DIFFERENTIAL INVARIANTS OF KUNDT SPACETIMES

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Abstract. We find generators for the algebra of rational differential invariants for general and degenerate Kundt spacetimes and relate this to other approaches to the equivalence problem for Lorentzian metrics. Special attention is given to dimensions three and four.

1. Introduction

There are various approaches for distinguishing and classifying pseudo-Riemannian metrics, and to determine their Killing vectors, important in mathematical relativity.

- **spi:** Scalar polynomial invariants are obtained by complete contractions of the Riemann tensor, its covariant derivatives and their tensor products.
- **cci:** Cartan curvature invariants are obtained from structure functions of the absolute parallelism on the reduced frame bundle, and their derivatives.
- **sdi:** Scalar differential invariants are obtained as the invariants of the diffeomorphism pseudogroup acting in the space of jets of metrics.

By a theorem of Weyl [31], spi are sufficient to distinguish Riemannian and generic pseudo-Riemannian metrics. However, there exist non-equivalent metrics of non-positive signature with the same spi. For instance, VSI spaces with vanishing scalar (polynomial) invariants [26] are indistinguishable from the Minkowski spacetime by spi, and this is not related to any symmetry of the problem [11]. In addition, a sufficient number of spi has never been specified in the literature[1]. For instance, while principally known to resolve the count of Killing vectors for Riemannian metrics [10, 28, 7], the number and complexity of the involved spi is beyond a reasonable computational capacity [17].

Cartan invariants, on the other hand, are universally applicable in the study of spacetimes in general relativity, since they do separate metrics. They are especially popular in the form of the Cartan-Karlhede algorithm, in the Penrose-Newman formalism, etc [9, 25, 12, 22]. In the original approach, the invariants live on the Cartan frame bundle [2], and they classically correspond to covariants. Those combinations that are invariant with respect to the structure group are actually cci, and can be treated on the base manifold M. They are obtained as components of the curvature tensor and its covariant derivatives by normalizations of the group parameters. The invariants are usually considered local and smooth.

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[1]Even to specify all invariants of the second order in dimension 4 required some efforts [32].

Scalar differential invariants, though a classical tool in differential geometry [30], have never been pivotal in relativity applications. An essential difference between cci and sdi is the following. With the approach of Élie Cartan, the frame bundle depends on the metric and the remaining freedom is a finite-dimensional structure group (a subgroup of the pseudo-orthogonal group). With the approach of Sophus Lie, an infinite-dimensional transformation pseudogroup (a subgroup of the diffeomorphism group) acts on the space of jets of all metrics in the class and sdi are invariant functions of this action. With mild assumptions, the differential invariants can be assumed rational in jet-variables [15] and global[2] since they do separate generic orbits of the action.

In this paper, we follow the latter (sdi) approach and apply it to Lorentzian metrics that are indistinguishable by spi. These metrics are known to be contained in the Kundt class [18, 5]. Furthermore, it was shown in [3] (in dimensions 3 and 4) that a Lorentzian spacetime is either weakly I-nondegenerate (a discrete point in the set of metrics with the same spi; thus locally characterized by them) or is a degenerate Kundt spacetime (this class contains the VSI spacetimes). For related results in higher dimensions see [4].

Our main goal is to distinguish degenerate Kundt spacetimes by describing the algebra of rational sdi following the theory from [15]. Simultaneously, we consider the case of general Kundt spacetimes since the equivalence problem of those can be approached in the same way. For all Kundt spacetimes (general and degenerate), there exist local coordinates in which their metric takes the form

$$g = du\left(dv + H(u,x,v)\,du + W_i(u,x,v)\,dx^i\right) + h_{ij}(u,x)\,dx^i dx^j.$$

The quotient of the space of all (local) Kundt metrics by the diffeomorphism pseudogroup coincides with the quotient of the space of metrics of the above form by the Lie pseudogroup preserving this form. We will describe the algebra $\mathcal{A}$ of rational sdi in the latter setting. In particular, we find a set of generators for $\mathcal{A}$. This implies the solution of equivalence problem through the method of signature manifolds.

Note that our results are adaptable to Kundt metrics (general or degenerate) in any coordinate system. We also find an invariant frame adapted to Kundt spacetimes, thus relating to the Cartan approach. Several subclasses of Kundt metrics have been discussed in the literature using the Cartan-Karlhede algorithm, see [21] and references therein. For Kundt waves in 4D we compared both approaches, via sdi and cci, in [16]. In the present paper we discuss the general situation in general dimension n, with special attention given to dimensions 3 and 4.

The structure of the paper is as follows. In Section 2, we introduce the necessary concepts and notations regarding Kundt spacetimes and their jets, and recall the Lie-Tresse approach to differential invariants. For Kundt spacetimes (both general and degenerate), we choose adapted coordinates that result in metric tensors of a particular shape, and we write down the Lie pseudogroup of transformations preserving this shape. In Section 3, we count the number of algebraically independent sdi depending on the jet-order, and

[2]This means they are only subjects to nonequalities but not to inequalities (which can happen for smooth cci). Contrary to spi, defined for all metrics, both cci and sdi have a domain of definition.

provide Hilbert functions and Poincaré functions for the algebras of differential invariants. Afterwards, in Section 4, we find generators for the algebra of differential invariants in general dimension. We finish by presenting a simplified version of invariants for three- and four-dimensional Kundt spacetimes. In particular, we write down a generating set of invariants in coordinates in dimension 3. In the appendix we demonstrate how the class of degenerate Kundt spacetimes arises by consideration of relative invariants of the Lie pseudogroup of shape preserving transformations.

2. Setup: Jets and Pseudogroups

We first review the general theory and then apply it to Kundt spacetimes.

2.1. **Jets and Differential invariants.** The notion of jet-space formalizes the computational devise of truncated Taylor polynomials; we refer for details to [14]. If x^i are coordinates on $X = \mathbb{R}^n$ and y^j are coordinates on $Y = \mathbb{R}^m$, then the jet-space $J^k = J^k(X,Y)$ of k-jet of maps from X to Y has coordinates y^j_σ for multi-indices $\sigma = (i_1, \dots, i_n)$, $i_s \geq 0$, $|\sigma| = \sum i_s \leq k$. The same applies for general X, Y with local coordinates x^i, y^j, called independent and dependent variables, respectively. Any map $\psi : X \to Y$, $x^i \mapsto y^j = y^j(x)$ lifts to the map $j_k\psi : X \to J^k$ given by $x^i \mapsto y^j_\sigma = \partial y^j(x)/\partial x^\sigma$.

The jet-space J^k is equipped with the Cartan distribution $\mathscr{C}_k \subset TJ^k$, where for a point $a_k \in J^k$ the space $\mathscr{C}_k(a_k)$ is spanned by all n-planes $T_{a_k} j_k\psi(X)$, $j_k\psi(X) \ni a_k$. A differential equation of order $\leq l$ can be geometrically interpreted as a submanifold $\mathcal{E}^l \subset J^l$, and its solutions are integral manifolds of $\mathscr{C}_l$.

By differentiating the defining equations for $\mathcal{E}^l$, we obtain the prolonged equations $\mathcal{E}^{l+i}$ for $i \geq 1$. A smooth solution of $\mathcal{E}^l$ is also a smooth solution of $\mathcal{E}^{l+i}$. For jet spaces, we have the projections $\pi_{j,i} \colon J^j \to J^i$ for $i < j$, and we define $\mathcal{E}^k = \pi_{l,k}(\mathcal{E}^l) \subset J^k$ for $k = 0, \dots, l-1$.

A transformation group G on $J^0 = X \times Y$ or a local Lie pseudogroup $G \subset \mathrm{Diff}_{\mathrm{loc}}(X \times Y)$ canonically lifts to a pseudogroup acting in J^k by the condition that the class of integral manifolds $j_k\psi(X)$ (or equivalently the Cartan distribution $\mathscr{C}_k$) is preserved. For $G \ni g : J^0 \to J^0$ the prolongation is denoted by $g^{(k)} : J^k \to J^k$. If G consists of symmetries of a PDE $\mathcal{E}$, then there is an induced action $g^{(k)} : \mathcal{E}^k \to \mathcal{E}^k$.

Differential invariants of order $\leq k$ are functions $f : \mathcal{E}^k \subset J^k \to \mathbb{R}$ that are constant on the orbits of the prolonged action: $f \circ g^{(k)} = f$ (when there is no PDE, one considers functions on J^k that are constant on orbits). The spaces of all such invariants $\mathcal{A}_k$ unite over k into the algebra of differential invariants

$$\mathcal{A} = \lim_{k \to \infty} \mathcal{A}_k.$$

One also exploits invariant derivations $\nabla : \mathcal{A}_k \to \mathcal{A}_{k+1}$ of this algebra, which are invariant horizontal vector fields. In other words, they are invariant operators of the form $a^i \mathcal{D}_{x^i}$, where a^i are functions on $\mathcal{E}^r \subset J^r$ for some r and $\mathcal{D}_{x^i}$ are total derivative operators: $\mathcal{D}_{x^i}(f)|_{j^{k+1}\psi} = \partial_{x^i}(f \circ j^k\psi)$ for every $f \in C^\infty(J^k)$ and $\psi \colon X \to Y$.

By [15], under mild assumptions[3], the invariants can be assumed rational in the jet-variables y^j_σ, $|\sigma| > 0$, and even polynomial in jets of sufficiently high order $|\sigma| > k_0$. From now on we will suppose that $\mathcal{A}$ consists of such rational-polynomial functions. Moreover, the main theorem of [15] gives Lie-Tresse type generation of $\mathcal{A}$ by a finite set of differential invariants I_a and invariant derivations ∇_b, so that generic G-orbits in $\mathcal{E}^\infty \subset J^\infty$ are separated by I_a and their derivatives

$$\nabla_B I_a = \nabla_{b_1} \cdots \nabla_{b_t} I_a.$$

Here B denotes the multi-index $B = (b_1, \dots, b_t)$. For every k the orbits that are separated by the invariants unite to a Zariski open set $\mathcal{E}^k \setminus \Sigma^k$ in $\mathcal{E}^k$. Here, the set Σ^k of singular points intersects any fiber of $\mathcal{E}^k \to J^0$ by a proper algebraic set, and starting from some jet-level k_1 no new singularities appear for $k > k_1$. More precisely, the set $\Sigma^k \subset \mathcal{E}^k$ of singular points satisfies $\Sigma^k \subset \pi^{-1}_{k,k_1}(\Sigma^{k_1})$.

In general, the invariant derivations need not commute, meaning that there are non-vanishing (invariant) structure functions c^k_{ij}, given by

$$[\nabla_i, \nabla_j] = c^k_{ij} \nabla_k.$$

When $\mathcal{A}$ contains n horizontally independent invariants $I_1, \dots, I_n$, i.e. such that their restriction to a generic holonomic jet-section $j^\infty\psi$ are functionally independent, the functions c^k_{ij} can be derived from the invariants $\nabla_B I_a$.

2.2. **The equivalence problem.** If two solutions ψ_1, ψ_2 to $\mathcal{E}$ are equivalent through a transformation in G, then the k-th order differential invariant $I_a \circ j^k\psi_1$ of one solution pulls back to differential invariants $I_a \circ j^k\psi_2$ of the other. In particular, if $I_a \circ j^k\psi_1$ is constant, then $I_a \circ j^k\psi_2$ is also constant (with the same value), and thus they are easily compared. On the other hand, two seemingly different nonconstant invariants do not at once obstruct equivalence. The equivalence problem for solutions of $\mathcal{E}$ is thus reduced to an equivalence problem for a set of functions. However, when we have several differential invariants, the relations between them are invariant, and these relations do obstruct equivalence in the sense that if the relations between the invariants in the set $\{I_a \circ j^k\psi_1\}$ are different from the relations between the invariants in $\{I_a \circ j^k\psi_2\}$, then ψ_1 and ψ_2 are not equivalent.

To be more precise, with the above approach to differential invariants the equivalence problem for generic solutions ψ of $\mathcal{E}$ with respect to G is solved as follows. Assume that $\mathcal{A}$ is generated by the differential invariants I_a and the invariant derivations ∇_b, and that there are n horizontally independent invariants in $\{I_a\}$ (this suffices for our considerations). Then we define the signature

$$\Psi : X \ni x \mapsto (I_a, \nabla_b I_a) \in \mathbb{R}^q$$

as the map with the components consisting of the basic invariants and their first derivatives (q is the total number) computed on the submanifold $j^\infty\psi(X) \subset \mathcal{E}^\infty \subset J^\infty$. Then

[3]The pseudogroup G acts transitively on J^0 (this can be further relaxed) and algebraically on the fibers of the projections $J^k \to J^0$; these assumptions will be satisfied in the case we study.

ψ_1 is G-equivalent to ψ_2 iff the *signature manifolds* $\Psi_1(X), \Psi_2(X)$ coincide, see [2, 23] (actually this statement can be localized in X). In general, the differential algebra of differential invariants is not freely generated; there are differential syzygies. In particular, this implies that not every n-dimensional submanifold of $\mathbb{R}^q$ is a signature manifold since the syzygies manifest as differential constraints on the signature manifolds.

We note that the definition of signature manifold depends on the chosen set of generators, and so does the integer q.

2.3. **General Kundt spacetimes.** An n-dimensional Lorentzian manifold (M, g) is a Kundt spacetime if it admits a null congruence that is geodesic, expansion-free, shear-free and twist-free. In other words, there exists a vector field ℓ such that[4]

$$\|\ell\|^2 = 0, \quad \mathbb{D}^g_\ell \ell = 0, \quad \mathrm{Tr}(\mathbb{D}^g \ell) = 0, \quad \|\mathbb{D}^g \ell^{\mathrm{sym}}\|^2 = 0, \quad \|\mathbb{D}^g \ell^{\mathrm{alt}}\|^2 = 0.$$

The twist-free condition is equivalent to Frobenius-integrability of $\ell^\perp$. Thus we have embedded integrable distributions $\mathbb{R} \cdot \ell \subset \ell^\perp$ of dimension 1 and codimension 1 on M. Let λ denote the foliation corresponding to $\mathbb{R} \cdot \ell$, and Λ the foliation corresponding to $\ell^\perp$. The (local) quotient of the corresponding foliations $\bar{M} = \Lambda/\lambda$ has dimension $n-2$; the other Kundt conditions translate to the claim that the degenerate symmetric bivector $g|_{\lambda^\perp}$ projects to a Riemannian metric $h = (h_{ij})$ on $\bar{M}$. Below we will denote by $x = (x^1, \dots, x^{n-2})$ both local coordinates on $\bar{M}$ and their pullback on M.

This implies the well-known claim [18, 12, 5] that in some local coordinates (u, x, v) on M any Kundt metric can be written as follows

$$g = du\,\big(dv + H(u,x,v)\,du + W_i(u,x,v)\,dx^i\big) + h_{ij}(u,x)\,dx^i dx^j. \tag{1}$$

In these coordinates $\ell = \partial_v$ and $\ell^\perp = \{du = 0\}$. We let $\bar{M}_u$ denote Λ_u/λ, where Λ_u is a leaf of Λ (leaves of Λ are parametrized by u, as they are given by $u = \text{const.}$).

2.4. **Shape-preserving transformations.** Now we determine the Lie pseudogroup $\mathcal{G}$ of diffeomorphisms preserving the class of Kundt metrics given by (1). In other words, we find the transformations preserving the shape of such metrics; in the language of [29], they are the transformations "preserving the functional form" (1). For the case $n = 4$, the shape-preserving transformations were found in [26].

Theorem 1. *The transformations preserving the shape of* (1) *take the form*

$$\mathcal{G} \ni \varphi : (u, x^i, v) \mapsto \left(C(u), A^i(u,x), \frac{v}{C'(u)} + B(u,x)\right), \quad \det[A^i_{x^j}] \neq 0, C'(u) \neq 0. \tag{2}$$

The Lie algebra $\mathfrak{g}$ of this Lie pseudogroup $\mathcal{G}$ consists of vector fields of the form

$$\xi = c(u)\partial_u + a^i(u,x)\partial_{x^i} + \big(b(u,x) - c'(u)v\big)\partial_v. \tag{3}$$

[4] All contractions, norms and raising-lowering are with respect to g. We write $\mathbb{D}^g$ for the Levi-Civita connection to distinguish from invariant derivations ∇_i exploited in generation of the algebra $\mathcal{A}$.

Proof. One approach for proving this is to require that for $\varphi \in \mathcal{G}$ the metric $\varphi^* g$ has form (1) with some other functions H, W_i, h_{ij} of the same type. This gives a PDE system that is easy to solve. Another approach is to note that transformations preserving the filtration $\lambda \subset \lambda^\perp$ have the form: $\varphi(u,x,v) = (U(u), X(u,x), V(u,x,v))$. Taking into account the condition $(\varphi^* g)(\partial_u, \partial_v) = 1/2$ yields explicit affine behaviour of V in v. $\square$

Note that the Lie pseudogroup $\mathcal{G}$ has four connected components in the smooth topology, but it is Zariski connected (hence $\mathcal{G}$ is the Zariski closure of the component given by $C' > 0$, $\det(A^i_{x^j}) > 0$). The formulas in Theorem 1 imply that the action of $\mathcal{G}$ on M is transitive. In the next section we will lift the transformation group to the bundle of metrics, and the explicit formulas will show that the action is transitive on the total space of this bundle and its prolongation to the space of jets is algebraic. This justifies the conditions of the global Lie-Tresse theorem [15]. Consequently, the algebra $\mathcal{A}$ of invariants can be assumed to consist of rational-polynomial functions, since such invariants separate orbits in general position.

The equivalence problem for metrics of the form (1) with respect to the Lie pseudogroup $\mathcal{G}$ action is related to the equivalence problem for Kundt metrics (in arbitrary coordinates) under the action of the pseudogroup of all local diffeomorphisms as follows. Let g_1, g_2 be two coordinate expressions of kundt metrics that are not necessarily of the form (1) and that are related by a diffeomorphism φ on M, so that $g_2 = \varphi^* g_1$. Let ρ_1, ρ_2 be diffeomorphisms that bring g_1 and g_2 to $g_1' = \rho_1^* g_1$ and $g_2' = \rho_2^* g_2$, respectively, of the form (1). Then $g_2' = (\rho_1^{-1}\varphi\rho_2)^* g_1'$. Clearly, the diffeomorphism $\varphi' = \rho_1^{-1}\varphi\rho_2$ preserves the foliations λ and $\lambda^\perp$, and it is therefore of the form $(u,x,v) \mapsto (U(u), X(u,x), V(u,x,v))$. With φ' of this form, it is also clear that if the equality $((\varphi')^* g_1')(\partial_u, \partial_v) = 1/2$ holds for some g_1' in aligned coordinates, then that implies $((\varphi')^* g)(\partial_u, \partial_v) = 1/2$ for every g of form (1). Therefore, $\varphi' \in \mathcal{G}$. The metrics g_1 and g_2 do not necessarily determine $\varphi' = \rho_1^{-1}\varphi\rho_2$ uniquely, as there may be several possible choices of ρ_1, ρ_2. However, in that case, different transformations φ' are related by a shape-preserving transformation.

Since all Kundt metrics can be locally brought to the form (1), this implies that the quotient of (local) Kundt metrics in arbitrary coordinates by the pseudogroup of local diffeomorphisms is equal to the quotient of metrics of the form (1) by the Lie pseudogroup $\mathcal{G}$. This justifies the claim made in the introduction.

Notice also that some symmetries of metrics may disappear when we transform the equivalence problem to adapted coordinates. For example, it is clear that $\mathcal{G}$ does not contain the whole Poincaré group, so some of the symmetries of the Minkowski metric have been lost (those that do not preserve the foliation λ). This does not contradict the above discussion, as the only equivalences we lose this way are self-equivalences.

2.5. **Lifts and jet-prolongations.** The image of (1) in S^2T^*M, the second symmetric tensor power of the cotangent bundle, determines a subbundle isomorphic to the trivial bundle

$$\pi : M \times F \to M, \quad \text{where} \quad F \subset \mathbb{R}^N, \; N = n - 1 + \binom{n-1}{2} = \binom{n}{2}, \tag{4}$$

whose sections are exactly Lorentzian metrics of the form (1). The coordinates on M are independent variables u, x^i, v, the coordinates on $\mathbb{R}^N$ are dependent variables H, W_i, h_{ij} $(1 \leq i \leq j \leq n-2)$, and the domain $F \subset \mathbb{R}^N$ is given by the requirement that the symmetric matrix defined by h_{ij} is positive definite.

Denote by $J^k\pi$ the k-th order jet bundle of π. It contains the subbundle of jets of Kundt metrics given by the equation

$$\mathcal{E}^1 = \{(h_{ij})_v = 0\} \subset J^1\pi$$

as well as its prolongations $\mathcal{E}^k \subset J^k\pi$ for $k > 0$, given by the $(k-1)$ differentiations of the above conditions. We use the notation $\mathcal{E}^0 = J^0\pi$. The infinitely prolonged Kundt equation is $\mathcal{E}^\infty \subset J^\infty\pi$.

The pseudogroup $\mathcal{G}$ (and its Lie algebra $\mathfrak{g}$) have the natural lift to $J^0\pi = M \times F$, $\mathcal{G} \ni \varphi \mapsto \varphi^{(0)} \in \mathcal{G}^{(0)} \subset \mathrm{Diff}_{\mathrm{loc}}(J^0)$, obtained from the requirement that

$$g = du\left(dv + H\,du + W_i dx^i\right) + h_{ij}\,dx^i dx^j \in \pi^* S^2 T^* M$$

is invariant with respect to every $\varphi^{(0)}$.

Remark 1. From here on, g is interpreted as a horizontal symmetric 2-form on π. The restriction of g to a section ψ of π given by $H = H(u,x,v), W_i = W_i(u,x,v), h_{ij} = h_{ij}(u,x)$ is exactly the metric (1). The invariant tensors associated with a metric (the Riemann tensor, Ricci tensor, etc.) can be defined for the horizontal form g so that the restriction of such a tensor to a section of π gives exactly the corresponding tensor field associated to the metric (1). We recall that a horizontal form on a fiber bundle is a form that contracts to zero with every vertical vector. The value of the horizontal form g evaluated on vectors along a section is thus completely determined by the restriction of g to the section.

To get the formula for the lift following the notations of Theorem 1 denote $A^i_j = \partial_{x^j} A^i$ and let $\check{A}^i_j$ be the inverse matrix. Denote also $B_j = \partial_{x^j} B$ and $\check{C}' = (C')^{-1}$. Then $\varphi^{(0)}$ maps the fiber as follows:

$$\begin{aligned} h_{ij} &\mapsto \check{A}^k_i \check{A}^l_j h_{kl}, \\ W_i &\mapsto \check{C}' \check{A}^j_i W_j - \check{A}^j_i B_j - 2\check{C}' \check{A}^k_i \check{A}^l_j A^j_u h_{kl}, \\ H &\mapsto \check{C}'^2 H + \check{C}'^3 C'' v - \check{C}' B_u + \check{C}' \check{A}^j_i A^i_u B_j - \check{C}'^2 \check{A}^j_i A^i_u W_j + \check{C}'^2 \check{A}^k_i \check{A}^l_j A^i_u A^j_u h_{kl}. \end{aligned}$$

The lift of vector fields from $\mathfrak{g}$, with $a = a(u,x), b = b(u,x), c = c(u)$, is (here and in what follows $a^j_i = a^j_{x^i}$, $b_i = b_{x^i}$, etc) given by

$$\begin{aligned} \xi^{(0)} = {} & c\partial_u + a^i \partial_{x^i} + (b - c'v)\partial_v - (a^l_i h_{lj} \partial_{h_{ij}} + a^l_i h_{li} \partial_{h_{ii}}) \\ & - (c' W_i + a^j_i W_j + b_i + 2a^j_u h_{ij})\partial_{W_i} - (2c' H - c'' v + b_u + a^j_u W_j)\partial_H. \end{aligned}$$

These prolong further to transformations $\varphi^{(k)}$ and vector fields $\xi^{(k)}$ on $J^k\pi$.[5] Moreover, by the construction of the lift, prolongations of the pseudogroup $\mathcal{G}$ preserve $\mathcal{E}$.

[5] For the general prolongation formula for vector fields, see for example (2) in [14].

Note that order k differential invariants of $\mathcal{G}$ are rational-polynomial functions f on $\mathcal{E}^k \subset J^k\pi$ that satisfy

$$\mathscr{L}_{\xi^{(k)}} f = 0 \quad \forall \xi \in \mathfrak{g}. \tag{5}$$

However, there exist functions that satisfy this system of equations, but are not invariant under the entire Lie pseudogroup $\mathcal{G}$. An example of this is mentioned in Section 4.4.

2.6. **Degenerate Kundt spacetimes.** Degenerate Kundt metrics are the Kundt metrics that satisfy the following additional conditions:

- The Riemann tensor Riem is aligned and of algebraically special type II.
- $\mathbb{D}^g$(Riem) is aligned and of algebraically special type II.

In terms of (1), the first condition implies $(W_i)_{vv} = 0$ while the second implies $H_{vvv} = 0$. It follows that $(\mathbb{D}^g)^{(k)}$(Riem) is aligned and of algebraically special type II for every positive integer k. (In all cases we understand the set of type II tensors to also include the more special types III, D, etc.)

In 4D these conditions imply that the metric g is I-degenerate [3]. The opposite, I-nondegeneracy of g, can be defined through the map I : (spacetimes) $\to$ (spi) as discreteness[6] of the set $I^{-1}(I(g))$ for germs of g (in localization of M). Note that for a generic metric, $I^{-1}(I(g)) = g$ is a one point set.

In any dimension n one can show (by varying H and W_i in lower v-degree terms) that any degenerate Kundt metric g can be smoothly deformed as a family g_τ with $I(g_\tau) = \text{const}$ and the deformation is not an isotopy.

Theorem 2. *The pseudogroup of local transformations preserving the degenerate Kundt spacetimes of shape (1) coincides with the pseudogroup $\mathcal{G}$.*

Proof. The conditions of degeneracy are natural (coordinate-independent) and therefore are respected by any pseudogroup of transformations on M. On the other hand, the class of degenerate Kundt metrics also specifies the filtration $\lambda \subset \lambda^\perp$ used in the preceding proof, and the shape is the same, whence the claim. □

Adding the degeneracy condition to Kundt spacetimes determines a new PDE, denoted by $\tilde{\mathcal{E}}$, which is specified by the equations

$$(h_{ij})_v = 0, \quad (W_i)_{vv} = 0, \quad H_{vvv} = 0.$$

Including the prolongations of those conditions (that is applying total derivatives of all orders and directions) we get the infinitely prolonged system $\tilde{\mathcal{E}}^\infty \subset J^\infty\pi$.

More precisely, we have $\tilde{\mathcal{E}}^k = \mathcal{E}^k$ for $k < 2$, the submanifold $\tilde{\mathcal{E}}^2 \subset \mathcal{E}^2$ is given by the additional equations $(W_i)_{vv} = 0$, and $\tilde{\mathcal{E}}^3 \subset \mathcal{E}^3$ by first derivatives of those plus the equation $H_{vvv} = 0$, etc.

By virtue of Theorem 2 the lift and prolongations of the pseudogroup $\mathcal{G}$ restrict to the equation $\tilde{\mathcal{E}}$ of degenerate Kundt metrics. Differential invariants of order k are rational-polynomial functions f on $\tilde{\mathcal{E}}^k$ satisfying Lie equation (5). Though the main target is the

[6]In [3] a weaker requirement is stated, but the proof implies the stated stronger property.

class of degenerate Kundt spacetimes, we can study simultaneously the class of general Kundt metrics.

3. Counting the invariants

Now we count the amount of (algebraically) independent differential invariants for both general and degenerate Kundt spacetimes, depending on the jet-order k.

3.1. Jets and differential equations. At first we determine dimensions of the involved jet-spaces and equation-manifolds. With $N = \binom{n}{2}$ from (4) we have

$$\dim J^k\pi = n + N\binom{n+k}{n}.$$

There are $\binom{n-1}{2}\binom{n+k-1}{n}$ equations of order $\leq k$ specifying Kundt spacetimes of the form (1). This number is the codimension of $\mathcal{E}^k \subset J^k\pi$. We use standard combinatorial identities in order to obtain

$$\begin{aligned}\dim \mathcal{E}^k &= n + (n-1)\binom{n+k}{n} + \binom{n-1}{2}\binom{n+k-1}{n-1} \\ &= n + (n-1)\binom{n+k-1}{n}\frac{n^2+2k}{2k} \quad \text{for} \quad k > 0\end{aligned}$$

and $\dim \mathcal{E}^0 = n + N = \binom{n+1}{2}$.

The equation-manifolds for degenerate Kundt metrics satisfy $\tilde{\mathcal{E}}^0 = \mathcal{E}^0$ and $\tilde{\mathcal{E}}^1 = \mathcal{E}^1$, while $\tilde{\mathcal{E}}^k \subset \mathcal{E}^k$ is given by $(n-2)$ additional constraints for $k = 2$ and by $(n-2)\binom{n+k-2}{n} + \binom{n+k-3}{n}$ constraints for $k \geq 3$. Thus

$$\begin{aligned}\dim \tilde{\mathcal{E}}^2 &= \left(\binom{n+1}{2}+1\right)\left(\binom{n}{2}+1\right), \\ \dim \tilde{\mathcal{E}}^k &= \dim \mathcal{E}^k - (n-2)\binom{n+k-2}{n} - \binom{n+k-3}{n} \quad \text{for} \quad k > 2.\end{aligned}$$

3.2. Orbit dimensions. The action of $\mathcal{G}$ on $J^0\pi$ is transitive, so that any point can be mapped to the point p_0 given by

$$u = 0,\ x^i = 0,\ v = 0,\ h_{ij} = \delta_{ij},\ W_i = 0,\ H = 0.$$

The stabilizer in $\mathfrak{g}$ of the point p_0 is given in terms of the functional parameters from Theorem 1 by

$$a^i = b = c = b_u = 0,\ b_i = -2a^i_u,\ a^i_j = -a^j_i.$$

These conditions imposed on the jets of pseudogroup elements preserving p_0 define an algebraic (finite-dimensional) group $\mathcal{G}_0^{(k)}$ acting in the fibers $J^k_0\pi \supset \mathcal{E}^k_0 \supset \tilde{\mathcal{E}}^k_0$ over p_0. The invariants of this action bijectively correspond to differential invariants of order k of $\mathcal{G}$.

For an algebraic group action, its field of rational invariants separates generic orbits, due to Rosenlicht's theorem [27, 8]. The transcendence degree of this field is equal to the codimension of a generic orbit. Of course, the codimension of orbits of $\mathcal{G}$ on $\mathcal{E}^k$ or $\tilde{\mathcal{E}}^k$ equals the codimension of orbits of $\mathcal{G}_0$ on $\mathcal{E}_0^k$ or $\tilde{\mathcal{E}}_0^k$, respectively.

Theorem 3. *For $k = 1$ the codimension of an orbit in general position in $\mathcal{E}^1 = \tilde{\mathcal{E}}^1$ is 1. For $k \geq 2$ the dimension of an orbit in general position both in $\mathcal{E}^k$ and in $\tilde{\mathcal{E}}^k$ is given by*

$$(n-1)\binom{n+k}{n-1} + k + 2.$$

Proof. Consider the action of the stabilizer $\mathcal{G}_0^{(1)}$ on $\mathcal{E}_0^1$. A straightforward verification shows that $\sum_{i=1}^{n-2}(W_i)_v^2$ is an invariant. Now we use the pseudogroup to normalize a point in $\mathcal{E}_0^1$ by sequentially fixing a set of coordinates, thus restricting to a sequence of submanifolds. We simultaneously fix parameters of the Lie algebra, so that the remaining vector fields are tangent to the current submanifold.

(1) Bring the point to the submanifold given by $(h_{ij})_k = 0$, $(h_{ij})_u = 0$. The Lie subalgebra preserving the submanifold is restricted further by $a^i_{jk} = 0, a^i_{ju} = -a^j_{iu}$.
(2) Fix $(W_i)_j = 0$. The stabilizer of this new submanifold is given by the additional equations $b_{ij} = (W_i)_v a^j_u + (W_j)_v a^i_u$, $a^i_{ju} = \frac{1}{2}((W_i)_v a^j_u - (W_j)_v a^i_u)$.
(3) Fix $(W_i)_u = 0$. The new stabilizer is given by $b_{iu} = -2a^i_{uu}$.
(4) Fix $H_u = H_i = H_v = 0$. The new stabilizer is given by $a^i_{uu} = b_{uu} = 0$, $c_{uu} = (W_i)_v a^i_u$.
(5) The remaining stabilizer is $CO(n-2) \ltimes \mathbb{R}^{n-2}$, and its subgroup $SO(n-2)$ acts nontrivially on the covector $(W_i)_v$, so we fix it so: $(W_2)_v = \cdots = (W_{n-2})_v = 0$. Then $(W_1)_v^2$ is the value of the above invariant.

For the action of $\mathcal{G}_0^{(1)}$ on $\mathcal{E}_0^1$ the stabilizer of a generic point p_1 has dimension $\binom{n-2}{2} + 2$, in particular the action is not free.

The same approach works in higher jets: by choosing a specific point $p_k \in \tilde{\mathcal{E}}_0^k$ we compute the rank of all k-jets of vector fields $\xi \in \mathfrak{g}$ at p_k. The totality of those fields may be thought to be the number of free jets of group parameters entering the fields $\xi^{(k)}$, which is $(n-1)\binom{n+k}{n-1} + k + 3$. However, over p_0 the coefficient of $c^{(k+2)}$ vanishes (since $v = 0$), the corresponding field is in the kernel of the action, and therefore the group $\mathcal{G}_0^{(k)}$ has dimension 1 less than the indicated number.

Now a tedious verification, which we omit, shows that these vector fields are actually independent, so the orbit has the dimension as stated, and the action is free for $k \geq 2$ (by definition this means that $\mathcal{G}_0^{(k)}$ acts freely).

An alternative route is to check (in the same manner as for 1-jets) that the action is free on a Zariski open subset of $\tilde{\mathcal{E}}_0^2$ (hence also on a Zariski open subset of $\mathcal{E}_0^2$). Therefore, from the persistence of freeness in prolongation [24], the claim follows. □

3.3. **Hilbert and Poincaré functions.** Let s_k^n denote the codimension of an orbit in general position in $\mathcal{E}^k$. This is equal to the transcendence degree of the field of rational differential invariants of order k. In other words, s_k^n is the number of algebraically independent differential invariants of order k. The Hilbert and Poincaré functions conveniently encode this sequence of numbers.

We define the Hilbert function for the action of $\mathcal{G}$ on $\mathcal{E}$ (or for the quotient $\mathcal{E}/\mathcal{G}$) by $\mathcal{H}_k^n = s_k^n - s_{k-1}^n$ and $\mathcal{H}_0^n = s_0^n$. From Theorem 3 we conclude:

Proposition 4. *The Hilbert function for $\mathcal{E}/\mathcal{G}$ is given by $\mathcal{H}_0^n = 0$, $\mathcal{H}_1^n = 1$,*

$$\begin{aligned}
\mathcal{H}_2^n &= n - 5 + (n-1)\left(\tbinom{n+2}{n} - \tbinom{n+2}{n-1}\right) + \tbinom{n-1}{2}\tbinom{n+1}{n-1} \\
&= \frac{n^4 - 4n^3 + 11n^2 + 16n - 72}{12}, \\
\mathcal{H}_k^n &= (n-1)\left(\tbinom{n+k-1}{n-1} - \tbinom{n+k-1}{n-2}\right) + \tbinom{n-1}{2}\tbinom{n+k-2}{n-2} - 1 \quad \text{for } k \geq 3.
\end{aligned}$$

Corollary 5. *For $k \geq 3$ the Hilbert function in dimensions $n = 3, 4, 5$ is given by*

$$\begin{aligned}
\mathcal{H}_k^3 &= k^2 + 2k - 2, \\
\mathcal{H}_k^4 &= \frac{1}{2}\left(k^3 + 6k^2 + 5k - 8\right), \\
\mathcal{H}_k^5 &= \frac{1}{6}\left(k^4 + 12k^3 + 35k^2 + 12k - 42\right).
\end{aligned}$$

For $k = 2$ we have $\mathcal{H}_2^3 = 4$, $\mathcal{H}_2^4 = 14$ and $\mathcal{H}_2^5 = 34$.

Similarly define $\tilde{s}_k^n$ and $\tilde{\mathcal{H}}_k^n = \tilde{s}_k^n - \tilde{s}_{k-1}^n$, $\tilde{\mathcal{H}}_0^n = \tilde{s}_0^n$ for the action of $\mathcal{G}$ on $\tilde{\mathcal{E}}^k$. In the same manner as Theorem 3 we conclude:

Proposition 6. *The Hilbert function for $\tilde{\mathcal{E}}/\mathcal{G}$ is given by $\tilde{\mathcal{H}}_0^n = 0$, $\tilde{\mathcal{H}}_1^n = 1$,*

$$\begin{aligned}
\tilde{\mathcal{H}}_2^n &= (n-1)\left(\binom{n+2}{2} - \binom{n+2}{3}\right) + \binom{n-1}{2}\binom{n+1}{2} - 3, \\
\tilde{\mathcal{H}}_k^n &= (n-1)\left(\binom{n+k-1}{n-1} - \binom{n+k-1}{n-2}\right) + \binom{n-1}{2}\binom{n+k-2}{n-2} \\
&\quad - (n-2)\binom{n+k-3}{n-1} - \binom{n+k-4}{n-1} - 1 \quad \text{for } k \geq 3.
\end{aligned}$$

Corollary 7. *For $k \geq 3$ the Hilbert function in dimensions $n = 3, 4, 5$ is given by*

$$\begin{aligned}
\tilde{\mathcal{H}}_k^3 &= 4k - 3, \\
\tilde{\mathcal{H}}_k^4 &= \frac{1}{2}(7k^2 + 5k - 8), \\
\tilde{\mathcal{H}}_k^5 &= \frac{1}{6}(11k^3 + 36k^2 + 13k - 42).
\end{aligned}$$

For $k = 2$ we have $\tilde{\mathcal{H}}_2^3 = 3$, $\tilde{\mathcal{H}}_2^4 = 12$ and $\tilde{\mathcal{H}}_2^5 = 31$.

Another way to encode the counting of invariants is through the Poincaré function

$$P_n(z) = \sum_{k=0}^{\infty} \mathcal{H}_k^n z^k.$$

Since the Hilbert function is polynomial in $k \geq k_0$, the Poincaré function is rational.

Corollary 8. *The Poincaré function in dimensions $n = 3, 4, 5$ is given by*

$$P_3(z) = \frac{(1 + z + 4z^2 - 6z^3 + 2z^4)z}{(1-z)^3},$$
$$P_4(z) = \frac{(1 + 10z - 6z^2 - 10z^3 + 11z^4 - 3z^5)z}{(1-z)^4},$$
$$P_5(z) = \frac{(1 + 29z - 41z^2 + 33z^4 - 23z^5 + 5z^6)z}{(1-z)^5}$$

for general Kundt spacetimes; for degenerate Kundt spacetimes it is

$$\tilde{P}_3(z) = \frac{(1 + z + 4z^2 - 2z^3)z}{(1-z)^2},$$
$$\tilde{P}_4(z) = \frac{(1 + 9z + 2z^2 - 8z^3 + 3z^4)z}{(1-z)^3},$$
$$\tilde{P}_5(z) = \frac{(1 + 27z - 15z^2 - 15z^3 + 18z^4 - 5z^5)z}{(1-z)^4}.$$

4. Computing the invariants

There are several approaches for describing the algebra of invariants by generators and syzygies in Lie-Tresse type framework discussed in Section 2.1. We first give a common scheme, and then specify it for general and degenerate Kundt spacetimes. Afterwards we provide an alternative approach with simpler computations in low dimensions $n = 3, 4$.

4.1. **The general scheme.** One general approach is to find n *horizontally independent*[7] rational differential invariants $I_1, \dots, I_n$, i.e. invariants satisfying

$$\det[\mathcal{D}_i I_s] \not\equiv 0. \tag{6}$$

The condition (6) means that for a generic section $\psi \in \Gamma(\pi)$, the restriction $\bar{I}_s = (j^\infty \psi)^* I_s$ of the above invariants to the holonomic jet-section $j^\infty \psi$ are functionally independent; since under this restriction they become functions on M this can be writen as follows:

$$\det[\partial_i \bar{I}_s] \not\equiv 0.$$

Next, derive the corresponding horizontal[8] coframe $\omega^i = \hat{d} I_i$ and its dual horizontal frame $\nabla_i = \mathcal{D}_{I_i}$. This particular set of invariant derivations ∇_i are called Tresse derivatives, and they are pairwise commuting. When restricted to a generic section of π, they

[7]Horizontal independence implies algebraic independence (in jets), i.e. $\operatorname{rank}(\partial_{J^{i,j}_\sigma} I_s) = n$, where $J^{i,j}_\sigma$ consists of the base variables x^i and the jet-variables y^j_σ. But n invariants can be algebraically independent without being horizontally independent.

[8]The horizontal differential is defined by the formula $\hat{d}f|_{j^{k+1}\psi} = d(f \circ j^k \psi)\ \forall f \in C^\infty(J^k \pi),\ \psi \in \Gamma(\pi)$.

reduce to partial derivatives with respect to $\bar{I}_i$. We express g in this frame:

$$g = G_{ij}\omega^i\omega^j, \qquad G_{ij} = g(\nabla_i, \nabla_j) \tag{7}$$

Now the algebra $\mathcal{A}$ of differential invariants is generated by I_i, G_{ij} and ∇_i.

Indeed, in any coordinate system (x^i) the invariants are obtained from the invariant combinations of the components g_{ij} of the metric, and their partial derivatives. If we choose invariant coordinates $\bar{I}_i$, then the metric components and their derivatives are also invariants. Since no invariants are lost during the change of coordinates, all invariants are obtained as derivatives of the components with respect to $\bar{I}_i$.

Note that the passage $(x^1, \dots, x^n) \mapsto (I_1, \dots, I_n)$ is a differential operator, without differential inverse in general. Therefore the count of invariants in Section 3 does not survive this transformation. However, the asymptotics of the Hilbert function[9] do survive. For general metrics the asymptotics are given by $d = n$, $\sigma = \binom{n}{2}$. For general Kundt spacetimes $d = n$, $\sigma = n - 1$. For degenerate Kundt spacetimes $d = n - 1$, $\sigma = \binom{n}{2} + 1$. This tells us that, modulo diffeomorphism (coordinate) freedom, the metrics in the class locally depend on σ arbitrary functions of d variables.

The requirement (6) allows for a wide variety of possibilities when it comes to choosing the n scalar differential invariants $I_1, \cdots, I_n$. For example, for generic Kundt metrics, they can be taken as normalized components[10] of the Riemann tensor (cci: Cartan invariants), i.e. through its Ricci or Weyl components, cf. [12]. They can also be taken as spi. For instance, following [19], choose

$$I_1 = \mathrm{Tr}(\mathrm{Ric}_g),\ \dots,\ I_n = \mathrm{Tr}(\mathrm{Ric}_g^n).$$

There are other possibilities, as we will show in detail for dimensions 3 and 4.

The differential invariants G_{ij} are rational functions (also when I_i are psi), and so are their Tresse derivatives. Notice however that we have good control of the domain where these rational invariants are defined. The condition (6) is equivalent to

$$\hat{d}I_1 \wedge \cdots \wedge \hat{d}I_n \neq 0.$$

Assume that the n horizontally independent invariants are of order k or less, and let $\Sigma \subset \mathcal{E}^{k+1}$ (or $\Sigma \subset \tilde{\mathcal{E}}^{k+1}$ in the case of degenerate Kundt) denote the set on which $\hat{d}I_1 \wedge \cdots \wedge \hat{d}I_n$ vanishes or diverges. Then G_{ij} are defined on $\mathcal{E}^{k+1} \setminus \Sigma$. Moreover, the derivatives $\nabla_{i_1} \circ \cdots \circ \nabla_{i_r}(G_{ij})$ are defined on $\pi^{-1}_{k+r+1,k+1}(\Sigma) \cap \mathcal{E}^{k+r+1}$, and their restrictions to fibers of $\mathcal{E}^{k+r+1} \to \mathcal{E}^{k+1}$ are polynomials. We refer to [15] for more details.

Another way of finding a generating set of invariants is to construct n independent invariant derivations $\nabla_1, \dots, \nabla_n$ that are not Tresse derivatives. These form a horizontal frame with dual horizontal coframe $\omega^1, \dots, \omega^n$, which in turn determines differential invariants G_{ij} via (7). This lets us again generate $\mathcal{A}$ if we include, in the set of generators,

[9] If $P(z) = \frac{R(z)}{(1-z)^d}$ is the Poincaré function, with a polynomial $R(z)$ not divisible by $(1 - z)$, then the asymptotic is encoded by the numbers d and $\sigma = R(1)$, see [13].

[10] Beware that normalization can result in elements of an algebraic extension of the field of rational invariants. In particular, invariants obtained in this way may contain roots.

the structure functions c_{ij}^k from $[\nabla_i, \nabla_j] = c_{ij}^k \nabla_k$. The analysis of singular points in the previous paragraph can be adapted to this setting.

4.2. **Invariants of Kundt spacetimes.** For general Kundt metrics we can, as discussed above, use the Ricci operator Ric_g and n second-order differential invariants $I_i = \mathrm{Tr}(\mathrm{Ric}_g^i)$, $1 \leq i \leq n$. (The restriction of the horizontal tensor field Ric_g to a Kundt spacetime is an operator $TM \to TM$.) The invariants $I_1, \dots, I_n$ are horizontally independent on a Zariski open set of 3-jets of Kundt metrics, and thus are sufficient to generate the entire algebra $\mathcal{A}$ of sdi as explained above.

However, for degenerate Kundt spacetimes there are less than n horizontally independent functions among I_i. Actually, the Ricci operator in the (u, x, v) coordinates adapted to Kundt alignment has the form

$$\mathrm{Ric}_g = \begin{bmatrix} \nu & 0 & 0 \\ * & R_h & 0 \\ * & * & \nu \end{bmatrix}$$

with ν being a double eigenvalue and R_h being determined by the Ricci operator for the Riemannian metric h_{ij} on $\bar{M}_u$ and the 2-jet of W_i (more precisely by $(W_i)_v$ and $(W_i)_{x^j v}$) in an invariant manner. When restricted to a degenerate Kundt spacetime, the block-diagonal entries of the operator depend only on (u, x). Thus, the eigenvalues are v-independent functions, and the maximal number of functionally independent eigenvalues is $(n-1)$. For generic degenerate Kundt metrics, this upper bound is reached and the rank of the total Jacobian matrix $[\mathcal{D}_i I_j]$ is equal to $n-1$.

Let $I_1, \dots, I_{n-1}$ be horizontally independent invariants chosen from the above set. For degenerate Kundt spacetimes we have $\partial_v \bar{I}_i = 0$. The annihilator of restricted invariants $d(I_i|_{j^\infty \psi}) = \hat{d} I_i|_{j^\infty \psi}$ integrates to the foliation λ of dimension 1; here $I_i|_{j^\infty \psi}$ is the pullback by the jet-section $j^\infty \psi$ of $J^\infty \pi$ (equivalently: evaluated on the Kundt metric defined by ψ), that we also denoted $\bar{I}_i$, and similar for 1-forms.

Consider the horizontal covectors $\hat{d} I_1, \dots, \hat{d} I_{n-1}$ and g-dual horizontal vector fields $\nabla_1, \dots, \nabla_{n-1}$ tangent to Λ (we remind that Λ is the foliation of codimension 1 with fibers tangent to $\lambda^\perp$). Since the restriction of g to Λ is non-negative definite with one-dimensional kernel, we can without restriction of generality assume that the vectors $\nabla_2, \dots, \nabla_{n-1}$ determine a spacelike subbundle of $\pi_\infty^* TM$ on a Zariski open set in jets. We claim that the $(n-2) \times (n-2)$ Gram matrix is non-degenerate (and hence positive definite):

$$\det[g(\nabla_i, \nabla_j)]_{i,j=2}^{n-1} \not\equiv 0.$$

Finally, we uniquely determine the last invariant derivation ∇_n by the conditions

$$g(\nabla_1, \nabla_n) = 1, \; g(\nabla_i, \nabla_n) = 0 \text{ for } 1 < i \leq n.$$

In fact, since restriction of g to the rank two distribution $\langle \nabla_2, \dots, \nabla_{n-1} \rangle^\perp$ is Lorentzian, it has precisely two null-directions at each point. One is $\mathcal{D}_v$, and ∇_1 is projected to it along $\langle \nabla_2, \dots, \nabla_{n-1} \rangle$ (its projection is also an invariant derivation). The other null-direction is spanned by ∇_n.

This gives an invariant frame, i.e. a basis of sections of the Cartan distribution $\mathscr{C} \simeq \pi_\infty^* TM$, and the algebra $\mathcal{A}$ is determined by this general scheme.

4.3. **The algebra of differential invariants in low dimensions.** The algorithm considered above provides a complete set of differential invariants, but the generators have high algebraic complexity. Therefore, in what follows, we provide an alternative simpler description of the algebra $\mathcal{A}$ in important dimensions $n = 3, 4$.

We begin with a general remark. As we saw in Section 3, the action of $\mathcal{G}$ is transitive on $J^0\pi$ and has precisely 1 differential invariant of order 1 for both $\mathcal{E}$ and $\tilde{\mathcal{E}}$ in any dimension n. We will recycle the notation I_i and ∇_i from Section 4.1 and Section 4.2, and we will continue doing so in Section 4.4 to Section 4.7.

Proposition 9. *Let $w = (W_i)_v dx^i$. The first-order differential invariant is given by*

$$I_1 = \|w\|_g^2 = (W_i)_v (W_j)_v h^{ij}.$$

This invariant corresponds to the invariant described at the beginning of the proof of Theorem 3. Here $[h^{ij}]$ is the inverse of the symmetric matrix consisting of fiber coordinates h_{ij}.

Since the foliation λ is internally invariant, it is reasonable to look for a derivation of the form

$$\nabla_1 = \gamma \mathcal{D}_v.$$

For general Kundt spacetimes, we have $\mathcal{D}_v(I_1) \not\equiv 0$, which means that the factor γ can be determined by the condition $\nabla_1(I_1) = 2$.[11] The invariant I_1 determines the invariant derivation $\nabla_2 = g^{-1}\hat{d}I_1$ which has, for general Kundt spacetimes, a nonzero $\mathcal{D}_u$-component.

If $n = 3$, we can complete the frame with a derivation ∇_3 which is determined (up to an overall sign) by the equations

$$g(\nabla_1, \nabla_3) = 0, \qquad g(\nabla_2, \nabla_3) = 0, \qquad g(\nabla_3, \nabla_3) = 4/I_1.$$

Note also that we have $g(\nabla_1, \nabla_1) = 0$ and $g(\nabla_1, \nabla_2) = \nabla_1(I_1) = 2$. Therefore, the determinant of the matrix with entries $g(\nabla_i, \nabla_j)$ is equal to $-8/I_1$, implying that $\nabla_1, \nabla_2, \nabla_3$ are independent for 2-jets in general position.

If $n = 4$, we find the third derivation ∇_3 in a different way. Let $\nabla_3 = \mathbb{D}^g_{\nabla_2} \nabla_1$, where $\mathbb{D}^g$ denotes the covariant derivative with respect to the Levi-Civita connection. The invariant horizontal frame can now be completed (up to an overall sign) by ∇_4 satisfying

$$g(\nabla_1, \nabla_4) = 0, \qquad g(\nabla_2, \nabla_4) = 0, \qquad g(\nabla_3, \nabla_4) = 0, \qquad g(\nabla_4, \nabla_4) = I_1.$$

In this case we have $g(\nabla_1, \nabla_3) = \frac{1}{2}\nabla_2 g(\nabla_1, \nabla_1) = 0$, and one can verify that $g(\nabla_3, \nabla_3) = I_1$ which implies that ∇_3 is independent of ∇_1, ∇_2 on 3-jets in general position. The four

[11]The constant 2 is a convenient choice with our coordinates. In principle, the right-hand-side can be set equal to any differential invariant.

derivations $\nabla_1, \nabla_2, \nabla_3, \nabla_4$ are then independent by the same argument as for $n = 3$, using the Gram matrix.

In the case of degenerate Kundt spacetimes, the above approach can still be used, but since $\mathcal{D}_v(I_1) \equiv 0$ for degenerate Kundt spacetimes, I_1 must be replaced with a different invariant I which satisfies $\mathcal{D}_v(I) \not\equiv 0$.

In what follows we write down explicitly a basis of invariant derivations for $n = 3$ and $n = 4$, slightly different than the ones suggested above. By the discussion in Section 4.1, such an invariant horizontal frame solves the equivalence problem. For $n = 3$ we also write down explicitly a transcendence basis for the field $\mathcal{A}_2$ of second-order differential invariants.

We remind that invariance of a function f on $\mathcal{E}^k$ (or $\tilde{\mathcal{E}}^k$) can be verified by using the equation $\mathcal{L}_{\xi^{(k)}} f = 0$ while invariance of a horizontal derivation ∇ is verified by using the equation $[\xi^{(\infty)}, \nabla] = 0$. In both cases, the equation must hold for every $\xi \in \mathfrak{g}$. In addition, one must verify that they are invariant under the prolongations of the discrete transformations

$$(u, x^1, \dots, x^{n-2}, v) \mapsto (-u, x^1, \dots, x^{n-2}, -v),$$
$$(u, x^1, \dots, x^{n-2}, v) \mapsto (u, -x^1, x^2, \dots, x^{n-2}, v),$$

that belong to the disconnected components of $\mathcal{G}$. Such computations are not technically difficult, but quite cumbersome, and they are thus better left to computer algebra systems. The same goes for verifying that a set of rational functions are functionally (and thus algebraically) independent[12], although in some of the cases we consider this can be verified by a straightforward hand-computation.

We have used Maple, with the `DifferentialGeometry` and `JetCalculus` packages, when computing with invariant derivations and differential invariants. These packages provide an easy way to verify the statements in this section that rely on symbolic computations. In several cases the invariants were found by using Maple's symbolic PDE solver `pdsolve`. When `pdsolve` failed to give the required solutions, we facilitated the computation by restricting to a simpler ansatz for the invariants. Because of the highly computational nature of these results, they are mostly stated without proof.

4.4. **General 3D Kundt spacetimes.** In this and the next subsection (when we consider $n = 3$) we simplify the notation: $W_1 = W, h_{11} = h, x^1 = x$. Then the invariant of Proposition 9 is given by

$$I_1 = \frac{W_v^2}{h}. \tag{8}$$

Note that the function $W_v/\sqrt{h}$ is invariant with respect to the connected component of $\mathcal{G}$ in the smooth topology, and it is rational on fibers of $\mathcal{E}^1 \to J^0\pi$. However, it changes sign under the transformation

$$(u, x, v, h, W, H) \mapsto (u, -x, v, h, -W, H)$$

[12]This can be checked by computing the rank of the corresponding Jacobian matrix.

which is contained the Zariski closure of the connected component.

The following proposition is easily verified.

Proposition 10. *The derivations*

$$\nabla_1 = \frac{W_v}{W_{vv}}\mathcal{D}_v, \qquad \nabla_2 = \frac{2}{W_v}\mathcal{D}_x + \frac{h_x W_v - 2hW_{xv}}{hW_vW_{vv}}\mathcal{D}_v,$$
$$\nabla_3 = \frac{1}{W_v}\left(H_{vv}\mathcal{D}_x - W_{vv}\mathcal{D}_u + (W_{uv} - H_{xv})\mathcal{D}_v\right)$$

are invariant, and they are independent on a Zariski open subset of $\mathcal{E}^2$.

We have $[\nabla_1, \nabla_2] = -\nabla_2$. The other commutation relations contain nontrivial structure functions (and new invariants), but we omit their explicit form due to their length.

Let α^j denote the elements of the dual horizontal coframe (defined by $\langle \nabla_i, \alpha^j\rangle = \delta_i^j$). The horizontal symmetric 2-form g written in terms of this coframe will have coefficients given by $g(\nabla_i, \nabla_j)$. It takes the form

$$g = I_1^{-1}\left((J_1\alpha^3 + J_2\alpha^2 - I_1\alpha^1)\alpha^3 + 4(\alpha^2)^2\right)$$

where

$$J_1 = \frac{HW_{vv}^2 + (-H_{vv}W + H_{xv} - W_{uv})W_{vv} + H_{vv}^2 h}{h},$$
$$J_2 = \frac{4H_{vv}h^2 + 2(W_{xv} - WW_{vv})h - W_v h_x}{h^2}.$$

Let us recall from Section 3 that there are 4 algebraically independent second-order invariants (excluding the one of first order).

Proposition 11. *The five differential invariants I_1, J_1, J_2 and*

$$\nabla_3(I_1) = 2\,\frac{H_{vv}W_{xv} - H_{xv}W_{vv}}{h} - \frac{W_v(H_{vv}h_x - W_{vv}h_u)}{h^2},$$
$$\begin{aligned} J_3 = {} & \frac{W_{vv}^2(h_u^2 - 2hh_{uu})}{h^3} - \frac{2W_{vv}(H_vW_{vv} - H_{vv}W_v)h_u}{h^2} \\ & - \frac{((H_vW - H_x + W_u)W_{vv}^2 - W_v(H_{vv}W - H_{xv} + W_{uv})W_{vv} + 2H_{vv}^2hW_v)h_x}{h^3} \\ & + \frac{(-2H_xW_v + 2H_vW_x + 2H_{xv}W - 2H_{xx} + 2W_{ux})W_{vv}^2}{h^2} \\ & + \frac{((-2H_{vv}W + 2H_{xv} - 2W_{uv})W_{xv} - 4H_{xv}H_{vv}h)W_{vv} + 4H_{vv}^2hW_{xv}}{h^2} \end{aligned}$$

constitute a transcendence basis for the field of second-order differential invariants on $\mathcal{E}^2$.

In this case algebraic independence can be verified by, for example, analyzing how the five differential invariants depend on the variables h_{uu}, h_u, h_x, H_{vv}.

Note that $\nabla_2(I_1) = 0$ and $\nabla_3(I_1) = 2I_1$. By differentiating $J_1, J_2, J_3, \nabla_1(I_1)$ with respect to $\nabla_1, \nabla_2, \nabla_3$, we get 12 differential invariants of order 3, while $\mathcal{H}_3^3 = 13$. The differential invariant

$$c_{13}^3 = \frac{W_{vvv}W_v}{W_{vv}^2} - 1$$

is algebraically independent from the others, and thus completes the transcendence basis for the field of third-order invariants. Here c_{13}^3 is one of the structure functions in the commutation relation $[\nabla_1, \nabla_3] = c_{13}^i \nabla_i$.

By adding to the 12 third-order invariants the 9 second-order derivatives of J_1,J_2, J_3, $\nabla_1(I_1)$ (36 in total), we get 40 algebraically independent differential invariants of order 4, which generate a transcendence basis for the field of fourth-order differential invariants.

Since $\hat{d}I_1 \wedge \hat{d}J_1 \wedge \hat{d}J_2 \not\equiv 0$, we can obtain all the structure functions from the commutation relations by differentiating these three invariants. Since the coefficients of the metric in the chosen frame are effectively I_1, J_1, J_2, we obtain the following statement.

Theorem 12. *For $n = 3$ the algebra $\mathcal{A}$ of differential invariants of the $\mathcal{G}$ action on $\mathcal{E}$ is generated by the differential invariants I_1, J_1, J_2 and the invariant derivations $\nabla_1, \nabla_2, \nabla_3$.*

This is a good place to point out some similarities and differences between this approach and the Cartan-Karlhede approach to differential invariants. In the Cartan-Karlhede algorithm, one chooses a frame $\hat{\nabla}_i$ such that $g(\hat{\nabla}_i, \hat{\nabla}_j)$ are constant, and then one expresses the curvature tensor in terms of this frame. When choosing the frame, there is some freedom corresponding to the Lorentz group $O(1, n-1)$. In order to find actual differential invariants, one removes this freedom by requiring a sufficient number of the components of the curvature tensor to be constant. When the group is used to fix as many of the components as possible, the remaining nonconstant components are invariant. Next, one computes the covariant derivative of the curvature tensor and repeat the process until the frame is fixed as much as possible. Note that one in general needs to compute the covariant derivative to a relatively high order.

In the approach to differential invariants adapted in this article we need an invariant horizontal frame ∇_i. This can be made from the Tresse derivatives corresponding to n horizontally independent differential invariants, or they can be of more general type such as $\nabla_1, \nabla_2, \nabla_3$ in this subsection. We do not require $g(\hat{\nabla}_i, \hat{\nabla}_j)$ to be constant. Instead, we use these derivations ∇_i and the invariants $g(\nabla_i, \nabla_j)$ to generate the algebra of differential invariants. If the number of horizontally independent invariants among those is $< n$, one should also add the structure functions c_{ij}^k, coming from the commutation relations, to the generating set of invariants. There exist different approaches for finding differential invariants and invariant derivations. In the previous subsections, we found them with geometrical arguments, using the Ricci tensor. In this section, and the next ones, we used symbolic software to help finding solutions to (5) with relatively compact coordinate expressions.

4.5. **Degenerate 3D Kundt spacetimes.** The function I_1 of the form (8) is a differential invariant also in the case of degenerate Kundt spacetimes, since $\tilde{\mathcal{E}}^1 = \mathcal{E}^1$ and the Lie pseudogroup action is the same. From Section 3.3 we know that there are, in addition, 3 algebraically independent differential invariants of order 2. It is possible to restrict the invariants on $\mathcal{E}^2$ to $\tilde{\mathcal{E}}^2$, but our transcendence basis on $\mathcal{E}^2$ does not restrict to a transcendence basis on $\tilde{\mathcal{E}}^2$.

Let us first define the following:

$$I_{2a} = H_{vv}, \quad I_{2b} = \frac{W_v h_x - 2hW_{xv}}{h^2}, \quad K_{2a} = \frac{H_{xv} - W_{uv}}{W}, \quad K_{2b} = \frac{W_v h_u - 2hW_{uv}}{Wh}.$$

The functions I_{2a} and I_{2b} are second-order differential invariants on $\tilde{\mathcal{E}}^2$. The functions K_{2a} and K_{2b} are not invariant, but will be convenient for simplifying the formulas in this subsection. For the same reason, we also introduce the (non-invariant) functions

$$\begin{aligned} Q &= \frac{(2I_{2a}K_{2b} + I_{2b}K_{2a} - I_{2a}I_{2b})W}{I_1}, \\ R &= \frac{I_{2b}HW_v^2}{I_1} - \frac{(I_{2b}I_{2a}^2 - 2K_{2a}(I_{2b} - 2K_{2b})I_{2a} + I_{2b}K_{2a}^2)W^2}{4I_{2a}^2}. \end{aligned}$$

A fourth second-order differential invariant is given by

$$\begin{aligned} I_{2c} =& \frac{1}{Q^2}\Big(\frac{(I_1^2 h_u(WW_v + h_u) - (h_x(H_vW - H_x + W_u)I_1^2 - W_v^4 I_{2b}H))I_{2a}I_{2b}}{W_v^2} \\ &-2\left(W_vH_x + (h_u - W_x)H_v + H_{xx} - W_{ux} + h_{uu}\right) I_1 I_{2a} I_{2b} \\ &-W^2 I_1(K_{2b}(I_{2b} - K_{2b})I_{2a} - 2I_{2b}K_{2a}^2)\Big). \end{aligned}$$

Proposition 13. *The differential invariants $I_1, I_{2a}, I_{2b}, I_{2c}$ constitute a transcendence basis for the field of second-order differential invariants on $\tilde{\mathcal{E}}^2$.*

Notice that $\mathcal{D}_v(I_1) = \mathcal{D}_v(I_{2a}) = \mathcal{D}_v(I_{2b}) = 0$ on $\tilde{\mathcal{E}}^3$. Therefore, $\hat{d}I_1 \wedge \hat{d}I_{2a} \wedge \hat{d}I_{2b} = 0$ everywhere. On the other hand, we have $\hat{d}I_1 \wedge \hat{d}I_{2a} \wedge \hat{d}I_{2c} \neq 0$ on a Zariski open set in $\tilde{\mathcal{E}}^3$. Since I_1, I_{2a}, I_{2c} are horizontally independent, we can write g in terms of them, as explained in Section 4.1, and in this way generate the whole algebra of differential invariants.

Alternatively, we can express the metric in terms of an invariant horizontal frame.

Proposition 14. *The derivations*

$$\begin{aligned} \nabla_1 &= \frac{I_1}{I_{2a}I_{2b}} \cdot \frac{Q}{W_v}\mathcal{D}_v, \qquad \nabla_2 = \frac{1}{W_v}\left(\mathcal{D}_x - \frac{K_{2a}}{I_{2a}}W\mathcal{D}_v\right), \\ \nabla_3 &= \frac{2I_{2a}}{I_1} \cdot \frac{1}{QW_v}\left(K_{2b}W\mathcal{D}_x - I_{2b}h\mathcal{D}_u + R\mathcal{D}_v\right) \end{aligned}$$

are invariant, and they are independent on a Zariski open subset of $\tilde{\mathcal{E}}^2$.

Notice that ∇_1 and ∇_2 can be simplified by multiplying by invariant functions. We have kept these factors because the metric has simple coefficients when expressed in terms of this horizontal frame. If we denote by $\alpha^1, \alpha^2, \alpha^3$ the horizontal coframe dual to the horizontal frame $\nabla_1, \nabla_2, \nabla_3$, we have

$$g = I_1^{-1}\left((-2\alpha^1 + 2\alpha^2 + \alpha^3)\alpha^3 + (\alpha^2)^2\right).$$

It follows that the algebra of differential invariants is generated by $I_1, \nabla_1, \nabla_2, \nabla_3$ and the structure functions in the commutation relations. Since $\hat{d}I_1 \wedge \hat{d}I_{2a} \wedge \hat{d}I_{2c} \not\equiv 0$, the structure functions can be recovered by applying $\nabla_1, \nabla_2, \nabla_3$ to I_1, I_{2a}, I_{2c}.

Theorem 15. *For $n = 3$ the algebra $\mathcal{A}$ of differential invariants of the $\mathcal{G}$ action on $\tilde{\mathcal{E}}$ is generated by the differential invariants I_1, I_{2a}, I_{2c} and the invariant derivations $\nabla_1, \nabla_2, \nabla_3$.*

Remark 2. Here, we have written a frame of invariant derivations with coefficients in $\tilde{\mathcal{E}}^2$. Allowing coefficients in $\tilde{\mathcal{E}}^k$ for higher k, may allow for invariant derivations in more compact form, such as

$$\frac{(WH_{vv} - H_{xv} + W_{uv})H_{xvv} - 2hH_{vv}H_{uvv}}{h}\mathcal{D}_v.$$

4.6. **General 4D Kundt spacetimes.** For general four-dimensional Kundt spacetimes, the invariant of Proposition 9 is given by

$$I_1 = \frac{(W_1)_v^2 h_{22} - 2(W_1)_v(W_2)_v h_{12} + (W_2)_v^2 h_{11}}{h_{11}h_{22} - h_{12}^2}. \tag{9}$$

Let us introduce the notation

$$A = (W_1)_v h_{22} - (W_2)_v h_{12}, \qquad B = (W_1)_v h_{12} - (W_2)_v h_{11},$$

and

$$T = \frac{\begin{array}{r}A^3(h_{11})_{x^1} - ((h_{11})_{x^2} + 2(h_{12})_{x^1})A^2B + ((h_{22})_{x^1} + 2(h_{12})_{x^2})AB^2 - B^3(h_{22})_{x^2} \\ - 2(h_{11}h_{22} - h_{12}^2)(A^2(W_1)_{x^1v} + B^2(W_2)_{x^2v} - AB((W_1)_{x^2v} + (W_2)_{x^1v}))\end{array}}{2(h_{11}h_{22} - h_{12}^2)(A(W_1)_{vv} - B(W_2)_{vv})}.$$

We have the following proposition.

Proposition 16. *The derivations*

$$\nabla_1 = \frac{(W_1)_v^2 h_{22} - 2(W_1)_v(W_2)_v h_{12} + (W_2)_v^2 h_{11}}{A(W_1)_{vv} - B(W_2)_{vv}}\mathcal{D}_v,$$
$$\nabla_2 = \frac{(W_2)_{vv}\mathcal{D}_{x^1} - (W_1)_{vv}\mathcal{D}_{x^2} + ((W_1)_{x^2v} - (W_2)_{x^1v})\mathcal{D}_v}{(W_1)_v(W_2)_{vv} - (W_2)_v(W_1)_{vv}},$$
$$\nabla_3 = \frac{A\mathcal{D}_{x^1} - B\mathcal{D}_{x^2} + T\mathcal{D}_v}{h_{11}h_{22} - h_{12}^2}, \qquad \nabla_4 = g^{-1}\hat{d}I_1$$

are invariant, and they are independent on a Zariski open subset of $\mathcal{E}^2$.

Notice that ∇_4 is the only derivation among these that have a non-zero $\mathcal{D}_u$-component. We have $g(\nabla_1, \nabla_i) = 0$ for $i = 1, 2, 3$, and

$$g(\nabla_2, \nabla_2) = I_{2a},\ g(\nabla_2, \nabla_3) = 1,\ g(\nabla_3, \nabla_3) = I_1,\ g(\nabla_1, \nabla_4) = 2I_1,\ g(\nabla_3, \nabla_4) = 0.$$

Here

$$I_{2a} = \frac{(W_1)_{vv}^2 h_{22} - 2(W_1)_{vv}(W_2)_{vv}h_{12} + (W_2)_{vv}^2 h_{11}}{((W_1)_v(W_2)_{vv} - (W_2)_v(W_1)_{vv})^2}$$

is one of the second-order differential invariants. The formulas for the differential invariants $g(\nabla_2, \nabla_4)$ and $g(\nabla_4, \nabla_4)$ are more complicated.

There are $\mathcal{H}_2^4 = 14$ algebraically independent differential invariants of order 2, so we will not attempt to write down all of them. Instead we will be satisfied with finding four horizontally independent differential invariants. The scalar curvature S_h of the (u-parametrized) metric h is an invariant function depending only on h_{ij} and their x^i-derivatives up to second order. A fourth differential invariant is given by

$$I_{2b} = \frac{\big(((W_2)_{uv} - H_{x^2v})(W_1)_{vv} - ((W_1)_{uv} - H_{x^1v})(W_2)_{vv} + ((W_1)_{x^2v} - (W_2)_{x^1v})H_{vv}\big)^2}{h_{11}h_{22} - h_{12}^2}.$$

Theorem 17. *The four differential invariants I_1, I_{2a}, I_{2b}, S_h are horizontally independent on a Zariski open subset in $\mathcal{E}^3$, and thus sufficient for solving the equivalence problem.*

4.7. **Degenerate 4D Kundt spacetimes.** The first-order invariant I_1 is the same as in the previous section. In total, there are $\tilde{\mathcal{H}}_2^4 = 12$ algebraically independent invariants of second order. We write down two of them:

$$I_{2a} = H_{vv}, \qquad I_{2b} = \frac{((W_1)_{x^2v} - (W_2)_{x^1v})^2}{h_{11}h_{22} - h_{12}^2}.$$

Let us find an invariant horizontal frame. The horizontal 1-forms $\hat{d}I_1, \hat{d}I_{2a}, \hat{d}I_{2b}$ are independent: $\hat{d}I_1 \wedge \hat{d}I_{2a} \wedge \hat{d}I_{2b} \not\equiv 0$. Since $\mathcal{D}_v(I_1) = \mathcal{D}_v(I_{2a}) = \mathcal{D}_v(I_{2b}) = 0$, the 1-forms have no dv-component. By solving the equations

$$(\hat{d}I_1 + a_1\hat{d}I_{2a} + a_2\hat{d}I_{2b})(\mathcal{D}_{x^1}) = 0, \qquad (\hat{d}I_1 + a_1\hat{d}I_{2a} + a_2\hat{d}I_{2b})(\mathcal{D}_{x^2}) = 0$$

for a_1 and a_2, we obtain an invariant 1-form which is proportional to du. We turn it into a horizontal vector field by using g, and denote the resulting invariant derivation, which is proportional to $\mathcal{D}_v$, by ∇_1. Next, we define

$$\nabla_2 = g^{-1}\hat{d}I_{2a}, \qquad \nabla_3 = g^{-1}\hat{d}I_{2b}.$$

We complete the invariant horizontal frame by requiring ∇_4 to satisfy

$$g(\nabla_1, \nabla_4) = 1, \qquad g(\nabla_2, \nabla_4) = 0, \qquad g(\nabla_3, \nabla_4) = 0, \qquad g(\nabla_4, \nabla_4) = 0.$$

Proposition 18. *The derivations $\nabla_1, \nabla_2, \nabla_3, \nabla_4$ are invariant, and independent on a Zariski open subset of $\tilde{\mathcal{E}}^3$.*

We have $\mathcal{D}_v(g(\nabla_i, \nabla_j)) \equiv 0$ for every i and j. We choose an invariant for which this is not the case from the commutation relations $[\nabla_i, \nabla_j] = c_{ij}^k \nabla_k$. For instance, we have $\mathcal{D}_v(c_{23}^1) \not\equiv 0$.

Theorem 19. *The differential invariants $I_1, I_{2a}, I_{2b}, c_{23}^1$ are horizontally independent, and thus sufficient for solving the equivalence problem.*

5. Conclusion

We considered the equivalence problem for general and degenerate Kundt metrics with respect to the action of the pseudogroup of local diffeomorphisms. Denoting these classes of spacetimes by $\mathcal{K}$ and $\tilde{\mathcal{K}}$, respectively, we have (many other important subclasses are omitted):

$$\mathcal{K} \supset \tilde{\mathcal{K}} \supset \text{VSI} \supset \text{Kundt waves}.$$

For general Kundt metrics the problem can be solved using scalar polynomial invariants, but even then it is a nontrivial task to specify the required invariants, cf. [32]. For degenerate Kundt metrics, the spi are insufficient for separating metrics.

We use instead rational differential invariants, which separate jets of metrics in general position within the class of degenerate Kundt metrics. By integrating the foliations (λ, Λ) internal to the class of Kundt metrics, one can normalize the set of admissible coordinates and reduce the pseudogroup $\mathrm{Diff}_{\mathrm{loc}}(M)$ to $\mathcal{G}$ consisting of transformations that preserve the form of Kundt metrics expressed in terms of admissible coordinates. The equivalence classes of Kundt metrics $\mathcal{K}$ (respectively degenerate Kundt metrics $\tilde{\mathcal{K}}$) with respect to all transformations are in bijective correspondence to those of form (1) with respect to the shape-preserving transformations:

$$\mathcal{K}/\,\mathrm{Diff}_{\mathrm{loc}}(M) = \mathcal{E}/\mathcal{G} \quad \text{and} \quad \tilde{\mathcal{K}}/\,\mathrm{Diff}_{\mathrm{loc}}(M) = \tilde{\mathcal{E}}/\mathcal{G}.$$

In order to be consistent, in these equalities we should interpret $\mathcal{K}$ and $\tilde{\mathcal{K}}$ to mean the corresponding spaces of jets of metrics. The algebras of differential invariants consist of functions on those spaces.

Since our invariants are rational functions in jet-variables of low order and polynomial in higher jet-variables, there is a Zariski closed subset of jets of (degenerate) Kundt spacetimes that are not separated by the invariants we have found. By restricting to this Zariski closed set, and considering the Lie pseudogroup action on this set, it is possible to repeat the procedure and find an algebra of rational invariants separating generic jets of metrics in this singular set, etc.

One should note that the coordinates used to create the signature manifold $\Psi(X)$ need not be adapted to (λ, Λ). For instance, none of the invariants I_i constructed in Section 4.1 were required to be constant along Λ. This however does not obstruct to solve the equivalence problem: the foliation λ is reconstructed from the first $(n-1)$ differential invariants and since the metric g is determined, $\Lambda = \lambda^\perp$ is recovered.

In principle, the Cartan invariants can be used for the same purposes, yet with the formalism for differential invariants we have a better control over the analytic properties of the functions in the algebra $\mathcal{A}$ of differential invariants. The normalization used in the Cartan-Karlhede algorithm, for example, requires to solve algebraic equations for group parameters. In particular, this often results in expressions involving radicals. The differential invariants we have computed are manifestly rational in jet-variables and thus defined on a Zariski open subset of the space of jets of Kundt metrics. This is one of the differences between the approaches. We refer to [16] for further comparisons.

Several classes of transformations were considered in the literature that are natural subgroups of $\mathcal{G}$. Reference [1] studied nil-Killing fields defined as those vector fields X on M that are aligned with respect to λ and $\mathscr{L}_X g$ is nilpotent wrt the filtration (λ, Λ). It was shown in [20] that nil-Killing vector fields wrt λ, preserving λ, form a Lie algebra:

$$\mathfrak{g}_\lambda = \{X \,:\, \mathscr{L}_X\lambda = \lambda \text{ and } \mathscr{L}_X g \text{ is of type III wrt } \lambda\}.$$

This is an infinite-dimensional Lie subalgebra of $\mathfrak{g}$ given by (3). The corresponding Lie pseudogroup of λ-aligned transformations, preserving spi, depends on 1 function of $(n-1)$ arguments (and other functions of fewer arguments), cf. [1, Proposition 6]. In fact, this pseudogroup consists of transformations (2) forming $\mathcal{G}$ such that the induced transformation of $(\bar{M}_u, h)$ is a u-parametric isometry.

A proper subalgebra of $\mathfrak{g}_\lambda$ is the Lie algebra of Kerr-Schild vector fields wrt λ, defined as those X, preserving λ, for which $\mathscr{L}_X g \in S^2\lambda^*$ (has type N), see [6]. This Lie algebra may be trivial, however if the 1-form $w = (W_i)_v dx^i$ on Λ^* (important in our computations of invariants, see Proposition 9) is exact, then any infinitesimal transformation $b(u)\partial_v$ is a Kerr-Schild vector field, so this algebra may also be infinite-dimensional.

The equivalence problem of classes of spacetimes wrt to those and other Lie sub-pseudogroups may be of interest in its own right.

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Appendix A. Relative differential invariants

Here we demonstrate that the class of degenerate Kundt spacetimes is singled out among all Kundt metrics by a relative invariant condition, so that the singular behavior can be observed by studying orbits of the diffeomorphism pseudogroup on the jets of metrics. Note that due to normalization of the vector ℓ as in (1) the diffeomorphism pseudogroup shrinks to the pseudogroup of shape-preserving transformations.

A function $f \in C^\infty(J^k\pi)$ is a relative differential invariant wrt a pseudogroup $\mathcal{G}$ if $\varphi^* f = s_\varphi \cdot f\ \forall\varphi \in \mathcal{G}$ for some nonzero function s_φ on $J^k\pi$. For the corresponding Lie algebra $\mathfrak{g}$ this translates into:

$$\mathscr{L}_{\xi^{(k)}} f = \omega(\xi) f \quad \forall \xi \in \mathfrak{g}$$

for some $\omega \in \mathfrak{g}^* \otimes C^\infty(J^k\pi)$. This ω is a 1-cocycle, i.e. it satisfies the equation (cf. [23])

$$\mathscr{L}_{\xi^{(k)}}\omega(\eta) - \mathscr{L}_{\eta^{(k)}}\omega(\xi) - \omega([\xi,\eta]) = 0 \quad \forall \xi,\eta \in \mathfrak{g}.$$

Cocycles of the type $\omega = dh$ are called trivial. Cocycles modulo trivial ones are called cohomology; in arbitrary order they form the group $H^1(\mathfrak{g}, C^\infty(J^\infty\pi))$. To catch the algebraic structure of the jet-fibers we consider only such cocycles that (modulo trivial) are polynomial in the jet-variables. Such relative invariants are not plentiful.

Of course, absolute differential invariants are relative. For any relative differential invariant f the equation given by $f = 0$ is $\mathcal{G}$-invariant. If the invariant is genuinely relative (not absolute), then $\{f = 0\}$ contains singular orbits. Recall that an orbit is regular if a neighborhood of it is fibred by orbits, and it is called singular otherwise.

Let us focus on the 3D case (as before we omit indices for W_i and h_{ij} here). Since the action of $\mathcal{G}$ is transitive on $J^0\pi$, as in Section 3.2, we can translate any point to

$$p_0 = \{u = 0, x = 0, v = 0, h = 1, H = 0, W = 0\}.$$

Consider the action of $\mathcal{G}_0^{(1)}$ on the fiber $\pi_{1,0}^{-1}(p_0) \cap \mathcal{E}_1$. Here W_v^2 is an absolute invariant, with the action transitive on its level sets[13]. As shown in Section 3.2, we can bring any point to the point (omitting equations of p_0)

$$p_1 = \{h_u = 0, h_x = 0, H_u = 0, H_x = 0, H_v = 0, W_u = 0, W_x = 0, W_v = c\}.$$

Here $c \geq 0$ is the level parameter, and $h_u, h_x, \dots$ are jet-variables. Note that the first seven of these are normalized by translations, after which we are left with the action of $O(n-2) = O(1) = \mathbb{Z}_2$ on W_v.

Next consider the action of the stabilizer pseudogroup $\mathcal{G}_1^{(2)}$ on 2-jets $\mathcal{E}_1^2 = \mathcal{E}^2 \cap \pi_{2,1}^{-1}(p_1)$. This space has dimension 15, while the group acting on it has dimension 11. Thus we get 4 absolute invariants, as established in Section 3.3 and explicitly given in Section 4.4. To get more precise structure of the orbit space note that the group consists of 9 translations and 2 affine transformations. The translations form a 9-dimensional Abelian group $\mathfrak{A}$ with the Lie algebra

$$\partial_{h_{xx}},\ \partial_{h_{ux}},\ \partial_{h_{uu}} + \partial_{W_{ux}},\ \partial_{H_{xx}} + \partial_{W_{ux}},\ \partial_{H_{ux}},\ \partial_{H_{uu}},\ \partial_{H_{uv}},\ \partial_{W_{xx}},\ \partial_{W_{uu}}.$$

We use them to set $h_{xx} = h_{ux} = h_{uu} = H_{xx} = H_{ux} = H_{uu} = H_{uv} = W_{xx} = W_{uu} = 0$. This global transversal to the action of $\mathfrak{A}$ can be identified with the quotient space $Q^6 = \mathcal{E}_1^2/\mathfrak{A}$. Let us introduce the coordinates $z_1 = -\frac{1}{2}W_{ux}$, $z_2 = -W_{uv}$, $z_3 = W_{xv}$, $z_4 = 2W_{vv}$, $z_5 = \frac{1}{3}W_{uv} - \frac{2}{3}H_{xv}$, $z_6 = \frac{2}{3}W_{xv} + \frac{4}{3}H_{vv}$ on Q. Then the infinitesimal affine transformations are

$$V_1 = 2z_1\partial_{z_1} + z_2\partial_{z_2} - z_4\partial_{z_4} + z_5\partial_{z_5}, \quad V_2 = z_2\partial_{z_1} + z_3\partial_{z_2} + z_4\partial_{z_3} + (z_3 - z_6)\partial_{z_5}.$$

They form a 2-dimensional solvable Lie algebra with $[V_1, V_2] = -V_2$. Since V_2 is nilpotent, any polynomial relative invariant must belong to its kernel. The linear polynomials in

[13] Note that W_v^2 is an absolute invariant only with respect to the stabilizer subgroup of the point p_0. If we restore the entire group action, then both W_v^2 and h are relative invariants of the same weight, so that their ratio I_1 is an absolute invariant.

the kernel are spanned by z_4 and z_6. Here z_6 is an absolute invariant, while z_4 has weight -1 with respect to V_1. The relative invariant z_4 gives the first condition for degenerate Kundt spacetimes: $W_{vv} = 0$.

By extending this analysis to $\mathcal{E}^3$, we see that the function H_{vvv} is not a relative invariant, but becomes so when we restrict to the subset in $\mathcal{E}^3$ given by $W_{vv} = 0$ and its differential consequences. This (conditional) relative invariant also has weight -1 with respect to (the prolongation of) V_1.

Note that there exist other nontrivial relative invariants, of higher degree. The invariant W_{vv} on $\mathcal{E}^2$ is singled out by having negative weight; all other relative invariants with negative weight have W_{vv} as a factor. The invariant H_{vvv} on the sub-PDE given by $W_{vv} = 0$ is determined uniquely in the same way. Thus the degenerate Kundt conditions $W_{vv} = 0, H_{vvv} = 0$ arise from investigations of singularities of the $\mathcal{G}$ action on $\mathcal{E}^2$ and $\mathcal{E}^3$.

Theorem 20. *The function W_{vv} is a relative invariant of the $\mathcal{G}$ action on $\mathcal{E}^2$. On the submanifold in $\mathcal{E}^3$ given by $W_{vv} = 0$ and its differential consequences, the function H_{vvv} is a relative invariant of the $\mathcal{G}$ action.*

The same idea can be applied in higher dimensions. The Lie algebra spanned by V_1 and V_2 is then replaced by an $(n-1)$-dimensional Lie algebra from which information about singular orbits can be read. In this case, W_{vv} should be considered as a tensorial relative invariant (covector), whose corresponding zero-set has codimension greater than 1. Then the theorem holds true in higher dimensions as well.

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