \cdot \mathfrak{z}(\mathfrak{g}^{x_s})_1)$  by Lemma 18. It follows that  $\bar{J} \cap \mathfrak{c} \subset W_{\text{Vin}} \cdot \mathfrak{z}(\mathfrak{g}^{x_s})_1$ , since the restriction of  $\varphi$  to  $\mathfrak{c}$  is just the natural projection to  $\mathfrak{c}/W_{\text{Vin}}$  and both sets are  $W_{\text{Vin}}$ -stable.

We prove (iii). By Proposition 14 we have that  $\mathfrak{z}(\mathfrak{g}^{x_s})_1^{\text{reg}} + x_n \subset J$ , so  $\mathfrak{z}(\mathfrak{g}^{x_s})_1 + x_n \subset \bar{J}$  and  $y'_s + x_n \in \bar{J}$ . Therefore the orbit  $\mathcal{O}_{y'_s + x_n}^{G_0}$  is contained in the closure

of an orbit  $\mathcal{O}$  in  $\mathcal{M}_{\bar{J}}$ . Since the fibers of the Steinberg map are closed and [31, Thm. 3] is in force,  $\mathcal{O}$  is represented by an element of the form  $y'_s + y'_n$  for some  $y'_n \in \mathfrak{g}^{y_s} \cap \mathcal{N}_V$ . Clearly  $\mathcal{O} \subset \bar{J}^\bullet$  by Proposition 19.

Finally, (iv) follows from the action of  $\mathfrak{z}(\mathfrak{g})_1$  on  $J$ , cf. Remark 1(2).  $\square$

**Corollary 21.** *The  $G_0$ -regular closure  $\bar{J}^\bullet$  of a  $G_0$ -Jordan class  $J$  contains at least a nilpotent  $G_0$ -orbit.*

## 2.2. A local study of the closure of a $G_0$ -Jordan class

We start with a local characterization of the closure of a  $G_0$ -Jordan class.

**Lemma 22.** *The following statements are equivalent for a  $G_0$ -Jordan class  $J$ :*

- (i)  $\bar{J}$  is a union of  $G_0$ -Jordan classes;
- (ii) For every  $y \in \bar{J}$  there exists a Zariski open neighbourhood  $U_y$  of  $y$  in  $J_{G_0}(y)$  such that  $U_y \subset \bar{J}$ .

*Proof.* The implication (i)  $\Rightarrow$  (ii) is immediate, since  $G_0$ -Jordan classes are disjoint and we may take  $U_y = J_{G_0}(y)$ . Assume now that (ii) holds. Let  $y \in \bar{J}$  and set  $J' = J_{G_0}(y)$ . Then  $J' \cap \bar{J}$  is a nonempty closed subset of  $J'$ . On the other hand, condition (ii) implies that any point of  $J' \cap \bar{J}$  has an open neighbourhood of  $J'$  therein, therefore  $J' \cap \bar{J}$  is also open in  $J'$ . Since  $J'$  is a Zariski irreducible variety, it is connected in the Zariski topology, thus  $J' \subset \bar{J}$  and (i) holds.  $\square$

By virtue of Lemma 22 we shall apply a local approach and look at the closure of a  $G_0$ -Jordan class in the neighbourhood of a point of  $V$ . For the rest of this subsection for any  $y_s \in \mathcal{S}_V$  we will use the following notation:  $\mathfrak{m} := \mathfrak{g}^{y_s}$ ;  $M := G^{y_s} \leq G$ ; and  $M_0 := M \cap G_0$  with identity component  $M_0^\circ$ . For any subset  $X \subset \mathfrak{m}_1$ , we will write  $X^{\text{reg}, M}$  to indicate the regular part of  $X$  for the action of  $M$ . We also recall that for any GIT quotient  $\pi: X \rightarrow X//H$  of a reductive algebraic group  $H$  acting on a variety  $X$ , a subset  $U$  of  $X$  is called  $\pi$ -saturated or  $H$ -saturated if  $U = \pi^{-1}\pi(U)$ . Saturated implies  $H$ -stable, but the converse is not necessarily true.

For  $\mathfrak{m}$  as above, we consider the  $M_0$ -stable subset of  $\mathfrak{m}_1$  defined as follows:

$$U_{\mathfrak{m}} = \{z \in \mathfrak{m}_1 \mid \mathfrak{g}^z \subset \mathfrak{m}\}.$$

**Lemma 23.** *With notation as above:*

- (i)  $U_{\mathfrak{m}}$  is  $M_0$ -saturated;
- (ii)  $U_{\mathfrak{m}}$  is open in  $\mathfrak{m}_1$ ;
- (iii) For all  $z = z_s + z_n \in U_{\mathfrak{m}}$  we have  $\mathfrak{z}(\mathfrak{g}^{z_s})_1^{\text{reg}} + z_n = (\mathfrak{z}(\mathfrak{m}^{z_s})_1^{\text{reg}, M} + z_n) \cap U_{\mathfrak{m}}$ ;
- (iv) For any  $G_0$ -Jordan class  $J$  such that  $J \cap U_{\mathfrak{m}} \neq \emptyset$ , we have

$$J \cap U_{\mathfrak{m}} = \bigcup_{i \in I_J} J_{M,i} \cap U_{\mathfrak{m}}, \quad (13)$$

where  $\{J_{M,i} \mid i \in I_J\}$  is the (finite) set of  $M_0^\circ$ -Jordan classes in  $\mathfrak{m}_1$  such that  $J_{M,i} \cap U_{\mathfrak{m}} \cap J \neq \emptyset$ . In addition,  $\dim J_{M,i} = \dim J_{M,j}$  for any  $i, j \in I_J$ , so the closures of the  $J_{M,i} \cap U_{\mathfrak{m}}$ 's in (13) are the irreducible components of  $J \cap U_{\mathfrak{m}}$ ;

(v) Let  $y_s \in \mathcal{S}_V$  and  $y = y_s + y_n$  for  $y_n \in \mathcal{N}_V \cap \mathfrak{m}$ . Then

$$J_{G_0}(y) \cap U_{\mathfrak{m}} = \mathfrak{z}(\mathfrak{m})_1^{\text{reg}} + \bigcup_{n_i \in N_{G_0}(\mathfrak{m})/M_0^\circ} n_i \cdot \mathcal{O}_{y_n}^{M_0^\circ} \quad (14)$$

and the locally closed sets  $\mathfrak{z}(\mathfrak{m})_1^{\text{reg}} + n_i \cdot \mathcal{O}_{y_n}^{M_0^\circ}$  are the irreducible components of  $J_{G_0}(y) \cap U_{\mathfrak{m}}$ .

*Proof.* For  $m = 2$ , parts (i)–(ii) are [8, Lem. 2.1]. We propose a slightly different proof for (i). Saturation is equivalent to say that  $\mathfrak{g}^z \subset \mathfrak{m}$  if and only if  $\mathfrak{g}^{z_s} \subset \mathfrak{m}$  for any  $z = z_s + z_n \in \mathfrak{m}_1$ . As  $\mathfrak{g}^z = \mathfrak{g}^{z_s} \cap \mathfrak{g}^{z_n}$ , one implication is immediate. We will now show that  $\mathfrak{g}^{z_s} \not\subset \mathfrak{g}^s$  implies  $\mathfrak{g}^z \not\subset \mathfrak{g}^s$  for any semisimple element  $s \in \mathfrak{g}$  and any  $z \in \mathfrak{g}^s$ , independently of the  $\mathbb{Z}_m$ -grading. Since  $z_s$  and  $s$  commute, we can always find a Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$  containing both. Then

$$\mathfrak{g}^s = \mathfrak{h} \oplus \left( \bigoplus_{\alpha \in \Phi(s)} \mathfrak{g}_\alpha \right), \quad \mathfrak{g}^{z_s} = \mathfrak{h} \oplus \left( \bigoplus_{\alpha \in \Phi(z_s)} \mathfrak{g}_\alpha \right),$$

where  $\Phi(h)$  is the set of roots vanishing on an element  $h \in \mathfrak{h}$ . Since

$$(\Phi(s) + (\Phi \setminus \Phi(s))) \cap \Phi \subset \Phi \setminus \Phi(s),$$

the reductive subalgebra  $\mathfrak{g}^s \cap \mathfrak{g}^{z_s} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Phi(s) \cap \Phi(z_s)} \mathfrak{g}_\alpha)$  stabilizes the subspace  $X = \bigoplus_{\alpha \in \Phi(z_s) \setminus \Phi(s)} \mathfrak{g}_\alpha$ . As  $z_n \in \mathfrak{g}^s \cap \mathfrak{g}^{z_s}$  acts nilpotently on  $X$ , there is a nonzero  $\xi$  in there such that  $[z_n, \xi] = 0$ . In other words  $\xi \in \mathfrak{g}^z \setminus \mathfrak{g}^s$ .

To prove (ii), we use the argument in [6, Lem. 2.1]. We may assume  $y_s \in \mathfrak{c}$  and that  $\mathfrak{h}$  is a Cartan subalgebra of  $\mathfrak{g}$ , hence of  $\mathfrak{m}$ , containing  $\mathfrak{c}$ . The product  $f = \prod_{\alpha \in \Phi \setminus \Phi(y_s)} \alpha$  is a homogeneous polynomial on  $\mathfrak{h}$  that is invariant for the Weyl group of  $\mathfrak{m}$ . By Chevalley's restriction theorem  $f$  extends to an  $M$ -invariant polynomial  $F$  on  $\mathfrak{m}$ . By (i),  $U_{\mathfrak{m}} = \{z \in \mathfrak{m}_1 \mid \mathfrak{g}^{z_s} \subset \mathfrak{m}\}$ , and it is not hard to verify that this is equal to  $\{z \in \mathfrak{m}_1 \mid F(z) \neq 0\}$ , hence it is open in  $\mathfrak{m}_1$ .

Since  $U_{\mathfrak{m}}$  is  $M_0$ -saturated, it is enough to prove (iii) for  $z = z_s \in U_{\mathfrak{m}}$ . We have  $\mathfrak{g}^{z_s} = \mathfrak{m}^{z_s}$ , so  $\mathfrak{z}(\mathfrak{g}^{z_s})_1 = \mathfrak{z}(\mathfrak{m}^{z_s})_1$ . If  $x \in \mathfrak{z}(\mathfrak{g}^{z_s})_1^{\text{reg}}$  then  $\mathfrak{g}^x = \mathfrak{g}^{z_s} = \mathfrak{m}^{z_s} \subset \mathfrak{m}$ , so  $x \in \mathfrak{z}(\mathfrak{m}^{z_s})_1^{\text{reg}, M} \cap U_{\mathfrak{m}}$ . Conversely, if  $x \in \mathfrak{z}(\mathfrak{m}^{z_s})_1^{\text{reg}, M} \cap U_{\mathfrak{m}}$ , then  $\mathfrak{g}^x \subset \mathfrak{m}$ , so  $\mathfrak{g}^x = \mathfrak{m}^x = \mathfrak{g}^{z_s}$  and  $x \in \mathfrak{z}(\mathfrak{g}^{z_s})_1^{\text{reg}}$ .

We prove (iv). Clearly  $J \cap U_{\mathfrak{m}} \subset \bigcup_{i \in I_J} J_{M,i} \cap U_{\mathfrak{m}}$ , and we now show the other inclusion. Let  $z = z_s + z_n \in J \cap J_{M,i} \cap U_{\mathfrak{m}}$  for some  $i \in I_J$ , so  $J = J_{G_0}(z)$  and  $J_{M,i} = J_{M_0^\circ}(z)$ . Combining the fact that  $U_{\mathfrak{m}}$  is  $M_0^\circ$ -stable with (iii) gives

$$\begin{aligned} J_{M,i} \cap U_{\mathfrak{m}} &= (M_0^\circ \cdot (\mathfrak{z}(\mathfrak{m}^{z_s})_1^{\text{reg}, M} + z_n)) \cap U_{\mathfrak{m}} = M_0^\circ \cdot ((\mathfrak{z}(\mathfrak{m}^{z_s})_1^{\text{reg}, M} + z_n) \cap U_{\mathfrak{m}}) \\ &= M_0^\circ \cdot (\mathfrak{z}(\mathfrak{g}^{z_s})_1^{\text{reg}} + z_n) \subset G_0 \cdot (\mathfrak{z}(\mathfrak{g}^{z_s})_1^{\text{reg}} + z_n) = J \end{aligned}$$

establishing (13). Corollary 16 then gives

$$\dim J_{M,i} = \dim M_0^\circ - \dim \mathfrak{m}_0^z + \dim \mathfrak{z}(\mathfrak{m}^{z_s})_1 = \dim M_0^\circ - \dim \mathfrak{g}_0^z + \dim \mathfrak{z}(\mathfrak{g}^{z_s})_1,$$

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which is independent of  $i \in I_J$ . Equation (13) expresses  $J \cap U_{\mathfrak{m}}$  as the finite disjoint union of irreducible locally closed subsets of the same dimension, and the claim on the irreducible components follows directly.

Finally, we prove (v). By construction,

$$\begin{aligned} \mathfrak{z}(\mathfrak{m})_1^{\text{reg}} + \bigcup_{n_i \in N_{G_0}(\mathfrak{m})/M_0^\circ} n_i \cdot \mathcal{O}_{y_n}^{M_0^\circ} &= U_{\mathfrak{m}} \cap (\mathfrak{z}(\mathfrak{m})_1^{\text{reg}} + N_{G_0}(\mathfrak{m}) \cdot y_n) \\ &= U_{\mathfrak{m}} \cap (N_{G_0}(\mathfrak{m}) \cdot (\mathfrak{z}(\mathfrak{m})_1^{\text{reg}} + y_n)) \\ &\subset U_{\mathfrak{m}} \cap J_{G_0}(y). \end{aligned}$$

Conversely, let  $z \in J_{G_0}(y) \cap U_{\mathfrak{m}}$ . Then there is  $g \in G_0$  such that  $\mathfrak{g}^{z_s} = g \cdot \mathfrak{m}$  and  $z_n = g \cdot y_n$ . Saturation of  $U_{\mathfrak{m}}$  gives  $\mathfrak{g}^{z_s} \subset \mathfrak{m}$ , hence  $g \cdot \mathfrak{m} \subset \mathfrak{m}$  and  $z$  lies in

$$N_{G_0}(\mathfrak{m}) \cdot (\mathfrak{z}(\mathfrak{m})_1^{\text{reg}} + y_n) = \mathfrak{z}(\mathfrak{m})_1^{\text{reg}} + N_{G_0}(\mathfrak{m}) \cdot y_n = \mathfrak{z}(\mathfrak{m})_1^{\text{reg}} + \bigcup_{n_i \in N_{G_0}(\mathfrak{m})/M_0^\circ} n_i \cdot \mathcal{O}_{y_n}^{M_0^\circ},$$

establishing (14). Last claim follows from (iv) once we prove that the sets  $\mathfrak{z}(\mathfrak{m})_1^{\text{reg}} + n_i \cdot \mathcal{O}_{y_n}^{M_0^\circ}$  are closed in  $J_{G_0}(y) \cap U_{\mathfrak{m}}$ . On the one hand

$$\overline{\mathfrak{z}(\mathfrak{m})_1^{\text{reg}} + n_i \cdot \mathcal{O}_{y_n}^{M_0^\circ}}^{J_{G_0}(y) \cap U_{\mathfrak{m}}} \subset \overline{\mathfrak{z}(\mathfrak{m})_1 + n_i \cdot \mathcal{O}_{y_n}^{M_0^\circ}}^{m_1} = \mathfrak{z}(\mathfrak{m})_1 + \overline{n_i \cdot \mathcal{O}_{y_n}^{M_0^\circ}}^{m_1}.$$

On the other hand, if  $z \in \overline{\mathfrak{z}(\mathfrak{m})_1^{\text{reg}} + n_i \cdot \mathcal{O}_{y_n}^{M_0^\circ}}^{J_{G_0}(y) \cap U_{\mathfrak{m}}}$ , then (14) gives  $z \in \overline{\mathfrak{z}(\mathfrak{m})_1^{\text{reg}} + n_j \cdot \mathcal{O}_{y_n}^{M_0^\circ}}$  for some  $j$ . Hence,  $z \in (\mathfrak{z}(\mathfrak{m})_1 + n_j \cdot \mathcal{O}_{y_n}^{M_0^\circ}) \cap \overline{(\mathfrak{z}(\mathfrak{m})_1 + n_i \cdot \mathcal{O}_{y_n}^{M_0^\circ})}^{m_1}$ . Since the two nilpotent orbits have the same dimension, we necessarily have  $i = j$ , completing the proof.  $\square$

Let  $H$  and  $L$  be reductive algebraic groups acting on an affine variety  $X$ , with  $H \subset L$ . Then  $H$  acts with trivial stabilizers on the product  $L \times X$  via  $h \cdot (l, x) = (lh^{-1}, h \cdot x)$ ; we set  $L \times^H X := (L \times X)/H \cong (L \times X)//H$  and note that  $L$  acts on  $L \times^H X$  by multiplication from the left. The class of  $(l, x) \in L \times X$  will be denoted by the symbol  $l * x \in L \times^H X$ . Since  $H$  acts on  $L \times X$  with trivial stabilizer, the projection  $L \times X \rightarrow L \times^H X$  is a principal  $H$ -bundle [16, III.1, Cor. 1]. In other words, there is a surjective étale map  $Y \rightarrow L \times^H X$  such that the base change of  $L \times X \rightarrow L \times^H X$  through  $Y \rightarrow L \times^H X$  is isomorphic to the projection  $X \times Y \rightarrow Y$ . Hence,  $A$  is an open subset of  $L \times^H X$  if and only if its pre-image is an open  $H$ -stable subset of  $L \times X$ . In particular,  $L \times^H A'$  is open in  $L \times^H X$  for any open  $H$ -stable subset  $A'$  of  $X$ .

We consider the natural action maps

$$\tilde{\mu}: G \times \mathfrak{m} \rightarrow \mathfrak{g}, \quad \tilde{\mu}_0: G_0 \times \mathfrak{m}_1 \rightarrow V, \quad (15)$$

and the induced maps  $\mu: G \times^M \mathfrak{m} \rightarrow \mathfrak{g}$  and  $\mu_0: G_0 \times^{M_0} \mathfrak{m}_1 \rightarrow V$ .

**Lemma 24.** *The maps  $\tilde{\mu}, \tilde{\mu}_0$  are smooth at  $(1, y_s)$  and the induced maps  $\mu, \mu_0$  are étale at  $1 * y_s$ .*

*Proof.* The differential of the map  $\tilde{\mu}$  at  $(1, y_s)$  maps any element  $(x', y') \in \mathfrak{g} \oplus \mathfrak{m}$  to  $[x', y_s] + y'$ , therefore it is surjective by (3) combined with  $[\mathfrak{g}, y_s] \cap \mathfrak{m} = 0$  which follows from semisimplicity of  $y_s$ . The differential of the induced map  $\mu$  at  $1 * y_s$  is also surjective, hence bijective by dimensional reasons. The restriction of the differential of the map  $\tilde{\mu}$  at  $(1, y_s)$  to the degree 1 terms readily implies the surjectivity of the differential of  $\tilde{\mu}_0$  at  $(1, y_s)$ , whence the differential of  $\mu_0$  at  $1 * y_s$  is bijective.  $\square$

We will also consider the GIT quotient maps

$$\pi_0: G_0 \times^{M_0} \mathfrak{m}_1 \rightarrow (G_0 \times^{M_0} \mathfrak{m}_1) // G_0, \quad \pi_{\mathfrak{m}_1}: \mathfrak{m}_1 \rightarrow \mathfrak{m}_1 // M_0$$

associated with multiplication from the left by  $G_0$ , and the adjoint action of  $M_0$ , respectively. Recall that we have a natural identification  $(G_0 \times^{M_0} \mathfrak{m}_1) // G_0 \cong \mathfrak{m}_1 // M_0$ . We will invoke a variant of Luna's étale slice Theorem [16] and its consequences to deduce properties of the closure of  $G_0$ -Jordan classes.

**Corollary 25.** *There exists an affine open neighbourhood  $\mathcal{U}$  of  $y_s$  in  $\mathfrak{m}_1$ , which is  $M_0$ -saturated and such that the restriction of  $\mu_0$  to  $G_0 \times^{M_0} \mathcal{U}$  is étale with Zariski open image  $G_0 \cdot \mathcal{U}$  in  $V$ .*

*Proof.* By Lemma 24, the map  $\mu_0$  is étale at  $1 * y_s$ . The orbit  $\mathcal{O}_{1 * y_s}^{G_0}$  is closed (since it is an orbit of minimal dimension  $\dim G_0 - \dim M_0$ ) and so is the semisimple orbit  $\mathcal{O}_{y_s}^{G_0}$ , cf. [31, Prop. 3]. It is also not hard to verify that the restriction of  $\mu_0$  to  $\mathcal{O}_{1 * y_s}^{G_0}$  is injective. By [16, Lemme Fondamental, Sect. II.2] applied to  $X = G_0 \times^{M_0} \mathfrak{m}_1$  and  $Y = V$  there exists an affine  $\pi_0$ -saturated open neighbourhood  $U_X$  of  $1 * y_s$  in  $G_0 \times^{M_0} \mathfrak{m}_1$  such that the restriction of  $\mu_0$  to it is étale and the image is an affine open subset of  $V$ , saturated for the Steinberg map  $\varphi: V \rightarrow V // G_0$ . In fact,  $U_X$  is  $G_0 \times^{M_0} \mathcal{U}$  where  $\mathcal{U} = \pi_{\mathfrak{m}_1}^{-1} \pi_0(U_X)$  is the desired  $M_0$ -saturated affine open neighbourhood of  $y_s$  in  $\mathfrak{m}_1$ .  $\square$

**Proposition 26.** *Let  $J$  be a  $G_0$ -Jordan class in  $V$  and  $y = y_s + y_n$ , where  $y_s \in \mathcal{S}_V$  and  $y_n \in \mathcal{N}_V \cap \mathfrak{m}$ . Then  $y \in \bar{J}$  if and only if  $y \in \overline{J_{M,l}}$  for some  $l \in I_J$ , where  $\{J_{M,i} \mid i \in I_J\}$  is as in Lemma 23(iv).*

*Proof.* Consider the restriction of  $\mu_0$  from  $G_0 \times^{M_0} \mathcal{U}$  to  $V$  determined in Corollary 25. Assume  $y \in \bar{J}$ , so  $y_s \in \bar{J}$  by Lemma 20. We consider the commutative diagram

$$\begin{array}{ccc} G_0 \times^{M_0} (\bar{J} \cap \mathcal{U}) & \longrightarrow & G_0 \cdot \mathcal{U} \cap \bar{J} \\ j \downarrow & & \downarrow \\ G_0 \times^{M_0} \mathcal{U} & \xrightarrow{\mu_0} & G_0 \cdot \mathcal{U} \end{array} .$$

We claim that it is a pull-back. Indeed, the pull-back of  $\mu_0$  through the closed inclusion  $\bar{J} \cap G_0 \cdot \mathcal{U} \rightarrow G_0 \cdot \mathcal{U}$  is the restriction  $X := \mu_0^{-1}(\bar{J}) \rightarrow \bar{J} \cap G_0 \cdot \mathcal{U}$ . Moreover,  $\bar{J}$  is a reduced subscheme and  $\mu_0$  is étale, so  $X$  is a  $G_0$ -stable closed subscheme of  $G_0 \times^{M_0} \mathcal{U}$ . By [16, Lem. 3], there exists a closed  $M_0$ -stable subscheme  $Y'$  of  $\mathcal{U}$  such that  $X = G_0 \times^{M_0} Y'$  and the natural morphism  $X \rightarrow G_0 \times^{M_0} \mathcal{U}$  is induced by the inclusion  $Y' \subset \mathcal{U}$ . Observe that  $Y' = \mu_0(1 * Y') \subset \bar{J} \cap \mathcal{U}$ , so the morphism  $X \rightarrow G_0 \times^{M_0} \mathcal{U}$  factors through the morphism  $f: X \rightarrow G_0 \times^{M_0} (\bar{J} \cap \mathcal{U})$ .



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Let  $f': G_0 \times^{M_0} (\overline{J} \cap \mathcal{U}) \rightarrow X$  be the unique morphism coming from the universal property of the pull-back. Now  $j \circ f \circ f' = j$ , hence  $f \circ f' = \text{id}$  whilst  $f' \circ f = \text{id}$  follows from the universal property. Hence, the restriction of  $\mu_0$  to  $G_0 \times^{M_0} (\overline{J} \cap \mathcal{U}) \rightarrow \overline{J}$  is again étale, whence open. As  $U_{\mathfrak{m}}$  is open in  $\mathfrak{m}$ ,  $U_{\mathfrak{m}} \cap \mathcal{U} \cap \overline{J}$  is open in  $\mathcal{U} \cap \overline{J}$ . Thus,  $G_0 \times^{M_0} (U_{\mathfrak{m}} \cap \mathcal{U} \cap \overline{J}) \rightarrow \overline{J}$  is an open morphism, too.

If  $C$  is an irreducible component of  $U_{\mathfrak{m}} \cap \mathcal{U} \cap \overline{J}$  containing  $y$ , then  $G_0 \times^{M_0} C$  dominates the irreducible set  $\overline{J}$ , so  $G_0 \cdot C \cap J \neq \emptyset$  because  $J$  is locally closed and thus  $C \cap J \neq \emptyset$  by  $G_0$ -stability. Note also that  $C \cap J$  is open in  $C$ , hence it is irreducible and dense in  $C$ . Now

$$C \cap J \subset J \cap U_{\mathfrak{m}} = \bigcup_{i \in I_J} J_{M,i} \cap U_{\mathfrak{m}}$$

by Lemma 23 (iv), so  $C \cap J \subset \overline{J_{M,l} \cap U_{\mathfrak{m}}}^{J \cap U_{\mathfrak{m}}} \subset \overline{J_{M,l}}$  for some  $l \in I_J$  by irreducibility. Finally  $\overline{C} = \overline{C \cap J} \subset \overline{J_{M,l}}$  by density and  $y \in \overline{J_{M,l}}$  follows since  $y \in C$ .

Assume now  $y \in \overline{J_{M,l}}$  for some  $l \in I_J$ . Then  $\overline{J_{M,l} \cap U_{\mathfrak{m}}} = \overline{J_{M,l}}$  since  $J_{M,l}$  is irreducible and  $U_{\mathfrak{m}}$  is open in  $\mathfrak{m}_1$ , so  $y \in \overline{J_{M,l}} = \overline{J_{M,l} \cap U_{\mathfrak{m}}} \subset \overline{J \cap U_{\mathfrak{m}}} \subset \overline{J}$ .  $\square$

**Corollary 27.** *Let  $y_s \in \mathcal{S}_V$  and let  $y = y_s + y_n$  for  $y_n \in \mathcal{N}_V \cap \mathfrak{m}$ . Then there exists an  $M_0^\circ$ -stable open neighbourhood  $U_1$  of  $y$  in  $\mathfrak{m}_1$  such that  $M_0 \cdot U_1 \subset U_{\mathfrak{m}}$  and*

$$J_{G_0}(y) \cap U_1 = J_{M_0^\circ}(y) \cap U_1. \quad (16)$$

*Proof.* Lemma 23 (v) gives

$$\begin{aligned} J_{G_0}(y) \cap U_{\mathfrak{m}} &= (\mathfrak{z}(\mathfrak{m})_1^{\text{reg}} + \bigcup_{n_i \in N_{G_0}(\mathfrak{m})/M_0^\circ} n_i \cdot \mathcal{O}_{y_n}^{M_0^\circ}) \\ &= (\mathfrak{z}(\mathfrak{m})_1 + \bigcup_{n_i \in N_{G_0}(\mathfrak{m})/M_0^\circ} n_i \cdot \mathcal{O}_{y_n}^{M_0^\circ}) \cap U_{\mathfrak{m}} \end{aligned}$$

since  $\mathfrak{z}(\mathfrak{m})_1^{\text{reg}, M} = \mathfrak{z}(\mathfrak{m})_1$ . The subsets  $(\mathfrak{z}(\mathfrak{m})_1 + n_i \cdot \mathcal{O}_{y_n}^{M_0^\circ}) \cap U_{\mathfrak{m}}$  are the irreducible components of  $J_{G_0}(y) \cap U_{\mathfrak{m}}$  by Lemma 23 (v). They are disjoint, whence open, and  $M_0^\circ$ -stable. Hence, there exists an  $M_0^\circ$ -stable open subset  $\mathcal{U}_y$  in  $\mathfrak{m}_1$  such that  $J_{G_0}(y) \cap U_{\mathfrak{m}} \cap \mathcal{U}_y = (\mathfrak{z}(\mathfrak{m})_1 + \mathcal{O}_{y_n}^{M_0^\circ}) \cap U_{\mathfrak{m}}$ . Then  $U_1 := \mathcal{U}_y \cap U_{\mathfrak{m}}$  is an  $M_0^\circ$ -stable Zariski open neighbourhood of  $y$  in  $\mathfrak{m}_1$  satisfying

$$J_{G_0}(y) \cap U_1 = (\mathfrak{z}(\mathfrak{m})_1 + \mathcal{O}_{y_n}^{M_0^\circ}) \cap U_1 = J_{M_0^\circ}(y) \cap U_1.$$

Finally  $M_0 \cdot U_1 \subset M_0 \cdot U_{\mathfrak{m}} = U_{\mathfrak{m}}$  since  $U_{\mathfrak{m}}$  is  $M_0$ -stable.  $\square$

**Theorem 28.** *Let  $J$  be a  $G_0$ -Jordan class in  $V$ . Then  $\overline{J}$  is a union of  $G_0$ -Jordan classes and it is decomposable: i.e., it contains the semisimple and nilpotent components of all its elements.*

*Proof.* We will show that condition (ii) in Lemma 22 is satisfied for any  $y = y_s + y_n \in \bar{J}$ . Let  $U_1$  be as in Corollary 27 and  $\mathcal{U}$  as in Corollary 25. By saturation of  $\mathcal{U}$  we have  $y \in U_1 \cap \mathcal{U}$ .

We consider the  $M_0$ -stable open subset  $U'_1 = M_0 \cdot U_1 \subset U_{\mathfrak{m}}$  of  $\mathfrak{m}_1$  and apply  $M_0$  to both sides of (16) to get

$$J_{G_0}(y) \cap U'_1 = M_0 \cdot (J_{G_0}(y) \cap U_1) = M_0 \cdot (J_{M_0^\circ}(y) \cap U_1) \subset (M_0 \cdot J_{M_0^\circ}(y)) \cap U'_1. \quad (17)$$

We then set  $\mathcal{V} = U'_1 \cap \mathcal{U}$  and obtain an  $M_0$ -stable open neighbourhood of  $y$  in  $\mathfrak{m}_1$ . Corollary 25 then guarantees that  $G_0 \cdot \mathcal{V}$  is an open neighbourhood of  $y$  in  $V$ . We will show that  $U_y := J_{G_0}(y) \cap G_0 \cdot \mathcal{V}$  is the sought neighbourhood of  $y$  in  $J_{G_0}(y)$  contained in  $\bar{J}$ .

Proposition 26 ensures that  $y \in \overline{J_{M,l}}$  for some  $l \in I_J$ . Now  $y_s \in \mathfrak{z}(\mathfrak{m})_1$ , so combining (17), Remark 1(2) and Lemma 20(iv) we get

$$\begin{aligned} J_{G_0}(y) \cap \mathcal{V} &\subset (M_0 \cdot J_{M_0^\circ}(y)) \cap \mathcal{V} = (M_0 \cdot (\mathfrak{z}(\mathfrak{m})_1 + \mathcal{O}_{y_n}^{M_0^\circ})) \cap \mathcal{V} \\ &\subset (M_0 \cdot \overline{J_{M,l}}) \cap \mathcal{V} = M_0 \cdot (\overline{J_{M,l}} \cap \mathcal{V}) \subset \bar{J} \cap \mathcal{V}, \end{aligned}$$

where for the last inclusion we invoke Proposition 26 once more. We finally obtain

$$J_{G_0}(y) \cap G_0 \cdot \mathcal{V} = G_0 \cdot (J_{G_0}(y) \cap \mathcal{V}) \subset G_0 \cdot (\bar{J} \cap \mathcal{V}) \subset \bar{J}.$$

This proves that  $\bar{J}$  is the union of  $G_0$ -Jordan classes.

We now prove that  $\bar{J}$  is decomposable. Let  $y = y_s + y_n \in \bar{J}$  and  $J_{G_0}(y)$  the corresponding  $G_0$ -Jordan class. Then  $y_s \in \bar{J}$  by Lemma 20 (i) and

$$y_n \in \mathfrak{z}(\mathfrak{m})_1 + y_n = \overline{\mathfrak{z}(\mathfrak{m})_1^{\text{reg}} + y_n} \subset \overline{J_{G_0}(y)} \subset \bar{J},$$

where we used our previous result  $J_{G_0}(y) \subset \bar{J}$ .  $\square$

**Theorem 29.** *Let  $J$  be a  $G_0$ -Jordan class and let  $S$  be a sheet in  $V$ . Then  $\bar{J}^\bullet$ ,  $\bar{J}^{\text{reg}}$  and  $S$  are unions of  $G_0$ -Jordan classes.*

*Proof.* By Theorem 28, the closure  $\bar{J}$  is a union of  $G_0$ -Jordan classes. Since all such classes are of constant  $G$ - and  $G_0$ -orbit dimension, it follows that also  $\bar{J}^\bullet$  and  $\bar{J}^{\text{reg}}$  are unions of  $G_0$ -Jordan classes. The statement for  $S$  is a direct consequence of Proposition 17.  $\square$

We conclude this subsection with the following important application of the local study of the closure of a  $G_0$ -Jordan class.

**Proposition 30.**  *$G_0$ -Jordan classes are smooth.*

*Proof.* Let  $J = J_{G_0}(y)$  be a  $G_0$ -Jordan class in  $V$  and  $\mathfrak{m} = \mathfrak{g}^{y_s}$ . We will show that  $y$  has a smooth Zariski open neighbourhood in  $J$ . Let  $U_{\mathfrak{m}}$  and  $\mathcal{U}$  be the open neighbourhoods of  $y_s$  and  $y$  in  $\mathfrak{m}_1$  as in Lemma 23 and Corollary 25, respectively.

By Lemma 23 (v), the intersection  $J \cap U_{\mathfrak{m}}$  is smooth, therefore  $J \cap U_{\mathfrak{m}} \cap \mathcal{U}$  is nonempty and smooth as well. Recall that  $p: G_0 \times \bar{J} \rightarrow G_0 \times^{M_0} \bar{J}$  is a principal  $M_0$ -bundle, so there is a surjective étale map  $f: Y \rightarrow G_0 \times^{M_0} \bar{J}$  such that the

base change  $X \rightarrow Y$  of  $G_0 \times \bar{J} \rightarrow G_0 \times^{M_0} \bar{J}$  is isomorphic to the projection  $\tilde{p}: M_0 \times Y \rightarrow Y$ . Being the base change of an étale map, the induced morphism  $\tilde{f}: M_0 \times Y \rightarrow G_0 \times \bar{J}$  is again étale. By [1, Éxp. 1, Cor. 9.2],  $G_0 \times (J \cap U_m \cap \mathcal{U})$  is smooth if and only if  $\tilde{f}\tilde{p}^{-1}f^{-1}(G_0 \times^{M_0} (J \cap U_m \cap \mathcal{U})) = p^{-1}(G_0 \times^{M_0} (J \cap U_m \cap \mathcal{U}))$  is so. One may verify that the scheme-theoretic fiber of  $G_0 \times^{M_0} (J \cap U_m \cap \mathcal{U})$  through  $p$  is  $G_0 \times (J \cap U_m \cap \mathcal{U})$  hence  $G_0 \times^{M_0} (J \cap U_m \cap \mathcal{U})$  is smooth.

Invoking again [1, Exp 1, Cor. 9.2] we conclude that  $\mu_0(G_0 \times^{M_0} (J \cap U_m \cap \mathcal{U}))$  is smooth and it is a smooth open neighbourhood of  $y$  in  $J$ .  $\square$

### 2.3. Regularity questions

Let  $J = J_{G_0}(x_s + x_n)$  be a  $G_0$ -Jordan class. Then  $\bar{J}^{\text{reg}} \subset \bar{J}^\bullet$  since  $J$ , hence  $\bar{J}$ , are irreducible and Lemma 12 is in force. Note that  $\bar{J}^\bullet = \bar{J}^{\text{reg}}$  whenever  $x_s = 0$ , because  $J = \mathfrak{z}(\mathfrak{g})_1 \times \mathcal{O}_{x_n}^{G_0}$  and orbits are locally closed, so  $J = \bar{J}^\bullet = \bar{J}^{\text{reg}}$ . The equality  $\bar{J}^\bullet = \bar{J}^{\text{reg}}$  is always satisfied in the symmetric case  $m = 2$  due to Corollary 11 and one may wonder if  $\bar{J}^\bullet = \bar{J}^{\text{reg}}$  also for  $m \geq 3$ , by combining Theorem 29 and the fact that  $G_0$ -Jordan classes are defined in terms of regular parts for the action of  $G$ , cf. Corollary 16.

However, this is not the case. A reason is that open  $G_0$ -orbits  $\mathcal{O}^{G_0}$  in irreducible components of the fibers of the Steinberg map  $\varphi: V \rightarrow V//G_0 \cong \mathfrak{c}/W_{\text{Vin}}$  do *not* give rise in general to open  $G$ -orbits  $G \cdot \mathcal{O}^{G_0}$  in the irreducible components of the Steinberg map  $p: \mathfrak{g} \rightarrow \mathfrak{g}/G \cong \mathfrak{h}/W$ . To make this more precise, we need some notions and results from [21], [22] and, for simplicity of exposition, we restrict to the case where  $\mathfrak{g}$  is semisimple.

**Definition 2.** A complex semisimple  $\mathbb{Z}_m$ -graded Lie algebra  $\{\mathfrak{g}, \theta, m\}$ , its grading, and the automorphism  $\theta$  are called:

- (i)  $\mathcal{S}$ -regular if  $\mathcal{S}_V \cap \mathfrak{g}^{\text{reg}} \neq \emptyset$ ;
- (ii)  $\mathcal{N}$ -regular if  $\mathcal{N}_V \cap \mathfrak{g}^{\text{reg}} \neq \emptyset$ ;
- (iii) *very*  $\mathcal{N}$ -regular if each irreducible component of  $\mathcal{N}_V$  intersects  $\mathfrak{g}^{\text{reg}}$  non-trivially.

Clearly (iii) implies (ii). It is an important result of L. V. Antonyan and D. I. Panyushev in [21] that if a connected component of  $\text{Aut}(\mathfrak{g})$  contains automorphisms of order  $m$ , then it contains a unique  $\mathcal{N}$ -regular automorphism of that order (up to conjugation by the group of inner automorphisms of  $\mathfrak{g}$ ). Moreover, as mentioned in the introduction of [21], the condition of  $\mathcal{S}$ -regularity is equivalent to  $\mathcal{N}$ -regularity in the symmetric case  $m = 2$ , but for  $m \geq 3$  neither of these properties implies the other. An example of  $\mathcal{S}$ -regular grading that is not  $\mathcal{N}$ -regular is given in [21, Example 4.5]. Here  $\mathfrak{g}$  is of type  $E_6$  with the inner automorphism of order  $m = 4$  described by the Kac diagram



This is the affine Dynkin diagram of  $\mathfrak{g}$  of type  $E_6$ , where the white and black nodes correspond to roots subspaces of degree 0 and 1, respectively. The semisimple part

of  $\mathfrak{g}_0$  is given by the subdiagram consisting of white nodes and the dimension of the centre of  $\mathfrak{g}_0$  is the number of black nodes minus 1. We have  $G_0 \cong \mathrm{SL}(4) \times \mathrm{SL}(2) \times (\mathbb{C}^*)^2$  up to local isomorphism, acting on  $V = \mathfrak{g}_1 \cong \mathbb{C}^4 \oplus (\mathbb{C}^4)^* \oplus (\wedge^2 \mathbb{C}^4 \boxtimes \mathbb{C}^2)$ . The reader is referred to [33, Chap. 3, Sect. 3] for a detailed treatment of periodic automorphisms and their associated Kac diagrams.

Now  $G_0$ -Jordan classes form a finite partition of  $V$ , which is irreducible, so there is one class  $J$  that is open in  $V$ . We call it the  $G_0$ -regular Jordan class of  $V$  and note that it is the unique  $G_0$ -Jordan class that is dense in the  $G_0$ -regular sheet  $S = V^\bullet$  of  $V$ . (See Example 3 for an explicit description of representatives of the  $G_0$ -orbits in the  $G_0$ -regular Jordan class.) Since the grading (18) is  $\mathcal{S}$ -regular, we have  $\overline{J}^{\mathrm{reg}} = V^{\mathrm{reg}} = \mathfrak{g}^{\mathrm{reg}} \cap V$  in this case. Let  $\mathcal{O}^{G_0}$  be the nilpotent  $G_0$ -orbit that is open in one of the irreducible components of  $\mathcal{N}_V$ . We have  $\mathcal{O}^{G_0} \subset \overline{J}^\bullet = V^\bullet$  by [31, Cor. 1 and 2], but  $\mathcal{O}^{G_0} \not\subset \overline{J}^{\mathrm{reg}}$  since the grading is not  $\mathcal{N}$ -regular.

The cone  $\mathcal{N}_V$  is often reducible and a larger class of examples for which  $\overline{J}^{\mathrm{reg}} \neq \overline{J}^\bullet$  comes from  $\mathcal{N}$ -regular gradings that are not very  $\mathcal{N}$ -regular: the  $G_0$ -regular Jordan class  $J$  satisfies  $\overline{J}^{\mathrm{reg}} = \mathfrak{g}^{\mathrm{reg}} \cap V$  and, by an argument as above, there is a nilpotent  $G_0$ -orbit contained in  $\overline{J}^\bullet$  but not in  $\overline{J}^{\mathrm{reg}}$ . Exceptional  $\mathcal{N}$ -regular gradings whose nodes are not all black are classified in [9], and very  $\mathcal{N}$ -regular gradings appear to occur very rarely. Inner exceptional gradings with all nodes black are  $\mathcal{N}$ -regular but not very  $\mathcal{N}$ -regular [21, Example 4.4] and the same is true for the outer grading of  $E_6$  with all nodes black (W. A. de Graaf, 05-05-2020, personal communication).

The following result is a consequence of these observations, and the tables are a specialization of Tables 2–7 of [9].

**Proposition 31.** *Let  $\{\mathfrak{g}, \theta, m\}$  be an exceptional complex simple  $\mathbb{Z}_m$ -graded Lie algebra,  $m \geq 3$ . Then  $\{\mathfrak{g}, \theta, m\}$  is  $\mathcal{N}$ -regular but not very  $\mathcal{N}$ -regular if and only if the associated Kac diagram has all the nodes black or is one in the following tables. In all these cases we have that  $\overline{J}^{\mathrm{reg}} \subset \overline{J}^\bullet$  properly, where  $J$  is the  $G_0$ -regular Jordan class of  $V$ .*

Table 1:  $\mathcal{N}$ -regular but not very  $\mathcal{N}$ -regular automorphisms of  $G_2$

$m$	Kac diagram	# orbits in $\mathcal{N}_V$	# components of $\mathcal{N}_V$	$\dim \mathcal{N}_V$	$\dim \mathfrak{c}$
3		6	2	4	1

Table 2:  $\mathcal{N}$ -regular but not very  $\mathcal{N}$ -regular automorphisms of  $F_4$

$m$	Kac diagram	# orbits in $\mathcal{N}_V$	# components of $\mathcal{N}_V$	$\dim \mathcal{N}_V$	$\dim \mathfrak{c}$
4		29	3	12	2
6		35	6	8	2
8		30	4	6	1

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Table 3:  $\mathcal{N}$ -regular but not very  $\mathcal{N}$ -regular inner automorphisms of  $E_6$

$m$	Kac diagram	# orbits in $\mathcal{N}_V$	# components of $\mathcal{N}_V$	$\dim \mathcal{N}_V$	$\dim \mathfrak{c}$
4		43	3	18	2
6		133	9	12	2
8		70	4	9	1
9		118	6	8	1

Table 4:  $\mathcal{N}$ -regular but not very  $\mathcal{N}$ -regular outer automorphisms of  $E_6$

$m$	Kac diagram	# orbits in $\mathcal{N}_V$	# components of $\mathcal{N}_V$	$\dim \mathcal{N}_V$	$\dim \mathfrak{c}$
6		34	5	12	3
8		22	3	9	1
10		25	2	8	1
12		30	4	6	1

Table 5:  $\mathcal{N}$ -regular but not very  $\mathcal{N}$ -regular automorphisms of  $E_7$

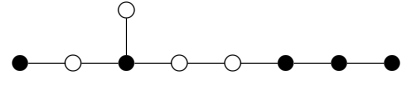
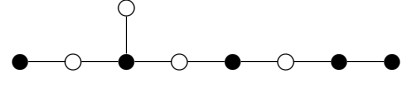
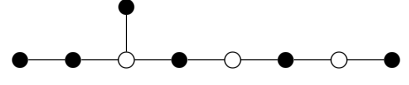
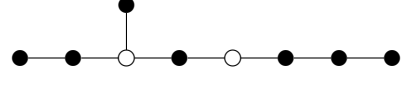
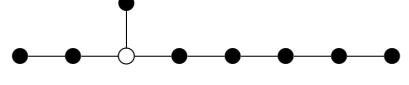
$m$	Kac diagram	# orbits in $\mathcal{N}_V$	# components of $\mathcal{N}_V$	$\dim \mathcal{N}_V$	$\dim \mathfrak{c}$
6		233	10	21	3
7		112	3	18	1

$\mathcal{N}$ -regular but not very $\mathcal{N}$ -regular automorphisms of $E_7$ (continued)					
8		163	2	17	1
9		132	4	14	1
10		199	4	13	1
12		217	5	11	1
14		238	7	9	1

Table 6:  $\mathcal{N}$ -regular but not very  $\mathcal{N}$ -regular automorphisms of  $E_8$

$m$	Kac diagram	orbits in $\mathcal{N}_V$	components of $\mathcal{N}_V$	$\dim \mathcal{N}_V$	$\dim \mathfrak{c}$
4		144	2	60	4
6		270	7	40	4
8		219	2	30	2
9		206	2	28	1
10		300	7	24	2
12		398	10	20	2

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$\mathcal{N}$ -regular but not very $\mathcal{N}$ -regular automorphisms of $E_8$ (continued)					
14		333	4	18	1
15		354	5	16	1
18		397	5	14	1
20		438	7	12	1
24		478	8	10	1

3. Slice-induction and parametrization of orbits and classes

3.1. Slice-induction

Theorem 28 shows that the closure of a  $G_0$ -Jordan class in  $V$  is a union of  $G_0$ -Jordan classes, generalising results of [5], [8]. We aim at detecting which  $G_0$ -Jordan classes lie in the closure of a given one. In the classical case, this can be described in terms of Lusztig-Spaltenstein's parabolic induction of adjoint orbits, [17], [4]. Slice induction is introduced in [8] to deal with the  $m = 2$  case, since orbit induction no longer works. We will briefly show how to extend to the case of general  $m$  the construction in [8], by combining some of its general arguments with our local approach.

Let  $\mathfrak{m}$  be now a  $\theta$ -stable reductive subalgebra of  $\mathfrak{g}$  and  $M$  the connected subgroup of  $G$  with  $\text{Lie}(M) = \mathfrak{m}$ . For a nilpotent element  $e \in \mathfrak{m}_1$ , we consider a graded  $\mathfrak{sl}(2)$ -triple  $\{e, h, f\}$  in  $\mathfrak{m}$ , so that  $h \in \mathfrak{m}_0$  and  $f \in \mathfrak{m}_{-1}$ , and the corresponding Slodowy slice  $S_{\mathfrak{m},e} = e + \mathfrak{m}^f \subset \mathfrak{m}$ . Since  $\mathfrak{m}^f$  is homogeneous, we can consider its intersection with  $V$ , obtaining  $S_{\mathfrak{m}_1,e} = e + \mathfrak{m}_1^f \subset \mathfrak{m}_1$ . If  $e = 0$ , we consider the trivial triple as an  $\mathfrak{sl}(2)$ -triple, so  $S_{\mathfrak{m}_1,0} = \mathfrak{m}_1$ . We start with two preliminary results in the case  $\mathfrak{m} = \mathfrak{g}$ .

**Lemma 32.** *Let  $\{e, h, f\}$  be a graded  $\mathfrak{sl}(2)$ -triple in  $\mathfrak{g}$  and let  $X \subset V$  be an irreducible locally closed  $G_0$ -stable subset such that  $X \cap S_{\mathfrak{g}_1,e} \neq \emptyset$ . Then the action morphism  $\psi: G_0 \times S_{\mathfrak{g}_1,e} \rightarrow V$  is smooth, its restriction  $\psi_X: G_0 \times (S_{\mathfrak{g}_1,e} \cap X) \rightarrow X$  is smooth and dominant, and more precisely  $\psi_X(G_0 \times C)$  is dense in  $X$  for any irreducible component  $C$  of  $S_{\mathfrak{g}_1,e} \cap X$ .*

*Proof.* For  $m = 2$ , this is part of [8, Prop. 2.4(i)], we record the proof for completeness. The action morphism  $\psi$  is  $G_0$ -equivariant with smooth domain and codomain,

hence it suffices to verify that the differential is surjective at any point of the form  $(1, y) \in G_0 \times S_{\mathfrak{g}_1, e}$ . We note that

$$\begin{aligned} d\psi|_{(1, y)} : \mathfrak{g}_0 \times \mathfrak{g}_1^f &\rightarrow V \\ (x, z) &\rightarrow [x, y] + z \end{aligned}$$

and by  $\mathfrak{sl}(2)$ -representation theory  $\mathfrak{g} = [\mathfrak{g}, e] \oplus \mathfrak{g}^f$ , which in degree 1 becomes  $V = [\mathfrak{g}_0, e] \oplus \mathfrak{g}_1^f$ , so the differential at  $(1, e)$  is surjective. The contracting  $\mathbb{C}^*$ -action argument in [27, 7.4, Cor. 1] carries over to the  $\mathbb{Z}_m$ -graded case because  $\{e, h, f\}$  is a graded  $\mathfrak{sl}(2)$ -triple and  $h \in \mathfrak{m}_0$ , so  $\psi$  is smooth at any point  $(1, y)$ , hence everywhere.

By [27, III.5, Lem. 2] applied to the  $G_0$ -equivariant morphism given by the inclusion of  $X$  in  $V$ , the restriction  $\psi_X$  is again smooth, whence open. Since  $X$  is irreducible, the image of any irreducible component of  $G_0 \times (S_{\mathfrak{g}_1, e} \cap X)$  is dense in  $X$ .  $\square$

**Lemma 33.** *Let  $J$  be a  $G_0$ -Jordan class in  $V$  and  $e \in \mathcal{N}_V$ . Then the following properties are equivalent:*

- (i)  $e \in \bar{J}$ ;
- (ii)  $J \cap S_{\mathfrak{g}_1, e} \neq \emptyset$ ;
- (iii)  $\bar{J} \cap S_{\mathfrak{g}_1, e} \neq \emptyset$ .

*Proof.* We note that  $J$  is a locally closed  $G_0$ -stable cone by Proposition 15 and Corollary 16, so when  $m = 2$  these are the equivalences (i) = (iv) = (v) in [8, Thm. 2.6]. The proof of [8, Lem. 2.3] shows the existence of a contracting  $\mathbb{C}^*$ -action on  $S_{\mathfrak{g}_1, e}$  and it carries over to the  $m > 2$  case. If (ii) holds, then each irreducible component of  $\bar{J} \cap S_{\mathfrak{g}_1, e}$  is stable under the  $\mathbb{C}^*$ -action, so  $e$  lies in each of them. This gives (i).

Clearly (i) implies (iii), so it remains to show that (iii) implies (ii). We follow the proof of [8, Prop. 2.5], establishing that  $J \cap S_{\mathfrak{g}_1, e}$  is dense in  $\bar{J} \cap S_{\mathfrak{g}_1, e}$ .

Since  $J$  is open in  $\bar{J}$ , the subset  $J \cap S_{\mathfrak{g}_1, e}$  is open in  $\bar{J} \cap S_{\mathfrak{g}_1, e}$  and therefore it is enough to prove that it meets every irreducible component  $C$  of  $\bar{J} \cap S_{\mathfrak{g}_1, e}$ . The latter follows then from the density of  $G_0 \cdot C$  in  $\bar{J}$ , guaranteed by Lemma 32 applied to  $X = \bar{J}$ .  $\square$

For the rest of the section we resume the convention from Subsection 2.2 that whenever we have  $y_s \in \mathcal{S}_V$ , then  $\mathfrak{m} := \mathfrak{g}^{y_s}$ , and  $M_0^\circ$  is the identity component of  $M_0 := G_0^{y_s} = G_0 \cap G^{y_s}$ .

**Theorem 34.** *Let  $J_1, J_2$  be  $G_0$ -Jordan classes in  $V$ . Then the following conditions are equivalent:*

- (i)  $J_2 \subset \bar{J}_1$ ;
- (ii)  $J_2 \cap \bar{J}_1 \neq \emptyset$ ;
- (iii) *There exist  $x \in J_1, y \in J_2$  such that  $\mathfrak{g}^{x_s} \subset \mathfrak{m}$  and  $J_{M_0^\circ}(x) \cap S_{\mathfrak{m}_1, y_n} \neq \emptyset$ ;*
- (iv) *There exist  $x \in J_1, y \in J_2$  such that  $\mathfrak{g}^{x_s} \subset \mathfrak{m}$  and  $y \in \overline{J_{M_0^\circ}(x)}$ .*



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*Proof.* This is the generalization of [8, Thm. 3.5] to the  $m > 2$  case, but our proof is slightly different and it combines Lemma 33 and Lemma 23 with our local approach.

The equivalence (i)  $\Leftrightarrow$  (ii) is immediate from Theorem 28. We prove the other ones.

*Claim* (iii)  $\Leftrightarrow$  (iv). Lemma 33 applied to  $\mathfrak{m}$ ,  $y_n$  and  $J_{M_0^\circ}(x)$  says that  $J_{M_0^\circ}(x) \cap S_{\mathfrak{m}_1, y_n} \neq \emptyset$  if and only if  $y_n \in \overline{J_{M_0^\circ}(x)}$ . Since  $y_s \in \mathfrak{z}(\mathfrak{m})_1$ , the latter condition is equivalent to  $y \in \overline{J_{M_0^\circ}(x)}$  by Lemma 20(iv).

*Claim* (iv)  $\Rightarrow$  (ii). Let  $x, y$  be as in (iv).

Since  $\mathfrak{g}^x \subset \mathfrak{g}^{x_s}$ , we have  $x \in U_{\mathfrak{m}}$  and hence  $J_{G_0}(x) \cap U_{\mathfrak{m}} \cap J_{M_0^\circ}(x) \neq \emptyset$ . Lemma 23(iv) gives

$$J_{G_0}(x) \cap U_{\mathfrak{m}} = \bigcup_{i \in I_{J_1}} J_{M,i} \cap U_{\mathfrak{m}} \quad (19)$$

and  $J_{M_0^\circ}(x)$  is, by construction, one of the  $M_0^\circ$ -Jordan classes occurring in the right-hand side. Applying Proposition 26 to  $J = J_{G_0}(x)$  and  $y \in \overline{J_{M_0^\circ}(x)}$ , we see that  $y \in \overline{J_{G_0}(x)}$ . This shows  $y \in J_2 \cap \overline{J_1} \neq \emptyset$ , proving (ii).

*Claim* (ii)  $\Rightarrow$  (iv).

Assume now  $y \in J_2 \cap \overline{J_1}$ . By Proposition 26, the element  $y$  lies in  $\overline{J_{M,l}}$  where  $J_{M,l}$  is an  $M_0^\circ$ -Jordan class such that  $\emptyset \neq J_{M,l} \cap U_{\mathfrak{m}} \subset J_1 \cap U_{\mathfrak{m}}$ . Then, for any  $x \in J_{M,l} \cap U_{\mathfrak{m}}$  the pair  $(x, y)$  satisfies (iv).  $\square$

Comparing dimensions of orbits in  $J_1$  and  $J_2$  we readily get the following.

**Corollary 35.** *Let  $J_1, J_2$  be  $G_0$ -Jordan classes in  $V$ . Then  $J_2 \subset \overline{J_1}^\bullet$  if and only if there exist  $x \in J_1, y \in J_2$  such that  $\mathfrak{g}^{x_s} \subset \mathfrak{m}$ ,  $J_{M_0^\circ}(x) \cap S_{\mathfrak{m}_1, y_n} \neq \emptyset$  and  $\dim \mathcal{O}_x^{M_0} = \dim \mathcal{O}_{y_n}^{M_0}$ .*

*Remark 2.* Condition (iii) from Theorem 34 is called weak slice-induction in [8]. If  $J_2$  is weakly slice-induced from  $J_1$  and satisfies the dimension condition in Corollary 35, then it is called slice-induced from  $J_1$ . Slice-induction is shown to coincide with parabolic induction in the ungraded case  $m = 1$  in [8, Cor. 3.7].

**Corollary 36.** *A  $G_0$ -Jordan class  $J = J_{G_0}(y)$  contained in  $V_{(d)}$  is dense in a sheet if and only if  $J_{M_0^\circ}(x) \cap S_{\mathfrak{m}_1, y_n} = \emptyset$  for any  $x \in V_{(d)} \setminus J$  such that  $\mathfrak{g}^{x_s} \subset \mathfrak{m}$ .*

*Proof.* First of all, the irreducible subset  $J$  is contained in some sheet  $S$  in  $V_{(d)}$  and there is a unique  $G_0$ -Jordan class  $J' \subset V_{(d)}$  such that  $S = \overline{J'}^\bullet$  by Proposition 17. The condition  $J_{M_0^\circ}(x) \cap S_{\mathfrak{m}_1, y_n} = \emptyset$  for any  $x \in V_{(d)} \setminus J$  such that  $\mathfrak{g}^{x_s} \subset \mathfrak{m}$  is equivalent to say that there are no  $G_0$ -Jordan classes  $\mathcal{J} \neq J$  such that  $J \subset \overline{\mathcal{J}}^\bullet$ , in other words, that  $J = J'$ .  $\square$

### 3.2. Parametrization of orbits and classes

We aim at a parametrization of the  $G_0$ -orbits contained in a  $G_0$ -Jordan class  $J_{G_0}(x) = G_0 \cdot (\mathfrak{z}(\mathfrak{g}^{x_s})_1^{\text{reg}} + x_n)$ . By Theorem 6, we may assume that  $x = x_s + x_n \in V$  with  $x_s \in \mathfrak{c}$ , so Corollary 8 ensures that  $\mathfrak{z}(\mathfrak{g}^{x_s})_1 \subset \mathfrak{c}$ . Let

$$\Gamma := N_{W_{\text{vin}}}(\mathfrak{z}(\mathfrak{g}^{x_s})_1),$$

the stabilizer of  $\mathfrak{z}(\mathfrak{g}^{x_s})_1$  in  $W_{\text{Vin}}$ .

*Remark 3.*

(1) Observe that  $x_s \in \mathfrak{c}$  implies  $Z_{G_0}(\mathfrak{c}) \subset G_0^{x_s} \subset N_{G_0}(\mathfrak{g}^{x_s})$ . Corollary 3 gives also  $N_{G_0}(\mathfrak{g}^{x_s}) = N_{G_0}(\mathfrak{z}(\mathfrak{g}^{x_s})_1) = N_{G_0}(\mathfrak{z}(\mathfrak{g}^{x_s}))$ , so

$$\Gamma \cong (N_{G_0}(\mathfrak{c}) \cap N_{G_0}(\mathfrak{g}^{x_s})) / Z_{G_0}(\mathfrak{c}).$$

In other words, if  $w \in \Gamma$ , then any of its representatives  $\dot{w} \in N_{G_0}(\mathfrak{c})$  lies in  $N_{G_0}(\mathfrak{g}^{x_s})$ .

(2) The group  $N_{G_0}(\mathfrak{c}) \cap N_{G_0}(\mathfrak{g}^{x_s})$  normalizes  $G_0^{x_s}$  and  $\mathfrak{g}_1^{x_s}$  and thus acts on the set of  $G_0^{x_s}$ -orbits in  $\mathfrak{g}_1^{x_s}$ . Since  $Z_{G_0}(\mathfrak{c}) \subset G_0^{x_s}$ , this action factors through an action of  $\Gamma$  on the set of  $G_0^{x_s}$ -orbits in  $\mathfrak{g}_1^{x_s}$  which preserves the set of nilpotent ones. Observe that  $G_0^{x_s}$  and its orbits need not be connected.

We shall need the stabilizer  $\Gamma_n$  in  $\Gamma$  of  $\mathcal{O}_{x_n}^{G_0^{x_s}}$  with respect to the action defined above:

$$\Gamma_n = \text{Stab}_{\Gamma}(\mathcal{O}_{x_n}^{G_0^{x_s}}).$$

**Proposition 37.** *Let  $x = x_s + x_n \in V$  with  $x_s \in \mathfrak{c}$ . The assignment  $\varphi$  from  $\mathfrak{z}(\mathfrak{g}^{x_s})_1^{\text{reg}}$  to the orbit set  $J_{G_0}(x)/G_0$  given by  $y_s \mapsto \mathcal{O}_{(y_s+x_n)}^{G_0}$  induces a homeomorphism  $\bar{\varphi}: \mathfrak{z}(\mathfrak{g}^{x_s})_1^{\text{reg}}/\Gamma_n \rightarrow J_{G_0}(x)/G_0$ , where the orbit set is endowed with the quotient topology.*

*Proof.* The map  $\varphi$  is well defined and surjective by Proposition 14. We prove injectivity. Let  $y_s, z_s \in \mathfrak{z}(\mathfrak{g}^{x_s})_1^{\text{reg}}$  be such that  $g \cdot (y_s + x_n) = z_s + x_n$  for some  $g \in G_0$ : i.e.,

$$g \cdot y_s = z_s, \tag{20}$$

$$g \cdot x_n = x_n, \tag{21}$$

and consider  $w \in W_{\text{Vin}}$  such that  $w \cdot y_s = z_s$ , cf. Theorem 9. Any representative  $\dot{w} \in N_{G_0}(\mathfrak{c})$  of  $w$  satisfies  $\dot{w} \cdot \mathfrak{g}^{x_s} = \dot{w} \cdot \mathfrak{g}^{y_s} = \mathfrak{g}^{z_s} = \mathfrak{g}^{x_s}$ , so  $w \in \Gamma$  by Remark 3. Moreover,  $\dot{w}g^{-1} \in G^{z_s} \cap G_0 = G_0^{x_s}$  by (20). It follows from (21) that

$$\dot{w} \cdot x_n \in \mathcal{O}_{x_n}^{G_0^{x_s}} \quad \text{so} \quad \dot{w} \cdot \mathcal{O}_{x_n}^{G_0^{x_s}} = \mathcal{O}_{x_n}^{G_0^{x_s}},$$

in other words  $w \in \Gamma_n$  and  $\bar{\varphi}$  is injective.

Let  $p: J_{G_0}(x) \rightarrow J_{G_0}(x)/G_0$  be the quotient map and  $U$  an open subset in  $J_{G_0}(x)/G_0$ . Then  $p^{-1}(U)$  is a  $G_0$ -stable open subset in  $J_{G_0}(x)$  and its intersection

$$p^{-1}(U) \cap (\mathfrak{z}(\mathfrak{g}^{x_s})_1^{\text{reg}} \times \mathcal{O}_{x_n}^{G_0^{x_s}})$$

is an open  $\Gamma_n$ -stable subset of  $\mathfrak{z}(\mathfrak{g}^{x_s})_1^{\text{reg}} \times \mathcal{O}_{x_n}^{G_0^{x_s}}$ . Its projection onto  $\mathfrak{z}(\mathfrak{g}^{x_s})_1^{\text{reg}}$  is again an open  $\Gamma_n$ -stable subset, and so is its image through the quotient map by the finite group  $\Gamma_n$ . We have therefore proved that  $\bar{\varphi}$  is a continuous bijection, and it remains to show that is open.

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By Corollary 16 and Proposition 30, the action morphism

$$G_0 \times (\mathfrak{z}(\mathfrak{g}^{x_s})_1^{\text{reg}} + x_n) \rightarrow J_{G_0}(x)$$

is a morphism of smooth varieties whose induced map on the tangent spaces is surjective. Hence it is smooth, and an open morphism in the Zariski topology (see [2, VII, Rem. 1.2] and [2, V, Thm. 5.1 and VII, Thm. 1.8]). From this, it is straightforward to see that  $\bar{\varphi}$  is open.  $\square$

We briefly turn to the parametrization of  $G_0$ -Jordan classes. Thanks to Theorem 9 and Example 2 describing the centralizer of an element of  $\mathfrak{c}$ , we easily establish the following

**Lemma 38.** *Let  $x_s$  and  $y_s$  be two elements in  $\mathfrak{c}$ . Then the centralizers  $\mathfrak{g}^{x_s}$  and  $\mathfrak{g}^{y_s}$  are  $G_0$ -conjugate if and only if there exists  $w \in W_{\text{Vin}}$  such that  $w \cdot \Sigma(x_s) = \Sigma(y_s)$ .*

The hyperplane arrangement on  $\mathfrak{c}$  determined by the restricted roots  $\sigma \in \Sigma$  admits an action of  $W_{\text{Vin}}$  and it induces a stratification on  $\mathfrak{c}$ , where two elements lie in the same stratum  $\mathfrak{z}(\mathfrak{g}^{x_s})_1^{\text{reg}} = \{y_s \in \mathfrak{c} \mid \Sigma(y_s) = \Sigma(x_s)\}$  if and only if their centralizers coincide. Equivalently, the stratum associated to a closed and symmetric subset  $\tilde{\Sigma} \subset \Sigma$  (as in of [33, p. 182]) is

$$S_{\tilde{\Sigma}} = \{x \in \mathfrak{c} \mid \Sigma(x) = \tilde{\Sigma}\}$$

and the collection of  $S_{\tilde{\Sigma}}$ 's is a finite partition of  $\mathfrak{c}$ . Already in the ungraded case, where the class of centralizers of semisimple elements coincides with the class of Levi subalgebras, not all closed and symmetric subsets  $\tilde{\Sigma}$  of  $\Sigma$  give rise to a nonempty stratum.

In the graded case, some information on stabilizers of generic elements in  $S_V$  is to be found in [31] under the assumption that  $V$  is a simple  $G_0$ -module. We refer to [24, Prop. 3.4] for an alternative general description of centralizers of semisimple elements. In view of Lemma 38, two strata  $S_{\tilde{\Sigma}}$  and  $S_{\tilde{\Sigma}'}$  are said to be equivalent if  $w \cdot \tilde{\Sigma} = \tilde{\Sigma}'$  for some  $w \in W_{\text{Vin}}$ .

Given  $\tilde{\Sigma} \subset \Sigma$ , we set  $\mathfrak{m}(\tilde{\Sigma})$  to be the  $\theta$ -stable Levi subalgebra of  $\mathfrak{g}$  constructed as in (6) and  $M(\tilde{\Sigma}) \subset G$ ,  $M(\tilde{\Sigma})_0 = M(\tilde{\Sigma}) \cap G_0 \subset G_0$  as usual.

**Proposition 39.** *Jordan classes in  $V$  are parametrized by  $W_{\text{Vin}}$ -classes of pairs  $(\tilde{\Sigma}, \mathcal{O})$  where  $\tilde{\Sigma} \subset \Sigma$  satisfies  $S_{\tilde{\Sigma}} \neq \emptyset$  and  $\mathcal{O}$  is a nilpotent orbit in  $\mathfrak{m}(\tilde{\Sigma})_1$  for the action of  $M(\tilde{\Sigma})_0$ .*

*Proof.* Observe that  $N_{G_0}(\mathfrak{c})$  acts on the set of pairs  $(\tilde{\Sigma}, \mathcal{O})$  as above and that if  $\mathfrak{m}(\tilde{\Sigma})$  is the centralizer of some  $x_s \in \mathfrak{c}$ , then  $Z_{G_0}(\mathfrak{c}) \subset M(\tilde{\Sigma})_0$ , hence it acts trivially on  $(\tilde{\Sigma}, \mathcal{O})$ . Thus the action of  $N_{G_0}(\mathfrak{c})$  induces an action of  $W_{\text{Vin}}$ .

Now recall that for  $x \in V$ , the assignment  $J_{G_0}(x) \mapsto (\mathfrak{g}^{x_s}, \mathcal{O}_{x_n}^{G_0^{x_s}})$  establishes a one-to one-correspondence between  $G_0$ -Jordan classes in  $V$  and  $G_0$ -classes of pairs  $(\mathfrak{l}, \mathcal{O})$  where  $\mathfrak{l}$  is the stabilizer of a semisimple element in  $V$  and  $\mathcal{O}$  a nilpotent orbit in  $\mathfrak{l}_1$  for the action of  $L_0$ . Theorem 6 guarantees that we can always find a pair in the  $G_0$ -orbit where  $\mathfrak{l} = \mathfrak{m}(\tilde{\Sigma})$  for some  $\tilde{\Sigma} \subset \Sigma$ . Assume that for two pairs  $(\mathfrak{m}(\tilde{\Sigma}), \mathcal{O})$

and  $(\mathfrak{m}(\tilde{\Sigma}'), \mathcal{O}')$  of this form there is  $g \in G_0$  such that  $(g \cdot \mathfrak{m}(\tilde{\Sigma}), g \cdot \mathcal{O}) = (\mathfrak{m}(\tilde{\Sigma}'), \mathcal{O}')$ . By Lemma 38 we can decompose  $g = g' \dot{w}$ , where  $g' \in N_{G_0}(\mathfrak{m}(\tilde{\Sigma}'))$  and  $\dot{w} \in N_{G_0}(\mathfrak{c})$ . In addition,  $g' = l \dot{\sigma}$  with  $l \in M(\tilde{\Sigma}')_0$  and  $\dot{\sigma} \in N_{G_0}(\mathfrak{m}(\tilde{\Sigma}')) \cap N_{G_0}(\mathfrak{c})$ . In other words, we may replace  $g$  by an element in  $N_{G_0}(\mathfrak{c})$ , so  $(\mathfrak{m}(\tilde{\Sigma}), \mathcal{O})$  and  $(\mathfrak{m}(\tilde{\Sigma}'), \mathcal{O}')$  lie in the same  $W_{\text{Vin}}$ -orbit.  $\square$

The results of [34] encompass a parametrization of the  $G_0$ -Jordan classes, where  $\theta$  is the automorphism of order  $m = 3$  of  $\mathfrak{g} = \mathbf{E}_8$  for which  $\mathfrak{g}_1 \cong \wedge^3 \mathbb{C}^9$ ,  $\mathfrak{g}_0 \cong \mathfrak{sl}(9)$  and  $\mathfrak{g}_{-1} \cong \wedge^3(\mathbb{C}^9)^*$  as in Example 1. This is shown in the following example.

**Example 4.** By the discussion in [34, Sect. 3.4], the seven “families” described in [34, Sect. 1] parametrize the Levi subalgebras  $\mathfrak{l} = \mathfrak{g}^{x_s}$  that arise from elements  $x_s \in \mathcal{S}_V$  up to  $G_0$ -conjugation, and the “classes” in Tables 1–6 of [34, Sect. 1] parametrize the nilpotent orbits in  $\mathfrak{l}_1$  for the action of  $G_0^{x_s}$ . (If  $x_s$  is in family I then  $\mathfrak{g}^{x_s} = \mathfrak{h}$ , there is no nontrivial nilpotent orbit and only one class.) By Proposition 39, our  $G_0$ -Jordan classes almost coincide with the classes of [34]: the finite group  $N_{G_0}(\mathfrak{l})/G_0^{x_s}$  acts on the set of nilpotent  $G_0^{x_s}$ -orbits in  $\mathfrak{l}_1$ , possibly glueing some of them.

Hence, some of the 164 classes of [34] may correspond to the same  $G_0$ -Jordan class. A look at Tables 1–6 tells us that this may happen only in a few cases, since centralizers of elements of a  $G_0$ -Jordan class are  $G_0$ -conjugate by Proposition 15 and  $N_{G_0}(\mathfrak{l}) = G_0^{x_s}$  in the VII family:

**III family:** Classes 2–3, 4–6, and 7–8;

**V family:** Classes 7–8, and 10–11;

**VI family:** Classes 5–6, 8–9, 11–12, and 17–18.

Recall that the support of a trivector  $\varphi \in \wedge^3 \mathbb{C}^9$  is the unique minimal subspace  $E \subset \mathbb{C}^9$  such that  $\varphi \in \wedge^3 E$ . Its dimension is the *rank* of  $\varphi$ , one of the simplest discrete  $G_0$ -invariants of a trivector. The nilpotent  $G_0^{x_s}$ -orbits associated to classes 7–8 in V family have different rank, so they are not  $G_0$ -related. Thus, they correspond to different  $G_0$ -Jordan classes. A similar observation works in all the remaining cases, except those of the III family and classes 5–6 of VI family, but it is not difficult to see that the nilpotent  $G_0^{x_s}$ -orbits of these last two classes are not  $G_0$ -related. It remains therefore to deal with the III family.

First of all, the rank of the nilpotent orbit in class 4 is strictly smaller than the rank of those in classes 5 and 6. However, the permutation matrix

$$g = - \left( \begin{array}{c|c|c} \text{Id}_{3 \times 3} & 0 & 0 \\ \hline 0 & 0 & \text{Id}_{3 \times 3} \\ \hline 0 & \text{Id}_{3 \times 3} & 0 \end{array} \right) \quad (22)$$

is an element of  $N_{G_0}(\mathfrak{l})$  and it *does* relate the nilpotent  $G_0^{x_s}$ -orbits associated with classes 5–6, which then correspond to a single  $G_0$ -Jordan class. The same is true for classes 2–3 and 7–8. In summary, the space  $\wedge^3 \mathbb{C}^9$  is partitioned into 161  $G_0$ -Jordan classes.

The quotient  $\Gamma/W_{x_s}$  of  $\Gamma$  with the stabilizer  $W_{x_s}$  of  $x_s \in \mathfrak{c}$  in  $W_{\text{Vin}}$  was found in [34, Sect. 3.4] for all families (see also the fourth and fifth columns of [34, Table 7]). In the case of III family, it is a group of order 72 generated by complex reflections.

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Consider, for example, the  $G_0$ -Jordan class III.5, represented by  $x = x_s + x_n$ . A simple check shows that  $g$  as in (22) also normalizes  $\mathfrak{c}$ , so  $g \in N_{G_0}(\mathfrak{z}(\mathfrak{g}^{x_s})) \cap N_{G_0}(\mathfrak{c})$  and, by our previous discussion, it is not in  $\Gamma_n$ . The  $G_0$ -orbits in the  $G_0$ -Jordan class III.5 are then parametrized by the quotient  $\mathfrak{z}(\mathfrak{g}^{x_s})_1^{\text{reg}}/\Gamma_n$  of  $\mathfrak{z}(\mathfrak{g}^{x_s})_1^{\text{reg}}$  by a group  $\Gamma_n$  of order 36.

We conclude with an application of Theorem 34. Let  $J_2 = J_{G_0}(y)$  be the  $G_0$ -Jordan class numbered III.7: i.e., the one represented by  $y = y_s + y_n$  with

$$y_s = (e_{123} + e_{456} + e_{789}) + i(e_{147} + e_{258} + e_{369}), \quad y_n = e_{159}. \quad (23)$$

The centralizer  $\mathfrak{m} = \mathfrak{g}^{y_s}$  is a reductive Lie algebra with semisimple part  $\mathfrak{r}$  of type  $A_2 \oplus A_2$ . More precisely, the center of  $\mathfrak{m}$  is 4-dimensional and sits in degrees  $\pm 1$ ; it consists of the two components in brackets that defines  $y_s$  in (23) and of their duals. The semisimple part  $\mathfrak{r} = \mathfrak{r}_{-1} \oplus \mathfrak{r}_0 \oplus \mathfrak{r}_1$  is graded as follows [34, Sect. 2.4]:

$$\begin{aligned} \mathfrak{r}_1 &= \text{span}\{e_{159}, e_{267}, e_{348}\} \oplus \text{span}\{e_{168}, e_{249}, e_{357}\}, \\ \mathfrak{r}_0 &= \text{span}\{d_{159}, d_{267}, d_{348}\} \oplus \text{span}\{d_{168}, d_{249}, d_{348}\}, \\ \mathfrak{r}_{-1} &= \text{span}\{e^{159}, e^{267}, e^{348}\} \oplus \text{span}\{e^{168}, e^{249}, e^{357}\}, \end{aligned} \quad (24)$$

where  $e^i$ , for  $1 \leq i \leq 9$ , is the dual basis of  $(\mathbb{C}^9)^*$ ,  $e^{ijl} := e^i \wedge e^j \wedge e^l$  and the elements  $d_{ijk} = [e_{ijk}, e^{ijk}]$  satisfy  $d_{159} + d_{267} + d_{348} = d_{168} + d_{249} + d_{348} = 0$ . The direct sums of vector spaces in (24) correspond to the Lie algebra decomposition of  $\mathfrak{r}$ .

Let  $J_1 = J_{G_0}(x)$  be any of the  $G_0$ -Jordan classes in the II family: i.e., one of II.1, II.2 or II.3. The choice of representative  $x = x_s + x_n$  given by

$$x_s = y_s + (e_{159} + e_{267} + e_{348}), \quad x_n = \begin{cases} e_{168} + e_{249} & \text{for II.1,} \\ e_{168} & \text{for II.2,} \\ 0 & \text{for II.3,} \end{cases} \quad (25)$$

easily allows to check that  $J_2 \subset \bar{J}_1$ . First of all  $\mathfrak{z}(\mathfrak{g}^{x_s})_1$  is generated by the 3 vectors in brackets in (23) and (25), hence  $y_s \in \mathfrak{z}(\mathfrak{g}^{x_s})_1$  and  $\mathfrak{g}^{x_s} \subset \mathfrak{m}$ . A graded  $\mathfrak{sl}(2)$ -triple  $\{e, h, f\}$  in  $\mathfrak{m}$  with  $e = y_n$  is provided by  $f = e^{159}$  and  $h = d_{159}$ , and the required Slodowy slice  $S_{\mathfrak{m}_1, e} = e + \mathfrak{m}_1^f$  is the affine subspace in  $\mathfrak{m}_1$  modeled on  $\mathfrak{m}_1^f = \text{span}\{e_{267}, e_{348}\} \oplus \text{span}\{e_{168}, e_{249}, e_{357}\} \oplus \mathfrak{z}(\mathfrak{m})_1$ . It is evident that  $x \in S_{\mathfrak{m}_1, e}$ , so  $J_2 \subset \bar{J}_1$  thanks to Theorem 34 (iii).

### A. Cartan, Levi and parabolic subalgebras in $\mathbb{Z}_m$ -graded Lie algebras

Let  $\{\mathfrak{g}, \theta\}$  be a reductive  $\mathbb{Z}_m$ -graded Lie algebra and  $\mathfrak{c} \subset V$  a fixed Cartan subspace. The existence of a homogeneous Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$  containing  $\mathfrak{c}$  is a result probably known to experts by a long time; the proof in [25, Sect. 4.1] is stated for  $\mathfrak{g}$  simple, but its proof carries over for any reductive  $\mathfrak{g}$ .

**Proposition 40.** *There exists a homogeneous Cartan subalgebra  $\mathfrak{h} = \bigoplus_{l \in \mathbb{Z}_m} \mathfrak{h}_l$  of  $\mathfrak{g}$  that satisfies  $\mathfrak{h} \supset \mathfrak{z}(\mathfrak{c}_{\mathfrak{g}}(\mathfrak{c}))$  and  $\mathfrak{h}_1 = \mathfrak{c}$ .*

*Remark 4.* By [31, Sect. 3.1], the Cartan subspace  $\mathfrak{c}$  is *not* an algebraic subalgebra in general, unless  $m \leq 2$ . On the other hand,  $\mathfrak{h}$  and  $\mathfrak{z}(\mathfrak{c}_{\mathfrak{g}}(\mathfrak{c}))$  are algebraic, hence  $\mathfrak{h} \supset \mathfrak{z}(\mathfrak{c}_{\mathfrak{g}}(\mathfrak{c})) \supset \bar{\mathfrak{c}}$ , where  $\bar{\mathfrak{c}}$  is the algebraic closure of  $\mathfrak{c}$ . It is clear that  $\mathfrak{h} = \mathfrak{z}(\mathfrak{c}_{\mathfrak{g}}(\mathfrak{c}))$  if and only if  $[\mathfrak{c}_{\mathfrak{g}}(\mathfrak{c}), \mathfrak{c}_{\mathfrak{g}}(\mathfrak{c})] = 0$  but we are not aware of any general condition under which  $\mathfrak{z}(\mathfrak{c}_{\mathfrak{g}}(\mathfrak{c})) = \bar{\mathfrak{c}}$ .

We will call *adapted* any Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$  as in Proposition 40. For such an  $\mathfrak{h}$ , let

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_{\alpha} \quad (26)$$

be the root space decomposition of  $\mathfrak{g}$  with respect to  $\mathfrak{h}$ , with associated set of roots  $\Phi \subset \mathfrak{h}^*$ . The automorphism  $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$  permutes the root spaces in (26):

**Lemma 41.** *For any  $\alpha \in \Phi$ , we have  $\alpha \circ \theta \in \Phi$  and  $\theta^{-1}(\mathfrak{g}_{\alpha}) = \mathfrak{g}_{\alpha \circ \theta}$ .*

We note that any root  $\alpha \in \Phi$  can be decomposed as  $\alpha = \alpha_0 + \alpha_1 + \cdots + \alpha_{m-2} + \alpha_{m-1}$ , where  $\alpha_l|_{\mathfrak{h}_l} = \alpha|_{\mathfrak{h}_l}$  and  $\text{Ker}(\alpha_l) = \bigoplus_{k \neq l} \mathfrak{h}_k$  for any  $l \in \mathbb{Z}_m$ . Repeatedly applying Lemma 41, we see that

$$\alpha \circ \theta^l = \alpha_0 + \omega^l \alpha_1 + \cdots + (\omega^l)^{m-2} \alpha_{m-2} + (\omega^l)^{m-1} \alpha_{m-1}$$

is a root too, for any  $l \in \mathbb{Z}_m$ . In other words, we may consider the equivalence class of roots given by  $[\alpha] = \{\alpha \circ \theta^l \mid l \in \mathbb{Z}_m\}$  for any  $\alpha \in \Phi$ . We let  $[\Phi] = \{[\alpha] \mid \alpha \in \Phi\}$  be the collection of such equivalence classes and note that the direct sum of root spaces

$$\mathfrak{g}_{[\alpha]} = \bigoplus_{l \in \mathbb{Z}_m} \mathfrak{g}_{\alpha \circ \theta^l}$$

is a homogeneous subspace of  $\mathfrak{g}$ , whence  $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{[\alpha] \in [\Phi]} \mathfrak{g}_{[\alpha]}$  is a decomposition of  $\mathfrak{g}$  into homogeneous subspaces.

Now, the centralizer  $\mathfrak{g}^x$  of any  $x \in \mathfrak{c}$  is a homogeneous Levi subalgebra containing  $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{c})$ . A natural question is whether there exists a parabolic subalgebra of  $\mathfrak{g}$  with Levi factor  $\mathfrak{g}^x$  that is also homogeneous; we will now see that this is rarely the case. For simplicity of exposition, we restrict to the case where  $\mathfrak{g}$  is semisimple.

Let  $\mathfrak{p}$  be a proper parabolic subalgebra of  $\mathfrak{g}$  with a homogeneous Levi factor  $\mathfrak{l}$  that contains  $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{c})$ . Then, there exists a  $\mathbb{Z}$ -grading

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}(j) \quad (27)$$

of  $\mathfrak{g}$  such that  $\mathfrak{p} = \mathfrak{g}(\geq 0) = \bigoplus_{j \geq 0} \mathfrak{g}(j)$  and  $\mathfrak{l} = \mathfrak{g}(0)$ . We let  $Z \in \mathfrak{g}$  be the grading element of (27), the unique element in  $\mathfrak{g}$  that satisfies  $[Z, X] = jX$  for all  $X \in \mathfrak{g}(j)$ ,  $j \in \mathbb{Z}$ : see, e.g., [30].

Now  $Z \in \mathfrak{z}(\mathfrak{c}_{\mathfrak{g}}(\mathfrak{c}))$ , so it belongs to the adapted Cartan subalgebra  $\mathfrak{h} = \bigoplus_{l \in \mathbb{Z}_m} \mathfrak{h}_l$  of  $\mathfrak{g}$  of Proposition 40. We will write  $Z = Z_0 + \cdots + Z_{m-1}$ , where  $Z_l \in \mathfrak{h}_l$  for all  $l \in \mathbb{Z}_m$ .

**Definition 3.** Let  $\alpha = \alpha_0 + \cdots + \alpha_{m-1} \in \Phi$  be a root with respect to  $\mathfrak{h}$  and  $l \in \mathbb{Z}_m$ . The  $l^{\text{th}}$  mode of  $\alpha$  is the complex number  $\lambda_l = \alpha_l(Z_l)$ .

We remark that  $\alpha(Z) = \sum_{l \in \mathbb{Z}_m} \lambda_l$ . Since the adjoint action of  $Z$  has integer eigenvalues, we may apply Lemma 41 repeatedly to the roots  $\alpha \circ \theta^l \in \Phi$  and get the following.

**Proposition 42.** *The modes of  $\alpha$  satisfy a system of linear equations of the form*

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{m-1} \\ 1 & \omega^2 & (\omega^2)^2 & \cdots & (\omega^{m-1})^2 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \omega^{m-1} & (\omega^2)^{m-1} & \cdots & (\omega^{m-1})^{m-1} \end{pmatrix} \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{m-1} \end{pmatrix} = \begin{pmatrix} n_0 \\ n_1 \\ n_2 \\ \vdots \\ n_{m-1} \end{pmatrix}, \quad (28)$$

where  $n_l = \alpha(\theta^l(Z)) \in \mathbb{Z}$  for any  $l \in \{0, \dots, m-1\}$ .

The  $m \times m$  matrix on the left-hand side of (28) is a symmetric matrix of Vandermonde type with coefficients in the cyclotomic field  $\mathbb{Q}(\omega)$ . We denote it by  $M(\omega)$  and compactly rewrite (28) as  $M(\omega)\vec{\lambda} = \vec{n}$ , where  $\vec{\lambda} \in \mathbb{C}^m$  is the vector of modes and  $\vec{n} \in \mathbb{Z}^m$ . Clearly all modes are elements of  $\mathbb{Q}(\omega)$ , but we have the following stronger result for  $\lambda_0$ .

**Proposition 43.** *The identity  $m\lambda_0 = \sum_{l \in \mathbb{Z}_m} n_l$  is satisfied for any  $\alpha$ , therefore  $\lambda_0 \in \frac{1}{m}\mathbb{Z}$ . If  $\mathfrak{h}_0 = 0$ , then  $\mathfrak{p}$  is not  $\theta$ -stable.*

*Proof.* Let  $W = \{\vec{y} \in \mathbb{C}^m \mid \sum_{l \in \mathbb{Z}_m} y_l = 0\}$ . All columns of  $M(\omega)$  but the first one lie in  $W$ , so

$$\vec{n} - \lambda_0 \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \in W. \quad (29)$$

Adding all entries of the above vector gives  $m\lambda_0 = \sum_{l \in \mathbb{Z}_m} n_l \in \mathbb{Z}$ . If  $\mathfrak{h}_0 = 0$ , then  $Z_0 = 0$ , so  $\lambda_0 = 0$  and  $\vec{n} \in W \cap \mathbb{Z}^m$ .

Now,  $\mathfrak{h} \subset \mathfrak{g}(0)$  and  $\mathfrak{p} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi, \alpha(Z) \geq 0} \mathfrak{g}_\alpha$ . If  $\mathfrak{g}_\alpha \subset \bigoplus_{j > 0} \mathfrak{g}(j)$ , then  $\alpha(Z) = n_0 > 0$  and, if  $\mathfrak{h}_0 = 0$ , there exists  $l \in \mathbb{Z}_m$  such that  $n_l < 0$ , i.e.,  $\theta^{-l}\mathfrak{g}_\alpha = \mathfrak{g}_{\alpha \circ \theta^l} \notin \mathfrak{p}$ .  $\square$

**Example 5.** The  $\mathbb{Z}_m$ -graded Lie algebra  $\{\mathfrak{g}, \theta, m\} = \{E_8, \theta, 3\}$  as in Examples 1 and 4 satisfies  $\mathfrak{h}_0 = 0$ . By Theorem 6 and Proposition 43, all centralizers  $\mathfrak{g}^x$  of nonzero  $x \in \mathcal{S}_V$  do not extend to  $\theta$ -stable parabolic subalgebras.

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