UiT The Arctic University of Norway

Faculty of Engineering Science and Technology
Department of Computer Science and Computational Engineering

A study of bounded operators on martingale Hardy spaces

Giorgi Tutberidze

A dissertation for the degree of Philosophiae Doctor

December 2021

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Abstract

The classical Fourier Analysis has been developed in an almost unbelievable way from the first fundamental discoveries by Fourier. Especially a number of wonderful results have been proved and new directions of such research has been developed e.g. concerning Wavelets Theory, Gabor Theory, Time-Frequency Analysis, Fast Fourier Transform, Abstract Harmonic Analysis, etc. One important reason for this is that this development is not only important for improving the "State of the art", but also for its importance in other areas of mathematics and also for several applications (e.g. theory of signal transmission, multiplexing, filtering, image enhancement, coding theory, digital signal processing and pattern recogni-tion).

The classical theory of Fourier series deals with decomposition of a function into sinusoidal waves. Unlike these continuous waves the Vilenkin (Walsh) functions are rectangular waves. The development of the theory of Vilenkin-Fourier series has been strongly influenced by the classical theory of trigonometric series. Because of this it is inevitable to compare results of Vilenkin series to those on trigonometric series. There are many similarities between these theories, but there exist differences also. Much of these can be explained by modern abstract harmonic analysis, which studies orthonormal systems from the point of view of the structure of a topological group.

The aim of my thesis is to discuss, develop and apply the newest developments of this fascinating theory connected to modern harmonic analysis. In particular, we investigate some strong convergence result of partial sums of Vilenkin-Fourier series. Moreover, we derive necessary and sufficient conditions for the modulus of continuity so that norm convergence of subsequences of Fejér means is valid. Furthermore, we consider Riesz and Nörlund logarithmic means. It is also proved that these results are the best possible in a special sense. As applications both some well-known and new results are pointed out. In addition, we investigate some T means, which are "inverse" summability methods of Nörlund, but only in the case when their coefficients are monotone.

The main body of the PhD thesis consists of seven papers (Papers A – G). We now continue by describing the main content of each of the papers.

In Paper A we investigate some new strong convergence theorems for partial sums with respect to Vilenkin system.

In Paper B we characterize subsequences of Fejér means with respect to Vilenkin systems, which are bounded from the Hardy space H_p to the Lebesgue space $L_p,$ for all $0<p<1/2.$ We also proved that this result is in a sense sharp.

In Paper C we find necessary and sufficient condition for the modulus of continuity for which subsequences of Fejér means with respect to Vilenkin systems are bounded from the Hardy space H_p to the Lebesgue space $L_p,$ for all $0 < p < 1/2$.

In Paper D we prove and discuss some new $(H_p, weak - L_p)$ type inequalities of maximal operators of T means with respect to Vilenkin systems with monotone coefficients. We also apply these results to prove a.e. convergence of such T means. It is also proved that these results are the best possible in a special sense. As applications, both some well-known and new results are pointed out.

In Paper E we prove and discuss some new (H_p, L_p) type inequalities of weighted maximal operators of T means with respect to the Vilenkin systems with monotone coefficients. We also show that these inequalities are the best possible in a special sense. Moreover, we apply these inequalities to prove strong convergence theorems of such T means. We also show that these results are the best possible in a special sense. As applications, both some well-known and new results are pointed out.

In Paper F we derive a new strong convergence theorem of Riesz logarithmic means of the one-dimensional Vilenkin-Fourier (Walsh-Fourier) series. The corresponding inequality is pointed out and it is also proved that the inequality is in a sense sharp, at least for the case with Walsh-Fourier series.

In Paper G we investigate (H_p, L_p)- type inequalities for weighted maximal operators of Nörlund logaritmic means, for $0 < p < 1$. Moreover, we apply these inequalities to prove strong convergence theorems of such Nörlund logaritmic means.

These new results are put into a more general frame in an Introduction, where, in particular, a comparison with some new international research and broad view of such interplay between applied mathematics and engineering problems is presented and discussed.

Preface

This PhD thesis is composed of seven papers [A] – [G] and a matching Introduction. In the Introduction the papers [A] – [G] are discussed and put into a more general frame. The Introduction is also of independent interest since it contains a brief discussion on the important definitions and notations in the theory of Fourier analysis and martingale Hardy spaces.

A very brief presentation of the main content of the seven papers can be found in the Abstract above and in a more general context at the end of the Introduction.

List of Papers

Paper A: G. Tutberidze. "A note on the strong convergence of partial sums with respect to Vilenkin system".

J. Contemp. Math. Anal., 54 (2019), no.6, 319–324.

Paper B: L. E. Persson, G. Tephnadze, G. Tutberidze, "On the boundedness of subsequences of Vilenkin-Fejér means on the martingale Hardy spaces".

Operators and Matrices, 14 (2020), no.1, 283–294.

Paper C: G. Tutberidze. "Modulus of continuity and boundedness of subsequences of Vilenkin- Fejér means in the martingale Hardy spaces". 13 pages *Submitted for publication.*

Paper D: G. Tutberidze. "Maximal operators of T means with respect to the Vilenkin system". *Nonlinear Studies, 27 (2020), no.4, 1–11.*

Paper E: G. Tutberidze. "Sharp (H_p, L_p) type inequalities of maximal operators of T means with respect to Vilenkin systems with monotone coefficients". *Submitted for publication*.

Paper F: D. Lukkassen, L.E. Persson, G. Tephnadze, G. Tutberidze. "Some inequalities related to strong convergence of Riesz logarithmic means of Vilenkin-Fourier series". *J. Inequal. Appl., 2020, DOI: https://doi.org/10.1186/s13660-020-02342-8*(17 pages).

Paper G: G. Tephnadze, G. Tutberidze. "A note on the maximal operators of the Nörlund logaritmic means of Vilenkin-Fourier series". *Trans. A. Razmadze Math. Inst., 174, (2020), no.1, 107–112.*

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Chapter 1

Introduction

1.1 Preliminaries

1.1.1 Vilenkin groups and functions

Denote by $\mathbb{N}_+$ the set of positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Let $m := (m_0, m_1, \ldots)$ be a sequence of positive integers not less than 2. Denote by

$$Z_{m_k} := \{0, 1, \ldots, m_k - 1\}$$

the additive group of integers modulo m_k.

Define the group G_m as the complete direct product of the groups Z_{m_k} with the product of the discrete topologies of Z_{m_k}.

The direct product μ of the measures

$$\mu_k(j) := 1/m_k \qquad (j \in Z_{m_k})$$

is the Haar measure on G_m with $\mu(G_m) = 1$.

If $\sup_{n\in\mathbb{N}} m_n < \infty$, then we call G_m a bounded Vilenkin group. If the generating sequence m is not bounded, then G_m is said to be an unbounded Vilenkin group.

In this PhD thesis we discuss bounded Vilenkin groups, i.e. the case when $\sup_{n\in\mathbb{N}} m_n < \infty$.

The elements of G_m are represented by sequences

$$x := (x_0, x_1, \ldots, x_j, \ldots) \qquad \left(x_j \in Z_{m_j}\right).$$

If we define the so-called generalized number system based on m in the following way :

$$M_0 := 1, \;\; M_{k+1} := m_k M_k \;\; (k \in \mathbb{N}),$$

then every $n \in \mathbb{N}$ can be uniquely expressed as

$$n = \sum_{j=0}^{\infty} n_j M_j,$$

where $n_j \in Z_{m_j} \;\; (j \in \mathbb{N}_+)$ and only a finite number of n_j's differ from zero.

Vilenkin group can be metrizable with the following metric:

$$\rho(x, y) := |x - y| := \sum_{k=0}^{\infty} \frac{|x_k - y_k|}{M_{k+1}}, \qquad (x \in G_m).$$

It is easy to give a base for the neighborhoods of G_m :

$$\begin{aligned} I_0(x) &: = G_m, \\ I_n(x) &: = \{y \in G_m \mid y_0 = x_0, \dots, y_{n-1} = x_{n-1}\} \qquad (x \in G_m,\ n \in \mathbb{N}). \end{aligned}$$

Let

$$e_n := (0, \dots, 0, x_n = 1, 0, \dots) \in G_m \qquad (n \in \mathbb{N}).$$

If we define $I_n := I_n(0)$, for $n \in \mathbb{N}$ and $\overline{I_n} := G_m \setminus I_n$, then

$$\overline{I_N} = \bigcup_{s=0}^{N-1} I_s \backslash I_{s+1} = \left(\bigcup_{k=0}^{N-2} \bigcup_{l=k+1}^{N-1} I_N^{k,l}\right) \bigcup \left(\bigcup_{k=1}^{N-1} I_N^{k,N}\right),$$

where

$$I_N^{k,l} := \begin{cases} I_N(0, \dots, 0, x_k \neq 0, 0, ..., 0, x_l \neq 0, x_{l+1}, \dots, x_{N-1}, \dots), \\ \text{for} \quad k < l < N, \\ I_N(0, \dots, 0, x_k \neq 0, x_{k+1} = 0, \dots, x_{N-1} = 0, x_N, \dots), \\ \text{for} \quad l = N. \end{cases}$$

The norm (or quasi-norm when $0 < p < 1$) of the Lebesgue space $L_p(G_m)$ $(0 < p < \infty)$ is defined by

$$\|f\|_p := \left(\int_{G_m} |f|^p \, d\mu\right)^{1/p}.$$

The space $weak - L_p(G_m)$ consists of all measurable functions f, for which

$$\|f\|_{weak-L_p} := \sup_{\lambda>0} \lambda\{\mu(f > \lambda)\}^{1/p} < +\infty.$$

The norm of the space of continuous functions $C(G_m)$ is defined by

$$\|f\|_C := \sup_{x \in G_m} |f(x)| < c < \infty.$$

The best approximation of $f \in L_p(G_m)$ $(1 \leq p \leq \infty)$ is defined as

$$E_n(f, L_p) := \inf_{\psi \in P_n} \|f - \psi\|_p,$$

where P_n is set of all Vilenkin polynomials of order less than $n \in \mathbb{N}$.

The modulus of continuity of functions in Lebesgue spaces $f \in L_p(G_m)$ and continuous functions $f \in C(G_m)$ are defined by

$$\omega_p\left(\frac{1}{M_n}, f\right) := \sup_{h \in I_n} \|f(\cdot - h) - f(\cdot)\|_p$$

and

$$\omega_C\left(\frac{1}{M_n}, f\right) := \sup_{h \in I_n} \|f(\cdot - h) - f(\cdot)\|_C,$$

respectively.

Next, we introduce on G_m orthonormal systems, which are called Vilenkin systems.

At first, we define the complex-valued function $r_k(x) : G_m \to \mathbb{C}$, the generalized Rademacher functions, by

$$r_k(x) := \exp(2\pi i x_k/m_k), \quad \left(i^2 = -1, \ x \in G_m, \ k \in \mathbb{N}\right).$$

Now, define Vilenkin systems $\psi := (\psi_n : n \in \mathbb{N})$ on G_m as:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x), \quad (n \in \mathbb{N}).$$

The Vilenkin systems are orthonormal and complete in $L_2(G_m)$ (for details see e.g. [1] , [61] and [108]).

It is well-known that for all $n \in \mathbb{N}$,

$$\begin{aligned}
|\psi_n(x)| &= 1, \\
\psi_n(x+y) &= \psi_n(x)\ \psi_n(y), \\
\psi_n(-x) &= \psi_{n^*}(x) = \overline{\psi}_n(x), \\
\psi_n(x-y) &= \psi_n(x)\ \overline{\psi}_n(y), \\
\psi_{n\widehat{+}k}(x) &= \psi_s\psi_n(x), \quad (s, n \in \mathbb{N}, \quad x, y \in G_m).
\end{aligned}$$

Specifically, we call this system the Walsh-Paley system when $m = 2$.

1.1.2 Partial sums and Fejér means with respect to the Vilenkin systems

Next, we introduce some analogues of the usual definitions in Fourier analysis. If $f \in L_1(G_m)$ we can define the Fourier coefficients, the partial sums of Vilenkin-Fourier series, the Dirichlet kernels, Fejér means, Dirichlet and Fejér kernels with respect to Vilenkin systems in the usual manner:

$$\begin{aligned}
\widehat{f}(n) &:= \int_{G_m} f\overline{\psi}_n d\mu, && (n \in \mathbb{N}), \\
S_n f &:= \sum_{k=0}^{n-1} \widehat{f}(k)\, \psi_k, && (n \in \mathbb{N}_+), \\
\sigma_n f &:= \frac{1}{n}\sum_{k=0}^{n-1} S_k f, && (n \in \mathbb{N}_+), \\
D_n &:= \sum_{k=0}^{n-1} \psi_k, && (n \in \mathbb{N}_+), \\
K_n &:= \frac{1}{n}\sum_{k=0}^{n-1} D_k, && (n \in \mathbb{N}_+).
\end{aligned}$$

respectively.

It is easy to see that

$$\begin{aligned} S_n f(x) &= \int_{G_m} f(t) \sum_{k=0}^{n-1} \psi_k(x-t)\, d\mu(t) \\ &= \int_{G_m} f(t) D_n(x-t)\, d\mu(t) \\ &= (f * D_n)(x). \end{aligned}$$

It is well-known that (for details see e.g. [1] , [61] and [108]) that for any $n \in \mathbb{N}$ and $1 \leq s_n \leq m_n - 1$ the following equalities holds:

$$D_{j+M_n} = D_{M_n} + \psi_{M_n} D_j = D_{M_n} + r_n D_j, \qquad j \leq (m_n - 1) M_n,$$

$$\begin{aligned} D_{M_n - j}(x) &= D_{M_n}(x) - \overline{\psi}_{M_n - 1}(-x) D_j(-x) \\ &= D_{M_n}(x) - \psi_{M_n - 1}(x) \overline{D}_j(x), \qquad j < M_n. \end{aligned}$$

$$D_{M_n}(x) = \begin{cases} M_n & x \in I_n \\ 0 & x \notin I_n \end{cases} \tag{1.1}$$

$$D_{s_n M_n} = D_{M_n} \sum_{k=0}^{s_n - 1} \psi_{k M_n} = D_{M_n} \sum_{k=0}^{s_n - 1} r_n^k \tag{1.2}$$

and

$$D_n = \psi_n \left(\sum_{j=0}^{\infty} D_{M_j} \sum_{k=m_j - n_j}^{m_j - 1} r_j^k \right).$$

By using (1.1) we immediately get that

$$\|D_{M_n}\|_1 = 1 < \infty.$$

It is obvious that

$$\begin{aligned} \sigma_n f(x) &= \frac{1}{n} \sum_{k=0}^{n-1} (D_k * f)(x) \\ &= \int_{G_m} f(t) K_n(x-t)\, d\mu(t) \\ &= (f * K_n)(x), \end{aligned}$$

where K_n are the so called Fejér kernels.

It is well-known that (for details see e.g. [42]) for every $n > t,\, t, n \in \mathbb{N}$ we have the following equality:

$$K_{M_n}(x) = \begin{cases} \frac{M_t}{1-r_t(x)}, & x \in I_t \backslash I_{t+1}, \qquad x - x_t e_t \in I_n, \\ \frac{M_n+1}{2}, & x \in I_n, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover,

$$s_n M_n K_{s_n M_n} = \sum_{l=0}^{s_n-1} \left(\sum_{i=0}^{l-1} r_n^i \right) M_n D_{M_n} + \left(\sum_{l=0}^{s_n-1} r_n^l \right) M_n K_{M_n}.$$

The next equality of Fejér kernels is very important for our further investigations (for details see Blahota and Tephnadze [26]). In particular, if $n = \sum_{i=1}^{r} s_{n_i} M_{n_i}$, where $n_1 > n_2 > \cdots > n_r \geq 0$ and $1 \leq s_{n_i} < m_{n_i}$ for all $1 \leq i \leq r$ as well as $n^{(k)} = n - \sum_{i=1}^{k} s_{n_i} M_{n_i}$, where $0 < k \leq r$, then

$$nK_n = \sum_{k=1}^{r} \left(\prod_{j=1}^{k-1} r_{n_j}^{s_{n_j}} \right) s_{n_k} M_{n_k} K_{s_{n_k} M_{n_k}} + \sum_{k=1}^{r-1} \left(\prod_{j=1}^{k-1} r_{n_j}^{s_{n_j}} \right) n^{(k)} D_{s_{n_k} M_{n_k}}.$$

It is well-known that

$$\|K_n\|_1 < c < \infty.$$

We define the maximal operators S^* and σ^* of partial sums and Féjer means by

$$S^* f := \sup_{n \in \mathbb{N}} |S_n f|,$$
$$\sigma^* f := \sup_{n \in \mathbb{N}} |\sigma_n f|.$$

Moreover, we define the restricted maximal operators $\widetilde{S}_{\#}^*$ and $\widetilde{\sigma}_{\#}^*$ of partial sums and Féjer means by

$$\widetilde{S}_{\#}^* f := \sup_{n \in \mathbb{N}} |S_{M_n} f|,$$
$$\widetilde{\sigma}_{\#}^* f := \sup_{n \in \mathbb{N}} |\sigma_{M_n} f|.$$

1.1.3 Character $\rho(n)$ and Lebesgue constants with respect to Vilenkin systems

Let us define

$$\langle n \rangle := \min\{j \in \mathbb{N} : n_j \neq 0\} \text{ and } |n| := \max\{j \in \mathbb{N} : n_j \neq 0\},$$

that is $M_{|n|} \leq n \leq M_{|n|+1}$. Set

$$\rho(n) := |n| - \langle n \rangle, \qquad \text{for all} \qquad n \in \mathbb{N}.$$

For the natural numbers $n=\sum_{j=1}^{\infty} n_j M_j$ and $k=\sum_{j=1}^{\infty} k_j M_j$ we define

$$n\widehat{+}k:=\sum_{i=0}^{\infty}(n_i\oplus k_i)M_{i+1}$$

and

$$n\widehat{-}k:=\sum_{i=0}^{\infty}(n_i\ominus k_i)M_{i+1},$$

where

$$a_i\oplus b_i:=(a_i+b_i)\text{mod}m_i, \qquad a_i,b_i\in Z_{m_i}$$

and $\ominus$ is the inverse operation for $\oplus$.

For the natural number $n=\sum_{j=1}^{\infty} n_j M_j$, we define functions v and v^* by

$$v(n):=\sum_{j=1}^{\infty}\left|\delta_{j+1}-\delta_j\right|+\delta_0, \qquad v^*(n):=\sum_{j=1}^{\infty}\delta_j^*,$$

where

$$\delta_j=sign\left(n_j\right)=sign\left(\ominus n_j\right) \text{ and } \delta_j^*=\left|\ominus n_j-1\right|\delta_j.$$

The n-th Lebesgue constant is defined in the following way:

$$L_n:=\left\|D_n\right\|_1.$$

For the trigonometric system it is important to note that the results of Fejér and Szego, latter on proved in [121] gives an explicit formula for the Lebesgue constants. The most properties of the Lebesgue constants with respect to the Walsh-Paley system were obtained by Fine in [36]. In [108], p. 34, the two-sided estimate is proved. In [76], Lukomskii presented the lower estimate with sharp constant $1/4$. Malykhin, Telyakovskii and Kholshchevnikova [77] (see also Astashkin and Semenov [8]) improved the estimation above and proved sharp estimate with factor 1. A new and shorter proof which improved upper bound and provide a similar lower bound can be found in [23]. In particular, for $\lambda:=\sup_{n\in\mathbb{N}}$ and for any $n=\sum_{i=1}^{\infty} n_i M_i$ and m_n we have the following two sided estimate:

$$\frac{1}{4\lambda}v(n)+\frac{1}{\lambda^2}v^*(n)\leq L_n\leq v(n)+v^*(n). \tag{1.3}$$

Moreover, it yields that (see Memic, Simon and Tephnadze [79]):

$$\frac{1}{nM_n}\sum_{k=1}^{M_n-1}v(k)\geq\frac{2}{\lambda^2}. \tag{1.4}$$

From the inequality (1.3) it immediately follows that for any $n\in\mathbb{N}$ there exists an absolute constant c, such that

$$\left\|D_n\right\|_1\leq c\log n.$$

For example, if we take $q_{n_k} = M_{2n_k} + M_{2n_k-2} + M_2 + M_0$, we have the following two-sided inequality

$$\frac{n_k}{2\lambda} \leq \left\| D_{q_{n_k}} \right\|_1 \leq \lambda n_k, \qquad \lambda := \sup_{n\in\mathbb{N}} m_n.$$

1.1.4 Definition and examples of Nörlund and T means and its maximal operators

Let $\{q_k : k \in \mathbb{N}\}$ be a sequence of nonnegative numbers. The n-th Nörlund means for the Fourier series of f is defined by

$$t_n f := \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} S_k f, \tag{1.5}$$

where

$$Q_n := \sum_{k=0}^{n-1} q_k.$$

A representation

$$t_n f(x) = \int_G f(t) A_n(x-t)\, d\mu(t)$$

plays a central role in the sequel, where

$$A_n := \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} D_k$$

is the so-called Nörlund kernel.

In Moore [80] (see also Tephnadze [130]) it was found necessary and sufficient conditions for regularity of Nörlund means. In particular, if $\{q_k : k \geq 0\}$ is a sequence of nonnegative numbers, $q_0 > 0$ and

$$\lim_{n\to\infty} Q_n = \infty,$$

then the summability method (1.5) generated by $\{q_k : k \geq 0\}$ is regular if and only if

$$\lim_{n\to\infty} \frac{q_{n-1}}{Q_n} = 0.$$

In addition, if the sequence $\{q_k : k \in \mathbb{N}\}$ is non-increasing, then the summability method generated by $\{q_k : k \in \mathbb{N}\}$ is regular, but if the sequence $\{q_k : k \in \mathbb{N}\}$ is non-decreasing, then the summability method generated by $\{q_k : k \in \mathbb{N}\}$ is not always regular.

Let $\{q_k : k \geq 0\}$ be a sequence of non-negative numbers. The n-th T mean T_n for a Fourier series of f is defined by

$$T_n f := \frac{1}{Q_n} \sum_{k=0}^{n-1} q_k S_k f,$$

where $Q_n := \sum_{k=0}^{n-1} q_k$. It is obvious that

$$T_n f(x) = \int_{G_m} f(t) F_n(x-t) d\mu(t),$$

where $F_n := \frac{1}{Q_n} \sum_{k=1}^{n} q_k D_k$ is called the kernel of T means.

We always assume that $\{q_k : k \geq 0\}$ is a sequence of non-negative numbers and $q_0 > 0$. Then the summability method (1.1.4) generated by $\{q_k : k \geq 0\}$ is regular if and only if $\lim_{n\to\infty} Q_n = \infty$.

Let t_n be Nörlund means with monotone and bounded sequence $\{q_k : k \in \mathbb{N}\}$, such that

$$q := \lim_{n\to\infty} q_n > c > 0.$$

If the sequence $\{q_k : k \in \mathbb{N}\}$ is non-decreasing, then we get that

$$nq_0 \leq Q_n \leq nq.$$

In the case when the sequence $\{q_k : k \in \mathbb{N}\}$ is non-increasing, we have that

$$nq \leq Q_n \leq nq_0.$$

In both cases we can conclude that

$$\frac{q_{n-1}}{Q_n} = O\left(\frac{1}{n}\right), \text{ when } n \to \infty.$$

One of the most well-known summability methods which is an example of Nörlund and T means are the so called Fejér means, which is given when $\{q_k = 1 : k \in \mathbb{N}\}$ as follows:

$$\sigma_n f := \frac{1}{n} \sum_{k=1}^{n} S_k f.$$

The (C, α)-means (Cesàro means) of the Vilenkin-Fourier series are defined by

$$\sigma_n^\alpha f := \frac{1}{A_n^\alpha} \sum_{k=1}^{n} A_{n-k}^{\alpha-1} S_k f,$$

where

$$A_0^\alpha := 0, \qquad A_n^\alpha := \frac{(\alpha+1)\ldots(\alpha+n)}{n!}.$$

It is well-known that (see e.g. Zygmund [186])

$$A_n^\alpha = \sum_{k=0}^{n} A_{n-k}^{\alpha-1},$$

$$A_n^\alpha - A_{n-1}^\alpha = A_n^{\alpha-1}, \ A_n^\alpha \backsim n^\alpha.$$

We also consider the "inverse" (C,α)-means U_n^α, which is an example of a T-mean:

$$U_n^\alpha f := \frac{1}{A_n^\alpha}\sum_{k=0}^{n-1} A_k^{\alpha-1} S_k f, \qquad 0<\alpha<1.$$

Let V_n^α denote the T mean, where $\left\{q_0=0,\ q_k=k^{\alpha-1} : k\in\mathbb{N}_+\right\}$, that is

$$V_n^\alpha f := \frac{1}{Q_n}\sum_{k=1}^{n-1} k^{\alpha-1} S_k f, \qquad 0<\alpha<1.$$

The n-th Nörlund logarithmic mean L_n and the Riesz logarithmic mean R_n are defined by

$$L_n f := \frac{1}{l_n}\sum_{k=1}^{n-1}\frac{S_k f}{n-k},$$
$$R_n f := \frac{1}{l_n}\sum_{k=1}^{n-1}\frac{S_k f}{k},$$

respectively, where

$$l_n := \sum_{k=1}^{n-1}\frac{1}{k}.$$

The kernels of the Nörlund logarithmic mean P_n and the Riesz logarithmic mean Y_n are, respectively, defined by

$$P_n f := \frac{1}{l_n}\sum_{k=1}^{n-1}\frac{D_k f}{n-k},$$
$$Y_n f := \frac{1}{l_n}\sum_{k=1}^{n-1}\frac{D_k f}{k}.$$

Up to now we have considered Nörlund and T means in the case when the sequence $\{q_k : k\in\mathbb{N}\}$ is bounded but now we consider Nörlund and T summabilities with unbounded sequence $\{q_k : k\in\mathbb{N}\}$.

Let $\alpha\in\mathbb{R}_+,\ \ \beta\in\mathbb{N}_+$ and

$$\log^{(\beta)} x := \overbrace{\log\ldots\log}^{\beta\text{times}} x.$$

If we define the sequence $\{q_k : k\in\mathbb{N}\}$ by

$$\left\{q_0=0 \text{ and } q_k=\log^{(\beta)} k^\alpha : k\in\mathbb{N}_+\right\},$$

then we get the class of Nörlund means $\kappa_n^{\alpha,\beta}$ with non-decreasing coefficients:

$$\kappa_n^{\alpha,\beta} f := \frac{1}{Q_n} \sum_{k=1}^{n} \log^{(\beta)} (n-k)^{\alpha} S_k f.$$

First we note that $\kappa_n^{\alpha,\beta}$ are well-defined for every $n \in \mathbb{N}_+$. It is obvious that

$$\frac{n}{2} \log^{(\beta)} \frac{n^{\alpha}}{2^{\alpha}} \leq Q_n \leq n \log^{(\beta)} n^{\alpha}.$$

It follows that

$$\begin{aligned} \frac{q_{n-1}}{Q_n} &\leq \frac{c \log^{(\beta)} (n-1)^{\alpha}}{n \log^{(\beta)} n^{\alpha}} \\ &= O\left(\frac{1}{n}\right) \to 0, \text{ as } n \to \infty. \end{aligned}$$

If we define the sequence $\{q_k : k \in \mathbb{N}\}$ by $\left\{q_0 = 0,\ q_k = \log^{(\beta)} k^{\alpha} : k \in \mathbb{N}_+\right\}$, then we get the class of T means $B_n^{\alpha,\beta}$ with non-decreasing coefficients:

$$B_n^{\alpha,\beta} f := \frac{1}{Q_n} \sum_{k=1}^{n-1} \log^{(\beta)} k^{\alpha} S_k f.$$

We note that $B_n^{\alpha,\beta}$ are well-defined for every $n \in \mathbb{N}$.

It is obvious that $\frac{n}{2} \log^{(\beta)} \frac{n^{\alpha}}{2^{\alpha}} \leq Q_n \leq n \log^{(\beta)} n^{\alpha} \to 0,$ as $n \to \infty$.

Let us define the maximal operators t^* and T^* of Nörlund and T means, respectively, by

$$t^* f := \sup_{n \in \mathbb{N}} |t_n f|,$$
$$T^* f := \sup_{n \in \mathbb{N}} |T_n f|.$$

The well-known examples of maximal operators of Nörlund and T means are maximal operator of Cesáro means $\sigma^{\alpha,*}$, Nörlund logarithmic mean L^* and Reisz logarithmic mean R^* which are defined by:

$$\begin{aligned} \sigma^{\alpha,*} f &:= \sup_{n \in \mathbb{N}} |\sigma_n^{\alpha} f|, \\ L^* f &:= \sup_{n \in \mathbb{N}} |L_n f|, \\ R^* f &:= \sup_{n \in \mathbb{N}} |R_n f|. \end{aligned}$$

We also define some new maximal operators $\kappa^{\alpha,\beta,*}$ and $\beta^{\alpha,*}$ as follows:

$$\begin{aligned} \kappa^{\alpha,\beta,*} f &:= \sup_{n \in \mathbb{N}} \left|\kappa_n^{\alpha,\beta} f\right|, \\ \beta^{\alpha,*} f &:= \sup_{n \in \mathbb{N}} |\beta_n^{\alpha} f|. \end{aligned}$$

1.1.5 Weak-type and strong-type inequalities and a.e convergence

The convolution of two functions $f, g \in L_1(G_m)$ is defined by

$$(f * g)(x) := \int_{G_m} f(x-t) g(t) \, dt \quad (x \in G_m).$$

It is easy to see that

$$(f * g)(x) = \int_{G_m} f(t) g(x-t) \, dt \quad (x \in G_m).$$

It is well-known (for details see e.g. [1] , [61] and [108]) that if $f \in L_p(G_m)$, $g \in L_1(G_m)$ and $1 \leq p < \infty$, then $f * g \in L_p(G_m)$ and

$$\|f * g\|_p \leq \|f\|_p \|g\|_1 ,$$

In classical Fourier analysis (see e.g. [186]), a point $x \in (-\infty, \infty)$ is called a Lebesgue point of an integrable function f if it yields that

$$\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} |f(t) - f(x)| \, d\mu(t) = 0.$$

On G_m we have the following definition of Lebesgue point: A point x on the Vilenkin group is called Lebesgue point of $f \in L_1(G_m)$, if

$$\lim_{n \to \infty} M_n \int_{I_n(x)} f(t) \, dt = f(x) \qquad a.e. \; x \in G_m.$$

It is well-known that if $f \in L_1(G_m)$, then

$$\lim_{n \to \infty} S_{M_n} f(x) = f(x) \text{ a.e. on } G_m,$$

where S_{M_n} is the M_n-th partial sum with respect to the Vilenkin system (for details see e.g. [1], [61] and [108]).

We introduce the operator W_A by

$$W_A f(x) := \sum_{s=0}^{A-1} M_s \sum_{r_s=1}^{m_s-1} \int_{I_A(x - r_s e_s)} |f(t) - f(x)| \, d\mu(t).$$

A point $x \in G_m$ is a Vilenkin-Lebesgue point of $f \in L_1(G_m)$, if

$$\lim_{A \to \infty} W_A f(x) = 0.$$

In most applications the a.e. convergence of $\{T_n : n \in \mathbb{N}\}$ can be established for f in some dense class of $L_1(G_m)$. In particular, the following result plays an important role for studying this type of questions (see e.g. the books [61], [108] and [186]).

Lemma 1.1.1. *Let $f \in L_1$ and $T_n : L_1 \to L_1$ be some sub-linear operators and*

$$T^* := \sup_{n\in\mathbb{N}} |T_n| .$$

If

$$T_n f \to f \text{ a.e. for every } f \in S,$$

where the set S is dense in the space L_1 and the maximal operator T^* is bounded from the space L_1 to the space $weak - L_1$, that is

$$\sup_{\lambda>0} \lambda\mu \{x \in G_m : \ |T^* f(x)| > \lambda\} \leq \|f\|_1 ,$$

then

$$T_n f \to f, \text{ a.e. for every } f \in L_1(G_m) .$$

Remark 1.1.2. Since the Vilenkin function ψ_m is constant on $I_n(x)$ for every $x \in G_m$ and $0 \leq m < M_n$, it is clear that each Vilenkin function is a complex-valued step function, that is, it is a finite linear combination of characteristic functions

$$\chi(E) = \begin{cases} 1, & x \in E, \\ 0, & x \notin E. \end{cases}$$

On the other hand, notice that, by (1.2), it yields that

$$\chi(I_n(t))(x) = \frac{1}{M_n} \sum_{j=0}^{M_n-1} \psi_j(x-t), \quad x \in I_n(t),$$

for each $x, t \in G_m$ and $n \in \mathbb{N}$. Thus each step function is a Vilenkin polynomial. Consequently, we obtain that the collection of step functions coincides with a collection of Vilenkin polynomials $\mathcal{P}$. Since the Lebesgue measure is regular it follows from the Lusin theorem that given $f \in L_1$ there exist Vilenkin polynomials $P_1, P_2...$, such that $P_n \to f$ a.e. when $n \to \infty$. This means that the Vilenkin polynomials are dense in the space L_1.

1.1.6 Basic notations concerning Walsh groups and functions

Let us define by Q_2 the set of rational numbers of the form $p2^{-n}$, where $0 \leq p \leq 2^n - 1$ for some $p \in \mathbb{N}$ and $n \in \mathbb{N}$.

Any $x \in [0,1]$ can be written in the form

$$x = \sum_{k=0}^{\infty} x_k 2^{-(k+1)},$$

where each $x_k = 0$ or 1. For each $x \in [0,1] \setminus Q_2$ there is only one expression of this form. We shall call it the dyadic expansion of x. When $x \in Q_2$ there are two expressions of this form, one which terminates in 0's and one which terminates in 1's. By the dyadic expansion of an $x \in Q_2$ we shall mean the one which terminates in 0's. Notice that $1 \nsim Q_2$ so the dyadic expansion of $x = 1$ terminates in 1's.

If $m_k = 2,$ for all $k \in \mathbb{N}$, we have dyadic group

$$G_2 = \prod_{j=0}^{\infty} Z_2,$$

which is called the Walsh group

Rademacher functions are defined by:

$$\rho_n(x) := (-1)^{x_n}.$$

We define Walsh functions w_n by

$$w_n := \prod_{k=0}^{\infty} \rho_k^{n_k}.$$

Let L^0 represent the collection of a.e. finite, Lebesgue measurable functions from G_2 into $[-\infty, \infty]$. For $0 < p < \infty$ let L^p represent the collection of $f \in L^0$ for which

$$\|f\|_p := \left(\int_{G_2} |f|^p\right)^{1/p}$$

is finite. Moreover, let L^∞ represent the collection of $f \in L^0$ for which

$$\|f\|_\infty := \inf\{y \in \mathbb{R} : |f(x)| \leq y \text{ for a.e. } \ x \in G_2\}$$

is finite. It is well known that L^p is a Banach space for each $1 \leq p \leq \infty$.

If $f \in L_1(G_2)$, then we can establish the Fourier coefficients, the partial sums of the Fourier series, the Fejér means, the Dirichlet and Fejér kernels with respect to the Walsh system w in the usual manner:

$$\begin{aligned}
\widehat{f}^w(k) \quad &: \quad = \int_{G_2} f\overline{\alpha}_k d\mu, \qquad (k \in \mathbb{N}),\\
S^w f \quad &: \quad = \sum_{k=0}^{n-1} \widehat{f}(k) w_k, \qquad (n \in \mathbb{N}_+,\ S_0^w f := 0),\\
D_n^w \quad &: \quad = \sum_{k=0}^{n-1} w_k, \qquad (n \in \mathbb{N}_+).
\end{aligned}$$

We state well-known equalities for Dirichlet kernels (for details see e.g. [61] and [108]):

$$D_{2^n}^w(x) = \begin{cases} 2^n, & \text{if } x \in I_n \\ 0, & \text{if } x \notin I_n \end{cases}$$

and

$$D_n^w = w_n \sum_{k=0}^{\infty} n_k r_k D_{2^k}^w = w_n \sum_{k=0}^{\infty} n_k \left(D_{2^{k+1}}^w - D_{2^k}^w\right), \quad \text{for} \quad n = \sum_{i=0}^{\infty} n_i 2^i.$$

Next we sketch the graph of some Dirichlet kernels on G_2:

The most properties of Lebesgue constants with respect to the Walsh-Paley system were obtained by Fine in [36]. Moreover, in [108], p. 34, the two-sided estimate

$$\frac{V(n)}{8} \leq L_n \leq V(n)$$

was proved, where $n = \sum_{j=1}^{\infty} n_j 2^j$ and $V(n)$ is defined by

$$V(n) := \sum_{j=1}^{\infty} |n_{j+1} - n_j| + n_0.$$

If $f \in L_1(G_2)$, then the Fejér means σ_n^w and Fejér kernels K_n^w with respect to the Walsh system w are, respectively, defined by

$$\begin{aligned} \sigma_n^w f \quad &: \quad = \frac{1}{n} \sum_{k=0}^{n-1} S_k^w f, \qquad (n \in \mathbb{N}_+), \\ K_n^w \quad &: \quad = \frac{1}{n} \sum_{k=0}^{n-1} D_k^w, \qquad (n \in \mathbb{N}_+). \end{aligned}$$

The n-th Nörlund logarithmic mean L_n^α and the Riesz logarithmic mean R_n^α with respect to the Walsh system ψ (Walsh system w) are defined by

$$L_n^w f \quad := \quad \frac{1}{l_n} \sum_{k=1}^{n-1} \frac{S_k^w f}{n-k}, \qquad (n \in \mathbb{N}_+), \qquad (n \in \mathbb{N}_+),$$

respectively, where

$$l_n := \sum_{k=1}^{n-1} \frac{1}{k}.$$

The kernels of the Nörlund logarithmic mean P_n^α and the Riesz logarithmic mean Y_n^α are, respectively, defined by

$$\begin{aligned} P_n^w f \quad &:= \quad \frac{1}{l_n} \sum_{k=1}^{n-1} \frac{D_k^w f}{n-k}, \qquad (n \in \mathbb{N}_+), \\ Y_n^w f \quad &:= \quad \frac{1}{l_n} \sum_{k=1}^{n-1} \frac{D_k^w f}{k} \qquad (n \in \mathbb{N}_+). \end{aligned}$$

1.1.7 On martingale Hardy spaces for $0 < p \leq 1$

The σ-algebra generated by the intervals

$$\{I_n(x) : x \in G_m\}$$

will be denoted by $F_n \, (n \in \mathbb{N})$.

A sequence $f = \left(f^{(n)} : n \in \mathbb{N}\right)$ of integrable functions $f^{(n)}$ is said to be a martingale with respect to the σ-algebras $F_n \, (n \in \mathbb{N})$ if (for details see e.g. Weisz [173] and Burkholder [31])

1) f_n is F_n measurable for all $n \in \mathbb{N}$,

2) $S_{M_n} f_m = f_n$ for all $n \leq m$.

The martingale $f = \left(f^{(n)}, n \in \mathbb{N}\right)$ is said to be L_p-bounded ($0 < p \leq \infty$) if $f^{(n)} \in L_p$ and

$$\|f\|_p := \sup_{n \in \mathbb{N}} \|f_n\|_p < \infty.$$

If $f \in L_1(G_m)$, then it is easy to show that the sequence $F = (S_{M_n} f : n \in \mathbb{N})$ is a martingale. This type of martingales is called regular. If $1 \leq p \leq \infty$ and $f \in L_p(G_m)$, then $f = \left(f^{(n)}, n \in \mathbb{N}\right)$ is L_p-bounded and

$$\lim_{n \to \infty} \|S_{M_n} f - f\|_p = 0$$

and consequently $\|F\|_p = \|f\|_p$ (see [90]). The converse of the latest statement holds also if $1 < p \leq \infty$ (see [90]): for an arbitrary L_p-bounded martingale $f = \left(f^{(n)}, n \in \mathbb{N}\right)$ there exists a function $f \in L_p(G_m)$ for which $f^{(n)} = S_{M_n} f$. If $p = 1$, then there exists a function $f \in L_1(G_m)$ of the preceding type if and only if f is uniformly integrable (see [90]), namely, if

$$\lim_{y \to \infty} \sup_{n \in \mathbb{N}} \int_{\{|f_n| > y\}} |f_n(x)| \, d\mu(x) = 0.$$

Thus the map $f \to f := (S_{M_n} f : n \in \mathbb{N})$ is isometric from L_p onto the space of L_p-bounded martingales when $1 < p \leq \infty$. Consequently, these two spaces can be identified with each other. Similarly, the space $L_1(G_m)$ can be identified with the space of uniformly integrable martingales.

Analogously, the martingale $f = \left(f^{(n)}, n \in \mathbb{N}\right)$ is said to be $weak - L_p$-bounded ($0 < p \leq \infty$) if $f^{(n)} \in L_p$ and

$$\|f\|_{weak-L_p} := \sup_{n \in \mathbb{N}} \|f_n\|_{weak-L_p} < \infty.$$

The maximal function f^* of a martingale f is defined by

$$f^* := \sup_{n \in \mathbb{N}} \left|f^{(n)}\right|.$$

In the case $f \in L_1(G_m)$, the maximal functions f^* are also given by

$$f^*(x) := \sup_{n \in \mathbb{N}} \frac{1}{|I_n(x)|} \left| \int_{I_n(x)} f(u) \, d\mu(u) \right|.$$

For $0 < p < \infty$ the Hardy martingale spaces H_p consist of all martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

Vilenkin-Fourier coefficients of the martingale $f = \left(f^{(n)} : n \in \mathbb{N}\right)$ must be defined in a slightly different manner:

$$\widehat{f}(i) := \lim_{k\to\infty} \int_{G_m} f^{(k)} \overline{\psi}_i d\mu.$$

Investigation of the classical Fourier analysis, definition of several variable Hardy spaces and real Hardy spaces and related theorems of atomic decompositions of these spaces can be found in Fefferman and Stein [34] (see also Later [73], Torchinsky [156], Wilson [175]).

A bounded measurable function a is a p-atom if there exist an interval I such that

$$\int_I a d\mu = 0, \quad \|a\|_\infty \le \mu(I)^{-1/p}, \quad \text{supp}(a) \subset I.$$

Explicit constructions of p-atoms can be found in the papers [18] and [19] by Blahota, Gát and Goginava.

Next, we note that the Hardy martingale spaces $H_p(G_m)$ for $0 < p \le 1$ have atomic characterizations:

The following useful lemma was proved by Weisz [171, 173] (see also Persson, Tephnadze and Weisz [105]):

Lemma 1.1.3. *A martingale $f = \left(f^{(n)} : n \in \mathbb{N}\right)$ is in $H_p\,(0 < p \le 1)$ if and only if there exist a sequence $(a_k, k \in \mathbb{N})$ of p-atoms and a sequence $(\mu_k : k \in \mathbb{N})$ of real numbers such that, for every $n \in \mathbb{N}$,*

$$\sum_{k=0}^{\infty} \mu_k S_{M_n} a_k = f^{(n)}, \quad \textit{a.e.,}$$

where

$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty.$$

Moreover,

$$\|f\|_{H_p} \backsim \inf \left(\sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p},$$

where the infimum is taken over all decomposition of $f = \left(f^{(n)} : n \in \mathbb{N}\right)$ of the form (1.1.3).

Explicit constructions of H_p martingales can be found in the papers [104], [105], [124], [125], [128], [131], [136], [138], [141], [142], [145], [149] and [150].

By using atomic characterization it can be easily proved that the following Lemmas hold:

Lemma 1.1.4. *Suppose that an operator T is sub-linear and for some $0 < p_0 \leq 1$*

$$\int_{\bar{I}} |Ta|^{p_0} \, d\mu \leq c_p < \infty$$

for every p_0-atom a, where I denotes the support of the atom. If T is bounded from L_{p_1} to L_{p_1}, $(1 < p_1 \leq \infty)$, then

$$\|Tf\|_{p_0} \leq c_{p_0} \|f\|_{H_{p_0}}.$$

Moreover, if $p_0 < 1$, then we have the weak (1,1) type estimate

$$\lambda\mu \{x \in G_m : \ |Tf(x)| > \lambda\} \leq \|f\|_1$$

for all $f \in L_1$.

A proof of Lemma 1.1.4 can be found in Weisz [171] (see also Persson, Tephnadze and Weisz [105]).

Lemma 1.1.5. *Suppose that an operator T is sub-linear and for some $0 < p_0 \leq 1$*

$$\sup_{\lambda > 0} \lambda^{p_0} \mu \left\{ x \in \bar{I} : |Tf| > \lambda \right\} \leq c_{p_0} < +\infty$$

for every p_0-atom a, where I denote the support of the atom. If T is bounded from L_{p_1} to L_{p_1}, $(1 < p_1 \leq \infty)$, then

$$\|Tf\|_{weak-L_{p_0}} \leq c_{p_0} \|f\|_{H_{p_0}}.$$

Moreover, if $p_0 < 1$, then

$$\lambda\mu \{x \in G_m : \ |Tf(x)| > \lambda\} \leq \|f\|_1,$$

for all $f \in L_1$.

The best approximation of $f \in L_p(G_m)$ $(1 \leq p \leq \infty)$ is defined as

$$E_n(f, L_p) := \inf_{\psi \in \mathbb{P}_n} \|f - \psi\|_p,$$

where $\mathcal{P}_n$ is set of all Vilenkin polynomials of order less than $n \in \mathbb{N}$.

The concept of modulus of continuity ω_{H_p} in martingale Hardy space H_p $(p > 0)$ is defined by

$$\omega_{H_p} \left(\frac{1}{M_n}, f \right) := \|f - S_{M_n} f\|_{H_p}.$$

We need to understand the meaning of the expression $f - S_{M_n} f$, where f is a martingale and $S_{M_n} f$ is function. Hence, we give an explanation in the following remark:

Remark 1.1.6. Let $0 < p \leq 1$. Since

$$S_{M_n} f = f^{(n)}, \text{ for } f = \left(f^{(n)} : n \in \mathbb{N}\right) \in H_p$$

and

$$\begin{aligned}\left(S_{M_k} f^{(n)} : k \in \mathbb{N}\right) &= \left(S_{M_k} S_{M_n} f, k \in \mathbb{N}\right) \\ &= \left(S_{M_0} f, \ldots, S_{M_{n-1}} f, S_{M_n} f, S_{M_n} f, \ldots\right) \\ &= \left(f^{(0)}, \ldots, f^{(n-1)}, f^{(n)}, f^{(n)}, \ldots\right),\end{aligned}$$

we obtain that

$$f - S_{M_n} f = \left(f^{(k)} - S_{M_k} f : k \in \mathbb{N}\right)$$

is a martingale, for which

$$\left(f - S_{M_n} f\right)^{(k)} = \begin{cases} 0, & k = 0, \ldots, n, \\ f^{(k)} - f^{(n)}, & k \geq n+1, \end{cases} \tag{1.6}$$

We also pronounce that Watari [167] showed that there are strong connections between the concepts

$$\omega_p\left(\frac{1}{M_n}, f\right), \quad E_{M_n}\left(L_p, f\right) \quad \text{and} \quad \left\|f - S_{M_n} f\right\|_p, \quad p \geq 1, \quad n \in \mathbb{N}.$$

In particular,

$$\frac{1}{2}\omega_p\left(\frac{1}{M_n}, f\right) \leq \left\|f - S_{M_n} f\right\|_p \leq \omega_p\left(\frac{1}{M_n}, f\right) \tag{1.7}$$

and

$$\frac{1}{2}\left\|f - S_{M_n} f\right\|_p \leq E_{M_n}\left(L_p, f\right) \leq \left\|f - S_{M_n} f\right\|_p.$$

The next lemma gives a deception what happens when $p > 1$. The proof can be found in Neveu [90] (see also Weisz [174]).

Lemma 1.1.7. *Let* $p > 1$. *Then*

$$H_p \sim L_p.$$

Remark 1.1.8. Since

$$\|f\|_{H_p} \sim \|f\|_p,$$

when $p > 1$, by applying (1.7), we obtain that

$$\omega_{H_p}\left(\frac{1}{M_n}, f\right) \sim \omega_p\left(\frac{1}{M_n}, f\right).$$

A proof of the next lemma can be found in [171] (see also book [108]).

Lemma 1.1.9. *If $f \in L_1$, then the sequence $F := (S_{M_n} f : n \in \mathbb{N})$ is a martingale and*

$$\|F\|_{H_p} \sim \left\| \sup_{n\in\mathbb{N}} |S_{M_n} f| \right\|_p .$$

Moreover, if $F := (S_{M_n} f : n \in \mathbb{N})$ is a regular martingale generated by $f \in L_1$, then

$$\widehat{F}(k) = \int_{G_m} f(x)\,\psi_k(x)\,d\mu(x) = \widehat{f}(k), \qquad k \in \mathbb{N}.$$

1.2 Some results on partial sums and classical summability methods of Vilenkin-Fourier series

In this part we have described a selected part of the area where the results in this PhD thesis belong to. We have also put these new results into this more general frame.

According to the Riemann-Lebesgue lemma (for details see e.g. the book [108]) we have that $\widehat{f}(k) \to 0,$ when $k \to \infty$, for each $f \in L_1$.

It is well-known (see e.g. the books [1] and [108]) that if $f \in L_1$ and the Vilenkin series $T(x) = \sum_{j=0}^{\infty} c_j \psi_j(x)$ convergences to f in L_1-norm, then $c_j = \int_{G_m} f \overline{\psi}_j d\mu := \widehat{f}(j)$, i.e. in this case the approximation series must be a Vilenkin-Fourier series. An analogous result is true also if the Vilenkin series convergences uniformly on G_m to an integrable function f .

By using the Lebesgue constants we easily obtain that $S_{n_k} f$ convergence to f in L_1-norm, for every integrable function f, if and only if $\sup_k L_{n_k} \leq c < \infty$. There are various results when $p > 1$.

It is also well-known that (see e.g. [106] and the books [105] and [108])

$$\|S_n f\|_p \leq c_p \|f\|_p, \text{ when } p > 1,$$

but it can be proved also a more stronger result (see e.g. [106] and the books [105] and [108]):

$$\|S^* f\|_p \leq c_p \|f\|_p, \text{ when } f \in L_p, \ p > 1.$$

Moreover, in the case $p = 1$ Watari [168] (see also Gosselin [62] and Young [180]) proved that there exists an absolute constant c such that, for $n = 1, 2, ...,$

$$\lambda\mu(|S_n f| > \lambda) \quad \leq \quad c\|f\|_1, \quad f \in L_1(G_m), \ \lambda > 0.$$

Uniform and point-wise convergence and some approximation properties of the partial sums with respect to the Vilenkin (Walsh) and trigonometric systems in L_1 norms were investigated by Antonov [7], Avdispahić and Memić [9], [11], Goginava [48, 49], Shneider [109], Sjölin [117], Onneweer and Waterman [94, 95]. Fine [36] derived sufficient conditions for the uniform convergence, which are in complete analogy with the Dini-Lipschitz conditions. Gulićev [63] estimated the rate of uniform convergence of a Walsh-Fourier series by using Lebesgue constants and modulus of continuity. Uniform convergence of subsequences of partial sums was investigated also in Goginava and Tkebuchava [53], Fridli [37] and Gát [41]. Approximation properties of the two-dimensional partial sums with respect to Vilenkin and trigonometric systems can be found [108] and [186].

Results on a.e. convergence of Vilenkin-Fourier series were proved in [106]. Important information concerning divergence of Vilenkin-Fourier series on the sets of measure zero and a.e. convergence can be found in Bitsadze [14, 15], Bugadze [29, 30], Fejér [35] Gosselin, [62] Kahane [66], Katznelson [70], Karagulian [67, 68], Kheladze [71, 72], Lebesgue [74] Stechkin [119] Young [178, 179, 181] and Zhizhiashvili [183].

Some estimates of Fourier coefficients and absolute convergence and divergence of Fourier Series with respect to complete orthonormal systems were studied in Bochkarev [28], Gogoladze and Tsagareishvili [55, 56, 57, 91], Kashin and Saakyan [69], Oniani [92, 93], Tsagareishvili and Tutberidze [157, 158, 159], Tetunashvili [151, 152, 153], Tevzadze [154], Tkebuchava [155] and Zhizhiashvili [182, 183, 184, 185] . Approximation of functions on locally compact Abelian groups was investigate by Ugulava [165, 166] (see also [32]).

Since $H_1 \subset L_1$, according to Riemann-Lebesgue theorem, it yields that $\widehat{f}(k) \to 0$ when $k \to \infty$, for every $f \in H_1$. The classical inequality of Hardy type is well known in the trigonometric as well as in the Vilenkin-Fourier analysis and was proved in the trigonometric case by Hardy and Littlewood [64] (see also the book [33]) and for the Walsh system it was proved in the book [108]. Some inequalities relative to Vilenkin-Fourier coefficients were considered in [97], [110], [114], [115], [122], [169], [172] and [173].

It is known (for details see e.g. the books [108] and [173]) that the subsequence S_{M_n} of the partial sums is bounded from the martingale Hardy space H_p to the Lebesgue space L_p, for all $p > 0$. However, (see Tephnadze [139]) there exists a martingale $f \in H_p$ $(0 < p < 1)$, such that

$$\sup_{n\in\mathbb{N}} \|S_{M_n+1}f\|_{weak-L_p} = \infty.$$

The reason of the divergence of $S_{M_n+1}f$ is that when $0 < p < 1$ the Fourier coefficients of $f \in H_p$ are not uniformly bounded (see Tephnadze [123]). On the other hand, there exists an absolute constant c_p, depending only on p, such that

$$\|S_{M_n}f\|_p \leq c_p \|f\|_{H_p}, \ \ p > 0, \ \ n \in \mathbb{N}_+.$$

Tephnadze [139] (see also [126] and [130]) proved that for every $0 < p < 1$, the maximal operator

$$\widetilde{S}_p^* f := \sup_{n\in\mathbb{N}} \frac{|S_n f|}{(n+1)^{1/p-1}}$$

is bounded from the Hardy space H_p to the Lebesgue space L_p. Moreover, the rate of the sequence $(n+1)^{1/p-1}$ is in the sense sharp.

It follow that for any $0<p<1$ and $f\in H_p$, there exists an absolute constant c_p, depending only on p, such that

$$\left\Vert S_nf\right\Vert_p\leq c_p\left(n+1\right)^{1/p-1}\left\Vert f\right\Vert_{H_p},\ \ n\in\mathbb{N}_+.$$

Blahota, Persson, Nagy and Tephnadze ([147]) proved that for any $0<p\leq 1$ and a sub-sequence of positive numbers $\{\alpha_k:k\in\mathbb{N}\}$, satisfying the condition

$$\sup_{k\in\mathbb{N}}\rho\left(\alpha_k\right)=\varkappa<\infty, \tag{1.8}$$

the maximal operator $\widetilde{S}^{*,\vartriangle}f:=\sup_{k\in\mathbb{N}}\left\vert S_{\alpha_k}f\right\vert$ is bounded from the Hardy space H_p to the space L_p. Moreover, for every $0<p<1$ and any sub-sequence of positive numbers $\{\alpha_k:k\in\mathbb{N}\}$ satisfying the condition

$$\sup_{k\in\mathbb{N}}\rho\left(\alpha_k\right)=\infty, \tag{1.9}$$

there exists a martingale $f\in H_p,\ (0<p<1)$ such that $\sup_{k\in\mathbb{N}}\left\Vert S_{\alpha_k}f\right\Vert_{weak-L_p}=\infty$.

It follows that for any $p>0$ and $f\in H_p$, the maximal operator $\widetilde{S}_{\#}^{*}$ defined by

$$\widetilde{S}_{\#}^{*}f:=\sup_{n\in\mathbb{N}}\left\vert S_{M_n}f\right\vert$$

is bounded from the Hardy space H_p to the space L_p. We also obtain that if $p>0$ and $f\in H_p$, then the maximal operator defined by

$$\sup_{n\in\mathbb{N}_+}\left\vert S_{M_n+1}f\right\vert$$

is not bounded from the Hardy space H_p to the space L_p.

It is well-known that (for details see [130])

$$\left\Vert S_{M_n}f-f\right\Vert_{H_p}\to 0,\quad f\in H_p\quad\left(p>0\right).$$

Tephnadze [139] (see also [126] and [130]) proved that for any $0<p<1$ and $f\in H_p$ there exists an absolute constant c_p depending only on p such that

$$\left\Vert S_nf\right\Vert_{H_p}\leq c_pn^{1/p-1}\left\Vert f\right\Vert_{H_p}.$$

In the some paper [139] Tephnadze proved that for any $0<p<1,\ f\in H_p$ and $M_k<n\leq M_{k+1}$ there is an absolute constant c_p depending only on p such that

$$\left\Vert S_nf-f\right\Vert_{H_p}\leq c_pn^{1/p-1}\omega_{H_p}\left(\frac{1}{M_k},f\right).$$

From this estimate it immediately follows that if $0<p<1,\ f\in H_p$ and

$$\omega_{H_p}\left(\frac{1}{M_n},f\right)=o\left(\frac{1}{M_n^{1/p-1}}\right),\ \text{when } n\to\infty,$$

then

$$\|S_k f - f\|_{H_p} \to 0, \text{ as } \quad k \to \infty.$$

Moreover, for every $0 < p < 1$ there exists a martingale $f \in H_p$, for which

$$\omega_{H_p}\left(\frac{1}{M_n}, f\right) = O\left(\frac{1}{M_n^{1/p-1}}\right), \text{ when } \quad n \to \infty$$

and

$$\|S_k f - f\|_{weak-L_p} \nrightarrow 0, \text{ when } k \to \infty.$$

Tephadze [140] proved that for any $0 < p < 1$ and $f \in H_p$ there exists an absolute constant c_p, depending only on p, such that

$$\|S_n f\|_{H_p} \leq \frac{c_p M_{|n|}^{1/p-1}}{M_{\langle n \rangle}^{1/p-1}} \|f\|_{H_p}.$$

Moreover, for every $0 < p < 1$ and any increasing sequence of nonnegative integers $\{n_k : k \in \mathbb{N}\}$ such that condition (1.9) is satisfied and for any non-decreasing sequence $\{\Phi_n : n \in \mathbb{N}\}$, satisfying the condition

$$\overline{\lim_{k\to\infty}} \frac{M_{|n_k|}^{1/p-1}}{M_{\langle n_k \rangle}^{1/p-1} \Phi_{n_k}} = \infty,$$

there exists a martingale $f \in H_p$, such that

$$\sup_{k\in\mathbb{N}} \left\|\frac{S_{n_k} f}{\Phi_{n_k}}\right\|_{L_{p,\infty}} = \infty.$$

Moreover, if $0 < p < 1$, $f \in H_p$ and $\{n_k : k \in \mathbb{N}\}$ is an increasing sequence of nonnegative integers, then $\|S_{n_k} f\|_{H_p} \leq c_p \|f\|_{H_p}$ holds true if and only if condition (1.8) is satisfied.

In [132] (see also [140]) it was proved that if $0 < p < 1$, $f \in H_p$ and $M_k < n \leq M_{k+1}$, then there exists an absolute constant c_p, depending only on p, such that

$$\|S_n f - f\|_{H_p} \leq \frac{c_p M_{|n|}^{1/p-1}}{M_{\langle n \rangle}^{1/p-1}} \omega_{H_p}\left(\frac{1}{M_k}, f\right), \quad (0 < p < 1).$$

It follows that if $\{n_k : k \in \mathbb{N}\}$ is an increasing sequence of nonnegative integers such that

$$\omega_{H_p}\left(\frac{1}{M_{|n_k|}}, f\right) = o\left(\frac{M_{\langle n_k \rangle}^{1/p-1}}{M_{|n_k|}^{1/p-1}}\right), \text{ as } k \to \infty,$$

then $\|S_{n_k} f - f\|_{H_p} \to 0$, as $k \to \infty$. Moreover, if $\{n_k : k \in \mathbb{N}\}$ is an increasing sequence of nonnegative integers such that condition (1.9) is satisfied, then there exists a martingale $f \in H_p$ and a subsequence $\{\alpha_k : k \in \mathbb{N}\} \subset \{n_k : k \in \mathbb{N}\}$, for which

$$\omega_{H_p}\left(\frac{1}{M_{|\alpha_k|}}, f\right) = O\left(\frac{M_{\langle\alpha_k\rangle}^{1/p-1}}{M_{|\alpha_k|}^{1/p-1}}\right), \text{ as } k \to \infty$$

and $\limsup_{k\to\infty} \|S_{\alpha_k} f - f\|_{weak-L_p} > c > 0, \text{ as } k \to \infty.$

In Tephnadze [139] (see also [130]) it was proved that for every $f \in H_1$, then the maximal operator, defined by

$$\widetilde{S}^* f := \sup_{n\in\mathbb{N}_+} \frac{|S_n f|}{\log(n+1)},$$

is bounded from the Hardy space H_1 to the space L_1. Moreover, the rate of the sequence $\log(n+1)$ is in the sense sharp. Hence, for any $f \in H_1$, there exists an absolute constant c, such that

$$\|S_n f\|_1 \le c \log(n+1) \|f\|_{H_1}, \ \ n \in \mathbb{N}_+.$$

From this estimate it immediately follows that if $f \in H_1$ and $M_k < n \le M_{k+1}$, then there is an absolute constant c such that

$$\|S_n f - f\|_{H_1} \le c \lg n \omega_{H_1}\left(\frac{1}{M_k}, f\right).$$

By using this estimate we obtain that if $f \in H_1$ and

$$\omega_{H_1}\left(\frac{1}{M_n}, f\right) = o\left(\frac{1}{n}\right), \text{ when } n \to \infty,$$

then $\|S_k f - f\|_{H_1} \to 0, \text{ when } k \to \infty$. Moreover (for details see [139]), there exists a martingale $f \in H_1$ for which

$$\omega_{H_1}\left(\frac{1}{M_{2M_n}}, f\right) = O\left(\frac{1}{M_n}\right), \quad \text{when } n \to \infty$$

and $\|S_k f - f\|_1 \nrightarrow 0, \text{ when } k \to \infty.$

In [132] (see also [140]) it was proved that if $f \in H_1$ and $M_k < n \le M_{k+1}$, then there exists an absolute constant c such that

$$\|S_n f\|_{H_1} \le c\left(v(n) + v^*(n)\right) \|f\|_{H_1}.$$

Moreover, if $\{\Phi_n : n \in \mathbb{N}\}$ is any non-decreasing and non-negative sequence satisfying $\lim_{n\to\infty} \Phi_n = \infty$ and $\{n_k \ge 2 : k \in \mathbb{N}\}$ is a subsequence such that

$$\lim_{k\to\infty} \frac{v(n_k) + v^*(n_k)}{\Phi_{n_k}} = \infty,$$

then there exists a martingale $f \in H_1$ such that

$$\sup_{k\in\mathbb{N}}\left\|\frac{S_{n_k}f}{\Phi_{n_k}}\right\|_1 \to \infty, \text{ as } k\to\infty.$$

In [132] (see also [140]) it was also proved that if $f \in H_1$ and $M_k < n \leq M_{k+1}$, then there exists an absolute constant c such that

$$\|S_n f - f\|_{H_1} \leq c\left(v\left(n\right)+v^*\left(n\right)\right)\omega_{H_1}\left(\frac{1}{M_k}, f\right).$$

It follows that if $f \in H_1$ and $\{n_k : k \in \mathbb{N}\}$ is a sequence of non-negative integers such that

$$\omega_{H_1}\left(\frac{1}{M_{|n_k|}}, f\right) = o\left(\frac{1}{v\left(n_k\right)+v^*\left(n_k\right)}\right), \text{ as } k\to\infty,$$

then $\|S_{n_k}f - f\|_{H_1} \to 0,$ when $k \to \infty$. Moreover, if $\{n_k : k \geq 1\}$ is a sequence of non-negative integers such that $\sup_{k\in\mathbb{N}}\left(v\left(n_k\right)+v^*\left(n_k\right)\right) = \infty$, then there exists a martingale $f \in H_1$ and a sequence $\{\alpha_k : k \in \mathbb{N}\} \subset \{n_k :\in \mathbb{N}\}$ for which

$$\omega_{H_1}\left(\frac{1}{M_{|\alpha_k|}}, f\right) = O\left(\frac{1}{v\left(\alpha_k\right)+v^*\left(\alpha_k\right)}\right)$$

and $\limsup_{k\to\infty}\|S_{\alpha_k}f - f\|_1 > c > 0$ when $k \to \infty$.

Simon [111] proved that for any $f \in H_p$, there exists an absolute constant c_p, depending only on p, such that

$$\sum_{k=1}^{\infty}\frac{\|S_k f\|_p^p}{k^{2-p}} \leq c_p\|f\|_{H_p}^p, \quad (0<p<1).$$

In Tephnadze [122]) it was proved sharpness of this result in a special sense. In particular, if $0 < p < 1$ and $\{\Phi_n : n \in \mathbb{N}\}$ is any non-decreasing sequence satisfying the condition $\overline{\lim_{n\to\infty}}\Phi_n = +\infty$, there exists a martingale $f \in H_p$ such that

$$\sum_{k=1}^{\infty}\frac{\|S_k f\|_{weak-L_p}^p\Phi_k}{k^{2-p}} = \infty.$$

In Gát [43] the following strong convergence result was obtained for all $f \in H_1$:

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{\|S_k f - f\|_1}{k} = 0$$

For the trigonometric analogue see Smith [118] (see also [54]) and for the Walsh-Paley system see Simon [113], for Vilenkin-like systems see Blahota [16] and for the two-dimentional diagonal partial sums by Goginava and Gogoladze [51]. Moreover, for all $f \in H_1$, there exists an absolute constant $c,$ such that

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{\|S_k f\|_1}{k} \leq c\,\|f\|_{H_1} \quad \text{and} \quad \lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{\|S_k f\|_1}{k} = \|f\|_{H_1} \quad (n=2,3...)\,.$$

In [160] (see paper A) was investigated some new Hardy type inequalities for partial sums of Vilenkin-Fourier series.

In the one-dimensional case Yano [177] proved that

$$\|K_n\| \leq 2 \quad (n \in \mathbb{N}).$$

Consequently,

$$\|\sigma_n f - f\|_p \to 0, \quad \text{when} \quad n \to \infty, \ \ (f \in L_p, \ \ 1 \leq p \leq \infty).$$

However (see [65] and [108]) the rate of convergence can not be better then $O\left(n^{-1}\right)$ $(n \to \infty)$ for non-constant functions. a.e, if $f \in L_p, 1 \leq p \leq \infty$ and

$$\|\sigma_{M_n} f - f\|_p = o\left(\frac{1}{M_n}\right), \ \text{when} \ \ n \to \infty,$$

then f is a constant function.

Fridli [38] used dyadic modulus of continuity to characterize the set of functions in the space L_p, whose Vilenkin-Fejér means converge at a given rate. It is also known that (see e.g books [1] and [108])

$$\|\sigma_n f - f\|_p$$

$$\leq c_p \omega_p\left(\frac{1}{M_N}, f\right) + c_p\sum_{s=0}^{N-1}\frac{M_s}{M_N}\omega_p\left(\frac{1}{M_s}, f\right), \quad \left(1 \leq p \leq \infty, \ \ n \in \mathbb{N}\right).$$

By applying this estimate we immediately obtain that if $f \in lip\left(\alpha, p\right)$, i.e.,

$$\omega_p\left(\frac{1}{M_n}, f\right) = O\left(\frac{1}{M_n^{\alpha}}\right), \quad n \to \infty,$$

then

$$\|\sigma_n f - f\|_p = \begin{cases} O\left(\frac{1}{M_N}\right), & \text{if } \alpha > 1, \\ O\left(\frac{N}{M_N}\right), & \text{if } \alpha = 1, \\ O\left(\frac{1}{M_N^{\alpha}}\right), & \text{if } \alpha < 1. \end{cases}$$

On the other hand, if $1 \leq p \leq \infty, \ f \in L_p$ and

$$\|\sigma_{M_n} f - f\|_p = o\left(1/M_n\right), \ \text{as } n \to \infty,$$

then f is a constant function.

Weisz [170] considered the norm convergence of Fejér means of Vilenkin-Fourier series and proved that

$$\|\sigma_k f\|_p \leq c_p \|f\|_{H_p}, \quad p > 1/2 \quad \text{and} \quad f \in H_p.$$

This result implies that

$$\frac{1}{n^{2p-1}} \sum_{k=1}^{n} \frac{\|\sigma_k f\|_p^p}{k^{2-2p}} \leq c_p \|f\|_{H_p}^p, \quad (1/2 < p < \infty). \tag{1.10}$$

If (1.10) holds for $0 < p \leq 1/2$, then we have that

$$\frac{1}{\log^{[1/2+p]} n} \sum_{k=1}^{n} \frac{\|\sigma_k f\|_p^p}{k^{2-2p}} \leq c_p \|f\|_{H_p}^p, \quad (0 < p \leq 1/2). \tag{1.11}$$

Furthermore, in Tephnadze [129] it was shown that the assumption $p > 1/2$ in (1.10) is essential. In particular, is was proved that there exists a martingale $f \in H_{1/2}$ such that

$$\sup_{n \in \mathbb{N}} \|\sigma_n f\|_{1/2} = +\infty.$$

For Vilenkin systems in [143] it was proved that (1.11) holds, though inequality (1.10) is not true for $0 < p \leq 1/2$.

Some new strong convergence result for Fejer means was considered in [58] and [164].

In the one-dimensional case the weak type inequality

$$\mu(\sigma^* f > \lambda) \leq \frac{c}{\lambda} \|f\|_1, \qquad (f \in L_1, \quad \lambda > 0)$$

can be found in Zygmund [186] for the trigonometric series, in Schipp [107] for Walsh series and in Pál, Simon [96] for bounded Vilenkin series. Fujji [40] and Simon [112] verified that σ^* is bounded from H_1 to L_1. Weisz [170] generalized this result and proved the boundedness of σ^* from the martingale space H_p to the Lebesgue space L_p for $p > 1/2$. Simon [111] gave a counterexample, which shows that boundedness does not hold for $0 < p < 1/2$. A corresponding counterexample for $p = 1/2$ is due to Goginava [20] (see also [18] and [19]). In [129] Tephnadze proved that there exist a martingale $f \in H_{1/2}$ such that

$$\sup_{n \in \mathbb{N}} \|\sigma_n f\|_{1/2} = +\infty.$$

Moreover, there exists a martingale $f \in H_p$, for $0 < p < 1/2$, such that

$$\sup_{n \in \mathbb{N}} \|\sigma_n f\|_{weak-L_p} = +\infty.$$

It follows that there exist a martingale $f \in H_{1/2}$ such that

$$\|\sigma^* f\|_{1/2} = +\infty.$$

Furthermore, there exists a martingale $f \in H_p$ for $0 < p < 1/2,$ such that

$$\|\sigma^* f\|_{weak-L_p} = +\infty.$$

Weisz [172] proved that σ^* is bounded from the Hardy space $H_{1/2}$ to the space $weak - L_{1/2}$. In [135] it was proved that the maximal operator $\widetilde{\sigma}_p^*$ with respect to Vilenkin systems defined by

$$\widetilde{\sigma}_p^* := \sup_{n\in\mathbb{N}} \frac{|\sigma_n|}{(n+1)^{1/p-2}},$$

where $0 < p < 1/2$, is bounded from the Hardy space H_p to the Lebesgue space L_p. Moreover, the order of deviant behavior of the n-th Fejér mean was given exactly. That is, for any non-decreasing sequence $\{\Phi_n : n \in \mathbb{N}\}$ satisfying the condition

$$\overline{\lim_{n\to\infty}} \frac{(n+1)^{1/p-2}}{\Phi_n} = +\infty,$$

we have that

$$\sup_{k\in\mathbb{N}} \frac{\left\| \frac{\sigma_{M_{2n_k}+1} f_k}{\Phi_{M_{2n_k}+1}} \right\|_{weak-L_p}}{\|f_k\|_{H_p}} = \infty.$$

As a consequence of this we immediately get that

$$\|\sigma_n f\|_p \le c_p (n+1)^{1/p-2} (n+1) \|f\|_{H_p},$$

but also a stronger result is known (for details see e.g. [130]). In particular, if $0 < p < 1/2$ and $f \in H_p$, there exists an absolute constant c_p, depending only on p, such that

$$\|\sigma_n f\|_{H_p} \le c_p n^{1/p-2} \|f\|_{H_p}.$$

In [134] (for Walsh system see [46]) it was proved that the maximal operator $\widetilde{\sigma}^*$ with respect to Vilenkin systems, defined by

$$\widetilde{\sigma}^* := \sup_{n\in\mathbb{N}} \frac{|\sigma_n|}{\log^2(n+1)},$$

is bounded from the Hardy space $H_{1/2}$ to the Lebesgue space $L_{1/2}$.

Moreover, for any non-decreasing sequence $\{\Phi_n : n \in \mathbb{N}\}$ satisfying the condition

$$\overline{\lim_{n\to\infty}} \frac{\log^2(n+1)}{\Phi_n} = +\infty,$$

we have that

$$\sup_{k\in\mathbb{N}} \frac{\left\| \frac{\sigma_{q_{n_k}} f_k}{\Phi_{q_{n_k}}} \right\|_{1/2}}{\|f_k\|_{H_{1/2}}} = \infty.$$

It follows from this that

$$\|\sigma_n f\|_{1/2} \leq c \log^2 (n+1) \|f\|_{H_{1/2}}.$$

but an even stronger result is known (for details see e.g. [130]). In particular, if $f \in H_{1/2}$, then there exists an absolute constant c, such that

$$\|\sigma_n f\|_{H_{1/2}} \leq c \log^2 (n+1) \|f\|_{H_{1/2}}.$$

Some analogical theorems for the Walsh-Kaczmarz system were proved in [52] and [122].

For the one-dimensional Vilenkin-Fourier series Weisz [170] proved that the maximal operator $\sigma^{\#}$, defined by

$$\sigma^{\#} f = \sup_{n \in \mathbb{N}} |\sigma_{M_n} f|,$$

is bounded from the martingale Hardy space H_p to the Lebesgue space L_p for $p > 0$. He also proved that

$$\|\sigma_{M_n} f - f\|_{H_p} \to 0, \quad f \in H_p \quad (p > 0).$$

On the other hand, the operator $|\sigma_{M_n} f|$ is not bounded from the space H_p to the space H_p, for $0 < p \leq 1$. This result for the Walsh system can be found in Goginava [49] and for bounded Vilenkin systems in Persson and Tephnadze [98].

Approximation properties of subsequences of Fejér means with respect to the one-dimensional Walsh-Fourier series was considered in Persson, Tephnadze and Tutberidze [100] (see paper B) and Tutberidze [162] (see paper C).

Tephnadze [99] proved that if $0 < p \leq 1/2$ and $\{\alpha_k : k \in \mathbb{N}\}$ is a subsequence of positive numbers such that

$$\sup_{k \in \mathbb{N}} \rho(\alpha_k) = \varkappa < c < \infty,$$

then the maximal operator $\widetilde{\sigma}^{*,\triangle}$, defined by

$$\widetilde{\sigma}^{*,\triangle} f := \sup_{k \in \mathbb{N}} |\sigma_{\alpha_k} f|,$$

is bounded from the Hardy space H_p to the Lebesgue space L_p.

Moreover, if $0 < p \leq 1/2$ and $\{\alpha_k : k \in \mathbb{N}\}$ is a subsequence of positive numbers satisfying the condition

$$\sup_{k \in \mathbb{N}} \rho(\alpha_k) = \infty,$$

then there exists a martingale $f \in H_p$ such that

$$\sup_{k \in \mathbb{N}} \|\sigma_{\alpha_k} f\|_{weak-L_p} = \infty, \quad (0 < p < 1/2).$$

It immediately follows that for $0 < p \leq 1/2$, and $f \in H_p$, there exists an absolute constant c_p, depending only on p, such that

$$\left\|\sigma_{n_k} f\right\|_p \leq c_p \left\|f\right\|_{H_p}, \quad k \in \mathbb{N},$$

if and only if

$$\sup_{k\in\mathbb{N}} \rho\left(n_k\right) < c < \infty.$$

As a consequence, if $p > 0$ and $f \in H_p$, then there exists an absolute constant c_p, depending only on p, such that

$$\left\|\sigma_{M_n} f\right\|_p \leq c_p \left\|f\right\|_{H_p}, \quad (p > 0).$$

In [123, 127] it was proved that if $0 < p < 1/2$, $f \in H_p$ and

$$\omega_p\left(\frac{1}{M_n}, f\right) = o\left(\frac{1}{M_n^{1/p-2}}\right) \text{when } n \to \infty,$$

then

$$\left\|\sigma_n f - f\right\|_{H_p} \to 0, \text{ when } n \to \infty.$$

Moreover, there exists a martingale $f \in H_p$ $(0 < p < 1/2)$ for which

$$\omega\left(\frac{1}{M_n}, f\right)_{H_p} = O\left(\frac{1}{M_n^{1/p-2}}\right) \text{ when } n \to \infty$$

and

$$\left\|\sigma_n f - f\right\|_{weak-L_p} \nrightarrow 0, \text{ when } n \to \infty.$$

When $p = 1/2$ we have the following results: If $f \in H_{1/2}$ and

$$\omega_{H_{1/2}}\left(\frac{1}{M_n}, f\right) = o\left(\frac{1}{n^2}\right), \text{ when } n \to \infty,$$

then

$$\left\|\sigma_n f - f\right\|_{H_{1/2}} \to 0, \text{ when } n \to \infty.$$

Moreover, there exists a martingale $f \in H_{1/2}$ for which

$$\omega_{H_{1/2}}\left(\frac{1}{M_n}, f\right) = O\left(\frac{1}{n^2}\right), \text{ when } n \to \infty$$

and

$$\left\|\sigma_n f - f\right\|_{1/2} \nrightarrow 0, \text{ when } n \to \infty.$$

We state some consequences of this result investigated in [123] for the Walsh system to clearly see the difference of divergence rates for various subsequences: Let $0 < p < 1/2$, $f \in H_p$. Then there exists an absolute constant c_p, depending only on p, such that

$$\left\|\sigma_{M_n+1} f\right\|_{H_p} \leq c_p M_n^{1/p-2} \left\|f\right\|_{H_p}, \quad n \in \mathbb{N} \tag{1.12}$$

and

$$\left\|\sigma_{M_n+M_{[n/2]}}f\right\|_{H_p} \leq c_p M_n^{1/2p-1}\|f\|_{H_p}, \quad n\in\mathbb{N}. \tag{1.13}$$

Moreover, the rates $M_n^{1/p-2}$ and $M_n^{1/2p-1}$ in inequalities (1.12) and (1.13) are sharp in the same sense.

Blahota and Tephnadze [26] proved that if $0<p<1/2$ and $f\in H_p$, then there exists an absolute constant c_p, depending only on p, such that

$$\sum_{k=1}^{\infty}\frac{\|\sigma_k f\|_p^p}{k^{2-2p}} \leq c_p\|f\|_{H_p}^p,$$

Moreover, if $0<p<1/2$ *and* $\{\Phi_k : k\in\mathbb{N}\}$ be any non-decreasing sequence satisfying the conditions $\Phi_n\uparrow\infty$ and

$$\overline{\lim_{k\to\infty}}\frac{k^{2-2p}}{\Phi_k}=\infty,$$

then there exists a martingale $f\in H_p$ such that

$$\sum_{k=1}^{\infty}\frac{\|\sigma_k f\|_{weak-L_p}^p}{\Phi_k}=\infty.$$

As a corollary we also get that if $0<p<1/2$ and $f\in H_p$, then there exists an absolute constant c_p, depending only on p, such that

$$\sum_{k=1}^{\infty}\frac{\|\sigma_k f\|_{H_p}^p}{k^{2-2p}} \leq c_p\|f\|_{H_p}^p,$$

$$\frac{1}{n}\sum_{k=1}^{n}\frac{\|\sigma_k f\|_{H_p}^p}{k^{1-2p}} \leq c_p\|f\|_{H_p}^p,$$

$$\frac{1}{n}\sum_{k=1}^{n}\frac{\|\sigma_k f-f\|_{H_p}^p}{k^{1-2p}}=0,$$

and

$$\frac{1}{n}\sum_{k=1}^{n}\frac{\|\sigma_k f\|_{H_p}^p}{k^{1-2p}}=\|f\|_{H_p}^p.$$

Blahota and Tephnadze [26] also considered the endpoint case $p=1/2$ and they proved that if $f\in H_{1/2}$, then there exists an absolute constant c such that

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{\|\sigma_k f\|_{1/2}^{1/2}}{k} \leq c\|f\|_{H_{1/2}}^{1/2}.$$

It follows from this that if $f\in H_{1/2}$, then

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{\|\sigma_k f\|_{H_{1/2}}^{1/2}}{k} \leq c\|f\|_{H_{1/2}}^{1/2},$$

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{\|\sigma_k f-f\|_{H_{1/2}}^{1/2}}{k}=0$$

and

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{\|\sigma_k f\|_{H_{1/2}}^{1/2}}{k}=\|f\|_{H_{1/2}}^{1/2}.$$

Approximation properties and strong convergence results of Marcinkiewicz-Fejer means with respect to Walsh and Kaczmarz systems were studied by Nagy and Tephnadze [85, 86, 87, 88, 89].

It is well-known that the so-called T means are generalizations of the Fejér, Reisz and logarithmic means. The T summation is a general summability method. Therefore it is of prior interest to study the behavior of operators related to Nörlund means of Fourier series with respect to orthonormal systems.

Since T means are inverse of Nörlund means we first state some interesting results concerning Nörlund summability, which has high influence on the new results for T means of Vilenkin-Fourier series.

In [47] Goginava investigated the behavior of Cesàro means of Walsh-Fourier series in detail. In the one-dimentional case approximation properties of Cesàro means was studied by Akhonadze [2, 3, 4, 5] (see also [6]) and two-dimensional case approximation properties of Nörlund and Cesàro means were considered by Nagy (see [82], [84] and [83]). The maximal operator $\sigma^{\alpha,*}$ $(0<\alpha<1)$ of the (C,α) means of Vilenkin systems was investigated by Weisz [169]. In this paper Weisz proved that $\sigma^{\alpha,*}$ is bounded from the martingale space H_p to the Lebesgue space L_p for $p>1/(1+\alpha)$. Goginava [50] gave a counterexample which shows that boundedness does not hold for $0<p\leq 1/(1+\alpha)$. Weisz and Simon [116] showed that the maximal operator $\sigma^{\alpha,*}$ is bounded from the Hardy space $H_{1/(1+\alpha)}$ to the space $weak-L_{1/(1+\alpha)}$.

Strong convergence theorems and boundedness of weighted maximal operators of the (C,α) means of Vilenkin systems on the Hardy spaces when $0<p\leq 1/(1+\alpha)$ were considered by Blahota and Tephnadze [25] and Blahota, Tephnadze and Toledo [27]. Summability of some general methods were considered by Blahota, Nagy and Tephnadze [21].

In [101] (see also [137]) the maximal operator of the Nörlund summation method (see (1.5)) was investigated. In particular, it was proved that the maximal operator t^* of the summability method (1.5) with non-decreasing sequence $\{q_k : k\in\mathbb{N}\}$ is bounded from the Hardy space $H_{1/2}$ to the space $weak-L_{1/2}$.

Moreover, for any $0<p<1/2$ and non-decreasing sequence $\{q_k : k\in\mathbb{N}\}$ satisfying the condition

$$\frac{q_0}{Q_n}\geq\frac{c}{n},\quad (c>0),$$

there exists a martingale $f\in H_p$, such that

$$\sup_{n\in\mathbb{N}}\|t_n f\|_{weak-L_p}=\infty.$$

In [102] it was proved that if $0 < p < 1/2$ and the sequence $\{q_k : k \in \mathbb{N}\}$ is non-decreasing, then the maximal operator $\widetilde{t}_{p,1}^{*}$, defined by

$$\widetilde{t}_{p,1}^{*} f := \sup_{n\in\mathbb{N}} \frac{|t_n f|}{(n+1)^{1/p-2}},$$

is bounded from the Hardy martingale space H_p to the Lebesgue space L_p.

Moreover, according to the fact that the Fejér means are examples of Nörlund means with non-decreasing sequence $\{q_k : k \in \mathbb{N}\}$ we immediately obtain that the asymptotic behaviour of the sequence of weights

$$\left\{1/(k+1)^{1/p-2} : k \in \mathbb{N}\right\}$$

in Nörlund means can not be improved.

Let the sequence $\{q_k : k \in \mathbb{N}\}$ be non-decreasing. Then the maximal operator $\widetilde{t}_1^{*}$, defined by

$$\tilde{t}{}_1^{*} f := \sup_{n\in\mathbb{N}} \frac{|t_n f|}{\log^2 (n+1)},$$

is bounded from the Hardy space $H_{1/2}$ to the Lebesgue space $L_{1/2}$. Furthermore, in view of the fact that the Fejér means are examples of Nörlund means with non-decreasing sequence $\{q_k : k \in \mathbb{N}\}$ we immediately obtain that the asymptotic behaviour of the sequence of weights

$$\left\{1/\log^2 (n+1) : n \in \mathbb{N}\right\}$$

in Nörlund means can not be improved.

In [101] it was proved that for all Nörlund means with non-increasing sequence $\{q_k : k \in \mathbb{N}\}$ there exists a martingale $f \in H_p$ such that

$$\sup_{n\in\mathbb{N}} \|t_n f\|_{weak-L_p} = \infty.$$

It follows that for any $0 < p < 1/2$ and Nörlund means t_n with non-increasing sequence $\{q_k : k \in \mathbb{N}\}$, the maximal operator t^* is not bounded from the martingale Hardy space H_p to the space $weak - L_p$, that is there exists a martingale $f \in H_p$, such that

$$\sup_{n\in\mathbb{N}} \|t^* f\|_{weak-L_p} = \infty.$$

In the same paper [101] it was also derived a corresponding necessary condition for the Nörlund means with non-increasing sequence $\{q_k : k \in \mathbb{N}\}$, when $1/2 \leq p < 1$. In particular, if $0 < p < 1/(1+\alpha)$, $0 < \alpha \leq 1$, and non-increasing sequence $\{q_k : k \in \mathbb{N}\}$ satisfying the condition

$$\overline{\lim_{n\to\infty}} \frac{n^\alpha}{Q_n} = c > 0, \; 0 < \alpha \leq 1, \tag{1.14}$$

then there exists a martingale $f \in H_p$ such that

$$\sup_{n\in\mathbb{N}} \|t_n f\|_{weak-L_p} = \infty.$$

Moreover, for any non-increasing sequence $\{q_k : k \in \mathbb{N}\}$ satisfying the condition

$$\overline{\lim_{n\to\infty}} \frac{n^\alpha}{Q_n} = \infty, \quad (0 < \alpha \leq 1), \tag{1.15}$$

there exists a martingale $f \in H_{1/(1+\alpha)}$, such that

$$\sup_{n\in\mathbb{N}} \|t_n f\|_{weak-L_{1/(1+\alpha)}} = \infty.$$

It follows that for any $0 < p < 1/(1+\alpha)$, $0 < \alpha \leq 1$ and non-increasing sequence $\{q_k : k \in \mathbb{N}\}$ satisfying the condition (1.14) there exists a martingale $f \in H_p$ such that

$$\|t^* f\|_{weak-L_p} = \infty.$$

Furthermore, if $\{q_k : k \in \mathbb{N}\}$ is a non-increasing sequence satisfying the condition (1.15), then there exists a martingale $f \in H_{1/(1+\alpha)}$ such that

$$\|t^* f\|_{weak-L_{1/(1+\alpha)}} = \infty.$$

In [78] it was proved that the maximal operator t^* of the Nörlund summability method with non-increasing sequence $\{q_k : k \in \mathbb{N}\}$, satisfying the condition

$$\frac{1}{Q_n} = O\left(\frac{1}{n^\alpha}\right), \quad \text{when} \quad n \to \infty \tag{1.16}$$

and

$$q_n - q_{n+1} = O\left(\frac{1}{n^{2-\alpha}}\right), \quad \text{when} \quad n \to \infty, \tag{1.17}$$

is bounded from the Hardy space $H_{1/(1+\alpha)}$ to the space $weak - L_{1/(1+\alpha)}$, for $0 < \alpha \leq 1$.

Moreover, for $0 < \alpha \leq 1$ and non-increasing sequence $\{q_k : k \in \mathbb{N}\}$ satisfying the conditions

$$\overline{\lim_{n\to\infty}} \frac{n^\alpha}{Q_n} \geq c_\alpha > 0 \tag{1.18}$$

and

$$|q_n - q_{n+1}| \geq c_\alpha n^{\alpha-2}, \;\; n \in \mathbb{N}. \tag{1.19}$$

there exists a martingale $f \in H_{1/(1+\alpha)}$ such that

$$\sup_{n\in\mathbb{N}} \|t_n f\|_{1/(1+\alpha)} = \infty.$$

In [101] (see also [24]) it was proved that if $f \in H_p$, where $0 < p < 1/(1+\alpha)$ for some $0 < \alpha \leq 1$, and $\{q_k : k \in \mathbb{N}\}$ is a sequence of non-increasing numbers satisfying the conditions (1.16) and (1.17), then the maximal operator $\tilde{t}^{*}_{p,\alpha}$, defined by

$$\tilde{t}^{*}_{p,\alpha} := \frac{|t_n f|}{(n+1)^{1/p-1-\alpha}},$$

is bounded from the martingale Hardy space H_p to the Lebesgue space L_p.

Moreover, if $\{\Phi_n : n \in \mathbb{N}_+\}$ is any non-decreasing sequence, satisfying the condition

$$\overline{\lim_{n\to\infty}} \frac{(n+1)^{1/p-1-\alpha}}{\Phi_n} = +\infty, \tag{1.20}$$

then there exists Nörlund means with non-increasing sequence $\{q_k : k \in \mathbb{N}\}$ satisfying the conditions (1.18) and (1.19) such that

$$\sup_{k\in\mathbb{N}} \frac{\left\| \frac{t_{M_{2n_k}+1} f_k}{\Phi_{M_{2n_k}+1}} \right\|_{weal-L_p}}{\|f_k\|_{H_p}} = \infty.$$

Let $0 < p < 1/(1+\alpha)$ and $f \in H_p$. Then there exists an absolute constant $c_{p,\alpha}$, depending only on p and α, such that

$$\|t_n f\|_p \le c_{p,\alpha} (n+1)^{1/p-1-\alpha} \|f\|_{H_p}, \ \ n \in \mathbb{N}_+.$$

On the other hand, if $\{\Phi_n : n \in \mathbb{N}\}$ is any non-decreasing sequence satisfying the condition (1.20), then there exists a martingale $f \in H_p$ such that

$$\sup_{n\in\mathbb{N}} \left\| \frac{t_n f}{\Phi_n} \right\|_{weak-L_p} = \infty.$$

Moreover, let $\{\Phi_n : n \in \mathbb{N}\}$ is any non-decreasing sequence satisfying the condition (1.20). Then the maximal operator

$$\sup_{n\in\mathbb{N}} \frac{|t_n f|}{\Phi_n}$$

is not bounded from the Hardy space H_p to the space $weak - L_p$.

In [22] (see also [24]) it was proved that if $f \in H_{1/(1+\alpha)}$, where $0 < \alpha \le 1$ and $\{q_k : k \in \mathbb{N}\}$ be a sequence of non-increasing numbers satisfying the conditions (1.16) and (1.17), then there exists an absolute constant c_α depending only on α such that the maximal operator

$$\tilde{t}^{*}_{\alpha} := \frac{|t_n f|}{\log^{1+\alpha}(n+1)}$$

is bounded from the martingale Hardy space $H_{1/(1+\alpha)}$ to the Lebesgue space $L_{1/(1+\alpha)}$.

Moreover, if $\{\Phi_n : n \in \mathbb{N}_+\}$ is any non-decreasing sequence satisfying the condition

$$\overline{\lim_{n\to\infty}} \frac{\log^{1+\alpha}(n+1)}{\Phi_n} = +\infty,$$

then there exists Nörlund means with non-increasing sequence $\{q_k : k \in \mathbb{N}\}$ satisfying the conditions (1.18) and (1.19) such that

$$\sup_{k\in\mathbb{N}} \frac{\left\| \sup_n \left| \frac{t_n f_k}{\Phi_n} \right| \right\|_{1/(1+\alpha)}}{\|f\|_{H_{1/(1+\alpha)}}} = \infty.$$

In [102] it was proved that if $0 < p < 1/2$, $f \in H_p$ and the sequence $\{q_k : k \in \mathbb{N}\}$ is non-decreasing, then there exists an absolute constant c_p depending only on p such that

$$\sum_{k=1}^{\infty} \frac{\|t_k f\|_p^p}{k^{2-2p}} \leq c_p \|f\|_{H_p}^p .$$

On the other hand, according the fact that Fejér means are examples of Nörlund means with non-decreasing sequence $\{q_k : k \in \mathbb{N}\}$ we immediately obtain that the asymptotic behaviour of the sequence of weights

$$\left\{1/k^{2-2p} : k \in \mathbb{N}\right\}$$

in Nörlund means can not be improved.

In [102] it was proved that if $f \in H_{1/2}$ and the sequence $\{q_k : k \in \mathbb{N}\}$ is non-decreasing satisfying condition (1.21) below, then there exists an absolute constant c, such that

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{\|t_k f\|_{1/2}^{1/2}}{k} \leq c \|f\|_{H_{1/2}}^{1/2} .$$

In Blahota and Tephnadze [24] was investigated Nörlund means with non-increasing sequence $\{q_k : k \in \mathbb{N}\}$ in the case $0 < p < 1/(1+\alpha)$ where $0 < \alpha < 1$. In particular, if $f \in H_p$, where $0 < p < 1/(1+\alpha)$, $0 < \alpha \leq 1$ and $\{q_k : k \in \mathbb{N}\}$, is a sequence of non-increasing numbers satisfying the conditions (1.16) and (1.17), then there exists an absolute constant $c_{\alpha,p}$, depending only on α and p such that

$$\sum_{k=1}^{\infty} \frac{\|t_k f\|_{H_p}^p}{k^{2-(1+\alpha)p}} \leq c_{\alpha,p} \|f\|_{H_p}^p .$$

In Blahota, Persson and Tephnadze [22] it was proved that if $f \in H_{1/(1+\alpha)}$, where $0 < \alpha \leq 1$ and $\{q_k : k \in \mathbb{N}\}$ is a sequence of non-increasing numbers satisfying the conditions (1.16) and (1.17), then there exists an absolute constant c_α depending only on α such that

$$\frac{1}{\log n} \sum_{m=1}^{n} \frac{\|t_m f\|_{H_{1/(1+\alpha)}}^{1/(1+\alpha)}}{m} \leq c_\alpha \|f\|_{H_{1/(1+\alpha)}}^{1/(1+\alpha)} .$$

In [161] (see paper D) we investigated the maximal operator T^* of the summability method (1.1.4) with non-increasing sequence $\{q_k : k \geq 0\}$. In particular, we proved that T^* is bounded from the Hardy space $H_{1/2}$ to the space $weak - L_{1/2}$.

Moreover, for any $0 < p < 1/2$ and non-increasing sequence $\{q_k : k \geq 0\}$ satisfying the condition

$$\frac{q_{n+1}}{Q_{n+2}} \geq \frac{c}{n}, \quad (c \geq 1),$$

then there exists a martingale $f \in H_p$, such that

$$\sup_{n\in\mathbb{N}} \|T_n f\|_{weak-L_p} = \infty.$$

We also proved that the maximal operator T^* of the summability method (1.1.4) with non-decreasing sequence $\{q_k : k \geq 0\}$ satisfying the condition

$$\frac{q_{n-1}}{Q_n} = O\left(\frac{1}{n}\right), \quad \text{as} \quad n \to \infty \tag{1.21}$$

is bounded from the Hardy space $H_{1/2}$ to the space $weak - L_{1/2}$.

Moreover, for any $0 < p < 1/2$ and non-decreasing sequence $\{q_k : k \geq 0\}$, there exists a martingale $f \in H_p$, such that

$$\sup_{n\in\mathbb{N}} \|T_n f\|_{weak-L_p} = \infty.$$

Similar problems for Walsh-Kaczmarz system were proved by Gogolashvili and Tephnadze [59, 60].

In [163] (see paper E) we proved that if $0 < p \leq 1/2$, $f \in H_p$ and a sequence $\{q_k : k \geq 0\}$ is either non-decreasing numbers (without any restrictions) or non-increasing numbers, satisfying the condition

$$\frac{1}{Q_n} = O\left(\frac{1}{n}\right), \quad \text{as} \quad n \to \infty, \tag{1.22}$$

then the maximal operator $\widetilde{T}_p^*$, defined by

$$\widetilde{T}_p^* f := \sup_{n\in\mathbb{N}_+} \frac{|T_n f|}{(n+1)^{1/p-2} \log^{2[1/2+p]}(n+1)} \tag{1.23}$$

is bounded from the Hardy space H_p to the space L_p.

Since the maximal operator $\widetilde{\sigma}_p^*$ defined by

$$\widetilde{\sigma}_p^* f := \sup_{n\in\mathbb{N}_+} \frac{|\sigma_n f|}{(n+1)^{1/p-2} \log^{2[1/2+p]}(n+1)}$$

is bounded from the martingale Hardy space H_p to the space L_p and the rate of denominator $(n+1)^{1/p-2} \log^{2[1/2+p]}$ is in a sense sharp and Fejer means is example of T means, for a non-decreasing and non-increasing sequence we obtain that these weights are also sharp in (1.23).

In [163] we also investigated strong convergence of T means with respect to Vilenkin systems. In particular, if $0<p<1/2,$ $f\in H_p$ and $\{q_k:k\geq 0\}$ is a sequence of non-increasing or non-decreasing numbers, then there exists an absolute constant c_p, depending only on p, such that the inequality

$$\sum_{k=1}^{\infty}\frac{\|T_kf\|_p^p}{k^{2-2p}}\leq c_p\left\|f\right\|_{H_p}^p$$

holds.

Moreover, if $f\in H_{1/2}$ and $\{q_k:k\geq 0\}$ is a sequence of non-increasing numbers, satisfying the condition (1.22), then there exists an absolute constant c, such that the inequality

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{\|T_kf\|_{1/2}^{1/2}}{k}\leq c\left\|f\right\|_{H_{1/2}}^{1/2} \tag{1.24}$$

holds.

If the sequence $\{q_k:k\geq 0\}$ is non-decreasing and satisfying condition, then the inequality (1.24) is true also for any $f\in H_{1/2}$.

Well-known examples of Nörlund and T means are Riesz and Nörlund logarithmic means.

Riesz logarithmic means with respect to the trigonometric system was studied by many authors. We mention, for instance, the papers by Szasz [120] and Yabuta [176]. These means with respect to the Walsh and Vilenkin systems were investigated by Baramidze, Gogolashvili, Nadirashvili [12] (see also [13]), Gàt [43] and Simon [111]. Blahota and Gát [17] considered norm summability of Nörlund logarithmic means and showed that Riesz logarithmic means R_n have better approximation properties on some unbounded Vilenkin groups than the Fejér means. Moreover, in [133] it was proved that the maximal operator of Riesz means is bounded from the Hardy space H_p to the Lebesgue space L_p for $p>1/2$ but not when $0<p\leq 1/2$. Strong convergence theorems and boundedness of weighted maximal operators of Riesz logarithmic means were considered in Lukkassen, Persson, Tutberidze, Tephnadze [75] (see paper F) and Tephnadze [133].

In [144] Tephnadze proved that the maximal operator of Riesz logarithmic means R^* is bounded from the Hardy space $H_{1/2}$ to the space $weak-L_{1/2}$. Moreover, there exists a martingale $f\in H_p$, where $0<p\leq 1/2$ such that

$$\left\|R^*f\right\|_p=+\infty.$$

In [133] Tephnadze proved that for any $0<p<1/2$, the maximal operator $\tilde{R}{}_p^*$, defined by

$$\tilde{R}{}_p^*:=\sup_{n\in\mathbb{N}}\frac{\log n\left|R_nf\right|}{\left(n+1\right)^{1/p-2}},$$

is bounded from the Hardy space H_p to the space L_p.

Moreover, for $0 < p < 1/2$ and non-decreasing function $\varphi : \mathbb{N}_+ \to [1, \infty)$ satisfying the condition

$$\frac{(n+1)^{1/p-2}}{\log(n+1)\varphi(n)} = \infty,$$

the maximal operator

$$\sup_{n\in\mathbb{N}} \frac{|R_n f|}{\varphi(n)}$$

is not bounded from the Hardy space H_p to the space $weak - L_p$.

In the case $p = 1/2$ he also proved that the maximal operator $\widetilde{R}^*$, defined by

$$\widetilde{R}^* f := \sup_{n\in\mathbb{N}} \frac{|R_n f|}{\log(n+1)},$$

is bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2}$.

Moreover, for any non-decreasing function $\varphi : \mathbb{N}_+ \to [1, \infty)$ satisfying the condition

$$\overline{\lim_{n\to\infty}} \frac{\log(n+1)}{\varphi(n)} = +\infty,$$

the maximal operator

$$\sup_{n\in\mathbb{N}} \frac{|R_n f|}{\varphi(n)}$$

is not bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2}$.

In [75] (see paper F) we also proved that if $0 < p < 1/2$ and $f \in H_p(G_m)$, then there exists an absolute constant c_p, depending only on p, such that the inequality

$$\sum_{n=1}^{\infty} \frac{\log^p n \|R_n f\|_{H_p}^p}{n^{2-2p}} \leq c_p \|f\|_{H_p}^p$$

holds, where $R_n f$ denotes the n-th Reisz logarithmic mean with respect to the Vilenkin-Fourier series of f.

Móricz and Siddiqi [81] investigated the approximation properties of some special Nörlund means of Walsh-Fourier series of L_p functions in norm. The case when $\{q_k = 1/k : k \in \mathbb{N}\}$ was excluded, since the methods of Móricz and Siddiqi are not applicable to Nörlund logarithmic means. Fridli, Manchanda and Siddiqi [39] improved and extended the results of Móricz and Siddiqi [81] to dyadic homogeneous Banach spaces and martingale Hardy spaces. In [45] Gát and Goginava proved some convergence and divergence properties of the Nörlund logarithmic means of functions in the class of continuous functions and in the Lebesgue space L_1. In particular, they gave a negative answer to the question of Móricz and Siddiqi [81]. Gát and Goginava [44] proved that for each measurable function satisfying $\phi(u) = o\left(u \log^{1/2} u\right),\ as\ u \to \infty$, there exists an integrable function f such that

$$\int_{G_m} \phi(|f(x)|)\, d\mu(x) < \infty$$

and that there exists a set with positive measure such that the Walsh-logarithmic means of the function diverges on this set. It follows that weak-(1,1) type inequality does not hold for the maximal operator of Nörlund logarithmic means L^*, defined by

$$L^*f \quad := \quad \sup_{n\in\mathbb{N}} |L_n f| \,.$$

On the other hand, there exists an absolute constant c_p such that

$$\|L^*f\|_p \leq c_p \|f\|_p \,, \text{ when } f \in L_p, \; p > 1.$$

If we consider the following restricted maximal operator $\widetilde{L}^*_{\#}$, defined by

$$\widetilde{L}^*_{\#}f \quad := \quad \sup_{n\in\mathbb{N}} |L_{M_n} f| \,, \quad (M_k := m_0...m_{k-1}, \quad k = 0, 1...)$$

then

$$\lambda\mu \left\{ \widetilde{L}^*_{\#}f > \lambda \right\} \leq c \|f\|_1 \,, \quad f \in L_1(G_m), \; \lambda > 0.$$

Hence, if $f \in L_1(G_m)$, then

$$L_{M_n} f \to f, \text{ a.e. on } G_m.$$

In [10] (see also [13]) it was proved that if $f \in L_1(G_m)$, then $L_{M_n} f(x) \to f(x)$ for all Lebesgue points.

In [122] (see also [148]) it was proved that there exists a martingale $f \in H_p$, $(0 < p \leq 1)$, such that the maximal operator of Nörlund logarithmic means L^* is not bounded in the Lebesgue space L_p. In particular, it was proved that there exists a martingale $f \in H_p$ such that

$$\|L^*f\|_p = +\infty.$$

Boundedness of weighted maximal operators of Nörlund logarithmic means was considered in [103]. In particular, it was proved that the maximal operator $\tilde{L}^*$, defined by

$$\tilde{L}^* f := \sup_{n\in\mathbb{N}} \frac{|L_n f|}{\log(n+1)},$$

is bounded from the Hardy space $H_1(G_m)$ to the space $L_1(G_m)$.

Moreover, if $\varphi : \mathbb{N}_+ \to [1, \infty)$ is a non-decreasing function satisfying the condition

$$\overline{\lim_{n\to\infty}} \frac{\log(n+1)}{\varphi(n)} = +\infty, \tag{1.25}$$

then there exists a martingale $f \in H_1(G_m)$, such that the maximal operator

$$\sup_{n\in\mathbb{N}} \frac{|L_n f|}{\varphi(n)}$$

is not bounded from the Hardy space $H_1(G_m)$ to the Lebesgue space $L_1(G_m)$.

In Tephnadze and Tutberidze [146] (see paper G) it was proved that the maximal operator $\widetilde{L}_p^*$, defined by

$$\widetilde{L}_p^* f := \sup_{n\in\mathbb{N}} \frac{|L_n f|}{(n+1)^{1/p-1}},$$

is bounded from the Hardy space $H_p(G_m)$ to the space $L_p(G_m)$.

We also proved that for $0 < p < 1$ and a non-decreasing function $\varphi : \mathbb{N}_+ \to [1,\infty)$ satisfying the condition

$$\overline{\lim_{n\to\infty}} \frac{n^{1/p-1}}{\log n \varphi(n)} = +\infty,$$

then there exists a martingale $f \in H_p(G_m)$, such that the maximal operator

$$\sup_{n\in\mathbb{N}} \frac{|L_n f|}{\varphi(n+1)}$$

is not bounded from the Hardy space $H_p(G_m)$ to the space $L_p(G_m)$.

In the same paper we also state the following problem:

Open Problem. For any $0 < p < 1$ and non-decreasing function $\Theta : \mathbb{N}_+ \to [1,\infty)$ is it true or not that the following maximal operator $\widetilde{L}_p^*$, defined by

$$\widetilde{L}_p^* f := \sup_{n\in\mathbb{N}} \frac{|L_n f|}{\Theta(n+1)}$$

is bounded from the Hardy space $H_p(G_m)$ to the Lebesgue space $L_p(G_m)$ and the rate of $\Theta : \mathbb{N}_+ \to [1,\infty)$ is sharp, that is, for any non-decreasing function $\varphi : \mathbb{N}_+ \to [1,\infty)$ satisfying the condition

$$\overline{\lim_{n\to\infty}} \frac{\Theta(n)}{\varphi(n)} = +\infty,$$

then there exists a martingale $f \in H_p(G_m)$, such that the maximal operator

$$\sup_{n\in\mathbb{N}} \frac{|L_n f|}{\varphi(n+1)}$$

is not bounded from the Hardy space $H_p(G_m)$ to the space $L_p(G_m)$.

According to the Theorems above we can conclude that there exist absolute constants C_1 and C_2 such that

$$\frac{C_1 n^{1/p-1}}{\log(n+1)} \leq \Theta(n) \leq C_2 n^{1/p-1}.$$

Later on, Memic generalized result of Tephnadze and Tutberidze [146] (see paper G) and proved that the maximal operator

$$\sup_{n\in\mathbb{N}}\frac{\log n\left|L_nf\right|}{(n+1)^{1/p-1}}$$

is bounded from the Hardy space $H_p\left(G_m\right)$ to the space $L_p\left(G_m\right)$.

Sharpness of this result immediately follows by using the negative result of Tephnadze and Tutberidze [146] (see paper G), which is already stated above.

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Papers

Paper A

A note on the strong convergence of partial sums with respect to Vilenkin system

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REAL AND COMPLEX ANALYSIS

A Note on the Strong Convergence of Partial Sums with Respect to Vilenkin System

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Abstract—In this paper we prove some strong convergence theorems for partial sums with respect to Vilenkin system.

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1. INTRODUCTION

It is well-known that the Vilenkin system does not form a basis in the space $L_1(G_m)$ (for details see [8] and [14]). Moreover, there is a function in the Hardy space $H_1(G_m)$ such that the partial sums of f are not bounded in L_1-norm (for details see [12, 13, 21, 22]) . However, a subsequence S_{M_n} of partial sums are bounded from the Hardy space $H_1(G_m)$ to the Lebesgue space $L_1(G_m)$ (see [2, 23]):

$$\|S_{M_k}f\|_{H_1} \leq c\,\|f\|_{H_1} \quad (k \in \mathbb{N}). \tag{1.1}$$

Moreover, in Gát [7] (see also Simon [18, 19]), it was proved the following strong convergence result: for all $f \in H_1$

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{\|S_kf-f\|_1}{k}=0,$$

where S_kf denotes the k-th partial sum of the Vilenkin-Fourier series of f.

It follows that there exists an absolute constant c such that

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{\|S_kf\|_1}{k} \leq c\,\|f\|_{H_1}, \quad (n=2,3...) \tag{1.2}$$

and for all $f \in H_1$

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{\|S_kf\|_1}{k}=\|f\|_{H_1}.$$

A similar result for trigonometric system was proved by Smith [20], and for Walsh-Paley system by Simon [17]. Observe that if the partial sums of Vilenkin-Fourier series will be bounded from H_1 to L_1, then we also would have

$$\sup_{n\in\mathbb{N}_+}\frac{1}{n}\sum_{m=1}^{n}\|S_mf\|_1 \leq c\,\|f\|_{H_1}, \tag{1.3}$$

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but, as it was mentioned above, the boundedness of the partial sums does not hold from H_1 to L_1. However, we have the inequality (1.2).

On the other hand, in the one-dimensional case, Fujii [6] and Simon [16] have proved that the maximal operator Fejér mean is bounded from H_1 to L_1, that is,

$$\sup_{n\in\mathbb{N}_+}\left\|\frac{1}{n}\sum_{m=1}^{n}S_m f\right\|_1 < c\left\|f\right\|_{H_1}. \tag{1.4}$$

So, a natural question that arises is that if the inequality (1.3) holds true, which would be a generalization of the inequality (1.4), or do we have a negative answer to this problem?

In this paper, we prove that there exists a function $f \in H_1$ such that

$$\sup_{n\in\mathbb{N}_+}\frac{1}{n}\sum_{m=1}^{n}\left\|S_m f\right\|_1 = \infty.$$

The paper is organized as follows: In Section 2 we present some necessary notation and definitions. In Section 3 we state the main results of the paper. The detailed proofs of the main results are given in Section 4.

2. DEFINITIONS AND NOTATION

Let $\mathbb{N}_+$ denote the set of the positive integers, $\mathbb{N} = \mathbb{N}_+ \cup \{0\}$, and let $m = (m_0, m_1, \dots)$ denote a sequence of positive integers not less than 2. Denote by $Z_{m_k} = \{0, 1, \dots, m_k - 1\}$ the additive group of integers modulo m_k, and define the group G_m to be the complete direct product of the group Z_{m_j} with the product of the discrete topologies of Z_{m_j}'s. The direct product μ of the measures $\mu_k(\{j\}) = 1/m_k$, $j \in Z_{m_k}$ is the Haar measure on G_m with $\mu(G_m) = 1$.

If $\sup_{n\in\mathbb{N}} m_n < \infty$, then we call G_m a bounded Vilenkin group. If the generating sequence m is not bounded, then G_m is said to be an unbounded Vilenkin group.

The elements of G_m are represented by sequences of the form $x = (x_0, x_1, \dots, x_k, \dots)$, $x_k \in Z_{m_k}$. It is easy to give a base for the neighborhood of G_m, namely we have

$$I_0(x) = G_m, \quad I_n(x) = \{y \in G_m \mid y_0 = x_0, \dots, y_{n-1} = x_{n-1}\}, \quad x \in G_m, \ n \in \mathbb{N}.$$

Denote $I_n = I_n(0)$ for $n \in \mathbb{N}$ and $\overline{I_n} = G_m \backslash I_n$. Let $e_n = (0, ..., 0, x_n = 1, 0, ...) \in G_m$, $n \in \mathbb{N}$. If we define the so-called generalized number system based on m in the following way: $M_0 = 1$, $M_{k+1} = m_k M_k$, $k \in \mathbb{N}$, then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j}$ $(j \in \mathbb{N})$, and only a finite number of n_j's differ from zero. Define $|n| = \max\{j \in \mathbb{N}; n_j \neq 0\}$.

Next, on the group G_m we introduce an orthonormal system, which is called the Vilenkin system. To this end, we first define the complex-valued functions $r_k(x) : G_m \to \mathbb{C}$, the generalized Rademacher functions:

$$r_k(x) = \exp(2\pi\imath x_k/m_k) \qquad \left(\imath^2 = -1, \ x \in G_m, \ k \in \mathbb{N}\right).$$

Now define the Vilenkin system $\psi = (\psi_n : n \in \mathbb{N})$ on G_m as follows:

$$\psi_n(x) = \prod_{k=0}^{\infty} r_k^{n_k}(x), \quad n \in \mathbb{N}.$$

Note that in the special case where $m = 2$, the above defined system is called the Walsh-Paley system.

The norm (or quasi norm) in the space $L_p(G_m)$ is defined by

$$\left\|f\right\|_p = \left(\int_{G_m} |f(x)|^p \, d\mu(x)\right)^{1/p}, \quad 0 < p < \infty.$$

Note that the Vilenkin system is orthonormal and complete in $L_2(G_m)$ (see [1, 25]).

For $f\in L_1(G_m)$ we define the Fourier coefficients, the partial sums of the Fourier series, the Fejér means, and the Dirichlet kernels with respect to the Vilenkin system in the usual manner:

$$\begin{aligned}\widehat{f}(k) &:= \int_{G_m} f\overline{\psi}_k d\mu, \quad k\in\mathbb{N},\\ S_n f &:= \sum_{k=0}^{n-1}\widehat{f}(k)\,\psi_k, \quad n\in\mathbb{N}_+, \quad S_0 f = 0,\\ \sigma_n f &:= \frac{1}{n}\sum_{k=0}^{n-1} S_k f, \quad D_n=\sum_{k=0}^{n-1}\psi_k, \quad n\in\mathbb{N}_+.\end{aligned}$$

Recall that

$$D_{M_n}(x)=\begin{cases} M_n, & x\in I_n\\ 0, & x\notin I_n.\end{cases} \tag{2.1}$$

and

$$D_{s_nM_n}=D_{M_n}\sum_{k=0}^{s_n-1}\psi_{kM_n}=D_{M_n}\sum_{k=0}^{s_n-1}r_n^k, \qquad 1\le s_n\le m_n-1. \tag{2.2}$$

The n-th Lebesgue constant is defined by $L_n=\|D_n\|_1$. It is well-known that (see [25]):

$$L_n=O(\log n), \ \ n\to\infty. \tag{2.3}$$

Moreover, there exist absolute constant c_1 and c_2 such that

$$c_1\log n\le \frac{1}{n}\sum_{k=1}^{n}L(k)\le c_2\log n, \quad n=2,3,.... \tag{2.4}$$

(For unbounded Vilenkin systems this result can be found in [5], while for bounded Vilenkin systems in [9] and [11, 24]).

The concept of the Hardy space (see [4]) can be defined in various manners, for instance, by a maximal function $f^*=\sup_{n\in\mathbb{N}}|S_{M_n}f|$, $f\in G_m$, saying that f belongs to the Hardy space if $f^*\in L^1(G_m)$. This definition is suitable if the sequence m is bounded. In this case a good property of the space $\{f\in L^1(G_m):\ f^*\in L^1(G_m)\}$ is the atomic structure (see [4]). To define the Hardy type space for an arbitrary m, we first introduce the concept of the atoms (see [16]). A set $I\subset G_m$ is called an interval if for some $x\in G_m$ and $n\in N$, I is of the form $I=\bigcup\limits_{k\in U} I_n(x,k)$, where U is obtained from Z_{m_n} by dyadic partition.

The sets $U_1,U_2,...\subset Z_{m_n}$, are obtained by means of such a partition as follows:

$$U_1=\left\{0,...,\left[\frac{m_n}{2}\right]-1\right\},\ \ U_2=\left\{\left[\frac{m_n}{2}\right],...,m_n-1\right\},$$

$$U_3=\left\{0,...,\left[\frac{[m_n/2]-1}{2}\right]-1\right\},\ \ U_4=\left\{\left[\frac{[m_n/2]-1}{2}\right],...,\left[\frac{m_n}{2}\right]-1\right\},...,$$

where $[a]$ denotes the integral part of a number a. We define the atoms as follows: a function $a\in L^\infty(G_m)$ is called an atom if either $a\equiv 1$ or there exists an interval I to satisfy $\sup a\subset I$, $|a|\le|I|^{-1}$ and $\int_I a=0$, where $|I|$ denotes the Haar measure of I .

Now we define the space $H_1(G_m)$ to be the set of all functions $f=\sum\limits_{i=0}^{\infty}\lambda_i a_i$, where a_i's are atoms and for the coefficients λ_i we have $\sum\limits_{i=0}^{\infty}|\lambda_i|<\infty$ (for details see [26, 27]). Observe that $H_1(G_m)$ is a Banach space with respect to the norm

$$\|f\|_{H_1} = \inf \sum_{k=0}^{\infty} |\lambda_k| < \infty, \tag{2.5}$$

where the infimum is taken over all decompositions $f = \sum_{i=0}^{\infty} \lambda_i a_i$. It is known (see [7]) that $\|f\|_{H_1}$ is equivalent to $\|f^{**}\|_1 \left(f \in L^1(G_m)\right)$, where $f^{**}(x) = \sup_I |I|^{-1} \left|\int_I f\right|$, ($x \in G_m$, $x \in I$ and I is an interval). Since by (2.1)

$$f^*(x) = \sup_{n\in\mathbb{N}} \frac{1}{|I_n(x)|} \left| \int_{I_n(x)} f(u)\,\mu(u) \right|,$$

we have $f^* \leq f^{**}$ and, thus, $H(G_m) \subset \left\{ f \in L^1(G_m) : \ f^* \in L^1(G_m) \right\}$. Moreover, these spaces coincide if the sequence m is bounded.

3. THE MAIN RESULT

Our main result is the following theorem.

Theorem 3.1. *a) Let $f \in H_1$. Then there exists an absolute constant c such that*

$$\sup_{n\in\mathbb{N}} \frac{1}{n \log n} \sum_{k=1}^{n} \|S_k f\|_1 \leq \|f\|_{H_1}.$$

b) Let $\varphi : \mathbb{N}_+ \to [1, \infty)$ be a nondecreasing function satisfying the condition:

$$\overline{\lim_{n\to\infty}} \frac{\log n}{\varphi_n} = +\infty. \tag{3.1}$$

Then there exists a function $f \in H_1$ such that

$$\sup_{n\in\mathbb{N}} \frac{1}{n\varphi_n} \sum_{k=1}^{n} \|S_k f\|_1 = \infty.$$

Corollary 3.1 (see [10, 16, 18]). *There exists a function $f \in H_1$ such that*

$$\sup_{n\in\mathbb{N}} \frac{1}{n} \sum_{k=1}^{n} \|S_k f\|_1 = \infty.$$

4. PROOF OF THEOREM 3.1

To prove assertion (a) of the theorem, we use (2.3) to conclude that

$$\frac{1}{n \log n} \sum_{k=1}^{n} \|S_k f\|_1 \leq \frac{c\,\|f\|_{H_1}}{n \log n} \sum_{k=1}^{n} \log k \leq c\,\|f\|_{H_1},$$

and the result follows.

Now we proceed to prove assertion (b). To this end, observe first that under the condition (3.1) there exists an increasing sequence of positive integers $\{\alpha_k : k \in \mathbb{N}\}$ such that

$$\overline{\lim_{k\to\infty}} \frac{\log M_{\alpha_k}}{\varphi_{2M_{\alpha_k}}} = +\infty$$

and

$$\sum_{k=0}^{\infty} \frac{\varphi_{2M_{\alpha_k}}^{1/2}}{\log^{1/2} M_{\alpha_k}} < c < \infty. \tag{4.1}$$

Let $f = \sum\limits_{k=1}^{\infty} \lambda_k a_k$, where $a_k = r_{\alpha_k} D_{M_{\alpha_k}} = D_{2M_{\alpha_k}} - D_{M_{\alpha_k}}$ and

$$\lambda_k = \frac{\varphi_{2M_{\alpha_k}}^{1/2}}{\log^{1/2} M_{\alpha_k}}.$$

Taking into account the definition of H_1 and (2.5) and applying (4.1) we can conclude that $f \in H_1$. Moreover, we have

$$\widehat{f}(j) = \begin{cases} \lambda_k, & j \in \{M_{\alpha_k}, ..., 2M_{\alpha_k} - 1\}, \ k \in \mathbb{N} \\ 0, & j \notin \bigcup\limits_{k=1}^{\infty} \{M_{\alpha_k}, ..., 2M_{\alpha_k} - 1\}. \end{cases} \tag{4.2}$$

Next, taking into account that $D_{j+M_{\alpha_k}} = D_{M_{\alpha_k}} + \psi_{M_{\alpha_k}} D_j$, when $j \le M_{\alpha_k}$, we can apply (4.2) to obtain that

$$\begin{aligned} S_j f &= S_{M_{\alpha_k}} f + \sum_{v=M_{\alpha_k}}^{j-1} \widehat{f}(v)\psi_v = S_{M_{\alpha_k}} f + \lambda_k \sum_{v=M_{\alpha_k}}^{j-1} \psi_v \\ &= S_{M_{\alpha_k}} f + \lambda_k \left(D_j - D_{M_{\alpha_k}}\right) = S_{M_{\alpha_k}} f + \lambda_k \psi_{M_{\alpha_k}} D_{j-M_{\alpha_k}} = I_1 + I_2. \end{aligned} \tag{4.3}$$

In view of (1.1) we obtain

$$\|I_1\|_1 \le \left\| S_{M_{\alpha_k}} f \right\|_1 \le c \|f\|_{H_1}. \tag{4.4}$$

By combining (2.4) and (4.4) we get

$$\|S_n f\|_1 \ge \|I_2\|_1 - \|I_1\|_1 \ge \lambda_k L\left(n - M_{\alpha_k}\right) - c\|f\|_{H_1}.$$

Therefore, we can write

$$\begin{aligned} &\sup_{n\in\mathbb{N}_+} \frac{1}{n\varphi_n} \sum_{k=1}^{n} \|S_k f\|_1 \ge \frac{1}{2M_{\alpha_k}\varphi_{2M_{\alpha_k}}} \sum_{\left\{M_{\alpha_k} \le l \le 2M_{\alpha_k}\right\}} \|S_l f\|_1 \\ &\ge \frac{1}{2M_{\alpha_k}\varphi_{2M_{\alpha_k}}} \sum_{\left\{M_{\alpha_k} \le l \le 2M_{\alpha_k}\right\}} \left(\frac{L\left(l - M_{\alpha_k}\right) \varphi_{2M_{\alpha_k}}^{1/2}}{\log^{1/2} M_{\alpha_k}} - c\|f\|_{H_1} \right) \\ &\ge \frac{c\varphi_{2M_{\alpha_k}}^{1/2}}{2M_{\alpha_k} \log^{1/2} M_{\alpha_k} \varphi_{2M_{\alpha_k}}} \sum_{l=1}^{M_{\alpha_k}-1} L(l) - c\|f\|_{H_1}^{1/2} \\ &\ge \frac{c\varphi_{2M_{\alpha_k}}^{1/2} \log M_{\alpha_k}}{\log^{1/2} M_{\alpha_k} \varphi_{2M_{\alpha_k}}} \ge \frac{c \log^{1/2} M_{\alpha_k}}{\varphi_{2M_{\alpha_k}}^{1/2}} \to \infty, \text{ as } k \to \infty. \end{aligned}$$

This completes the proof of assertion (b). Theorem 3.1 is proved.

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Paper B

On the boundedness of subsequences of Vilenkin-Fejér means on the martingale Hardy spaces

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[2266] First Galley Proofs

ON THE BOUNDEDNESS OF SUBSEQUENCES OF VILENKIN–FEJÉR MEANS ON THE MARTINGALE HARDY SPACES

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Abstract. In this paper we characterize subsequences of Fejér means with respect to Vilenkin systems, which are bounded from the Hardy space H_p to the Lebesgue space L_p, for all $0 < p < 1/2$. The result is in a sense sharp.

1. Introduction

In the one-dimensional case the weak (1,1)-type inequality for the maximal operator of Fejér means

$$\sigma^* f := \sup_{n\in\mathbb{N}} |\sigma_n f|$$

can be found in Schipp [12] for Walsh series and in Pál, Simon [10] for bounded Vilenkin series. Here, as usual, the symbol σ_n denotes the Fejér mean with respect to the Vilenkin system (and thus also called the Vilenkin-Fejér means, see Section 2).

Fujji [6] and Simon [14] verified that σ^* is bounded from H_1 to L_1. Weisz [23] generalized this result and proved boundedness of σ^* from the martingale space H_p to the Lebesgue space L_p for $p > 1/2$. Simon [13] gave a counterexample, which shows that boundedness does not hold for $0 < p < 1/2$. A counterexample for $p = 1/2$ was given by Goginava [8] (see also [2] and [3]). Weisz [24] proved that the maximal operator of the Fejér means σ^* is bounded from the Hardy space $H_{1/2}$ to the space $weak-L_{1/2}$. The boundedness of weighted maximal operators are considered in [9], [16] and [17].

Weisz [22] (see also [21]) also proved that the following theorem is true:

THEOREM W:(WEISZ). *Let* $p > 0$. *Then the maximal operator*

$$\sigma^{\nabla,*} f = \sup_{n\in\mathbb{N}} |\sigma_{M_n} f| \tag{1}$$

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where $M_0 := 1,\ M_{n+1} := m_n M_n\ (n \in \mathbb{N})$ *and* $m := (m_0, m_1, \ldots)$ *be a sequences of the positive integers not less than 2, which generate Vilenkin systems, is bounded from the Hardy space* H_p *to the space* L_p.

In [11] the result of Weisz was generalized and it was found the maximal subspace $S \subset \mathbb{N}$ of positive numbers, for which the restricted maximal operator on this subspace $\sup_{n \in S \subset \mathbb{N}} |\sigma_n f|$ of Fejér means is bounded from the Hardy space H_p to the space L_p for all $0 < p \leqslant 1/2$. The new theorem (Theorem 1) in this paper show in particular that this result is in a sense sharp. In particular, for every natural number $n = \sum_{k=0}^{\infty} n_k M_k$, where $n_k \in Z_{m_k}\ (k \in \mathbb{N}_+)$ we define numbers

$$\langle n \rangle := \min\{j \in \mathbb{N} : n_j \neq 0\}, \quad |n| := \max\{j \in \mathbb{N} : n_j \neq 0\}, \quad \rho(n) = |n| - \langle n \rangle$$

and prove that

$$S = \{n \in \mathbb{N} : \rho(n) \leqslant c < \infty.\}$$

Since $\rho(M_n) = 0$ for all $n \in \mathbb{N}$ we obtain that $\{M_n : n \in \mathbb{N}\} \subset S$ and that follows i.e. that result of Weisz [22] (see also [21]) that restricted maximal operator (1) is bounded from the Hardy space H_p to the space L_p.

The main aim of this paper is to generalize Theorem W and find the maximal subspace of positive numbers, for which the restricted maximal operator of Fejér means in this subspace is bounded from the Hardy space H_p to the space L_p for all $0 < p \leqslant 1/2$. As applications, both some well-known and new results are pointed out.

This paper is organized as follows: In order not to disturb our discussions later on some preliminaries (definitions, notations and lemmas) are presented in Section 2. The main result (Theorem 1) and some of its consequences can be found in Section 3. The detailed proof of Theorem 1 is given in Section 4.

2. Preliminaries

Denote by $\mathbb{N}_+$ the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Let $m := (m_0, m_1, \ldots)$ be a sequence of the positive integers not less than 2. Denote by $Z_{m_n} := \{0, 1, \ldots, m_n - 1\}$ the additive group of integers modulo m_n. Define the group G_m as the complete direct product of the groups Z_{m_n} with the product of the discrete topologies of Z_{m_n}'s. In this paper we discuss bounded Vilenkin groups, i.e. the case when $\sup_{n \in \mathbb{N}} m_n < \infty$.

The direct product μ of the measures $\mu_n(\{j\}) := 1/m_n,\ (j \in Z_{m_n})$ is the Haar measure on G_m with $\mu(G_m) = 1$.

The elements of G_m are represented by sequences

$$x := (x_0, x_1, \ldots, x_n, \ldots),\ (x_n \in Z_{m_n}).$$

It is easy to give a base for the neighbourhood of G_m :

$$I_0(x) := G_m,\ I_n(x) := \{y \in G_m \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\}\ (x \in G_m,\ n \in \mathbb{N}).$$

Set $I_n := I_n(0)$, for $n \in \mathbb{N}_+$ and

$$e_n := (0,\ldots,0,x_n = 1,0,\ldots) \in G_m \qquad (n \in \mathbb{N}).$$

Denote

$$I_N^{k,l} := \begin{cases} I_N(0,\ldots,0,x_k \neq 0,0,\ldots,0,x_l \neq 0,x_{l+1,\ldots,}\, x_{N-1}), & k < l < N, \\ I_N(0,\ldots,0,x_k \neq 0,0,\ldots,0), & l = N. \end{cases}$$

It is easy to show that

$$\overline{I_N} = \left(\bigcup_{i=0}^{N-2} \bigcup_{j=i+1}^{N-1} I_N^{i,j} \right) \bigcup \left(\bigcup_{i=0}^{N-1} I_N^{i,N} \right), \quad n = 2,3,... \tag{2}$$

If we define the so-called generalized number system based on m in the following way :

$$M_0 := 1,\ M_{n+1} := m_n M_n \quad (n \in \mathbb{N}),$$

then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_k M_k,$ where $n_k \in Z_{m_k}$ $(k \in \mathbb{N}_+)$ and only a finite number of n_k's differ from zero. Let

$$\langle n \rangle := \min\{j \in \mathbb{N} : n_j \neq 0\} \quad \text{and} \quad |n| := \max\{j \in \mathbb{N} : n_j \neq 0\},$$

that is $M_{|n|} \leqslant n \leqslant M_{|n|+1}$. Set $\rho(n) = |n| - \langle n \rangle,$ for all $n \in \mathbb{N}.$

Next, we introduce on G_m an orthonormal system, which is called the Vilenkin system. At first, we define the complex-valued function $r_k(x) : G_m \to \mathbb{C},$ the generalized Rademacher functions, by

$$r_k(x) := \exp(2\pi i x_k/m_k), \ \left(i^2 = -1, x \in G_m,\ k \in \mathbb{N}\right).$$

Now, define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on G_m as:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \quad (n \in \mathbb{N}).$$

Specifically, we call this system the Walsh-Paley system, when $m \equiv 2$.

The norms (or quasi-norms) of the spaces $L_p(G_m)$ and $weak-L_p(G_m)$ $(0 < p < \infty)$ are respectively defined by

$$\|f\|_p^p := \int_{G_m} |f|^p \, d\mu, \quad \|f\|_{weak-L_p}^p := \sup_{\lambda > 0} \lambda^p \mu(f > \lambda) < \infty.$$

The Vilenkin system is orthonormal and complete in $L_2(G_m)$ (see [20]).

If $f \in L_1(G_m)$ we can define Fourier coefficients, partial sums, Dirichlet kernels, Fejér means, Fejér kernels with respect to the Vilenkin system in the usual manner:

$$\widehat{f}(k) := \int_{G_m} f \overline{\psi}_k d\mu \quad (k \in \mathbb{N}),$$

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}(k)\,\psi_k, \qquad D_n := \sum_{k=0}^{n-1} \psi_k \qquad (n \in \mathbb{N}_+),$$

$$\sigma_n f := \frac{1}{n}\sum_{k=0}^{n-1} S_k f, \qquad K_n := \frac{1}{n}\sum_{k=0}^{n-1} D_k \quad (n \in \mathbb{N}_+).$$

Recall that (see e.g. [1])

$$D_{M_n}(x) = \begin{cases} M_n, & \text{if } x \in I_n, \\ 0, & \text{if } x \notin I_n, \end{cases} \tag{3}$$

and

$$D_{s_n M_n} = D_{s_n M_n} \sum_{k=0}^{s_n-1} \psi_{kM_n} = D_{M_n} \sum_{k=0}^{s_n-1} r_n^k, \tag{4}$$

where $n \in \mathbb{N}$ and $1 \leqslant s_n \leqslant m_n - 1$.

The σ-algebra generated by the intervals $\{I_n(x) : x \in G_m\}$ will be denoted by $\digamma_n$ $(n \in \mathbb{N})$. Denote by $f = \left(f^{(n)}, n \in \mathbb{N}\right)$ a martingale with respect to $\digamma_n$ $(n \in \mathbb{N})$ (for details see e.g. [21]). The maximal function of a martingale f is defined by

$$f^* = \sup_{n \in \mathbb{N}} \left| f^{(n)} \right|.$$

In the case $f \in L_1(G_m)$, the maximal functions are just also given by

$$f^*(x) = \sup_{n \in \mathbb{N}} \frac{1}{|I_n(x)|} \left| \int_{I_n(x)} f(u)\,\mu(u) \right|.$$

For $0 < p < \infty$ the Hardy martingale spaces $H_p(G_m)$ consist of all martingales f, for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

If $f \in L_1(G_m)$, then it is easy to show that the sequence $(S_{M_n}(f) : n \in \mathbb{N})$ is a martingale. If $f = \left(f^{(n)}, n \in \mathbb{N}\right)$ is a martingale, then the Vilenkin-Fourier coefficients must be defined in a slightly different manner:

$$\widehat{f}(i) := \lim_{k \to \infty} \int_{G_m} f^{(k)}(x)\,\overline{\psi}_i(x)\,d\mu(x).$$

The Vilenkin-Fourier coefficients of $f \in L_1(G_m)$ are the same as those of the martingale $(S_{M_n} f : n \in \mathbb{N})$ obtained from f.

A bounded measurable function a is said to be a p-atom if there exists an interval I, such that

$$\int_I a\,d\mu = 0, \quad \|a\|_\infty \leqslant \mu(I)^{-1/p}, \quad \operatorname{supp}(a) \subset I.$$

For the proof of the main result (Theorem 1) we need the following Lemmas:

LEMMA 1. (see e.g. [22]) *A martingale* $f=\left(f^{(n)},n\in\mathbb{N}\right)$ *is in* $H_p\,(0<p\leqslant 1)$ *if and only if there exist a sequence* $(a_k,k\in\mathbb{N})$ *of p-atoms and a sequence* $(\mu_k,k\in\mathbb{N})$ *of real numbers such that for every* $n\in\mathbb{N}$:

$$\sum_{k=0}^{\infty}\mu_k S_{M_n}a_k=f^{(n)} \tag{5}$$

and

$$\sum_{k=0}^{\infty}|\mu_k|^p<\infty.$$

Moreover, $\|f\|_{H_p}\backsim\inf\left(\sum_{k=0}^{\infty}|\mu_k|^p\right)^{1/p}$, *where the infimum is taken over all decomposition of* f *of the form (5).*

LEMMA 2. (see e.g. [22]) *Suppose that an operator* T *is* σ*-linear and for some* $0<p\leqslant 1$

$$\int_{\bar{I}}|Ta|^p\,d\mu\leqslant c_p<\infty,$$

for every p*-atom* a, *where* I *denotes the support of the atom. If* T *is bounded from* L_∞ *to* L_∞, *then*

$$\|Tf\|_p\leqslant c_p\|f\|_{H_p}.$$

LEMMA 3. (see [7]) *Let* $n>t,\ t,n\in\mathbb{N},\ x\in I_t\setminus\ I_{t+1}$. *Then*

$$K_{M_n}(x)=\begin{cases}0, & \text{if } x-x_te_t\notin I_n,\\ \frac{M_t}{1-r_t(x)}, & \text{if } x-x_te_t\in I_n.\end{cases}$$

LEMMA 4. (see [17]) *Let* $x\in I_N^{i,j},\ i=0,\dots,N-1,\ j=i+1,\dots,N$. *Then*

$$\int_{I_N}|K_n(x-t)|\,d\mu(t)\leqslant\frac{cM_iM_j}{M_N^2},\quad \text{for } n\geqslant M_N.$$

LEMMA 5. (see [11]) *Let* $n\in\mathbb{N}$. *Then*

$$|K_n(x)|\leqslant\frac{c}{n}\sum_{l=\langle n\rangle}^{|n|}M_l\left|K_{M_l}\right|\leqslant c\sum_{l=\langle n\rangle}^{|n|}\left|K_{M_l}\right| \tag{6}$$

and

$$|nK_n|\geqslant\frac{M_{\langle n\rangle}^2}{2\pi\lambda},\quad x\in I_{\langle n\rangle+1}\left(e_{\langle n\rangle-1}+e_{\langle n\rangle}\right), \tag{7}$$

where $\lambda:=\sup m_n$.

3. The main result and applications

Our main result reads:

THEOREM 1. *a) Let $0<p<1/2,\ f\in H_p$. Then there exists an absolute constant c_p, depending only on p, such that*

$$\left\|\sigma_{n_k} f\right\|_{H_p} \leqslant \frac{c_p M_{|n_k|}^{1/p-2}}{M_{\langle n_k\rangle}^{1/p-2}} \|f\|_{H_p}.$$

b) (sharpness) Let $0<p<1/2$ and $\Phi(n)$ be any nondecreasing function, such that

$$\sup_{k\in\mathbb{N}} \rho(n_k)=\infty, \quad \overline{\lim_{k\to\infty}} \frac{M_{|n_k|}^{1/p-2}}{M_{\langle n_k\rangle}^{1/p-2}\Phi(n_k)}=\infty. \tag{8}$$

Then there exists a martingale $f\in H_p$, such that

$$\sup_{k\in\mathbb{N}} \left\|\frac{\sigma_{n_k} f}{\Phi(n_k)}\right\|_{weak-L_p}=\infty.$$

COROLLARY 1. *Let $0<p<1/2$, and $f\in H_p$. Then there exists an absolute constant c_p, depending only on p, such that*

$$\left\|\sigma_{n_k} f\right\|_{H_p} \leqslant c_p \|f\|_{H_p}, \quad k\in\mathbb{N}$$

if and only if

$$\sup_{k\in\mathbb{N}} \rho(n_k)<c<\infty.$$

As an application we also obtain the previous mentioned result by Weisz [21], [22] (Theorem W).

COROLLARY 2. *Let $0<p<1/2,\ f\in H_p$. Then there exists an absolute constant c_p, depending only on p, such that*

$$\|\sigma_{M_n} f\|_{H_p} \leqslant c_p \|f\|_{H_p}, \quad n\in\mathbb{N}.$$

On the other hand, the following unexpected result is true:

COROLLARY 3. *a) Let $0<p<1/2,\ f\in H_p$. Then there exists an absolute constant c_p, depending only on p, such that*

$$\|\sigma_{M_n+1} f\|_{H_p} \leqslant c_p M_n^{1/p-2} \|f\|_{H_p}, \quad n\in\mathbb{N}.$$

b) Let $0<p<1/2$ and $\Phi(n)$ be any nondecreasing function, such that

$$\overline{\lim_{k\to\infty}} \frac{M_k^{1/p-2}}{\Phi(k)}=\infty.$$

Then there exists a martingale $f \in H_p$, *such that*

$$\sup_{k\in\mathbb{N}}\left\|\frac{\sigma_{M_k+1}f}{\Phi(k)}\right\|_{weak-L_p}=\infty.$$

REMARK 1. From Corollary 2 we obtain that σ_{M_n} are bounded from H_p to H_p, but from Corollary 3 we conclude that σ_{M_n+1} are not bounded from H_p to H_p. The main reason is that Fourier coefficients of martingales $f\in H_p$ are not uniformly bounded (for details see e.g. [18]).

In the next corollary we state some estimates for the Walsh system only to clearly see the difference of divergence rates for the various subsequences:

COROLLARY 4. *a)Let* $0<p<1/2,\ \ f\in H_p$. *Then there exists an absolute constant* c_p, *depending only on* p, *such that*

$$\|\sigma_{2^n+1}f\|_{H_p}\leqslant c_p 2^{(1/p-2)n}\|f\|_{H_p},\quad n\in\mathbb{N} \tag{9}$$

and

$$\left\|\sigma_{2^n+2^{[n/2]}}f\right\|_{H_p}\leqslant c_p 2^{\frac{(1/p-2)n}{2}}\|f\|_{H_p},\quad n\in\mathbb{N}, \tag{10}$$

where $[n/2]$ *denotes an integer part of* $n/2$.

b) The rates $2^{(1/p-2)n}$ *and* $2^{\frac{(1/p-2)n}{2}}$ *in inequalities* (9) *and* (10) *are sharp in the same sense as in Theorem* 1.

4. Proof of Theorem 1

Proof. a) Since

$$\sup_{n\in\mathbb{N}}\int_{G_m}|K_n(x)|\,d\mu(x)\leqslant c<\infty, \tag{11}$$

we obtain that

$$\frac{M_{\langle n_k\rangle}^{1/p-2}\left|\sigma_{n_k}a(x)\right|}{M_{|n_k|}^{1/p-2}}$$

is bounded from L_∞ to L_∞. According to Lemma 2 we find that the proof of Theorem 1 will be complete, if we show that

$$\int_{\overline{I_N}}\left|\frac{M_{\langle n_k\rangle}^{1/p-2}\sigma_{n_k}a(x)}{M_{|n_k|}^{1/p-2}}\right|^p<c<\infty,$$

for every p-atom a, with support I and $\mu(I)=M_N^{-1}$. We may assume that $I=I_N$. It is easy to see that $\sigma_{n_k}(a)=0$ when $n_k\leqslant M_N$. Therefore, we can suppose that $n_k>M_N$.

Since $\|a\|_\infty \leqslant M_N^{1/p}$ we find that

$$\frac{M_{\langle n_k\rangle}^{1/p-2}\left|\sigma_{n_k}a(x)\right|}{M_{|n_k|}^{1/p-2}} \leqslant \frac{M_{\langle n_k\rangle}^{1/p-2}}{M_{|n_k|}^{1/p-2}}\int_{I_N}|a(t)|\left|K_{n_k}(x-t)\right|d\mu(t) \tag{12}$$

$$\leqslant \frac{M_{\langle n_k\rangle}^{1/p-2}\|a\|_\infty}{M_{|n_k|}^{1/p-2}}\int_{I_N}\left|K_{n_k}(x-t)\right|d\mu(t) \leqslant \frac{M_{\langle n_k\rangle}^{1/p-2}M_N^{1/p}}{M_{|n_k|}^{1/p-2}}\int_{I_N}\left|K_{n_k}(x-t)\right|d\mu(t)$$

$$\leqslant M_{\langle n_k\rangle}^{1/p-2}M_{|n_k|}^{2}\int_{I_N}\left|K_{n_k}(x-t)\right|d\mu(t).$$

Without loss the generality we may assume that $i<j$. Let $x\in I_N^{i,j}$ and $j<\langle n_k\rangle$. Then $x-t\in I_N^{i,j}$ for $t\in I_N$ and, according to Lemma 3, we obtain that

$$\left|K_{M_l}(x-t)\right|=0,\ \text{ for all } \langle n_k\rangle\leqslant\ l\leqslant|n_k|.$$

By applying (12) and (6) in Lemma 5, for $x\in I_N^{i,j},\ 0\leqslant i<j<\langle n_k\rangle$ we get that

$$\frac{M_{\langle n_k\rangle}^{1/p-2}\left|\sigma_{n_k}a(x)\right|}{M_{|n_k|}^{1/p-2}} \leqslant M_{\langle n_k\rangle}^{1/p-2}M_{|n_k|}^{2}\sum_{l=\langle n_k\rangle}^{|n_k|}\int_{I_N}\left|K_{M_l}(x-t)\right|d\mu(t)=0. \tag{13}$$

Let $x\in I_N^{i,j}$, where $\langle n_k\rangle\leqslant j\leqslant N$. Then, in the view of Lemma 4, we have that

$$\int_{I_N}\left|K_{n_k}(x-t)\right|d\mu(t)\leqslant\frac{cM_iM_j}{M_N^2}.$$

By using again (12) we find that

$$\frac{M_{\langle n_k\rangle}^{1/p-2}\left|\sigma_{n_k}a(x)\right|}{M_{|n_k|}^{1/p-2}} \leqslant \frac{M_{\langle n_k\rangle}^{1/p-2}M_N^{1/p}}{M_{|n_k|}^{1/p-2}}\int_{I_N}\left|K_{n_k}(x-t)\right|d\mu(t) \tag{14}$$

$$\leqslant \frac{M_{\langle n_k\rangle}^{1/p-2}M_N^{1/p}}{M_{|n_k|}^{1/p-2}}\frac{M_iM_j}{M_N^2}\leqslant M_{\langle n_k\rangle}^{1/p-2}M_iM_j.$$

By combining (2) and (12)-(14) we get that

$$\int_{\overline{I_N}}\left|\frac{M_{\langle n_k\rangle}^{1/p-2}\left|\sigma_{n_k}a(x)\right|}{M_{|n_k|}^{1/p-2}}\right|^p d\mu$$

$$=\sum_{i=0}^{N-2}\sum_{j=i+1}^{N-1}\int_{I_N^{i,j}}\left|\frac{M_{\langle n_k\rangle}^{1/p-2}\left|\sigma_{n_k}a(x)\right|}{M_{|n_k|}^{1/p-2}}\right|^p d\mu+\sum_{i=0}^{N-1}\int_{I_N^{k,N}}\left|\frac{M_{\langle n_k\rangle}^{1/p-2}\left|\sigma_{n_k}a(x)\right|}{M_{|n_k|}^{1/p-2}}\right|^p d\mu$$

$$\leqslant\sum_{i=0}^{\langle n_k\rangle-1}\sum_{j=\langle n_k\rangle}^{N-1}\int_{I_N^{i,j}}\left|\frac{M_{\langle n_k\rangle}^{1/p-2}\left|\sigma_{n_k}a(x)\right|}{M_{|n_k|}^{1/p-2}}\right|^p d\mu+\sum_{i=\langle n_k\rangle}^{N-2}\sum_{j=i+1}^{N-1}\int_{I_N^{i,j}}\left|\frac{M_{\langle n_k\rangle}^{1/p-2}\left|\sigma_{n_k}a(x)\right|}{M_{|n_k|}^{1/p-2}}\right|^p d\mu$$

$$+\sum_{i=0}^{N-1}\int_{I_N^{i,N}}\left|\frac{M_{\langle n_k\rangle}^{1/p-2}\left|\sigma_{n_k}a(x)\right|}{M_{|n_k|}^{1/p-2}}\right|^p d\mu$$

$$\leqslant \sum_{i=0}^{\langle n_k\rangle-1}\sum_{j=\langle n_k\rangle}^{N-1}\int_{I_N^{i,j}}\left|M_{\langle n_k\rangle}^{1/p-2}M_iM_j\right|^p d\mu+\sum_{i=\langle n_k\rangle}^{N-2}\sum_{j=i+1}^{N-1}\int_{I_N^{i,j}}\left|M_{\langle n_k\rangle}^{1/p-2}M_iM_j\right|^p d\mu$$

$$+\sum_{i=0}^{N-1}\int_{I_N^{i,N}}\left|M_{\langle n_k\rangle}^{1/p-2}M_iM_N\right|^p d\mu$$

$$\leqslant c_pM_{\langle n_k\rangle}^{1-2p}\sum_{i=0}^{\langle n_k\rangle-1}\sum_{j=\langle n_k\rangle}^{N-1}\frac{(M_iM_j)^p}{M_j}+c_pM_{\langle n_k\rangle}^{1-2p}\sum_{i=\langle n_k\rangle}^{N-2}\sum_{j=i+1}^{N-1}\frac{(M_iM_j)^p}{M_j}+c_pM_{\langle n_k\rangle}^{1-2p}\sum_{i=0}^{N-1}\frac{(M_iM_N)^p}{M_N}$$

$$\leqslant c_pM_{\langle n_k\rangle}^{1-2p}\sum_{i=0}^{\langle n_k\rangle}M_i^p\sum_{j=\langle n_k\rangle+1}^{N-1}\frac{1}{M_j^{1-p}}+M_{\langle n_k\rangle}^{1-2p}\sum_{i=\langle n_k\rangle}^{N-2}M_i^p\sum_{j=i+1}^{N-1}\frac{1}{M_j^{1-p}}+c_p\sum_{i=0}^{N-1}\frac{M_i^p}{M_N^p}$$

$$\leqslant c_pM_{\langle n_k\rangle}^{1-2p}M_{\langle n_k\rangle}^p\frac{1}{M_{\langle n_k\rangle}^{1-p}}+c_pM_{\langle n_k\rangle}^{1-2p}\sum_{i=\langle n_k\rangle}^{N-2}\frac{1}{M_i^{1-2p}}+c_p\leqslant c_p<\infty.$$

The proof of the a) part is complete.

b) Let $\{n_k : k\geqslant 0\}$ be a sequence of positive numbers, satisfying condition (8). Then

$$\sup_{k\in\mathbb{N}}\frac{M_{|n_k|}}{M_{\langle n_k\rangle}}=\infty. \tag{15}$$

Under condition (15) there exists a sequence $\{\alpha_k :\ k\geqslant 0\}\subset\{n_k :\ k\geqslant 0\}$ such that $\alpha_0\geqslant 3$ and

$$\sum_{k=0}^{\infty}\frac{M_{\langle\alpha_k\rangle}^{(1-2p)/2}\Phi^{p/2}(\alpha_k)}{M_{|\alpha_k|}^{(1-2p)/2}}<c<\infty. \tag{16}$$

Let

$$f^{(n)}=\sum_{\{k;\ |\alpha_k|<n\}}\lambda_k a_k,$$

where

$$\lambda_k=\frac{\lambda M_{\langle\alpha_k\rangle}^{(1/p-2)/2}\Phi^{1/2}(\alpha_k)}{M_{|\alpha_k|}^{(1/p-2)/2}}$$

and

$$a_k=\frac{M_{|\alpha_k|}^{1/p-1}}{\lambda}\left(D_{M_{|\alpha_k|+1}}-D_{M_{|\alpha_k|}}\right).$$

Here $\lambda=\sup_{n\in\mathbb{N}}m_n$. By applying Lemma 1 we can conclude that $f\in H_p$.

It is evident that

$$\widehat{f}(j)=\begin{cases} M_{|\alpha_k|}^{1/2p}M_{\langle\alpha_k\rangle}^{(1/p-2)/2}\Phi^{1/2}(\alpha_k), \\ \text{if } \ j\in\left\{M_{|\alpha_k|},..., \ M_{|\alpha_k|+1}-1\right\},\ k=0,1,2..., \\ 0\,, \\ \text{if } \ j\notin\bigcup\limits_{k=0}^{\infty}\left\{M_{|\alpha_k|},..., \ M_{|\alpha_k|+1}-1\right\}. \end{cases} \tag{17}$$

Moreover,

$$\frac{\sigma_{\alpha_k}f}{\Phi(\alpha_k)}=\frac{1}{\alpha_k\Phi(\alpha_k)}\sum_{j=1}^{M_{|\alpha_k|}}S_jf+\frac{1}{\alpha_k\Phi(\alpha_k)}\sum_{j=M_{|\alpha_k|}+1}^{\alpha_k}S_jf:=I+II.$$

Let $M_{|\alpha_k|}<j\leqslant\alpha_k$. Then, by applying (17) we get that

$$S_jf=S_{M_{|\alpha_k|}}f+M_{|\alpha_k|}^{1/2p}M_{\langle\alpha_k\rangle}^{(1/p-2)/2}\Phi^{1/2}(\alpha_k)\left(D_j-D_{M_{|\alpha_k|}}\right). \tag{18}$$

By using (18) we can rewrite II as

$$\begin{aligned} II \ &= \ \frac{\alpha_k-M_{|\alpha_k|}}{\alpha_k\Phi(\alpha_k)}S_{M_{|\alpha_k|}}f+\frac{M_{|\alpha_k|}^{1/2p}M_{\langle\alpha_k\rangle}^{(1/p-2)/2}}{\alpha_k\Phi^{1/2}(\alpha_k)}\sum_{j=M_{|\alpha_k|}}^{\alpha_k}\left(D_j-D_{M_{|\alpha_k|}}\right) \\ &:= II_1+II_2. \end{aligned}$$

Since (for details see e.g. [5] and [19])

$$\left\|S_{M_{|\alpha_k|}}f\right\|_{weak-L_p}\leqslant c_p\left\|f\right\|_{H_p}$$

we obtain that

$$\left\|II_1\right\|_{weak-L_p}^p\leqslant\left(\frac{\alpha_k-M_{|\alpha_k|}}{\alpha_k\Phi(\alpha_k)}\right)^p\left\|S_{M_{|\alpha_k|}}f\right\|_{weak-L_p}^p\leqslant\left\|S_{M_{|\alpha_k|}}f\right\|_{weak-L_p}^p\leqslant c_p\left\|f\right\|_{H_p}^p<\infty.$$

By using part a) of Theorem 1 (see also Corollary 2) we find that

$$\left\|I\right\|_{weak-L_p}^p=\left(\frac{M_{|\alpha_k|}}{\alpha_k\Phi(\alpha_k)}\right)^p\left\|\sigma_{M_{|\alpha_k|}}f\right\|_{weak-L_p}^p\leqslant c_p\left\|f\right\|_{H_p}^p<\infty.$$

Let $x\in I_{\langle\alpha_k\rangle+1}^{\langle\alpha_k\rangle-1,\langle\alpha_k\rangle}$. Under condition (8) we can conclude that $\langle\alpha_k\rangle\neq|\alpha_k|$ and $\left\langle\alpha_k-M_{|\alpha_k|}\right\rangle=\langle\alpha_k\rangle$. Since

$$D_{j+M_n}=D_{M_n}+\psi_{M_n}D_j=D_{M_n}+r_nD_j,\ \text{when}\ \ j<M_n \tag{19}$$

if we apply estimate (7) in Lemma 5 for II_2 we obtain that

$$\left|II_2\right|=\frac{M_{|\alpha_k|}^{1/2p}M_{\langle\alpha_k\rangle}^{(1/p-2)/2}}{\alpha_k\Phi^{1/2}(\alpha_k)}\left|\sum_{j=1}^{\alpha_k-M_{|\alpha_k|}}\left(D_{j+M_{|\alpha_k|}}-D_{M_{|\alpha_k|}}\right)\right|$$

$$= \frac{M_{|\alpha_k|}^{1/2p} M_{\langle\alpha_k\rangle}^{(1/p-2)/2}}{\alpha_k \Phi^{1/2}(\alpha_k)} \left| \psi_{M_{|\alpha_k|}} \sum_{j=1}^{\alpha_k - M_{|\alpha_k|}} D_j \right|$$

$$\geqslant \frac{c_p M_{|\alpha_k|}^{1/2p-1} M_{\langle\alpha_k\rangle}^{(1/p-2)/2}}{\Phi^{1/2}(\alpha_k)} \left(\alpha_k - M_{|\alpha_k|}\right) \left| K_{\alpha_k - M_{|\alpha_k|}} \right| \geqslant \frac{c_p M_{|\alpha_k|}^{1/2p-1} M_{\langle\alpha_k\rangle}^{(1/p+2)/2}}{\Phi^{1/2}(\alpha_k)}.$$

It follows that

$$\|II_2\|_{weak-L_p}^p$$

$$\geqslant c_p \left(\frac{M_{|\alpha_k|}^{(1/p-2)/2} M_{\langle\alpha_k\rangle}^{(1/p+2)/2}}{\Phi^{1/2}(\alpha_k)} \right)^p \mu \left\{ x \in G_m : |IV_2| \geqslant c_p M_{|\alpha_k|}^{(1/p-2)/2} M_{\langle\alpha_k\rangle}^{(1/p+2)/2} \right\}$$

$$\geqslant c_p \frac{M_{|\alpha_k|}^{1/2-p} M_{\langle\alpha_k\rangle}^{1/2+p} \mu \left\{ I_{\langle\alpha_k\rangle+1}^{\langle\alpha_k\rangle-1,\langle\alpha_k\rangle} \right\}}{\Phi^{p/2}(\alpha_k)} \geqslant \frac{c_p M_{|\alpha_k|}^{1/2-p}}{M_{\langle\alpha_k\rangle}^{1/2-p} \Phi^{p/2}(\alpha_k)}.$$

Hence, if we apply (16) for large k,

$$\left\| \sigma_{\alpha_k} f \right\|_{weak-L_p}^p \geqslant \|II_2\|_{weak-L_p}^p - \|II_1\|_{weak-L_p}^p - \|I\|_{weak-L_p}^p$$

$$\geqslant \frac{1}{2} \|II_2\|_{weak-L_p}^p \geqslant \frac{c_p M_{|\alpha_k|}^{1/2-p}}{2 M_{\langle\alpha_k\rangle}^{1/2-p} \Phi^{p/2}(\alpha_k)} \to \infty, \text{ as } k \to \infty.$$

The proof is complete.

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Paper C

Modulus of continuity and boundedness of subsequences of Vilenkin- Fejér means in the martingale Hardy spaces

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C

MODULUS OF CONTINUITY AND CONVERGENCE OF SUBSEQUENCES OF VILENKIN-FEJÉR MEANS IN THE MARTINGALE HARDY SPACES

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ABSTRACT. In this paper we find necessary and sufficient condition for the modulus of continuity for which subsequences of Fejér means with respect to Vilenkin systems are bounded from the Hardy space H_p to the Lebesgue space L_p, for all $0<p<1/2$.

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1. INTRODUCTION

It is known (for details see e.g. and [12] and the books [16] and [31, 34]) that the subsequence S_{M_n} of the partial sums are bounded from the martingale Hardy space H_p to the Lebesgue space L_p, for all $p>0$. It follows that for any $F\in H_p$,

$$\|S_{M_k}F-F\|_p\to 0, \quad \text{as} \quad k\to\infty$$

and

$$\|S_{M_k}F-F\|_{L_{p,\infty}}\to 0, \quad \text{as} \quad k\to\infty, \tag{1}$$

However, (see Tephnadze [12, 27]) there exists a martingale $F\in H_p$ $(0<p<1)$, such that

$$\sup_{n\in\mathbb{N}}\|S_{M_n+1}F\|_{L_{p,\infty}}=\infty.$$

The reason of the divergence of $S_{M_n+1}f$ is that when $0<p<1$ the Fourier coefficients of $f\in H_p$ are not uniformly bounded (see Tephnadze [26, 27]). In particular, for $f\in H_p(G_m)$ where $0<p<1$,

$$\|S_{n_k}f-f\|_p\to 0, \text{ as } k\to\infty$$

holds if and only if

$$\sup_{k\in\mathbb{N}}d(n_k)\leq c<\infty, \tag{2}$$

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where $d(n_k)$ is defined by (6).

In the one-dimensional case the weak-(1,1)-type inequality for the maximal operator of Fejér means $\sigma^* f := \sup_{n\in\mathbb{N}} |\sigma_n f|$ can be found in Schipp [15] for Walsh series and in Pál, Simon [9] for bounded Vilenkin series. Fujji [5] and Simon [18] verified that σ^* is bounded from H_1 to L_1. Weisz [33] generalized this result and proved boundedness of σ^* from the martingale space H_p to the Lebesgue space L_p for $p > 1/2$. Simon [17] gave a counterexample, which shows that boundedness does not hold for $0 < p < 1/2$. A counterexample for $p = 1/2$ was given by Goginava [7] (see also [2, 3] and [14]). Weisz [34] (see also [11] and [29]) proved that the maximal operator of the Fejér means σ^* is bounded from the Hardy space $H_{1/2}$ to the space $weak - L_{1/2}$. The boundedness of weighted maximal operators are considered in [20, 21], [28]. Similar problems for Walsh-Kaczmarz-Fejér means were considered in [8], [22, 23].

Weisz [32] (see also [31]) also proved that for any $p > 0$ the maximal operator

$$\sigma^{\nabla,*} f = \sup_{n\in\mathbb{N}} |\sigma_{M_n} f|$$

is bounded from the Hardy space H_p to the space L_p. It follows that for $F \in H_p$ we get

$$\|\sigma_{M_k} F - F\|_p \to 0, \quad \text{as} \quad k \to \infty$$

and

$$\|\sigma_{M_k} F - F\|_{L_{p,\infty}} \to 0, \quad \text{as} \quad k \to \infty, \tag{3}$$

Moreover, Weisz [32] (see also [31]) also proved that for any $f \in H_p$,

$$\|\sigma_{M_k} f - f\|_{H_p} \to 0, \quad \text{as} \quad k \to \infty. \tag{4}$$

In [10] was generalized result of Weisz (see Theorem W) and was proved that if $0 < p \le 1/2$ and $\{n_k : k \ge 0\}$ be a sequence of positive numbers, such that condition (6) is fulfilled. Then the maximal operator

$$\widetilde{\sigma}^{*,\nabla} f = \sup_{k\in\mathbb{N}} |\sigma_{n_k} f|$$

is bounded from the Hardy space H_p to the space L_p. Moreover, under condition (2) there exists an absolute constant c_p, depending only on p, such that

$$\|\sigma_{n_k} f\|_{H_p} \le c_p \|f\|_{H_p}.$$

It was also proved that these results are sharp.

In [13] was considered case when $\sup_{k\in\mathbb{N}} d(n_k) = \infty$ and was proved that the following is true:

Theorem PTT: (Persson, Tephnadze, Tutberidze)
a) Let $0 < p < 1/2$, $f \in H_p$. Then there exists an absolute constant c_p,

depending only on p, such that

$$\|\sigma_{n_k} f\|_{H_p} \leq \frac{c_p M_{|n_k|}^{1/p-2}}{M_{\langle n_k \rangle}^{1/p-2}} \|f\|_{H_p}.$$

b) (sharpness) Let $0 < p < 1/2$ *and* $\Phi(n)$ *be any nondecreasing function, such that*

$$\sup_{k\in\mathbb{N}} d(n_k) = \infty, \quad \overline{\lim_{k\to\infty}} \frac{M_{|n_k|}^{1/p-2}}{M_{\langle n_k \rangle}^{1/p-2} \Phi(n_k)} = \infty. \tag{5}$$

Then there exists a martingale $f \in H_p$*, such that*

$$\sup_{k\in\mathbb{N}} \left\| \frac{\sigma_{n_k} f}{\Phi(n_k)} \right\|_{L_{p,\infty}} = \infty.$$

Similar problems for Walsh system when $0 < p \leq 1/2$ was proved in [24]. Moreover, it was found necessary and sufficient condition for the modulus of continuity for which subsequences of Fejér means with respect to Walsh system are bounded from the Hardy space H_p to the Lebesgue space L_p, for all $0 < p \leq 1/2$.

The main aim of this paper is to generalized results considered in [24] for bounded Vilenkin systems when $0 < p < 1/2$. As applications, both some well-known and new results are pointed out.

We note that analogical results for Vilenkin systems when $p = 1/2$ are open problems.

This paper is organized as follows: in order not to disturb our discussions later on some definitions and notations are presented in Section 2. The main results and some of its consequences can be found in Section 3. For the proofs of the main results we need some auxiliary Lemmas. These results are presented in Section 4. The detailed proof of the mine result is given in Section 5.

2. Definitions and Notations

Denote by $\mathbb{N}_+$ the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Let $m := (m_0, m_1, \dots)$ be a sequence of the positive integers not less than 2. Denote by $Z_{m_n} := \{0, 1, \dots, m_n - 1\}$ the additive group of integers modulo m_n. Define the group G_m as the complete direct product of the groups Z_{m_n} with the product of the discrete topologies of Z_{m_n}'s.

In this paper we discuss bounded Vilenkin groups, i.e. the case when $\sup_{n\in\mathbb{N}} m_n < \infty$.

The direct product μ of the measures $\mu_n(\{j\}) := 1/m_n$, $(j \in Z_{m_n})$ is the Haar measure on G_m with $\mu(G_m) = 1$.

The elements of G_m are represented by sequences

$$x := (x_0, x_1, \dots, x_n, \dots), \ (x_n \in Z_{m_n}).$$

It is easy to give a base for the neighbourhood of G_m :

$I_0(x) := G_m,\ I_n(x) := \{y \in G_m \mid y_0 = x_0, \dots, y_{n-1} = x_{n-1}\}\ (x \in G_m,\ n \in \mathbb{N})$.

Set $I_n := I_n(0)$, for $n \in \mathbb{N}_+$ and

$$e_n := (0, \dots, 0, x_n = 1, 0, \dots) \in G_m \qquad (n \in \mathbb{N}).$$

Denote

$$I_N^{k,l} := \begin{cases} I_N(0, \dots, 0, x_k \neq 0, 0, \dots, 0, x_l \neq 0, x_{l+1,\dots,} x_{N-1}), & k < l < N, \\ I_N(0, \dots, 0, x_k \neq 0, 0, \dots, 0), & l = N. \end{cases}$$

It is easy to show that

$$\overline{I_N} = \left(\bigcup_{i=0}^{N-2} \bigcup_{j=i+1}^{N-1} I_N^{i,j}\right) \bigcup \left(\bigcup_{i=0}^{N-1} I_N^{i,N}\right).$$

If we define the so-called generalized number system based on m in the following way :

$$M_0 := 1,\ M_{n+1} := m_n M_n \quad (n \in \mathbb{N}),$$

then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_k M_k$, where $n_k \in Z_{m_k}$ $(k \in \mathbb{N}_+)$ and only a finite number of n_k's differ from zero. Let

$$\langle n \rangle := \min\{j \in \mathbb{N} : n_j \neq 0\} \quad \text{and} \quad |n| := \max\{j \in \mathbb{N} : n_j \neq 0\},$$

that is $M_{|n|} \leq n \leq M_{|n|+1}$.

Set

$$d(n) = |n| - \langle n \rangle \ \textit{for all} \ n \in \mathbb{N}. \tag{6}$$

Next, we introduce on G_m an orthonormal system, which is called the Vilenkin system. At first, we define the complex-valued function $r_k(x) : G_m \to \mathbb{C}$, the generalized Rademacher functions, by

$$r_k(x) := \exp(2\pi i x_k / m_k), \ \left(i^2 = -1, x \in G_m,\ k \in \mathbb{N}\right).$$

Now, define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on G_m as:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \quad (n \in \mathbb{N}).$$

Specifically, we call this system the Walsh-Paley system, when $m \equiv 2$.

The norms (or quasi-norms) of the spaces $L_p(G_m)$ and $weak - L_p(G_m)$ $(0 < p < \infty)$ are respectively defined by

$$\|f\|_p^p := \int_{G_m} |f|^p \, d\mu, \quad \|f\|_{L_{p,\infty}}^p := \sup_{\lambda > 0} \lambda^p \mu(|f| > \lambda) < \infty.$$

The Vilenkin system is orthonormal and complete in $L_2(G_m)$ (see [30]).

If $f\in L_1(G_m)$ we can define Fourier coefficients, partial sums, Dirichlet kernels, Fejér means, Fejér kernels with respect to the Vilenkin system in the usual manner:

$$\widehat{f}(k):=\int_{G_m} f\overline{\psi}_k d\mu \quad (k\in\mathbb{N}),$$

$$S_n f := \sum_{k=0}^{n-1}\widehat{f}(k)\psi_k, \qquad D_n:=\sum_{k=0}^{n-1}\psi_k \qquad (n\in\mathbb{N}_+),$$

$$\sigma_n f := \frac{1}{n}\sum_{k=0}^{n-1}S_k f, \qquad K_n:=\frac{1}{n}\sum_{k=0}^{n-1}D_k \quad (n\in\mathbb{N}_+).$$

Recall that (see e.g. [1])

$$D_{M_n}(x)=\begin{cases} M_n, & \text{if } x\in I_n,\\ 0, & \text{if } x\notin I_n.\end{cases}$$

The σ-algebra generated by the intervals $\{I_n(x): x\in G_m\}$ will be denoted by F_n $(n\in\mathbb{N})$. Denote by $F=\left(F^{(n)}, n\in\mathbb{N}\right)$ a martingale with respect to F_n $(n\in\mathbb{N})$ (for details see e.g. [31]). The maximal function of a martingale F is defined by

$$F^*=\sup_{n\in\mathbb{N}}\left|F^{(n)}\right|.$$

In the case $f\in L_1(G_m)$, the maximal functions are also be given by

$$f^*(x)=\sup_{n\in\mathbb{N}}\frac{1}{|I_n(x)|}\left|\int_{I_n(x)} f(u)\,\mu(u)\right|.$$

For $0<p<\infty$ the Hardy martingale spaces $H_p(G_m)$ consist of all martingales F, for which

$$\|F\|_{H_p}:=\|F^*\|_p<\infty.$$

If $f\in L_1(G_m)$, then it is easy to show that the sequence $(S_{M_n}f: n\in\mathbb{N})$ is a martingale. If $F=\left(F^{(n)}, n\in\mathbb{N}\right)$ is martingale, then the Vilenkin-Fourier coefficients must be defined in a slightly different manner:

$$\widehat{F}(i):=\lim_{k\to\infty}\int_{G_m}F^{(k)}\overline{\psi}_i d\mu.$$

The Vilenkin-Fourier coefficients of $f\in L_1(G_m)$ are the same as those of the martingale $(S_{M_n}f: n\in\mathbb{N})$ obtained from f.

A bounded measurable function a is said to be a p-atom if there exists an interval I, such that

$$\int_I a\,d\mu=0, \quad \|a\|_\infty\leq\mu(I)^{-1/p}, \quad \operatorname{supp}(a)\subset I.$$

The modulus of continuity of the function $f\in L_p(G_m)$, is defined by

$$\omega_p(1/M_n,f):=\sup_{h\in I_n}\|f(\cdot+h)-f(\cdot)\|_p.$$

The concept of modulus of continuity in $H_p(G_m)$ $(p>0)$ is defined in the following way

$$\omega_{H_p}\left(1/M_n, F\right) := \|F - S_{M_n}F\|_{H_p}.$$

We need to understand the meaning of the expression $F - S_{M_n}F$ where F is a martingale and $S_{M_n}F$ is function. Since

$$S_{M_n}F = F^{(n)}, \quad \text{for} \quad F = \left(F^{(n)} : n \in \mathbb{N}\right) \in H_p$$

and

$$\begin{aligned}
\left(S_{M_k}F^{(n)} : k \in \mathbb{N}\right) &= \left(S_{M_k}S_{M_n}F, k \in \mathbb{N}\right) \\
&= \left(S_{M_0}F, \ldots, S_{M_{n-1}}F, S_{M_n}F, S_{M_n}F, \ldots\right) \\
&= \left(f^{(0)}, \ldots, f^{(n-1)}, f^{(n)}, f^{(n)}, \ldots\right)
\end{aligned}$$

we obtain that $F - S_{M_n}F$ is a martingale, for which

$$(F - S_{M_n}F)_k = \begin{cases} 0, & k = 0, \ldots, n, \\ F_k - F_n, & k \geq n+1, \end{cases} \tag{7}$$

Since $\|F\|_{H_p} \sim \|F\|_p$, for $p>1$, we obtain that

$$\omega_{H_p}\left(1/M_n, F\right) \sim \|F - S_{M_n}F\|_p, \quad p > 1.$$

On the other hand, there are strong connection among this definitions:

$$\omega_p\left(1/M_n, f\right)/2 \leq \|f - S_{M_n}f\|_p \leq \omega_p\left(1/M_n, f\right),$$

and

$$\|f - S_{M_n}f\|_p/2 \leq E_{M_n}\left(f, L_p\right) \leq \|f - S_{M_n}f\|_p.$$

3. The Main Result and applications

Our main result reads:

Theorem 1. *a) Let* $0<p<1/2$, $F \in H_p(G_m)$, $\sup_{k\in\mathbb{N}} d\left(n_k\right) = \infty$ *and*

$$\omega_{H_p}\left(1/M_{|n_k|}, F\right) = o\left(\frac{M_{\langle n_k\rangle}^{1/p-2}}{M_{|n_k|}^{1/p-2}}\right), \quad \text{as} \quad k \to \infty. \tag{8}$$

Then

$$\|\sigma_{n_k}F - F\|_{H_p} \to 0, \quad \text{as} \quad k \to \infty. \tag{9}$$

b) Let $\sup_{k\in\mathbb{N}} d\left(n_k\right) = \infty$. *Then there exists a martingale* $F \in H_p(G_m)$ $(0<p<1/2)$, *for which*

$$\omega_{H_p}\left(1/M_{|n_k|}, F\right) = O\left(\frac{M_{\langle n_k\rangle}^{1/p-2}}{M_{|n_k|}^{1/p-2}}\right), \quad \text{as} \quad k \to \infty \tag{10}$$

and

$$\|\sigma_{n_k}F - F\|_{L_{p,\infty}} \nrightarrow 0, \quad as \ k \to \infty. \tag{11}$$

Corollary 1. *Let $0 < p < 1/2$, and $F \in H_p(G_m)$. Then there exists an absolute constant c_p, depending only on p, such that*

$$\|\sigma_{n_k}F\|_{H_p} \leq c_p \|F\|_{H_p}, \quad k \in \mathbb{N}$$

if and only if when $\sup_{k\in\mathbb{N}} d(n_k) < c < \infty$.

As a application we also obtain the previous mentioned result by Weisz [31], [32]:

Corollary 2. *Let $0 < p < 1/2$, $F \in H_p(G_m)$. Then there exists an absolute constant c_p, depending only on p, such that*

$$\|\sigma_{M_n}F\|_{H_p} \leq c_p \|F\|_{H_p}, \quad n \in \mathbb{N}.$$

On the other hand, the following unexpected new result is also obtained:

Corollary 3. *a)Let $0 < p < 1/2$, $F \in H_p$. Then there exists an absolute constant c_p, depending only on p, such that*

$$\|\sigma_{M_n+1}F\|_{H_p} \leq c_p M_n^{1/p-2} \|F\|_{H_p}, \quad n \in \mathbb{N}.$$

b) Let $0 < p < 1/2$ and $\Phi(n)$ be any nondecreasing function, such that

$$\overline{\lim_{k\to\infty}} \frac{M_k^{1/p-2}}{\Phi(k)} = \infty.$$

Then there exists a martingale $F \in H_p$, such that

$$\sup_{k\in\mathbb{N}} \left\| \frac{\sigma_{M_k+1}F}{\Phi(k)} \right\|_{L_{p,\infty}} = \infty.$$

Remark 1. *From Corollary 2 we obtain that σ_{M_n} are bounded from $H_p(G_m)$ to $H_p(G_m)$, but from Corollary 3 we conclude that σ_{M_n+1} are not bounded from $H_p(G_m)$ to $H_p(G_m)$. The main reason is that Fourier coefficients of martingale $f \in H_p(G_m)$, $(0 < p < 1)$ are not uniformly bounded (for details see e.g.* [25]*).*

In the next corollary we state theorem for Walsh system only to clearly see difference of divergence rates for the various subsequences:

Corollary 4. *a) Let $0 < p < 1/2$, $F \in H_p(G_m)$. Then there exists an absolute constant c_p, depending only on p, such that*

$$\|\sigma_{2^n+1}F\|_{H_p} \leq c_p 2^{n(1/p-2)} \|F\|_{H_p}, \quad n \in \mathbb{N} \tag{12}$$

and

$$\|\sigma_{2^{2n}+2^n}F\|_{H_p} \leq c_p 2^{n(1/p-2)} \|F\|_{H_p}, \quad n \in \mathbb{N}. \tag{13}$$

Here σ_{2^n+1} and $\sigma_{2^{2n}+2^n}$ are Fejér means of Walsh-Fourier series.

b) The rates $2^{n(1/p-2)}$ and $2^{n(1/2p-1)}$ in inequalities (12) and (13) are sharp in the same sense as in Theorem 1.

4. AUXILIARY LEMMAS

For the proof of Theorem 1 we need the following Lemmas:

Lemma 1 (see e.g. [32]). *A martingale* $F=\left(F^{(n)},n\in\mathbb{N}\right)$ *is in* $H_p(G_m)\,(0<p\leq 1)$ *if and only if there exist a sequence* $(a_k,k\in\mathbb{N})$ *of p-atoms and a sequence* $(\mu_k,k\in\mathbb{N})$ *of real numbers such that for every* $n\in\mathbb{N}$ *:*

$$\sum_{k=0}^{\infty}\mu_k S_{M_n}a_k=F^{(n)} \tag{14}$$

and

$$\sum_{k=0}^{\infty}|\mu_k|^p<\infty.$$

Moreover, $\|F\|_{H_p(G_m)}\backsim\inf\left(\sum_{k=0}^{\infty}|\mu_k|^p\right)^{1/p}$, *where the infimum is taken over all decomposition of* f *of the form (14).*

Lemma 2 (see e.g. [32]). *Suppose that an operator* T *is* σ*-linear and for some* $0<p\leq 1$

$$\int_{\overline{I}}|Ta|^p\,d\mu\leq c_p<\infty$$

for every p-atom a*, where* I *denote the support of the atom. If* T *is bounded from* L_∞ *to* L_∞*, then*

$$\|TF\|_p\leq c_p\|F\|_{H_p}.$$

Lemma 3 (see [6]). *Let* $n>t$, $t,n\in\mathbb{N}$, $x\in I_t\backslash I_{t+1}$. *Then*

$$K_{M_n}(x)=\begin{cases}0, & \text{if } x-x_te_t\notin I_n,\\ \frac{M_t}{1-r_t(x)}, & \text{if } x-x_te_t\in I_n.\end{cases}$$

Lemma 4 (see [21]). *Let* $x\in I_N^{i,j}$, $i=0,\ldots,N-1$, $j=i+1,\ldots,N$. *Then*

$$\int_{I_N}|K_n(x-t)|\,d\mu(t)\leq\frac{cM_iM_j}{M_N^2},\qquad \text{for } n\geq M_N.$$

Lemma 5 (see [10]). *Let* $n\in\mathbb{N}$. *Then exists an absolute constant* c, *such that the following upper estimation holds true*

$$|K_n(x)|\leq\frac{c}{n}\sum_{l=\langle n\rangle}^{|n|}M_l\,|K_{M_l}|\leq c\sum_{l=\langle n\rangle}^{|n|}|K_{M_l}|. \tag{15}$$

Moreover, we have the following lower estimation:

$$|nK_n|\geq\frac{M_{\langle n\rangle}^2}{2\pi\lambda},\quad \text{for } x\in I_{\langle n\rangle+1}\left(e_{\langle n\rangle-1}+e_{\langle n\rangle}\right),\quad \text{where } \lambda:=\sup_{n\in\mathbb{N}}m_n. \tag{16}$$

5. Proof

Proof of Theorem 1. Let $0 < p < 1/2$, $f \in H_p(G_m)$ and $M_k < n \leq M_{k+1}$. By applying part a) of Theorem PTT we can conclude that

$$\|\sigma_n F - F\|_{H_p}^p$$
$$\leq \|\sigma_n F - \sigma_n S_{M_k} F\|_{H_p}^p + \|\sigma_n S_{M_k} F - S_{M_k} F\|_{H_p}^p + \|S_{M_k} F - F\|_{H_p}^p$$
$$= \|\sigma_n (S_{M_k} F - F)\|_{H_p}^p + \|S_{M_k} F - F\|_{H_p}^p + \|\sigma_n S_{M_k} F - S_{M_k} F\|_{H_p}^p$$
$$\leq c_p \left(\frac{M_{|n|}^{1-2p}}{M_{\langle n \rangle}^{1-2p}} + 1 \right) \omega_{H_p}^p (1/M_n, F) + \|\sigma_n S_{M_k} F - S_{M_k} F\|_{H_p}^p .$$

By simple calculation we get that

$$\sigma_n S_{M_k} F - S_{M_k} F = \frac{M_k}{n} (S_{M_k} \sigma_{M_k} F - S_{M_k} F) = \frac{M_k}{n} S_{M_k} (\sigma_{M_k} F - F) .$$

Let $p > 0$. Then (see inequality (4))

$$\|\sigma_n S_{M_k} F - S_{M_k} F\|_{H_p}^p \tag{17}$$
$$\leq \frac{2^{M_k}}{n^p} \|S_{M_k} (\sigma_{M_k} F - F)\|_{H_p}^p \leq c_p \|\sigma_{M_k} F - F\|_{H_p}^p \to 0, \text{ as } k \to \infty.$$

On the other hand, under the condition (8) we also get that

$$c_p \left(\frac{M_{|n|}^{1-2p}}{M_{\langle n \rangle}^{1-2p}} + 1 \right) \omega_{H_p}^p (1/M_n, F) \to 0 \tag{18}$$

by combining (17) and (18) we complete the proof of part a).

Now, prove second part of theorem. Since $\sup_{k \in \mathbb{N}} d(n_k) = \infty$, we obtain that, for $0 < p < 1/2$ there exists a subsequence $\{s_k : k \geq 1\} \subset \{n_k : k \geq 1\}$ such that $\lim_{k \to \infty} d(s_k) = \infty$ and

$$\frac{M_{\langle s_k \rangle}^{1/p-2}}{M_{|s_k|}^{1/p-2}} = \frac{1}{\left(m_{\langle s_k \rangle} \dots m_{|s_k|-1}\right)^{1/p-2}} \leq \frac{1}{2^{d(s_k)(1/p-2)}} \to 0.$$

It follows that there exists $\{\alpha_k : k \geq 1\} \subset \{s_k : k \geq 1\}$ such that $|\alpha_0| \neq \langle \alpha_0 \rangle$, $d(\alpha_k)$ is an increasing sequence satisfying $\lim_{k \to \infty} d(\alpha_k) = \infty$ and

$$\frac{M_{\langle \alpha_k \rangle}^{1/p-2}}{M_{|\alpha_k|}^{1/p-2}} \leq \left(\frac{M_{\langle \alpha_{k-1} \rangle}^{1/p-2}}{M_{|\alpha_{k-1}|}^{1/p-2}} \right)^2 \quad \text{for all} \quad k \in \mathbb{N}. \tag{19}$$

By using (19) we get that

$$\frac{M_{\langle \alpha_k \rangle}^{1/p-2}}{M_{|\alpha_k|}^{1/p-2}} \leq \left(\frac{M_{\langle \alpha_{k-1} \rangle}^{1/p-2}}{M_{|\alpha_{k-1}|}^{1/p-2}} \right)^2 \leq \dots \leq \left(\frac{M_{\langle \alpha_0 \rangle}^{1/p-2}}{M_{|\alpha_0|}^{1/p-2}} \right)^{k+1} \leq \frac{1}{2^{(k+1)(|\alpha_0| - \langle \alpha_0 \rangle)(1/p-2)}}$$

and

$$\sum_{k=0}^{\infty}\left(\frac{M_{\langle\alpha_k\rangle}^{1/p-2}}{M_{|\alpha_k|}^{1/p-2}}\right)^p \leq \sum_{k=0}^{\infty}\frac{1}{2^{(k+1)(|\alpha_0|-\langle\alpha_0\rangle)(1-2p)}} < c < \infty. \tag{20}$$

For $\lambda = \sup_{k\in\mathbb{N}} m_k$ we set $F = \left(F^{(n)}, n \in \mathbb{N}\right)$ where

$$F^{(n)} = \sum_{\{i:\ |\alpha_i|<n\}} \frac{\lambda M_{\langle\alpha_i\rangle}^{(1/p-2)}}{M_{|\alpha_i|}^{(1/p-2)}} a_i^{(p)}, \qquad a_k^{(p)} := \frac{M_{|\alpha_k|}^{1/p-1}}{\lambda}\left(D_{M_{|\alpha_k|+1}} - D_{M_{|\alpha_k|}}\right)$$

Since $a_i^{(p)}(x)$ is p-atom if we use equality (7) we find that

$$\left(F - S_{M_{|\alpha_n|}}F\right)_k = \begin{cases} 0, & k = 0,\ldots,|\alpha_n|, \\ F^{(k)} - F^{(|\alpha_n|)}, & k \geq |\alpha_n| + 1, \end{cases}$$

and

$$F - S_{M_{|\alpha_n|}}F = \left(0,...,0,\sum_{i=n}^{n+s}\frac{M_{\langle\alpha_i\rangle}^{1/p-2}}{M_{|\alpha_i|}^{1/p-2}}a_i^{(p)},...\right), \quad s \in \mathbb{N}_+$$

is martingale. On the other hand, according that $d(\alpha_n)$ is increasing and $d(\alpha_0) \neq 0$ we obtain that $d(\alpha_n) \neq 0$, for all $n \in \mathbb{N}_+$. Hence, by combining (19) and Lemma 1 we get that

$$\omega_{H_p}(1/M_{|\alpha_n|}, F) = \|F - S_{M_{|\alpha_n|}}F\|_{H_p}$$

$$\leq \sum_{i=n}^{\infty}\frac{M_{\langle\alpha_i\rangle}^{1/p-2}}{M_{|\alpha_i|}^{1/p-2}} \leq \sum_{i=1}^{\infty}\left(\frac{M_{\langle\alpha_n\rangle}^{1/p-2}}{M_{|\alpha_n|}^{1/p-2}}\right)^i = O\left(\frac{M_{\langle\alpha_n\rangle}^{1/p-2}}{M_{|\alpha_n|}^{1/p-2}}\right), \quad \text{as } n \to \infty.$$

It is easy to show that

$$\widehat{F}(j) = \begin{cases} M_{|\alpha_k|}M_{\langle\alpha_k\rangle}^{1/p-2}, & j \in \left\{M_{|\alpha_k|},...,M_{|\alpha_k|+1}-1\right\},\ k = 0,1,... \\ 0\,, & j \notin \bigcup\limits_{i=0}^{\infty}\left\{M_{|\alpha_k|},...,M_{|\alpha_k|+1}-1\right\}. \end{cases} \tag{21}$$

Let $M_{|\alpha_k|} < j < \alpha_k$. By using (21) we get that

$$S_jF = S_{M_{|\alpha_k|}}F + \sum_{v=M_{|\alpha_k|}}^{j-1}\widehat{F}(v)w_v = S_{M_{|\alpha_k|}}F + M_{|\alpha_k|}M_{\langle\alpha_k\rangle}^{1/p-2}\left(D_j - D_{M_{|\alpha_k|}}\right).$$

Hence,

$$\sigma_{\alpha_k}F = \frac{1}{\alpha_k}\sum_{j=1}^{M_{|\alpha_k|}}S_jF + \frac{1}{\alpha_k}\sum_{j=M_{|\alpha_k|}+1}^{\alpha_k}S_jF \tag{22}$$

$$= \frac{M_{|\alpha_k|}}{\alpha_k}\sigma_{M_{|\alpha_k|}}F + \frac{\left(\alpha_k - M_{|\alpha_k|}\right)S_{M_{|\alpha_k|}}F}{\alpha_k}$$

$$+\quad \frac{M_{|\alpha_k|}M_{\langle\alpha_k\rangle}^{1/p-2)}}{\alpha_k}\sum_{j=M_{|\alpha_k|}+1}^{\alpha_k}\left(D_j-D_{M_{|\alpha_k|}}\right)=I+II+III.$$

Since$D_{j+M_n}=D_{M_n}+\psi_{M_n}D_j$, when $j<M_{n+1}$ we obtain that

$$
\begin{aligned}
(23)\qquad |III| &= \frac{M_{|\alpha_k|}M_{\langle\alpha_k\rangle}^{1/p-2}}{\alpha_k}\left|\sum_{j=1}^{\alpha_k-M_{|\alpha_k|}}\left(D_{j+M_{|\alpha_k|}}-D_{M_{|\alpha_k|}}\right)\right| \\
&= \frac{M_{|\alpha_k|}M_{\langle n_k\rangle}^{1/p-2}}{\alpha_k}\left|\sum_{j=1}^{\alpha_k-M_{|\alpha_k|}}D_j\right| \\
&= \frac{M_{|\alpha_k|}M_{\langle\alpha_k\rangle}^{1/p-2}}{\alpha_k}\left(\alpha_k-M_{|\alpha_k|}\right)\left|K_{\alpha_k-M_{|\alpha_k|}}\right| \\
&\geq cM_{\langle\alpha_k\rangle}^{1/p-2}\left(\alpha_k-M_{|\alpha_k|}\right)\left|K_{\alpha_k-M_{|\alpha_k|}}\right|.
\end{aligned}
$$

By combining (22) and (23) we can conclude that

$$
\begin{aligned}
&\|\sigma_{\alpha_k}F-F\|_{L_{p,\infty}}^p=\|I+II+III-F\|_{L_{p,\infty}}^p \\
= &\ \|III+\frac{M_{|\alpha_k|}}{\alpha_k}\sigma_{M_{|\alpha_k|}}F+\frac{\left(\alpha_k-M_{|\alpha_k|}\right)S_{M_{|\alpha_k|}}F}{\alpha_k}-F\|_{L_{p,\infty}}^p \\
= &\ \|III+\frac{M_{|\alpha_k|}}{\alpha_k}\left(\sigma_{M_{|\alpha_k|}}F-F\right)+\frac{\alpha_k-M_{|\alpha_k|}}{\alpha_k}\left(S_{M_{|\alpha_k|}}F-F\right)\|_{L_{p,\infty}}^p \\
\geq &\ \|III\|_{L_{p,\infty}}^p-\left(\frac{M_{|\alpha_k|}}{\alpha_k}\right)^p\|\sigma_{M_{|\alpha_k|}}F-F\|_{L_{p,\infty}}^p \\
- &\ \left(\frac{\alpha_k-M_{|\alpha_k|}}{\alpha_k}\right)^p\|S_{M_{|\alpha_k|}}F-F\|_{L_{p,\infty}}^p \\
\geq &\ \|III\|_{L_{p,\infty}}^p-\|\sigma_{M_{|\alpha_k|}}F-F\|_{L_{p,\infty}}^p-\|S_{M_{|\alpha_k|}}F-F\|_{L_{p,\infty}}^p.
\end{aligned}
$$

By combining (1) and (3) we find that

$$\|S_{M_{|\alpha_k|}}F-F\|_{L_{p,\infty}}^p\to 0,\qquad \|\sigma_{M_{|\alpha_k|}}F-F\|_{L_{p,\infty}}^p\to 0,\quad \text{as}\quad k\to\infty,$$

and

$$\|\sigma_{\alpha_k}F-F\|_{L_{p,\infty}}^p\geq\|III\|_{L_{p,\infty}}^p-o(1),\quad \text{as}\quad k\to\infty.$$

Let $x\in I_{\langle\alpha_k\rangle+1}\left(e_{\langle\alpha_k\rangle-1}+e_{\langle\alpha_k\rangle}\right)$. By using Lemma 5 we have that

$$
\begin{aligned}
&\mu\left\{x\in G_m:\left(\alpha_k-M_{|\alpha_k|}\right)\left|K_{\alpha_k-M_{|\alpha_k|}}\right|\geq cM_{\langle\alpha_k\rangle}^2\right\} \\
\geq &\ \mu\left(I_{\langle\alpha_k\rangle+1}\left(e_{\langle\alpha_k\rangle-1}+e_{\langle\alpha_k\rangle}\right)\right)\geq\frac{c}{M_{\langle\alpha_k\rangle}},
\end{aligned}
$$

and

$$\begin{aligned} & \| \left(\alpha_k - M_{|\alpha_k|}\right) K_{\alpha_k - M_{|\alpha_k|}} \|_{L_{p,\infty}}^p \\ \geq & \ cM_{\langle\alpha_k\rangle}^{2p} \mu \left\{ x \in G_m : \left(\alpha_k - M_{|\alpha_k|}\right) \left| K_{\alpha_k - M_{|\alpha_k|}} \right| \geq cM_{\langle\alpha_k\rangle}^2 \right\} \\ \geq & \ cM_{\langle\alpha_k\rangle}^{2p} \frac{1}{M_{\langle\alpha_k\rangle}} = cM_{\langle\alpha_k\rangle}^{2p-1}. \end{aligned}$$

It follows that

$$\|III\|_{L_{p,\infty}}^p \geq M_{\langle\alpha_k\rangle}^{1-2p} \| \left(\alpha_k - M_{|\alpha_k|}\right) K_{\alpha_k - M_{|\alpha_k|}} \|_{L_{p,\infty}}^p \geq c > 0.$$

Hence, for sufficiently large k, we can write that

$$\|\sigma_{\alpha_k} F - F\|_{L_{p,\infty}}^p \geq \|III\|_{L_{p,\infty}}^p - o(1) \geq \frac{1}{2} \|III\|_{L_{p,\infty}}^p > \frac{c}{2} \nrightarrow 0, \text{ as } k \to \infty$$

and proof is complete. □

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Paper D

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Maximal operators of T means with respect to the Vilenkin system

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Abstract. In this paper we prove and discuss some new $(H_p, weak-L_p)$ type inequalities of maximal operators of T means with respect to Vilenkin systems with monotone coefficients. We also apply these results to prove a.e. convergence of such T means. It is also proved that these results are the best possible in a special sense. As applications, both some well-known and new results are pointed out.

1 Introduction

For the notation used in this introduction see Section 2.

Weisz [20] proved boundedness of σ^* from the martingale space H_p to the space L_p, for $p > 1/2$. Simon [13] gave a counterexample, which shows that boundedness does not hold for $0 < p < 1/2$. A counterexample for $p = 1/2$ was given by Goginava [6] (see also [15, 16] and [12]). Moreover, Weisz [22] proved that the following is true:

Theorem W1. The maximal operator of the Fejeans σ^* is bounded from the Hardy space $H_{1/2}$ to the space weak-$L_{1/2}$.

Riesz's logarithmic means with respect to the Walsh and Vilenkin systems were investigated by Simon [13], Blahota and Gbg. For the Vilenkin systems in [17] and for the Walsh system in [14] it were proved that the maximal operator of Riesz's means R^* is bounded from the Hardy space $H_{1/2}$ to the space $weak-L_{1/2}$, but is not bounded from the Hardy space H_p to the space L_p, when

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$0 < p \leqslant 1/2$. Since the set of Vilenkin polynomials are dense in L_1, by well-known density argument due to Marcinkiewicz and Zygmund [7] we have that $R_n f \to f$, a.e. for all $f \in L_1$.

Mz and Siddiqi [8] investigated the approximation properties of some special Nrlund means of Walsh-Fourier series of L_p function in norm. In the two-dimensional case similar problems was studied by Nagy [9, 10]. In [11] (see also [1, 5]) it was proved some (H_p, L_p)-type inequalities for the maximal operators of Nrlund means, when $0 < p \leqslant 1$.

In [3] and [4] were investigated T means and studied some approximation properties of these summability methods in the Lebesgue spaces for L_p, $1 \leqslant p \leqslant \infty$. In this paper we prove analogous theorems considered in [11] and derive some new (H_p, L_p)-type inequalities for the maximal operators of T means, when $0 < p \leqslant 1$. We also apply these results to prove a.e. convergence of such T means. It is also proved that these results are the best possible in a special sense. As applications, both some well-known and new results are pointed out.

The paper is organized as follows: In Section 3 we present and discuss the main results and in Section 4 the proofs can be found. Moreover, in order not to disturb our discussions in these Sections some preliminaries are given in Section 2.

2 Preliminaries

Denote by $\mathbb{N}_+$ the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Let $m := (m_0, m_1, ...)$ be a sequence of the positive integers not less than 2. Denote by

$$Z_{m_k} := \{0, 1, ..., m_k - 1\}$$

the additive group of integers modulo m_k.

Define the group G_m as the complete direct product of the groups Z_{m_i} with the product of the discrete topologies of Z_{m_j}'s.

The direct product μ of the measures

$$\mu_k(\{j\}) := 1/m_k \quad (j \in Z_{m_k})$$

is the Haar measure on G_m with $\mu(G_m) = 1$.

In this paper we discuss bounded Vilenkin groups, i.e. the case when $\sup_n m_n < \infty$.

The elements of G_m are represented by sequences

$$x := (x_0, x_1, ..., x_j, ...), \ (x_j \in Z_{m_j}).$$

It is easy to give a base for the neighborhood of G_m :

$$I_0(x) := G_m, \ \ I_n(x) := \{y \in G_m \mid y_0 = x_0, ..., y_{n-1} = x_{n-1}\},$$

where $x \in G_m$, $n \in \mathbb{N}$.

Denote $I_n := I_n(0)$ for $n \in \mathbb{N}_+$, and $\overline{I_n} := G_m \setminus I_n$.

If we define the so-called generalized number system based on m in the following way :

$$M_0 := 1, \ M_{k+1} := m_k M_k \quad (k \in \mathbb{N}),$$

then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j}$ $(j \in \mathbb{N}_+)$ and only a finite number of n_j's differ from zero.

Next, we introduce on G_m an orthonormal system which is called the Vilenkin system. At first, we define the complex-valued function $r_k(x) : G_m \to \mathbb{C}$, the generalized Rademacher functions, by

$$r_k(x) := \exp(2\pi i x_k/m_k), \quad \left(i^2 = -1, x \in G_m, \ k \in \mathbb{N}\right).$$

Now, define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on G_m as:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x), \quad (n \in \mathbb{N}).$$

Specifically, we call this system the Walsh-Paley system when $m \equiv 2$.

The norms (or quasi-norms) of the spaces $L_p(G_m)$ and $weak-L_p(G_m)$ $(0 < p < \infty)$ are respectively defined by

$$\|f\|_p^p := \int_{G_m} |f|^p \, d\mu \quad \text{and} \quad \|f\|_{weak-L_p}^p := \sup_{\lambda>0} \lambda^p \mu(f > \lambda) < +\infty.$$

The Vilenkin system is orthonormal and complete in $L_2(G_m)$ (see [18]).

Now, we introduce analogues of the usual definitions in Fourier-analysis. If $f \in L_1(G_m)$ we can define the Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels with respect to the Vilenkin system in the usual manner:

$$\widehat{f}(n) := \int_{G_m} f \overline{\psi}_n d\mu \qquad (n \in \mathbb{N}),$$

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}(k) \psi_k, \quad D_n := \sum_{k=0}^{n-1} \psi_k, \quad (n \in \mathbb{N}_+)$$

respectively. Recall that

$$D_{M_n}(x) = \begin{cases} M_n, & \text{if } x \in I_n, \\ 0, & \text{if } x \notin I_n. \end{cases} \tag{2.1}$$

The σ-algebra generated by the intervals $\{I_n(x) : x \in G_m\}$ will be denoted by $\digamma_n$ $(n \in \mathbb{N})$. Denote by $f = \left(f^{(n)}, n \in \mathbb{N}\right)$ a martingale with respect to $\digamma_n$ $(n \in \mathbb{N})$ (for details see e.g. [19]).

For $0 < p < \infty$ the Hardy martingale spaces $H_p(G_m)$ consist of all martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty, \quad \text{where} \quad f^* := \sup_{n \in \mathbb{N}} \left|f^{(n)}\right|.$$

A bounded measurable function a is called a p-atom, if there exists an interval I, such that

$$\int_I a \, d\mu = 0, \quad \|a\|_\infty \leqslant \mu(I)^{-1/p}, \quad \operatorname{supp}(a) \subset I.$$

Weisz [21] proved that the Hardy spaces H_p have atomic characterizations. In particular the following is true:

Proposition 2.1. A martingale $f = \left(f^{(n)}, n \in \mathbb{N}\right)$ is in H_p $(0 < p \leqslant 1)$ if and only if there exists a sequence $(a_k, k \in \mathbb{N})$ of p-atoms and a sequence $(\mu_k, k \in \mathbb{N})$, of real numbers, such that, for every $n \in \mathbb{N}$,

$$\sum_{k=0}^{\infty}\mu_k S_{M_n} a_k = f^{(n)}, \qquad \sum_{k=0}^{\infty}|\mu_k|^p < \infty. \tag{2.2}$$

Moreover,

$$\|f\|_{H_p} \backsim \inf\left(\sum_{k=0}^{\infty}|\mu_k|^p\right)^{1/p},$$

where the infimum is taken over all decomposition of f of the form (2.2). We also need the following result of Weisz [21]:

Proposition 2.2. Suppose that the operator T is σ-linear and for some $0 < p < 1$

$$\|Tf\|_{weak-L_p} \leqslant c_p \|f\|_{H_p},$$

then T is of weak type-(1,1) i.e.

$$\|Tf\|_{weak-L_1} \leqslant c\|f\|_1.$$

If $f = \left(f^{(n)}, n \in \mathbb{N}\right)$ is a martingale, then the Vilenkin-Fourier coefficients must be defined in a slightly different manner:

$$\widehat{f}(i) := \lim_{k\to\infty}\int_{G_m} f^{(k)}\overline{\Psi}_i d\mu.$$

Let $\{q_k : k \geqslant 0\}$ be a sequence of non-negative numbers. The n-th Nrlund and T means for a Fourier series of f are respectively defined by

$$t_n f = \frac{1}{Q_n}\sum_{k=1}^{n} q_{n-k} S_k f,$$

and

$$T_n f := \frac{1}{Q_n}\sum_{k=0}^{n-1} q_k S_k f, \tag{2.3}$$

where $Q_n := \sum_{k=0}^{n-1} q_k$. It is obvious that

$$T_n f(x) = \int_{G_m} f(t) F_n(x-t)\, d\mu(t),$$

where $F_n := \frac{1}{Q_n}\sum_{k=1}^{n} q_k D_k$ is called the T kernel.

We always assume that $\{q_k : k \geqslant 0\}$ is a sequence of non-negative numbers and $q_0 > 0$. Then the summability method (2.3) generated by $\{q_k : k \geqslant 0\}$ is regular if and only if $\lim_{n\to\infty} Q_n = \infty$.

If we invoke Abel transformation we get the following identities, which are very important for the investigations of T summability:

$$Q_n := \sum_{j=0}^{n-1} q_j = \sum_{j=0}^{n-2}(q_j - q_{j+1})\, j + q_{n-1}(n-1) \tag{2.4}$$

and

$$F_n = \frac{1}{Q_n}\left(\sum_{j=0}^{n-2}(q_j - q_{j+1})\, jK_j + q_{n-1}(n-1)K_{n-1}\right). \tag{2.5}$$

The well-known example of Nörlund summability is the so-called (C,α)-mean (Cesàro means), which are defined by

$$\sigma_n^{\alpha} f := \frac{1}{A_n^{\alpha}} \sum_{k=1}^{n} A_{n-k}^{\alpha-1} S_k f, \qquad 0<\alpha<1,$$

where

$$A_0^{\alpha} := 0, \qquad A_n^{\alpha} := \frac{(\alpha+1)\dots(\alpha+n)}{n!}, \qquad \alpha \neq -1,-2,\dots$$

We also consider the "inverse" (C,α)-means, which is an example of a T-means:

$$U_n^{\alpha} f := \frac{1}{A_n^{\alpha}} \sum_{k=0}^{n-1} A_k^{\alpha-1} S_k f, \qquad 0<\alpha<1.$$

Let V_n^{α} denote the T mean, where $\left\{q_0=1,\ q_k=k^{\alpha-1} : k\in\mathbb{N}_+\right\}$, that is

$$V_n^{\alpha} f := \frac{1}{Q_n} \sum_{k=1}^{n} k^{\alpha-1} S_k f, \qquad 0<\alpha<1.$$

The n-th Riesz‘s logarithmic mean R_n and the Nrlund logarithmic mean L_n are defined by

$$R_n f := \frac{1}{l_n} \sum_{k=1}^{n-1} \frac{S_k f}{k} \quad \text{and} \quad L_n f := \frac{1}{l_n} \sum_{k=1}^{n-1} \frac{S_k f}{n-k},$$

respectively, where $l_n := \sum_{k=1}^{n-1} 1/k$.

Up to now we have considered T means in the case when the sequence $\{q_k : k\in\mathbb{N}\}$ is bounded but now we consider T summabilities with unbounded sequence $\{q_k : k\in\mathbb{N}\}$. Let $\alpha\in\mathbb{R}_+,\ \beta\in\mathbb{N}_+$ and

$$\log^{(\beta)} x := \overbrace{\log\dots\log}^{\beta \text{times}} x.$$

If we define the sequence $\{q_k : k\in\mathbb{N}\}$ by $\left\{q_0=0,\ q_k=\log^{(\beta)} k^{\alpha} : k\in\mathbb{N}_+\right\}$, then we get the class of T means with non-decreasing coefficients:

$$B_n^{\alpha,\beta} f := \frac{1}{Q_n} \sum_{k=1}^{n} \log^{(\beta)} k^{\alpha} S_k f.$$

We note that $B_n^{\alpha,\beta}$ are well-defined for every $n\in\mathbb{N}$

$$B_n^{\alpha,\beta} f = \sum_{k=1}^{n} \frac{\log^{(\beta)} k^{\alpha}}{Q_n} S_k f.$$

It is obvious that $\frac{n}{2}\log^{(\beta)} \frac{n^{\alpha}}{2^{\alpha}} \leqslant Q_n \leqslant n\log^{(\beta)} n^{\alpha}$. It follows that

$$\frac{q_{n-1}}{Q_n} \leqslant \frac{c\log^{(\beta)} n^{\alpha}}{n\log^{(\beta)} n^{\alpha}} = O\left(\frac{1}{n}\right) \to 0, \ \text{as} \ n\to\infty. \tag{2.6}$$

We also define the maximal operator T^* of T means by

$$T^* f := \sup_{n\in\mathbb{N}} |T_n f|.$$

Some well-known examples of maximal operators of T means are the maximal operator of Fejσ^* and Riesz R^* logarithmic means, which are defined by:

$$\sigma^* f := \sup_{n\in\mathbb{N}} |\sigma_n f|, \quad R^* f := \sup_{n\in\mathbb{N}} |R_n f|.$$

We also define some new maximal operators $U^{\alpha,*}, V^{\alpha,*}, B^{\alpha,\beta,*}$, $(\alpha \in \mathbb{R}_+, \beta \in \mathbb{N}_+)$ by:

$$U^{\alpha,*} f := \sup_{n\in\mathbb{N}} |U_n^\alpha f|, \quad V^{\alpha,*} f := \sup_{n\in\mathbb{N}} |V_n^\alpha f|, \quad B^{\alpha,\beta,*} f := \sup_{n\in\mathbb{N}} \left|B_n^{\alpha,\beta} f\right|.$$

3 The Main Results

First we state our main result concerning the maximal operator of the summation method (2.3), which we also show is in a sense sharp.

Theorem 3.1. a) The maximal operator T^* of the summability method (2.3) with non-increasing sequence $\{q_k : k \geqslant 0\}$, is bounded from the Hardy space $H_{1/2}$ to the space $weak-L_{1/2}$.

The statement in a) is sharp in the following sense:

b) Let $0 < p < 1/2$ and $\{q_k : k \geqslant 0\}$ be a non-increasing sequence, satisfying the condition

$$\frac{q_{n+1}}{Q_{n+2}} \geqslant \frac{c}{n}, \quad (c \geqslant 1). \tag{3.1}$$

Then there exists a martingale $f \in H_p$, such that

$$\sup_{n\in\mathbb{N}} \|T_n f\|_{weak-L_p} = \infty.$$

A number of special cases of our results are of particular interest and give both well-known and new information. We just give the following examples of such T means with non-increasing sequence $\{q_k : k \geqslant 0\}$:

Corollary 3.1. The maximal operators $U^{\alpha,*}$, $V^{\alpha,*}$ and R^* are bounded from the Hardy space $H_{1/2}$ to the space $weak-L_{1/2}$ but are not bounded from H_p to the space $weak-L_p$, when $0 < p < 1/2$.

Corollary 3.2. Let $f \in L_1$ and T_n be the T means with non-increasing sequence $\{q_k : k \geqslant 0\}$. Then $T_n f \to f$, a.e., as $n \to \infty$.

Corollary 3.3. Let $f \in L_1$. Then

$$R_n f \to f, \quad \text{a.e.,} \quad \text{as } n \to \infty,$$
$$U_n^\alpha f \to f, \quad \text{a.e.,} \quad \text{as } n \to \infty,$$
$$V_n^\alpha f \to f, \quad \text{a.e.,} \quad \text{as } n \to \infty,$$

Our next main result reads:

Theorem 3.2. a) The maximal operator T^* of the summability method (2.3) with non-decreasing sequence $\{q_k : k \geqslant 0\}$ satisfying the condition

$$\frac{q_{n-1}}{Q_n} = O\left(\frac{1}{n}\right) \tag{3.2}$$

is bounded from the Hardy space $H_{1/2}$ to the space $weak-L_{1/2}$.

b) Let $0 < p < 1/2$. For any non-decreasing sequence $\{q_k : k \geqslant 0\}$, there exists a martingale $f \in H_p$, such that

$$\sup_{n\in\mathbb{N}} \|T_n f\|_{weak-L_p} = \infty.$$

A number of special cases of our results are of particular interest and give both well-known and new information. We just give the following examples of such T means with non-decreasing sequence $\{q_k : k \geqslant 0\}$:

Corollary 3.4. The maximal operator $B^{\alpha,\beta,*}$ is bounded from the Hardy space $H_{1/2}$ to the space $weak-L_{1/2}$ but is not bounded from H_p to the space $weak-L_p$, when $0 < p < 1/2$.

Corollary 3.5. Let $f \in L_1$ and T_n be the T means with *non-decreasing sequence* $\{q_k : k \geqslant 0\}$ *and* satisfying condition (3.2). Then

$$T_n f \to f, \quad \text{a.e.,} \quad \text{as } n \to \infty.$$

Corollary 3.6. Let $f \in L_1$. Then $B_n^{\alpha,\beta} f \to f, \quad$ a.e., $\quad$ as $n \to \infty$.

4 Proofs

Proof of Theorem 3.1 a). Let the sequence $\{q_k : k \geqslant 0\}$ be non-increasing. By combining (2.4) with (2.5) and using Abel transformation we get that

$$\begin{aligned}
|T_n f| &\leqslant \left| \frac{1}{Q_n} \sum_{j=1}^{n-1} q_j S_j f \right| \\
&\leqslant \frac{1}{Q_n} \left(\sum_{j=1}^{n-2} \left| q_j - q_{j+1} \right| j \left| \sigma_j f \right| + q_{n-1}(n-1) \left| \sigma_n f \right| \right) \\
&\leqslant \frac{1}{Q_n} \left(\sum_{j=1}^{n-2} (q_j - q_{j+1}) j + q_{n-1}(n-1) \right) \sigma^* f \leqslant \sigma^* f
\end{aligned}$$

so that

$$T^* f \leqslant \sigma^* f. \tag{4.1}$$

If we apply (4.1) and Theorem W1 we can conclude that the maximal operators T^* of all T means with non-increasing sequence $\{q_k : k \geqslant 0\}$, are bounded from the Hardy space $H_{1/2}$ to the space $weak-L_{1/2}$. The proof of part a) of Theorem 1 is complete.

b) Let $0 < p < 1/2$ and $\{\alpha_k : k \in \mathbb{N}\}$ be an increasing sequence of positive integers such that:

$$\sum_{k=0}^{\infty} 1/\alpha_k^p < \infty, \tag{4.2}$$

$$\lambda\sum_{\eta=0}^{k-1}\frac{M_{\alpha_\eta}^{1/p}}{\alpha_\eta}<\frac{M_{\alpha_k}^{1/p}}{\alpha_k}, \tag{4.3}$$

$$\frac{32\lambda M_{\alpha_{k-1}}^{1/p}}{\alpha_{k-1}}<\frac{M_{\alpha_k}^{1/p-2}}{\alpha_k}, \tag{4.4}$$

where $\lambda=\sup_n m_n$.

We note that such an increasing sequence $\{\alpha_k : k\in\mathbb{N}\}$ which satisfies conditions (4.2)-(4.4) can be constructed.

Let

$$f^{(A)}=\sum_{\{k;\ \lambda_k<A\}}\lambda_k a_k, \tag{4.5}$$

where

$$\lambda_k=\frac{\lambda}{\alpha_k}\quad and \quad a_k=\frac{M_{\alpha_k}^{1/p-1}}{\lambda}\left(D_{M_{\alpha_k+1}}-D_{M_{\alpha_k}}\right).$$

By using Proposition 2.1, it is easy to show that the martingale $f=\left(f^{(1)},f^{(2)}...f^{(A)}...\right)\in H_{1/2}$. Moreover, it is easy to show that

$$\widehat{f}(j)=\begin{cases}\frac{M_{\alpha_k}^{1/p-1}}{\alpha_k}, & \text{if } j\in\{M_{\alpha_k},...,M_{\alpha_k+1}-1\},\ k=0,1,2...,\\ 0, & \text{if } j\notin\bigcup_{k=1}^{\infty}\{M_{\alpha_k},...,M_{\alpha_k+1}-1\}.\end{cases} \tag{4.6}$$

We can write

$$T_{M_{\alpha_k}+2}f=\frac{1}{Q_{M_{\alpha_k}+2}}\sum_{j=0}^{M_{\alpha_k}}q_jS_jf+\frac{q_{M_{\alpha_k}+1}}{Q_{M_{\alpha_k}+2}}S_{M_{\alpha_k}+1}f:=I+II. \tag{4.7}$$

Let $M_{\alpha_s}\leqslant j\leqslant M_{\alpha_s+1}$, where $s=0,...,k-1$. Moreover,

$$\left|D_j-D_{M_{\alpha_s}}\right|\leqslant j-M_{\alpha_s}\leqslant\lambda M_{\alpha_s},\quad (s\in\mathbb{N})$$

so that, according to (2.1) and (4.6), we have that

$$\begin{aligned}\left|S_jf\right|&=\left|\sum_{v=0}^{M_{\alpha_{s-1}+1}-1}\widehat{f}(v)\psi_v+\sum_{v=M_{\alpha_s}}^{j-1}\widehat{f}(v)\psi_v\right| \qquad (4.8)\\ &\leqslant\left|\sum_{\eta=0}^{s-1}\sum_{v=M_{\alpha_\eta}}^{M_{\alpha_\eta+1}-1}\frac{M_{\alpha_\eta}^{1/p-1}}{\alpha_\eta}\psi_v\right|+\frac{M_{\alpha_s}^{1/p-1}}{\alpha_s}\left|\left(D_j-D_{M_{\alpha_s}}\right)\right|\\ &=\left|\sum_{\eta=0}^{s-1}\frac{M_{\alpha_\eta}^{1/p-1}}{\alpha_\eta}\left(D_{M_{\alpha_\eta+1}}-D_{M_{\alpha_\eta}}\right)\right|+\frac{M_{\alpha_s}^{1/p-1}}{\alpha_s}\left|\left(D_j-D_{M_{\alpha_s}}\right)\right|\\ &\leqslant\lambda\sum_{\eta=0}^{s-1}\frac{M_{\alpha_\eta}^{1/p}}{\alpha_\eta}+\frac{\lambda M_{\alpha_s}^{1/p}}{\alpha_s}\leqslant\frac{2\lambda M_{\alpha_{s-1}}^{1/p}}{\alpha_{s-1}}+\frac{\lambda M_{\alpha_s}^{1/p}}{\alpha_s}\leqslant\frac{4\lambda M_{\alpha_{k-1}}^{1/p}}{\alpha_{k-1}}.\end{aligned}$$

Let $M_{\alpha_{s-1}+1}+1 \leqslant j \leqslant M_{\alpha_s}$, where $s=1,...,k$. Analogously to (4.8) we can prove that

$$\begin{aligned} \left|S_j f\right| &= \left|\sum_{v=0}^{M_{\alpha_{s-1}+1}-1} \widehat{f}(v)\psi_v\right| = \left|\sum_{\eta=0}^{s-1} \sum_{v=M_{\alpha_\eta}}^{M_{\alpha_\eta+1}-1} \frac{M_{\alpha_\eta}^{1/p-1}}{\alpha_\eta}\psi_v\right| \\ &= \left|\sum_{\eta=0}^{s-1} \frac{M_{\alpha_\eta}^{1/p-1}}{\alpha_\eta}\left(D_{M_{\alpha_\eta+1}} - D_{M_{\alpha_\eta}}\right)\right| \leqslant \frac{2\lambda M_{\alpha_{s-1}}^{1/p}}{\alpha_{s-1}} \leqslant \frac{4\lambda M_{\alpha_{k-1}}^{1/p}}{\alpha_{k-1}}. \end{aligned}$$

Hence

$$|I| \leqslant \frac{1}{Q_{M_{\alpha_k}+2}} \sum_{j=0}^{M_{\alpha_k}} q_j \left|S_j f\right| \leqslant \frac{4\lambda M_{\alpha_{k-1}}^{1/p}}{\alpha_{k-1}} \frac{1}{Q_{M_{\alpha_k}+2}} \sum_{j=0}^{M_{\alpha_k}} q_j \leqslant \frac{4\lambda M_{\alpha_{k-1}}^{1/p}}{\alpha_{k-1}}. \tag{4.9}$$

If we now apply (4.6) and (4.8) we get that

$$\begin{aligned} |II| &= \frac{q_{M_{\alpha_k}+1}}{Q_{M_{\alpha_k}+2}} \left|\frac{M_{\alpha_k}^{1/p-1}}{\alpha_k}\psi_{M_{\alpha_k}} + S_{M_{\alpha_k}} f\right| \\ &= \frac{q_{M_{\alpha_k}+1}}{Q_{M_{\alpha_k}+2}} \left|\frac{M_{\alpha_k}^{1/p-1}}{\alpha_k}\psi_{M_{\alpha_k}} + S_{M_{\alpha_{k-1}+1}} f\right| \\ &\geqslant \frac{q_{M_{\alpha_k}+1}}{Q_{M_{\alpha_k}+2}} \left(\left|\frac{M_{\alpha_k}^{1/p-1}}{\alpha_k}\psi_{M_{\alpha_k}}\right| - \left|S_{M_{\alpha_{k-1}+2}} f\right|\right) \\ &\geqslant \frac{q_{M_{\alpha_k}+1}}{Q_{M_{\alpha_k}+2}} \left(\frac{M_{\alpha_k}^{1/p-1}}{\alpha_k} - \frac{4\lambda M_{\alpha_{k-1}}^{1/p}}{\alpha_{k-1}}\right) \\ &\geqslant \frac{q_{M_{\alpha_k}+1}}{Q_{M_{\alpha_k}+2}} \frac{M_{\alpha_k}^{1/p-1}}{4\alpha_k}. \end{aligned} \tag{4.10}$$

Without lost the generality we may assume that $c=1$ in (3.1). By combining (4.9) and (4.10) we get

$$\begin{aligned} \left|T_{M_{\alpha_k}+2} f\right| &\geqslant |II| - |I| \geqslant \frac{q_{M_{\alpha_k}+1}}{Q_{M_{\alpha_k}+2}} \frac{M_{\alpha_k}^{1/p-1}}{4\alpha_k} - \frac{4\lambda M_{\alpha_{k-1}}^{1/p}}{\alpha_{k-1}} \\ &\geqslant \frac{M_{\alpha_k}^{1/p-2}}{4\alpha_k} - \frac{4\lambda M_{\alpha_{k-1}}^{1/p}}{\alpha_{k-1}} \geqslant \frac{M_{\alpha_k}^{1/p-2}}{16\alpha_k}. \end{aligned} \tag{4.11}$$

On the other hand,

$$\mu\left\{x \in G_m : \left|T_{M_{\alpha_k}+2} f(x)\right| \geqslant \frac{M_{\alpha_k}^{1/p-2}}{16\alpha_k}\right\} = \mu(G_m) = 1. \tag{4.12}$$

Let $0<p<1/2$. Then

$$\begin{aligned} &\frac{M_{\alpha_k}^{1/p-2}}{16\alpha_k} \cdot \left(\mu\left\{x \in G_m : \left|T_{M_{\alpha_k}+2} f(x)\right| \geqslant \frac{M_{\alpha_k}^{1/p-2}}{16\alpha_k}\right\}\right)^{1/p} \\ &= \frac{M_{\alpha_k}^{1/p-2}}{16\alpha_k} \to \infty, \text{ as } k \to \infty. \end{aligned} \tag{4.13}$$

The proof is complete.

Proof of Corollary 3.1. Since R_n, U_n^{α} and V_n^{α} are the T means with non-increasing sequence $\{q_k : k \geqslant 0\}$, then the proof of this corollary is direct consequence of Theorem 3.1.

Proof of Corollary 3.2. According to Theorem 1 a) and Proposition 2.2 we also have weak $(1,1)$ type inequality and by well-known density argument due to Marcinkiewicz and Zygmund [7] we have $T_n f \to f$, a.e., for all $f \in L_1$. Which follows proof of Corollary 3.2.

Proof of Corollary 3.3. Since R_n, U_n^{α} and V_n^{α} are the T means with non-increasing sequence $\{q_k : k \geqslant 0\}$, then the proof of this corollary is direct consequence of Corollary 3.2.

Proof of Theorem 3.2. Let the sequence $\{q_k : k \geqslant 0\}$ be non-decreasing. By combining (2.4) with (2.5) and using Abel transformation we get that

$$\begin{aligned}
|T_n f| &\leqslant \left| \frac{1}{Q_n} \sum_{j=1}^{n-1} q_j S_j f \right| \\
&\leqslant \frac{1}{Q_n} \left(\sum_{j=1}^{n-2} \left| q_j - q_{j+1} \right| j \left| \sigma_j f \right| + q_{n-1}(n-1) \left| \sigma_n f \right| \right) \\
&\leqslant \frac{1}{Q_n} \left(\sum_{j=1}^{n-2} -\left(q_j - q_{j+1} \right) j - q_{n-1}(n-1) + 2q_{n-1}(n-1) \right) \sigma^* f \\
&\leqslant \frac{1}{Q_n} \left(2q_{n-1}(n-1) - Q_n \right) \sigma^* f \leqslant c\sigma^* f
\end{aligned}$$

so that

$$T^* f \leqslant c\sigma^* f. \tag{4.14}$$

If we apply (4.14) and Theorem W1 we can conclude that the maximal operators T^* are bounded from the Hardy space $H_{1/2}$ to the space $weak - L_{1/2}$. The proof of part a) is complete.

To prove part b) of Theorem 2 we use the martingale, defined by (4.5) where α_k satisfy conditions (4.2)-(4.4). It is easy to show that for every non-increasing sequence $\{q_k : k \geqslant 0\}$ it automatically holds that

$$q_{M_{\alpha_k+1}} / Q_{M_{\alpha_k+2}} \geqslant c / M_{\alpha_k}.$$

According to (4.7)-(4.11) we can conclude that

$$\left| T_{M_{\alpha_k}+2} f \right| \geqslant |II| - |I| \geqslant \frac{M_{\alpha_k}^{1/p-2}}{8\alpha_k}.$$

Analogously to (4.12) and (4.13) we then get that

$$\sup_{k \in \mathbb{N}} \left\| T_{M_{\alpha_k}+2} f \right\|_{weak-L_p} = \infty.$$

The proof is complete.

Proof of Corollary 3.4. Since $B^{\alpha,\beta,*}$ are the T means with non-decreasing sequence $\{q_k : k \geqslant 0\}$, then the proof of this corollary is direct consequence of Theorem 3.2.

Proof of Corollary 3.5. According to Proposition 2.2 we can conclude that T^* has weak type-(1,1) and by well-known density argument due to Marcinkiewicz and Zygmund [7] we also have $T_n f \to f$, a.e.. Which follows proof of Corollary 3.5.

Proof of Corollary 3.6. Since $B^{\alpha,\beta,*}$ are the T means with non-decreasing sequence $\{q_k : k \geqslant 0\}$, then the proof of this corollary is direct consequence of Corollary 3.5.

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Paper E

Sharp (H_p, L_p) type inequalities of maximal operators of T means with respect to Vilenkin systems with monotone coefficients

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E

SHARP (H_p, L_p) TYPE INEQUALITIES OF MAXIMAL OPERATORS OF T MEANS WITH RESPECT TO VILENKIN SYSTEMS

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Abstract. We prove and discuss some new (H_p, L_p) type inequalities of maximal operators of T means with respect to the Vilenkin systems with monotone coefficients. We also show that these inequalities are the best possible in a special sense. Moreover, we apply these inequalities to prove strong convergence theorems of such T means. We also show that these results are the best possible in a special sense. As applications, both some well-known and new results are pointed out.

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Key words and phrases: Vilenkin groups, Vilenkin systems, partial sums of Vilenkin-Fourier series, T means, Vilenkin-Nörlund means, Fejér mean, Riesz means, martingale Hardy spaces, L_p spaces, $weak - L_p$ spaces, maximal operator, strong convergence, inequalities.

1. Introduction

The definitions and notations used in this introduction can be found in our next Section.

It is well-known that Vilenkin systems do not form bases in the space L_1. Moreover, there is a function in the Hardy space H_p, such that the partial sums of f are not bounded in L_p-norm, for $0 < p \leq 1$. Approximation properties of Vilenkin-Fourier series with respect to one- and two-dimensional cases can be found in [17] and [32]. Simon [24] proved that there exists an absolute constant c_p, depending only on p, such that the inequality

$$\frac{1}{\log^{[p]} n}\sum_{k=1}^{n}\frac{\|S_k f\|_p^p}{k^{2-p}} \leq c_p \|f\|_{H_p}^p \quad (0 < p \leq 1)$$

holds for all $f \in H_p$ and $n \in \mathbb{N}_+$, where $[p]$ denotes the integer part of p. For $p = 1$ analogous results with respect to more general systems were proved in Blahota [2] and Gát [4] and for $0 < p < 1$ a simpler proof was given in Tephnadze [31]. Some new strong convergence results for partial sums with respect to Vilenkin system were considered in Tutberidze [33].

In the one-dimensional case the weak (1,1)-type inequality for the maximal operator of Fejér means $\sigma^* f := \sup_{n\in\mathbb{N}} |\sigma_n f|$ can be found in Schipp [21] for Walsh series and in Pál, Simon [15] for bounded Vilenkin series. Fujji [8] and Simon [23] verified that σ^* is bounded from H_1 to L_1. Weisz [38] generalized this result and proved boundedness of σ^* from the martingale space H_p to the space L_p, for $p > 1/2$. Simon [22] gave a counterexample, which shows that boundedness does not hold for $0 < p < 1/2$. A counterexample for $p = 1/2$ was given by Goginava [6] (see also Tephnadze [25]). Moreover, Weisz [40] proved that the

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maximal operator of the Fejér means σ^* is bounded from the Hardy space $H_{1/2}$ to the space $weak-L_{1/2}$. In [26] and [27] the following result was proved:

Theorem T1: Let $0<p\leq 1/2$. Then the weighted maximal operator of Fejér means $\widetilde{\sigma}_p^*$ defined by

$$\widetilde{\sigma}_p^* f := \sup_{n\in\mathbb{N}_+} \frac{|\sigma_n f|}{(n+1)^{1/p-2}\log^{2[1/2+p]}(n+1)}$$

is bounded from the martingale Hardy space H_p to the Lebesgue space L_p.

Moreover, the rate of the weights $\left\{1/\left(n+1\right)^{1/p-2}\log^{2[p+1/2]}\left(n+1\right)\right\}_{n=1}^{\infty}$ in n-th Fejér mean was given exactly.

Similar results with respect to Walsh-Kachmarz systems were obtained in [7] for $p=1/2$ and in [28] for $0<p<1/2$. Approximation properties of Fejér means with respect to Vilenkin and Kaczmarz systems can be found in Tephnadze [29], Tutberidze [34], Persson, Tephnadze and Tutberidze [19].

In [3] it was proved that there exists an absolute constant c_p, depending only on p, such that the inequality

$$\frac{1}{\log^{[1/2+p]} n}\sum_{k=1}^{n}\frac{\|\sigma_k f\|_p^p}{k^{2-2p}} \leq c_p \|f\|_{H_p}^p \quad (0<p\leq 1/2,\ n=2,3,\dots). \tag{1}$$

holds. Some new strong convergence results for Vilenkin-Fejér means were derived in [20].

Móricz and Siddiqi [11] investigated the approximation properties of some special Nörlund means of Walsh-Fourier series of L_p function in norm. In the two-dimensional case approximation properties of Nörlund means were considered by Nagy [12, 13, 14]. In [16] it was proved that the maximal operators of Nörlund means t^* defined by $t^* f := \sup_{n\in\mathbb{N}}|t_n f|$, either with non-decreasing coefficients, or non-increasing coefficients, satisfying the condition

$$\frac{1}{Q_n} = O\left(\frac{1}{n}\right), \quad \text{as} \quad n\to\infty \tag{2}$$

are bounded from the Hardy space $H_{1/2}$ to the space $weak-L_{1/2}$ and are not bounded from the Hardy space H_p to the space L_p, when $0<p\leq 1/2$.

In [18] it was proved that for $0<p<1/2$, $f\in H_p$ and non-decreasing sequence $\{q_k : k\geq 0\}$ there exists an absolute constant c_p, depending only on p, such that the inequality

$$\sum_{k=1}^{\infty}\frac{\|t_k f\|_p^p}{k^{2-2p}} \leq c_p \|f\|_{H_p}^p$$

holds.

Moreover, if $f\in H_{1/2}$ and $\{q_k : k\geq 0\}$ is a sequence of non-decreasing numbers, satisfying the condition

$$\frac{q_{n-1}}{Q_n} = O\left(\frac{1}{n}\right), \quad \text{as} \quad n\to\infty, \tag{3}$$

then there exists an absolute constant c, such that the inequality

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{\|t_k f\|_{1/2}^{1/2}}{k} \leq c\|f\|_{H_{1/2}}^{1/2}$$

holds.

In [35] was proved that the maximal operators T^* defined by $T^*f := \sup_{n\in\mathbb{N}} |T_n f|$ of T means either with non-increasing coefficients, or non-decreasing sequence satisfying condition (3) are bounded from the Hardy space $H_{1/2}$ to the space $weak-L_{1/2}$. Moreover, there exists a martingale and such T means for which boundedness from the Hardy space H_p to the space L_p does not hold when $0 < p \leq 1/2$.

One of the most well-known mean of T means is the Riesz summability. In [30] it was proved that the maximal operator R^* of Riesz means is bounded from the Hardy space $H_{1/2}$ to the space $weak - L_{1/2}$ and is not bounded from H_p to the space L_p, for $0 < p \leq 1/2$. There was also proved that Riesz summability has better properties than Fejér means. In particular, the following weighted maximal operators

$$\frac{\log n |R_n f|}{(n+1)^{1/p-2} \log^{2[1/2+p]}(n+1)}$$

are bounded from H_p to the space L_p, for $0 < p \leq 1/2$ and the rate of weights are sharp. Moreover, in [9] was also proved that if $0 < p < 1/2$ and $f \in H_p(G_m)$, then there exists an absolute constant c_p, depending only on p, such that the following inequality holds:

$$\sum_{n=1}^{\infty} \frac{\log^p n \, \|R_n f\|_{H_p}^p}{n^{2-2p}} \leq c_p \|f\|_{H_p}^p \tag{4}$$

If we compare strong convergence results, given by (1) and (4), we obtain that Riesz means has better properties then Fejér means, for $0 < p < 1/2$, but in the case $p = 1/2$ is was not possible to show even similar result for Riesz means as it was proved for Fejér means given by inequality (1).

In this paper we prove and discuss some new (H_p, L_p) type inequalities of maximal operators of T means with respect to the Vilenkin systems with monotone coefficients. Moreover, we apply these inequalities to prove strong convergence theorems of such T means. In particular, we investigate strong convergence of T means with non-increasing sequences in the case $p = 1/2$, but under the condition (2). For example, this condition is fulfilled for Fejér means but does not hold for Riesz means. We also show that these inequalities are the best possible in a special sense. As applications, both some well-known and new results are pointed out.

This paper is organized as follows: In order not to disturb our discussions later on some definitions and notations are presented in Section 2. The main results and some of its consequences can be found in Section 3. For the proofs of the main results we need some auxiliary Lemmas, some of them are new and of independent interest. These results are presented in Section 4. The detailed proofs of the main results are given in Section 5.

2. Definitions and Notation

Denote by $\mathbb{N}_+$ the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Let $m := (m_0, m_1, ...)$ be a sequence of the positive integers not less than 2. Denote by

$$Z_{m_k} := \{0, 1, ..., m_k - 1\}$$

the additive group of integers modulo m_k.

Define the group G_m as the complete direct product of the groups Z_{m_i} with the product of the discrete topologies of Z_{m_j}'s.

The direct product μ of the measures $\mu_k(\{j\}) := 1/m_k \quad (j \in Z_{m_k})$ is the Haar measure on G_m with $\mu(G_m) = 1$.

In this paper we discuss bounded Vilenkin groups, i.e. the case when $\sup_n m_n < \infty$.

The elements of G_m are represented by sequences

$$x := (x_0, x_1, ..., x_j, ...), \ \left(x_j \in Z_{m_j}\right).$$

Set $e_n := (0, ..., 0, 1, 0, ...) \in G_m$, the n-th coordinate of which is 1 and the rest are zeros $(n \in \mathbb{N})$. It is easy to give a basis for the neighborhoods of G_m :

$$I_0(x) := G_m, \quad I_n(x) := \{y \in G_m \mid y_0 = x_0, ..., y_{n-1} = x_{n-1}\},$$

where $x \in G_m$, $n \in \mathbb{N}$.

If we define $I_n := I_n(0)$, for $n \in \mathbb{N}$ and $\overline{I_n} := G_m \setminus I_n$, then

$$\overline{I_N} = \left(\bigcup_{k=0}^{N-2}\bigcup_{l=k+1}^{N-1} I_N^{k,l}\right) \bigcup \left(\bigcup_{k=1}^{N-1} I_N^{k,N}\right), \tag{5}$$

where

$$I_N^{k,l} := \begin{cases} I_N(0, ..., 0, x_k \neq 0, 0, ..., 0, x_l \neq 0, x_{l+1}, ..., x_{N-1}, ...), & \text{for } k < l < N, \\ I_N(0, ..., 0, x_k \neq 0, 0, ..., , x_{N-1} = 0, \ x_N, ...), & \text{for } l = N. \end{cases}$$

If we define the so-called generalized number system based on m in the following way :

$$M_0 := 1, \ M_{k+1} := m_k M_k \quad (k \in \mathbb{N}),$$

then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j}$ $(j \in \mathbb{N}_+)$ and only a finite number of n_j's differ from zero.

We introduce on G_m an orthonormal system which is called the Vilenkin system. At first, we define the complex-valued function $r_k(x) : G_m \to \mathbb{C}$, the generalized Rademacher functions, by

$$r_k(x) := \exp(2\pi i x_k / m_k), \ \left(i^2 = -1, x \in G_m, \ k \in \mathbb{N}\right).$$

Next, we define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on G_m by:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x), \quad (n \in \mathbb{N}).$$

Specifically, we call this system the Walsh-Paley system when $m \equiv 2$.

The norms (or quasi-norms) of the spaces $L_p(G_m)$ and $weak - L_p(G_m)$ $(0 < p < \infty)$ are respectively defined by

$$\|f\|_p^p := \int_{G_m} |f|^p \, d\mu, \ \ \|f\|_{weak-L_p}^p := \sup_{\lambda>0} \lambda^p \mu(f > \lambda) < +\infty.$$

The Vilenkin system is orthonormal and complete in $L_2(G_m)$ (see [36]).

Now, we introduce analogues of the usual definitions in Fourier-analysis. If $f \in L_1(G_m)$ we can define Fourier coefficients, partial sums and Dirichlet kernels with respect to the Vilenkin system in the usual manner:

$$\widehat{f}(n) := \int_{G_m} f \overline{\psi}_n d\mu, \qquad S_n f := \sum_{k=0}^{n-1} \widehat{f}(k) \psi_k, \quad D_n := \sum_{k=0}^{n-1} \psi_k, \quad (n \in \mathbb{N}_+).$$

Let us define the Fejér means σ_n and Kernels K_n as follows:

$$\sigma_n f := \frac{1}{n}\sum_{k=1}^{n} S_k f\,, \quad K_n := \frac{1}{n}\sum_{k=1}^{n} D_k.$$

It is well known that if $n \in \mathbb{N}$, then

$$D_{M_n}(x) = \begin{cases} M_n, & x \in I_n, \\ 0, & x \notin I_n. \end{cases} \tag{6}$$

Moreover, if $n = \sum_{i=0}^{\infty} n_i M_i$, and $1 \le s_n \le m_n - 1$, then we have the following identity:

$$D_n = \psi_n \left(\sum_{j=0}^{\infty} D_{M_j} \sum_{k=m_j-n_j}^{m_j-1} r_j^k \right), \tag{7}$$

The σ-algebra generated by the intervals $\{I_n(x) : x \in G_m\}$ will be denoted by $F_n\,(n \in \mathbb{N})$. Denote by $f = \left(f^{(n)}, n \in \mathbb{N}\right)$ a martingale with respect to $F_n\,(n \in \mathbb{N})$. (for details see e.g. [37]). The maximal function of a martingale f is defined by $f^* := \sup_{n\in\mathbb{N}} \left|f^{(n)}\right|$. For $0 < p < \infty$ the Hardy martingale spaces H_p consist of all martingales f for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

A bounded measurable function a is called a p-atom, if there exists an interval I, such that

$$\int_I a\,d\mu = 0, \quad \|a\|_\infty \le \mu(I)^{-1/p}, \quad \operatorname{supp}(a) \subset I.$$

If $f = \left(f^{(n)}, n \in \mathbb{N}\right)$ is a martingale, then the Vilenkin-Fourier coefficients must be defined in a slightly different manner:

$$\widehat{f}(i) := \lim_{k\to\infty} \int_{G_m} f^{(k)} \overline{\psi}_i \,d\mu.$$

Let $\{q_k : k \ge 0\}$ be a sequence of non-negative numbers. The n-th T means T_n for a Fourier series of f are defined by

$$T_n f := \frac{1}{Q_n}\sum_{k=0}^{n-1} q_k S_k f, \quad \text{where} \quad Q_n := \sum_{k=0}^{n-1} q_k. \tag{8}$$

It is obvious that $T_n f(x) = \int\limits_{G_m} f(t)\, F_n(x-t)\, d\mu(t)$, where $F_n := \frac{1}{Q_n}\sum\limits_{k=0}^{n-1} q_k D_k$ is called the T kernel.

We always assume that $\{q_k : k \ge 0\}$ is a sequence of non-negative numbers and $q_0 > 0$. Then the summability method (8) generated by $\{q_k : k \ge 0\}$ is regular if and only if $\lim_{n\to\infty} Q_n = \infty$.

It is easy to show that, for any real numbers $a_1, \ldots, a_m$, $b_1, \ldots, b_m$ and $a_k = A_k - A_{k-1}$, $k = n, \ldots, m$, we have so called Abel transformation:

$$
\begin{aligned}
\sum_{k=m}^{n} a_k b_k &= \sum_{k=m}^{n} (A_k - A_{k-1}) b_k = \sum_{k=m}^{n} A_k b_k - \sum_{k=m}^{n} A_{k-1} b_k \\
&= \sum_{k=m}^{n} A_k b_k - \sum_{k=n-1}^{m-1} A_k b_{k+1} = \sum_{k=m}^{n-1} A_k b_k + A_n b_n - \sum_{k=m}^{n-1} A_k b_{k+1} - A_{m-1} b_m \\
&= A_n b_n - A_{m-1} b_m + \sum_{k=m}^{n-1} A_k (b_k - b_{k+1}).
\end{aligned}
$$

For $a_j = A_j - A_{j-1},\ j = 1, ..., n$, if we invoke Abel transformations

$$\sum_{j=1}^{n-1} a_j b_j = A_{n-1} b_{n-1} - A_0 b_1 + \sum_{j=1}^{n-2} A_j (b_j - b_{j+1}), \tag{9}$$

$$\sum_{j=M_N}^{n-1} a_j b_j = A_{n-1} b_{n-1} - A_{M_N - 1} b_{M_N} + \sum_{j=M_N}^{n-2} A_j (b_j - b_{j+1}), \tag{10}$$

for $b_j = q_j$, $a_j = 1$ and $A_j = j$ for any $j = 0, 1, ..., n$ we get the following identity:

$$Q_n := \sum_{j=0}^{n-1} q_j = q_0 + \sum_{j=1}^{n-1} q_j = q_0 + \sum_{j=1}^{n-2} (q_j - q_{j+1})\, j + q_{n-1}(n-1), \tag{11}$$

$$\sum_{j=M_N}^{n-1} q_j = \sum_{j=M_N}^{n-2} (q_j - q_{j+1})\, j + q_{n-1}(n-1) - (M_N - 1) q_{M_N}, \tag{12}$$

Moreover, if use $D_0 = K_0 = 0$ for any $x \in G_m$ and invoke Abel transformations (9) and (10) for $b_j = q_j$, $a_j = D_j$ and $A_j = jK_j$ for any $j = 0, 1, ..., n-1$ we get identities:

$$F_n = \frac{1}{Q_n} \sum_{j=1}^{n-1} q_j D_j = \frac{1}{Q_n} \left(\sum_{j=1}^{n-2} (q_j - q_{j+1})\, j K_j + q_{n-1}(n-1) K_{n-1} \right), \tag{13}$$

$$
\begin{aligned}
&\frac{1}{Q_n} \sum_{j=M_N}^{n-1} q_j D_j \\
&= \frac{1}{Q_n} \left(\sum_{j=M_N}^{n-2} (q_j - q_{j+1})\, j K_j + q_{n-1}(n-1) K_{n-1} - q_{M_N}(M_N - 1) K_{M_N - 1} \right).
\end{aligned} \tag{14}
$$

Analogously, if use $S_0 f = \sigma_0 f = 0$, for any $x \in G_m$ and invoke Abel transformations (9) and (10) for $b_j = q_j$, $a_j = S_j$ and $A_j = j\sigma_j$ for any $j = 0, 1, ..., n-1$ we get identities:

$$T_n f = \frac{1}{Q_n} \sum_{j=1}^{n-1} q_j S_j f = \frac{1}{Q_n} \left(\sum_{j=1}^{n-2} (q_j - q_{j+1})\, j \sigma_j f + q_{n-1}(n-1) \sigma_{n-1} f \right), \tag{15}$$

$$
\begin{aligned}
&\frac{1}{Q_n} \sum_{j=M_N}^{n-1} q_j S_j f \\
&= \frac{1}{Q_n} \left(\sum_{j=M_N}^{n-2} (q_j - q_{j+1})\, j \sigma_j f + q_{n-1}(n-1) \sigma_{n-1} f - q_{M_N}(M_N - 1) \sigma_{M_N - 1} f \right).
\end{aligned} \tag{16}
$$

Let $\{q_k : k \geq 0\}$ be a sequence of nonnegative numbers. The n-th Nörlund mean t_n for a Fourier series of f is defined by

$$t_n f = \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} S_k f, \tag{17}$$

where $Q_n := \sum_{k=0}^{n-1} q_k$.

If $q_k \equiv 1$ in (8) and (17) we respectively define the Fejér means σ_n and Kernels K_n as follows:

$$\sigma_n f := \frac{1}{n} \sum_{k=1}^{n} S_k f, \quad K_n := \frac{1}{n} \sum_{k=1}^{n} D_k.$$

It is well-known that (for details see [1])

$$n |K_n| \leq c \sum_{l=0}^{|n|} M_l |K_{M_l}| \tag{18}$$

and

$$\|K_n\|_1 \leq c < \infty. \tag{19}$$

The well-known example of Nörlund summability is the so-called (C, α) mean (Cesàro means) for $0 < \alpha < 1$, which are defined by

$$\sigma_n^\alpha f := \frac{1}{A_n^\alpha} \sum_{k=1}^{n} A_{n-k}^{\alpha-1} S_k f,$$

where

$$A_0^\alpha := 0, \qquad A_n^\alpha := \frac{(\alpha+1)\ldots(\alpha+n)}{n!}.$$

We also consider the "inverse" (C, α) means, which is an example of T means:

$$U_n^\alpha f := \frac{1}{A_n^\alpha} \sum_{k=0}^{n-1} A_k^{\alpha-1} S_k f, \qquad 0 < \alpha < 1.$$

Let V_n^α denote the T mean, where $\{q_0 = 0,\ q_k = k^{\alpha-1} : k \in \mathbb{N}_+\}$, that is

$$V_n^\alpha f := \frac{1}{Q_n} \sum_{k=1}^{n-1} k^{\alpha-1} S_k f, \qquad 0 < \alpha < 1.$$

The n-th Riesz logarithmic mean R_n and the Nörlund logarithmic mean L_n are defined by

$$R_n f := \frac{1}{l_n} \sum_{k=1}^{n-1} \frac{S_k f}{k} \quad \text{and} \quad L_n f := \frac{1}{l_n} \sum_{k=1}^{n-1} \frac{S_k f}{n-k},$$

respectively, where $l_n := \sum_{k=1}^{n-1} 1/k$.

Up to now we have considered T means in the case when the sequence $\{q_k : k \in \mathbb{N}\}$ is bounded but now we consider T summabilities with unbounded sequence $\{q_k : k \in \mathbb{N}\}$.

Let $\alpha\in\mathbb{R}_+,\quad \beta\in\mathbb{N}_+$ and $\log^{(\beta)}x:=\overbrace{\log\ldots\log}^{\beta-\text{times}}x$. If we define the sequence $\{q_k:k\in\mathbb{N}\}$ by $\left\{q_0=0,\ q_k=\log^{(\beta)}k^\alpha : k\in\mathbb{N}_+\right\}$, then we get the class $B_n^{\alpha,\beta}$ of T means with non-decreasing coefficients:

$$B_n^{\alpha,\beta}f:=\frac{1}{Q_n}\sum_{k=1}^{n-1}\log^{(\beta)}k^\alpha S_kf.$$

We note that $B_n^{\alpha,\beta}$ are well-defined for every $n\in\mathbb{N}$

$$B_n^{\alpha,\beta}f=\sum_{k=1}^{n-1}\frac{\log^{(\beta)}k^\alpha}{Q_n}S_kf.$$

It is obvious that $\frac{n}{2}\log^{(\beta)}\frac{n^\alpha}{2^\alpha}\leq Q_n\leq n\log^{(\beta)}n^\alpha$. It follows that

$$\frac{q_{n-1}}{Q_n}\leq\frac{c\log^{(\beta)}n^\alpha}{n\log^{(\beta)}n^\alpha}=O\left(\frac{1}{n}\right)\to0,\quad\text{as}\quad n\to\infty. \tag{20}$$

We also define the maximal operator T^* of T and Nörlund means by

$$T^*f:=\sup_{n\in\mathbb{N}}\left|T_nf\right|,\quad t^*f:=\sup_{n\in\mathbb{N}}\left|t_nf\right|.$$

Some well-known examples of maximal operators of T means are the maximal operator of Fejér σ^* and Riesz R^* logarithmic means, which are respectively defined by:

$$\sigma^*f:=\sup_{n\in\mathbb{N}}\left|\sigma_nf\right|,\qquad R^*f:=\sup_{n\in\mathbb{N}}\left|R_nf\right|.$$

3. The Main Results and Applications

Our first main result reads:

Theorem 1. *Let $0<p\leq1/2$, $f\in H_p$ and $\{q_k:k\geq0\}$ be a sequence of non-increasing numbers. Then the maximal operator $\widetilde{T}_p^*$ defined by*

$$\widetilde{T}_p^*f:=\sup_{n\in\mathbb{N}_+}\frac{|T_nf|}{(n+1)^{1/p-2}\log^{2[1/2+p]}(n+1)} \tag{21}$$

is bounded from the Hardy space H_p to the space L_p.

Corollary 1. *Let $0<p\leq1/2$ and $f\in H_p$. Then the maximal operator $\widetilde{R}_p^*$ defined by*

$$\widetilde{R}_p^*f:=\sup_{n\in\mathbb{N}_+}\frac{|R_nf|}{(n+1)^{1/p-2}\log^{2[1/2+p]}(n+1)}$$

is bounded from the Hardy space H_p to the space L_p.

Corollary 2. *Let $0<p\leq1/2$ and $f\in H_p$. Then the maximal operator $\widetilde{U}_p^{\alpha,*}$ defined by*

$$\widetilde{U}_p^{\alpha,*}f:=\sup_{n\in\mathbb{N}_+}\frac{|U_n^\alpha f|}{(n+1)^{1/p-2}\log^{2[1/2+p]}(n+1)}$$

is bounded from the Hardy space H_p to the space L_p.

Corollary 3. *Let $0<p\leq 1/2$ and $f\in H_p$. Then the maximal operator $\widetilde{V}_p^{\alpha,*}$ defined by*

$$\widetilde{V}_p^{\alpha,*}f:=\sup_{n\in\mathbb{N}_+}\frac{|V_n^{\alpha}f|}{(n+1)^{1/p-2}\log^{2[1/2+p]}(n+1)}$$

is bounded from the Hardy space H_p to the space L_p.

Next, we consider maximal operators of T means with non-decreasing sequence:

Theorem 2. *Let $0<p\leq 1/2$, $f\in H_p$ and $\{q_k:k\geq 0\}$ be a sequence of non-decreasing numbers, satisfying the condition*

$$\frac{q_{n-1}}{Q_n}=O\left(\frac{1}{n}\right),\quad as\quad n\to\infty. \tag{22}$$

Then the maximal operator $\widetilde{T}_p^{*}$ defined by

$$\widetilde{T}_p^{*}f:=\sup_{n\in\mathbb{N}_+}\frac{|T_nf|}{(n+1)^{1/p-2}\log^{2[1/2+p]}(n+1)} \tag{23}$$

is bounded from the martingale Hardy space H_p to the space L_p.

Corollary 4. *Let $0<p\leq 1/2$, $f\in H_p$ and $\{q_k:k\geq 0\}$ be a sequence of non-decreasing numbers, such that*

$$\sup_{n\in\mathbb{N}}q_n<c<\infty. \tag{24}$$

Then

$$\frac{q_{n-1}}{Q_n}\leq\frac{c}{Q_n}\leq\frac{c}{q_0n}=\frac{c_1}{n}=O\left(\frac{1}{n}\right),\ \ as\ n\to 0,$$

and weighted maximal operators of such T means, given by (23), $\widetilde{T}_p^{*}$ are bounded from the Hardy space H_p to the space L_p.

Corollary 5. *Let $0<p\leq 1/2$ and $f\in H_p$. Then the maximal operator $\widetilde{T}_p^{*}$ defined by*

$$\widetilde{T}_p^{*}f:=\sup_{n\in\mathbb{N}_+}\frac{\left|B_n^{\alpha,\beta}f\right|}{(n+1)^{1/p-2}\log^{2[1/2+p]}(n+1)}$$

is bounded from the martingale Hardy space H_p to the space L_p.

Remark 1. *According to Theorem T1 we obtain that the weights in (21) and (23) are sharp.*

Theorem 3. *a) Let $0<p<1/2$, $f\in H_p$ and $\{q_k:k\geq 0\}$ be a sequence of non-increasing numbers. Then there exists an absolute constant c_p, depending only on p, such that the following inequality holds:*

$$\sum_{k=1}^{\infty}\frac{\|T_kf\|_p^p}{k^{2-2p}}\leq c_p\|f\|_{H_p}^p$$

b)Let $f\in H_{1/2}$ and $\{q_k:k\geq 0\}$ be a sequence of non-increasing numbers, satisfying the condition

$$\frac{1}{Q_n}=O\left(\frac{1}{n}\right),\quad as\quad n\to\infty. \tag{25}$$

Then there exists an absolute constant c, such that the following inequality holds:

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{\|T_kf\|_{1/2}^{1/2}}{k}\leq c\|f\|_{H_{1/2}}^{1/2} \tag{26}$$

Corollary 6. *Let $0<p\leq 1/2$ and $f\in H_p$. Then there exists absolute constant c_p, depending only on p, such that the following inequality holds:*

$$\frac{1}{\log^{[1/2+p]}n}\sum_{k=1}^{n}\frac{\|\sigma_kf\|_p^p}{k^{2-2p}}\leq c_p\|f\|_{H_p}^p.$$

Corollary 7. *Let $0<p\leq 1/2$ and $f\in H_p$. Then there exists an absolute constant c_p, depending only on p, such that the following inequalities hold:*

$$\sum_{k=1}^{\infty}\frac{\|U_k^{\alpha}f\|_p^p}{k^{2-2p}}\leq c_p\|f\|_{H_p}^p,\qquad \sum_{k=1}^{\infty}\frac{\|V_k^{\alpha}f\|_p^p}{k^{2-2p}}\leq c_p\|f\|_{H_p}^p,\qquad \sum_{k=1}^{\infty}\frac{\|R_kf\|_p^p}{k^{2-2p}}\leq c_p\|f\|_{H_p}^p.$$

Theorem 4. *a) Let $0<p<1/2$, $f\in H_p$ and $\{q_k:k\geq 0\}$ be a sequence of non-decreasing numbers. Then there exists an absolute constant c_p, depending only on p, such that the following inequality holds:*

$$\sum_{k=1}^{\infty}\frac{\|T_kf\|_p^p}{k^{2-2p}}\leq c_p\|f\|_{H_p}^p$$

b)Let $f\in H_{1/2}$ and $\{q_k:k\geq 0\}$ be a sequence of non-increasing numbers, satisfying the condition (22)*. Then there exists an absolute constant c, such that the inequality holds:*

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{\|T_kf\|_{1/2}^{1/2}}{k}\leq c\|f\|_{H_{1/2}}^{1/2}. \tag{27}$$

Corollary 8. *Let $0<p\leq 1/2$, $f\in H_p$ and $\{q_k:k\geq 0\}$ be a sequence of non-decreasing numbers, such that $\sup_{n\in\mathbb{N}}q_n<c<\infty$. Then condition (22) is satisfied and for all such T means there exists an absolute constant c, such that the inequality (27) holds.*

We have already considered the case when the sequence $\{q_k:k\geq 0\}$ is bounded. Now, we consider some Nörlund means, which are generated by a unbounded sequence $\{q_k:k\geq 0\}$.

Corollary 9. *Let $0<p\leq 1/2$ and $f\in H_p$. Then there exists an absolute constant c_p, depending only on p, such that the following inequality holds:*

$$\frac{1}{\log^{[1/2+p]}n}\sum_{k=1}^{n}\frac{\left\|B_k^{\alpha,\beta}f\right\|_p^p}{k^{2-2p}}\leq c_p\|f\|_{H_p}^p.$$

4. Auxiliary lemmas

We need the following auxiliary Lemmas:

Lemma 1 (see e.g. [39])**.** *A martingale $f=\left(f^{(n)},n\in\mathbb{N}\right)$ is in $H_p\,(0<p\leq 1)$ if and only if there exists a sequence $(a_k,k\in\mathbb{N})$ of p-atoms and a sequence $(\mu_k,k\in\mathbb{N})$ of real numbers such that, for every $n\in\mathbb{N}$,*

$$\sum_{k=0}^{\infty}\mu_kS_{M_n}a_k=f^{(n)},\quad a.e.,\quad where\quad \sum_{k=0}^{\infty}|\mu_k|^p<\infty. \tag{28}$$

Moreover, $\|f\|_{H_p} \backsim \inf\left(\sum_{k=0}^{\infty}|\mu_k|^p\right)^{1/p}$ *, where the infimum is taken over all decompositions of* f *of the form (28).*

Lemma 2 (see e.g. [39])**.** *Suppose that an operator* T *is* σ*-sublinear and for some* $0 < p \leq 1$

$$\int_{\bar{I}} |Ta|^p \, d\mu \leq c_p < \infty,$$

for every p*-atom* a*, where* I *denotes the support of the atom. If* T *is bounded from* L_∞ *to* L_∞*, then*

$$\|Tf\|_p \leq c_p \|f\|_{H_p}, \ \ 0 < p \leq 1.$$

Lemma 3 (see [5])**.** *Let* $n > t$*,* $t, n \in \mathbb{N}$*. Then*

$$K_{M_n}(x) = \begin{cases} \frac{M_t}{1-r_t(x)}, & x \in I_t \backslash I_{t+1}, \ x - x_t e_t \in I_n, \\ \frac{M_n - 1}{2}, & x \in I_n, \\ 0, & \text{otherwise.} \end{cases}$$

For the proof of our main results we also need the following new Lemmas:

Lemma 4. *Let* $n \in \mathbb{N}$ *and* $\{q_k : k \in \mathbb{N}\}$ *be a sequence either of non-increasing numbers, or non-decreasing numbers satisfying condition* (22)*. Then*

$$\|T_n f\|_1 < c. \tag{29}$$

Proof: Let $n \in \mathbb{N}$ and $\{q_k : k \in \mathbb{N}\}$ be a sequence of non-increasing numbers. By combining (11) and (15) with (19) we can conclude that

$$\begin{aligned} \|T_n f\|_1 &\leq \frac{1}{Q_n}\left(\sum_{j=1}^{n-2}|q_j - q_{j+1}|\, j\|\sigma_j f\|_1 + q_{n-1}(n-1)\|\sigma_{n-1}f\|_1\right) \\ &\leq \frac{c}{Q_n}\left(\sum_{j=1}^{n-2}(q_j - q_{j+1})\, j + q_{n-1}(n-1)\right) \leq c < \infty. \end{aligned}$$

Let $n \in \mathbb{N}$ and $\{q_k : k \in \mathbb{N}\}$ be a sequence non-decreasing satisfying condition (22). Then, by using again (11) and (15) combined with (19) we find that

$$\begin{aligned} \|T_n f\|_1 &\leq \frac{1}{Q_n}\left(\sum_{j=1}^{n-2}|q_j - q_{j+1}|\, j\|\sigma_j f\|_1 + q_{n-1}(n-1)\|\sigma_{n-1}f\|_1\right) \\ &\leq \frac{c}{Q_n}\left(\sum_{j=1}^{n-2}(q_{j+1} - q_j)\, j + q_{n-1}(n-1)\right) \\ &= \frac{c}{Q_n}\left(2q_{n-1}(n-1) - \left(\sum_{j=1}^{n-2}(q_j - q_{j+1})\, j + q_{n-1}(n-1) + q_0\right)\right) + \frac{cq_0}{Q_n} \\ &= \frac{c}{Q_n}(2q_{n-1}(n-1) - Q_n) + \frac{cq_0}{Q_n} < c. \end{aligned}$$

The proof is complete.

Lemma 5. *Let $\{q_k : k \in \mathbb{N}\}$ be a sequence of non-increasing numbers and $n > M_N$. Then*

$$\left|\frac{1}{Q_n}\sum_{j=M_N}^{n-1} q_j D_j(x)\right| \leq \frac{c}{M_N}\sum_{j=0}^{|n|} M_j \left|K_{M_j}\right|,$$

Proof. Since the sequence is non-increasing we get that

$$\begin{aligned} &\frac{1}{Q_n}\left(q_{M_N} + \sum_{j=M_N}^{n-2} |q_j - q_{j+1}| + q_{n-1}\right) \\ \leq\ &\frac{1}{Q_n}\left(q_{M_N} + \sum_{j=M_N}^{n-2} (q_j - q_{j+1}) + q_{n-1}\right) = \frac{2q_{M_N}}{Q_n} \leq \frac{2q_{M_N}}{Q_{M_N+1}} \leq \frac{c}{M_N}. \end{aligned} \tag{30}$$

If we apply Abel transformation (14) combine with (18) and (30) we get that

$$\begin{aligned} &\left|\frac{1}{Q_n}\sum_{j=M_N}^{n-1} q_j D_j\right| \\ =\ &\frac{1}{Q_n}\left(\sum_{j=M_N}^{n-2} (q_j - q_{j+1})\, jK_j + q_{n-1}(n-1)K_{n-1} - q_{M_N}(M_N - 1)K_{M_N-1}\right) \\ \leq\ &\frac{c}{Q_n}\left(q_{M_n} + \sum_{j=M_N}^{n-2} |q_j - q_{j+1}| + q_{n-1}\right)\sum_{i=0}^{|n|} M_i \left|K_{M_i}\right| \leq \frac{c}{M_N}\sum_{i=0}^{|n|} M_i \left|K_{M_i}\right|. \end{aligned}$$

The proof is complete. □

Lemma 6. *Let $x \in I_N^{k,l}$, $k = 0, \dots, N-1$, $l = k+1, \dots, N$ and $\{q_k : k \in \mathbb{N}\}$ be a sequence of non-increasing numbers. Then there exists an absolute constant c, such that*

$$\int_{I_N}\left|\frac{1}{Q_n}\sum_{j=M_N}^{n-1} q_j D_j(x-t)\right| d\mu(t) \leq \frac{cM_l M_k}{M_N^2}.$$

Proof: Let $x \in I_N^{k,l}$, for $0 \leq k < l \leq N-1$ and $t \in I_N$. First, we observe that $x - t \in I_N^{k,l}$. Next, we apply Lemmas 3 and 5 to obtain that

$$\begin{aligned} &\int_{I_N}\left|\frac{1}{Q_n}\sum_{j=M_N}^{n-1} q_j D_j(x-t)\right| d\mu(t) \\ \leq\ &\frac{c}{M_N}\sum_{i=0}^{|n|} M_i \int_{I_N} \left|K_{M_i}(x-t)\right| d\mu(t) \\ \leq\ &\frac{c}{M_N}\int_{I_N}\sum_{i=0}^{l} M_i M_k \, d\mu(t) \leq \frac{cM_k M_l}{M_N^2} \end{aligned} \tag{31}$$

and the first estimate is proved.

Now, let $x \in I_N^{k,N}$. Since $x - t \in I_N^{k,N}$ for $t \in I_N$, by combining (6) and (7) we have that

$$|D_i(x-t)| \leq M_k$$

and

$$\text{(32)} \qquad \int_{I_N}\left|\frac{1}{Q_n}\sum_{j=M_N}^{n-1}q_jD_j\left(x-t\right)\right|d\mu\left(t\right)$$

$$\leq \frac{c}{Q_n}\sum_{i=0}^{|n|}q_i\int_{I_N}\left|D_i\left(x-t\right)\right|d\mu\left(t\right)$$

$$\leq \frac{c}{Q_n}\sum_{i=0}^{|n|-1}q_i\int_{I_N}M_kd\mu\left(t\right)\leq\frac{cM_k}{M_N}.$$

According to (31) and (32) the proof is complete.

Lemma 7. *Let $n>M_N$ and $\{q_k:k\in\mathbb{N}\}$ be a sequence of non-increasing numbers, satisfying condition (25). Then*

$$\left|\frac{1}{Q_n}\sum_{j=M_N}^{n-1}q_jD_j\right|\leq\frac{c}{n}\sum_{j=0}^{|n|}M_j\left|K_{M_j}\right|,$$

where c is an absolute constant.

Proof. Since the sequence is non-increasing and satisfying condition (25), we get that

$$\frac{1}{Q_n}\left(q_{M_n}+\sum_{j=M_N}^{n-2}\left|q_j-q_{j+1}\right|+q_{n-1}\right)$$

$$=\frac{1}{Q_n}\left(q_{M_n}+\sum_{j=M_N}^{n-2}\left(q_j-q_{j+1}\right)+q_{n-1}\right)$$

$$\leq\frac{2q_{M_N}}{Q_n}\leq\frac{2q_0}{Q_n}\leq\frac{c}{Q_n}\leq\frac{c}{n}.$$

Hence, if we apply Abel transformation (14) and estimate (18) we find that

$$\left|\frac{1}{Q_n}\sum_{j=M_N}^{n-1}q_jD_j\right|$$

$$\leq\left(\frac{c}{Q_n}\left(q_{M_n}+\sum_{j=M_N}^{n-2}\left|q_j-q_{j+1}\right|+q_{n-1}\right)\right)\sum_{i=0}^{|n|}M_i\left|K_{M_i}\right|$$

$$\leq\frac{c}{n}\sum_{i=0}^{|n|}M_i\left|K_{M_i}\right|.$$

The proof is complete. □

Lemma 8. *Let $x\in I_N^{k,l},\quad k=0,\dots,N-2,\quad l=k+1,\dots,N-1$ and $\{q_k:k\in\mathbb{N}\}$ be a sequence of non-increasing numbers, satisfying condition (25). Then, for some $c>0$*

$$\int_{I_N}\left|\frac{1}{Q_n}\sum_{j=M_N}^{n-1}q_jD_j\left(x-t\right)\right|d\mu\left(t\right)\leq\frac{cM_lM_k}{nM_N}.$$

Let $x \in I_N^{k,N}, \quad k = 0, \dots, N-1$. *Then*

$$\int_{I_N} \left| \frac{1}{Q_n} \sum_{j=M_N}^{n-1} q_j D_j (x-t) \right| d\mu(t) \leq \frac{cM_k}{M_N}.$$

Proof: Let $x \in I_N^{k,l}$, for $0 \leq k < l \leq N-1$ and $t \in I_N$. First, we observe that $x - t \in I_N^{k,l}$. Next, we apply Lemmas 3 and 7 to obtain that

$$\begin{aligned} &\int_{I_N} \left| \frac{1}{Q_n} \sum_{j=M_N}^{n-1} q_j D_j (x-t) \right| d\mu(t) \qquad (33) \\ \leq\ & \frac{c}{n} \sum_{i=0}^{|n|} M_i \int_{I_N} |K_{M_i}(x-t)| \, d\mu(t) \\ \leq\ & \frac{c}{n} \int_{I_N} \sum_{i=0}^{l} M_i M_k d\mu(t) \leq \frac{cM_k M_l}{nM_N} \end{aligned}$$

and the first estimate is proved.

Now, let $x \in I_N^{k,N}$. Since $x - t \in I_N^{k,N}$ for $t \in I_N$, by combining again Lemmas 3 and 7 we have that

$$\begin{aligned} &\int_{I_N} \left| \frac{1}{Q_n} \sum_{j=M_N}^{n-1} q_j D_j (x-t) \right| d\mu(t) \qquad (34) \\ \leq\ & \frac{c}{n} \sum_{i=0}^{|n|} M_i \int_{I_N} |K_{M_i}(x-t)| \, d\mu(t) \\ \leq\ & \frac{c}{n} \sum_{i=0}^{|n|-1} M_i \int_{I_N} M_k d\mu(t) \leq \frac{cM_k}{M_N}. \end{aligned}$$

By combining (33) and (34) we complete the proof.

Lemma 9. *Let* $n \geq M_N, \quad x \in I_N^{k,l}, \quad k = 0, \dots, N-1, \quad l = k+1, \dots, N$ *and* $\{q_k : k \in \mathbb{N}\}$ *be a sequence of non-increasing numbers, satisfying condition* (25). *Then*

$$\int_{I_N} \left| \frac{1}{Q_n} \sum_{j=M_N}^{n-1} q_j D_j (x-t) \right| d\mu(t) \leq \frac{cM_l M_k}{M_N^2},$$

where c is an absolute constant.

Proof: Since $n \geq M_N$ if we apply Lemma 8 we immediately get the proof.

Lemma 10. *Let* $\{q_k : k \in \mathbb{N}\}$ *be a sequence of non-decreasing numbers satisfying (22). Then*

$$|F_n| \leq \frac{c}{n} \sum_{j=0}^{|n|} M_j \left| K_{M_j} \right|,$$

where c is an absolute constant.

Proof. Since the sequence $\{q_k : k \in \mathbb{N}\}$ is non-decreasing if we apply condition (22) we can conclude that

$$\begin{aligned} &\frac{1}{Q_n}\left(\sum_{j=1}^{n-2}|q_j-q_{j+1}|+q_{n-1}+q_0\right)\\ &\leq \frac{1}{Q_n}\left(\sum_{j=1}^{n-2}(q_{j+1}-q_j)+q_{n-1}+q_0\right)\\ \leq\ &\frac{2q_{n-1}+q_0}{Q_n}\leq\frac{3q_{n-1}}{Q_n}\leq\frac{c}{n}.\end{aligned}$$

Therefore, if we apply Abel transformation (13) and (18) we get that

$$\begin{aligned}|F_n| &\leq \left(\frac{c}{Q_n}\left(\sum_{j=1}^{n-2}|q_j-q_{j+1}|+q_{n-1}+q_0\right)\right)\sum_{i=0}^{|n|}M_i\left|K_{M_i}\right|\\ &\leq \frac{c}{n}\sum_{i=0}^{|n|}M_i\left|K_{M_i}\right|.\end{aligned}$$

The proof is complete. □

Lemma 11. *Let* $x\in I_N^{k,l},\quad k=0,\dots,N-2,\quad l=k+1,\dots,N-1$ *and* $\{q_k : k\in\mathbb{N}\}$ *be a sequence of non-decreasing numbers satisfying condition (22). Then*

$$\int_{I_N}|F_n(x-t)|\,d\mu(t)\leq\frac{cM_lM_k}{nM_N}.$$

Let $x\in I_N^{k,N},\quad k=0,\dots,N-1$. *Then*

$$\int_{I_N}|F_n(x-t)|\,d\mu(t)\leq\frac{cM_k}{M_N}.$$

Here c *is an absolute constant.*

Proof: The proof is quite analogously to the proof of Lemma 8, so we leave out the details.

Lemma 12. *Let* $n\geq M_N,\quad x\in I_N^{k,l},\quad k=0,\dots,N-1,\quad l=k+1,\dots,N$ *and* $\{q_k : k\in\mathbb{N}\}$ *be a sequence of non-decreasing numbers, satisfying condition (22). Then*

$$\int_{I_N}|F_n(x-t)|\,d\mu(t)\leq\frac{cM_lM_k}{M_N^2}.$$

Proof: Since $n\geq M_N$ if we apply Lemma 11 we immediately get the proof.

5. Proofs of the main result

Proof of Theorem 1. Let $0 < p \leq 1/2$ and the sequence $\{q_k : k \geq 0\}$ be non-increasing. By using (11) and (15) we get that

$$\begin{aligned}
&\frac{|T_n f|}{(n+1)^{1/p-2}\log^{2[1/2+p]}(n+1)} \\
\leq\ & \frac{1}{(n+1)^{1/p-2}\log^{2[1/2+p]}(n+1)}\left|\frac{1}{Q_n}\sum_{j=1}^{n-1} q_j S_j f\right| \\
\leq\ & \frac{1}{(n+1)^{1/p-2}\log^{2[1/2+p]}(n+1)}\frac{1}{Q_n}\left(\sum_{j=1}^{n-2}|q_j-q_{j+1}|\,j\,|\sigma_j f| + q_{n-1}(n-1)\,|\sigma_n f|\right) \\
\leq\ & \frac{1}{Q_n}\left(\sum_{j=1}^{n-2}\frac{|q_j-q_{j+1}|\,j\,|\sigma_j f|}{(j+1)^{1/p-2}\log^{2[1/2+p]}(j+1)} + \frac{q_{n-1}(n-1)\,|\sigma_n f|}{(n+1)^{1/p-2}\log^{2[1/2+p]}(n+1)}\right) \\
\leq\ & \frac{1}{Q_n}\left(\sum_{j=1}^{n-2}(q_j-q_{j+1})\,j + q_{n-1}(n-1)\right)\sup_{n\in\mathbb{N}_+}\frac{|\sigma_n f|}{(n+1)^{1/p-2}\log^{2[1/2+p]}(n+1)} \\
\leq\ & \sup_{n\in\mathbb{N}_+}\frac{|\sigma_n f|}{(n+1)^{1/p-2}\log^{2[1/2+p]}(n+1)} = \widetilde{\sigma}_p^* f,
\end{aligned}$$

so that $\widetilde{T}_p^* f \leq \widetilde{\sigma}_p^* f$. Hence, if we apply Theorem T1 we can conclude that the maximal operators $\widetilde{T}_p^*$ of T means with non-increasing sequence $\{q_k : k \geq 0\}$ are bounded from the Hardy space H_p to the space L_p for $0 < p \leq 1/2$. The proof is complete. □

Proof of Theorem 2. Let $0 < p \leq 1/2$ and the sequence $\{q_k : k \geq 0\}$ be non-decreasing satisfying the condition (22). By applying (11) and (15) we find that

$$\begin{aligned}
&\frac{|T_n f|}{(n+1)^{1/p-2}\log^{2[1/2+p]}(n+1)} \\
\leq\ & \frac{1}{(n+1)^{1/p-2}\log^{2[1/2+p]}(n+1)}\left|\frac{1}{Q_n}\sum_{j=1}^{n-1} q_j S_j f\right| \\
\leq\ & \frac{1}{(n+1)^{1/p-2}\log^{2[1/2+p]}(n+1)}\frac{1}{Q_n}\left(\sum_{j=1}^{n-2}|q_j-q_{j+1}|\,j\,|\sigma_j f| + q_{n-1}(n-1)\,|\sigma_n f|\right) \\
\leq\ & \frac{1}{Q_n}\left(\sum_{j=1}^{n-2}\frac{|q_j-q_{j+1}|\,j\,|\sigma_j f|}{(j+1)^{1/p-2}\log^{2[1/2+p]}(j+1)} + \frac{q_{n-1}(n-1)\,|\sigma_n f|}{(n+1)^{1/p-2}\log^{2[1/2+p]}(n+1)}\right) \\
\leq\ & \frac{1}{Q_n}\left(\sum_{j=1}^{n-2}(q_{j+1}-q_j)\,j + q_{n-1}(n-1)\right)\sup_{n\in\mathbb{N}_+}\frac{|\sigma_n f|}{(n+1)^{1/p-2}\log^{2[1/2+p]}(n+1)} \\
\leq\ & \frac{2q_{n-1}(n-1)-Q_n+q_0}{Q_n}\sup_{n\in\mathbb{N}_+}\frac{|\sigma_n f|}{(n+1)^{1/p-2}\log^{2[1/2+p]}(n+1)} \\
\leq\ & c\sup_{n\in\mathbb{N}_+}\frac{|\sigma_n f|}{(n+1)^{1/p-2}\log^{2[1/2+p]}(n+1)} = c\widetilde{\sigma}_p^* f.
\end{aligned}$$

so that

$$\widetilde{T}_p^* f \leq c\widetilde{\sigma}_p^* f \tag{35}$$

If we apply (35) and Theorem T1 we can conclude that the maximal operators $\widetilde{T}_p^*$ of T means with non-decreasing sequence $\{q_k : k \geq 0\}$, are bounded from the Hardy space H_p to the space L_p for $0 < p \leq 1/2$. The proof is complete. □

Proof of Theorem 3. Let $0 < p < 1/2$ and the sequence $\{q_k : k \geq 0\}$ be non-increasing. By Lemma 1, the proof of part a) will be complete, if we show that

$$\sum_{k=1}^{\infty} \frac{\|T_k a\|_{H_p}^p}{k^{2-2p}} \leq c_p, \tag{36}$$

for every p-atom a, with support I, $\mu(I) = M_N^{-1}$. We may assume that $I = I_N$. It is easy to see that $S_k(a) = T_k(a) = 0$, when $k \leq M_N$. Therefore, we can suppose that $k > M_N$.

Let $x \in I_N$. Since T_k is bounded from L_∞ to L_∞ (boundedness follows Lemma 4) and $\|a\|_\infty \leq M_N^{1/p}$ we obtain that

$$\int_{I_N} |T_k a|^p \, d\mu \leq \frac{\|a\|_\infty^p}{M_N} \leq c < \infty.$$

Hence,

$$\sum_{k=1}^{\infty} \frac{\int_{I_N} |T_k a|^p \, d\mu}{k^{2-2p}} \leq \sum_{k=1}^{\infty} \frac{c}{k^{2-2p}} \leq c < \infty, \quad 0 < p < 1/2. \tag{37}$$

It is easy to see that

$$\begin{aligned} (38) \qquad & |T_k a(x)| \\ = & \left| \int_{I_N} a(t) F_k(x-t) \, d\mu(t) \right| = \left| \int_{I_N} a(t) \frac{1}{Q_k} \sum_{l=M_N}^{k} q_l D_l(x-t) \, d\mu(t) \right| \\ \leq & \|a\|_\infty \int_{I_N} \left| \frac{1}{Q_k} \sum_{l=M_N}^{k} q_l D_l(x-t) \right| d\mu(t) \leq c M_N^{1/p} \int_{I_N} \left| \frac{1}{Q_k} \sum_{l=M_N}^{k} q_l D_l(x-t) \right| d\mu(t). \end{aligned}$$

Let T_k be T means, with non-decreasing coefficients $\{q_k : k \geq 0\}$ and $x \in I_N^{i,j}$, $0 \leq i < j \leq N$. Then, in the view of Lemma 6 we get that

$$|T_k a(x)| \leq c M_i M_j M_N^{1/p-2}, \text{ for } 0 < p < 1/2. \tag{39}$$

Let $0 < p < 1/2$. By using (5), (38) and (39) we find that

$$\begin{aligned} (40) \quad \int_{\overline{I_N}} |T_k a|^p \, d\mu & = \sum_{i=0}^{N-2} \sum_{j=i+1}^{N-1} \sum_{x_j=0,\ j\in\{i+1,\dots,N-1\}}^{m_{j-1}} \int_{I_N^{i,j}} |T_k a|^p \, d\mu + \sum_{i=0}^{N-1} \int_{I_N^{i,N}} |T_k a|^p \, d\mu \\ & \leq c \sum_{i=0}^{N-2} \sum_{j=i+1}^{N-1} \frac{m_{j+1} \cdots m_{N-1}}{M_N} (M_i M_j)^p M_N^{1-2p} + c \sum_{i=0}^{N-1} \frac{1}{M_N} M_i^p M_N^{1-p} \\ & \leq c M_N^{1-2p} \sum_{i=0}^{N-2} \sum_{j=i+1}^{N-1} \frac{(M_j M_k)^p}{M_j} + c \sum_{i=0}^{N-1} \frac{M_i^p}{M_N^p} \leq c M_N^{1-2p}. \end{aligned}$$

Moreover, according to (40) we get that

$$\sum_{k=M_N+1}^{\infty}\frac{\int_{\overline{I_N}}|T_k a|^p\,d\mu}{k^{2-2p}}\leq\sum_{k=M_N+1}^{\infty}\frac{cM_N^{1-2p}}{k^{2-2p}}<c,\quad(0<p<1/2)\,.\tag{41}$$

The proof of (36), and thus of part a), is complete by just combining (37) and (41).

Let $p=1/2$ and T_k be T means, with non-increasing coefficients $\{q_k : k\geq 0\}$, satisfying condition (25). By Lemma 1, the proof of part b) will be complete, if we show that

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{\|T_k a\|_{H_{1/2}}^{1/2}}{k}\leq c,\tag{42}$$

for every 1/2-atom a, with support I, $\mu(I)=M_N^{-1}$. We may assume that $I=I_N$. It is easy to see that $S_k(a)=T_k(a)=0$, when $k\leq M_N$. Therefore, we can suppose that $k>M_N$.

Let $x\in I_N$. Since T_n is bounded from L_∞ to L_∞ (boundedness follows from Lemma 4) and $\|a\|_\infty\leq M_N^2$ we obtain that

$$\int_{I_N}|T_k a|^{1/2}\,d\mu\leq\frac{\|a\|_\infty^{1/2}}{M_N}\leq c<\infty.$$

Hence,

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{\int_{I_N}|T_k a|^{1/2}\,d\mu}{k}\leq\frac{c}{\log n}\sum_{k=1}^{n}\frac{1}{k}\leq c<\infty.\tag{43}$$

Analogously to (38) we find that

$$\begin{aligned}|T_k a(x)| &= \left|\int_{I_N}a(t)\frac{1}{Q_k}\sum_{l=M_N}^{k}q_l D_l(x-t)\,d\mu(t)\right|\\ &\leq \|a\|_\infty\int_{I_N}\left|\frac{1}{Q_k}\sum_{l=M_N}^{k}q_l D_l(x-t)\right|d\mu(t)\\ &\leq M_N^2\int_{I_N}\left|\frac{1}{Q_k}\sum_{l=M_N}^{k}q_l D_l(x-t)\right|d\mu(t)\,.\end{aligned}\tag{44}$$

Let $x\in I_N^{i,j}$, $0\leq i<j<N$. Then, in the view of Lemma 8 we get that

$$|T_k a(x)|\leq\frac{cM_iM_jM_N}{k}.\tag{45}$$

Let $x\in I_N^{i,N}$. Then, according to Lemma 8 we obtain that

$$|T_k a(x)|\leq cM_iM_N.\tag{46}$$

By combining (5), (44), (45) and (46) we obtain that

$$\begin{aligned}
&\int_{\overline{I_N}} |T_k a(x)|^{1/2} d\mu(x) \\
\leq\ & c\sum_{i=0}^{N-2}\sum_{j=i+1}^{N-1}\frac{m_{j+1}\cdots m_{N-1}}{M_N}\frac{(M_iM_j)^{1/2}M_N^{1/2}}{k^{1/2}}+c\sum_{i=0}^{N-1}\frac{1}{M_N}M_i^{1/2}M_N^{1/2} \\
\leq\ & cM_N^{1/2}\sum_{i=0}^{N-2}\sum_{j=i+1}^{N-1}\frac{(M_iM_j)^{1/2}}{k^{1/2}M_j}+c\sum_{i=0}^{N-1}\frac{M_i^{1/2}}{M_N^{1/2}}\leq\frac{cM_N^{1/2}N}{k^{1/2}}+c.
\end{aligned}$$

It follows that

$$\frac{1}{\log n}\sum_{k=M_N+1}^{n}\frac{\int_{\overline{I_N}}|T_k a(x)|^{1/2}d\mu(x)}{k}\leq\frac{1}{\log n}\sum_{k=M_N+1}^{n}\left(\frac{cM_N^{1/2}N}{k^{3/2}}+\frac{c}{k}\right)<c<\infty. \tag{47}$$

The proof of (42), and thus of part b), is completed by just combining (43) and (47). □

Proof of Theorem 4. If we use Lemmas 11 and 12 and follow analogical steps as in the proof of Theorem 3 we get the proof of Theorem 4. Hence, we leave out the details. □

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Paper F

Some inequalities related to strong convergence of Riesz logarithmic means of Vilenkin-Fourier series

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Some inequalities related to strong convergence of Riesz logarithmic means

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Abstract

In this paper we derive a new strong convergence theorem of Riesz logarithmic means of the one-dimensional Vilenkin–Fourier (Walsh–Fourier) series. The corresponding inequality is pointed out and it is also proved that the inequality is in a sense sharp, at least for the case with Walsh–Fourier series.

MSC: 26D10; 26D20; 42B25; 42C10

Keywords: Inequalities; Vilenkin systems; Walsh system; Riesz logarithmic means; Martingale Hardy space; Strong convergence

1 Introduction

Concerning definitions used in this introduction we refer to Sect. 2. Weisz [47] proved the boundedness of the maximal operator of Fejér means $\sigma^{\psi,*}$ with respect to bounded Vilenkin systems from the martingale Hardy space $H_p(G_m)$ to the space $L_p(G_m)$, for $p > 1/2$. Simon [31] gave a counterexample, which shows that boundedness does not hold for $0 < p < 1/2$. The corresponding counterexample for $p = 1/2$ is due to Goginava [14]. Moreover, Weisz [50] proved the following result.

Theorem W *The maximal operator of Fejér means $\sigma^{\psi,*}$ is bounded from the Hardy space $H_{1/2}(G_m)$ to the space weak-$L_{1/2}(G_m)$.*

In [35] and [36] it was proved that the maximal operator $\widetilde{\sigma}_p^{\psi,*}$ defined by

$$\widetilde{\sigma}_p^{\psi,*} := \sup_{n\in\mathbb{N}} \frac{|\sigma_n^{\psi}|}{(n+1)^{1/p-2}\log^{2[1/2+p]}(n+1)},$$

where $0 < p \leq 1/2$ and $[1/2+p]$ denotes the integer part of $1/2+p$, is bounded from the Hardy space $H_p(G_m)$ to the space $L_p(G_m)$. Moreover, for any nondecreasing function $\varphi : \mathbb{N}_+ \to [1,\infty)$ satisfying the condition

$$\overline{\lim_{n\to\infty}} \frac{(n+1)^{1/p-2}\log^{2[1/2+p]}(n+1)}{\varphi(n)} = +\infty, \tag{1.1}$$

there exists a martingale $f \in H_p(G_m)$, *such that*

$$\sup_{n\in\mathbb{N}} \left\| \frac{\sigma_n^{\psi} f}{\varphi(n)} \right\|_p = \infty.$$

For Walsh–Kaczmarzi system some analogical results were proved in [16] and [37].

Weisz [47] considered the norm convergence of the Fejér means of a Vilenkin–Fourier series and proved the following result.

Theorem W1 (Weisz) *Let* $p > 1/2$ *and* $f \in H_p(G_m)$. *Then there exists an absolute constant* c_p, *depending only on* p, *such that for all* $k = 1, 2, \ldots$ *and* $f \in H_p(G_m)$ *the following inequality holds*:

$$\left\| \sigma_k^{\psi} f \right\|_p \le c_p \|f\|_{H_p(G_m)}.$$

Moreover, in [34] it was proved that the assumption $p > 1/2$ in Theorem W1 is essential. In fact, the following is true.

Theorem T1 *There exists a martingale* $f \in H_{1/2}(G_m)$ *such that*

$$\sup_{n\in\mathbb{N}} \left\| \sigma_n^{\psi} f \right\|_{1/2} = +\infty.$$

Theorem W1 implies that

$$\frac{1}{n^{2p-1}} \sum_{k=1}^{n} \frac{\|\sigma_k^{\psi} f\|_p^p}{k^{2-2p}} \le c_p \|f\|_{H_p(G_m)}^p, \quad 1/2 < p < \infty, n = 1, 2, \ldots.$$

If Theorem W1 holds for $0 < p \le 1/2$, then we would have

$$\frac{1}{\log^{[1/2+p]} n} \sum_{k=1}^{n} \frac{\|\sigma_k^{\psi} f\|_p^p}{k^{2-2p}} \le c_p \|f\|_{H_p(G_m)}^p, \quad 0 < p \le 1/2, n = 2, 3, \ldots. \tag{1.2}$$

For the Walsh system in [38] and for the bounded Vilenkin systems in [37] were proved that (1.2) holds, though Theorem T1 is not true for $0 < p < 1/2$.

Some results concerning summability of the Fejér means of a Vilenkin–Fourier series can be found in [10, 12, 16, 25, 28, 30].

The Riesz logarithmic means with respect to the Walsh system was studied by Simon [31], Goginava [15], Gát, Nagy [13] and for Vilenkin systems by Gát [11] and Blahota, Gát [3], Persson, Ragusa, Samko, Wall [26]. Moreover, in [27] it was proved that the maximal operator of the Riesz logarithmic means of a Vilenkin–Fourier series is bounded from the martingale Hardy space $H_p(G_m)$ to the space $L_p(G_m)$ when $p > 1/2$ and is not bounded from the martingale Hardy space $H_p(G_m)$ to the space $L_p(G_m)$ when $0 < p \le 1/2$.

In [35] and [36] it was proved that the Riesz logarithmic means has better properties than the Fejér means. In particular, one considered the maximal operator $\widetilde{R}_p^{\psi,*}$ of a Riesz logarithmic means $\widetilde{R}_p^{\psi,*}$ defined by

$$\widetilde{R}_p^{\psi,*} := \sup_{n\in\mathbb{N}} \frac{|R_n^{\psi}| \log(n+1)}{(n+1)^{1/p-2} \log^{2[1/2+p]}(n+1)},$$

where $0 < p \leq 1/2$ and $[1/2 + p]$ denotes the integer part of $1/2 + p$, which is bounded from the Hardy space $H_p(G_m)$ to the space $L_p(G_m)$.

Moreover, this result is sharp in the following sense: For any nondecreasing function $\varphi : \mathbb{N}_+ \to [1, \infty)$ satisfying the condition

$$\overline{\lim_{n\to\infty}} \frac{(n+1)^{1/p-2} \log^{2[1/2+p]}(n+1)}{\varphi(n)\log(n+1)} = \infty, \tag{1.3}$$

there exists a martingale $f \in H_p(G_m)$, such that

$$\sup_{n\in\mathbb{N}} \left\| \frac{R_n^{\psi} f}{\varphi(n)} \right\|_p = \infty.$$

The main aim of this paper is to derive a new strong convergence theorem of the Riesz logarithmic means of one-dimensional Vilenkin–Fourier (Walsh–Fourier) series (see Theorem 1). The corresponding inequality is pointed out. The sharpness is proved in Theorem 2, at least for the case with Walsh–Fourier series.

The paper is organized as follows: In Sect. 2 some definitions and notations are presented. The main results are presented and proved in Sect. 3. Section 4 is reserved for some concluding remarks and open problems.

2 Definitions and notations

Let $\mathbb{N}_+$ denote the set of positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$.

Let $m := (m_0, m_1, \dots)$ denote a sequence of positive integers not less than 2.

Denote by

$$Z_{m_k} := \{0, 1, \dots m_k - 1\}$$

the additive group of integers modulo m_k.

Define the group G_m as the complete direct product of the group Z_{m_j} with the product of the discrete topologies of the Z_{m_j}.

The direct product μ of the measures

$$\mu_k(\{j\}) := 1/m_k \quad (j \in Z_{m_k})$$

is a Haar measure on G_m with $\mu(G_m) = 1$.

If $\sup_{n\in\mathbb{N}} m_n < \infty$, then we call G_m a bounded Vilenkin group. If the generating sequence m is not bounded, then G_m is said to be an unbounded Vilenkin group. In this paper we discuss only bounded Vilenkin groups.

The elements of G_m are represented by the sequences

$$x := (x_0, x_1, \dots, x_j, \dots) \quad (x_k \in Z_{m_k}).$$

It is easy to give a base for the neighborhood of G_m, namely

$$I_0(x) := G_m,$$

$$I_n(x) := \{y \in G_m \mid y_0 = x_0, \dots y_{n-1} = x_{n-1}\} \quad (x \in G_m, n \in \mathbb{N}).$$

Denote $I_n := I_n(0)$ for $n \in \mathbb{N}$ and $\overline{I_n} := G_m \backslash I_n$.

Let

$$e_n := (0, 0, \dots, x_n = 1, 0, \dots) \in G_m \quad (n \in \mathbb{N}).$$

It is evident that

$$\overline{I_M} = \left(\bigcup_{k=0}^{M-2} \bigcup_{x_k=1}^{m_k-1} \bigcup_{l=k+1}^{M-1} \bigcup_{x_l=1}^{m_l-1} I_{l+1}(x_k e_k + x_l e_l) \right) \cup \left(\bigcup_{k=1}^{M-1} \bigcup_{x_k=1}^{m_k-1} I_M(x_k e_k) \right). \tag{2.1}$$

If we define the so-called generalized number system based on m in the following way:

$$M_0 := 1, \qquad M_{k+1} := m_k M_k \quad (k \in \mathbb{N}),$$

then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j}$ $(j \in \mathbb{N})$ and only a finite number of the n_j differ from zero. Let $|n| := \max\{j \in \mathbb{N}; n_j \neq 0\}$.

The norm (or quasi-norm when $p < 1$) of the space $L_p(G_m)$ is defined by

$$\|f\|_p := \left(\int_{G_m} |f|^p \, d\mu \right)^{1/p} \quad (0 < p < \infty).$$

The space weak-$L_p(G_m)$ consists of all measurable functions f for which

$$\|f\|_{\text{weak-}L_p(G_m)} := \sup_{\lambda > 0} \lambda^p \mu(f > \lambda) < +\infty.$$

Next, we introduce on G_m an orthonormal system which is called the Vilenkin system.

Let us define complex valued function $r_k(x) : G_m \to \mathbb{C}$, the generalized Rademacher functions, as

$$r_k(x) := \exp(2\pi i x_k / m_k) \quad \left(i^2 = -1, x \in G_m, k \in \mathbb{N}\right).$$

Now, define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on G_m as

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \quad (n \in \mathbb{N}).$$

The Vilenkin systems are orthonormal and complete in $L_2(G_m)$ (for details see e.g. [1]).

Specifically, we call this system Walsh–Paley if $m_k = 2$, for all $k \in \mathbb{N}$. In this case we have the dyadic group $G_2 = \prod_{j=0}^{\infty} Z_2$, which is called the Walsh group and the Vilenkin system coincides with the Walsh functions defined by (for details see e.g. [17] and [29])

$$w_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \quad (n \in \mathbb{N}),$$

where $n_k = 0 \vee 1$ and $x_k = 0 \vee 1$.

Now, we introduce analogues of the usual definitions in Fourier analysis.

If $f \in L_1(G_m)$, then we can establish the Fourier coefficients, the partial sums of the Fourier series, the Fejér means, the Dirichlet and Fejér kernels with respect to the Vilenkin system ψ (Walsh system w) in the usual manner:

$$\widehat{f}^{\alpha}(k) := \int_{G_m} f\overline{\alpha}_k \, d\mu \quad (\alpha_k = w_k \text{ or } \psi_k)\ (k \in \mathbb{N}),$$

$$S_n^{\alpha} f := \sum_{k=0}^{n-1} \widehat{f}(k)\alpha_k \quad (\alpha_k = w_k \text{ or } \psi_k)\ \big(n \in \mathbb{N}_+, S_0^{\alpha} f := 0\big),$$

$$\sigma_n^{\alpha} f := \frac{1}{n}\sum_{k=0}^{n-1} S_k^{\alpha} f \quad (\alpha = w \text{ or } \psi)\ (n \in \mathbb{N}_+),$$

$$D_n^{\alpha} := \sum_{k=0}^{n-1} \alpha_k \quad (\alpha = w \text{ or } \psi)\ (n \in \mathbb{N}_+),$$

$$K_n^{\alpha} := \frac{1}{n}\sum_{k=0}^{n-1} D_k^{\alpha} \quad (\alpha = w \text{ or } \psi)\ (n \in \mathbb{N}_+).$$

It is well known that (see e.g. [1])

$$\sup_{n\in\mathbb{N}} \int_{G_m} \left|K_n^{\alpha}\right| d\mu \le c < \infty, \quad \text{where } \alpha = w \text{ or } \psi. \tag{2.2}$$

The σ-algebra generated by the intervals $\{I_n(x) : x \in G_m\}$ will be denoted by $\digamma_n$ $(n \in \mathbb{N})$. Denote by $f = (f^{(n)}, n \in \mathbb{N})$ a martingale with respect to $\digamma_n$ $(n \in \mathbb{N})$ (for details see e.g. [5, 23, 46]). The maximal function of a martingale f is defend by

$$f^* = \sup_{n\in\mathbb{N}} \left|f^{(n)}\right|.$$

In the case $f \in L_1(G_m)$, the maximal functions are also given by

$$f^*(x) = \sup_{n\in\mathbb{N}} \frac{1}{|I_n(x)|}\left|\int_{I_n(x)} f(u)\mu(u)\right|.$$

For $0 < p < \infty$ the Hardy martingale spaces $H_p(G_m)$ consist of all martingales for which

$$\|f\|_{H_p(G_m)} := \left\|f^*\right\|_p < \infty.$$

If $f \in L_1(G_m)$, then it is easy to show that $S_{M_n} f$ is $\digamma_n$ measurable and the sequence $(S_{M_n} f : n \in \mathbb{N})$ is a martingale. If $f = (f^{(n)}, n \in \mathbb{N})$ is a martingale, then the Vilenkin–Fourier (Walsh–Fourier) coefficients must be defined in a slightly different manner, namely

$$\widehat{f}(i) := \lim_{k\to\infty} \int_{G_m} f^{(k)}(x)\overline{\alpha}_i(x)\, d\mu(x), \quad \text{where } \alpha = w \text{ or } \psi.$$

The Vilenkin–Fourier coefficients of $f \in L_1(G_m)$ are the same as those of the martingale $(S_{M_n} f : n \in \mathbb{N})$ obtained from f.

In the literature, there is the notion of the Riesz logarithmic means of a Fourier series. The nth Riesz logarithmic means of the Fourier series of an integrable function f is defined by

$$R_n^\alpha f := \frac{1}{l_n}\sum_{k=1}^{n}\frac{S_k^\alpha f}{k}, \quad \text{where } \alpha = w \text{ or } \psi,$$

with

$$l_n := \sum_{k=1}^{n}\frac{1}{k}.$$

The kernels of Riesz's logarithmic means are defined by

$$L_n^\alpha := \frac{1}{l_n}\sum_{k=1}^{n}\frac{D_k^\alpha}{k}, \quad \text{where } (\alpha = w \text{ or } \psi).$$

For the martingale f we consider the following maximal operators:

$$\sigma^{\alpha,*} f : \sup_{n\in\mathbb{N}}\left|\sigma_n^\alpha f\right| \quad (\alpha = w \text{ or } \psi),$$

$$R^* f := \sup_{n\in\mathbb{N}}\left|R_n^\alpha f\right| \quad (\alpha = w \text{ or } \psi),$$

$$\widetilde{R}^{\alpha,*} f := \sup_{n\in\mathbb{N}}\frac{|R_n^\alpha f|}{\log(n+1)} \quad (\alpha = w \text{ or } \psi),$$

$$\widetilde{R}_p^{\alpha,*} f := \sup_{n\in\mathbb{N}}\frac{\log(n+1)|R_n^\alpha f|}{(n+1)^{1/p-2}} \quad (\alpha = w \text{ or } \psi).$$

A bounded measurable function a is a p-atom, if there exists an interval I, such that

$$\int_I a\,d\mu = 0, \qquad \|a\|_\infty \le \mu(I)^{-1/p}, \qquad \operatorname{supp}(a) \subset I.$$

In order to prove our main results we need the following lemma of Weisz (for details see e.g. Weisz [49]).

Proposition 1 *A martingale $f = (f^{(n)}, n \in \mathbb{N})$ is in $H_p(G_m)$ $(0 < p \le 1)$ if and only if there exist a sequence $(a_k, k \in \mathbb{N})$ of p-atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of a real numbers such that for every $n \in \mathbb{N}$*

$$\sum_{k=0}^{\infty}\mu_k S_{M_n} a_k = f^{(n)} \tag{2.3}$$

and

$$\sum_{k=0}^{\infty}|\mu_k|^p < \infty.$$

Moreover, $\|f\|_{H_p(G_m)} \backsim \inf(\sum_{k=0}^{\infty}|\mu_k|^p)^{1/p}$, where the infimum is taken over all decompositions of f of the form (2.3).

By using atomic characterization (see Proposition 1) it can be easily proved that the following statement holds (see e.g. Weisz [50]).

Proposition 2 *Suppose that an operator T is sub-linear and for some $0 < p_0 \leq 1$*

$$\int_{\bar{I}} |Ta|^{p_0}\, d\mu \leq c_p < \infty$$

for every p_0-atom a, where I denotes the support of the atom. If T is bounded from L_{p_1} to L_{p_1} $(1 < p_1 \leq \infty)$, then

$$\|Tf\|_{p_0} \leq c_{p_0} \|f\|_{H_{p_0}(G_m)}. \tag{2.4}$$

Let us define classical Hardy spaces (see e.g. [44]). Let $H_p(D)$, $p > 0$ be the one-dimensional complex quasi-Banach space of analytic functions f on the unit disc $D := (z : |z| < 1)$ for which

$$\|f\|_{H_p(D)} = \sup_{r<1} \frac{1}{2\pi} \left(\int_{[-\pi,\pi]} \left|f\left(re^{it}\right)\right|^p dt \right)^{1/p}.$$

Now, we define real Hardy spaces. A real-valued distributions $f(t) \in D'(T)$ belongs to $H_p(T)$ where $T = (-\pi, \pi]$ if and only if there exists a function $F(z) \in H_p(D)$ with the properties $\operatorname{Im}(F(0)) = 0$ and $f(t) = \lim_{r\to 1} \operatorname{Re} F(re^{it})$ in the sense of distributions. Equipped with quasi-norm $\|f(z)\|_{H_p(T)} = \|F(z)\|_{H_p(D)}$ the class obviously becomes a real quasi-Banach space with quite the same properties as $H_p(D)$. Atomic decomposition of classical Hardy spaces and real Hardy spaces can be found e.g. in Fefferman and Stein [6] (see also Later [19], Torchinsky [44], Wilson [51]).

3 Main results

Our first main result reads as follows.

Theorem 1 *Let $0 < p < 1/2$ and $f \in H_p(G_m)$. Then there exists an absolute constant c_p, depending only on p, such that the inequality*

$$\sum_{n=1}^{\infty} \frac{\log^p n \|R_n^{\psi} f\|_{H_p(G_m)}^p}{n^{2-2p}} \leq c_p \|f\|_{H_p(G_m)}^p \tag{3.1}$$

holds, where $R_n^{\psi} f$ denotes the nth Riesz logarithmic mean with respect to the Vilenkin–Fourier series of f.

For the proof of Theorem 1 we will use the following lemmas.

Lemma 1 (see [38]) *Let $x \in I_N(x_k e_k + x_l e_l)$, $1 \leq x_k \leq m_k - 1$, $1 \leq x_l \leq m_l - 1$, $k = 0, \dots, N-2$, $l = k+1, \dots, N-1$. Then*

$$\int_{I_N} \left|K_n^{\psi}(x-t)\right| d\mu(t) \leq \frac{c M_l M_k}{n M_N}, \quad \textit{when } n \geq M_N.$$

Let $x \in I_N(x_k e_k)$, $1 \le x_k \le m_k - 1$, $k = 0, \dots, N-1$. *Then*

$$\int_{I_N} \left|K_n^{\psi}(x-t)\right| d\mu(t) \le \frac{cM_k}{M_N}, \quad \textit{when } n \ge M_N.$$

Lemma 2 (see [39]) *Let* $x \in I_N(x_k e_k + x_l e_l)$, $1 \le x_k \le m_k - 1$, $1 \le x_l \le m_l - 1$, $k = 0, \dots, N-2$, $l = k+1, \dots, N-1$. *Then*

$$\int_{I_N} \sum_{j=M_N+1}^{n} \frac{|K_j^{\psi}(x-t)|}{j+1} d\mu(t) \le \frac{cM_k M_l}{M_N^2}.$$

Let $x \in I_N(x_k e_k)$, $1 \le x_k \le m_k - 1$, $k = 0, \dots, N-1$. *Then*

$$\int_{I_N} \sum_{j=M_N+1}^{n} \frac{|K_j^{\psi}(x-t)|}{j+1} d\mu(t) \le \frac{cM_k}{M_N} l_n.$$

Proof By using an Abel transformation, the kernels of the Riesz logarithmic means can be rewritten as (see also [39])

$$L_n^{\psi} = \frac{1}{l_n} \sum_{j=1}^{n-1} \frac{K_j^{\psi}}{j+1} + \frac{K_n^{\psi}}{l_n}. \tag{3.2}$$

Hence, according to (2.2) we get

$$\sup_{n \in \mathbb{N}} \int_{G_m} \left|L_n^{\alpha}\right| d\mu \le c < \infty, \quad \text{where } \alpha = w \text{ or } \psi$$

and it follows that R_n^{ψ} is bounded from L_∞ to L_∞. By Proposition 2, the proof of Theorem 1 will be complete, if we show that

$$\sum_{n=1}^{\infty} \frac{\log^p n \int_{\bar{I}} |R_n^{\psi} a|^p \, d\mu}{n^{2-2p}} \le c_p < \infty, \quad \text{for } 0 < p < 1/2, \tag{3.3}$$

for every p-atom a, where I denotes the support of the atom.

Let a be an arbitrary p-atom with support I and $\mu(I) = M_N^{-1}$. We may assume that $I = I_N$. It is easy to see that $R_n^{\psi} a = \sigma_n^{\psi}(a) = 0$, when $n \le M_N$. Therefore we suppose that $n > M_N$.

Since $\|a\|_\infty \le cM_N^2$ if we apply (3.2), then we can conclude that

$$\begin{aligned}
&\left|R_n^{\psi} a(x)\right| \\
&\quad = \int_{I_N} \left|a(t)\right| \left|L_n^{\psi}(x-t)\right| d\mu(t) \\
&\quad \le \|a\|_\infty \int_{I_N} \left|L_n^{\psi}(x-t)\right| d\mu(t) \\
&\quad \le \frac{cM_N^{1/p}}{l_n} \int_{I_N} \sum_{j=M_N+1}^{n-1} \frac{|K_j^{\psi}(x-t)|}{j+1} d\mu(t) \\
&\qquad + \frac{cM_N^{1/p}}{l_n} \int_{I_N} \left|K_n^{\psi}(x-t)\right| d\mu(t).
\end{aligned} \tag{3.4}$$

Let $x \in I_N(x_k e_k + x_l e_l)$, $1 \le x_k \le m_k - 1$, $1 \le x_l \le m_l - 1$, $k = 0, \dots, N-2$, $l = k+1, \dots, N-1$. From Lemmas 1 and 2 it follows that

$$\left|R_n^{\psi} a(x)\right| \le \frac{c M_l M_k M_N^{1/p-2}}{\log(n+1)}. \tag{3.5}$$

Let $x \in I_N(x_k e_k)$, $1 \le x_k \le m_k - 1$, $k = 0, \dots, N-1$. Applying Lemmas 1 and 2 we can conclude that

$$\left|R_n^{\psi} a(x)\right| \le M_N^{1/p-1} M_k. \tag{3.6}$$

By combining (2.1) and (3.4)–(3.6) we obtain

$$\begin{aligned}
&\int_{\overline{I_N}} \left|R_n^{\psi} a(x)\right|^p d\mu(x) \\
&= \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \sum_{x_j=0, j\in\{l+1,\dots,N-1\}}^{m_j-1} \int_{I_N^{k,l}} \left|R_n^{\psi} a\right|^p d\mu + \sum_{k=0}^{N-1} \int_{I_N^{k,N}} \left|R_n^{\psi} a\right|^p d\mu \\
&\le c \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{m_{l+1} \dots m_{N-1}}{M_N} \frac{(M_l M_k)^p M_N^{1-2p}}{\log^p(n+1)} + \sum_{k=0}^{N-1} \frac{1}{M_N} M_k^p M_N^{1-p} \\
&\le \frac{c M_N^{1-2p}}{\log^p(n+1)} \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{(M_l M_k)^p}{M_l} + \sum_{k=0}^{N-1} \frac{M_k^p}{M_N^p} \\
&\le \frac{c M_N^{1-2p}}{\log^p(n+1)} + c_p.
\end{aligned} \tag{3.7}$$

It is easy to see that

$$\sum_{n=M_N+1}^{\infty} \frac{1}{n^{2-2p}} \le \frac{c}{M_N^{1-2p}}, \quad \text{for } 0 < p < 1/2. \tag{3.8}$$

By combining (3.7) and (3.8) we get

$$\begin{aligned}
&\sum_{n=M_N+1}^{\infty} \frac{\log^p n \int_{\overline{I_N}} |R_n a|^p d\mu}{n^{2-2p}} \\
&\le \sum_{n=M_N+1}^{\infty} \left(\frac{c_p M_N^{1-2p}}{n^{2-p}} + \frac{c_p}{n^{2-p}} \right) + c_p \\
&\le c_p M_N^{1-2p} \sum_{n=M_N+1}^{\infty} \frac{1}{n^{2-2p}} + \sum_{n=M_N+1}^{\infty} \frac{1}{n^{2-p}} + c_p \le C_p < \infty.
\end{aligned}$$

It means that (3.3) holds true and the proof is complete. □

Our next main result shows in particular that the inequality in Theorem 1 is in a special sense sharp at least in the case of Walsh–Fourier series (cf. also Problem 2 in the next section).

Theorem 2 *Let* $0 < p < 1/2$ *and* $\Phi : \mathbb{N} \to [1, \infty)$ *be any nondecreasing function, satisfying the condition*

$$\lim_{n\to\infty} \Phi(n) = +\infty. \tag{3.9}$$

Then there exists a martingale $f \in H_p(G_2)$ *such that*

$$\sum_{n=1}^{\infty} \frac{\log^p n \|R_n^w f\|_p^p \Phi(n)}{n^{2-2p}} = \infty, \tag{3.10}$$

where $R_n^w f$ *denotes the nth Riesz logarithmic means with respect to Walsh–Fourier series of* f.

Proof It is evident that if we assume that $\Phi(n) \geq cn$, where c is some positive constant then

$$\frac{\log^p n\Phi(n)}{n^{2-2p}} \geq n^{1-2p} \log^p n \to \infty, \quad \text{as } n \to \infty,$$

and also (3.10) holds. So, without loss of generality we may assume that there exists an increasing sequence of positive integers $\{\alpha'_k : k \in \mathbb{N}\}$ such that

$$\Phi(\alpha'_k) = o(\alpha'_k), \quad \text{as } k \to \infty. \tag{3.11}$$

Let $\{\alpha_k : k \in \mathbb{N}\} \subseteq \{\alpha'_k : k \in \mathbb{N}\}$ be an increasing sequence of positive integers such that $\alpha_0 \geq 2$ and

$$\sum_{k=0}^{\infty} \frac{1}{\Phi^{1/2}(2^{2\alpha_k})} < \infty, \tag{3.12}$$

$$\sum_{\eta=0}^{k-1} \frac{2^{2\alpha_\eta/p}}{\Phi^{1/2p}(2^{2\alpha_\eta})} \leq \frac{2^{2\alpha_{k-1}/p+1}}{\Phi^{1/2p}(2^{2\alpha_{k-1}})}, \tag{3.13}$$

$$\frac{2^{2\alpha_{k-1}/p+1}}{\Phi^{1/2p}(2^{2\alpha_{k-1}})} \leq \frac{1}{128\alpha_k} \frac{2^{2\alpha_k(1/p-2)}}{\Phi^{1/2p}(2^{2\alpha_k})}. \tag{3.14}$$

We note that under condition (3.11) we can conclude that

$$\frac{2^{2\alpha_\eta/p}}{\Phi^{1/2p}(2^{2\alpha_\eta})} \geq \left(\frac{2^{2\alpha_\eta}}{\Phi(2^{2\alpha_\eta})}\right)^{1/2p} \to \infty, \quad \text{as } \eta \to \infty$$

and it immediately follows that such an increasing sequence $\{\alpha_k : k \in \mathbb{N}\}$, which satisfies conditions (3.12)–(3.14), can be constructed.

Let

$$f^{(A)}(x) := \sum_{\{k;2\alpha_k<A\}} \lambda_k a_k,$$

where

$$\lambda_k = \frac{1}{\Phi^{1/2p}(2^{2\alpha_k})}$$

and

$$a_k = 2^{2\alpha_k(1/p-1)}(D_{2^{2\alpha_k+1}} - D_{2^{2\alpha_k}}).$$

From (3.12) and Lemma 1 we can conclude that $f = (f^{(n)}, n \in \mathbb{N}) \in H_p(G_2)$.
It is easy to show that

$$\widehat{f}^w(j) = \begin{cases} \frac{2^{2\alpha_k(1/p-1)}}{\Phi^{1/2p}(2^{2\alpha_k})}, & \text{if } j \in \{2^{2\alpha_k}, \dots, 2^{2\alpha_k+1}-1\}, k \in \mathbb{N}, \\ 0, & \text{if } j \notin \bigcup_{k=1}^{\infty}\{2^{2\alpha_k}, \dots, 2^{2\alpha_k+1}-1\}. \end{cases} \tag{3.15}$$

For $n = \sum_{i=1}^{s} 2^{n_i}$, $n_1 < n_2 < \cdots < n_s$ we denote

$$\mathbb{A}_{0,2} := \left\{ n \in \mathbb{N} : n = 2^0 + 2^2 + \sum_{i=3}^{s_n} 2^{n_i} \right\}.$$

Let $2^{2\alpha_k} \le j \le 2^{2\alpha_k+1} - 1$ and $j \in \mathbb{A}_{0,2}$. Then

$$R_j^w f = \frac{1}{l_j} \sum_{n=1}^{2^{2\alpha_k}-1} \frac{S_n f}{n} + \frac{1}{l_j} \sum_{n=2^{2\alpha_k}}^{j} \frac{S_n f}{n} := I + II. \tag{3.16}$$

Let $n < 2^{2\alpha_k}$. Then from (3.13), (3.14) and (3.15) we have

$$\begin{aligned} \left|S_n^w f(x)\right| &\le \sum_{\eta=0}^{k-1} \sum_{v=2^{2\alpha_\eta}}^{2^{2\alpha_\eta+1}-1} \left|\widehat{f}^w(v)\right| \le \sum_{\eta=0}^{k-1} \sum_{v=2^{2\alpha_\eta}}^{2^{2\alpha_\eta+1}-1} \frac{2^{2\alpha_\eta(1/p-1)}}{\Phi^{1/2p}(2^{2\alpha_\eta})} \\ &\le \sum_{\eta=0}^{k-1} \frac{2^{2\alpha_\eta/p}}{\Phi^{1/2p}(2^{2\alpha_\eta})} \le \frac{2^{2\alpha_{k-1}/p+1}}{\Phi^{1/2p}(2^{2\alpha_{k-1}})} \le \frac{1}{128\alpha_k} \frac{2^{2\alpha_k(1/p-2)}}{\Phi^{1/2p}(2^{2\alpha_k})}. \end{aligned}$$

Consequently,

$$\begin{aligned} |I| &\le \frac{1}{l_j} \sum_{n=1}^{2^{2\alpha_k}-1} \frac{|S_n^w f(x)|}{n} \\ &\le \frac{1}{l_{2^{2\alpha_k}}} \frac{1}{128\alpha_k} \frac{2^{2\alpha_k(1/p-2)}}{\Phi^{1/2p}(2^{2\alpha_k})} \sum_{n=1}^{2^{2\alpha_k}-1} \frac{1}{n} \le \frac{1}{128\alpha_k} \frac{2^{2\alpha_k(1/p-2)}}{\Phi^{1/2p}(2^{2\alpha_k})}. \end{aligned} \tag{3.17}$$

Let $2^{2\alpha_k} \le n \le 2^{2\alpha_k+1} - 1$. Then we have the following:

$$\begin{aligned} S_n^w f &= \sum_{\eta=0}^{k-1} \sum_{v=2^{2\alpha_\eta}}^{2^{2\alpha_\eta+1}-1} \widehat{f}^w(v) w_v + \sum_{v=2^{2\alpha_k}}^{n-1} \widehat{f}^w(v) w_v \\ &= \sum_{\eta=0}^{k-1} \frac{2^{2\alpha_\eta(1/p-1)}}{\Phi^{1/2p}(2^{2\alpha_\eta})} \left(D^w_{2^{2\alpha_\eta+1}} - D^w_{2^{2\alpha_\eta}}\right) + \frac{2^{2\alpha_k(1/p-1)}}{\Phi^{1/2p}(2^{2\alpha_k})} \left(D_n^w - D^w_{2^{2\alpha_k}}\right). \end{aligned}$$

This gives

$$II = \frac{1}{l_j}\sum_{n=2^{2\alpha_k}}^{2^{2\alpha_k+1}}\frac{1}{n}\left(\sum_{\eta=0}^{k-1}\frac{2^{2\alpha_\eta(1/p-1)}}{\Phi^{1/2p}(2^{2\alpha_\eta})}\left(D^w_{2^{2\alpha_\eta+1}}-D^w_{2^{2\alpha_\eta}}\right)\right)$$
$$+\frac{1}{l_j}\frac{2^{2\alpha_k(1/p-1)}}{\Phi^{1/2p}(2^{2\alpha_k})}\sum_{n=2^{2\alpha_k}}^{j}\frac{(D^w_n-D^w_{2^{2\alpha_k}})}{n}$$
$$:= II_1 + II_2. \tag{3.18}$$

Let $x\in I_2(e_0+e_1)\in I_0\backslash I_1$. We use well-known equalities for Dirichlet kernels (for details see e.g. [17] and [29]): recall that

$$D^w_{2^n}(x) = \begin{cases} 2^n, & \text{if } x\in I_n,\\ 0, & \text{if } x\notin I_n,\end{cases} \tag{3.19}$$

and

$$D^w_n = w_n\sum_{k=0}^{\infty}n_k r_k D^w_{2^k} = w_n\sum_{k=0}^{\infty}n_k\left(D^w_{2^{k+1}}-D^w_{2^k}\right),\quad \text{for } n=\sum_{i=0}^{\infty}n_i 2^i, \tag{3.20}$$

so we can conclude that

$$D^w_n(x) = \begin{cases} w_n, & \text{if } n \text{ is odd number},\\ 0, & \text{if } n \text{ is even number}.\end{cases}$$

Since $\alpha_0\geq 2$, $k\in\mathbb{N}$ we obtain $2\alpha_k\geq 4$, for all $k\in\mathbb{N}$ and if we apply (3.19) we get

$$II_1 = 0 \tag{3.21}$$

and

$$II_2 = \frac{1}{l_j}\frac{2^{2\alpha_k(1/p-1)}}{\Phi^{1/2p}(2^{2\alpha_k})}\sum_{n=2^{2\alpha_k-1}}^{(j-1)/2}\frac{w_{2n+1}}{2n+1} = \frac{1}{l_j}\frac{2^{2\alpha_k(1/p-1)}r_1}{\Phi^{1/2p}(2^{2\alpha_k})}\sum_{n=2^{2\alpha_k-1}}^{(j-1)/2}\frac{w_{2n}}{2n+1}.$$

Let $x\in I_2(e_0+e_1)$. Then, by the definition of Walsh functions, we get

$$w_{4n+2} = r_1 w_{4n} = -w_{4n}$$

and

$$|II_2| = \frac{1}{l_j}\frac{2^{2\alpha_k(1/p-1)}}{\Phi^{1/2p}(2^{2\alpha_k})}\left|\sum_{n=2^{2\alpha_k-1}}^{(j-1)/2}\frac{w_{2n}}{2n+1}\right|$$
$$= \frac{1}{l_j}\frac{2^{2\alpha_k(1/p-1)}}{\Phi^{1/2p}(2^{2\alpha_k})}\left|\frac{w_{j-1}}{j}+\sum_{n=2^{2\alpha_k-2}+1}^{(j-1)/4}\left(\frac{w_{4n-4}}{4n-3}+\frac{w_{4n-2}}{4n-1}\right)\right|$$

$$= \frac{1}{l_j}\frac{2^{2\alpha_k(1/p-1)}}{\Phi^{1/2p}(2^{2\alpha_k})}\left|\frac{w_{j-1}}{j}+\sum_{n=2^{2\alpha_k-2}+1}^{(j-1)/4}\left(\frac{w_{4n-4}}{4n-3}-\frac{w_{4n-2}}{4n-1}\right)\right|$$
$$\geq \frac{c}{\log(2^{2\alpha_k+1})}\frac{2^{2\alpha_k(1/p-1)}}{\Phi^{1/2p}(2^{2\alpha_k})}\left(\left|\frac{w_{j-1}}{j}\right|-\sum_{n=2^{2\alpha_k-2}+1}^{(j-1)/4}|w_{4n-4}|\left(\frac{1}{4n-3}-\frac{1}{4n-1}\right)\right)$$
$$\geq \frac{1}{4\alpha_k}\frac{2^{2\alpha_k(1/p-1)}}{\Phi^{1/2p}(2^{2\alpha_k})}\left(\frac{1}{j}-\sum_{n=2^{2\alpha_k-2}+1}^{(j-1)/4}\left(\frac{1}{4n-3}-\frac{1}{4n-1}\right)\right). \tag{3.22}$$

By a simple calculation we can conclude that

$$\sum_{n=2^{2\alpha_k-2}+1}^{(j-1)/4}\left(\frac{1}{4n-3}-\frac{1}{4n-1}\right)$$
$$= \sum_{n=2^{2\alpha_k-2}+1}^{(j-1)/4}\frac{2}{(4n-3)(4n-1)}$$
$$\leq \sum_{n=2^{2\alpha_k-2}+1}^{(j-1)/4}\frac{2}{(4n-4)(4n-2)} = \frac{1}{2}\sum_{n=2^{2\alpha_k-2}+1}^{(j-1)/4}\frac{1}{(2n-2)(2n-1)}$$
$$\leq \frac{1}{2}\sum_{n=2^{2\alpha_k-2}+1}^{(j-1)/4}\frac{1}{(2n-2)(2n-2)} = \frac{1}{8}\sum_{n=2^{2\alpha_k-2}+1}^{(j-1)/4}\frac{1}{(n-1)(n-1)}$$
$$\leq \frac{1}{8}\sum_{n=2^{2\alpha_k-2}+1}^{(j-1)/4}\frac{1}{(n-1)(n-2)} = \frac{1}{8}\sum_{l=2^{2\alpha_k-2}+1}^{(j-1)/4}\left(\frac{1}{n-2}-\frac{1}{n-1}\right)$$
$$\leq \frac{1}{8}\left(\frac{1}{2^{2\alpha_k-2}-1}-\frac{4}{j-5}\right) \leq \frac{1}{8}\left(\frac{1}{2^{2\alpha_k-2}-1}-\frac{4}{j}\right).$$

Since $2^{2\alpha_k} \leq j \leq 2^{2\alpha_k+1}-1$, where $\alpha_k \geq 2$, we obtain

$$\frac{2}{2^{2\alpha_k}-4} \leq \frac{2}{2^4-4} = \frac{1}{6}$$

and

$$|II_2| \geq \frac{1}{4\alpha_k}\frac{2^{2\alpha_k(1/p-1)}}{\Phi^{1/2p}(2^{2\alpha_k})}\left(\frac{1}{j}-\frac{1}{8}\left(\frac{1}{2^{2\alpha_k-2}-1}-\frac{4}{j}\right)\right) \tag{3.23}$$
$$\geq \frac{1}{4\alpha_k}\frac{2^{2\alpha_k(1/p-1)}}{\Phi^{1/2p}(2^{2\alpha_k})}\left(\frac{3}{2j}-\frac{1}{2^{2\alpha_k+1}-8}\right)$$
$$\geq \frac{1}{4\alpha_k}\frac{2^{2\alpha_k(1/p-1)}}{\Phi^{1/2p}(2^{2\alpha_k})}\left(\frac{3}{4}\frac{1}{2^{2\alpha_k}}-\frac{1}{2}\frac{1}{2^{2\alpha_k}-4}\right)$$
$$\geq \frac{1}{4\alpha_k}\frac{2^{2\alpha_k(1/p-1)}}{\Phi^{1/2p}(2^{2\alpha_k})}\left(\frac{1}{4}\frac{1}{2^{2\alpha_k}}+\frac{1}{2}\frac{1}{2^{2\alpha_k}}-\frac{1}{2}\frac{1}{2^{2\alpha_k}-4}\right)$$
$$= \frac{1}{4\alpha_k}\frac{2^{2\alpha_k(1/p-1)}}{\Phi^{1/2p}(2^{2\alpha_k})}\left(\frac{1}{4}\frac{1}{2^{2\alpha_k}}-\frac{2}{2^{2\alpha_k}(2^{2\alpha_k}-4)}\right)$$
$$\geq \frac{1}{4\alpha_k}\frac{2^{2\alpha_k(1/p-1)}}{\Phi^{1/2p}(2^{2\alpha_k})}\left(\frac{1}{4}\frac{1}{2^{2\alpha_k}}-\frac{1}{6}\frac{1}{2^{2\alpha_k}}\right)$$

$$\geq \frac{1}{48\alpha_k}\frac{2^{2\alpha_k(1/p-2)}}{\Phi^{1/2p}(2^{2\alpha_k})} \geq \frac{1}{64\alpha_k}\frac{2^{2\alpha_k(1/p-2)}}{\Phi^{1/2p}(2^{2\alpha_k})}.$$

By combining (3.14), (3.16)–(3.23) for $\in I_2(e_0+e_1)$ and $0<p<1/2$ we find that

$$\begin{aligned}\left|R_j^w f(x)\right| &\geq |II_2| - |II_1| - |I| \\ &\geq \frac{1}{64\alpha_k}\frac{2^{2\alpha_k(1/p-2)}}{\Phi^{1/2p}(2^{2\alpha_k})} - \frac{1}{128\alpha_k}\frac{2^{2\alpha_k(1/p-2)}}{\Phi^{1/2p}(2^{2\alpha_k})} = \frac{1}{128\alpha_k}\frac{2^{2\alpha_k(1/p-2)}}{\Phi^{1/2p}(2^{2\alpha_k})}.\end{aligned}$$

Hence,

$$\begin{aligned}&\left\|R_j^w f\right\|_{\text{weak-}L_p(G_2)}^p \\ &\quad\geq \frac{1}{128\alpha_k^p}\frac{2^{2\alpha_k(1-2p)}}{\Phi^{1/2}(2^{2\alpha_k})}\mu\left\{x\in G_2 : \left|R_j^w f\right| \geq \frac{1}{128\alpha_k}\frac{2^{2\alpha_k(1/p-2)}}{\Phi^{1/2p}(2^{2\alpha_k})}\right\}^{1/p} \\ &\quad\geq \frac{1}{128\alpha_k^p}\frac{2^{2\alpha_k(1-2p)}}{\Phi^{1/2}(2^{2\alpha_k})}\mu\left\{x\in I_2(e_0+e_1) : \left|R_j^w f\right| \geq \frac{1}{128\alpha_k}\frac{2^{2\alpha_k(1/p-2)}}{\Phi^{1/2p}(2^{2\alpha_k})}\right\} \\ &\quad\geq \frac{1}{128\alpha_k^p}\frac{2^{2\alpha_k(1-2p)}}{\Phi^{1/2}(2^{2\alpha_k})}\left(\mu\left(x\in I_2(e_0+e_1)\right)\right) > \frac{1}{516\alpha_k^p}\frac{2^{2\alpha_k(1-2p)}}{\Phi^{1/2}(2^{2\alpha_k})}.\end{aligned} \tag{3.24}$$

Moreover,

$$\begin{aligned}&\sum_{j=1}^{\infty}\frac{\|R_j^w f\|_{\text{weak-}L_p(G_2)}^p \log^p(j)\Phi(j)}{j^{2-2p}} \\ &\quad\geq \sum_{\{j\in\mathbb{A}_{0,2}:2^{2\alpha_k}<j\leq 2^{2\alpha_k+1}-1\}}\frac{\|R_j^w f\|_{\text{weak-}L_p}^p \log^p(j)\Phi(j)}{j^{2-2p}} \\ &\quad\geq \frac{c}{\alpha_k^p}\frac{2^{2\alpha_k(1-2p)}}{\Phi^{p/2}(2^{2\alpha_k})}\sum_{\{j\in\mathbb{A}_{0,2}:2^{2\alpha_k}<j\leq 2^{2\alpha_k+1}-1\}}\frac{\log^p(j)\Phi(j)}{j^{2-2p}} \\ &\quad\geq \frac{c\Phi(2^{2\alpha_k})\log^p(2^{2\alpha_k})}{\alpha_k^p}\frac{2^{2\alpha_k(1-2p)}}{\Phi^{1/2}(2^{2\alpha_k})}\sum_{\{j\in\mathbb{A}_{0,2}:2^{2\alpha_k}<j\leq 2^{2\alpha_k+1}-1\}}\frac{1}{j^{2-2p}} \\ &\quad\geq \Phi^{1/2}\left(2^{2\alpha_k}\right)\to\infty, \quad \text{as } k\to\infty.\end{aligned}$$

The proof is complete. □

4 Final remarks and open problems

In this section we present some final remarks and open problems, which might be interesting for further research. The first problem reads as follows.

Problem 1 For any $f\in H_{1/2}$, is it possible to find strong convergence theorems for Riesz means R_m^w, where $\alpha = w$ or $\alpha=\psi$?

Remark 1 Similar problems for Fejér means with respect to Walsh and Vilenkin systems can be found in [2, 4, 40] (see also [45] and [48]). Our method and estimations of Riesz and Fejér kernels (see Lemmas 1 and 2) do not give an opportunity to prove even similar strong convergence result as for the case of Fejer means. In particular, for any $f \in H_{1/2}$ is it possible to prove the following inequality:

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{\|R_k^{\alpha}f\|_{1/2}^{1/2}}{k}\leq c\|f\|_{H_{1/2}}^{1/2},\quad \text{where } \alpha = w \text{ or } \alpha = \psi?$$

It is interesting to generalize Theorem 2 for Vilenkin systems.

Problem 2 For $0 < p < 1/2$ and any nondecreasing function $\Phi : \mathbb{N} \to [1,\infty)$ satisfying the conditions $\lim_{n\to\infty}\Phi(n) = +\infty$, is it possible to find a martingale $f \in H_p(G_m)$ such that

$$\sum_{n=1}^{\infty}\frac{\log^p n\|R_n^{\psi}f\|_p^p\Phi(n)}{n^{2-2p}} = \infty,$$

where $R_n^{\psi}f$ denotes the nth Riesz logarithmic means with respect to the Vilenkin–Fourier series of f?

Problem 3 Is it possible to find a martingale $f \in H_{1/2}$, such that

$$\sup_{n\in\mathbb{N}}\left\|R_n^{\alpha}f\right\|_{1/2} = \infty,$$

where $\alpha = w$ or $\alpha = \psi$?

Remark 2 For $0 < p < 1/2$, divergence in the space L_p of Riesz logarithmic means with respect to Walsh and Vilenkin systems of martingale $f \in H_p$ was already proved in [27].

Problem 4 For any $f \in H_p$ $(0 < p \leq 1/2)$, is it possible to find necessary and sufficient conditions for the indices k_j for which

$$\left\|R_{k_j}^{\alpha}f - f\right\|_{H_p} \to 0, \quad \text{as } j \to \infty,$$

where $\alpha = w$ or $\alpha = \psi$?

Remark 3 Similar problem for partial sums and Fejer means with respect to Walsh and Vilenkin systems can be found in Tephnadze [41, 42] and [43].

Problem 5 Is it possible to find necessary and sufficient conditions in terms of the one-dimensional modulus of continuity of martingale $f \in H_p$ $(0 < p \leq 1/2)$, for which

$$\left\|R_j^{\alpha}f - f\right\|_{H_p} \to 0, \quad \text{as } j \to \infty,$$

where $\alpha = w$ or ψ?

Remark 4 Approximation properties of some summability methods in the classical and real Hardy spaces were considered by Oswald [24], Kryakin and Trebels [18], Storoienko [32, 33] and for martingale Hardy spaces in Fridli, Manchanda and Siddiqi [9] (see also [7, 8]), Nagy [20–22], Tephnadze [41–43].

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Paper G

A note on the maximal operators of the Nörlund logaritmic means of Vilenkin-Fourier series

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A NOTE ON THE MAXIMAL OPERATORS OF THE NÖRLUND LOGARITMIC MEANS OF VILENKIN-FOURIER SERIES

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Abstract. The main aim of this paper is to investigate the (H_p, L_p)- type inequalities for the maximal operators of Nörlund logarithmic means for $0 < p < 1$.

1. Introduction

It is well-known that (see e.g., [1], [8] and [16]) Vilenkin systems do not form bases in the Lebesgue space $L_1(G_m)$. Moreover, there exists a function in the Hardy space H_1 such that the partial sums of f are not bounded in L_1-norm.

In [19] (see also [21]), it was proved that the following is true:

Theorem T1. Let $0 < p < 1$. Then the maximal operator

$$\widetilde{S}_p^* f := \sup_{n\in\mathbb{N}} \frac{|S_n f|}{(n+1)^{1/p-1}}$$

is bounded from the Hardy space $H_p(G_m)$ to the space $L_p(G_m)$. Here, S_n denotes the n-th partial sum with respect to the Vilenkin system. Moreover, it was proved that the rate of the factor $(n+1)^{1/p-1}$ is in a sense sharp.

In the case $p = 1$, it was proved that the maximal operator $\widetilde{S}^*$ defined by

$$\widetilde{S}^* := \sup_{n\in\mathbb{N}} \frac{|S_n|}{\log(n+1)}$$

is bounded from the Hardy space $H_1(G_m)$ to the space $L_1(G_m)$. Moreover, the rate of the factor $\log(n+1)$ is in a sense sharp. Similar problems for the Nörlund logarithmic means in the case, where $p = 1$, was considered in [15].

Móricz and Siddiqi [9] investigated the approximation properties of some special Nörlund means of Walsh-Fourier series of $L_p(G_m)$ functions in L_p-norm. Fridli, Manchanda and Siddiqi [5] improved and extended the results of Móricz and Siddiqi [9] to the Martingale Hardy spaces. However, the case for $\{q_k = 1/k : k \in \mathbb{N}_+\}$ was excluded, since the methods are not applicable to the Nörlund logarithmic means. In [6], Gt and Goginava proved some convergence and divergence properties of Walsh-Fourier series of the Nörlund logarithmic means of functions in the Lebesgue space $L_1(G_m)$. In particular, they proved that there exists a function in the space $L_1(G_m)$ such that

$$\sup_{n\in\mathbb{N}} \|L_n f\|_1 = \infty.$$

In [2] (see also [15, 17]), it was proved that there exists a martingale $f \in H_p(G_m)$, $\quad (0 < p < 1)$ such that

$$\sup_{n\in\mathbb{N}} \|L_n f\|_p = \infty.$$

Analogous problems for the Nörlund means with respect to Walsh, Kaczmarz and unbounded Vilenkin systems were considered in Blahota, and Tephnadze, [3,4], Goginava and Nagy [7], Nagy and Tephnadze [10–12], Persson, Tephnadze and Wall [13,14], Tephnadze [18,20,21], Tutberidze [22].

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In this paper, we discuss the boundedness of the weighted maximal operators from the Hardy space $H_p(G_m)$ to the Lebesgue space $L_p(G_m)$ for $0<p<1$.

2. Definitions and Notation

Let $\mathbb{N}_+$ denote the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$.

Let $m := (m_0, m_1, \dots)$ denote a sequence of the positive integers, not less than 2.

Denote by

$$Z_{m_k} := \{0, 1, \dots, m_k - 1\}$$

the additive group of integers modulo m_k.

Define the group G_m as the complete direct product of the group Z_{m_j} with the product of the discrete topologies of Z_{m_j}.

The direct product μ of the measures

$$\mu_k(\{j\}) := 1/m_k \qquad (j \in Z_{m_k})$$

is the Haar measure on G_m with $\mu(G_m) = 1$.

If $\sup_{n\in\mathbb{N}} m_n < \infty$, then we call G_m a bounded Vilenkin group. If the generating sequence m is not bounded, then G_m is said to be an unbounded one. **In this paper we discuss the bounded Vilenkin groups only.**

The elements of G_m are represented by the sequences

$$x := (x_0, x_1, \dots, x_j, \dots) \qquad (x_k \in Z_{m_k}).$$

It is easy to give a base for the neighborhood of G_m,

$$I_0(x) := G_m,$$

$$I_n(x) := \{y \in G_m \mid y_0 = x_0, \dots, y_{n-1} = x_{n-1}\} \ (x \in G_m, \ n \in \mathbb{N})$$

Denote $I_n := I_n(0)$, for $n \in \mathbb{N}$ and $\overline{I_n} := G_m \backslash I_n$.

If we define the so-called generalized number system based on m in the following way :

$$M_0 := 1, \qquad M_{k+1} := m_k M_k \qquad (k \in \mathbb{N})$$

then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j}$ $(j \in \mathbb{N})$ and only a finite number of n_j's differs from zero. Let $|n| := \max\{j \in \mathbb{N}; \quad n_j \neq 0\}$.

The norm (or quasi-norm) of the space $L_p(G_m)$ is defined by

$$\|f\|_p^p := \int_{G_m} |f|^p \, d\mu \qquad (0 < p < \infty).$$

The space $weak-L_p(G_m)$ consists of all measurable functions f for which

$$\|f\|_{weak-L_p(G_m)}^p := \sup_{\lambda>0} \lambda^p \mu(x : |f(x)| > \lambda) < +\infty.$$

Next, we introduce on G_m an orthonormal system which is called the Vilenkin system. First we define the complex-valued function $r_k(x) : G_m \to C$, the generalized Rademacher functions as

$$r_k(x) := \exp(2\pi i x_k / m_k) \qquad \left(i^2 = -1, \quad x \in G_m, \quad k \in \mathbb{N}\right).$$

Now, define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on G_m as:

$$\psi_n := \prod_{k=0}^{\infty} r_k^{n_k}, \qquad (n \in \mathbb{N}).$$

Specifically, we call this system the Walsh-Paley one if m=2.

The Vilenkin system is orthonormal and complete in $L_2(G_m)$ [1,23].

Now we introduce analogues of the usual definitions in the Fourier analysis.

If $f\in L_1(G_m)$, we can establish the Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels with respect to the Vilenkin system ψ in the usual manner:

$$\widehat{f}(k):=\int_{G_m} f\overline{\psi}_k d\mu, \qquad (k\in\mathbb{N}),$$
$$S_nf:=\sum_{k=0}^{n-1}\widehat{f}(k)\psi_k, \qquad (n\in\mathbb{N}_+, \quad S_0f:=0),$$
$$D_n:=\sum_{k=0}^{n-1}\psi_k, \qquad (n\in\mathbb{N}_+).$$

Recall that (for details see e.g. [1])

$$D_{M_n}(x)=\begin{cases}M_n & x\in I_n\\ 0 & x\notin I_n.\end{cases} \tag{1}$$

The σ-algebra generated by the intervals $\{I_n(x):x\in G_m\}$ will be denoted by $\digamma_n$ $(n\in\mathbb{N})$. Denote by $f=(f_n:n\in\mathbb{N})$ a martingale with respect to $\digamma_n$ $(n\in\mathbb{N})$ (for details see e.g. [24,25]). The maximal function of a martingale f is defined by

$$f^*=\sup_{n\in\mathbb{N}}|f_n|.$$

In the case, where $f\in L_1$, the maximal function is also given by

$$f^*(x)=\sup_{n\in\mathbb{N}}\frac{1}{|I_n(x)|}\left|\int_{I_n(x)} f(u)\mu(u)\right|.$$

For $0<p<\infty$, the Hardy martingale spaces $H_p(G_m)$ consist of all martingales for which

$$\|f\|_{H_p}:=\|f^*\|_p<\infty.$$

If $f\in L_1$, then it is easy to show that the sequence $(S_{M_n}f:n\in\mathbb{N})$ is a martingale. If $f=(f_n:n\in\mathbb{N})$ is a martingale, then the Vilenkin-Fourier coefficients should be defined in a slightly different manner:

$$\widehat{f}(i):=\lim_{k\to\infty}\int_{G_m} f_k\overline{\psi}_i d\mu.$$

The Vilenkin-Fourier coefficients of $f\in L_1(G_m)$ are the same as those of the martingale $(S_{M_n}f:n\in\mathbb{N})$ obtained from f.

Let $\{q_k:k>0\}$ be a sequence of non-negative numbers. The n-th Nörlund means for the Fourier series of f is defined by

$$\frac{1}{Q_n}\sum_{k=1}^{n}q_{n-k}S_kf, \quad \text{where} \quad Q_n:=\sum_{k=1}^{n}q_k.$$

If $q_k=1/k$, then we get the Nörlund logarithmic means

$$L_nf:=\frac{1}{l_n}\sum_{k=0}^{n-1}\frac{S_kf}{n-k}, \quad \text{where} \quad l_n=\sum_{k=0}^{n-1}\frac{1}{n-k}=\sum_{j=1}^{n}\frac{1}{j}.$$

A bounded measurable function a is p-atom, if there exists a dyadic interval I such that

$$\int_I a\,d\mu=0, \quad \|a\|_\infty\le\mu(I)^{-1/p}, \quad \operatorname{supp}(a)\subset I.$$

3. Formulation of Main Results

Theorem 1. a) *Let* $0<p<1$. *Then the maximal operator*

$$\tilde{L}{}_p^* f := \sup_{n\in\mathbb{N}} \frac{|L_n f|}{(n+1)^{1/p-1}}$$

is bounded from the Hardy space $H_p(G_m)$ *to the space* $L_p(G_m)$.

b) *Let* $0<p<1$ *and* $\varphi:\mathbb{N}_+\to[1,\infty)$ *be a non-decreasing function satisfying the condition*

$$\overline{\lim_{n\to\infty}} \frac{n^{1/p-1}}{\log n\varphi(n)} = +\infty.$$

Then there exists a martingale $f\in H_p(G_m)$ *such that the maximal operator*

$$\sup_{n\in\mathbb{N}} \frac{|L_n f|}{\varphi(n+1)}$$

is not bounded from the Hardy space $H_p(G_m)$ *to the space* $L_p(G_m)$.

4. Proof of the Theorem

Proof. Since

$$\frac{|L_n f|}{(n+1)^{1/p-1}} \le \frac{1}{(n+1)^{1/p-1}} \sup_{1\le k\le n} |S_k f| \le \sup_{1\le k\le n} \frac{|S_k f|}{(k+1)^{1/p-1}} \le \sup_{n\in\mathbb{N}} \frac{|S_n f|}{(n+1)^{1/p-1}},$$

if we use Theorem T1, we obtain

$$\sup_{n\in\mathbb{N}} \frac{|L_n f|}{(n+1)^{1/p-1}} \le \sup_{n\in\mathbb{N}} \frac{|S_n f|}{(n+1)^{1/p-1}}$$

and

$$\left\| \sup_{n\in\mathbb{N}} \frac{|L_n f|}{(n+1)^{1/p-1}} \right\|_p \le \left\| \sup_{n\in\mathbb{N}} \frac{|S_n f|}{(n+1)^{1/p-1}} \right\|_p \le c_p \|f\|_{H_p}.$$

Now, prove part b) of the Theorem. Let

$$f_{n_k} = D_{M_{2n_k+1}} - D_{M_{2n_k}}.$$

It is evident that

$$\widehat{f}_{n_k}(i) = \begin{cases} 1, & \text{if } \quad i = M_{2n_k},\dots,M_{2n_k+1}-1, \\ 0, & \text{otherwise.} \end{cases}$$

Then we can write that

$$S_i f_{n_k} = \begin{cases} D_i - D_{M_{2n_k}}, & \text{if } \quad i = M_{2n_k}+1,\dots,M_{2n_k+1}-1, \\ f_{n_k}, & \text{if } \quad i \ge M_{2n_k+1}, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

From (1), we get

$$\begin{aligned} \|f_{n_k}\|_{H_p} &= \left\| \sup_{n\in\mathbb{N}} S_{M_n} f_{n_k} \right\|_p = \left\| D_{M_{2n_k+1}} - D_{M_{2n_k}} \right\|_p \\ &\le \left\| D_{M_{2n_k+1}} \right\|_p + \left\| D_{M_{2n_k}} \right\|_p \le c M_{2n_k}^{1-1/p} < c < \infty. \end{aligned} \tag{3}$$

Let $0<p<1$ and $\{\lambda_k : k\in\mathbb{N}_+\}$ be an increasing sequence of the positive integers such that

$$\lim_{k\to\infty} \frac{\lambda_k^{1/p-1}}{\varphi(\lambda_k)} = \infty.$$

Let $\{n_k : k \in \mathbb{N}_+\} \subset \{\lambda_k : k \in \mathbb{N}_+\}$ such that

$$\lim_{k\to\infty} \frac{\left(M_{2n_k}+2\right)^{1/p-1}}{\log\left(M_{2n_k}+2\right)\varphi\left(M_{2n_k+2}\right)} \geq c \lim_{k\to\infty} \frac{\lambda_k^{1/p-1}}{\varphi\left(\lambda_k\right)} = \infty.$$

According to (2), we can conclude that

$$\left|\frac{L_{M_{2n_k}+2} f_{n_k}}{\varphi\left(M_{2n_k+2}\right)}\right| = \frac{\left|D_{M_{2n_k}+1} - D_{M_{2n_k}}\right|}{l_{M_{2n_k}+1}\varphi\left(M_{2n_k+1}\right)}$$

$$= \frac{\left|\psi_{M_{2n_k}}\right|}{l_{M_{2n_k}+2}\varphi\left(M_{2n_k+1}\right)} = \frac{1}{l_{M_{2n_k}+1}\varphi\left(M_{2n_k+2}\right)}.$$

Hence,

$$\mu\left\{x \in G_m : \left|L_{M_{2n_k}+2} f_{n_k}\right| \geq \frac{1}{l_{M_{2n_k}+2}\varphi\left(M_{2n_k+2}\right)}\right\} = \mu\left(G_m\right) = 1. \tag{4}$$

By combining (3) and (4), we get

$$\frac{\frac{1}{l_{M_{2n_k}+2}\varphi\left(M_{2n_k+2}\right)}\left(\mu\left\{x \in G_m : \quad \left|L_{M_{2n_k}+2} f_{n_k}\right| \geq \frac{1}{l_{M_{2n_k}+2}\varphi\left(M_{2n_k+2}\right)}\right\}\right)^{1/p}}{\left\|f_{n_k}\right\|_p}$$

$$\geq \frac{M_{2n_k}^{1/p-1}}{l_{M_{2n_k}+2}\varphi\left(M_{2n_k+2}\right)} \geq \frac{c\left(M_{2n_k}+2\right)^{1/p-1}}{\log\left(M_{2n_k}+2\right)\varphi\left(M_{2n_k+2}\right)} \to \infty, \quad \text{as} \quad k \to \infty. \qquad \square$$

Open Problem. For any $0 < p < 1$, let us find a non-decreasing function $\Theta : \mathbb{N}_+ \to [1, \infty)$ such that the following maximal operator

$$\tilde{L}{}_p^* f := \sup_{n\in\mathbb{N}} \frac{|L_n f|}{\Theta\left(n+1\right)}$$

is bounded from the Hardy space $H_p\left(G_m\right)$ to the Lebesgue space $L_p\left(G_m\right)$ and the rate of $\Theta : \mathbb{N}_+ \to [1, \infty)$ is sharp, that is, for any non-decreasing function $\varphi : \mathbb{N}_+ \to [1, \infty)$ satisfying the condition

$$\overline{\lim_{n\to\infty}} \frac{\Theta\left(n\right)}{\varphi\left(n\right)} = +\infty,$$

there exists a martingale $f \in H_p\left(G_m\right)$ such that the maximal operator

$$\sup_{n\in\mathbb{N}} \frac{|L_n f|}{\varphi\left(n+1\right)}$$

is not bounded from the Hardy space $H_p\left(G_m\right)$ to the space $L_p\left(G_m\right)$.

Remark 1. *According to Theorem 1, we can conclude that there exist absolute constants C_1 and C_2 such that*

$$\frac{C_1 n^{1/p-1}}{\log(n+1)} \leq \Theta\left(n\right) \leq C_2 n^{1/p-1}.$$

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