



Information in Financial Markets

How private information affects prices,
how it can be revealed and
how it may be used

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**To Katja,
and my parents**

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Introduction

The literature on financial markets is vast and it is probably safe to say that all tools in the economists' tool case have been applied to this field. In this dissertation I will present three papers that are very diverse in their approach to the subject of finance, but have an important common theme; asymmetric information and efficiency in financial markets.

A. The noisy rational equilibrium debate

In the first paper, "Are Noise Traders Really Necessary? A General Approach" I construct a general model which facilitates a better understanding of the relationship between two important models on how information is integrated into prices. First we look at the seminal paper of Grossman and Stiglitz (1980). They showed that information must be worthless in an efficient market for an equilibrium to obtain. A market is defined to be efficient if any private information is reflected in the market price. Grossman and Stiglitz chose to attribute the inefficiency in prices to "noise traders". These traders are "stupid" traders in the sense that they persistently place losing bets in the market. Since investors cannot distinguish between such trades and those that are motivated by private information, noise traders ensure information is valuable.

The problem for an investor with private information when no noise traders are present, is that prices will adjust as soon as she acts on it and starts to trade, and the information is revealed to everyone. Therefore, there

is no incentive for anyone to trade on private information. If that is the case however, prices cannot reflect any information. This important result is called the Grossman-Stiglitz Paradox, and leads to their conclusion: Informationally efficient markets are impossible.

The paper started an extensive debate since asset markets are generally regarded as very efficient. As always in economics the question was which important assumptions were questionable? A couple of candidates quickly emerged.

One reason that the Grossman-Stiglitz result may not hold is if there are other motives to trade than information. The point is best illustrated with a simple example. Assume there are two identical agents, A and B, who possess one asset each. Agent A has, however, better information on what return the asset will give than B. It is quite obvious in this example that B would not be willing to sell the asset at any price to A, since any price would imply that A would profit from the transaction at B's expense. Hence A's information is both worthless and will not be reflected in any "market price".

Let us now change the asset holdings, so that A has one asset and B has three. Being identical in all respects, B will now realize that A has legitimate reasons to demand one asset from him. After all, with the same information they should have the same number of assets. The question now is only at which price the transaction should clear. However A has more information than B, and knows exactly the price needed for her to break even. The problem for B is then to infer what information A has from the bid she offers B. A difficult task indeed. With no further information about which type of bidding procedure and game rules they have agreed upon it is in fact impossible for us to say anything about what the "market price" reflects, what the information is worth or what the utilities will be.

Introducing auxiliary incentives for trades therefore pretty much "solves" the paradox. In addition as we saw from the above example, any such model must have a quite detailed specification, such as the market institution, the

rules of the game and the number of the different players, and so they cannot be very general. There are a large number of these papers, since the different ways to specify such a market is infinite.

This "solution" is however not satisfactory. My interpretation of the Grossman and Stiglitz paper, although I am not sure it even corresponds with the authors' original intent, is to question what kind of model is best suited to explain how financial markets operate and work. It is then found that a model with no noise is not a very productive starting point, since the price then really reflects nothing. Adding additional incentives to trade may of course be interesting to analyze in itself, but theoretically it is no different than adding noise traders. Whether the incentive to trade is white noise or given by the modeler is only a matter of specification. In fact, since the noise traders are an unbiased error term it actually has some very desirable properties relative to many alternative specifications.

Let me be perfectly clear, there are a lot of eminent papers in this literature. They are, however, good for other reasons than "disproving" Grossman and Stiglitz, and for the most part that is not intended either.

Alongside this debate a different assumption was questioned. The Grossman-Stiglitz model is static. Would the results change if we looked at trading over time? The uninformed traders in the original model look at the price and immediately react to it to form new demands in zero time. That is surely not a very realistic assumption. In fact it is outright impossible in any real financial market. One could of course argue that the traders' post continuous *functions* of the price to the market, but trading costs make this a purely theoretical construction. As if that is not enough, this problem also ensures that the equilibrium obtained by Grossman and Stiglitz is not consistent with a Nash Equilibrium (Dubey, Geanakoplos, and Shubik (1987)).

A solution to these problems was clearly required, and a couple of years after the original paper, Hellwig (1982) proposed the reasonable assumption that current prices cannot be observed. Hellwig showed that information

would be valuable in this case, even when the time between trading approached zero.

A major problem with Hellwig's paper is however that it assumes a demand function that cannot be derived from an expected utility function. The uninformed in Hellwig's model treat the current price as a cost, even though it is not observed. This means they persistently make losing bets, in addition to having exactly the same deficiency as the Grossman-Stiglitz model in that it requires the uninformed to post bids and asks in terms of a *function* of the price.

Dubey, Geanakoplos, and Shubik (1987) noted something along these lines in a footnote, and showed as mentioned that such models in general do not even have a Nash Equilibrium. Thus for any such model to be consistent, the price to condition demand on must be realized before the demand is posted. An alternative was therefore proposed based on the model of Shapley and Shubik (1977). In the Shapley-Shubik model there are no noise traders, but since current prices cannot be observed, prices do not reflect all available information. The implications of this model have been derived further by Jackson and Peck (1999). A disadvantage is however that it is difficult to compare with that of Grossman and Stiglitz.

In the first paper I attempt to remedy this problem by setting up a quite general model, that allows the agents to condition on any past price. In addition I assume a general supply function that can work for both the Shapley-Shubik model as well as for the Grossman and Stiglitz model. Setting these models side by side, reveals that the Shapley-Shubik model requires that the uninformed are forced to make state dependent demands that are negatively correlated with expected profits. Such demands need to be forced by the market institution since uninformed investors would strictly prefer to demand fixed quantities.

It is further found that the original Grossman-Stiglitz model can easily be altered to overcome the problems of the static model by disallowing obser-

vations of current prices. The model will yield the exact same results, since the ability to observe current prices is not a vital assumption in the original model. However, in a dynamic setting, a model that allows observation of current prices has unwanted properties such as weak form inefficient prices and intractable equilibrium solutions.

In my view therefore, it seems the real insight of the Grossman-Stiglitz model has not been appreciated sufficiently by later researchers. If we for some reason were to regard noise traders as inappropriate for financial models, the last two papers in this dissertation would in fact be invalid. In my mind it seems a better idea for a researcher to construct models that work well with the data observed, as opposed to artificially assuming that every thing in a financial market can be explained with certainty.

There seem to be some inherent unwillingness to accept that our models will never explain every aspects of the asset market. Randomness that conceals information can be added to models in many ways however. It might not be in demand, but in unobservable income shocks to a subset of investors or shocks to beliefs. The specification that requires the smallest set of assumptions is however randomness in supply.

It is therefore my opinion that the Grossman-Stiglitz model is still the most valid general equilibrium model for asset pricing under asymmetric information. It tells us that by making models that do not allow for unexplained unbiased trading, for which ever reason may be creating and not solving problems.

B. How to protect yourself from private information

In the first paper that is presented, and commented on above, I conclude that financial models should not try to explain every trader's behavior. In "Optimal Order Submission" I follow a long line of microstructure literature, and build the model around the assumption that there are traders seeking liquidity who trade randomly for whichever reason. The uninformed traders

can then make consistent profits from these noise traders. Much of the microstructure theory is built around the idea that a market maker determines the market spread, and profits from this by buying low and selling high to noise traders. The spread is often set to the level where the market maker makes no expected profit, and so if there are only noise traders in the market, the spread would be zero.

There are however two other factors the market maker needs to take into account. First there might be informed traders in the market who will trade only if the market maker makes a corresponding loss. Second, it is expensive to be a market maker because inventories tend to build up in the short term. This exposes the market maker to a lot of idiosyncratic risk. The spread therefore needs to be set so wide that it covers these two costs, in addition to the direct costs such as trading fees and operating costs.

My initial intention when starting on this paper was to model such market maker behavior. There is quite an extensive microstructure literature on how spreads are set, and which of the three costs are most important for determining the spread. In addition quite a lot of literature evolves around the informativeness of volume observations in the market. However, what has not received as much attention is which order sizes should be set in order to minimize exposure to informed traders.

The data that was available for me was from the Norwegian Stock Exchange (OSE). That is, however, not a good place to study market maker behavior, as there are almost none there. Fortunately, the model that will be presented is flexible enough to explain both market maker behavior as well as optimal order submission by ordinary traders. The mechanisms are the same, since any trader would make an effort to protect herself from informed traders. Posting small orders at a time is one way to obtain such protection. When uncertainty with respect to fundamentals is high, the probability of trading with an informed increases, and the order size should be reduced. The optimal order size function is therefore a decreasing convex function

of volatility, where volatility is measured in number of standard deviations. Thus, the more likely it is that the market price is far from the underlying fundamentals, the smaller order size that should be submitted at a time. At some point the volatility may be so large that submitting any order will be unprofitable, and hence the optimal order size is zero.

The model is then developed further to find the optimal price adjustment and an expression for the long term equilibrium volatility level.

As one will notice in the empirical part, the model is overspecified in the sense that it is impossible to test all parameters simultaneously. What is found is that the general shape of the optimal order size function fits well with the data. It is also found that the model with all parameters displays a lot of multicollinearity. For estimation this is a problem, but it also shows that the ability of the model to describe the trading is not too sensitive to different parameter values.

C. How to profit from private information

In the last paper of this dissertation, "Optimal Distribution of Information by an Information Monopolist: A Generalization", I present a model directly descending from that of Grossman and Stiglitz, with a noisy demand element in a rational expectations model. The main issue of the model is as its title says, how an information monopolist can maximize profits by selling the information to investors.

Admati and Pfleiderer (1986) found that a seller should sell independent signals with identical distributions to a fraction of the traders. In another paper addressing this issue Admati and Pfleiderer (1990) found that it would be even better to sell exactly the same information to everybody if it could be done through a mutual fund. In any case, one general conclusion one can draw from these papers is that all the buyers should be treated equally and receive the same type of information. Given the symmetry of the problem (all traders had the same risk aversion) the symmetry of the solution would

be expected. The model with direct sale (Admati and Pfleiderer (1986)) is however not fully symmetric when we take into account those who do not get to buy the information, since only a fraction of the traders are informed in the optimum.

It would however be of general interest first to examine whether this symmetric result would really hold for any given distribution of signals. Second, it would be of interest to know under which specific circumstances the symmetric results of Admati and Pfleiderer would not hold.

I therefore approached the problem studied by Admati and Pfleiderer in a very general way, in order to obtain a proof that is as general as possible. The smallest set of conditions for the proof would then serve as cases where non-symmetric solutions could be expected to be found.

I find that the information monopolist may select an asymmetric solution if either the cost of selling information depends directly on the number that receive it or if they are heterogeneous (i.e. have different risk aversion coefficients) and if the cost function can have local maxima. In the first case it is important to note that the cost of selling too many investors is an explicit cost. The dilution of information value that occurs through prices when many are informed is part of the model specification. Such a direct cost may for example be the possibility of an insider being caught when more investors receive the same information.

In the second case, if investors are heterogeneous then an asymmetric solution is expected. The general proof is based on control theory, so in the case of heterogeneous agents if the model is rigorously enough specified an explicit solution may be possible. A solution strategy is therefore suggested.

A nested information structure is then suggested to solve the problem for the seller that buyer may pool their information in order to increase precision.

D. How the models relate to each other

The models are different with respect to a number of characteristics. Only one is for instance truly dynamic, and one of them is not related to the theoretical Grossman-Stiglitz framework. All models are however attempts to characterize the role of information in asset markets. None of them assume full market efficiency per se, which as I conclude in the first paper may be an inappropriate assumption. All the papers rely on a setting where some traders receive signals that better enables them to estimate the true value of the asset.

In general, the first paper discusses the issue of which types of models are productive in financial modeling. This is then applied in the two remaining papers. The second paper has an empirical part, and the model is constructed carefully to be able to reflect some features of real market institutions. The third paper is at the other end of the spectrum, where a proof is obtained with the maximum amount of generalization.

References

- Admati, Anat R., and Paul Pfleiderer, 1986, A Monopolistic Market for Information, *Journal of Economic Theory* 39, 400–438.
- Admati, Anat R., and Paul Pfleiderer, 1990, Direct and Indirect Sale of Information, *Econometrica* 58, 901–928.
- Dubey, Pradeep, John Geanakoplos, and Martin Shubik, 1987, The Revelation of Information in Strategic Market Games: A Critique of Rational Expectations Equilibrium, *Journal of Mathematical Economics* 16, 105–137.

- Grossman, S. J., and J. E. Stiglitz, 1980, On the Impossibility of Informationally Efficient Markets, *The American Economic Review* 70, No. 3, 393–408.
- Hellwig, Martin F., 1982, Rational Expectations Equilibrium with Conditioning on Past Prices: A Mean-Variance Example, *Journal of Economic Theory* 26, 279–312.
- Jackson, Matthew O., and James Peck, 1999, Asymmetric Information in a Competitive Market Game: Reexamining the Implications of Rational Expectations, *Economic Theory* 13, 603–28.
- Shapley, Lloyd, and Martin Shubik, 1977, Trade Using One Commodity as a Means of Payments, *The Journal of Political Economy* 85, 937–968.

Are Noise Traders Really Necessary? A General Approach

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Abstract

In this paper it is shown that noise traders in dynamic equilibrium models with asymmetric information are necessary for information to have value under fairly general assumptions, unless uninformed investors are forced to make state dependent bids. The result is obtained by setting up a general linear model where investors are allowed to condition on any previous price in history and where the supply function has a general form. This enables us to compare the very different models of Shapley and Shubik (SS) and Grossman and Stiglitz (GS) and allows a comprehensive study of the effect of past prices on conditional expectations. It is found that; 1) if uninformed investors cannot condition on current prices, they will not use past prices, 2) this dynamic version of GS with unobservable current prices has a Nash Equilibrium, 3) the SS model requires state dependent bids, e.g. bids in terms of portfolio cost. 4) if current prices are observable then investors may condition on the complete price history and as proved by Dubey, Geanakoplos, and Shubik (1987) there is no NE.

Keywords: finance, asset pricing, information

JEL Classification: G12, G14

Are noise traders really necessary in order for information to have value? A number of authors have investigated this issue. In this paper it is shown

that under very general assumptions, they are indeed a necessary condition, unless the uninformed investors are forced to make state dependent bids.

This general result is presented at the end of the paper, since we first need a framework where we can compare very different models. We therefore present a short term linear model where investors can condition on any price in the entire price history. This allows us to study different regimes of price observation. Furthermore, the supply is characterized as a general function of the stochastic variables in the system. As we will see this enables us to compare a couple of very different models, and thereby obtain fairly general results.

The model presented here thus allows us to study a number of interesting features of dynamic financial markets with asymmetric information. In particular it is found that

1) In a noisy rational equilibrium model, if uninformed investors cannot condition on current prices they will not use past prices.

3) This dynamic version of GS with unobservable current prices has a Nash Equilibrium, in contrast to the original one, as proved by Dubey, Geanakoplos, and Shubik (1987).

2) The SS model requires state dependent bids, e.g. bids in terms of portfolio cost.

4) If current prices are observable then investors will condition on the complete price history and as proved by Dubey, Dubey, Geanakoplos, and Shubik (1987) there is no NE.

The paper is built around three examples of a simple dynamic Grossman and Stiglitz (1980) (GS) type Rational Expectation Equilibrium (REE) model, assuming CARA utility functions, linear demand and price functions and myopic agents. The original Grossman and Stiglitz (1980) model is not consistent with a Nash Equilibrium, as noted by Dubey, Geanakoplos, and Shubik (1987), which is generally acknowledged as a major problem. In this paper a simple model is developed that incorporates three different

approaches to information asymmetry in a competitive financial market.

In Example 1 investors are unable to observe current prices. It is found that their best response is then to hold fixed portfolios and not condition on past prices either. This slight modification of GS is shown to have a NE and the main results of GS are not affected. Although it is an obvious point, it seems not to have been made before.

In Example 2 we allow the uninformed to condition on current prices, and we obtain a model similar to those of Brown and Jennings (1989) and Grundy and McNichols (1989). The results are consistent with that literature in that prices are not weak form efficient in such a market. If investors can condition on current prices, then uninformed investors will use past and current prices to predict future returns. The resulting equilibrium is not a NE as noted by Dubey et. al. though. It is therefore argued that this type of technical analysis may only be possible in a model where the price setting mechanism is not consistent with a NE.

In Example 3 we consider the Shapley-Shubik (SS) model that Dubey, Geanakoplos, and Shubik (1987) proposed as an alternative to GS. The model is adapted to the GS framework presented here in order to better compare it with the two other examples. It is found that the SS model corresponds to a market where investors are restricted to bid in terms of cost and not units of the asset. The results of the modified model are identical to those of Jackson and Peck (1999), who did a comprehensive comparison of SS and the efficient REE model of GS.

As mentioned Dubey, Geanakoplos, and Shubik (1987) found that the GS model as originally described was not consistent with a Nash Equilibrium (NE). The problem is that demand both generate and determine equilibrium prices at the same time. Dubey et. al. then considered the possibility of submitting entire demand functions, and proved that the resulting rational expectation equilibrium (REE) could not be implemented as a NE. In addition Dubey et. al. argued that submitting an entire demand function would

be impractical and not consistent with how actual asset markets work. They therefore concluded that for a market game to be consistent with NE one need to model in more detail how information is put into prices. The problems with the original GS model noted by Dubey et. al. are now generally acknowledged as major drawbacks of the REE approach.

Dubey et. al. then presented some examples in their paper, based on the Shapely-Shubik model (Shapley and Shubik (1977)), which do have NE. The competitive example they presented was then further developed by Jackson and Peck (1999), who pointed out the major differences between the SS model and the efficient rational expectation model of GS. Also Goenka (2003) have applied the Shapely-Shubik model to financial markets.

This paper relies heavily on the results of Dubey et. al. Due to them we know that a market where current prices are observable does not in general have a NE. In order for such an equilibrium to exist the strategies of the uninformed must be independent of the current price. It is however not always necessary to forbid current price observations for this to be the case, since as we will see the optimal strategy may be to not condition on any price.

As mentioned, we will also see in this paper that if the uninformed cannot observe current prices they will just demand a fixed number of assets. The result of Hellwig (1982) is quite different. Hellwig's paper is frequently cited and applied on different areas (for example Boswijk, Hommes, and Manzan (2003), Kirchler and Huber (2005), Chamley (2003) , Blume, Easley, and O'Hara (1994)). However, it requires very special assumptions about the demand functions of the uninformed. This makes Hellwig's model incomparable to those of GS and SS, and so we will not spend much time on it. The proximity and popularity of this work does however require a few comments on the main problem of the Hellwig model, and why these problems are not present in Shapley-Shubik.

In Hellwig (1982) uninformed investors trade actively even though they

cannot condition on current prices and noise traders are kept out of the market place. As the time difference between price observations goes to zero, Hellwig show that there are benefits to being informed, as the market approaches full efficiency in the sense that current prices can be observed.

What drives these results are inconsistent demand functions not related to a utility function. This has also been noted by Dubey, Geanakoplos, and Shubik (1987). The demand is assumed to be proportional to the difference between expected price and the unobserved current price. This has the effect that the uninformed in Hellwig's model reduce demand when "good news" push up the current price and vice versa. Admitting to such a demand function thus ensures that the uninformed always make losing bets, which is clearly not rational. Mathematically the problem is that the current price can not be present in the expected utility, and thereby in the demand function, unless it is a known variable or a known variable depends on it.

In the Shapley–Shubik model one can argue that institutional constraints determine how bids can be made. In particular investors are restricted to bid in terms of costs and so their strategy (the amount of money they will invest) is independent of the current price. Furthermore, even though the uninformed do worse than the informed they do not persistently lose, but rather earns a little less than the informed.

The aim of this paper is somewhat different than learning models such as Blume and Easley (1984), Bray and Kreps (1987), Feldman (1987) and Routledge (1999), and surveyed in Blume and Easley (1992). In such models the objective is often to show how prices converge to the fundamentals over time. Although the model lends itself to such analysis with some extra assumptions about the fundamental process, that will not be an issue here.

In the models presented here it is assumed a continuum of competitive investors. Different results will apply if that assumption is relaxed, such as in Milgrom (1981), Jackson (1991) and Gottardi and Serrano (2006). Also Dubey, Geanakoplos, and Shubik (1987) have an example where investors

are strategic.

The plan of the paper is as follows: First we give a motivation for the model presented here, as it diverges from previous literature in some key aspects. In the subsequent section, the model is presented. In section three, the tree examples are presented and commented on. In the final section a short summary is given and conclusions drawn.

I Important features of the model

The model presented here sets it apart from other previous work by some special features commented on here

A. A short term model with no dividends

The model assumes a very short time span, because we are investigating the notion that investor may not be able to observe current prices. The idea that investors cannot observe current prices does not seem appropriate if each period is, say, one year. It might happen that an "annual trader" does not observe his transaction price, but when a year has gone by that really does not matter much.

We therefore assume that no dividend payments occur within the time span of the model.

It does however seem common in the literature to model the uncertainty in dynamic models as a dividend process which uninformed investors then try to predict (for example Hellwig (1982), Singleton (1987) and Routledge (1999)). This may be mathematically convenient and it works fine in a long term model, but it is not a very reasonable assumption in a short term model where each period is, say, one day. It does not work as an abstraction either, unless one could easily abandon explicit dividend payments without affecting the main results. This is however usually not the case, so for a very short

term model it seems more appropriate and safer to discard dividend payments entirely.

B. Fundamentals

Since there are no dividend payments, uncertainty stems from an underlying fundamental process running in a finite time span. At the terminal date the asset pays an amount equal to the fundamental process. Different interpretations can be made here. One is that a growing informational imbalance in the market is initiated at time $t = 1$, for example right after a quarterly result has been announced. Then at $t = T$ a new quarterly result is presented and all information is again public. This interpretation requires the conjuncture that the market is efficient at time T in the sense that when the fundamental process is public knowledge, then the price is equal to the fundamental value with probability one.

The martingale property of asset prices means that the finite time span is a valid simplification of the model. In addition it is also an exact representation of many derivatives.

C. Conditioning on the full history of prices

The assumption of a finite time span of course implies that the history that the investors can condition on is assumed to be finite. We do however allow investors to condition on the full history of prices, and the number of periods can be any positive integer. The results are therefore fairly general in this respect.

II The model

The market consists of two types of risk averse investors, informed and uninformed. We assume for simplicity a zero interest rate, although changing

this would not affect the main results. The myopic investors have demand functions that are proportional to the expected payoff:

$$z_{i,t} = \alpha_{i,t} \mathbf{E} [\Delta p_{t+1} | F_{i,t}] \quad (1)$$

Where $\Delta p_{t+1} = p_{t+1} - p_t$ is the absolute price difference, the expected excess capital gain and $F_{i,t}$ the information available to investor i at time t . (1) is a well established demand function in asset pricing literature (Grossman and Stiglitz (1980)). Usually it is derived from the CARA utility function, so that $\alpha_{i,t} = 1/\gamma_{i,t} \text{var} [\Delta p_{t+1} | F_{i,t}]$, where $\gamma_{i,t}$ is the coefficient of risk aversion. It is however mathematically much more convenient to not to explicitly let all parameters determining the conditional variance enter the demand functions and the equilibrium conditions. In the end, the equilibrium is determined by the relative weights of the random variables in the demand functions involved. Therefore explicitly solving for the variance parameters would require us to solve for parameters that are inherently not important for the equilibrium solutions.

This simplification means in effect that the solutions for the parameters in the model are not explicit solutions. This is not necessary though since, as we will see, any equilibrium can be determined by just assuming that $\alpha_{i,t}$ is some positive real number.

A. *The fundamental process and price process*

Define the fundamental value of the asset as

$$v_t = \mu + \boldsymbol{\theta}'_t \mathbf{1} \quad (2)$$

where $\boldsymbol{\theta}_t \sim N(0, \mathbf{I}\sigma_\theta^2)$ is a vector of independent random variables and $\mathbf{1}$ is a vector of ones of appropriate dimension, and μ is the terminal payoff expected at time $t = 1$. Assume further that at some final date T the asset

pays v_T .

$\boldsymbol{\theta}_t$ here is thus a vector of the independent increments in the assets value up to the current period t . We assume linear demand functions, and so at any time in the process up to t an equilibrium market price is established which is linear in the information available to some or all of the market participants:

$$p_t = a_t + \boldsymbol{\theta}'_t \mathbf{m}_{t,t} + \boldsymbol{\varepsilon}'_t \mathbf{s}_{t,t} \quad (3)$$

$\boldsymbol{\varepsilon}_t \sim N(0, \mathbf{I}\sigma_\varepsilon^2)$ are demands from noise traders up to time t and independent of $\boldsymbol{\theta}_t$. $\mathbf{m}_{j,t}$ and $\mathbf{s}_{j,t}$ will be referred to as the "price vectors", with prefix "fundamental" and "noise" respectively. We will see shortly that it is an advantage to use the notation $\mathbf{m}_{j,t}$ and $\mathbf{s}_{j,t}$ with two subscripts, where the first one denotes the length of the vector. Thus $\mathbf{m}_{j,t}$ is a vector at time t determining the impact of the first j fundamentals on the price. $\mathbf{s}_{j,t}$ likewise determine the impact of noise trading occurring in the first j periods, on the price at time t .

We allow the price to depend on all stochastic variables that have been observed by at least some investors, and we allow for the parameters to change over time. Furthermore, the price is allowed to depend on all random variables back to period $t = 1$. a_t is set endogenously, and takes account of risk aversion.

The realized profit in trading period $t + 1$ is then

$$\begin{aligned} \Delta p_{t+1} = & \Delta a_{t+1} + \boldsymbol{\theta}'_t \Delta \mathbf{m}_{t,t+1} + \boldsymbol{\varepsilon}'_t \Delta \mathbf{s}_{t,t+1} \\ & + s_{t+1,t+1} \varepsilon_{z,t+1,t+1} + m_{t+1,t+1} \theta_{t+1} \end{aligned} \quad (4)$$

where Δ is a difference operator yielding the difference between coefficients in the current period t and the last period $t - 1$. For example $\Delta \mathbf{m}_{t,t+1} = \mathbf{m}_{t,t+1} - \mathbf{m}_{t,t}$ is the change in the fundamental price vector.

B. *Informed investors*

Informed traders know $\boldsymbol{\theta}_t$ and total demand is observable. Knowing their own demand and that of the less informed, they are able to figure out the demand from noise traders $\boldsymbol{\varepsilon}_t$, which is equivalent to knowing p_t , but only the sufficient information set $\{\boldsymbol{\theta}_t, \boldsymbol{\varepsilon}_t\}$ is used.

The total demand from informed traders, after integrating (1) over the set I of such investors, is then $z_{I,t} = \alpha_{I,t} \mathbb{E}[\Delta p_{t+1} | \boldsymbol{\theta}_t, \boldsymbol{\varepsilon}_t]$ where $\alpha_{I,t} = \int_I \alpha_{i,t} d\mu(i)$. The expected return for these traders is

$$\mathbb{E}[\Delta p_{t+1} | \boldsymbol{\theta}_t, \boldsymbol{\varepsilon}_t] = \Delta a_{t+1} + \boldsymbol{\theta}'_t \Delta \mathbf{m}_{t,t+1} + \boldsymbol{\varepsilon}'_t \Delta \mathbf{s}_{t,t+1} \quad (5)$$

since the last two terms in (4) have expectation zero. Although it is assumed here that the informed observe the fundamentals $\boldsymbol{\theta}_t$ at date t , it does not matter much whether the actual realization of these fundamentals occur before or after this date. That will affect the date of the final payment v_T relative to the last period of the market, but this would merely be a mathematical technicality.

C. *Uninformed investors*

Uninformed investors know only the first $t - \tau$ prices. A fraction of the market are uninformed investors. Integrating (1) over the set U of such investors then gives the total demand of $z_{U,t} = \alpha_{U,t} \mathbb{E}[\Delta p_t | \mathbf{p}_{t-\tau}]$ where $\alpha_{U,t} = \int_U \alpha_{i,t} d\mu(i)$. $\tau = 0$ if the uninformed can observe current prices, and $\tau > 0$ otherwise. Furthermore \mathbf{p}_t is the vector of all previous prices up to t defined as:

$$\mathbf{p}_t = \mathbf{a}_t + \boldsymbol{\theta}'_t \mathbf{M}_t + \boldsymbol{\varepsilon}'_t \mathbf{S}_t \quad (6)$$

where $\mathbf{M}_t = \{\mathbf{m}_{0,0}, \mathbf{m}_{1,1}, \dots, \mathbf{m}_{t,t}\}$ and $\mathbf{S}_t = \{\mathbf{s}_{0,0}, \mathbf{s}_{1,1}, \dots, \mathbf{s}_{t,t}\}$ are matrices of the price vectors with redundant elements set to zero, so that

$$\mathbf{M}_t = \begin{pmatrix} m_{0,0} & m_{0,1} & \cdots & m_{0,t} \\ 0 & m_{1,1} & \cdots & m_{1,t} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & m_{t,t} \end{pmatrix}, \quad \mathbf{S}_t = \begin{pmatrix} s_{0,0} & s_{0,1} & \cdots & s_{0,t} \\ 0 & s_{1,1} & \cdots & s_{1,t} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & s_{t,t} \end{pmatrix} \quad (7)$$

and $\mathbf{a}_t = \{a_0, \dots, a_t\}$. \mathbf{M}_t and \mathbf{S}_t will be denoted "price matrices". The uninformed now assigns weights $\mathbf{g}_{t-\tau}$ to the prices that she observe throughout history¹. We will denote these weights as the "regression coefficients". Thus

$$\mathbb{E}[\Delta p_{t+1} | \mathbf{p}_{t-\tau}] = b_t + (\mathbf{p}_{t-\tau} - \mathbf{a}_{t-\tau}) \mathbf{g}_{t-\tau} \quad (8)$$

where the constant terms \mathbf{a}_t are removed from the prices for notational convenience. b_t is a deterministic term allowing for risk aversion. Since we do not solve for the variance, we assume of course that it is known by all market participants at any time t , but we allow it to vary arbitrarily over time. Hence b_t may not be constant.

We see from (8) that the uninformed is assigning coefficients to the information available in the market at $t - \tau$, with the restriction that the relationship between $\boldsymbol{\theta}'_{t-\tau}$ and $\boldsymbol{\varepsilon}'_{t-\tau}$ is given by the price matrices. It is mathematically easier to define these coefficients. We therefore define a vector $\mathbf{h}_{t-\tau}$ that represents the impact of fundamentals on the expectation (8). A vector $\mathbf{c}_{t-\tau}$, which is a linear function of $\mathbf{h}_{t-\tau}$, then represents the associated impact from noise. We can now write the expected return (8) as

$$\mathbb{E}[\Delta p_{t+1} | \mathbf{p}_{t-\tau}] = b_t + (\boldsymbol{\theta}'_{t-\tau} \mathbf{h}_{t-\tau} + \boldsymbol{\varepsilon}'_{t-\tau} \mathbf{c}_{t-\tau}) \quad (9)$$

¹One can argue that the uninformed should condition on the price *changes*. However, allowing the investors to freely choose a weight $g_{t,k}$ on each price is less restrictive. \mathbf{g}_t could anyway be set so that it implied differences in prices if this was optimal.

where $\mathbf{h}_{t-\tau}$ will be referred to as the "direct regression coefficients". If we substitute (6) into (8) and compare that to the equivalent expectation (9), we see that $\mathbf{h}_{t-\tau} = \mathbf{M}_{t-\tau}\mathbf{g}_{t-\tau}$ and $\mathbf{c}_{t-\tau} = \mathbf{S}_{t-\tau}\mathbf{g}_{t-\tau}$. This² in turn implies

$$\mathbf{g}_{t-\tau} = \mathbf{M}_{t-\tau}^{-1}\mathbf{h}_{t-\tau} \quad (10)$$

$$\mathbf{c}_{t-\tau} = \mathbf{S}_{t-\tau}\mathbf{M}_{t-\tau}^{-1}\mathbf{h}_{t-\tau} \quad (11)$$

assuming $\mathbf{M}_{t-\tau}$ is not singular. We can now restate price expectation of the uninformed as

$$\mathbb{E}[\Delta p_{t+1}|\mathbf{p}_{t-\tau}] = b_t + (\boldsymbol{\theta}'_{t-\tau} + \boldsymbol{\varepsilon}'_{t-\tau}\mathbf{S}_{t-\tau}\mathbf{M}_{t-\tau}^{-1})\mathbf{h}_{t-\tau} \quad (12)$$

D. Efficient profit estimate

The uninformed set the coefficients b_t and $\mathbf{h}_{t-\tau}$ by minimizing the expected squared difference between the expected and realized price, e.g. obtaining the least squares coefficients:

$$\min_{b_t, \mathbf{h}_{t-\tau}} L = \mathbb{E}(\mathbb{E}[\Delta p_{t+1}|\mathbf{p}_{t-\tau}] - \Delta p_{t+1})^2 \quad (13)$$

It can be found that the optimal parameters that minimizes this are

$$b_t^* = \Delta a_t \quad (14)$$

$$\begin{aligned} \mathbf{h}_{t-\tau}^* &= \mathbf{M}_{t-\tau} (\mathbf{M}'_{t-\tau}\mathbf{M}_{t-\tau}\sigma_\theta^2 + \mathbf{S}'_{t-\tau}\mathbf{S}_{t-\tau}\sigma_\varepsilon^2)^{-1} \\ &\quad \cdot (\mathbf{M}'_{t-\tau}\Delta\mathbf{m}_{t-\tau,t+1}\sigma_\theta^2 + \mathbf{S}'_{t-\tau}\Delta\mathbf{s}_{t-\tau,t+1}\sigma_\varepsilon^2) \end{aligned} \quad (15)$$

The expected profit and hence demand from the uninformed is now found by substituting (14) and (15) into (12). A proof is found in the Appendix.

²There are no restrictions on b_t since the constant term was removed from the price in (8)

E. Equilibrium condition

Define the total demand as

$$D_{t-\tau,t} = \alpha_{U,t} \mathbb{E} [\Delta p_{t+1} | \mathbf{p}_{t-\tau}] + \alpha_{I,t} \mathbb{E} [\Delta p_{t+1} | \boldsymbol{\theta}_t, \boldsymbol{\varepsilon}_t] \quad (16)$$

The total supply is a linear function of $\boldsymbol{\theta}_t$ and $\varepsilon_{z,t}$, $Z_t(\boldsymbol{\theta}_t, \varepsilon_{z,t})$, and will be defined explicitly in each example. In GS it typically depends only on the current noisy demand $\varepsilon_{z,t}$. In equilibrium, supply equals demand, so

$$D_{t-\tau,t} = Z_t(\boldsymbol{\theta}_t, \varepsilon_t) \quad (17)$$

The general specification of the supply side makes our model very general. By specifying different assumptions about the supply side of the market $Z_t(\boldsymbol{\theta}_t, \varepsilon_{z,t})$ and the price observation lag τ we can identify exactly the reason for different results in a range of models. In this paper we will compare two alternative specifications of $Z_t(\boldsymbol{\theta}_t, \varepsilon_{z,t})$, and models with positive and zero τ .

Since the supply side of the market is a model choice, it is assumed to only depend on current noise traders ε_t here. A model where it also depends on previous noise trading can easily be incorporated if that would be of any interest though.

The equilibrium condition must hold for any realization of the random variables. A necessary condition, for (17) to hold for any realization of $\boldsymbol{\theta}'_t$ and $\boldsymbol{\varepsilon}_t$ is that it holds for any marginal change in the random variables. The equilibrium condition therefore implies:

$$\partial D_{t-\tau,t} / \partial \{\boldsymbol{\theta}'_t, \boldsymbol{\varepsilon}_t\} = Z_t(\boldsymbol{\theta}_t, \varepsilon_t) / \partial \{\boldsymbol{\theta}'_t, \boldsymbol{\varepsilon}_t\} \quad (18)$$

There will in general be an equilibrium as long as there are at least $2t+1$ parameters, less the number of cases where both sides are zero, since one can always define some price function that satisfies any equilibrium given that it can span all relevant variables. It is however more convenient to prove

existence in each example below, when the function $Z_t(\boldsymbol{\theta}_t, \varepsilon_t)$ is defined.

F. The value of information

Since

1. Profits are normally distributed and hence completely described by the first two central moments of Δp_{t+1} given some information set $F_{i,t}$ known to investor of type $i \in \{U, I\}$ at time t .

2. Before observation of the information $F_{i,t}$, the first moment is the same for the informed and uninformed due to the law of iterated expectations, $\mathbb{E}[\mathbb{E}[\Delta p_{t+1}|F_{I,t}]|F_{U,t}] = \mathbb{E}[\Delta p_{t+1}|F_{U,t}]$, where in this case $F_{I,t} = \{\theta_t, \varepsilon_t\}$ and $F_{U,t} = \mathbf{p}_{t-\tau}$.

3. By standard assumptions all higher central moments, e.g. the conditional variances, are public knowledge, and more information cannot decrease precision, so $\mathbb{E}[\text{var}[\Delta p_{t+1}|F_{I,t}]|F_{I,t}] = \text{var}[\Delta p_{t+1}|F_{I,t}] \leq \text{var}[\Delta p_{t+1}|F_{U,t}]$.

It follows that the difference in expected utilities prior to observing the information $F_{I,t}$, are completely determined by the second central moments $\text{var}[\Delta p_{t+1}|F_{i,t}]$. Since investors are risk averse we need only to consider

$$c_{I,t} = \text{var}(\Delta p_{t+1}|F_{U,t}) - \text{var}(\Delta p_{t+1}|F_{I,t}) \quad (19)$$

as a measure of information cost. Note that this does not require identical risk aversion, since the cost of information is what the same individual would be willing to pay to become informed. This measure is simple, sufficient for our purposes, and equivalent to the accurate cost (as derived in GS) for ordering.

III The examples

We now have the basic model in place, so that the three examples can be presented.

A. *Example 1 - Grossman-Stiglitz with unobservable current prices*

In this example we assume that investors cannot condition on current prices. It will be shown that this modification has no impact on the results originally found by GS, and that prices carry no payoff relevant information. In this example there is a fixed number of assets in supply z , in addition to some demand from noise traders ε_t , so that $Z_t(\boldsymbol{\theta}_t, \varepsilon_{z,t}) = z + \varepsilon_t$.

Using the equilibrium condition (18) we can now calculate the *first* $t - \tau$ derivatives after inserting the expectations (5) and (12) into (16) and substituting the solution for $\mathbf{h}_{t-\tau}^*$ from (15). We then get that the equilibrium condition requires that for the *first* $t - \tau$ periods, provided $\tau \geq 1$, we must have

$$\Delta \mathbf{m}_{t-\tau, t+1} = -\frac{\alpha_{U,t}}{\alpha_{I,t}} \mathbf{h}_{t-\tau}^* \quad (20)$$

$$\Delta \mathbf{s}_{t-\tau, t+1} = \mathbf{S}_{t-\tau} \mathbf{M}_{t-\tau}^{-1} \left(-\frac{\alpha_{U,t}}{\alpha_{I,t}} \mathbf{h}_{t-\tau}^* \right) \quad (21)$$

We can now state the following proposition:

Proposition 1 *If investors are unable to observe current prices, $\tau \geq 1$, no uninformed investor will condition on past prices*

Proof: Since (20) and (21) have common terms, we can write $\Delta \mathbf{s}_{t-\tau, t+1}$ in terms of $\Delta \mathbf{m}_{t-\tau, t+1}$ as

$$\Delta \mathbf{s}_{t-\tau, t+1} = \mathbf{S}_{t-\tau} \mathbf{M}_{t-\tau}^{-1} \Delta \mathbf{m}_{t-\tau, t+1} \quad (22)$$

Substituting this expression for $\Delta \mathbf{m}_{t-\tau, t+1}$ into the expression for $\mathbf{h}_{t-\tau}^*$ (15) yields $\mathbf{h}_{t-\tau}^* = \Delta \mathbf{m}_{t-\tau, t+1}$. Using (21) we find that in equilibrium $\Delta \mathbf{m}_{t-\tau, t+1} = -\Delta \mathbf{m}_{t-\tau, t+1} (\alpha_{U,t}/\alpha_{I,t})$. This can only hold if $\Delta \mathbf{m}_{t-\tau, t+1} = \mathbf{0}$, since $\alpha_{U,t} > 0$ and $\alpha_{I,t} > 0$. By (22) it follows that $\Delta \mathbf{s}_{t-\tau, t+1} = \mathbf{0}$ as well ■

1 General solution

From the equilibrium condition (18) and Proposition 1 it follows that we can rewrite the entire price vectors at any date t up to $t - 1$ as

$$\Delta \mathbf{m}_{t-1,t} = \mathbf{0} \quad (23)$$

$$\Delta \mathbf{s}_{t-1,t} = \frac{1}{\alpha_{I,t}} \mathbf{e} \quad (24)$$

where $\mathbf{e}' = \{0, 0, \dots, 0, 1\}$.

Since at the terminal date $p_T = v_T = \mu + \boldsymbol{\theta}'_T \mathbf{1}$, it must be the case that $\mathbf{s}_{T,T} = \mathbf{0}$ and $\mathbf{m}_{T,T} = \mathbf{1}$. By backwards induction using (23) and (24) it follows that the price matrix \mathbf{M}_T is a matrix with the upper right triangular filled with ones. That is

$$\mathbf{m}_{t,t} = \mathbf{1} \forall t \leq T \quad (25)$$

The important point here is that except for the impact of noise trading, current prices will always reflect current fundamentals perfectly. The fundamental price vector is always $\mathbf{m}_{t,t} = \mathbf{1}$, which is the same as the associated vector in the fundamental process v_t .

Furthermore \mathbf{S}_T must be a diagonal matrix with $-1/\alpha_{I,t}$ along the diagonal, except $s_{T,T} = 0$. The elements in the diagonal are found by rewriting the last element of (21) as $s_{t,t} = 1/\alpha_{I,t} + s_{t,t+1}$. For it to be the case that $\mathbf{s}_{T,T} = \mathbf{0}$, backwards induction therefore again implies that

$$s_{T,T} = 0, \mathbf{s}_{t,t} = -\frac{1}{\alpha_{I,t}} \mathbf{e} \forall t < T \quad (26)$$

The intuition behind this is that as soon as the informed gets to know θ_t , prices map this perfectly due to competition among the informed. Therefore, the next period profit, Δp_{t+1} , will be completely independent of current private information $\boldsymbol{\theta}_t$ and perfectly incorporate the unknown next period innovation in the fundamental θ_{t+1} . Thus, even though the past prices do

depend on the first $t - \tau$ elements of $\boldsymbol{\theta}_t$ and so a fairly good estimate of $\boldsymbol{\theta}_t$ can be made, that does not help the uninformed a bit since it is not related to next period profits.

The next period profits are however affected by the noise trading by a factor of $-1/\alpha_{I,t}$, as noise trading push up current prices. That will however only affect the unobserved current price, and so the uninformed traders cannot participate in the exploitation of noise traders.

Interestingly this is not the case in the original GS, where the uninformed observe current prices and therefore can participate in noise trader exploitation.

2 Nash equilibrium

We will here show that a model with a GS type supply does not suffer from the game theoretical problems described in Dubey, Geanakoplos, and Shubik (1987). These problems occur in the demand of the uninformed, when there are all ready some fraction of informed and uninformed traders in the market. The problems are not directly related to the decision to buy information. We will therefore consider the game where the strategies available for the uninformed are the determination of the regression coefficients $\mathbf{g}_{t-\tau}$. For the informed the strategy is to select fundamental and noise parameters. The payoff is the next period per share price increment.

As shown, the optimal strategy is for the informed to choose $\Delta \mathbf{m}_{t,t+1}$ and $\Delta \mathbf{s}_{t,t+1}$, and for the uninformed to choose $\mathbf{g}_{t-\tau}^* = 0$ and so the demand by the uninformed will be some fixed, state independent amount $b_t^* = \frac{\alpha_{I,t}}{\alpha_{U,t}} \Delta a_{t+1}$ each period. Since the demand of the uninformed is independent of the price, this is a fixed point determining a given price by the informed demand for the realization of $(\boldsymbol{\theta}_t, \varepsilon_{z,t})$, which no individual investor can improve upon (by the previous proof and derivation), and hence the equilibrium is a pure strategy Nash equilibrium. Further more it follows from the proof that this is a unique Nash equilibrium.

3 The value of information

The informed return estimate $E[\Delta p_{t+1}|\theta_t, \varepsilon_t]$ in (5) and the realized price (4) differ only by the last two terms, for which coefficients are known by (25) and (26). Thus the conditional variance of the informed is $\text{var}[\Delta p_{t+1}|\theta_t, \varepsilon_t] = \sigma_z^2/\alpha_{I,t}^2 + \sigma_\theta^2$. The uninformed demands a constant quantity $b_t^* = \Delta a_{t+1}$, and hence using (4) it can be found that $\text{var}[\Delta p_{t+1}|\mathbf{p}_{t-\tau}] = 2\sigma_z^2/\alpha_{I,t}^2 + \sigma_\theta^2$. It follows that using our measure (19) the cost of information is

$$c_{I,t} = \sigma_z^2/\alpha_{I,t}^2 \quad (27)$$

Thus, as long as we have noise traders and $\sigma_z^2 > 0$, there is an advantage in being informed. If however the market is efficient in the sense that $\sigma_z^2 = 0$, then the Grossman-Stiglitz paradox arises. We can now restate the findings of the current and the last section as

Proposition 2 *If investors cannot condition on current prices, they will submit fixed demands Δa_{t+1} and the REE is consistent with a pure strategy NE. In absence of noise traders, $\sigma_z^2 = 0$, the Grossman-Stiglitz paradox still arises.*

The proof follows from the previous discussion and so is omitted.

B. Example 2 - Grossman-Stiglitz with observable current prices

This example corresponds to a dynamic version of the original GS model. Similar models were introduced by Brown and Jennings (1989) and Grundy and McNichols (1989). They found that past prices do in fact carry information in a noisy REE model. The reason is that the fundamental price vector will now change over time, so that the price maps the fundamentals differently in each period. Each price observation therefore improves the estimate of the fundamental value v_t . If the total number of independent fundamental

increments and noise terms are less than t , then past prices reveal all private information.

Importantly, the equilibrium obtained will not generally be a NE. Comparing the result of this example with Example 1 therefore suggests that the results of Brown and Jennings (1989) and Grundy and McNichols (1989) may be incompatible with a NE.

If uninformed can observe current prices, then $\tau = 0$. As in Example 1 supply depends solely on noise trading, $Z_t(\boldsymbol{\theta}_t, \varepsilon_{z,t}) = z + \varepsilon_{z,t}$. The equilibrium conditions (18) then requires

$$\Delta \mathbf{m}_{t,t+1} = -\frac{\alpha_{U,t}}{\alpha_{I,t}} \mathbf{h}_t^* \quad (28)$$

$$\Delta \mathbf{s}_{t,t+1} - \frac{1}{\alpha_{I,t}} \mathbf{e} = \mathbf{S}_t \mathbf{M}_t^{-1} \left(-\frac{\alpha_{U,t}}{\alpha_{I,t}} \mathbf{h}_t^* \right) \quad (29)$$

We note the similarity between the equilibrium conditions where only past prices are observed (20) and (21) with the conditions (28) and (29) above. The main difference is that in the last period t the disturbance term $\varepsilon_{z,t}$ enters so that we get an additional term on the left side of the last equation in (29) of $1/\alpha_{I,t}$.

As in the previous case we can now substitute (28) into (29) by the common term, to get

$$\Delta \mathbf{s}_{t,t+1} = \frac{1}{\alpha_{I,t}} \mathbf{e} + \mathbf{S}_t \mathbf{M}_t^{-1} \Delta \mathbf{m}_{t,t+1} \quad (30)$$

Substituting this into the expression for \mathbf{h}_t^* in (15) using (28) yields

$$\mathbf{h}_t^* = \Delta \mathbf{m}_{t,t+1} + \mathbf{M}_t (\mathbf{M}'_t \mathbf{M}_t \sigma_\theta^2 + \mathbf{S}'_t \mathbf{S}_t \sigma_\varepsilon^2)^{-1} \mathbf{S}'_t \mathbf{e} \frac{1}{\alpha_{I,t}} \sigma_\varepsilon^2 \quad (31)$$

If we now pre-multiply (28) with \mathbf{M}_t^{-1} we can use that $\mathbf{M}_t^{-1} \mathbf{h}_t^* = \mathbf{g}_t$ from

(10) to find that

$$\mathbf{M}_t^{-1} \Delta \mathbf{m}_{t,t+1} = -\frac{\alpha_{U,t}}{\alpha_{I,t}} \mathbf{g}_t \quad (32)$$

Therefore in equilibrium, after pre-multiplying (31) with \mathbf{M}_t^{-1} and rearranging, the regression coefficients in period t can be found to be

$$\mathbf{g}_t = \frac{1}{\alpha_{I,t} + \alpha_{U,t}} (\mathbf{M}'_t \mathbf{M}_t \sigma_\theta^2 + \mathbf{S}'_t \mathbf{S}_t \sigma_\varepsilon^2)^{-1} \mathbf{S}'_t \mathbf{e} \sigma_\varepsilon^2 \quad (33)$$

$$(m_{t,t}^2 \sigma_\theta^2 + s_{t,t}^2 \sigma_\varepsilon^2) g_t = \frac{1}{\alpha_{I,t} + \alpha_{U,t}} g_t s_{t,t} \sigma_\varepsilon^2 \quad (34)$$

where we have substituted $\mathbf{M}_t^{-1} \Delta \mathbf{m}_{t,t+1}$ for $-\frac{\alpha_{U,t}}{\alpha_{I,t}} \mathbf{g}_t$ by using (28).

We see that (33) does not immediately seem to be a vector proportional to \mathbf{e} . That is, the uninformed do not seem to condition only on the current price. This is not the case either, and so we have the following proposition

Proposition 3 *If uninformed investors can observe current prices, they will also condition on past prices in equilibrium and past prices have informational value. That is $\mathbf{g}_t \neq \mathbf{e}k$ for any k .*

The reader is referred to the Appendix for the proof. See Brown and Jennings (1989) or Grundy and McNichols (1989) for proofs of the three period case.

Interestingly it is not the fundamentals that initially makes the current price useful for the uninformed in equilibrium, but the noise trading. Say that the uninformed did not condition on any price, as in Example 1 so that $\Delta \mathbf{m}_{t,t+1} = \mathbf{0}$. In that case the *informed* investors would in effect establish a price that depends solely on the noise trading. Thus the advantage of the informed in Example 1 is not the ability to predict next period prices, but to exploit the noise traders. In that case any noise trading is revealed by the price.

Therefore, trying to reveal the noise trading is in effect an incentive for the uninformed to mess up the nice constant price vector process we saw

in Example 1. If $\mathbf{m}_{t,t}$ is not constant, i.e. $\mathbf{m}_{t-1,t} \neq \mathbf{m}_{t-1,t-1}$, it has two important effects. First it means that current price innovations are no longer independent of past and current fundamentals, and so predicting $\boldsymbol{\theta}_t$ becomes *useful*. Second if the price vectors at different dates become independent, it also *enables* such prediction.

1 The value of information

As in the previous example, due to the common parameters in (5) and (4) it is the case that $\text{var}[\Delta p_{t+1}|\theta_t, \boldsymbol{\varepsilon}_t] = s_{t+1,t+1}\sigma_z^2 + m_{t+1,t+1}\sigma_\theta^2$. It can be found that the additional variance that the uninformed face is

$$c_{I,t} = \frac{\sigma_\varepsilon}{\alpha_{I,t}^2} \frac{|(\mathbf{M}'_t \mathbf{M}_t \sigma_\theta^2 + \mathbf{S}'_t \mathbf{S}_t \sigma_\varepsilon^2) - \sigma_\varepsilon^2 \mathbf{S}'_t \mathbf{e} \mathbf{e}' \mathbf{S}_t|}{|\mathbf{M}'_t \mathbf{M}_t \sigma_\theta^2 + \mathbf{S}'_t \mathbf{S}_t \sigma_\varepsilon^2|} > 0 \quad (35)$$

As we can see, the information has value as long as there are noise traders, $\sigma_\varepsilon > 0$. An interesting point here is however that if $\sigma_\varepsilon = 0$, it is not only the value of information that becomes zero. So too do the regression coefficients \mathbf{g}_t , by (33) and so uninformed investors will demand the fixed quantity $b_t^* = \Delta a_{t+1}$. Thus, if the market is fully efficient, it is actually consistent with a NE.

The reason is that there is no need for the investors in an efficient market to even consider the current price. Optimal bids by the informed ensure that the price always incorporates all available information. Uninformed investors just calculate a fixed number of assets and buy these, using a market order, at any price. This is of course possible to see from the original Grossman-Stiglitz model, although they did not explicitly state it since it was not an issue then.

The fact that investors in this model should condition on past prices in the presence of noise traders, has the further implication that holding fixed index portfolios is not optimal for the uninformed unless the market is fully efficient. How exactly these portfolios should be formed is however not easy

to say, even when the distribution of the stochastic variables are known.

The reason is that a general solution to the system involves $T(T - 1)$ equations of fourth degree polynomials. There might therefore very well exist multiple equilibria here as Grundy and McNichols (1989) found in the simpler three period case. In the general case these equilibria may possibly be indistinguishable for the market participants. To sort out how the market actually works in general is therefore a difficult task, even in the simple world considered here. What this model tells us is therefore that the uninformed should condition their demand on past prices, but not how to do it. For practical advice it may therefore be better to rely on the model from Example 1 and hold the market portfolio, unless you are informed.

C. Example 3 - The Shapley–Shubik model adapted to the Grossman–Stiglitz framework

In the original paper Shapley–Shubik was used to characterize an economy where payments of one good are done in terms of other goods. The model is however somewhat difficult to interpret when applied to financial markets, since it was not specifically designed to resemble such market institutions. The fact that results are so different from a market with delayed price information, such as Example 1 in this paper, does however suggest that the Shapley–Shubik model assumes a very different kind of bidding procedure.

It will be argued here that a reasonable interpretation of the model is that investors are restricted to bid in terms of total cost of the assets, and not units. This interpretation rests on three arguments. First, this is consistent with a simplified Shapley–Shubik model. Second, this interpretation is also relatively easy to incorporate in the dynamic GS model presented here. Third, the main results of the Shapley–Shubik model, such as the relationship with REE models found in Jackson and Peck (1999), are preserved. It should be noted that in contrast to this paper, there is a countable number of traders who are risk neutral in Jackson and Peck (1999). Their main results

are however not sensitive to this difference in specification.

In order to see that the interpretation made is consistent with the original Shapley–Shubik market, we will take a look at a very simplified version of it. Notation in the next section will be made similar to that used by Shapley and Shubik (1977), and therefore deviates from the rest of the paper.

1 A simplified Shapley–Shubik for financial markets

The Shapley-Shubik model can be used to find both strategic and competitive equilibria. Here agents are assumed to be price takers, so the asset price P is independent of their actions. The results from the Shapley-Shubik model that conflict with the REE as noted by Dubey, Geanakoplos, and Shubik (1987) relates to the competitive case, so only price taking behavior is of interest here.

There are only two goods, and consumption occurs only next period. The first good is cash, indicated by subscript c , the other is a financial asset, indicated by subscript θ . The cash pays no return, while the financial asset pays the random amount $V_2 = \Theta_1 \Theta_2 e^\mu$ of cash next period, where and

$$\ln \Theta_i \equiv \theta_i \sim N(0, \sigma_\theta^2), i \in \{1, 2\} \quad (36)$$

independently and $v_2 = \theta_1 + \theta_2 + \mu$. Next period returns Θ_2 is not known by anyone, but some informed traders know Θ_1 and therefore also the current price (since this is determined by the information in the market). The current price is not known to the uninformed.

"Cash" plays the role of the numeraire commodity. It can be stored for next period and used for consumption, but pays no return. The initial endowments of cash are zero, $a_{c,i} = 0$, since we do not restrict credit and the initial endowment of the numeraire commodity therefore does not matter.

Initially all endowments of assets, $a_{\theta,i}$, are put out for sale in the market. The total supply of assets is therefore $\sum_{i=1}^N a_{\theta,i} = \bar{a}_\theta$. Investors then bid

in terms of cash for these assets, each placing a bid of $b_{\theta,i}$. The total bids amount to $\sum_{i=1}^N b_{\theta,i} = \bar{b}_\theta$ cash. Since credit is not restricted, the total bids do not (necessarily) sum to zero³. As is standard in the Shapley-Shubik model the price P is the number of cash needed to buy one asset. This price will depend on the realization of Θ_1 through demand \bar{b}_θ and is hence a random variable.

After trade, the uninformed receives $x_{\theta,i} = b_{\theta,i}/P$ units of the asset. Investors also receive $a_{\theta,i}P$ cash for their initial asset endowments, and pays $b_{\theta,i}$ cash for the assets, so the number of cash held after trading is $x_{c,i} = a_{\theta,i}P - b_{\theta,i}$.

Next period wealth in terms of cash is therefore the payment from the $x_{\theta,i} = b_{\theta,i}/P$ units of the asset, each paying $\Theta_1\Theta_2e^\mu$. In addition the agents have $x_{c,i}$ of cash, so next period wealth is

$$\omega_i = b_{\theta,i} \left(\frac{\Theta_1\Theta_2e^\mu}{P} - 1 \right) + a_{\theta,i}P \quad (37)$$

As a sum of a constant and log normal variables, neither the distribution of ω_i nor $\Theta_1\Theta_2e^\mu/P - 1$ have closed forms. This precludes a closed form solution without simplifying approximations. In fact deriving closed form solutions to the equilibrium in the Shapley-Shubik with continuous distributions is always a problem due to the inverted price in (37). This is usually overcome by assuming risk neutrality or discrete distributions.

The problem is handled here by assuming as in the previous sections that investors are mean-variance maximizers and hence the demand functions in terms of cash $b_{\theta,i}^*$ is proportional to the expected return on the investment, which is the conditional expectation of $\Theta_1\Theta_2e^\mu/P - 1$. For the informed the demand is then

$$b_{\theta,i \in I}^* = \alpha_{I,i} \left(\exp \left(\mathbb{E} (v_2 - p|\theta_1) - \frac{1}{2}\sigma_{\theta_1}^2 \right) - 1 \right) \quad (38)$$

³We do not consider the case with a zero price and no trade here.

where $\alpha_{I,i}$ is determined by attitude to risk and $\text{var}(\Theta_1\Theta_2e^\mu/P - 1|\theta_1 + \theta_2)$ which has no closed form expression. For the uninformed the demand is

$$b_{\theta,i \in U}^* = \alpha_{U,i} (\exp(\mathbb{E}(v_2 - p) - \sigma_{\theta_1}^2) - 1) \quad (39)$$

In equilibrium the sum of assets purchased by each agent, $x_{\theta,i}^* = b_{\theta,i}^*/P$, has to equal the total of endowments, which is \bar{a}_θ . Thus, we have the following equilibrium condition in number of assets is:

$$\int_I (b_{\theta,i}^*/P) d\mu(i) = \bar{a}_\theta \quad (40)$$

The usual solution for the price in a Shapley-Shubik model is then as we see the ratio of demand in cash \bar{b}_θ relative to the total endowments, \bar{a}_θ , so that $P = \bar{b}_\theta/\bar{a}_\theta$.

It is however an advantage to solve for the price function that satisfies (40) for any realization of the private information θ_1 . The easiest way to do this is to multiply the equilibrium in terms of assets (40) by the price P . Integrating over investors as shown in sections II.C and D this gives us the following equilibrium condition in terms of cash:

$$\alpha_I e^{\mathbb{E}(v_2|\theta_1)-p} + \alpha_U e^{\mathbb{E}(v_2-p) - \frac{1}{2}\sigma_{\theta_1}^2} = (\bar{a}_\theta P + \alpha_I + \alpha_U) e^{\frac{1}{2}\sigma_{\theta_1}^2} \quad (41)$$

By the equilibrium condition (18) it follows that the equilibrium price function is given by the solvable differential equation⁴

$$\alpha_I e^{\mu + (\theta_1 - p)} (1 - p'(\theta_1)) = e^{\frac{1}{2}\sigma_{\theta_1}^2} \bar{a}_\theta e^p p'(\theta_1) \quad (42)$$

Assume now that $\bar{a}_\theta = 0$. The conjecture that $P = \bar{b}_\theta/\bar{a}_\theta$ will of course not be valid in that case, but there will exist a price. It can be found that the unique real solution to the differential equation (18) in this case is

⁴With the unique real solution $p(\theta_1) = -\frac{1}{2}\sigma_{\theta_1}^2 + \ln \frac{C + \sqrt{e^{\frac{1}{2}\sigma_{\theta_1}^2 + \theta_1 + \mu} \alpha_I + C^2}}{2\bar{a}_\theta}$

$$p(\theta_1) = \theta_1 + C.$$

Proposition 4 *In the competitive Shapley-Shubik model with zero net supply of the asset and unobservable prices, the value of information is zero.*

Proof is omitted, as it is clear from the preceding that if the expected return is always constant, $p(\theta_1) - \theta_1 = C$, then the information θ_1 is redundant.

The result does not depend on the exact formulation of the Shapley-Shubik model here, except for the assumption of competitiveness. The reason one gets different results in the competitive Shapley-Shubik compared with REE models is that demands are stated in the numeraire commodity (e.g. dollars) rather than in units of the asset. When net supply is zero, the effect of this model feature disappears. Thus any model where demand is stated in terms of total value and not units will produce similar results, which we will see when we now extend this notion to the REE framework.

2 The Shapley-Shubik model adapted to the REE framework

Requiring demand to be stated in terms of total value in an REE model of the type presented here, is equivalent to defining the total supply to be proportional to the price. Hence we define the supply function to be

$$Z_t(\boldsymbol{\theta}_t, \varepsilon_t) = z_t p_t = z_t (\Delta a_t + \boldsymbol{\theta}'_t \mathbf{m}_{t,t}) \quad (43)$$

where we now assume no noise traders as in the Shapley-Shubik model. This specification will from here on be referred to as the Shapley-Shubik model.

Except for the noise trader terms, the optimal portfolios will be formed in the same way as before. It can be shown that it is optimal for the uninformed to set the optimal direct regression coefficients $\mathbf{h}_{t-\tau}^*$ to $\Delta \mathbf{m}_{t-\tau, t+1}$ (set $\sigma_\varepsilon^2 = 0$ in (15)) and the constant term to $b_t^* = \Delta a_{t+1}$, since observing the price at

any date t is now equivalent to observing θ_t . Thus expectations of the return can be formulated for both the informed and the uninformed as

$$\mathbb{E} [\Delta p_{t+1} | \mathbf{p}_{t-\tau}] = \Delta a_{t+1} + \boldsymbol{\theta}'_{t-\tau} \Delta \mathbf{m}_{t-\tau, t+1} \quad (44)$$

where $\tau > 0$ for the uninformed and $\tau = 0$ for the informed. Applying the equilibrium condition in (18) with respect to the observable prices τ periods ago, as we did in Example 1, we obtain the following solution for the fundamental coefficients:

$$\mathbf{m}_{t-\tau, t+1} = \mathbf{m}_{t-\tau, t} \frac{z_t + \alpha_{t,U} + \alpha_{t,I}}{\alpha_{t,I} + \alpha_{t,U}} \quad (45)$$

Define $\bar{\mathbf{m}}_{t-\tau, t}$ as the last τ elements of $\mathbf{m}_{t,t}$. Then total differentiation of (17) with respect to fundamentals only observable to the informed, implies that in this case

$$\bar{\mathbf{m}}_{t-\tau, t} = \bar{\mathbf{m}}_{t-\tau, t+1} \frac{\alpha_{t,I}}{z_t + \alpha_{t,I}} \quad (46)$$

Two important results now emerge

Proposition 5 *With the Shapley-Shubik model with positive net supply, $z_t > 0$, and a positive fraction of informed traders, $\alpha_{t,I} > 0$, it is the case that*

a) the price vector at date $t < T$ is strictly positive, but less than the unit vector, $\mathbf{0} \ll \mathbf{m}_{t,t} \ll \mathbf{1}$

b) its elements do not all have the same values, $\mathbf{m}_{t,t} \neq \mathbf{1}k$ for some scalar $k > 0$.

Proof: The proposition follows directly from the fact that at the terminal date v_T is paid, so $\mathbf{m}_{T,T} = \mathbf{1}$. At the preceding dates the price coefficients decrease from one, as we move backwards in time, as described by (45) and (46), but remain positive. Thus, a) in Proposition 5 must hold. The

coefficients do however change at a different pace depending on whether the associated fundamental value is publicly known or not. This ensures that b) in Proposition 5 holds ■

As in Jackson and Peck (1999) we will now take a look at the differences between the SS and GS.

3 Price efficiency

Jackson and Peck (1999) found that while prices, always perfectly reflect fundamentals in the noiseless REE of GS, this is not the case in the SS model. This is the case in the version of SS presented here too.

Prices can be said to perfectly reflect the fundamentals if the correlation between these is 100%. The equilibrium price in SS is $p_t = a_t + \boldsymbol{\theta}'_t \mathbf{m}_{t,t}$ and fundamentals are $v_t = \mu_t + \boldsymbol{\theta}'_t \mathbf{1}$. It can be found then that the correlation is

$$\rho_{p,\theta,t} = \frac{\mathbf{1}'\mathbf{m}_{t,t}}{\sqrt{t\mathbf{m}'_{t,t}\mathbf{m}_{t,t}}} \quad (47)$$

It can further be found that the only real solution to $(\mathbf{1}'\mathbf{m}_{t,t})^2 / \mathbf{m}'_{t,t}\mathbf{m}_{t,t} = t$, is $\mathbf{m}_{t,t} = k\mathbf{1}$ for some positive scalar k . For prices to reflect fundamentals perfectly, it therefore needs to be the case that $\mathbf{m}_{t,t} = k\mathbf{1}$. This is however not true due to Proposition 5 b). Thus, in contrast to Example 1 and 2 when there is no noise trading, prices in the Shapley-Shubik model do not reflect fundamentals perfectly.

The reason for this is that when uninformed investors are forced to post their bids in terms of total cost, they will consistently increase demand when the current fundamental θ_t is low and vice versa. This smoothes the total demand. The informed would otherwise have bid up the price to a level where next period profits was a predictable constant. With the counter effect from the total cost bidding of the uninformed that will not happen, and not all private information is incorporated into prices.

4 The value of information

For the informed in the SS model it can be found that the conditional variance is $m_{t+1,t+1}\sigma_\theta^2$, while for the uninformed it can be found to also include the term

$$c_{I,t} = \Delta\bar{\mathbf{m}}'_{t-\tau,t}\Delta\bar{\mathbf{m}}_{t-\tau,t}\sigma_\theta^2 > 0 \quad (48)$$

This implies that there is in deed an advantage in becoming informed. Since investors are willing to pay for information, there also exists an equilibrium in the market for information. In GS and the first two examples that is not the case when $\sigma_\varepsilon^2 = 0$.

5 Volatility of equilibrium prices and the fundamentals

Jackson and Peck (1999) also make the point that in SS it is possible for equilibrium prices to be more volatile than dividends. The same point can be made here, although we will look at the volatility of the underlying fundamentals θ_t , and not the dividends since such are not paid in this model until the terminal period.

The unconditional volatility of the price process is

$$\text{var} [\Delta p_{t+1}] = (\Delta\mathbf{m}'_{t,t}\Delta\mathbf{m}_{t,t} + m_{t+1,t+1}) \sigma_\theta^2 \quad (49)$$

while the underlying fundamentals change with θ_t each period, so $\text{var} [\Delta v_t] = \sigma_\theta^2$. We know from Proposition 5 however that $0 \ll \Delta\mathbf{m}_{t,t+1} \ll 1$ and $m_{t+1,t+1} < 1$. Therefore $\text{var} [\Delta p_{t+1}] > \text{var} [\Delta v_t]$ is possible, although it depends on the relationship between $m_{t+1,t+1}$ and $\Delta\mathbf{m}'_{t,t}\Delta\mathbf{m}_{t,t}$.

In the GS model without noise traders in Example 1 and 2 it can be found that $\text{var} [\Delta p_{t+1}] = \text{var} [\Delta v_t]$.

6 Conflicting results in the Shapley-Shubik and Grossman-Stiglitz models

In absence of noise traders the GS and SS models with unobservable current prices yield conflicting results. In both models the uninformed agents bid amounts that are independent of the current fundamentals. In SS however a good signal pushes up prices so that the uninformed receives fewer units for the cash they bid. In the end they lose relative to the informed, since they receive less of an asset that is expected to pay a higher return, and vice versa.

This mechanism then ensures that there is an incentive in SS to acquire information and hence to become informed, even without noise traders. Furthermore, since holding some of the risky asset does provide the uninformed with a higher expected return in exchange for some risk taking, they nevertheless prefer to participate in the market.

If the uninformed were able to fix the *number of units* bought though, they would clearly do that. This would in effect reduce SS to an efficient GS model where uninformed are just as well off as the informed.

D. Generalizing the results

Although some of the results found so far relies on the specification of the model, the most important results holds under very general assumptions. Assume now only that:

- 1) All higher central moments of the price process are public knowledge⁵
- 2) demand functions are known⁶
- 3) demand is strictly increasing in expected returns

This implies that given all public information, demand by informed traders

⁵If these are normally distributed it is of course sufficient that variance is public knowledge.

⁶This does of course not rule out individual random differences in demand as long as they are independent between traders and the distribution is common knowledge.

is uniquely determined by expected returns and vice versa. Total demand must equal total supply. The payoff relevant information in total supply Z_t and in θ_t is therefore identical in equilibrium. This is the case for any equilibrium model given assumptions 1)-3), including Shapley-Shubik.

In GS total supply depends only on $\varepsilon_{z,t}$ so that $Z_t(\theta_t, \varepsilon_{z,t}) = \varepsilon_{z,t}$. If $\varepsilon_{z,t}$ is known by the uninformed, information cannot have value because Z_t always reveals θ_t as mentioned.

On the other hand, if neither $\varepsilon_{z,t}$ nor any variable depending on $\varepsilon_{z,t}$ is observed, e.g. current prices are unobservable, the uninformed will always strictly prefer to a post state independent demand (Proposition 1). This is so because demand from informed traders can only decrease if returns increase, by 3). Thus any state dependent demand from the uninformed will have to be negatively correlated with returns. This contradicts 3).

If now the uninformed are forced to make state dependent demands, we may denominate supply in the currency of their orders as in Shapley-Shubik. In that case total supply depends on the private information, $Z_t(\theta_t, \varepsilon_{z,t}) = Z_t(\theta_t)$, and uninformed will be unable to observe this total supply. As shown initially, if Z_t is not observed, then neither is the private information. Thus with state dependent demands, information has value and the results from the Shapley-Shubik model follow.

IV Summary and conclusion

In Example 1 we have seen that a dynamic REE model can be specified so that it has a NE by restricting traders to observe only past prices. The results are pretty much the same as those observed by Grossman and Stiglitz, except that the uninformed hold fixed amounts of the asset in equilibrium.

In Example 2 a model similar to Brown and Jennings (1989) and Grundy and McNichols (1989) was presented where investors could observe current prices. In that case, prices map the fundamentals differently each period,

which means that the price history contains information. However, as found by Dubey, Geanakoplos, and Shubik (1987), a market where current demand depends on current prices does not have a NE. Thus, if investors can condition on current prices, the market is neither weak form efficient nor does it have a Nash Equilibrium.

In the last example we have modified the Shapley-Shubik model so that we better can compare it with the Grossman-Stiglitz model with unobservable current prices in Example 1. It is suggested that to understand the results in Dubey, Geanakoplos, and Shubik (1987) and Jackson and Peck (1999) we need to specify exactly which kind of market mechanism this model entails. Using the two-good examples of these authors, it is found that a bidding mechanism where bids are made in terms of costs and not units of the assets correspond well to the original model and produce the same results.

In order for cost bidding to be an appropriate assumption, it would either need to be optimal for the agents, or it would have to be imposed as a restriction by the market institution. Neither of these options seems to be an accurate description of reality. It is not common in most asset markets to restrict bidding in this way. Furthermore, uninformed investors themselves would clearly prefer to buy a fixed number of assets in an efficient market since it would guaranty them the same expected utility as the informed.

Based on this argument, I will argue that the noisy rational equilibrium model with delayed price information is the best way to explain the existence of positive information value. The well acknowledged drawback is of course the presence of "stupid" noise traders.

Whether these noise traders exist or not is subject to some debate. There are very good theoretical reasons for dismissing them as unsustainable. As noted by Fama (1965) and Friedman (1953) the loss-making noise traders should disappear by them selves after some time. The problem is however that we observe that investors do persistently hold different portfolios, and that inside information is profitable (Fama (1991)). This is not consistent

with the absence of noise traders in the GS model.

When a pair of theoretical arguments are mutually incompatible with the empirical evidence, either one or both must be false. We do not, however, at present have a generally accepted alternative that can explain this inconsistency, although a number of theories have been proposed.

One particularly intriguing idea is that the unpredictable shocks observed in the market, which we call noise trading, is in fact the effect of people changing their positions in risky assets due to income and consumption shocks outside the model. This may happen at different dates for different traders, and so it need not be the case that the same noise traders place losing bets over and over again. Rather it may be the case that when the financial situation of some uninformed traders requires it, they willingly pay a premium or accepts a discount in order to shift between more and less liquid assets. If the portfolio is not rebalanced too often, the cost may be quite acceptable. This idea is supported by the fact that at least theoretically, asset prices are ultimately determined by aggregate consumption and income (Breedon (1979) and Loewenstein and Willard (2006)).

The main theoretical argument against this idea is that such shocks would be predictable since it is the effect of a large number of individual agents holding market portfolios on average. Therefore investors should over time learn how these shocks affect each asset and form optimal portfolios based on this. In equilibrium therefore, the market index would reveal any such noise. In addition, if the market index is not sufficient to predict the shocks, investors can choose between a large number of economic factors in order to predict them, using multi-factor models (Merton (1973), Ross (1976), Fama and French (1989)), consumption betas (Breedon (1979)) or both.

The main problem with the above argument is its initial assumption, that such shocks must be predictable. Econometric models with zero residual variance do not exist. Nevertheless we expect market participants to be able to explain with 100% accuracy the shocks that come from agents that unex-

pectedly rebalance their portfolios due to consumption and income shocks. That seems to be a bit too optimistic with respect to the abilities of these agents. In fact, since different groups of investors may have consumption and income paths that depend very differently on the factors in the market, these shocks need not even be fully predictable at the individual stock level.

We might accept that changes in liquidity needs are predictable in the long run, though. However, for sufficiently short time periods which may be when private information is most important, the number of trades may be too small to be perfectly offset against each other. Thus even if such effects net to zero for the entire market over time, it may be unlikely that such demand and supply offset each other exactly in a single period for a single asset. That is, in the very short run it seems reasonable that individual differences in rationally motivated transactions actually matters.

In addition there is a serious problem in the assumption that either the market index or other factors can be used to predict such noisy demand. Even the use of the frequently published market index would be problematic, since filtering such trades out using the index would lead us right into the same problems that Dubey, Geanakoplos, and Shubik (1987) found to be inconsistent with a NE. It is just not possible for investors to calculate demand by conditioning on an index that is constructed from that same demand. In addition other important economic factors such as gross consumption are perhaps published only a few times a year.

In the end, there are a number of reasons why investors may behave differently but rationally. Even an assumption of different priors (different opinions) does not per se imply irrationality, it just poses severe game theoretical problems Harsanyi (1968). Our models are therefore not even close to explaining all that goes on in an asset market. Since assigning arbitrary demand schedules to different investors is not a very scientific approach, one usually deals with such problems by adding a noise term to the equation. The idea is then to improve the model in order to explain this random term

the best we can. For some reason that approach seems to be a problem in financial models.

The major problem with the original noisy REE model of Grossman and Stiglitz is that it did not have a Nash Equilibrium. What has been shown in this paper is that by altering the model slightly by assuming unobservable current prices, the original results remain the same. Furthermore, the alternative SS model where there is no noise seems to assume a price setting mechanism not found in most financial markets. After three decades one might therefore argue that the noisy REE model of Grossman and Stiglitz explains the value of information in financial markets best. Perhaps with the inconsequential modification that current prices are not observable.

References

- Blume, Lawrence, and David Easley, 1984, Rational Expectation Equilibrium: An Alternative Approach, *Journal of Economic Theory* 34, 116–129.
- Blume, Lawrence, David Easley, and Maureen O’Hara, 1994, Market Statistics and Technical Analysis: The Role of Volume, *The Journal of Finance* 49, 153–181.
- Blume, Lawrence E., and David Easley, 1992, Rational Expectations and Rational Learning, Cornell University.
- Boswijk, H.P., C.H. Hommes, and S. Manzan, 2003, Mean Reversion, Bubbles and Heterogeneous Beliefs in Stock Prices, *Computing in Economics and Finance*.
- Bray, M., and D. Kreps, 1987, Rational Learning and Rational Expectations, in George Feiwel, eds.: *Arrow and the Ascent of Modern Economic Theory* (New York University Press, New York).

- Breeden, Douglas, 1979, An Intertemporal Asset Pricing Model with Stochastic Consumption and Investment Opportunities, *Journal of Financial Economics* 7, 265–296.
- Brown, D. P., and R. H. Jennings, 1989, On Technical Analysis, *Review of Financial Studies* 2, 527–551.
- Chamley, Christophe, 2003, Dynamic Speculative Attacks, *The American Economic Review* 93, 603–621.
- Dubey, Pradeep, John Geanakoplos, and Martin Shubik, 1987, The Revelation of Information in Strategic Market Games: A Critique of Rational Expectations Equilibrium, *Journal of Mathematical Economics* 16, 105–137.
- Fama, Eugene, and Kenneth R. French, 1989, Business Conditions and Expected Returns on Stocks and Bonds, *Journal of Financial Economics* 25, 23–49.
- Fama, Eugene F., 1965, The Behaviour of Stock Market Prices, *Journal of Business* 38, 34–105.
- Fama, Eugene F., 1991, Efficient Capital Markets II, *The Journal of Finance* 46, 1575–1617.
- Feldman, M., 1987, An Example of Convergence to Rational Expectations with Heterogeneous Beliefs, *International Economic Review* 28, 635–650.
- Friedman, Milton, 1953, The Case for Flexible Exchange Rates", in *Essays Om Positive Economics* (Univ. Chicago Press, Chicago).
- Goenka, Aditya, 2003, Informed Trading and the Leakage of Information, *Journal of Economic Theory* 109, 360–377.

- Gottardi, Piero, and Roberto Serrano, 2006, Market Power And Information Revelation In Dynamic Trading, *European Economic Association* 3, 1279–1317.
- Grossman, S. J., and J. E. Stiglitz, 1980, On the Impossibility of Informationally Efficient Markets, *The American Economic Review* 70, No. 3, 393–408.
- Grundy, Bruce D., and Maureen McNichols, 1989, Trade and the Revelation of Information Through Prices and Direct Disclosure, *The Review of Financial Studies* 2, 495–526.
- Harsanyi, John C., 1968, Games with Incomplete Information Played by "Bayesian" Players: Part III. The Basic Probability Distribution of the Game, *Management Science* 14, 486–502.
- Hellwig, Martin F., 1982, Rational Expectations Equilibrium with Conditioning on Past Prices: A Mean-Variance Example, *Journal of Economic Theory* 26, 279–312.
- Jackson, Matthew O., 1991, Equilibrium, Price Formation, and the Value of Private Information, *The Review of Financial Studies* 4, 1–16.
- Jackson, Matthew O., and James Peck, 1999, Asymmetric Information in a Competitive Market Game: Reexamining the Implications of Rational Expectations, *Economic Theory* 13, 603–28.
- Kirchler, Michael, and Jürgen Huber, 2005, Testing for Stylized Facts in Experimental Financial Markets, in *Working Paper* (University of Innsbruck,).
- Loewenstein, Mark, and Gregory A. Willard, 2006, The Limits of Investor Behaviour, *The Journal of Finance* 61, 231–258.

- Merton, Robert C., 1973, An Intertemporal Capital Asset Pricing Model, *Econometrica* 41, 867–887.
- Milgrom, Paul, 1981, Rational Expectations, Information Acquisition, and Competitive Bidding, *Econometrica* 49, 921–943.
- Ross, Stephen A., 1976, The Arbitrage Theory of Capital Asset Pricing, *Journal of Economic Theory* 13, 341–60.
- Routledge, Bryan R., 1999, Adaptive Learning in Financial Markets, *Review-of-Financial-Studies* 12, 1165–1202.
- Shapley, Lloyd, and Martin Shubik, 1977, Trade Using One Commodity as a Means of Payments, *The Journal of Political Economy* 85, 937–968.
- Singleton, Kenneth J., 1987, Asset Prices in a Time Series Model with Disparately Informed, Competitive Traders, *NBER Working Paper No. 1897*.

V Appendix

A. Optimal $\mathbf{h}_{t-\tau}$ and b_t

The problem is to

$$\min_{b_{t,0}, \mathbf{c}_{t-\tau}} L = \mathbf{E}(\mathbf{E}(\Delta p_{t+1} | \Delta \mathbf{p}_{t-\tau}) - \Delta p_{t+1})^2 \quad (50)$$

Expanding (50) and taking expectations provides us with the expression

$$L = b_{t,0}^2 + \mathbf{h}'_{t-\tau} (\mathbf{I}\sigma_\theta^2 + \mathbf{M}_{t-\tau}^{-1'} \mathbf{S}'_{t-\tau} \mathbf{S}_{t-\tau} \mathbf{M}_{t-\tau}^{-1} \sigma_\varepsilon^2) \mathbf{h}_{t-\tau} \quad (51)$$

$$-2\Delta a_{t,0} b_{t,0} \quad (52)$$

$$-2\mathbf{h}'_{t-\tau} (\Delta \mathbf{m}_{t-\tau, t+1} \sigma_\theta^2 + \mathbf{M}_{t-\tau}^{-1'} \mathbf{S}'_{t-\tau} \Delta \mathbf{s}_{t-\tau, t+1} \sigma_\varepsilon^2) \quad (53)$$

L is globally convex in $\mathbf{h}_{t-\tau}$ and b_t . The derivatives are

$$\frac{\partial L}{\partial \mathbf{h}'_{t-\tau}} = 2 \left(\mathbf{I}\sigma_\theta^2 + \mathbf{M}_{t-\tau}^{-1'} \mathbf{S}'_{t-\tau} \mathbf{S}_{t-\tau} \mathbf{M}_{t-\tau}^{-1} \sigma_\varepsilon^2 \right) \mathbf{h}_{t-\tau} \quad (54)$$

$$-2 \left(\Delta \mathbf{m}_{t-\tau, t+1} \sigma_\theta^2 + \mathbf{M}_{t-\tau}^{-1'} \mathbf{S}'_{t-\tau} \Delta \mathbf{s}_{t-\tau, t+1} \sigma_\varepsilon^2 \right) \quad (55)$$

$$\frac{\partial L}{\partial b_{t,0}} = 2b_{t,0} - 2\Delta a_{t,0} \quad (56)$$

and (15) and (14) follows from this.

B. Proof of Proposition 3

Assume $\mathbf{g}_t = \mathbf{e}k_t$. From (28) we know that $\mathbf{h}_t^* = -\frac{\alpha_{I,t}}{\alpha_{U,t}} \Delta \mathbf{m}_{t,t+1}$, and from (10) $\mathbf{e}k_t = \mathbf{g}_t = \mathbf{M}_t^{-1} \mathbf{h}_t$. Therefore equilibrium requires

$$\mathbf{M}_t^{-1} \Delta \mathbf{m}_{t,t+1} = -\mathbf{e}k_t \frac{\alpha_{U,t}}{\alpha_{I,t}} \quad (57)$$

Inserting this into (29) premultiplying the result with \mathbf{S}_t and rearranging gives us

$$\frac{1}{\alpha_{I,t}} \mathbf{S}_t^{-1} \mathbf{e} = \mathbf{S}_t^{-1} \Delta \mathbf{s}_{t,t+1} + k_t \frac{\alpha_{U,t}}{\alpha_{I,t}} \mathbf{e} \quad (58)$$

Consider now period $T-1$ where $\mathbf{s}_{T-1,T} = \mathbf{0}$. Since $\mathbf{s}_{T-1,T-1}$ is the last column of \mathbf{S}_{T-1} , we know that $\mathbf{S}_{T-1}^{-1} \mathbf{s}_{T-1,T-1} = \mathbf{e}$. Using this, it can be found that the last column of \mathbf{S}_{T-1}^{-1} ($\mathbf{S}_{T-1}^{-1} \mathbf{e}$) is

$$\mathbf{S}_{T-1}^{-1} \mathbf{e} = (k_{T-1} \alpha_{U,T-1} - \alpha_{I,T-1}) \mathbf{e} \quad (59)$$

hence, the first $T-2$ equations of (59) says that the corresponding elements of $\mathbf{S}_{T-1}^{-1} \mathbf{e}$ must be zero. Since \mathbf{S}_{T-1} is some upper right triangular matrix, it is easiest to first solve for element $T-2$ of $\mathbf{S}_{T-1}^{-1} \mathbf{e}$, which is

$s_{T-2,T-1}/(s_{T-1,T-1}s_{T-2,T-2})$, which implies $s_{T-1,T-2} = 0$. Since this simplifies the $T - 3$ equation to $s_{T-3,T-1}/(s_{T-1,T-1}s_{T-4,T-4})$ and so on, we can continue this way and subsequently solve for each element in $\mathbf{S}_{T-1}^{-1}\mathbf{e}$. By induction then, $\mathbf{s}_{T-2,T-1} = \mathbf{0}$.

Considering one period ahead ($T - 2$) the above procedure can be used to find $\mathbf{s}_{T-3,T-2}$ as well since we now know that $\mathbf{s}_{T-1,T} = \mathbf{0}$. Therefore $\mathbf{s}_{T-3,T-2} = \mathbf{0}$, and so on. Thus, by induction, it can be found that \mathbf{S}_t must be a diagonal matrix.

Now pre-multiplying (57) with \mathbf{M}_t and using the fact that $\mathbf{M}_t^{-1}\mathbf{e} = \mathbf{m}_{t,t}$ we can rewrite (57) as

$$\mathbf{m}_{t,t} = \frac{\alpha_{I,t}}{\alpha_{I,t} - k_t \alpha_{U,t}} \mathbf{m}_{t,t+1} \quad (60)$$

Since in the terminal period $\mathbf{m}_{T-1,T} = \mathbf{1}$ it follows that all $\mathbf{m}_{t,t}$ are multiples of $\mathbf{1}$, and so we can write

$$\mathbf{m}_{t,t} = m_t \mathbf{1}, \text{ where } m_t = \prod_{j=t}^{T-1} \alpha_{I,j} / (\alpha_{I,j} - k_j \alpha_{U,j}) \quad (61)$$

Now consider the equilibrium equation for \mathbf{g}_t in (33). Since \mathbf{S}_t is a diagonal matrix and the vectors of \mathbf{M}_t are $m_j \mathbf{1}$ for $j \leq t$, we can rewrite it

$$(\mathbf{M}'_t \mathbf{1} m_t \sigma_\theta^2 + \mathbf{e} s_{t,t}^2 \sigma_\varepsilon^2) k_t = \frac{s_{t,t} \sigma_\varepsilon^2}{\alpha_{I,t} + \alpha_{U,t}} \mathbf{e} \quad (62)$$

In the first $t - 1$ equations, the right hand side is zero, so we can write these as $j \sigma_\theta^2 m_j m_t = 0$. This implies that either in period $t = T - 1$ or period $t = T - 2$, we would have $\mathbf{S}_t = \mathbf{0}$ and $\mathbf{M}_t = \mathbf{0}$. That would however violate the equilibrium condition (29), since it would in the end imply that informed traders do not take into account their private information. If this was the case then there would be no way for the market to absorb the noise trading, and the market would not be in equilibrium. It can therefore not be the case that $\mathbf{g}_t = \mathbf{e} k_t \blacksquare$

Optimal Order Submission

Espen Sirnes

Abstract

In this paper a model for optimal order submission is derived. Previous literature has focused on the spread set by market makers in order to cover inventory and adverse selection costs. In this paper the spread is assumed constant. However, uninformed investors choose an optimal amount of limit orders to post in the market. If all these orders are absorbed, that is a signal of an informed trader in the market, and expectations are adjusted accordingly. The model predicts a specific decreasing convex relationship between transaction volume and market variance, which is tested on data from Oslo Stock Exchange. The data supports this prediction. The optimal order submission scheme is valid for both market makers as well as uninformed traders who need to rebalance their portfolio. In the latter case the incentive for trade is assumed to be negative inventory costs.

Keywords: Asset pricing, market microstructure

JEL Classification:

The contribution of this paper is to derive a model where price adjustments are made in a discontinuous fashion as a function of the transaction volume in the market. Transaction volume above a specific level, set to maximize profits, becomes a signal of informed trading which causes a subsequent price adjustment in order to limit losses. Transactions at lower volumes are however assumed by the market maker to be caused by uninformed noise traders, which are strictly profitable to trade with. The model is tested on

data from Oslo Stock Exchange. The data from Oslo Stock Exchange confirms that price changes can in deed be described by such a discontinuous process as mentioned.

Oslo Stock Exchange (OSE), as an increasing number of international stock exchanges, is an order-driven electronic market with a lot of the smaller trades being done through electronic brokers. There are no dealers and in general no official market makers at OSE¹, such as at NYSE and NASDAQ. The market system at OSE is however quite similar to the current NASDAQ system, as NASDAQ has moved towards a more integrated European style market with a central order book, the last few years.

The decentralized structure and low transaction costs of such trading systems often lead to substantial limit order competition on the bid and ask prices, which narrows the spread. According to a professional in liquidity provision at one of the largest member firms of OSE, the spread today is too small for any profitable market making, even for the biggest players, when one takes into account the inventory risk.

Market making at the Oslo Stock Exchange is therefore limited to a few illiquid stocks where the company enters into an agreement with a member of the exchange which provides quotes at some maximum spread with a minimum amount of volume. Except for such agreements, the only advantage a potential market maker in Oslo has over ordinary traders is the lower transaction cost faced by members of the exchange.

The reason that market makers are not driven out of the market in NASDAQ may be the pricing system. At NASDAQ, order submission fees are set to the advantage of the market makers. For instance², you pay \$0.003 per share to accept a limit order, but you get a rebate of \$0.002 if you add liquidity to the market by posting a limit order. You will however only obtain this credit if you have costs exceeding it. Since each executed limit order

¹Except for a very few illiquid companies, which may enter into a contract with a financial institution, who will supply a minimum amount of liquidity.

²<http://www.nasdaqtrader.com/trader/tradingservices/productservices/pricesheet/pricing.stm>

must necessarily be matched by an equal number of market orders in total³, ordinary clients cannot on average profit from this as the rebate will always be smaller than the costs.

A market maker however, trade at his own account as well as matching quotes for customers. This means that the market maker can incur costs that he does not pay by executing market orders from customers and pass these on. Thus, the special role of a market maker makes it possible to turn the rebate into a profit. The system at NASDAQ may therefore work as a subsidy of market making that might otherwise not be profitable⁴.

There does not seem to be any other recent international comparisons of spreads and trading costs than Ian Domowitz and Madhavan (2001). Such a study would be of great interest due to the substantial changes that has occurred in internet based trading and the dramatic reduction in fees that has followed this. There is however some evidence indicating that order driven markets in general have lower spreads (Didier Davydoff and Grillet-Aubert (2002)), although if we look at total trading costs the picture is not that clear. One might however argue that easier access to the order book and order submission we have seen in recent years may have been of particular benefit to the order-driven trading systems in terms of transaction costs.

Combined with low trading fees by the electronic brokers it is therefore very easy for any small trader in Oslo to marginally undercut a potential market maker in order to obtain time priority. If the order book is not fully visible and you are not allowed to enter the orders directly into the trading system, then there is a risk in entering a limit order with too low time priority. The ease at which you can enter limit orders in Oslo therefore reduces the spread and eliminates the need for market makers in most cases.

Most of the associated literature assume that spreads and limit order size

³Limit orders outside the spread are counted as market orders.

⁴There is a sort of market maker subsidy in the OSE market as well, since non-members must go through a member to trade, and hence pay a commission. Limit orders and market orders are charged the same though.

is set by market makers. The model presented here will however describe optimal behavior of both market makers as well as other traders who need to rebalance their portfolios for some reason.

In the empirical testing it will be assumed that the spread is constant for each asset, possibly set competitively to a zero profit level as described in for example by competition among market makers as in Glosten and Milgrom (1985) and Easley and O'Hara (1992). Implementing both endogenous order size and spread was found to make the model too complicated.

The model by Easley and O'Hara (1992) and applied empirically in Easley, Kiefer, O'Hara, and Paperman (1996) is similar to the one presented here in that trading is taken as a signal of private information, although their focus was on the frequency of trading. In their model as here the spread is set to a zero profit level for a market maker. More frequent trading then signals that new information has arrived, and so affects the spread positively. The spread is then compared across different assets.

Checking the impact of volume on prices is of course not new. Easley and O'Hara (1987) (EO) did a theoretical analysis of the effect of trade volume on prices. Since there is an adverse selection problem with respect to volume as informed traders typically wish to trade large quantities, large trades would typically be made at less favorable prices. The model presented here differ from Easley and O'Hara (1987) in three important aspects. First there is a single spread in the market, and so the price discrimination scheme central in the EO paper is not considered. Second in this paper the trade size is optimally chosen by the uninformed. Furthermore the probability of an informed trader is not determined by the frequency of large trades as in EO, as the probability of an informed trader is assumed constant. Where the model coincides is the idea that high volumes are a signal of informed trading. In this paper it will be argued that the signal is perceived when the volume reaches some threshold value.

Also in Kyle (1985) the market price depends on past volumes. In Kyle

the expectations of the uninformed market makers is conditioned on the observed quantity traded. This is the case in this paper as well, but in this model a maximum quantity is set by the market makers as limit orders. Further in Kyle the informed and noise traders initiate the trading, but in the model presented here the market is initiated by limit orders by the uninformed.

Also in Blume, Easley, and O'Hara (1994) it is argued that the information from observed volume may be used to predict price changes. In contrast to this paper and Easley and O'Hara (1987), the market structure in Blume, Easley, and O'Hara (1994) is not defined in great detail, as is the case in microstructure models. Furthermore the volume is not dependent on the probability of informed traders in the market.

Related empirical research have in particular been devoted to the bid-ask spread. Glosten and Harris (1988) for instance, use the average transaction volume to obtain an estimate of the adverse selection component of the bid-ask spread. Huang and Stoll (1997) suggest an improved decomposition of the spread, and finds that volume has significant impact on the components of the bid-ask spread. Although the data collected for this paper certainly can be used for such testing, the bid-ask spread is not the issue in this paper.

The model presented here is inherently short run, and so in order to test its predictions it was important to use detailed high frequency data. This sets the paper apart from some other papers where daily volumes and returns are sufficient (for example Simon Gervais and Mingelgrin (2001), AR Gallant and Tauchen (1992), Stickel and Verrecchia (1994)).

A main feature of this paper is the threshold volume set by the market maker. Allowing the market makers to determine the trade size optimally has previously been suggested by Easley and O'Hara (1987) and Dennert (1993). There are also other works where the quantity plays a role, but often this quantity is determined by the demand functions of the noise traders or liquidity traders. In this paper the main focus is on the total amount of

volume absorbed by the uninformed traders. To which extent this quantity is actually absorbed by the market then depends on the volatility of noise trading.

The model of Dennert (1993) is similar to the one presented here in some aspects. In Dennert's model the spread and size are set simultaneously by all market makers. Due to this simultaneous move feature there is no pure strategy equilibrium. The mixed strategy equilibrium where each market maker sets a spread according to a common distribution implies that most market makers will trade only with the informed trader since their bids are not competitive with respect to noise traders. Dennert therefore finds that more market makers increase the mean spread. Evidence from the NASDAQ does however point in the opposite direction (Klock (1999), Ellis, Michaely, and O'Hara (2002)). The most critical assumption of Dennert is the simultaneous move assumption. In many markets bid and ask schedules are set according to time priority, so that the bid/ask by any given market maker depends on the bid/asks already placed. Thus a simultaneous bidding procedure may not be as realistic as a sequential one unless the order book is not publicly available.

This model together with others such as Glosten and Milgrom (1985) and Easley and O'Hara (1992) may be used by market makers themselves in order to design optimal algorithms for liquidity provision. The contribution of this paper is then to suggest how optimal limit volumes may be set by a market maker.

The literature of microstructure has been growing and become more important in recent years as noted by O'Hara (2003). Much of it relates to the spreads and tick-sizes in financial markets. O'Hara (1997) is recommended for a more comprehensive review of the area.

The plan of the paper is to present the model in the next section. In the following section the data is presented and the model operationalized. Results are then presented. Conclusions are then drawn in the last section.

I The model

There are three main types of participants in the market; uninformed traders, noise traders and informed traders.

Uninformed traders may require or be willing to pay a premium on each transaction to compensate for inventory risk, which enters as a component of the transaction cost. For simplicity we assume that investors maximize expected profits rather than utilities, after trade specific costs have been deducted. The risk premium is thus treated as an exogenous cost of unknown magnitude, which we will allow to vary across traders. This simplification is consistent with the literature on the components of the bid-ask spread though (Huang and Stoll (1997)), where risk aversion is often taken into account explicitly as inventory costs.

The market is assumed to be symmetric so that the optimal strategies for sellers and buyers are in effect the same. It is therefore often not necessary to distinguish between selling and buying, and so for clarity we will mostly consider the case where the uninformed buys assets by posting bid orders.

Prices and fundamentals are expressed in logs.

Market liquidity is used in various ways in economic literature. In this model it is specifically defined as follows

Definition 1 *Market liquidity is the amount of outstanding limit orders at the highest bid price and at the lowest ask price.*

A. The market

The market works as a double auction where market participants can place either *limit orders* or *market orders*. A limit order is an offer to buy or sell a certain quantity at a specific price. A market order is an acceptance of the best limit order available.

Each period starts when uninformed place limit orders. These are then absorbed by informed traders, who may place both limit and market orders,

and noise traders who place market orders only.

The market consists of one risky asset. This pays \tilde{u}_T in some final period, for example the end of the trading day. The process up to \tilde{u}_T is given by

$$\tilde{u}_t = \tilde{u}_{t-1} + \tilde{\varepsilon}_t \quad (1)$$

where $\tilde{\varepsilon}_t \sim N(0, \sigma_\varepsilon^2)$. Tilde will throughout the paper indicate random variables. \tilde{u}_t will be referred to as the fundamentals and is known only by the informed. Public information is left out from the model. Since the uninformed and informed traders are risk neutral here, there is no need for risk compensation. So in order to simplify, the expected payoff from the asset is zero.

The *market expectation* p_t is the expected terminal value⁵ of the asset, given the information available to the uninformed h_{t-1} . Given that u_t is a random walk, this means

$$p_t = E[u_t | h_{t-1}] \quad (2)$$

where h_{t-1} is the payoff relevant information available to the uninformed *before trade* in period t . We will be more specific about the values that h_{t-1} may take later in the paper.

The market clears in two stages. Let s represent the spread. First uninformed investors post bids at some price lower than the current market expectation, $p_t - \frac{1}{2}s$, and asks at some price higher than this, $p_t + \frac{1}{2}s$. In this way the uninformed profits unless the counterpart is better informed than them. Since we are only considering the side of the market where limit orders are bought, the transaction price we look at is the bid price

$$p_t^* = p_t - \frac{1}{2}s \quad (3)$$

⁵One can view p_t as the expectation of the assets' present value, or the expectation under a risk neutral measure.

In the next stage, noise traders and possibly informed traders arrive and choose whether to accept this price. If the amount of demand or supply is higher than some critical value, discussed later in the paper, this provides the uninformed with a signal that the trading activity is possibly not just due to noise traders.

As implied above it is assumed that current market price p_t is always the middle value of the upper and lower prices. This is not a realistic assumption, but a vastly simplifying one.

B. Trading costs

As mentioned we know that one of the largest member firms at Oslo Stock Exchange do not participate in market making⁶ due to the low spread relative to their own transaction costs, which we can assume are among the lowest in the market. It therefore seems unlikely that there is widespread market making activity in this market place. We do however observe limit orders at quite narrow spread in the market. As will be described in more detail, will attribute the posting of these limit orders to investors driven by the need for temporary portfolio rebalancing.

In the literature trading costs are typically decomposed into three components; Order processing costs, inventory costs and adverse selection costs. Order processing costs are assumed constant across traders, while inventory costs are allowed to vary across traders. The total of these costs per trade for trader i will be denoted k_i . Adverse selection costs are endogenous in this model, and will therefore equal the spread less the smallest trading costs over the uninformed investors, k_{\min} , that is $\frac{1}{2}s - k_{\min}$.

The informed does not face inventory costs. Furthermore, trading fees for limit orders placed by the uninformed and the market orders needed to accept the limit orders may not be the same.

⁶Except for special liquidity provision contracts with specific low liquidity companies.

We will define trading cost of the informed as a fixed quantity k_I . If there is a subsidy scheme such as at NASDAQ, then k_I may actually be quite large relative to k_i .

1 Inventory costs

Inventory costs may differ across investors due to portfolio heterogeneity, which is a rationale for posting limit orders⁷. If the portfolio is not well balanced (that is, it is not optimal), then inventory costs will typically turn into inventory benefits for trade in the direction that improves the portfolio. We will refer to such benefits as negative costs for generality.

We will distinguish between the inventory cost and order processing costs for trader i by denoting the risk premium of inventory cost $k_{R,i}$, which may be negative, and the order processing cost k_C , so that $k_i = k_C + k_{R,i}$. This inventory cost is then the negative (positive) amount $k_{R,i}$ the trader is willing to pay (requires) in order to avoid (take) risk by improving (worsen) their portfolios.

In the literature inventory costs are often just regarded as some general cost of holding an asset short or long for a period of time. There have been theoretical attempts to quantify them though, such as Ho and Stoll (1981) and O'Hara and Oldfield (1986). For this model it will suffice with a much simpler specification of these costs. It will be defined as follows

Definition 2 *Inventory cost $k_{R,i}$ equals the certainty equivalent compensation needed to offset a change in the current portfolio.*

Inventory costs, thus vary across investors depending on the effect of a trade on the portfolio risk and return. With a CARA utility function with unit risk aversion we know that for small changes this approximately amount to the present value of the derivative of the mean-variance difference with respect to the portfolio weight of this asset. That is, assuming a zero interest

⁷a similar argument was proposed by among others Bhushan (1991)

rate the inventory cost (the current wealth compensation dw_0) per share for a small trade of $da_{j,i}$ in asset j given a portfolio \mathbf{a}_i with covariances $\sigma_{j,l}$ is for a buyer approximately

$$k_{R,i} = I_i \frac{dw_0}{da_{j,i}} = I_i \left(\mu_j - \gamma \sum_{l=1}^N a_{l,i} \sigma_{j,l} \right) \quad (4)$$

where $I_i = 1$ if trader i is a seller and $I_i = -1$ for a buyer, and γ is the risk aversion coefficient. The return μ_j will of course depend on the current market price and transaction price.

If the portfolio is optimal, inventory costs are effectively zero, since (4) is the first order condition of the portfolio optimization problem for asset j . Furthermore, if the investor has too little of the asset relative to the optimal level, $k_{R,i}$ is negative for a buyer and positive for a seller. If the spread is too narrow for market making, then bid orders are typically placed by investors with too little of the asset in their portfolios. Those with positive inventory costs with respect to a purchase would have negative costs with respect to selling, and would therefore place ask orders.

On average the portfolio of the investors will be optimal, but only a tiny fraction of outstanding assets are usually traded within, say, any given hour. Those with the smallest transaction costs in any short time interval may therefore be considered extreme observations and may have portfolios that deviate quite substantially from the optimal one. Thus, since we are looking at very short time periods here, there might be a substantial amount of trading even if most investors hold well balanced portfolios.

The heterogeneity of portfolios may stem from a variety of different sources. There is some resemblance between the specification here and the approach taken by Glosten (1994), who attributed the motive for trading to unspecified shocks to investor characteristics. It might be due to differences in opinion. This explanation, different priors, has some game theoretical problems associated with it (Harsanyi (1968)). It does not necessarily breach

the assumption of rationality though. A more consistent explanation would be that individual differences in consumption and income generate heterogeneous portfolios, and shocks throws portfolios out of balance. In any case, different portfolios are an empirical fact, which implies that any shock to the economy will impact the individual asset holders differently.

A major advantage with the approach taken here is that the model can be used to predict optimal behavior by both market makers as well as investors seeking to rebalance their portfolios. If the investor has high risk bearing capacity and low order processing costs so that $k_{R,i} \approx 0$, then it might be an advantage to act as a market maker.

The definition of the inventory cost $k_{R,i}$ in (4) in this paper is for illustrative purposes. All we need to know is that investors may hold different portfolios, some better balanced than others. The ones with the smallest costs on each side of the market then post limit orders, and the sign of the inventory cost $k_{R,i}$ then determines whether it is a seller or a buyer. Those with the smallest inventory costs will determine the maximum amount of liquidity offered in the market, as we will see.

C. The noise traders

There are also noise traders in the market who at time $t \in \{1, \dots, T\}$ have a net demand of $q_t \sigma_q$ where q_t is a standard normally distributed random variable

$$\tilde{q}_t \sim N(0, 1) \tag{5}$$

$q_t \sigma_q$ is sold from the noise traders at the lowest ask price $\tilde{p}_{t,L}$ when there is a net supply in the market and bought at the highest bid price $p_{t,H}$ when there is a net demand.

Trading with the noise traders constitutes the incentive for the uninformed to post limit orders. Without them, positing a limit order would imply a sure loss. The noise traders are thus investors who need to sell and

buy at unfavorable prices for some reason. As described in the previous section, investors may for some reasons need to rebalance their portfolios. Some traders do however value immediacy more than others, which is why market orders are placed. One might therefore view these noise traders as "impatient" uninformed traders, such as in Foucault, Kadan, and Kandel (2005).

Since s is the cost of shifting from less liquid to more liquid assets, the level of noise trading q_t is typically regarded to be negatively related to the spread. In this paper we will not spend much time on this side of the market, but it is worth noting that we could model this by letting the volatility of noise trading σ_q , which determines volume in absolute terms, be a decreasing function of the spread s .

In any case, the only reason that limit orders are placed in the market is the existence of noise traders without superior information. Unless these are present, any limit order would incur losses. Therefore, it does not really matter why the noise traders are in the market. For a meaningful discussion of the strategies of limit order submission the existence of noise traders is a fundamental assumption.

D. The informed

At any time there is some given probability λ that some traders are informed of the underlying fundamentals u_t and so λ is the arrival rate of these traders.

The informed traders act non strategic in the sense that if there are possibilities for profitable trades in the market, the informed will trade until no such possibilities are available. Problems associated with this assumption are discussed in O'Hara (1997) p. 74. The easiest way to justify such lack of strategy is to make the following assumption, which is common in related literature:

Assumption 1 *The informed investors in the market act competitively*

If all the informed investors act competitively by buying or selling all available quantities on the market, then this is also a Nash equilibrium in this model, even with only two informed investors. The reason for this is that there is a finite amount of limit orders posted in the market, as we will see. If this volume is absorbed the uninformed concludes that there are informed traders in the market and the informed are revealed. Thus, if at least one informed investor in the market has a competitive strategy of accepting all available limit orders, then this will be the optimal strategy for all other investors since revelation occurs with probability one in any case. Therefore by Assumption 1 the informed will not strategically ration their demand, but rather absorb all limit orders and their presence is subsequently revealed in the same period.

As we will also see, this causes the uninformed market makers to change their expectations about the fundamentals p_t . At the new price level however, the probability of an informed trader remains the same, λ .

Given that the informed investors act competitively then, they will buy or sell the maximum amount available. Let $c_t\sigma_q$ be the total amount that the uninformed post at p_t^* . $c_t\sigma_q$ is thus the total liquidity of the market as defined in Definition 1. How c_t it is set will be explained in the next section. The informed traders absorb the entire volume when they trades, so the amount bought by them is $c_t\sigma_q - \tilde{q}_t\sigma_q$.

If $\tilde{u}_t < p_t + \frac{1}{2}s + k_I$ then the information of the informed cannot be used to exploit uninformed investors by buying limit orders. This will occur with a certain probability $1 - \gamma_t$. It will further be assumed for simplicity that if the informed is able to exploit the sellers, that is if $\tilde{u}_t > p_t - \frac{1}{2}s - k_I$, then she will absorb any noise trading that would otherwise go to the buyers, and so the uninformed buyer does not trade in this case either. The total probability that the uninformed gets to trade with noise traders even when there is an informed trader in the market is then $1 - 2\gamma_t$. Two times γ_t is subtracted since the probability that the informed will trade is twice the probability

that she will either buy or sell.

E. The uninformed traders

Each uninformed trader posts limit orders of $c_{t,i}\sigma_q$ at either the bid or ask price in the market. $c_{t,i}$ is normalized by the volatility of noise trading σ_q , which we will see is convenient.

Let the normalized supply of limit orders by the first i traders be

$$C_{t,i} = \sum_{j=1}^i c_{t,j} \quad (6)$$

There is now a given probability that all available limit orders up to trader i , $C_{t,i}\sigma_q$, are absorbed by accident by noise traders in the market. This probability is

$$\alpha [C_{t,i}] = (1 - \Phi [C_{t,i}]) \quad (7)$$

where Φ is the normal CDF. This is the probability of observing $\tilde{q}_t \geq C_{t,i}$. Thus $\alpha_t = \alpha [C_{t,i}]$ is the significance level of a one sided test performed by the uninformed on the presence of informed traders.

F. The expected profit function

In calculating the expected profit, the volatility of the current market expectation as an estimate on the fundamentals u_t matters. We define the market variance of p_t as the variance of the difference between the known market price p_t and the random fundamentals \tilde{u}_t , that is

$$v_t^2 = \text{E} [(\tilde{u}_t - p_t)^2 | h_{t-1}] \quad (8)$$

As in (2) h_{t-1} is the information available to the uninformed before trades in period t . To be specific, $h_{t-1} \in \{0, 1\}$, where $h_{t-1} = 0$ if total demand

(supply) did not exceed $C_{t-1,N}$ in period $t - 1$ and $h_{t-1} = 1$ otherwise. In the latter case the market will change its expectation so that $p_t \neq p_{t-1}$.

This represents the uncertainty of the current market estimate p_t with respect to the underlying fundamentals \tilde{u}_t . It turns out to be an advantage to express this market variance in terms of the spread faced by the informed when she takes into account trading costs, $s + 2k_I$. If her valuation lies within this spread, she cannot use her information. Thus we define the normalized market variance as

$$\sigma_{m,t} = v_t / (s + 2k_I) \quad (9)$$

Define the probability that the private information is actually useful with respect to a limit buy order as $\gamma_t = P(\tilde{u}_t < p_t^* - k_I)$. It can be found that this probability is

$$\gamma_t[\sigma_{m,t}] = 1 - \Phi\left[\frac{1}{2\sigma_{m,t}}\right] \quad (10)$$

Noise traders sell or buy the quantity q_t . If there are no informed traders in the market the realized profit for trader i from trading with the noise traders is the volume that falls to him times the profit per share. If he is a buyer then this profit from trading with noise traders only will be

$$\pi_{t,N} = \sigma_q \max(\tilde{q}_t - C_{t,i-1}, 0) (\tilde{u}_t - p_t^* - k_i) \quad (11)$$

If there are informed investors in the market, they will accept all limit orders, including $c_{t,i}\sigma_q$ for the price p_t^* while the true value of the asset is \tilde{u}_t . The negative profit for trader i when buying from an informed trader is therefore

$$\pi_{t,I} = \sigma_q c_{t,i} (\tilde{u}_t - p_t^* - k_i) \quad (12)$$

Since \tilde{q}_t and \tilde{u}_t are independent, we can calculate the expectation in the

two terms of (11) separately. Define the expected noise trading going to trader i as

$$\bar{q}_{t,i} = \sigma_q \mathbb{E} \max(\tilde{q}_t - C_{t,i-1}, 0) \quad (13)$$

$$= \sigma_q (c_{t,i} \alpha [C_{t,i}] + C_{t,i-1} (\alpha [C_{t,i}] - \alpha [C_{t,i-1}])) \quad (14)$$

$$+ \sigma_q (\phi [C_{t,i-1}] - \phi [C_{t,i}]) \quad (15)$$

which follows from the formula of the expectation of a truncated normal random variable, where by definition $c_{t,i} = C_{t,i} - C_{t,i-1}$.

The expected profit per share when trading with noise traders is half the spread, less trading costs, $\frac{1}{2}s - k_i$. As mentioned, this happens when either there is no informed trader in the market, or the informed trader does not trade. The probability of this is $1 - 2\lambda\gamma_t$, assuming the noise trader orders are absorbed by the informed both when she buys and sells, which occurs with probability $2\gamma_t$ when she is in the market.

The (negative) expected profit per share given that the uninformed trades⁸ with an informed trader is

$$\mathbb{E} [\tilde{u}_t - p_t^* - k_i | \tilde{u}_t < p_t^* - k_I] = \frac{1}{2}s - k_i - \frac{2\phi\left[\frac{1}{2\sigma_{m,t}}\right]\sigma_{m,t}}{\gamma_t[\sigma_{m,t}]} \left(\frac{1}{2}s + k_I\right) \quad (16)$$

The expected total profit is then obtained by multiplying the expected profits per share with the associated expected volume ($c_{t,i}$ if the informed trades and $\bar{q}_{t,i}$ otherwise) and the associated probabilities. This gives us the total expected profit

$$\mathbb{E} \pi_t = \lambda\gamma_t \sigma_q c_{t,i} \mathbb{E} [\tilde{u}_t - p_t^* - k_i | \tilde{u}_t < p_t^* - k_I] \quad (17)$$

$$+ (1 - 2\lambda\gamma_t) \bar{q}_{t,i} \left(\frac{1}{2}s - k_i\right) \quad (18)$$

⁸the problem is symmetric, so the formula apply for both bid and ask orders.

G. Optimal order submission

It can be found that $\partial \bar{q}_{t,i} / \partial c_{t,i} = \sigma_q \alpha [C_{t,i}]$. Therefore the first order condition for maximization of the profit function is

$$\begin{aligned} \frac{\partial \mathbf{E} \pi_t}{\partial c_{t,i}} &= \lambda \gamma_t \sigma_q \mathbf{E} [\tilde{u}_t - p_t^* - k_i | \tilde{u}_t < p_t^* - k_i] \\ &+ (1 - 2\lambda \gamma_t) \sigma_q \alpha [C_{t,i}] \left(\frac{1}{2} s - k_i \right) = 0 \end{aligned} \quad (19)$$

The expected profit is globally concave in $c_{t,i}$, so the optimal order submitted is uniquely determined by this first order condition. Let $C_{t,i}^*$ be the optimal total order submission after trader i has submitted his optimal order size $c_{t,i}^*$. Then the total optimal amount of orders solves

$$\alpha [C_{t,i}^*] = - \frac{\lambda \gamma_t \mathbf{E} [\tilde{u}_t - p_t^* - k_i | \tilde{u}_t < p_t^* - k_i]}{(1 - 2\lambda \gamma_t) \left(\frac{1}{2} s - k_i \right)} \quad (20)$$

The right hand side is also the probability that the uninformed will mistakenly take high noise trading to be the activity of an informed trader.

There are two important conclusions we can draw from this optimal strategy. First, as we see from the solution (20) the optimal order submission for a single investors gives a unique solution for the total order submission $C_{t,i}^*$. That is if trading costs are constant across investors, then trader 1 would submit the total order size, $c_{t,1}^* = C_{t,1}^*$ and no further orders are submitted.

Furthermore we see from (19) that

$$\frac{\partial^2 \mathbf{E} \pi_t}{\partial c_{t,i} \partial k_i} = - (1 - 2\lambda \gamma_t) \sigma_q \alpha [C_{t,i}] < 0 \quad (21)$$

Since the profit function is globally concave in $c_{t,i}$, (21) means that if the FOC (19) initially holds then an increase in k_i makes it negative, and so $c_{t,i}$ must be reduced. The optimal amount submitted is therefore decreasing in

trading costs k_i . For a trader j with the smallest trading cost k_j , it will therefore always be optimal to top up the total order size to $C_{t,j}^*$, and no other investor will be willing to submit any additional volume.

Thus the following proposition holds

Proposition 1 *The optimal total order submission $C_{t,i}^*$ is independent of the number of uninformed investors and is determined by the investor with the smallest trading cost k_{\min} .*

The proof follows from the above discussion.

What is important with Proposition 1 is that it states that the optimal order submission is the same for any given minimum trading cost k_{\min} .

The inventory cost (4) will however decrease as an investor balances his portfolio with increasing order submission, because the portfolio will become closer to optimum. Fortunately this is not a problem as long as there are sufficiently many uninformed investors in the market. In that case we only need to consider the transaction cost of the last marginal order, which by Proposition 1 must be the smallest of all k_i . We will from now on assume that this transaction cost is constant equal to k_{\min} and that the total order size submitted by the last trader is c_t^* . Calculating the right hand side of (20) we can now restate that the total amount of orders optimally submitted satisfies⁹

$$\alpha [c_t^*] = \frac{1}{2} - \frac{1 - 4\lambda\phi\left[\frac{1}{2\sigma_{m,t}}\right]\sigma_{m,t}K}{2(1 - 2\lambda\gamma_t)} \quad (22)$$

where ϕ is the standard normal density function and

$$K = \frac{\frac{1}{2}s + k_I}{\frac{1}{2}s - k_{\min}} \quad (23)$$

is the relative advantage for the uninformed relative to the informed with respect to trading costs. If the right hand side of (22) is less than $\frac{1}{2}$ then the

⁹Which also can be written $\Phi(c_t^*) = \frac{1}{1 - \lambda\gamma_t + \lambda 2\phi\left[\frac{1}{2\sigma_t}\right]\sigma_t\frac{s+2k_I}{s-2k}}$

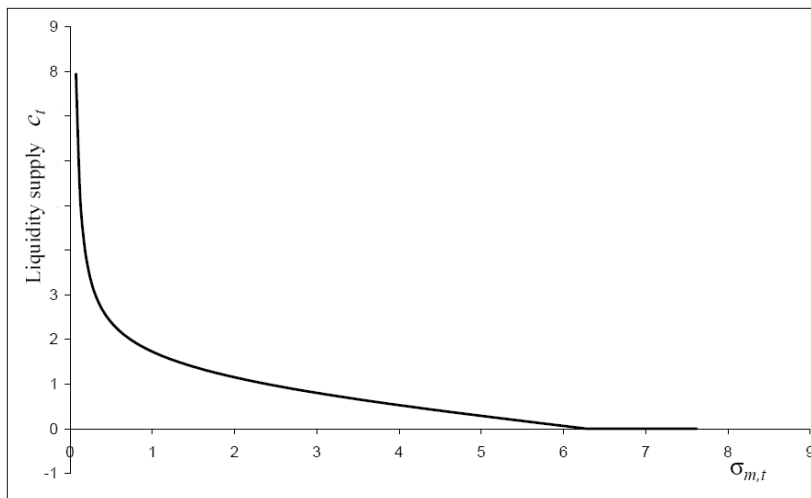


Figure 1: Optimal order submission with $K = 1$ and $\lambda = 0.1$. We see that $\sigma_{\max} \approx 6.286$.

normalized volatility $\sigma_{m,t}$ is simply too large for it to be worth submitting any orders. The point at which zero liquidity is provided is

$$\sigma_{\max} [K\lambda] = \frac{1}{2} \text{LambertW} [2 (K\lambda)^2 / \pi]^{-\frac{1}{2}} \quad (24)$$

where $\text{LambertW}(x)$ solves $W e^W = x$. The optimal liquidity function $c_t^* [\sigma_{m,t}]$ that solves (22) with respect to c_t^* can now be written

$$c_t^* [\sigma_{m,t}] \equiv \Phi^{-1} \left[\frac{1}{2} + \frac{1 - 4\lambda\phi \left[\frac{1}{2\sigma_{m,t}} \right] \sigma_{m,t} K}{2(1 - 2\lambda\gamma_t)} \right] \quad (25)$$

where $c_t^* [\sigma_{m,t}] = 0$ if $\sigma_{m,t} > \sigma_{\max} [K\lambda]$. Φ^{-1} is the inverse of the standard cumulative normal distribution function.

A graph of optimal liquidity provided is depicted in Figure 1.

1 Market makers

As mentioned the above argument will determine optimal order submission by a market maker too, due to Proposition 1. The market maker submits simultaneous bid and ask orders. Since the events that the market absorbs the bids or absorbs the asks are mutually exclusive by construction, the expected profit of a market maker will simply be twice the expected profit of an uninformed investor who is buying (or selling). Optimal order submission for the market maker is therefore exactly the same as above.

In this paper we will take a closer look at a stock market where the trading fees are the same for limit orders and market orders. In that case a market maker can however only submit simultaneous orders if his inventory costs are not too high. This requires that any market maker have very low order processing costs and high risk bearing capacity so that the inventory cost $k_{R,i} \approx 0$ for both selling and buying even as inventory builds up. Even then, the market maker may become outcompeted by any trader at each side of the market with sufficiently negative inventory costs. At Oslo Stock Exchange there does not seem to be much market making, which is possibly due to the inability of any potential market maker to compete with the trader who has the most misaligned portfolio in the market. These traders have negative inventory costs as opposed to the market maker, and so competition may drive the spread to a level where only such investors can place limit orders.

At NASDAQ market makers have an advantage with respect to trading costs. In the model, this would imply considerably smaller trading costs k_i for such traders. In addition the market makers have better information about the full order book than the rest of the market, and hence the adverse selection cost may be smaller. That might be enough to have the smallest trading cost of all traders, and thus explain why market makers are widespread at NASDAQ but not at Oslo Stock Exchange.

H. Expectation updating

Assuming the order book is open, significant trading volumes absorbing all that is posted c_t^* , will be observed by market participants. As previously stated, the informed will absorb any standing orders in the market, and so c_t^* will work as a "critical transaction volume". If all outstanding orders c_t^* are absorbed, then this is a signal to the uninformed that there might be an informed trader in the market, with a significance level of $\alpha [c_t^*]$. Since observing trading c_t^* always occur if there is an active informed trader in the market, the uninformed will optimally change their expectations, p_t , in the face of this event. We will now take a look at the magnitude of this adjustment in expectations, which will be called $z_t (s + 2k_I)$. z_t is thus normalized by the spread of the informed as $\sigma_{m,t}$ is.

z_t is now chosen so that the next period market variance $\sigma_{m,t+1}^2$ is minimized. As mentioned earlier expectations p_t and volatility $\sigma_{m,t}$ are based on the information $h_{t-1} \in \{0, 1\}$ which is the realization of the random variable \tilde{h}_t , where $\tilde{h}_t = 1$ if total demand or supply exceeded c_t^* and the market adjusts, and $\tilde{h}_t = 0$ if it did not. The probability of adjustment given the current market volatility is then

$$\delta [\sigma_{m,t}] = P(\tilde{h}_t = 1) = 2\lambda\gamma_t + (1 - 2\lambda\gamma_t) 2\alpha_t^* \quad (26)$$

$$= 4\lambda\phi \left[\frac{1}{2\sigma_{m,t}} \right] \sigma_{m,t} K \quad (27)$$

where the last equality follows from the expression of the optimal significance level $\alpha_t^* = \alpha [c_t^*]$ in (22).

The realization h_t depends on the realization of two random variables; the amount of noise trading \tilde{q}_t and whether an informed trader is in the market or not. The variance after observing trades is then the random variable

$$\begin{aligned} \sigma_{m,t|\tilde{h}_t}^2 &= \text{E} \left[(u_t - p_t)^2 | \tilde{h}_t \right] / (s + 2k_I)^2 \\ &= \sigma_{m,t}^2 - \tilde{h}_t w_t [\sigma_{m,t}] \end{aligned} \quad (28)$$

where $w_t [\sigma_{m,t}]$ is to be determined, and represents the change in variance if adjustment to expectations are made. Since we know the probability of an adjustment from (26), we know that the expected market variance after trade is

$$E [\sigma_{m,t}^2 | \tilde{h}_t] = \sigma_{m,t}^2 - \delta [\sigma_{m,t}] w_t [\sigma_{m,t}] \quad (29)$$

Observing significant demand or supply does however not necessarily imply an informed trader. Define the events in case of adjustments accordingly as

1. Event I_t : It is correctly determined that there is an informed investor in the market who trades (in any direction¹⁰). This happens with probability

$$P(I_t) = 2\lambda\gamma_t \quad (30)$$

2. Event IN_t : There is an informed investor in the market, but she does not trade. Excess demand/supply from noise traders however generates a signal (in any direction). This happens with probability

$$P(IN_t) = \lambda(1 - 2\gamma_t) 2\alpha_t^* \quad (31)$$

3. Event N_t : There is no informed trader in the market but excess demand/supply from noise traders generates a signal. This happens with probability

$$P(N_t) = (1 - \lambda) 2\alpha_t^* \quad (32)$$

As one can confirm, the probabilities above sums up to $\delta [\sigma_{m,t}]$. We multiply α_t^* by 2 in event IN_t and N_t , since the noise traders may generate both a sell and a buy signal. We can now derive variance in the current period, $\sigma_{m,t}^2$, given an adjustment has been made. The variances conditional on each of the three events 1.-3. defined above can be found to be

¹⁰Since the trade can go in any direction, the total probability is twice the one sided probability.

$$\sigma_{m,t}^2|I_t = \sigma_{m,t}^2 + z_t^2 - \frac{1}{2}\sigma_{m,t}(4z_t - 1) \frac{\phi[1/2\sigma_{m,t}]}{\gamma_t} \quad (33)$$

$$\sigma_{m,t}^2|IN_t = \sigma_{m,t}^2 + z_t^2 - \frac{1}{2}\sigma_{m,t} \frac{\phi[1/2\sigma_{m,t}]}{1-\gamma_t} \quad (34)$$

$$\sigma_{m,t}^2|N_t = \sigma_{m,t}^2 + z_t^2 \quad (35)$$

Total variance is then found by multiplying the variances under these mutually exclusive events by their respective probabilities (30)-(32), divided by the total probability of adjustment $\delta[\sigma_{m,t}]$. Using the expression for $\alpha[c_t^*]$ giving the optimal order submission in (22), it can be found that the variance in case of adjustment in period t is

$$\sigma_{m,t}^2|\tilde{h}_t=1 = \sigma_{m,t}^2 + z_t^2 - \frac{(4z_t - 1 + 2\alpha[c_t^*])}{4K} \quad (36)$$

We now assume that the uninformed based on this result set an appropriate adjustment z_t that minimizes this distance from the fundamentals¹¹. $\sigma_{m,t}^2|\tilde{h}_t=1$ is convex in z_t . Using the definition of K in (23), the corresponding FOC therefore implies that the optimal adjustment is

$$z^* = \frac{1}{2K} \quad (37)$$

which in turn implies (29) that the market variance $\sigma_{m,t}^2$ is reduced by a factor of

$$w[\sigma_{m,t}] = \frac{1}{4K} \left(\frac{1}{K} - 1 + 2\alpha[c_t^*] \right) \quad (38)$$

when expectations are updated. $w[\sigma_{m,t}]$ may be negative if $\alpha[c_t^*]$ is small.

Define the normalized price variance as¹² $\sigma_{p,t}^2 = \mathbf{E}[\Delta p_t^2] / (s + 2k_I)^2$. The price remain constant if no adjustment is made, and changes by z^* otherwise. Since the probability of an adjustment is $\delta[\sigma_{m,t}]$, the price variance can be

¹¹Of course other norms could have been used, but variance minimization is computationally convenient and conventional.

¹² $\Delta p_t = p_{t+1} - p_t$

expressed as

$$\sigma_{p,t}^2 = \frac{4\lambda}{K} \phi \left[\frac{1}{2\sigma_{m,t}} \right] \sigma_{m,t} \quad (39)$$

after substituting for z^* and dividing the right hand side by $(s + 2k_I)^2$. This relationship between the price variance and the underlying market volatility $\sigma_{m,t}$ will be important when estimating the model.

I. Variance process

In the next period the normalized variance $\sigma_{m,t}^2$ will increase by the normalized innovation $\sigma_\varepsilon^2 / (s + 2k_I)^2$ regardless of adjustment or not. This is due to the fundamental process (1) and the definition of $\sigma_{m,t}$ in (9). Thus the expected next period variance is

$$\mathbb{E} [\tilde{\sigma}_{m,t+1}^2 | \sigma_{m,t}, \frac{\sigma_\varepsilon}{s}] = \mathbb{E} [\sigma_{m,t}^2 | \tilde{h}_t] + (\sigma_\varepsilon^2 / (s + 2k_I)^2) \quad (40)$$

or

$$\mathbb{E} [\tilde{\sigma}_{m,t+1}^2 | \sigma_{m,t}, \frac{\sigma_\varepsilon}{s}] = \sigma_{m,t}^2 - \delta [\sigma_{m,t}] w [\sigma_{m,t}] + \frac{\sigma_\varepsilon^2}{(\frac{1}{2}s - k_{\min})^2} \quad (41)$$

(41) describes the conditional next period expectation of a random dynamic variance process.

The stochastic process $\tilde{\sigma}_{m,t}^2$ is then given by

$$\tilde{\sigma}_{m,t+1}^2 = \tilde{\sigma}_{m,t}^2 - \tilde{h}_t w [\tilde{\sigma}_{m,t}] + \frac{\sigma_\varepsilon^2}{(\frac{1}{2}s - k_{\min})^2} \quad (42)$$

The path of this process is determined by the realized frequency at which updating occur, which again is determined by the realization of \tilde{q}_t and the presence of informed traders at rate λ .

(42) is a stochastic variance process much like conditional heteroscedasticity models such as GARCH. The expected change in variance is not a linear function of current volatility though. The expected next period variance

is then obtained by substituting the random variable \tilde{h}_t for its expectation $\delta[\sigma_{m,t}]$. Doing this, we can find the "equilibrium variance", which we may define as the variance required for $E \tilde{\sigma}_{m,t+1}^2 = \sigma_{m,t}^2$ to hold. Thus, the equilibrium variance σ_m^2 solves

$$\delta[\sigma_m] w[\sigma_m] = \frac{\sigma_\varepsilon^2}{\left(\frac{1}{2}s - k_{\min}\right)^2} \quad (43)$$

which has no explicit solution.

An important implication of the variance process is that though $w[\sigma_{m,t}]$ may be negative so that the variance increases even when an optimal adjustment is made, this cannot continue forever. As the variance $\sigma_{m,t}^2$ increases, c_t^* will approach zero and $\alpha[c_t^*]$ will approach $\frac{1}{2}$. If $\alpha[c_t^*] = \frac{1}{2}$ we see from (38) that $w[\sigma_{m,t}] > 0$ always, and hence abnormal trading and the following adjustment will always reduce variance. Thus at some point variance will cease to increase.

As we will assume that the market volatility is constant, we are implicitly assuming that it varies around some equilibrium level σ_m . We will use (39) to infer the underlying equilibrium volatility σ_m from the price variance $\sigma_{p,t}^2$.

II Testing the model

A. The data

In order to test some predictions of the model, detailed market data was extracted from the web-site of a Norwegian on-line business newspaper¹³. Both order and transaction data were obtained from the 71 most liquid stocks (measured in daily traded volume) at the Oslo Stock exchange in the period November 9th through December 3rd 2007. Due to low trading activity in some stocks and some technical problems associated with the data extraction

¹³Data was extracted automatically from "<http://www.dn.no/finans/>" by the use of a custom made computer algorithm.

only 66 of these stocks were used in estimations. For the same reason the data used in the estimations do not include all the trading days (the number of trading days is reported in the Appendix).

The data is organized in two databases for orders and trades. The order database consisted of some 5 million observations of minute-by-minute best ask and bid quotes, number of best ask and bid quotes, total number of ask and bid quotes, time and ticker (name). The data usually contained more than one recorded order depth during any given minute, but the seconds were not registered in the data base. The quotes that gave the narrowest spread were therefore used.

The trades data base consisted of 1.1 million trade observations with exact time (second), price, volume, buyer (broker) and ticker.

These two databases were then combined into one by assigning recorded trades to either the bid or ask side depending on whether the transaction price was below or above the mid quote in the order data base at the same time. On inspection it was found that there seemed to be a variable time lag between the order and trade data. For that reason all orders and trades within each five minute trading period during the day were aggregated. This increased the probability that removal of any given order was attributed to the trading volume in the same time period. Thus the data used has a maximum frequency of 12 trades per hour when there is successive trading in the stock and a trading period in the model is defined to be five minutes. Preliminary estimations suggested that this improved the reliability of the data too, which might suggest that a five minute trading interval is a suitable definition of "one period". Characteristics of the companies used in the sample are given in the Appendix.

Prices are measured in first differences, excluding the innovation from one day to another. The number of initial observations equal to the number of lags was further deleted for each day, so that the return one day did not depend on variables the previous day.

The member transaction fee at OSE is 0.011% for a \$10 000 transaction. This is comparable to similar stock markets such as NASDAQ.

B. Tests

We will pursue two main objectives when estimating the model. First we will test the main prediction of the model, the general shape of the optimal liquidity function $c^*[\sigma_{m,t}]$, and see if it fits with the data. Second we will estimate the parameters, given that the model is true. This approach is taken since we do not have detailed enough data to test all its predictions against the null that the model is false. There are two main obstacles for a more rigorous testing.

First we assume constant parameters across time and individual stocks in this study due to a limited number of liquid stocks at OSE. In particular we will assume that the market variance σ_m^2 is constant as mentioned and that the optimal liquidity function $c^*[\sigma_m]$ is the same for all assets. If we had more companies in the sample, say a few thousand, this could possibly be solved by dividing the assets into several classes by some criterion and vary parameters across time as well. More trading in each stock than in the OSE sample would also possibly improve estimates¹⁴.

Second there is an identification problem which essentially can only be solved by specifying some parameters exogenously. The reason we nevertheless have presented such a rich model is to reduce the number of assumptions needed in the derivation of the model.

C. Estimation of liquidity supply $c^[\sigma_{m,t}]$ and market volatility $\sigma_{m,t}$ for each stock*

The model presented was operationalized by estimating the following model for a single asset (dropping individual subscripts for simplicity):

¹⁴observations for each company in the sample are given in the Appendix

1 The regression

First we needed to estimate the liquidity supply \hat{C} , for each asset, which will be normalized by an estimate of the noise trading volatility, $\hat{\sigma}_q$. As mentioned we have classified volume traded in each period as either bids or ask orders depending on the transaction price relative to the mean price in the period. Let $V_{B,t}$ be the bid volume and $V_{A,t}$ be the ask volume. In order to estimate the liquidity supply \hat{C} we now define the dummies

$$h_{i,t} = \begin{cases} 1 & \text{if } \hat{C} < V_{i,t} \\ 0 & \text{if } \hat{C} \geq V_{i,t} \end{cases} \quad (44)$$

for $i = \{B, A\}$. These dummies indicates if trading exceeds some critical volume \hat{C} . Their coefficients will determine the effect of the observed volume as a signal to revise expectations, and hence be useful to estimate the adjustment z_t . We will estimate \hat{C} by choosing the level for each stock that maximizes the likelihood function.

We do however need to control for the direct effect that volume has on the price. That is, we do not want to find a threshold value \hat{C} unless it adds to the explanatory power of the regression. In order to allow for the possibility that the market just does not regard large volumes as a signal according to our model, we add the volume $V_{i,t}$ to our regression. Multicollinearity with $h_{i,t}$ is then avoided by only using $V_{i,t}$ in the regression when $h_{i,t} = 0$, and so we defined the truncated volume.

$$\underline{V}_{i,t} = \begin{cases} 0 & \text{if } V_{i,t} > \hat{C} \\ V_{i,t} & \text{if } V_{i,t} \leq \hat{C} \end{cases} \quad (45)$$

we will of course allow for the possibility that $h_{i,t} = 0$ always and $\underline{V}_{i,t} = V_{i,t}$. The volume often tend to take extreme values, which we take into account by using the transformation $\underline{v}_{B,t} = \ln(\underline{V}_{B,t} + 1)$ and $v_{A,t} = \ln(\underline{V}_{A,t} + 1)$, the natural logs of the total number of asset traded at bid and ask prices sub-

sequently, with 1 added to ensure finite values.

Since the time it takes before an observation has effect on the market price is not known, we allow for up to $L = 5$ lags for the variables defined above. The regression we will estimate then becomes

$$\Delta p_t = \sum_{j=0}^{L-1} (B_{V,j} \underline{v}_{B,t-j} + B_{h,j} h_{B,t-j} + A_{V,j} \underline{v}_{A,t-j} + A_{h,j} h_{A,t-j}) + \mathbf{x}'_t \boldsymbol{\beta}_S + \zeta_t \quad (46)$$

\mathbf{x}'_t are the structural variables, which in this case are just dummy variables for each day, mapping the return on the asset at each date. A measure of the stock index was also initially added¹⁵. This did however not contribute much in terms of explanation, possibly due to the high frequency of the time series, and was therefore discarded due to reduced adjusted R^2 . It is possible that the reason for this was that the daily trends captured essentially the same market trend as the market index.

(46) was estimated by GLS in order to take account of order one autocorrelation in the error terms, which was apparent in most of the time series. A grid search procedure was used to find the coefficient that minimized the residual sum of squares.

2 Estimating the liquidity supply $c^*[\sigma_{m,t}]$

Assuming a constant level of market volatility for each asset $\sigma_{m,t}$, the unnormalized liquidity supply $c^*[\sigma_{m,t}] \sigma_q$ was estimated by picking the threshold level for transaction volume \hat{C} that maximized the log-likelihood of the regression. This was done conditional on that there was not severe multicollinearity problems (condition index with multiple correlated variables < 500). If such problems were apparent, the number of lags was

¹⁵High frequency data on the index was not available. Therefore an index for each individual stock was constructed as an average weighted by market capitalization from all the other stocks in the sample. Thus for stock i , the index included all other stocks than i .

reduced until the problematic condition index was reduced to an acceptable level.

For each stock the main assumption of \hat{C} as a threshold variables was tested, as mentioned, by estimating a simpler regression with the restriction $B_{h,j} = A_{h,j} = 0$. If the adjusted R^2 was higher for that simpler model, then it was determined that $\hat{C} = 0$ and that all trades were perceived by the market to be potential informed trading.

In order to obtain an estimate of the normalized liquidity supply, $\hat{c} = \hat{C}/\hat{\sigma}_q$, an estimate of the noise trading volatility σ_q was needed. For this the square root of the sum of squared differences between ask and bid volumes for each five minute interval was used as a proxy. Specifically, the proxy for noise trading volatility was estimated as

$$\hat{\sigma}_q^2 = \frac{1}{T} \sum_{j=1}^T (V_{B,t} - V_{A,t})^2 \quad (47)$$

where T is the total number of five minute intervals in the sample. There is an implicit assumption here that the expectation of the difference is zero. We then estimate the normalized liquidity supply as $\hat{c} = \hat{C}/\hat{\sigma}_q$. Since $\hat{\sigma}_q$ is only a proxy however, we will not restrict the relationship between the estimated and fitted liquidity to be one. We will see later what this means.

3 Estimate of the normalized market volatility σ_m

The normalized market volatility is also unobservable. However we can use (39) to infer it from the observed price volatility. Let $\hat{\sigma}_p^2$ be the estimated price variance normalized by the mean spread \hat{s} , over all five minute intervals in the sample. The function $\phi\left[\frac{1}{2\sigma_m}\right]\sigma_m$ in (39) can be approximated fairly good by a linear function¹⁶. We therefore estimate σ_m as a linear function of

¹⁶2.3% standard deviation and 7.2% max deviation in the range 0.07-10 of the inverse function.

$\hat{\sigma}_p^2$. That is

$$\hat{\sigma}_m = g_0 + g_1 \hat{\sigma}_p^2 \quad (48)$$

As seen from (39) we can in principle calculate the coefficients of (48). However, the estimate of the price volatility will typically contain additional noise not captured by the model. For this reason, g_0 and g_1 are assumed to be free coefficients in the estimation.

By normalizing the price variance with the average spread of the sample, \hat{s} , we implicitly assume that the transaction costs of the informed equals the spread, $k_I = \frac{1}{2}\hat{s}$. In other words we assume that the only reason for the informed to trade is the informational advantage. \hat{s} is measured as the mean percentage spread, since the price difference Δp_t is in logs.

4 Estimate of price adjustment z

z is the adjustment to expectations given that a significant amount of trading, $c^*[\sigma_{m,t}]$, has occurred. We can calculate this effect of transaction volume as a signal by taking the sum of the difference between ask and bid of associated coefficients $B_{h,j}$ and $A_{h,j}$, in the regression (46). The estimate for adjustment per percentage spread in asset j is then

$$\hat{z}_j = \frac{1}{2L} \sum_{j=0}^{L-1} (A_{h,j} - B_{h,j}) / \hat{s}_j \quad (49)$$

where we have normalized by the average spread as we do with $\hat{\sigma}_m$ above, and $L = 5$ as in (46). Since this includes up to five lags, the immediate temporary effect that volume has on prices should be canceled out by a price reversal. \hat{z}_j is therefore the "permanent" effect of significant volume, to the extent that 25 minutes can be called permanent.

If the market is inefficient in the sense that price reversals happen slowly, we might not detect it with only five lags. Furthermore autocorrelation in transaction volumes may lead to a high adjustment factor even when we sum

over all the lags. The estimation was however done allowing for first order autocorrelation. If $\Delta p_t - \hat{\rho}\Delta p_{t-1}$ is stationary, the autocorrelation coefficient $\hat{\rho}$ should take care of this problem. A Durbin Watson test on $\Delta p_t - \hat{\rho}\Delta p_{t-1}$ indicates stationary, and hence it seems that autocorrelation in volumes may not be a problem here.

Assuming a common normalized adjustment factor, we can now estimate this as the average adjustment relative to the spread, \hat{z} , for all assets in the sample. We take into account heteroscedasticity by weighting each observation \hat{z}_j by the associated inverse of the estimated standard deviation of the parameter estimates of (49) from the regression (46).

D. Fitted regression model

We will now find the parameters that make the optimal liquidity function fit the data as close as possible. The intention is to give the reader an idea of which parameter values are reasonable in the model. A more general test of the models main prediction is done by a less restrictive linear regression in the next section.

We will now use the estimated critical volume \hat{c}_j and market volatility $\hat{\sigma}_{m,j} = g_0 + g_1\hat{\sigma}_{p,j}^2$ for stock j to estimate the parameters in the model. We will from here on use subscripts j to indicate stock j .

From a plot of \hat{c}_j and $\hat{\sigma}_p^2$ (Figure 2) we see that there is substantial heteroscedasticity in these data, with variance declining in $\hat{\sigma}_p^2$. For this reason we assume a heteroscedasticity function for the residual term η_j , which is specified¹⁷ as $\sigma_{\eta,j}^2 = a_0e^{a_1\hat{\sigma}_{p,j}^2}$. This adds an additional pair of variables, a_0 and a_1 , to the model.

¹⁷an example of this specification is found in for instance in Greene (2000), p. 515

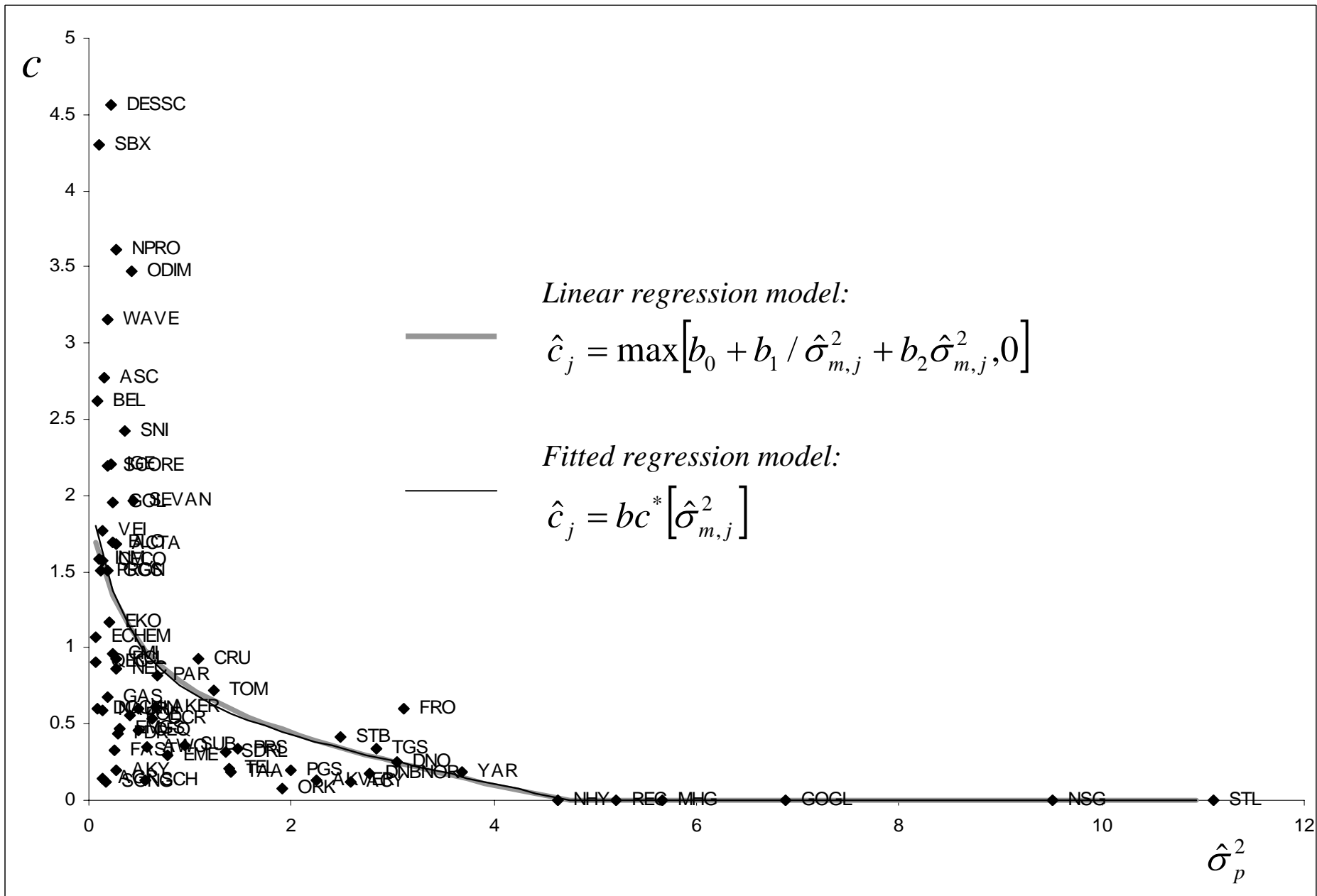


Figure 2: Estimated normalized critical value \hat{c}_j and normalized price volatility $\hat{\sigma}_{p,m}^2$ for each company, with the predicted liquidity for the the fitted regression (grey thick line) and the simple regression (thin black line)

1 The regression

Writing the optimal liquidity function c^* from (25) with all its arguments, we estimate the parameters of the following regression

$$\hat{c}_j = bc^* [g_0 + g_1 \hat{\sigma}_{p,j}^2, \lambda, K] + \eta_j \quad (50)$$

where η_j is the disturbance term. Estimates of the parameters a_0 , a_1 , g_0 , g_1 , K and b are obtained by maximizing the likelihood with respect to these variables. It is assumed that $\lambda = 1$ due to a multicollinearity problem with K , which we will come back to. Since the estimate of \hat{c}_j relies on the estimate of the unobserved noise trading volatility $\hat{\sigma}_{q,j}$, there may not be a one to one relationship between \hat{c}_j and c^* . Hence we allow for the coefficient $b \neq 1$.

The hessian matrix with respect to all variables was analytically computed from the likelihood function, in order to calculate the covariance matrix and maximize the log-likelihood function.

The calculated hessian matrix was also used to detect multicollinearity in the model. It was found that this was specifically a problem when jointly estimating λ and K . This seems reasonable when we look at the optimal liquidity function in (25). The problem is of course that since we do not observe the underlying market volatility, our data does not permit estimation of all these variables from (50).

The model is in other words too complex. We could of course have introduced assumptions about the parameters in the beginning of the theoretical part of the paper, but this would also have concealed the effect of these assumptions. We will however deal with this problem by assuming $\lambda = 1$. Thus we assume that there are always informed traders in the market, but they do not trade with probability $1 - 2\gamma [\hat{\sigma}_{m,j}]$ because their information lies within the spread.

2 Results from the fitted regression

The following parameters estimates were obtained from the regression (50):

Table 1: Estimated parameters fitted regression

	Param. est.	95% confidence interval		Standard deviation
\hat{a}_0	2.275	0.908	3.64	0.594
\hat{a}_1	-1.84	-2.16	-1.52	0.140
λ	1 [§]	-	-	-
\hat{K}	1.12	0.00150	2.24	0.487
\hat{g}_0	0.0612	0.00199	0.120	0.0257
\hat{g}_1	0.139	0.0333	0.246	0.0460
\hat{b}	0.242	0.144	0.342	0.0431

[§]Assumed

Predicted values from the fitted regression is given in Figure 2, as the black thin line. This is essentially the same function as the one depicted in Figure 1 in the theoretical part of the paper.

The adjusted R^2 was 0.611. As we see from Table 1, the confidence intervals are quite large though. With a level of 95% confidence all we can say about the relative trading cost parameter K is that it is a positive number not much larger than 2. This is because we have too many parameters to estimate. A high condition index for a couple of variables (179.7) suggests that confidence intervals would be considerably narrower if we had assumed specific values for more variables. This does however only emphasize the main problem: As we will see, the general shape of c_j^* can be estimated with a higher level of precision when we come to the linear regression. Which part the different parameters play here is however less clear.

The estimate of K has as mentioned a wide range. In order to get an idea of what this number means, assume that the informed cannot profit from market making¹⁸, $k_I = \frac{1}{2}s$, and that the smallest trading costs of the

¹⁸That is, when simultaneous selling at the ask price and buying at the bid price yields

uninformed is zero $k_{\min} = 0$. In that case $K = 2$, near the upper bound of the confidence interval. This would also mean that k_{\min} is negative for $K > 2$.

K is determined by lowest transaction costs k_{\min} , and hence by the need for uninformed investors to rebalance their portfolios as described in (4). If inventory costs are negative, the total trading costs may be zero or even negative. For sufficiently negative trading costs (k_{\min} approaching $-\frac{1}{2}s$) K will approach infinity. The results in Table 1 does therefore tell us that the minimum trading costs are most likely not very negative. There is however a different way of obtaining and estimate for K .

In the previous section the procedure for estimating \hat{z} was described. Doing this, it can be found that $\hat{z} = 0.681$, with a 95% confidence interval of $z \in [0.55, 0.82]$. Assuming that the uninformed investors adjust their expectations optimally, we can use (37) to find $\hat{K} = 1.47$, with the confidence interval $K \in [1.22, 1.83]$. Under the assumption $k_I = \frac{1}{2}s$, this estimate suggests that the minimum transaction costs are not negative.

Using the estimate of \hat{z} then seems to indicate that negative inventory costs are not sufficient to offset the overall trading costs. The adjustments to volume exceeding the threshold value \hat{c}_j thus seems to suggest that trade is generated by only moderate differences in inventory costs.

3 Linear regression model

We will now proceed to test whether liquidity is a decreasing convex function of the market volatility $\hat{\sigma}_{m,j} = g_0 + g_1 \hat{\sigma}_{p,j}^2$. We do this by estimating the following truncated linear regression model.

$$\hat{c}_j = \max \left(b_0 + b_1 \frac{1}{\hat{\sigma}_{m,j}} + b_2 \hat{\sigma}_{m,j}, 0 \right) + \eta_j \quad (51)$$

We will assume the same heteroscedasticity structure as before, $\sigma_{\eta,j}^2 =$

a sure loss.

$a_0 e^{a_1 \hat{\sigma}_{p,j}^2}$. Due to this and that negative \hat{c}_j are not allowed, (51) is estimated with maximum likelihood, as in the previous section.

We call this a linear regression model, even though it is truncated and depends on a functional form of $\hat{\sigma}_{m,j}$, in order to emphasize that it is a linear regression for positive values in two different functions of $\hat{\sigma}_{m,j}$. The model thus allows us to separate the convexity and linearity of \hat{c}_j with respect to $\hat{\sigma}_{m,j}$.

In order to reduce the number of parameters in the model, it is assumed that g_0 and g_1 are known and equal to the estimates in the fitted model. This is not a critical assumption however. What is important is that the second term in (51) is decreasing and convex in the observed price volatility $\hat{\sigma}_{p,j}^2$. We use the previous estimates of the market volatility, $\sigma_{m,j}$, since this would be our best guess of it, and g_0 and g_1 are therefore treated as constants since we are not interested in which values they take. Similar results would be obtained by the parameterization $g_0 = 0$ and $g_1 = 1$.

Maximizing the likelihood function then produced the following parameter estimates:

Table 2: Estimated parameters simple regression

	Param. est.	95% conf. interv.	St. dev.	t-stat	sign.	
\hat{a}_0	2.29	0.902	3.68	0.603	2.49	0.0157
\hat{a}_1	-1.82	-2.16	-1.49	0.144	-12.7	0.0000
\hat{g}_0	0.0612 [§]	-	-	-	-	-
\hat{g}_1	0.139 [§]	-	-	-	-	-
\hat{b}_0	0.448	-0.0169	0.913	0.202	2.22	0.0304
\hat{b}_1	0.0907	0.0210	0.160	0.0303	2.99	0.0040
\hat{b}_2	-0.797	-1.37	-0.225	0.249	-3.21	0.0021

[§]Assumed constant from the fitted regression (Table 1).

What is important here is to note that both $\hat{b}_1 > 0$ and $\hat{b}_2 < 0$, and significantly different from zero. This shows that the estimated critical vol-

umes c_j^* are in deed best described by a convex decreasing function in market volatility $\hat{\sigma}_{m,j}$.

The adjusted R^2 was 0.605. Predicted values for the linear model are plotted in Figure 2 as a black thin line. We see that the predictions of the simpler linear regression are quite close to the predicted values of the fitted regression.

III Conclusion

The model presented has two main implications. First it shows how an uninformed market participant should optimally place orders and how he should subsequently adjust expectations if all standing orders are absorbed. The model works for both investors who need to balance their portfolios for some reason, and therefore have a negative inventory cost, as well as for market makers.

In the Norwegian stock market, there seems to be little market making. Therefore it would be very interesting to perform the tests described above on markets where market making is more apparent, such as NASDAQ. Whether market makers actually use models like the one described here is uncertain, but a market participant who posts on both sides of the market may benefit by optimally setting the size of his orders and change expectations accordingly when the orders are absorbed.

Second the model provides us with a theoretical explanation for how transaction volume relates to the adverse selection problem of uninformed traders. The empirical investigation shows that there is a significant negative convex relationship between critical order volumes, normalized by average order volumes, and price variance per squared percentage spread unit as a measure of market volatility.

What seems to be the case then, is that when there is not much scope for an informed trader to exploit the uninformed (low market volatility relative

to the spread), the volume traded needs to be several times the average volume for the market to perceive it as a signal of informed trading. If on the other hand there is substantial fundamental uncertainty, then only quite small transactions may trigger significant price adjustment. If the normalized price volatility is bigger than about two times the average spread ($\hat{\sigma}_{p,j}^2 > 4$), then $c^* = 0$ and expectations will be adjusted for any transaction volume. In the estimation, these were the cases where continuous specification of the volume effect worked best, that is where $c^* = 0$.

It should be noted that the fact that volume affects prices per se is not a surprising result. If large orders arrive at the market, they will necessarily move prices. We have taken account of this effect by allowing for a continuous effect of volume on prices.

The main point in this paper is however that in addition there exist some threshold level c^* . Transaction volumes above this level work as a signal to market participants to change their expectations by z . The empirical evidence here supports this idea.

References

- AR Gallant, PE Rossi, and G Tauchen, 1992, Stock Prices and Volume, *Review of Financial Studies* 5, 199–242.
- Bhushan, Ravi, 1991, Trading Costs, Liquidity, and Asset Holdings, *The Review of Financial Studies* 4, 343–360.
- Blume, Lawrence, David Easley, and Maureen O’Hara, 1994, Market Statistics and Technical Analysis: The Role of Volume, *The Journal of Finance* 49, 153–181.
- Dennert, Jürgen, 1993, Price Competition Between Market Makers, *The Review of Economic Studies* 60, 735–751.

- Didier Davydoff, Jean-François Gajewski, Carole Gresse, and Laurent Grillet-Aubert, 2002, Trading Costs Analysis: A Comparison of Euronext Paris and the London Stock Exchange, <http://www.oee.fr/>.
- Easley, David, Nicholas M. Kiefer, Maureen O'Hara, and Joseph B. Paperman, 1996, Liquidity, Information, and Infrequently Traded Stocks, *The Journal of Finance* 51, 1405–1436.
- Easley, David, and Maureen O'Hara, 1987, Price, Trade, and Information in Securities Markets, *Journal of Financial Economics* 19, 96–90.
- Easley, David, and Maureen O'Hara, 1992, Time and the Process of Security Price Adjustment, *The Journal of Finance* 47, 577–605.
- Ellis, Katrina, Roni Michaely, and Maureen O'Hara, 2002, The Making of a Dealer Market: From Entry to Equilibrium in the Trading of Nasdaq Stocks, *The Journal of Finance* 57, 2289–2316.
- Foucault, Thierry, Ohad Kadan, and Eugene Kandel, 2005, Limit Order Book as a Market for Liquidity, *The Review of Financial Studies* 18, 1171–1217.
- Glosten, Lawrence R., 1994, Is the Electronic Open Limit Order Book Inevitable?, *The Journal of Finance* 49, 1127–1161.
- Glosten, Lawrence R., and Lawrence E. Harris, 1988, Estimating the Components of the Bid/Ask Spread, *Journal of Financial Economics* 21, 123–142.
- Glosten, Lawrence R., and Paul R. Milgrom, 1985, Bid, Ask and Transaction Prices in a Specialist Market with Heterogeneously Informed Traders, *Journal of Financial Economics* 14, 71–100.
- Greene, William H., 2000, *Econometric Analysis*. (Prentice Hall New Jersey).
- Harsanyi, John C., 1968, Games with Incomplete Information Played by "Bayesian" Players: Part III. The Basic Probability Distribution of the Game, *Management Science* 14, 486–502.

- Ho, Thomas, and Hans R. Stoll, 1981, Optimal Dealer Pricing under Transactions and Return Uncertainty, *Journal of Financial Economics* 9, 47–73.
- Huang, Roger D., and Hans R. Stoll, 1997, The Components of the Bid-Ask Spread: A General Approach, *The Review of Financial Studies* 10, 995–1034.
- Ian Domowitz, Jack Glen, and Ananth Madhavan, 2001, Liquidity, Volatility and Equity Trading Costs Across Countries and Over Time, *International Finance* 4, 221–255.
- Klock, Mark, 1999, The Impact of Market Maker Competition on Nasdaq Spreads, *The Financial Review* 34, 55–74.
- Kyle, Albert S., 1985, Continuous Auctions and Insider Trading, *Econometrica* 53, 1315–1336.
- O’Hara, Maureen, 1997, *Market Microstructure Theory*. (Blackwell Publishing Malden).
- O’Hara, Maureen, 2003, Presidential Address: Liquidity and Price Discovery, *Journal of Finance* 58, 1335.
- O’Hara, Maureen, and George S. Oldfield, 1986, The Microeconomics of Market Making, *The Journal of Financial and Quantitative Analysis* 21, 361–376.
- Simon Gervais, Ron Kaniel, and Dan H. Mingelgrin, 2001, The High-Volume Return Premium, *The Journal of Finance* 56, 877–919.
- Stickel, Scott E., and Robert E. Verrecchia, 1994, Evidence That Trading Volume Sustains Stock Price Changes, *Financial Analysts Journal* 50, 57–67.

IV Appendix

Table A1(a): Regression results and characteristics for the companies in the sample

Ticker	\hat{s} : av spread, % of price	\hat{C} : est. critical vol.	SD vol	\hat{c} : est. normalized critical vol.	$\sigma_{m,j}$: est. normalized mkt.vol	z/s : est. total effect of vol > \hat{C} per average spread $0.5\text{Sum}(A_{h,j}-B_{h,j})/s$	SD. z	$\sigma_{p,j}$: intraday SD of price	δ : frequency of significant trades
ACTA	0.52 %	302 345	180 025	1.68	0.099	1.61	0.20	0.27 %	0.4 %
ACY	0.19 %	9 357	76 211	0.12	0.422	1.22	0.10	0.30 %	52.6 %
AGR	0.70 %	1 365	9 293	0.15	0.080	0.19	0.02	0.25 %	18.8 %
AKER	0.22 %	3 358	5 493	0.61	0.152	0.78	0.06	0.17 %	7.4 %
AKVER	0.17 %	5 840	44 856	0.13	0.374	1.83	0.11	0.26 %	46.4 %
AKY	0.37 %	6 421	33 018	0.19	0.099	0.60	0.03	0.19 %	14.0 %
ASC	0.76 %	51 695	18 639	2.77	0.082	0.96	0.13	0.29 %	1.2 %
AWO	0.28 %	11 593	32 896	0.35	0.141	0.73	0.04	0.21 %	17.6 %
BEL	1.85 %	8 402	3 210	2.62	0.072	0.55	0.12	0.51 %	1.0 %
BLO	0.90 %	5 786	3 427	1.69	0.094	1.21	0.09	0.43 %	2.4 %
CECO	1.64 %	2 688	1 708	1.57	0.080	0.90	0.07	0.59 %	2.8 %
CEQ	0.50 %	6 792	14 747	0.46	0.129	0.72	0.04	0.35 %	9.7 %
CMI	1.14 %	22 645	23 562	0.96	0.094	0.59	0.05	0.55 %	5.1 %
CRU	0.35 %	108 079	115 911	0.93	0.211	2.48	0.08	0.37 %	6.4 %
DESSC	0.50 %	96 041	21 024	4.57	0.091	-1.36	0.21	0.23 %	0.3 %
DNBNOR	0.12 %	12 926	73 865	0.17	0.447	0.95	0.09	0.19 %	62.3 %
DNO	0.18 %	49 731	200 545	0.25	0.485	2.14	0.10	0.31 %	35.4 %
DOCK	1.84 %	21 852	36 329	0.60	0.073	0.25	0.04	0.53 %	2.8 %
ECHEM	0.93 %	62 633	58 330	1.07	0.071	0.11	0.14	0.25 %	0.5 %
EKO	0.61 %	34 936	29 812	1.17	0.089	0.68	0.19	0.27 %	0.3 %
EME	0.38 %	7 512	25 502	0.29	0.168	0.95	0.04	0.33 %	29.2 %
EMGS	0.83 %	5 077	10 723	0.47	0.104	0.63	0.04	0.46 %	8.8 %
FAST	0.61 %	23 829	73 450	0.32	0.096	0.55	0.03	0.30 %	14.8 %
FOE	0.26 %	2 248	4 043	0.56	0.118	0.80	0.03	0.16 %	18.2 %
FRO	0.18 %	13 689	22 690	0.60	0.496	2.79	0.09	0.32 %	14.9 %
GAS	0.53 %	5 941	8 775	0.68	0.087	0.67	0.05	0.23 %	5.0 %
GGG	0.79 %	257 290	170 754	1.51	0.086	0.69	0.05	0.34 %	2.5 %
GOGL	0.12 %	14 499	213 569	-	1.019	1.56	0.27	0.32 %	78.2 %
GOL	0.86 %	12 780	6 542	1.95	0.095	1.91	0.12	0.42 %	1.3 %
IGE	0.95 %	259 631	117 989	2.20	0.092	2.57	0.10	0.45 %	1.6 %
INM	1.55 %	18 272	11 527	1.59	0.076	0.93	0.05	0.51 %	3.2 %
JIN	0.50 %	6 446	10 740	0.60	0.130	0.97	0.04	0.35 %	13.9 %
MHG	0.22 %	78 617	1 658 336	-	0.850	1.62	0.21	0.52 %	74.8 %
NAUR	1.21 %	10 779	18 438	0.58	0.081	0.71	0.04	0.45 %	8.0 %
NEC	0.44 %	66 698	77 645	0.86	0.098	0.64	0.04	0.23 %	7.3 %

Table A1(b): Regression results and characteristics for the companies in the sample

Ticker	\hat{s} : av spread, % of price	\hat{C} : est. critical vol.	SD vol	\hat{c} : est. normalized critical vol.	$\sigma_{m,j}$: est. normalized mkt.vol	z/s : est. total effect of vol > \hat{C} per average spread $0.5\text{Sum}(A_{h,j}-B_{h,j})/s$	SD. z	$\sigma_{p,j}$: intraday SD of price	δ : frequency of significant trades
NHY	0.11 %	10 022	116 964	-	0.707	1.32	0.24	0.24 %	81.9 %
NPRO	0.73 %	83 584	23 125	3.61	0.099	1.68	0.16	0.38 %	0.7 %
NSG	0.14 %	6 381	126 466	-	1.387	1.88	0.43	0.44 %	83.4 %
OCR	0.30 %	22 093	41 347	0.53	0.149	0.85	0.04	0.24 %	9.9 %
ODIM	0.64 %	26 278	7 568	3.47	0.119	2.22	0.21	0.42 %	0.6 %
ORK	0.13 %	7 372	102 288	0.07	0.327	0.66	0.15	0.17 %	76.8 %
PAR	0.35 %	20 767	25 245	0.82	0.157	1.39	0.05	0.29 %	10.2 %
PDR	0.53 %	586 243	1 329 491	0.44	0.102	-0.21	0.22	0.29 %	0.3 %
PGS	0.17 %	6 878	34 826	0.20	0.338	1.44	0.10	0.24 %	50.8 %
PRON	1.15 %	132 630	88 095	1.51	0.078	0.02	0.20	0.39 %	0.5 %
PRS	0.19 %	12 646	36 934	0.34	0.265	1.52	0.08	0.23 %	10.2 %
QEC	2.04 %	13 961	15 443	0.90	0.070	0.34	0.03	0.51 %	6.3 %
RCL	0.40 %	3 406	3 654	0.93	0.098	0.90	0.05	0.20 %	5.1 %
REC	0.12 %	2 758	55 229	-	0.787	1.08	0.40	0.28 %	90.0 %
SBX	1.34 %	51 766	12 043	4.30	0.076	1.43	0.15	0.44 %	0.4 %
SCH	0.37 %	1 701	13 023	0.13	0.140	0.67	0.04	0.28 %	22.5 %
SCORE	0.58 %	16 325	7 427	2.20	0.088	0.54	0.07	0.25 %	1.6 %
SDRL	0.19 %	17 670	55 728	0.32	0.251	0.93	0.05	0.23 %	38.7 %
SEVAN	0.40 %	99 349	50 594	1.96	0.124	-0.31	0.18	0.27 %	0.6 %
SNI	0.47 %	20 009	8 241	2.43	0.111	2.96	0.19	0.28 %	0.7 %
SONG	0.50 %	4 793	39 932	0.12	0.085	0.28	0.02	0.21 %	13.4 %
STB	0.20 %	19 405	46 260	0.42	0.406	1.69	0.09	0.31 %	29.2 %
STL	0.04 %	16 710	230 900	-	1.609	4.15	0.89	0.14 %	88.7 %
SUB	0.27 %	5 553	15 363	0.36	0.194	0.79	0.05	0.26 %	24.6 %
TEL	0.18 %	18 489	88 611	0.21	0.255	0.82	0.06	0.21 %	49.9 %
TGS	0.20 %	9 159	27 502	0.33	0.456	0.96	0.08	0.34 %	25.9 %
TOM	0.24 %	13 728	19 078	0.72	0.234	1.73	0.06	0.27 %	15.0 %
TAA	0.21 %	10 248	55 314	0.19	0.257	0.90	0.08	0.25 %	45.1 %
VEI	0.70 %	11 741	6 626	1.77	0.079	0.05	0.06	0.25 %	2.2 %
WAVE	1.40 %	68 818	21 804	3.16	0.087	0.21	0.11	0.61 %	0.6 %
YAR	0.11 %	13 168	71 674	0.18	0.575	1.94	0.13	0.22 %	53.7 %

Table A2(a): Regression results and characteristics for the companies in the sample

Ticker	Company name	Sector	Average price	Av daily turnover m NOK	# of days sampled	# of obs. in the sample	Condition Index	Adjusted R2	Auto-correlation ρ
ACTA	Acta Holding	Financial Services	20.99	69.31	13	1083	457	0.15	0.15
ACY	Acergy S.A	Energy	128.77	432.92	15	1269	463	0.13	0.16
AGR	Ability Group	Energy	49.16	11.10	10	623	-	0.11	0.17
AKER	Aker	Energy	351.00	79.22	14	1176	154	0.09	0.16
AKVER	Aker Kværner	Energy	170.20	343.21	14	1180	492	0.16	0.23
AKY	Aker Yards	Industry	84.33	80.28	15	1243	127	0.12	0.25
ASC	ABG Sundal Collier	Financial Services	13.16	5.90	7	366	-	0.15	0.19
AWO	Awilco Offshore	Energy	59.70	72.48	10	845	160	0.21	0.13
BEL	Belships Co.	Industry	24.49	1.47	9	439	123	0.26	0.09
BLO	Blom	Industry	67.62	7.89	10	764	-	0.15	0.14
CECO	Camillo Eitzen & Co	Industry	72.09	2.43	7	316	-	0.08	-0.27
CEQ	Cermaq	Food	69.38	31.92	12	995	149	0.15	0.11
CMI	Intex Resources	Materials	8.95	6.42	12	845	-	0.17	0.22
CRU	Crew Gold Corporation	Materials	8.39	39.91	14	1162	202	0.23	0.25
DESSC	Deep Sea Supply Plc	Energy	24.25	19.03	12	982	395	0.12	0.18
DNBNOR	DnB NOR	Financial Services	84.67	459.59	17	1445	491	0.17	0.10
DNO	DNO	Energy	9.24	91.56	15	1263	491	0.17	0.19
DOCK	DOCKWISE LTD.	#N/A	21.97	14.73	9	407	-	0.10	0.02
ECHEM	Eitzen Chemical	Industry	25.61	8.26	12	577	176	0.08	0.08
EKO	Ekornes	Consumer products	89.19	20.11	10	672	127	0.06	0.12
EME	Ementor	Information Technology	37.36	46.16	11	922	124	0.17	0.12
EMGS	ElectroMagnetic GeoServices	Energy	73.83	22.27	12	854	-	0.15	0.17
FAST	Fast Search & Transfer	Information Technology	10.92	26.88	15	1132	-	0.14	0.14
FOE	Fred. Olsen Energy	Energy	273.06	60.42	14	1175	133	0.19	0.09
FRO	Frontline	Energy	234.74	296.69	15	1271	494	0.23	0.22
GAS	BW Gas	Energy	67.94	12.90	12	802	-	0.08	0.17
GGS	Global Geo Services	Energy	1.50	8.58	12	952	126	0.16	0.05
GOGL	Golden Ocean Group Limited	Industry	35.57	422.66	15	1264	491	0.11	0.16
GOL	Golar LNG	Energy	116.85	19.70	9	667	187	0.22	0.09
IGE	Int. Gold Exploration	Materials	1.88	10.12	10	825	256	0.23	0.04
INM	Inmeta	Information Technology	5.78	1.61	8	516	-	0.19	0.12
JIN	Jinhui Shipping	Industry	65.04	33.78	14	1168	139	0.19	0.20
MHG	Marine Harvest	Food	3.75	353.99	15	1274	493	0.06	0.19
NAUR	Northland Resources	Materials	17.42	9.83	9	667	-	0.15	0.13
NEC	Norse Energy Corp.	Energy	3.94	11.88	10	793	-	0.20	0.10

Table A2(b): Regression results and characteristics for the companies in the sample

Ticker	Company name	Sector	Average price	Av daily turnover m NOK	# of days sampled	# of obs. in the sample	Condition Index	Adjusted R2	Auto-correlation ρ
NHY	Norsk Hydro	Materials	74.50	669.11	17	1445	494	0.11	0.17
NPRO	Norwegian Property	Financial Services	65.42	38.66	9	682	210	0.21	0.05
NSG	Norske Skogindustrier	Materials	37.05	270.57	15	1265	492	0.11	0.14
OCR	Ocean Rig	Energy	41.16	62.49	15	1248	235	0.17	0.06
ODIM	Odim	Industry	79.62	23.10	10	810	233	0.17	0.20
ORK	Orkla	Industry	98.17	678.29	16	1358	498	0.11	0.18
PAR	PA Resources	Energy	49.08	53.97	12	1009	137	0.19	0.22
PDR	Petrolia Drilling	Energy	2.59	34.75	11	876	360	0.16	0.14
PGS	Petroleum Geo-Services	Energy	150.76	292.19	12	1011	459	0.20	0.18
PRON	Pronova BioPharma ASA	#N/A	22.88	14.40	13	658	234	0.02	0.14
PRS	ProSafe	Energy	94.56	94.23	13	1092	353	0.18	0.24
QEC	Questerre Energy Corporation	Energy	3.89	1.18	11	605	-	0.11	0.00
RCL	Royal Caribbean Cruises	Consumer products	211.80	22.24	11	798	-	0.14	0.12
REC	Renewable Energy Corporation	Industry	261.69	1 036.44	17	1438	476	0.03	0.15
SBX	SeaBird Exploration	Energy	20.49	5.21	8	506	155	0.16	0.12
SCH	Schibsted	Consumer products	253.09	67.37	11	914	114	0.24	0.15
SCORE	Scorpion Offshore	Energy	72.56	20.48	12	835	-	0.07	0.15
SDRL	SeaDrill Ltd	Energy	117.75	403.90	17	1443	499	0.18	0.13
SEVAN	Sevan Marine	Energy	72.89	84.91	14	1168	347	0.12	0.16
SNI	Stolt-Nielsen	Industry	140.24	35.00	12	984	246	0.17	0.25
SONG	Songa Offshore	Energy	62.84	41.50	14	1095	-	0.11	0.12
STB	Storebrand	Financial Services	62.34	180.41	14	1180	432	0.18	0.25
STL	StatoilHydro	Energy	173.93	3 014.77	16	1392	491	0.08	0.20
SUB	Subsea 7	Energy	133.43	101.50	10	840	165	0.15	0.19
TEL	Telenor	Telecommunications	127.13	699.16	17	1444	486	0.13	0.13
TGS	TGS-NOPEC Geophysical Co	Energy	77.66	105.80	15	1257	350	0.11	0.11
TOM	Tomra Systems	Industry	35.44	46.40	13	1093	290	0.19	0.03
TAA	Tandberg	Information Technology	119.62	334.28	15	1266	491	0.17	0.14
VEI	Veidekke	Industry	46.22	13.34	12	895	-	0.07	0.14
WAVE	Wavefield Inseis	Energy	42.31	32.07	10	774	208	0.06	0.06
YAR	Yara International	Materials	196.27	800.51	15	1275	497	0.17	0.14

Optimal distribution of information by an information monopolist: A generalization

Espen Sirnes

Abstract

In this paper the results of Admati and Pfleiderer (1986) and Admati and Pfleiderer (1990) are generalized, and the cases where their main results do not hold are revealed. Admati and Pfleiderer show that an information monopolist should sell information with the same precision to buyers. This is proven in a very general model, and it is shown that the cases where this is not true are when agents are heterogeneous and when supplying the same information to many investors have a cost.

Keywords: Distribution of information, finance, asset pricing.

JEL Classification: G12, G14

The main contribution of this paper will be to pinpoint two conditions that must be met in order for a seller of information in a financial market to sell different precisions to different investors. The problem of a seller is that the information sold will be used in the market and hence be reflected in the price. This dilutes its value, and so it is important for the seller to control the information in one way or the other.

Admati and Pfleiderer (1986) shows that an information monopolist who sells information directly should add independent personalized noise to the signals he sells to each investor. This reduces the impact of the information in the market and thereby enables the seller to control its use. The optimal

distribution of information is however symmetric in the sense that all *buyers* receive the same *type* of signal, i.e. the same precision, Although they get different realizations. This notion of symmetry will be used throughout the paper.

Admati and Pfleiderer (1990) finds that if the information is sold through a mutual fund, then the seller can control the use of the information more effectively by pricing the fund appropriately. In that case it is therefore not necessary to add noise to the signals or to sell independent signals. Thus with indirect sale of information through a fund, the investors receive identical information, i.e. a single fund is sold. Furthermore, there is not added noise to the signal, so the buyers receive signals with the same precision as that observed by the seller.

In this paper a general proof of the symmetric solution is provided. The proof is of course useful in generalizing the results of Admati and Pfleiderer. Its main contribution does however lie in the two main conditions required for such a solution.

It is found, not surprisingly, that a symmetric equilibrium does require identical agents. A solution strategy is then suggested in the case of heterogeneous agents.

It is however also found that if there are costs associated with issuing the same information to many agents, then it may also be optimal with an asymmetric solution. Thus, the information monopolist may prefer to sell different precisions and/or different signals even though investors are identical.

An explanation for such costs could be a cost incurred by an insider, who increases his chances of being exposed when he sells the same information to many traders. Supplying different information to different individuals may therefore be safer.

If it is optimal for the seller to sell different types of information to different traders, then he faces the problem that the buyers may collude by

pooling the information. It is suggested that this can be encountered by selling nested information.

How an information monopolist can exploit information is a recurring issue in financial literature. Ozerturk (2004) shows that when precision is unobservable, an information seller can verify his precision by a non-linear contract. In this paper precision is known by both the seller and the buyer. Of recent research Biais and Germain (2002) also looks at contracts between an information seller and a client. What is studied is there is the optimal contract for the buyer that makes the fund manager behaves in the interest of the client.

The proof that will be provided is very general, but in order to not detach ourselves completely from the original papers of Admati and Pfleiderer, we do derive an equilibrium condition which the profit function depends on. The proof does however hold for any type of profit function, and will therefore hold even if contracts cannot be written to capture all consumer surplus. For instance according to Brennan and Chordia (1993) it might be argued that it is unclear how contracts can be written in this way.

Since the profit function is defined fairly general in the end, the model does not necessarily assume that the seller can not trade on his own account. This kind of setup with a trading information monopolist was suggested by Admati and Pfleiderer (1988). This seem to be possible only when a market maker sets prices such as in Kyle (1985) so that the monopolist can act more strategic in the market and avoid the effect seen in Grossman and Stiglitz (1980) as well as Admati and Pfleiderer (1986) and Admati and Pfleiderer (1990). In such models, information sold is partially transferred to uninformed investors immediately through prices, reducing its value.

The paper is organized as follows. In the first section we present the setup along the lines of Admati and Pfleiderer, and state the profit maximization problem. The setup will however be formulated more general in order for the proof in section II to be so too.

In Section II we start by proving that in general the solution is symmetric with respect to the information type. We then continue to investigate the two cases where the seller will distribute different precision to different traders. A short concluding comment is then given in the final section.

I Continuous information distribution

Let there be a risk free asset which yields a zero rate of return and a risky asset which returns \tilde{W} in the next period. The risky return is stochastic, but realized before the decision of investment. \tilde{W} is normally distributed as

$$\tilde{W} \sim N(\mu_W, \sigma_W^2) \quad (1)$$

\tilde{W} is however not directly observable. However, an information seller can observe an estimate of the return, $\tilde{W} + \tilde{\theta}$, where

$$\tilde{\theta} \sim N(0, \sigma_\theta^2) \quad (2)$$

Based on this estimate, the seller sells information signals in the form of the vector

$$\tilde{s}_{t,\alpha,n} \sim N(\mu_{t,\alpha,n}, \Gamma_{t,\alpha,n}) \quad (3)$$

to investors of type

$$(t, \alpha, n) \in [0, 1] \times \mathbb{R}^3 \quad (4)$$

where t characterizes the information "type" (we will define this concept later), α the investor type (e.g. risk aversion), and n a realization of the signal. The intention by having the realizations enumerated is to allow for independent realizations of $\tilde{s}_{t,\alpha,n}$ among investors.

$\tilde{s}_{t,\alpha,n}$ contains V signals, $\tilde{s}_{t,\alpha,n,v}$ for $v \in \{1, \dots, V\}$.

Denote the general set of distributions in the market as an "information structure". The exact definition of this term will be given later.

As the issuer of information, the seller can design any information structure. This in turn means that the seller has full discretion to set Γ_t as any symmetric non-singular matrix¹, with the restriction that the precision level of this covariance matrix is not higher than the original estimate of the seller. We can formulate this requirement formally by defining the precision level of by Γ_t as

$$\psi_t = \mathbf{1}\Gamma_t^{-1}\mathbf{1} \quad (5)$$

This measure is equal to the highest precision that is possible to obtain by combining the signals \tilde{s}_t by any kernel. Since this level of precision cannot be higher than that of the original estimate

Definition 1 Γ_t are admissible if $\psi_t = \mathbf{1}\Gamma_t^{-1}\mathbf{1} \leq 1/(\sigma_W^2 + \sigma_\theta^2)$

The precision level of ψ_t and the value of information is obviously closely related, but the value of the signals also depends on how they interact with the price and so these concepts are not equivalent. How this works, is determined by the market equilibrium

A. Market equilibrium

The market equilibrium condition is that the total demand must equal supply \tilde{Z}

$$\int_0^1 \int_0^1 f_\alpha f_t \left(x'_{t,\alpha} \tilde{s}_{t,\alpha} + k_{t,\alpha}^* \tilde{P} \right) dt d\alpha, \alpha, ndr = \tilde{Z} \quad (6)$$

Explanation of the parameters:

1) \tilde{Z} is the random supply of the asset from noise traders $\tilde{Z} \sim N(0, \sigma_Z^2)$. The constant term of demand will equal any constant supply, and so we can ignore these constant terms without loss of generalization.

¹A singular matrix would of course just imply redundant signals, and hence that the seller should sell fewer signals.

2) $\tilde{s}_{t,\alpha} = \int_0^1 \tilde{s}_{t,\alpha,n} dn$ since individual realizations of the signals disappears as we integrate over all investors.

3) $x_{t,\alpha}$ is the vector of portfolio weights assigned to each signal in $\tilde{s}_{t,\alpha}$. This may be determined by the optimality conditions of the buyer (Admati and Pfleiderer (1986)), or by strategic considerations by the seller (Admati and Pfleiderer (1990)).

4) $k_{t,\alpha}^*$ is the portfolio weight that investors puts on prices, and is set optimally by the buyer. If not, then $k_{t,\alpha}^* = 0$ by the seller, and the model becomes trivial.

5) f_α and f_t are densities assigned by the seller for investor types and information types. It is assumed that the assignment can be done independently, and hence the joint distribution is separable. Denote the distribution of information $\zeta = (f_\alpha, f_t) \forall (\alpha, t) \in \mathbb{R}^3 \times [0, 1]$

B. The profit function

The profit function can now be defined. Due to the equilibrium condition, any profit function will depend on the covariance matrix Γ_t and the covariances between the price and the signals, which will be denoted $\gamma_{P,t}$. If the seller can manipulate the use of the signal for all agents, $x_{t,\alpha}$, this will also affect profit from each of these buyers. Since each buyer is atomic, this will however only happen through $\gamma_{P,t}$, so the profit for each buyer can be written as

$$c_{t,\alpha}(\Gamma_t, \gamma_{P,t}) \tag{7}$$

We will refer to this as simply the profit function.

As found by Admati and Pfleiderer (1990) the seller can either sell information directly or indirectly through a fund. In either case, it is advantageous for the seller limit the use $x_{t,\alpha}$ of the information $s_{t,\alpha}$. If the sale is done directly to the investor, it is optimal to restrict $x_{t,\alpha}$ by adding noise to the signal, as found by Admati and Pfleiderer (1986). If information is

sold through a fund, then $x_{t,\alpha}$ is the actual amount of the fund bought by investor (t, α) . In that case the investor can charge a fee per share in the fund and thereby induce which ever level of investment $x_{t,\alpha}$ he wants. Since indirect sale of information means direct control over its use, as opposed to direct sale, selling the information through a fund is strictly more profitable than selling noisy versions of it directly. For the same reason, there is no point in adding noise to the signal if the seller has control over it use $x_{t,\alpha}$. How information is sold therefore determines how the signal impact $x_{t,\alpha}$ can be set by the seller. If the seller does not have full control over $x_{t,\alpha}$, he might choose to sell signals with less than full precision.

We may however assume that $\gamma_{P,t}$ is determined by optimization of $x_{t,\alpha}$, either by the buyer or the seller. The model that will be presented is therefore not specifically related to neither direct nor indirect sale, and so the results hold for both. Since we will not specify the functional form, we might as well just drop the argument specification since for a given distribution of information ζ , profits are given by (t, α) .

Since $c_{t,\alpha}$ may take almost any functional form, this model also covers the cases where perhaps not all of the consumer surplus can be extracted by the seller. In this continuous setting we do however need to make the following assumptions about $c_{t,\alpha}$

Assumption 1 $c_{t,\alpha}$ is continuous and $c_0 = 0$.

We will also make the following convenient and generalizing assumption

Assumption 2 If $c_{t,\alpha} = c_{r,\alpha}$, then $t = r$

Assumption 2 implies that the information type t is defined by the profit it generates. This sidesteps a considerable amount complexity, without being either restrictive or unrealistic. To see this, note that if t is taken to be signal precision, then both models of Admati and Pfleiderer satisfies Assumption 2 by their definition of information structure.

The problem for the information seller now is to

$$\max_{f_t, f_\alpha} \Pi = \int_0^1 \int_0^1 f_t f_\alpha c_{t,\alpha} dt d\alpha \quad (8)$$

Now, make the following definitions

Definition 2 *The matrix of covariances of two information types t and r , $\Gamma_{(t,\alpha),(r,\alpha)} = \text{cov}(\tilde{s}_{t,\alpha}, \tilde{s}_{r,\alpha})$, are admissible if $\psi_{(t,\alpha),(r,\alpha)} = \mathbf{1}\Gamma_{(t,\alpha),(r,\alpha)}^{-1}\mathbf{1} \leq 1/(\sigma_W^2 + \sigma_\theta^2)$*

Definition 2 ensures that the seller cannot sell signals that in sum are better than the original information. Note that $\Gamma_{(t,\alpha),(t,\alpha)}$ and $\Gamma_{t,\alpha}$ are not the same if traders receive independent personalized information. In that case, the covariance of that signal between two different buyers would be zero.

Further we define

Definition 3 *An information structure is an admissible set of $\Gamma_{(t,\alpha),(k,\alpha)}$, $\Gamma_{t,\alpha} \forall \{(t, \alpha), (r, \alpha)\}$.*

Information structures may differ with respect to how easy it is for agents to pool information. Two buyers may for example share their information. For some information structures this will lead to increased precision. It is therefore useful to define information structures that are robust for such pooling. We define this as

Definition 4 *An information structure is said to be nested if whenever $\psi_{t,\alpha} > \psi_{r,\alpha}$, then $\psi_{t,\alpha} = \psi_{(t,\alpha),(r,\alpha)}$, $\forall \{(t, \alpha), (r, \alpha)\}$*

What Definition 4 says, is that if one buyer is better informed than the other, then he will gain no precision by observing the signal of the less informed.

A particularly simple nested information structure is obtained if those with the best information gets to observe all the signals of those with inferior information. Say each "information bit" is an independent draw from the same normal distribution, that type t indicates t such draws, and that all investors have the same sequence of draws. Then it follows that the best precision is obtained² by taking the average of all observations. If the number of draws is a continuous variable, than the distribution resulting from such an information structure is given by the process

$$U = \left\{ \tilde{u}_t : \tilde{u}_t = \tilde{W} + \frac{1}{t} \tilde{\varepsilon}_t \right\}, t \in (0, 1] \quad (9)$$

where the error increments are distributed independently according to a Brownian motion

$$d\varepsilon_t \sim N(0, \sigma_\varepsilon) \quad (10)$$

It can be found that $\text{var}(u_t) = \sigma_W^2 + \frac{1}{t} \sigma_\varepsilon^2$ and $\text{cov}(u_t, u_s) = \sigma_W^2 + \frac{1}{\max(t,s)} \sigma_\varepsilon^2$. Using Definition 4 it can be found that (9) is a nested information structure.

In order to give some intuition behind the distribution (9), think of an unpublished quarterly report. A non-nested information structure would then correspond to selling different pages³ to different investors. This makes it however possible for all buyers to come together and reconstruct the report, and hence obtain full information for the average payment of a single page.

A nested information structure on the other hand, correspond to selling a set of consecutive pages from the report up to a given page number. This means that those with the most information has no incentive what so ever to waste time comparing pages with the less informed. A nested information structure is thus an efficient way to avoid pooling of information.

²regardless of whether Bayesian updating or sampling theory is used.

³of course assuming each page adds the same precision

II Profit maximization

We will here prove that the distribution ζ the uninformed should optimally select is one with symmetric information distribution among buyers. We will then look at the conditions for the proof, and see when such a symmetric equilibrium may not apply.

A. *The general case*

From the maximization problem (8), we see that the maximization problem of the seller is really quite simple. Setting the densities $f_t f_\alpha$ as high as possible wherever $c_{t,\alpha}$ is at its maximum, maximizes profits. Since we have defined all signals that generate the same income to the seller as belonging to the same "type" of information, and α is a characteristic of the individual buyer, there can be only one such maximum point.

Of course one has to take into account that the distribution ζ itself affects $c_{t,\alpha}$, so which t and α we should select is not a trivial problem, as is evident from Admati and Pfleiderer (1986) and Admati and Pfleiderer (1990). What the maximization problem (8) reveals though, is that the problem of selecting the optimal distribution simplifies to finding the point (t, α) where all information sold is concentrated. (8) does however not determine the total mass sold at this point. As in Admati and Pfleiderer (1986) it might be a good idea not to sell information to everybody.

Thus, we can state the following proposition:

Proposition 1 *For any admissible information structure*

- a. *the densities f_t and f_α can take any positive value*
- b. *the profit function independent of f_t and f_α*
- c. *the market is not fully efficient ($\sigma_Z^2 > 0$) and the information is relevant*
- d. *c_t is continuous*

it is always optimal to sell information to only one type of investors.

Proof: (8) is a typical control theory problem. Control theory defines a set of necessary (and sufficient) conditions for profit maximization that, if satisfied, guarantees an optimal solution. The task is therefore to assume a solution, and check if it satisfies the standard necessary conditions. Since the distributions of t and α are independent, it holds to prove Proposition 1 for f_t , and drop the α subscript.

Our control variable is f_t , which is assumed to be a density with a mass function $F_t = \int_0^t f_t dt$, where $F_1 = \lambda$ and $1 - \lambda$ is any mass assigned to $t = 0$. λ can for simplicity be taken as exogenous, and set to maximize the profits after the control problem is solved since Proposition 1 will hold for any value of λ . F_t is thus our state variable. The Hamilton function is the sum of the integrand of (8) and some coefficient p times the change in the state variable:

$$H_t = f_t (c_t + p) \tag{11}$$

The necessary conditions require that at optimum the change in the coefficient, \dot{p} and the effect of the state variable on H_t^* sums to zero, where asterisk * indicates evaluation for optimal f_t^* . Now, since H_t is independent of F_t , it follows that p is a constant.

The necessary conditions require that f_t is bounded, so let $f_t \in [0, \frac{1}{\Delta}]$. Define some interval where $H_t^* > 0$ as $I^* \in [t^* - \Delta, t^*]$. It follows from the optimality condition, in addition to being quite reasonable, that in order to maximize H_t^* it is optimal to set $f_t^* = \frac{1}{\Delta}$ in this interval.

For optimality a necessary condition is that p is set so that when the total mass of investors F_t is exhausted, we stop. Denote this point t^* . This control problem with a free stopping time and a single state variable has the transversality condition that $p = H_{t^*}^*$.

From this, and that c_t is continuous and has a single maximum by construction, it follows that for some arbitrarily small Δ there exist only one interval I^* where $H_t^* > 0$. Letting $\Delta \rightarrow 0$ concludes the proof ■

Proposition 1 means that any problem (8) can be simplified by assuming

a single type of information, and so the results of Admati and Pfleiderer is more general than previously thought.

Further, Proposition 1 and its proof is powerful because it includes two important sufficient conditions. This enables us to identify the under which circumstance an asymmetric solution may obtain. Not all the conditions are important in this respect though. Condition c. is trivial. Condition d. can be made redundant by solving the discrete optimal control problem, which would be very similar to the proof above, and is therefore not included in this paper. The really interesting conditions are thus a. and b..

Thus a symmetric solution requires that the densities are unrestricted. That is a fine assumption about f_t , but f_α would typically have a maximum value equal to the total number of investors with characteristics α . Perhaps the seller could meet with them and try to persuade them to change types, but that would quite possibly be a expensive strategy considering the infinite number of buyers. Thus f_α should be set in an interval, and for simplicity one might consider a uniform distribution.

Second, we see that a symmetric solution is unlikely if there are extra costs associated with assigning large densities to the same types. We will now take a closer look at the implications of breaching conditions a. and b. of Proposition 1. We will however only suggest how one might approach the problems, and not solve for optimal distributions.

B. Heterogeneous traders

Assume all traders have been assigned information of type t . and for illustrative purposes we will drop the t subscript and assume that c_α is globally concave in α . Furthermore $f_\alpha \in [0, \bar{f}]$. As shown by Verrecchia (1982), traders with different risk aversion will optimally buy different information from the seller. The problem now is however how the seller should assign information to them, and so the buyers have no impact on which information they receive. Their willingness to pay for the information does however vary across α . We

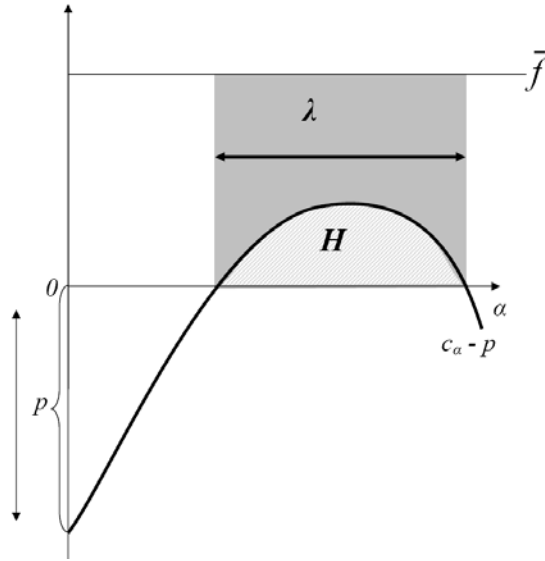


Figure 1: How to set optimal densities when there are only a given number of traders of each type

do of course here implicitly assume that the seller can discriminate perfectly between the customers. If this was not the case, the distribution of α would be endogenous, and the symmetric result in the Proposition 1 would hold.

Figure 1 should give the reader an idea of how the problem in principle could be approached. The seller now sets $f_\alpha^* = \bar{f}$ whenever $H_t^*/f_\alpha^* > 0$ and $f_\alpha^* = 0$ whenever $H^*/f_\alpha^* < 0$.

We see that as the seller changes p in the Hamilton function (11), the mass that gets to buy information λ changes. However, so will the parameters of the function $c_{t,\alpha,n}$. Thus, Figure 1 holds for only one specific distribution of information. This is what makes an explicit solution quite complicated. The Hamilton function with a uniform distribution is given by the area H . Figure 1 does however show the problem of the uninformed is to set some p that maximizes the area H by selling different information to different traders.

The problem was simpler when the seller could set an infinite f_α , because

the graph $c_\alpha - p$ then would be below 0 for any α except at the maximum of the function, α^* .

As we see it is quite likely that with such a restriction on the density, the seller will sell information with different precision to different investors.

C. Cost of assigning density

The results from Proposition 1 and the previous discussion does seem quite obvious in a way. What was proven in Proposition 1 was that in a symmetric setting we would have a symmetric solution. Accordingly we have shown tentatively in the previous section that with an asymmetric setup the solution is asymmetric. We will however here show that an asymmetric solution is possible with identical agents, that is for a given α by breaching condition b.. Thus we will not assume that the profit function is independent of f_t .

Assume instead that there is some cost that increases with the density assigned information type t . Note that this cost can not be attributed to the negative effect density has on the value of information through prices. This dilution of information value is all ready be taken into account by the moments of the price function. Therefore an explicit cost of selling the same type of information t to a high density of investor would have to be motivated in other ways.

One explanation for such costs could be that an insider might increase the risk of being exposed as he sells the same information to an increasing number of agents. In any case, assume now for whatever reason that there are such costs. For simplicity, we will just assume the cost is f_t , and deduct it from c_t . Thus, we have a new version of (8) to maximize:

$$\max_{f_t \forall t \in [0,1]} \Pi = \int_0^1 f_t (c_t - f_t) dt \quad (12)$$

This implies that the Hamilton function (11) now is quadratic in f_t . Therefore the optimal is no longer to set f_t to its maximum whenever $c_t - p$

is positive. In stead for a given optimal distribution $f_{t,\alpha,n}^*$, the densities must satisfy an internal solution $f_t^* = \frac{1}{2}c_t$. Finding the optimal distribution is quite complicated, so we will not do that here. It is sufficient to note that it will not in general be optimal to concentrate all information at one point t by letting $f_{t^*}^* \rightarrow \infty$ as in Proposition 1. Thus in this case, it is likely that an asymmetric distribution of information will be optimal.

D. Preventing collusion among the buyers

Assume now that different investors are assigned different information types to different investors, either because condition a. or condition b. in Proposition 1 is breached. Depending on the information structure it may then be advantageous for investors to collude in order to increase their precisions without paying the seller extra. In such a situation it is also clearly not advantageous to sell different realizations of the signal $\tilde{s}_{t,\alpha,n}$ to different traders, so we may assume at least that all buyers of a specific information type t receives identical photocopied signals. The problem for the seller is however to find an information structure that prevents too much collusion.

A nested information structure, as defined in Definition 4 will accomplish that. It can be shown that by selling signals of the type presented in (9), the precision (5) of a signal vector $\tilde{s}_{t,\alpha,n}$ consisting of any set of nested information signals, will always equal that of the most precise signal.

Note that when using the nested signal, investors will use the knowledge of inferior signals to calculate the part of the price that is generated by known information. This means that the price that is conditioned on will be that generated by only the last part of the integrals in the equilibrium condition (6). That is, such investors with information t will condition on a price based on the integral in (6) from t and not 0.

III Conclusion

A general proof is given for a symmetric solution with homogenous agents and unlimited density. The result does require that the profit function c_t is continuous in t . However the control variable f_t need not be continuous, so a discontinuous set of types t can be selected.

Furthermore it has been shown that if either investors are heterogeneous so that an information monopolist cannot sell information to any density of investors, or if there are explicit costs associated with high densities, then it might not be optimal to sell the same information structure to everyone. Thus, it seems that one of these two conditions must be breached in order for an asymmetric solution to emerge.

The critical conditions for a symmetric solution may be helpful for future research on the area, since it gives a clear answer to which conditions must at least be met for a asymmetric solution to obtain.

References

- Admati, Anat R., and Paul Pfleiderer, 1986, A Monopolistic Market for Information, *Journal of Economic Theory* 39, 400–438.
- Admati, Anat R., and Paul Pfleiderer, 1988, Selling and Trading on Information in Financial Markets, *American Economic Review* 78, 96–103.
- Admati, Anat R., and Paul Pfleiderer, 1990, Direct and Indirect Sale of Information, *Econometrica* 58, 901–928.
- Biais, Bruno, and Laurent Germain, 2002, Incentive-Compatible Contracts for the Sale of Information, *Review of Financial Studies* 15, 987–1003.
- Brennan, Michael J., and Tarun Chordia, 1993, Brokerage Commission Schedules, *The Journal of Finance* 48, 1379–1402.

Grossman, S. J., and J. E. Stiglitz, 1980, On the Impossibility of Informationally Efficient Markets, *The American Economic Review* 70, No. 3, 393–408.

Kyle, Albert S., 1985, Continuous Auctions and Insider Trading, *Econometrica* 53, 1315–1336.

Ozerturk, Saltuk, 2004, Direct Sale of Information When Precision is Unobservable, *Canadian Journal of Economics* 37, 269–293.

Verrecchia, Robert E., 1982, Information Acquisition in a Noisy Rational Expectations Economy, *Econometrica* 50, 1415–1430.

IV Appendix

A. The precision measure

$$\min_{x_{t,\alpha,n}} x'_{t,\alpha,n} \Gamma_{t,\alpha,n} x_{t,\alpha,n}, \text{ s.t. } x'_{t,\alpha,n} \mathbf{1} = 1 \quad (\text{A.1})$$

$$\frac{dL}{dx'_{t,\alpha,n}} = 2\Gamma_{t,\alpha,n} x_{t,\alpha,n} - \lambda \mathbf{1} \quad (\text{A.2})$$

$$2x'_{t,\alpha,n} \Gamma_{t,\alpha,n} x_{t,\alpha,n} = \lambda \quad (\text{A.3})$$

$$2\Gamma_{t,\alpha,n} x_{t,\alpha,n} = 2x'_{t,\alpha,n} \Gamma_{t,\alpha,n} x_{t,\alpha,n} \mathbf{1} \quad (\text{A.4})$$

$$\Rightarrow x'_{t,\alpha,n} \Gamma_{t,\alpha,n} x_{t,\alpha,n} = \mathbf{1} \Gamma_{t,\alpha,n}^{-1} \mathbf{1} \quad (\text{A.5})$$

Conclusion

I have presented three papers, where the first one, "Are Noise Traders Really Necessary? A General Approach", is a general asymmetric information model that enables comparison between very different financial models. It is argued that assuming noise traders is a productive way of building financial models, since it does not require special assumptions on how non-informational trade is generated. Unless strong empirical evidence exists, such assumptions may sometimes obscure the models. This notion is then applied to the two other papers in two different directions.

The second paper, "Optimal Order Submission", is a relatively specific model aiming to reveal features of a specific type of market institution. Using this model, I test its predictions on data from Oslo Stock Exchange, and find that the model does a fairly good job at predicting order size.

In the third paper, "Optimal Distribution of Information by an Information Monopolist: A Generalization", noise traders are necessary to analyze the problem, since if they do not exist no information would be sold at a positive price. A general characterization of the conditions under which asymmetric solutions are obtained is then found, which may be helpful for future research in the area. This and the preceding paper are however very different in the sense that one is very specific, the other very general.

I am not arguing that all models should be created in a specific manner here though. If more deterministic models have theoretical advantages, then that will of course be an appropriate approach. It does however seem to me that allowing for traders to behave in ways not explained by the model may sometimes be more realistic.