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Coloring the Voronoi tessellation of lattices

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Abstract

In this paper we define the chromatic number of a lattice: It is the least number of colors one needs to color the interiors of the cells of the Voronoi tessellation of a lattice so that no two cells sharing a facet are of the same color.

We compute the chromatic number of the root lattices, their duals, and of the Leech lattice, we consider the chromatic number of lattices of Voronoi's first kind, and we investigate the asymptotic behavior of the chromatic number of lattices when the dimension tends to infinity.

We introduce a spectral lower bound for the chromatic number of lattices in spirit of Hoffman's bound for finite graphs. We compute this bound for the root lattices and relate it to the character theory of the corresponding Lie groups.

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1. *Introduction*

Let $\Lambda \subseteq \mathbb{R}^n$ be an n-dimensional lattice in n-dimensional Euclidean space. One can tessellate space by lattice translates of the lattice' Voronoi cell, which is defined as

$$V(\Lambda) = \{x \in \mathbb{R}^n : \|x\| \leqslant \|x - v\| \text{ for all } v \in \Lambda\}.$$

By $V(\Lambda)^\circ$ we denote the topological interior of $V(\Lambda)$. Now we consider translates $v + V(\Lambda)^\circ$, with $v \in L$, as colored tiles of an n-dimensional mosaic in which one has infinitesimal small interstices between the mosaic tiles. How many colors does one need at least to get a colorful mosaic? In a colorful mosaic two neighboring tiles receive different colors. This defines the *chromatic number* $\chi(\Lambda)$ of the lattice.

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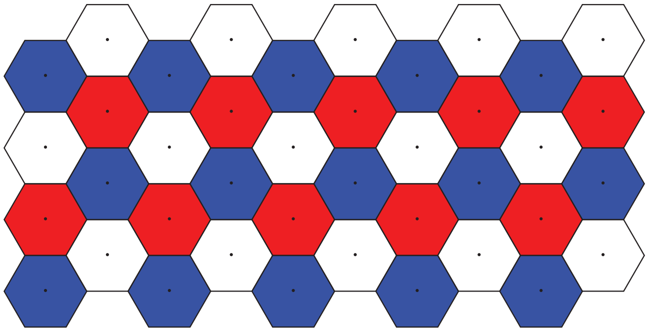


FIGURE 1 (colour online). *Optimal coloring of the hexagonal lattice,* $\chi(\mathsf{A}_2) = 3$.

More formally, we can also define the chromatic number of a lattice in graph-theoretical terms: Two distinct lattice translates of Voronoi cells $v + V(\Lambda)$ and $w + V(\Lambda)$, with $v \neq w$, are defining neighboring tiles whenever they share a facet, that is, their intersection is a polytope of maximal dimension $n - 1$. The differences $v - w$ are called *strict Voronoi vectors* and the set of these vectors is denoted by $\mathrm{Vor}(\Lambda)$. Now the chromatic number of Λ equals the chromatic number of the Cayley graph on the additive group Λ with generating set $\mathrm{Vor}(\Lambda)$:

$$\chi(\Lambda) = \chi(\mathrm{Cayley}(\Lambda, \mathrm{Vor}(\Lambda))).$$

Here, the set of vertices of the Cayley graph are all elements of Λ and two vertices v, w are adjacent whenever the difference $v - w$ lies in the set of strict Voronoi vectors $\mathrm{Vor}(\Lambda)$. Note that the Cayley graph is an r-regular infinite graph with $r = |\mathrm{Vor}(\Lambda)|$.

In Section 2.1 we recall all the definitions and properties of lattices, their Voronoi cells, and the Voronoi vectors, which we need later.

The chromatic number of a lattice seems to be a natural parameter. However, to the best of the authors' knowledge, see [**39**] and [**25**], $\chi(\Lambda)$ has not been considered before. The aim of this paper is to start a systematic investigation of it. Then, the following questions immediately come to mind.

1.1. *Determination of the chromatic number*

What is the chromatic number of some interesting lattices? How to find lower and upper bounds? Is there an algorithm to determine $\chi(\Lambda)$ *for a given lattice* Λ?

For instance, it is obvious that the chromatic number of the integer lattice $\mathbb{Z}^n$ is two, an optimal coloring is given by the black/white checkerboard pattern; see also Theorem 2.5.

We discuss simple lower and upper bounds for the chromatic number of a general lattice in Sections 2.2 and 2.3. For instance, we show that $\chi(\Lambda)$ is at most 2^n.

All two- and 3-dimensional lattices are of Voronoi's first kind. We consider the chromatic number of this class of lattices in Section 3 where we compute the chromatic number of all 3-dimensional lattices. It would be interesting to have a better understanding of the chromatic number of this class of lattices.

One of the most important classes of lattices are the root lattices. We recall the definitions and classification in Section 2.4. One main result of our paper is the determination of the chromatic number of all root lattices and their duals. Table 1 summarizes our results.

Note that we currently do not know the numerical value of $\chi(\mathsf{D}_n)$. We only know that it is equal to the chromatic number of the (finite) vertex–edge graph of the half-cube polytope

$$\frac{1}{2}H_n = \mathrm{conv}\left\{x \in \{0,1\}^n : \sum_{k=1}^{n} x_k \text{ is even}\right\},$$

which at the moment is only known up to dimension $n = 9$: For $n = 4,5,6,7,8,9$ we have $\chi(\frac{1}{2}H_n) = 4$, 8, 8, 8, 8 and 13 (see [35]).

For the proof we use a generalization of a lower bound for the chromatic number of finite graphs originally due to Hoffman [28]. Hoffman's bound is based on spectral considerations: Let $A \in \mathbb{R}^{V \times V}$ be the adjacency matrix of a finite graph $G = (V, E)$. Let $m(A)$ be the smallest eigenvalue of A and, respectively, let $M(A)$ be the largest eigenvalue of A, then

$$\chi(G) \geqslant 1 - \frac{M(A)}{m(A)}.$$

Bachoc, DeCorte, Oliveira, and Vallentin [2] showed how to generalize the spectral bound (and its weighted variant due to Lovász [38]) from finite to infinite graphs. In Section 5.1 we review this generalization and specialize it to $\chi(\Lambda)$. Here, classical Fourier analysis is used. We show in Corollary 5.2 that

$$\chi(\Lambda) \geqslant 1 - \left(\inf_{x \in \mathbb{R}^n} \frac{1}{|\operatorname{Vor}(\Lambda)|} \sum_{u \in \operatorname{Vor}(\Lambda)} e^{2\pi i u \cdot x} \right)^{-1},$$

holds.

In Section 5 we compute this bound for all irreducible root lattices. Surprisingly, the result of this computation can already be found in an Oberwolfach report by Serre [49] albeit in a different language and with a different motivation. In his report Serre computed all critical values of the characters of the adjoint representation of compact Lie groups. However, the report does not contain proofs. In Section 5.2 we provide proofs for the easy cases A_n and D_n. The cases E_6, E_7, and E_8 are much harder and we give Serre's proof in Appendix B after recalling relevant facts about compact Lie groups in Appendix A. We sketch an alternative, computational proof, which is based on optimization, in particular using sum of squares for the cases E_7 and E_8 at the end of Section 5.2. The case E_6 is easier and does not require computer assistance.

Then, in Section 6, we construct several efficient colorings of irreducible root lattices.

It would be nice to know the chromatic number of more important lattices. Following the book [14] by Conway and Sloane the next candidates, one should consider are the 12-dimensional Coxeter–Todd lattice K_{12} and the 16-dimensional Barnes–Wall lattice BW_{16}. We expect that the spectral lower bound gives a close approximation to the chromatic number.

We show in Section 4.1 that the chromatic number of the Leech lattice Λ_{24} in 24 dimensions is 4096. This is a consequence of the sphere packing optimality of Λ_{24}. It would be nice to have an independent (spectral) proof of this fact.

TABLE 1. *The chromatic number of important lattices, in particular the (irreducible) root lattices and their duals.*

Lattice	*Chromatic number*	
$\mathbb{Z}^n$	2	Section 1, Theorem 2.5, Section 3
A_n	$n+1$	Theorem 3.6, Theorem 5.3
A_n^*	$n+1$	Theorem 3.6
D_n	$\chi(\frac{1}{2}H_n)$	Theorem 6.1
D_n^*	4	Section 1
E_6	9	Section 5.2.4, Theorem 6.3, Theorem B.2
E_6^*	16	Theorem 6.4
E_7	14	Section 5.2.3, Theorem 6.3, Theorem B.4
E_7^*	16	Theorem 6.4
E_8	16	Section 4.1, Section 5.2.3, Theorem 6.3, Theorem B.3
Λ_{24}	4096	Section 4.1

Going back to general lattices: At the moment we do not know whether there is a finite algorithm to compute the chromatic number of a lattice which is given for example by a basis. Determining the strict Voronoi vectors — and thus the Cayley graph $\mathrm{Cayley}(\Lambda, \mathrm{Vor}(\Lambda))$ and the Voronoi cell $V(\Lambda)$ — is possible by a finite algorithm, see, for example, [20], although it can occupy exponential space (and therefore needs exponential time). The theorem of de Bruijn and Erdös [10] implies that the chromatic number of Λ is equal to the largest chromatic number of all finite subgraphs of Λ. This shows that the decision problem: "Is $\chi(\Lambda) \leqslant k$?" is at least semidecidable.

1.2. *Generic and extremal behavior of the chromatic number*

What is $\chi(\Lambda)$ of a random n-dimensional lattice? How fast can $\chi(\Lambda)$ grow depending on the dimension n?

In Section 4.2 we prove that the chromatic number of a generic n-dimensional lattice grows exponentially with the dimension. There we show that there are n-dimensional lattices Λ_n with

$$\chi(\Lambda_n) \geqslant 2 \cdot 2^{(0.0990\ldots - o(1))n}.$$

It would be very interesting to understand the extremal behavior.

2. *Background and first observations*

2.1. *Lattices, Voronoi cells, and Voronoi vectors*

A *lattice* Λ is a discrete free $\mathbb{Z}$-module in an n-dimensional Euclidean space. If its rank is strictly lower than n, then Λ also defines a lattice in its linear span over $\mathbb{R}$. We implicitly identify these two lattices, and assume for the following definitions that Λ is a full-rank lattice in $\mathbb{R}^n$. We denote by Λ^* the *dual lattice* of Λ:

$$\Lambda^* = \{x \in \mathbb{R}^n : x \cdot y \in \mathbb{Z} \text{ for all } y \in \Lambda\},$$

where $x \cdot y$ denotes the standard Euclidean scalar product between x and y. A *fundamental region* of Λ is a region $\mathcal{R} \subset \mathbb{R}^n$ such that for any $u \neq v \in \Lambda$, the volume of $(u + \mathcal{R}) \cap (v + \mathcal{R})$ is 0, and $\mathbb{R}^n = \bigcup_{v \in \Lambda}(v + \mathcal{R})$. The *volume* $\mathrm{vol}(\mathbb{R}^n/\Lambda)$ of Λ is defined as the volume of any of its fundamental region. A fundamental region of particular interest is the *Voronoi cell* of Λ:

$$V(\Lambda) = \{x \in \mathbb{R}^n : \|x\| \leqslant \|x - v\| \text{ for all } v \in \Lambda\}.$$

A vector $u \in \Lambda \setminus \{0\}$ is called a *strict Voronoi vector*, or sometimes a "relevant" vector, if the intersection $(u + V(\Lambda)) \cap V(\Lambda)$ is a facet, a face of dimension $n - 1$, of $V(\Lambda)$. By a well-known characterization of Voronoi (see for example [14, Chapter 21, Theorem 10] or [13]), the set of these vectors is

$$\mathrm{Vor}(\Lambda) = \{u \in \Lambda \setminus \{0\} : \pm u \text{ only shortest vectors in } u + 2\Lambda\}. \tag{1}$$

Now the *chromatic number* of Λ equals the chromatic number of the Cayley graph on the additive group Λ with generating set $\mathrm{Vor}(\Lambda)$:

$$\chi(\Lambda) = \chi(\mathrm{Cayley}(\Lambda, \mathrm{Vor}(\Lambda))).$$

Here, the set of vertices of the Cayley graph are all elements of Λ and two vertices v, w are adjacent whenever the difference $v - w$ lies in the set of strict Voronoi vectors $\mathrm{Vor}(\Lambda)$.

2.2. *Simple upper bounds for the chromatic number*

One can color a lattice Λ periodically by using translates of one of its sublattices Λ' which does not contain Voronoi vectors. More precisely it is enough to color the vertices of the graph $G = (V, E)$ with

$$V = \Lambda/\Lambda' \quad \text{and} \quad E = \{\{v + \Lambda', w + \Lambda'\} : v - w + u \in \mathrm{Vor}(\Lambda) \text{ for some } u \in \Lambda'\}.$$

This immediately gives the following upper bound on $\chi(\Lambda)$.

LEMMA 2.1. *Let $\Lambda' \subset \Lambda$ be a sublattice of Λ with $\Lambda' \cap \mathrm{Vor}(\Lambda) = \emptyset$. Then, $\chi(\Lambda)$ is at most $|\Lambda/\Lambda'|$.*

Sometimes, we can improve this bound by coloring the vertices of the graph $G = (V, E)$ greedily. This shows, see, for example, [6, Chapter V.1], that

$$\chi(\Lambda) \leqslant \chi(G) \leqslant \Delta(G) + 1, \tag{2}$$

where $\Delta(G)$ is the largest degree of a vertex in G.

Now we take $\Lambda' = 2\Lambda$. Lemma 2.1 implies that $\chi(\Lambda) \leqslant 2^n$. If the number of Voronoi vectors is not maximal, if $|\mathrm{Vor}(\Lambda)| < 2(2^n - 1)$, then we can improve this bound by using (2).

LEMMA 2.2. *The chromatic number of Λ is at most $|\mathrm{Vor}(\Lambda)|/2 + 1$.*

For generic lattices, the number of Voronoi vectors is $2(2^n - 1)$, so that Lemma 2.2 also gives an upper bound of 2^n for the chromatic number of Λ.

2.3. *Simple lower bounds for the chromatic number*

For a general graph G one has $\chi(G) \geqslant \chi(H)$ for every induced subgraph H of G. In particular, when choosing H to be a largest complete subgraph of G, we have $\chi(G) \geqslant \omega(G)$, where $\omega(G)$ is the clique number of G.

Canonical finite induced subgraphs of $\mathrm{Cayley}(\Lambda, \mathrm{Vor}(\Lambda))$ are the vertex-edge graphs of Delaunay polytopes of Λ. A *Delaunay polytope* of the lattice Λ is defined as follows: Let x be a vertex of the Voronoi cell $V(\Lambda)$. Consider all vectors $v_1, \dots, v_m \in \Lambda$ so that x is contained in all the translates $v_1 + V(\Lambda), \dots, v_m + V(\Lambda)$. Then, the convex hull $P = \mathrm{conv}\{v_1, \dots, v_m\}$ of $v_1, \dots, v_m$ is a Delaunay polytope of Λ. Clearly, all edges of P lie in $\mathrm{Vor}(\Lambda)$.

LEMMA 2.3. *The chromatic number of a lattice Λ is at least the chromatic number of the vertex–edge graph of any Delaunay polytope of Λ.*

2.4. *Root lattices and their duals*

One of the most important classes of lattices are the root lattices. Assume that $\Lambda \subseteq \mathbb{R}^n$ is an even lattice, that is, we have $v \cdot v \in 2\mathbb{Z}$ for all $v \in \Lambda$. Lattice vectors $v \in \Lambda$ with $v \cdot v = 2$ are called *root vectors*, or simply *roots*. A *root lattice*$\Lambda \subseteq \mathbb{R}^n$ is an even lattice which is spanned by roots. Root lattices have been classified by Witt in 1941, see, for example, [22, Section 1.4], and they are orthogonal direct sums of the irreducible root lattices A_n, D_n, E_6, E_7, and E_8. The strict Voronoi vectors of root lattices are precisely the root vectors [14, Chapter 21, § 3.A] (and, in fact, this condition that only the shortest nonzero vectors are relevant characterizes root lattices, see [47]). Also, the combinatorial description of the Voronoi cells of root lattices is well known: it is described in more detail in [41]. Here we recall the definitions of the irreducible root lattices and their duals.

Living inside the hyperplane $\Pi := \{x \in \mathbb{R}^{n+1} : \sum_{k=0}^{n} x_k = 0\}$ (the coordinates being here numbered 0 through n), the irreducible root lattice A_n is defined as

$$\mathsf{A}_n = \left\{ x \in \mathbb{Z}^{n+1} : \sum_{k=0}^{n} x_k = 0 \right\}.$$

The dual lattice A_n^* naturally lives in the vector space Π^* dual to Π, which can be identified with the quotient of $\mathbb{R}^{n+1}$ by the diagonal line $\{(t, \ldots, t) : t \in \mathbb{R}\}$. But we can identify Π^* with Π itself by choosing the representatives $(x_0, \ldots, x_n)$ of $\mathbb{R}^{n+1}$ modulo the diagonal which belong to Π.

For every $n \geqslant 4$, the irreducible root lattice D_n is defined as

$$\mathsf{D}_n = \left\{ x \in \mathbb{Z}^n : \sum_{k=1}^{n} x_k \text{ is even} \right\}$$

and its dual lattice D_n^* equals

$$\mathsf{D}_n^* = \mathbb{Z}^n \cup ((1/2, \ldots, 1/2) + \mathbb{Z}^n).$$

The root lattice E_8 can be constructed as the union of two translates of the lattice D_8:

$$\mathsf{E}_8 = \mathsf{D}_8 \cup ((1/2, \ldots, 1/2) + \mathsf{D}_8).$$

The lattice E_8 is *unimodular*, that is, $\mathsf{E}_8^* = \mathsf{E}_8$. The lattice E_7 (respectively, E_6) can be defined as a 7-dimensional (respectively, 6-dimensional) sublattice of E_8:

$$\mathsf{E}_7 = \{(x_1, \ldots, x_8) \in \mathsf{E}_8 : x_7 = x_8\}$$

and

$$\mathsf{E}_6 = \{(x_1, \ldots, x_8) \in \mathsf{E}_8 : x_6 = x_7 = x_8\}.$$

Then, if we define

$$u = \frac{1}{4}(1, 1, 1, 1, 1, 1, -3, -3) \quad \text{and} \quad v = \frac{1}{3}(0, -2, -2, 1, 1, 1, 1, 0),$$

the dual lattices of E_7 and E_6 are

$$\mathsf{E}_7^* = \mathsf{E}_7 \cup (u + \mathsf{E}_7) \quad \text{and} \quad \mathsf{E}_6^* = \mathsf{E}_6 \cup (v + \mathsf{E}_6) \cup (-v + \mathsf{E}_6).$$

2.5. *The chromatic number of an orthogonal sum of lattices*

Eichler [**24**] showed that one can decompose every lattice as a pairwise orthogonal sum of indecomposable lattices and that this decomposition is unique up to permutation of the summand; Kneser [**34**] gave a constructive and much simpler proof of Eichler's result.

We prove that the chromatic number of a lattice is the maximum of the chromatic numbers of its orthogonal summands.

This reduces in particular the study of the chromatic number of root lattices to the irreducible root lattices A_n, D_n, E_6, E_7, and E_8.

LEMMA 2.4. *Let $\Lambda \subseteq \mathbb{R}^n$ be a lattice which can be written as the orthogonal direct sum of lattices $\Lambda_1, \ldots, \Lambda_m \subseteq \mathbb{R}^n$:*

$$\Lambda = \Lambda_1 \perp \Lambda_2 \perp \ldots \perp \Lambda_m, \quad \text{with } m \in \mathbb{N},$$

so that every lattice vector $v \in \Lambda$ can be uniquely decomposed as $v = v_1 + \cdots + v_m$ with $v_i \in \Lambda_i$ and v_i is orthogonal to v_j whenever $i \neq j$. Then,

$$\mathrm{Vor}(\Lambda) = \bigcup_{i=1}^{m} \mathrm{Vor}(\Lambda_i).$$

Proof. By induction, we may assume that $\Lambda = \Lambda_1 \perp \Lambda_2$.

Every $v \in \mathrm{Vor}(\Lambda)$ we can write as $v = v_1 + v_2$ with $v_i \in V_i$. If both v_1 and v_2 are nonzero, then $w = v_1 - v_2$ is different from $\pm v$. It lies in $v + 2\Lambda$ and satisfies $\|w\| = \|v\|$, yielding a contradiction. So $v = v_i$ for $i \in \{1, 2\}$. In particular, $\pm v_i$ must be the only minimal vectors in $v_i + 2\Lambda_i$, so that $v \in \mathrm{Vor}(\Lambda_i)$.

Conversely, let, for instance, $v_1 \in \mathrm{Vor}(\Lambda_1)$, and let $w \in v_1 + 2\Lambda$. Let us write $w = v_1 + 2(u_1 + u_2) = (v_1 + 2u_1) + 2u_2$ for $u_i \in \Lambda_i$. Since $v_1 \in \mathrm{Vor}(\Lambda_1)$, we have $\|v_1 + 2u_1\| \geqslant \|v_1\|$, with equality if and only if $v_1 + 2u_1 = \pm v_1$. Thus,

$$\|w\|^2 = \|v_1 + 2u_1\|^2 + \|2u_2\|^2 \geqslant \|v_1 + 2u_1\|^2 \geqslant \|v_1\|^2$$

with equality if and only if $u_2 = 0$ and $v_1 + 2u_1 = \pm v_1$, namely, $w = \pm v_1$. So $v_1 \in \mathrm{Vor}(\Lambda)$. □

THEOREM 2.5. *Let Λ be a lattice such that*

$$\Lambda = \Lambda_1 \perp \Lambda_2 \perp \ldots \perp \Lambda_m, \quad \text{with } m \in \mathbb{N}.$$

Then,

$$\chi(\Lambda) = \max_{i \in \{1,\ldots,m\}} \chi(\Lambda_i).$$

Proof. We again assume $\Lambda = \Lambda_1 \perp \Lambda_2$.

By Lemma 2.4, $\mathrm{Vor}(\Lambda) = \mathrm{Vor}(\Lambda_1) \cup \mathrm{Vor}(\Lambda_2)$, and so $\mathrm{Cayley}(\Lambda, \mathrm{Vor}(\Lambda_i))$ is a subgraph of $\mathrm{Cayley}(\Lambda, \mathrm{Vor}(\Lambda))$. Hence, $\chi(\Lambda) \geqslant \max\{\chi(\Lambda_1), \chi(\Lambda_2)\}$.

Conversely, let $k = \max\{\chi(\Lambda_1), \chi(\Lambda_2)\}$. By definition, for $i \in \{1, 2\}$, there is a proper coloring $c_i : \Lambda_i \to \mathbb{Z}/k\mathbb{Z}$ such that if $v_i - v_i' \in \mathrm{Vor}(\Lambda_i)$, then $c_i(v_i) \neq c_i(v_i')$. We shall show that

$$\begin{array}{cccc} c & : & \Lambda & \to & \mathbb{Z}/k\mathbb{Z} \\ & & v_1 + v_2 & \mapsto & c_1(v_1) + c_2(v_2) \bmod k \end{array}$$

is a proper coloring of Λ. For this let $u, v \in \Lambda$ such that $v = u + w$ with $w \in \mathrm{Vor}(\Lambda)$. Following Lemma 2.4, $w \in \mathrm{Vor}(\Lambda_1) \cup \mathrm{Vor}(\Lambda_2)$. Assume, for instance, that $w = w_1 \in \mathrm{Vor}(\Lambda_1)$. Then, we write $u = u_1 + u_2$ with $u_i \in \Lambda_i$, and

$$c(v) = c_1(u_1 + w_1) + c_2(u_2) \neq c_1(u_1) + c_2(u_2) = c(u) \bmod k.$$

So c is a proper coloring of Λ and $\chi(\Lambda) \leqslant k$. □

3. *On the chromatic number of lattices of Voronoi's first kind*

In this section we give lower and upper bounds for the chromatic number of lattices of Voronoi's first kind. Lattices of Voronoi's first kind form a nice class of lattices: All lattices in dimensions 2 and 3 belong to this class as well as A_n and A_n^*. Our lower and upper bounds coincide for all these cases. For dimension 4 and greater the bounds can differ. We like to pose the question of computing the chromatic number of a lattice of Voronoi's first kind as an open problem.

3.1. *Definitions and first examples*

Lattices of Voronoi's first kind are treated in detail for example in [**15**]. Here we start by collecting some facts about them.

DEFINITION 3.1. A lattice Λ is called a lattice *of Voronoi's first kind* if it admits an *obtuse superbasis*: There exist lattice vectors $v_0, v_1, \ldots, v_n$ such that:

(1) the set $\{v_1, \ldots, v_n\}$ forms a basis of Λ,
(2) we have $v_0 + v_1 + \cdots + v_n = 0$,
(3) for every $0 \leqslant i < j \leqslant n$, the vectors v_i and v_j enclose an obtuse angle, $v_i \cdot v_j \leqslant 0$.

The lattices A_n and A_n^* are of Voronoi's first kind. The lattice A_n possesses an obtuse superbasis. Let $e_1, e_2, \ldots, e_{n+1}$ be the canonical basis of $\mathbb{R}^{n+1}$. For $1 \leqslant i \leqslant n$, let $v_i = e_i - e_{i+1}$, and let $v_0 = e_{n+1} - e_1$. Then $\{v_0, \ldots, v_n\}$ is an obtuse superbasis of A_n. The dual lattice A_n^* possesses a *strictly* obtuse superbasis. Let

$$v_0 = \left(-\frac{n}{n+1}, \frac{1}{n+1}, \ldots, \frac{1}{n+1}\right), \ldots, v_n = \left(\frac{1}{n+1}, \ldots, \frac{1}{n+1}, -\frac{n}{n+1}\right).$$

Then $\{v_0, \ldots, v_n\}$ is an obtuse superbasis:

$$v_i \cdot v_j = -1 \text{ for every } 0 \leqslant i < j \leqslant n.$$

If Λ is a lattice of Voronoi's first kind, then it is known (see [**15**]) that

$$\mathrm{Vor}(\Lambda) \subseteq \left\{ v_I = \sum_{i \in I} v_i : I \subseteq \{0, \ldots, n\}, I \neq \emptyset, I \neq \{0, \ldots, n\} \right\}.$$

This immediately gives an upper bound for the chromatic number.

LEMMA 3.2. *Let Λ be a lattice of Voronoi's first kind, with an obtuse superbasis $\{v_0, \ldots, v_n\}$. Then, $\chi(\Lambda)$ is at most $n+1$.*

Proof. By definition $\{v_1, \ldots, v_n\}$ is a basis of Λ. Let us show that the linear map

$$\begin{array}{cccc} c & : & \Lambda & \to & \mathbb{Z}/(n+1)\mathbb{Z} \\ & & \sum_{i=1}^{n} x_i v_i & \mapsto & \sum_{i=1}^{n} x_i \bmod (n+1) \end{array}$$

is a proper coloring of Λ. Because of linearity it is enough to check that it does not vanish on the strict Voronoi vectors. Let v_I be such a vector. If $0 \in I$ we replace v_I by $-v_I = v_{\{0,\ldots,n\}\setminus I}$ (by Definition 3.1 (2)) to make sure that $0 \notin I$. Since I is nontrivial, $0 < |I| < n+1$. In other words, $c(v_I) \neq 0$. □

With this lemma it is easy to see that the chromatic numbers of A_n and of its dual A_n^* are both equal to $n+1$. For A_n and for every $1 \leqslant i < j \leqslant n$, the vector $v_{\{i,i+1,\ldots,j\}} = e_i - e_{j+1}$ is a minimal vector, and thus is a strict Voronoi vector. So,

$$\{0, v_{\{1\}}, v_{\{1,2\}}, \ldots, v_{\{1,\ldots,n\}}\} \tag{3}$$

is a clique in $\mathrm{Cayley}(\mathsf{A}_n, \mathrm{Vor}(\mathsf{A}_n))$ and so $\chi(\mathsf{A}_n) \geqslant n+1$. For A_n^* we know (see [**15**]) that every v_I is a strict Voronoi vector. Again, (3) is a clique, and $\chi(\mathsf{A}_n^*) \geqslant n+1$.

3.2. *Interpretation in terms of graphs and more general results*

In order to get a better understanding of the chromatic number of lattices of Voronoi's first kind, we need to know which vectors v_I are strict Voronoi vectors.

Let Λ be a lattice of Voronoi's first kind with superbasis $\{v_0, \ldots, v_n\}$.

DEFINITION 3.3. The *Delaunay graph* $D(\Lambda, \{v_0, \ldots, v_n\})$ is an undirected graph with vertex set $\{0, \ldots, n\}$ and where i and j are connected by an edge whenever $v_i \cdot v_j < 0$.

The combinatorics of the Delaunay graph $D(\Lambda, \{v_0, \ldots, v_n\})$ determines the Cayley graph $\mathrm{Cayley}(\Lambda, \mathrm{Vor}(\Lambda))$. Recall some standard terminology in graph theory. Let $G = (V, E)$ be a graph, a subset of the vertex set $U \subseteq V$ defines a *cut* by

$$\delta(U) = \{e \in E : |e \cap U| = 1\}.$$

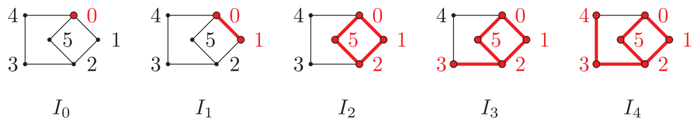


FIGURE 2 (colour online). *Constructing a clique in* Cayley$(\Lambda, \mathrm{Vor}(\Lambda))$ *from the Delaunay graph* $D(\Lambda, \{v_0, \dots, v_n\})$.

The strict Voronoi vectors of Λ are the v_I such that the cut $\delta(I)$ is minimal with respect to inclusion, see, for example, [**20**] or [**52**]. The minimal cuts are also known to be the ones such that, when removing the edges of the cut, the number of connected components in the graph increases by one.

A connected graph $G = (V, E)$ is called *biconnected* (*a block*) if it remains connected when we remove any of its vertices. One can decompose every connected graph into biconnected components (block decomposition). The set of edges E can be uniquely written as a disjoint union $E = \bigcup_{i=1}^{m} E_i$ such that the subgraph G_i of G induced by E_i is a maximal biconnected subgraph of G. Let G_i, with $i = 1, \dots, m$, be the biconnected components of the Delaunay graph $D(\Lambda, \{v_0, \dots, v_n\})$. Then, there exist lattices Λ_i of Voronoi's first kind with obtuse superbasis $\mathcal{B}_i$ such that

$$\Lambda = \Lambda_1 \perp \Lambda_2 \perp \dots \perp \Lambda_m \quad \text{and} \quad D(\Lambda_i, \mathcal{B}_i) = G_i$$

holds, see, for example, [**45**, Chapter 4, Chapter 5]. Then Theorem 2.5 and Lemma 3.2 yield the following upper bound for the chromatic number of Λ.

COROLLARY 3.4. *Let Λ be a lattice of Voronoi's first kind with obtuse superbasis $\{v_0, \dots, v_n\}$. Let G_i, $1 \leqslant i \leqslant m$, be the biconnected components of the Delaunay graph $D(\Lambda, \{v_0, \dots, v_n\})$. Then,*

$$\chi(\Lambda) \leqslant \max_{i=1,\dots,m} |V(G_i)| + 1.$$

Now we go for lower bounds. Recall that a *cycle* in a graph is a collection of vertices $i_1, \dots, i_l$ such that $|\{i_1, \dots, i_l\}| = l$ and such that i_j is connected to i_{j+1}, where indices are computed modulo l. Its length is equal to l.

THEOREM 3.5. *Let Λ be a lattice of Voronoi's first kind with obtuse superbasis $\{v_0, \dots, v_n\}$. Then, the chromatic number of Λ is at least the maximal length of a cycle in the Delaunay graph $D(\Lambda, \{v_0, \dots, v_n\})$.*

Proof. Up to a permutation of the indices, we may assume that $\{0, 1, \dots, \sigma - 1\}$ is a cycle C in $D(\Lambda, \{v_0, \dots, v_n\})$. We shall construct a clique of Cayley$(\Lambda, \mathrm{Vor}(\Lambda))$ of size σ.

For any k in $\{0, \dots, n\}$, we say that $c \in C$ is a *connector* between k and C if there exists a path $\gamma = (k_1 = k, k_2, \dots, k_s = c)$ in $D(\Lambda, \{v_0, \dots, v_n\})$ such that c is the only vertex on the path that belongs to C.

Let $0 \leqslant \ell \leqslant \sigma - 1$. We define the set I_ℓ as the subset of all vertices k in $\{0, \dots, n\}$ such that every connector from k to C is in $\{0, 1, \dots, \ell\}$. In particular, for $\ell = \sigma - 1$, $I_{\sigma-1} = \{0, \dots, n\}$. An example is depicted in Figure 2. Then, if we define $u_\ell = v_{I_\ell}$ for all $0 \leqslant \ell \leqslant \sigma - 1$, then for every $0 \leqslant i < j \leqslant \sigma - 1$,

$$u_j - u_i = v_{I_j \setminus I_i}.$$

In order to show that the set $\{u_1, u_2, \dots, u_\sigma\}$ is a clique in Cayley$(\Lambda, \mathrm{Vor}(\Lambda))$, we need to prove that for every $0 \leqslant i < j \leqslant \sigma - 1$, the vector $v_{I_j \setminus I_i}$ is in $\mathrm{Vor}(\Lambda)$. Equivalently, since $D(\Lambda, \{v_0, \dots, v_n\})$ is connected, we need to check that both $I_j \setminus I_i$ and its complementary in $\{0, \dots, n\}$ induce connected subgraphs of $D(\Lambda, \{v_0, \dots, v_n\})$.

$\sigma = 4, n+1 = 5, \chi = 4$ $\qquad$ $\sigma = 5, n+1 = 6, \chi = 6$

FIGURE 3. *In the inequality $\sigma \leqslant \chi \leqslant n+1$, both bounds can be sharp.*

Take k in $I_j \setminus I_i$. Since k is not in I_i, there is a connector between k and C which is not in $\{0, \ldots, i\}$; but since k is in I_j, this connector must be in $\{i+1, \ldots, j\}$. So k, as well as all the vertices in this path, are in $I_j \setminus I_i$ and are connected to $\{i+1, \ldots, j\}$, which is obviously connected and included in $I_j \setminus I_i$. Regarding the complementary set, take k not in $I_j \setminus I_i$. Then either k is not in I_j, and there is a connector between k and C in $\{j+1, \ldots, \sigma-1\}$, or k is in I_i, and every connector between k and C is in $\{0, \ldots, i\}$. Thus for every such k, one can find a path made of vertices not in $I_j \setminus I_i$, going to $\{j+1, \ldots, \sigma-1\} \cup \{0, \ldots, i\}$. $\square$

THEOREM 3.6. $\chi(\mathsf{A}_n) = \chi(\mathsf{A}_n^*) = n+1$.

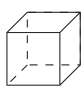
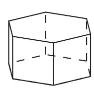
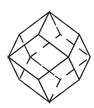
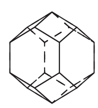
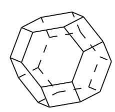
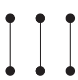
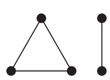
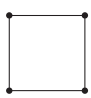
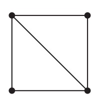
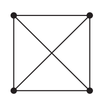
Proof. For A_n^* the Delaunay graph $D(\mathsf{A}_n^*, \{v_0, \ldots, v_n\})$ is the complete graph K_{n+1}. For A_n the Delaunay graph $D(\mathsf{A}_n, \{v_0, \ldots, v_n\})$ is a cycle of length $n+1$. In both cases, our upper bound and lower bound coincide. $\square$

EXAMPLE 3.7. In general, our upper bound differs from the lower bound. Both bounds can be attained: On the left of Figure 3, the longest cycle has size 4 and one can find a coloring of the corresponding 5-dimensional lattice with four colors, whereas the lattice associated with the graph depicted on the right of Figure 3, whose longest has size 5, has chromatic number 6. This can be seen by computing the chromatic number of a small subgraph of $\mathrm{Cayley}(\Lambda, \mathrm{Vor}(\Lambda))$.

3.3. *Application: the chromatic number of 3-dimensional lattices*

Every lattice in dimensions 2 and 3 is of Voronoi's first kind, see [15]. We can compute the chromatic number of these lattices by applying our bounds which coincide in these cases, see Table 2.

TABLE 2. *The chromatic number of 3-dimensional lattices.*

lattice	$\mathbb{Z}^3$	$\mathsf{A}_2 \perp \mathbb{Z}$	A_3	$\mathbb{Z}\begin{pmatrix}2\\0\\0\end{pmatrix} \oplus \mathbb{Z}\begin{pmatrix}0\\2\\0\end{pmatrix} \oplus \mathbb{Z}\begin{pmatrix}-1\\-1\\2\end{pmatrix}$	A_3^*
Voronoi cell	 cube	 hexagonal prism	 rhombic dodecahedron	 elongated dodecahedron	 truncated octahedron
Delaunay graph					
chromatic number	2	3	4	4	4

4. *Sphere packing lower bounds*

In this section we prove lower bounds for the chromatic number of a lattice by considering connections to the classical sphere packing problem.

A subset $\mathcal{P}$ of $\mathbb{R}^n$ defines a packing of unit spheres if the distance between all pairs of distinct points in $\mathcal{P}$ is at least 2. Define the center density of $\mathcal{P}$ as the number of points in $\mathcal{P}$ per unit volume, more precisely the (*upper*) *center density* of $\mathcal{P}$ is defined as

$$\delta(\mathcal{P}) = \limsup_{R\to\infty} \frac{|\mathcal{P}\cap[-R,R]^n|}{\operatorname{vol}[-R,R]^n},$$

where $[-R,R]^n$ is the regular n-dimensional cube with side length $2R$. The largest center density of a packing of unit spheres in $\mathbb{R}^n$ is

$$\delta_{\mathbb{R}^n} = \sup\{\delta(\mathcal{P}) : \mathcal{P} \subseteq \mathbb{R}^n \text{ defines a packing of unit spheres}\}.$$

THEOREM 4.1. *Let Λ be an n-dimensional lattice which defines a packing of unit spheres. Let ρ be a positive real number so that all strict Voronoi vectors of Λ have length strictly less than ρ. Then,*

$$\chi(\Lambda) \geqslant \left(\frac{\rho}{2}\right)^n \frac{\delta(\Lambda)}{\delta_{\mathbb{R}^n}}.$$

Proof. Suppose that the chromatic number of Λ equals k. Then one can decompose Λ into k color classes $C_1,\ldots,C_k$. We may assume that the first color class C_1 has the largest density among these color classes. In particular, inequality

$$k\delta(C_1) \geqslant \delta(\Lambda)$$

holds. Then for all $v,w \in C_1$ with $v \neq w$ we have $\|v-w\| \geqslant \rho$. Hence, $\frac{2}{\rho}C_1$ defines a packing of unit spheres. So,

$$\delta_{\mathbb{R}^n} \geqslant \delta\left(\frac{2}{\rho}C_1\right) = \left(\frac{\rho}{2}\right)^n \delta(C_1),$$

and the claim of the theorem follows by combining the two inequalities above. □

The following lemma gives a lower bound for ρ.

LEMMA 4.2. *Let Λ be an n-dimensional lattice which defines a packing of unit spheres. If $v \in \Lambda \setminus \{0\}$ is not a strict Voronoi vector, then $\|v\| \geqslant \sqrt{8}$.*

Proof. By Voronoi's characterization of strict Voronoi vectors (1), there is a lattice vector $w \in v + 2\Lambda$ with $w \neq \pm v$ and $\|w\| \leqslant \|v\|$. We may assume that $v \cdot w \geqslant 0$; otherwise, we replace w by its negative $-w$. Define $u = \frac{1}{2}(v-w) \in \Lambda \setminus \{0\}$. Then,

$$4\|u\|^2 = \|v-w\|^2 = \|v\|^2 - 2v\cdot w + \|w\|^2 \leqslant \|v\|^2 + \|w\|^2 \leqslant 2\|v\|^2.$$

So, $\|v\| \geqslant \sqrt{2}\|u\| \geqslant \sqrt{8}$, since $\|u\| \geqslant 2$. □

4.1. *First application: chromatic number of E_8 and of the Leech lattice*

THEOREM 4.3. *The chromatic number of the Leech lattice Λ_{24} equals* 4096.

Proof. It is known that the strict Voronoi vectors of Λ_{24} are all vectors $v \in \Lambda_{24}$ with $v \cdot v \in \{4,6\}$, see [**14**]. Dong, Li, Mason, and Norton showed [**18**, Theorem 4.1] that one can find an isometric copy of $\sqrt{2}\Lambda_{24}$ as a sublattice Γ of Λ_{24}; in [**43**] Nebe and Parker classified all 16 orbits of such sublattices under the action of the automorphism group of Λ_{24}. Clearly, such a sublattice Γ has index $2^{12} = 4096$ and any nonzero vector v in this sublattice satisfies $v \cdot v \geqslant 8$. Hence, we can color Λ_{24} by using the 4096 cosets $v + \Gamma$ as color classes, with $v \in \Lambda_{24}$. Thus, $\chi(\Lambda_{24}) \leqslant 4096$.

For the lower bound, we apply Theorem 4.1. The Leech lattice defines the densest sphere packing in dimension 24, see [12], $\delta(\Lambda_{24}) = \delta_{\mathbb{R}^{24}}$. We can apply Theorem 4.1 with $\rho = \sqrt{8}$ and get

$$\chi(\Lambda_{24}) \geqslant \left(\frac{\sqrt{8}}{2}\right)^{24} = 4096. \qquad \square$$

Similarly, one can show $\chi(\mathsf{E}_8) = 16$ by using the fact that E_8 is the densest sphere packing in dimension 8, see [55].

4.2. *Second application: exponential growth of the chromatic number*

In this section we investigate the asymptotic behavior of the chromatic number of lattices when the dimension tends to infinity.

For a lattice $\Lambda \subset \mathbb{R}^n$, we denote by $\mu = \mu(\Lambda)$ the norm of its smallest nonzero vector. Then $\frac{2}{\mu}\Lambda$ defines a packing of unit spheres, and we extend the notation $\delta(\Lambda)$ for the center density of this packing by

$$\delta(\Lambda) := \delta\left(\frac{2}{\mu}\Lambda\right) = \frac{1}{\operatorname{vol}\left(\mathbb{R}^n/(\frac{2}{\mu}\Lambda)\right)} = \frac{\mu^n}{2^n \cdot \operatorname{vol}(\mathbb{R}^n/\Lambda)}.$$

The best upper bound known for $\delta_{\mathbb{R}^n}$ is the Kabatiansky–Levenshtein bound, [30], which states

$$\delta_{\mathbb{R}^n} \leqslant 2^{(-0.5990\ldots+o(1))n} \cdot V_n^{-1},$$

where V_n is the volume of the n-dimensional unit ball. So by using Theorem 4.1 and Lemma 4.2, we get, for any lattice $\Lambda \subset \mathbb{R}^n$,

$$\chi(\Lambda) \geqslant (\sqrt{2})^n \cdot 2^{(0.5990\ldots-o(1))n} \cdot V_n \cdot \delta(\Lambda) = 2^{(1.0990\ldots-o(1))n} \cdot V_n \cdot \delta(\Lambda). \tag{4}$$

Let us now recall Siegel's mean value theorem (see [50]): For any lattice $\Lambda \subset \mathbb{R}^n$ and any $r > 0$, we denote by $N_\Lambda(r)$ the number of nonzero lattice vectors of Λ in the open ball $B(r)$ having radius r. The Siegel mean value theorem states that the expected value of $N_\Lambda(r)$ in a random lattice Λ of volume 1 is

$$\mathbb{E}[N_\Lambda(r)] = \operatorname{vol}(B(r)).$$

To compute the expectation one uses the Haar measure on $\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$.

This equality implies two remarkable consequences: First, by choosing r_n such that $\operatorname{vol}(B(r_n)) = 2$, it proves the existence of a lattice Λ_n with strictly less than two nonzero vectors in $B(r_n)$. Since such a vector would come with its opposite, the minimum of Λ_n has to be at least r_n, and therefore the density of Λ_n satisfies

$$\delta(\Lambda_n) \geqslant \left(\frac{r_n}{2}\right)^n = \frac{\operatorname{vol}(B(r_n))}{2^n \cdot V_n} = \frac{2}{2^n \cdot V_n},$$

which essentially is the Minkowski–Hlawka lower bound for lattice sphere packings (see [27]). The second consequence concerns the density of a random lattice: Let us fix $\varepsilon > 0$, and r_n such that $\operatorname{vol}(B(r_n)) = 2 \cdot (1+\varepsilon)^{-n}$. Following the previous idea, whenever $N_{\Lambda_n}(r_n) < 2$, the density of Λ_n satisfies

$$\delta(\Lambda_n) \geqslant \frac{2}{(2(1+\varepsilon))^n \cdot V_n}.$$

By using Siegel's mean value theorem and Markov's inequality, we prove that this happens with high probability when n grows:

$$\mathbb{P}[N_{\Lambda_n}(r_n) \geqslant 2] \leqslant \frac{\mathbb{E}[N_{\Lambda_n}(r_n)]}{2} = \frac{1}{(1+\varepsilon)^n} \to 0,$$

if n tends to infinity.

Combined with (4), these observations provide:

THEOREM 4.4. *With high probability, the chromatic number of a random n-dimensional lattice grows exponentially in n. Moreover, there are n-dimensional lattices Λ_n with*

$$\chi(\Lambda_n) \geqslant 2 \cdot 2^{(0.0990\ldots-o(1))n}.$$

Note that the Minkowski–Hlawka lower bound has been improved, in such a way that the constant 2 in the numerator could be replaced with some quasi-linear function in n, see [**48**], [**4**], [**53**], [**54**]. Even though any random lattice should be dense, and consequently should have an exponential chromatic number, to date there is no efficient way to construct such a lattice: The only algorithms for this purpose run in exponential time with respect to the dimension (see [**42**]).

However, explicit examples of subexponential growth are known. For $n \geqslant 3$ the cut polytope CUT_n (see [**17**]) is an $n(n-1)/2$-dimensional polytope which is a Delaunay polytope of lattice (see [**16**]). The vertex–edge graph of CUT_n is the complete graph on 2^{n-1} vertices. Thus we get an infinite family of Delaunay polytope with chromatic number lower bounded by $2^{O(\sqrt{n})}$.

We think that it is an interesting question to construct explicit families of lattices whose chromatic number grows exponentially with the dimension.

5. *Spectral lower bounds*

In this section we derive a lower bound for the chromatic number of a lattice where we apply the generalization of Hoffman's bound as developed by Bachoc, DeCorte, Oliveira, and Vallentin [**2**]. In Section 5.1 we recall some terminology and facts from [**2**]. Then we compute the spectral bound for the irreducible root lattices case by case in Section 5.2. Table 3 summarizes the results obtained.

5.1. *Setup*

For an n-dimensional lattice $\Lambda \subseteq \mathbb{R}^n$ define the (complex) Hilbert space

$$\ell^2(\Lambda) = \left\{ f\colon \Lambda \to \mathbb{C} : \sum_{u\in\Lambda} |f(u)|^2 < \infty \right\}$$

TABLE 3. *The spectral lower bounds on the chromatic number for the irreducible root lattices.*

Lattice	*Spectral lower bound*	
A_n	$n+1$	Theorem 5.3
D_n	n, when n even	Theorem 5.4
	$n+1$, when n odd	Theorem 5.4
E_6	9	Theorem B.2, Section 5.2.4
E_7	10	Theorem B.4, Section 5.2.3
E_8	16	Theorem B.3, Section 5.2.3

which has inner product

$$\langle f, g\rangle = \sum_{u\in\Lambda} f(u)\overline{g(u)}.$$

The *convolution* of two elements $f, g \in \ell^2(\Lambda)$ is $f * g \in \ell^2(\Lambda)$ defined by

$$(f * g)(v) = \sum_{u\in\Lambda} f(u)g(v-u).$$

Assume that $\mu \in \ell^2(\Lambda)$ is real-valued, that its support is contained in $\mathrm{Vor}(\Lambda)$, and that it satisfies $\mu(v) = \mu(-v)$ for all $v \in \Lambda$. Consider the convolution operator

$$A_\mu : \ell^2(\Lambda) \to \ell^2(\Lambda)$$

defined by

$$A_\mu f(v) = \sum_{u\in\mathrm{Vor}(\Lambda)} \mu(u)f(v-u) = (\mu * f)(v).$$

In a certain sense, A_μ is a weighted adjacency operator of $\mathrm{Cayley}(\Lambda, \mathrm{Vor}(\Lambda))$. It is easy to verify that A_μ is a bounded and self-adjoint operator. Its *numerical range* is

$$W(A_\mu) = \{\langle A_\mu f, f\rangle : f \in \ell^2(\Lambda), \langle f, f\rangle = 1\}.$$

The numerical range is known to be an interval in $\mathbb{R}$. By

$$m(A_\mu) = \inf\{\langle A_\mu f, f\rangle : f \in \ell^2(\Lambda), \langle f, f\rangle = 1\},$$

$$M(A_\mu) = \sup\{\langle A_\mu f, f\rangle : f \in \ell^2(\Lambda), \langle f, f\rangle = 1\}$$

we denote the endpoints of the interval $W(A_\mu)$.

We say that a subset $I \subseteq \Lambda$ is an *independent set* of the operator A_μ if $\langle A_\mu f, f\rangle = 0$ for each $f \in \ell^2(\Lambda)$ which vanishes outside of I. The *chromatic number* of A_μ is the smallest number k so that one can partition Λ into k independent sets. By [**2**, Theorem 2.3] one has the following lower bound for $\chi(A_\mu)$:

$$1 - \frac{M(A_\mu)}{m(A_\mu)} \leqslant \chi(A_\mu).$$

Since every independent set of $\mathrm{Cayley}(\Lambda, \mathrm{Vor}(\Lambda))$ is also an independent set of the operator A_μ, we see that

$$\chi(A_\mu) \leqslant \chi(\mathrm{Cayley}(\Lambda, \mathrm{Vor}(\Lambda))) = \chi(\Lambda).$$

Therefore, we are interested in determining the parameters $m(A_\mu)$ and $M(A_\mu)$ and in choosing μ so that the bound becomes as large as possible.

For determining $m(A_\mu)$ and $M(A_\mu)$ for a given convolution operator A_μ, we apply standard facts from Fourier analysis, in particular the Parseval identity, the theorem of Riesz–Fischer, and the fact that the Fourier transform of a convolution is a product.

The only nonstandard item here is that in standard texts on Fourier analysis, see, for example, [**21**], the role of primal and dual spaces is interchanged. In order not to confuse the reader (and to some extend not to confuse the authors) we consider a new n-dimensional lattice Γ. In our context Γ will play the role of the dual lattice Λ^*. When $\Gamma = \Lambda^*$, then $\Gamma^* = (\Lambda^*)^* = \Lambda$.

Consider the Hilbert space of square-integrable Γ-periodic functions

$$L^2(\mathbb{R}^n/\Gamma) = \left\{F\colon \mathbb{R}^n/\Gamma \to \mathbb{C} : \int_{\mathbb{R}^n/\Gamma} |F(x)|^2\, dx < \infty\right\}$$

with inner product

$$(F, G) = \int_{\mathbb{R}^n/\Gamma} F(x)\overline{G(x)}\,dx,$$

where we normalize the Lebesgue measure dx so that the n-dimensional volume of a fundamental domain $\mathrm{vol}(\mathbb{R}^n/\Gamma)$ equals 1. The exponential functions

$$E_v : x \mapsto e^{2\pi i v\cdot x} \quad \text{with} \quad v \in \Gamma^*$$

form a complete orthonormal system for $L^2(\mathbb{R}^n/\Gamma)$. We define the *Fourier coefficient* of F at v by

$$\widehat{F}(v) = (F, E_v) \quad \text{with} \quad v \in \Gamma^*.$$

By Parseval's identity and by the Riesz–Fischer theorem, the map

$$\widehat{\ }: L^2(\mathbb{R}^n/\Gamma) \to \ell^2(\Gamma^*), \quad \widehat{F}(v) = (F, E_v)$$

is an isometry:

$$(F, G) = \langle \widehat{F}, \widehat{G}\rangle \quad \text{for all } F, G \in L^2(\mathbb{R}^n/\Gamma).$$

We consider two functions $f, g \in \ell^2(\Gamma^*)$. By the isometry of $\widehat{\ }$ there are functions $F, G \in L^2(\mathbb{R}^n/\Gamma)$ with

$$\widehat{F} = f \quad \text{and} \quad \widehat{G} = g.$$

Furthermore,

$$(f * g)(v) = \widehat{F \cdot G}(v).$$

Back to the lattice $\Lambda = \Gamma^*$ and the convolution operator A_μ. For $\mu = \widehat{G}$, $f = \widehat{F} \in \ell^2(\Lambda)$, we have

$$G(x) = \sum_{u \in \mathrm{Vor}(\Lambda)} \mu(u) e^{2\pi i u\cdot x},$$

and

$$\begin{aligned}\langle A_\mu f, f\rangle = \langle \mu * f, f\rangle = \langle \widehat{G\cdot F}, \widehat{F}\rangle = (GF, F)\\ = \int_{\mathbb{R}^n/\Lambda^*} \sum_{u\in \mathrm{Vor}(\Lambda)} \mu(u) e^{2\pi i u\cdot x} |F(x)|^2\,dx.\end{aligned}$$

Hence, by choosing two appropriate sequences (see [2, Section 3.1])) approximating the corresponding Dirac measures, we see

$$m(A_\mu) = \inf\left\{ \sum_{u\in\mathrm{Vor}(\Lambda)} \mu(u) e^{2\pi i u\cdot x} : x \in \mathbb{R}^n/\Lambda^* \right\},$$

$$M(A_\mu) = \sup\left\{ \sum_{u\in\mathrm{Vor}(\Lambda)} \mu(u) e^{2\pi i u\cdot x} : x \in \mathbb{R}^n/\Lambda^* \right\}.$$

We summarize our considerations in the following theorem.

THEOREM 5.1. *Let $\Lambda \subseteq \mathbb{R}^n$ be an n-dimensional lattice. Suppose that $\mu \in \ell^2(\Lambda)$ is real-valued, $\mu(v) = \mu(-v)$ for all $v \in \Lambda$, and the support of μ is contained in* $\operatorname{Vor}(\Lambda)$. *Then,*

$$\chi(\Lambda) \geqslant 1 - \frac{\sup_{x \in \mathbb{R}^n/\Lambda^*} \sum_{u \in \operatorname{Vor}(\Lambda)} \mu(u) e^{2\pi i u \cdot x}}{\inf_{x \in \mathbb{R}^n/\Lambda^*} \sum_{u \in \operatorname{Vor}(\Lambda)} \mu(u) e^{2\pi i u \cdot x}}.$$

If we uniformly choose

$$\mu(v) = \begin{cases} 1/|\operatorname{Vor}(\Lambda)|, & \text{for } v \in \operatorname{Vor}(\Lambda), \\ 0, & \text{otherwise,} \end{cases}$$

then the bound in the previous theorem simplifies to the following generalization of the Hoffman bound for infinite graphs.

COROLLARY 5.2. *Let $\Lambda \subseteq \mathbb{R}^n$ be an n-dimensional lattice. Then,*

$$\chi(\Lambda) \geqslant 1 - \left(\inf_{x \in \mathbb{R}^n/\Lambda^*} \frac{1}{|\operatorname{Vor}(\Lambda)|} \sum_{u \in \operatorname{Vor}(\Lambda)} e^{2\pi i u \cdot x} \right)^{-1}.$$

5.2. *Computing the spectral bound for irreducible root lattices*

In this section we compute the lower bound given by Corollary 5.2 for each of the irreducible root lattices. By the classification of Witt, these are the families of lattices A_n, D_n and the three sporadic lattices E_6, E_7, and E_8.

As recalled in Section 2.4, the set $\operatorname{Vor}(\Lambda)$ of strict Voronoi vectors for a root lattice Λ is simply its set of roots. Now the latter constitutes a *root system* (sometimes known as a "crystallographic" root system, but these are the only ones which we will consider): see A.9; the root systems are themselves classified, [**29**, § 11], and the ones which arise from root lattices are known as "simply laced" or "A-D-E" root systems. (We will review the simply laced irreducible root systems below along with the corresponding lattices.)

Computing the lower bound of Corollary 5.2 for an irreducible root lattice is therefore equivalent to finding the smallest value attained by the Fourier transform of a simply laced irreducible root system. Here, a finite set $\Phi \subseteq \mathbb{R}^n$ defines the function $\mathscr{F}_\Phi \colon x \mapsto \sum_{u \in \Phi} e^{2\pi i u \cdot x}$ on $\mathbb{R}^n$ which is the Fourier transform of the sum of delta measures concentrated on the elements of Φ. Let us emphasize that since Φ is a (crystallographic!) root system, this function is, in fact, a trigonometric polynomial; and since Φ is symmetric ($\Phi = -\Phi$), it is real.

Now this reformulation affords a link with representation theory (the required facts of which are recalled in Appendix A). Namely, if Φ is a (reduced, but not necessarily simply laced) root system of rank n, then the function $n + \mathscr{F}_\Phi$ is "essentially" the character of the adjoint representation of the — say, simply connected — compact real Lie group G_Φ associated with Φ (namely SL_{n+1} in the case of A_n, or Spin_{2n} in the case of D_n, or one of the exceptional Lie groups $\mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$ in the case of the correspondingly named $\mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$); the precise statement and explanation of why the two problems are equivalent is given in Proposition A.13.

The problem of computing our spectral lower bound is therefore essentially that of computing the least value attained by the adjoint character of a simple compact real Lie group of type A-D-E. The values in question have been considered and computed by Serre in [**49**, Theorem 3']. In Table 3 we state Serre's result in the form in which it is useful for the main part of the paper.

Again, the equivalence of the result as stated here with that as stated in Serre's note is provided by Proposition A.13.

Serre's note does not contain a proof of the stated result.† In this paper we provide one for the A-D-E case. We treat A_n and D_n in Theorem 5.3 and Theorem 5.4 below, and we defer the case E_n to Appendix B. For E_7 and E_8 we suggest an alternative proof technique based on sums of squares and semidefinite optimization in Section 5.2.3. Using the link to the chromatic number of lattices, the case E_6 turns out to be the easiest of the E_n cases. We treat it in Section 5.2.4.

5.2.1. A_n. The irreducible root lattice A_n has $n(n+1)$ roots

$$R(\mathsf{A}_n) = \{e_r - e_s : 0 \leqslant r, s \leqslant n, r \neq s\},$$

where e_r denotes the rth standard unit vector in $\mathbb{R}^{n+1}$.

THEOREM 5.3. *The critical values of $\mathcal{F}_{\mathsf{A}_n}$ are $n(n+1)$ and $-(n+1)$. In particular, $\chi(\mathsf{A}_n) \geqslant n+1$.*

Proof. Given $x \in \mathbb{R}^{n+1}$ such that $x_0 + \cdots + x_n = 0$, we define $z_0, \ldots, z_n$ by $z_r = e^{2\pi i x_r}$, so that $z_0 \cdots z_n = 1$. The sum

$$S = \mathcal{F}_{\mathsf{A}_n}(x) = \sum_{u \in R(\mathsf{A}_n)} e^{2\pi i u \cdot x}$$

is equal to

$$\begin{aligned} S &= \sum_{r \neq s} z_r / z_s \\ &= \sum_{r,s} z_r / z_s - (n+1) \\ &= \left(\sum_r z_r\right)\left(\sum_s 1/z_s\right) - (n+1) \\ &= \left(\sum_r z_r\right)\overline{\left(\sum_s z_s\right)} - (n+1) \\ &= \left|\sum_r z_r\right|^2 - (n+1). \end{aligned}$$

Clearly $|\sum_r z_r|^2$ critical values are 0 (when $\sum_i z_i = 0$) and $(n+1)^2$ (when all z_i are identical). The critical values can be obtained when $z_0 \cdots z_n = 1$, for example by the $(n+1)$-th roots of unity‡ or by $z_0 = \cdots = z_n = 1$. Then, the lower bound on $\chi(\mathsf{A}_n)$ follows from Corollary 5.2. □

5.2.2. D_n. The irreducible root lattice D_n has $2n(n-1)$ roots

$$R(\mathsf{D}_n) = \{\pm(e_r + e_s) : 1 \leqslant r, s \leqslant n, r \neq s\} \cup \{\pm(e_r - e_s) : 1 \leqslant r, s \leqslant n, r \neq s\}.$$

†Serre writes: "The classical groups are easy enough, but F_4, E_6, E_7 and E_8 are not (especially E_6, which I owe to Alain Connes). I hope there is a better proof."

‡For example, we have $z_r = e^{2\pi i/(n+1)}$ for all r, by letting $x_1 = \cdots = x_n = 1/(n+1)$ and $x_0 = 1/(n+1) - 1 = -n/(n+1)$. More explicitly, we have a bijection between the set of all $x \in \mathbb{R}^{n+1}$ with $x_0 + \cdots + x_n = 0$, modulo the lattice dual to A_n and the set of complex vectors $(z_0, \ldots, z_n)$ which all lie on the unit circle, and whose product equals 1, modulo $(\zeta, \ldots, \zeta)$ where ζ is some $(n+1)$th root of unity obtained by taking $(x_0, \ldots, x_n)$ to $z_r = e^{2\pi i x_r}$.

THEOREM 5.4. *The critical values of $\mathcal{F}_{\mathsf{D}_n}$ are:*

$$\left\{\begin{matrix} -2(n-1) \text{ if} \\ n \text{ is odd} \end{matrix}\right\} \cup \left\{\begin{matrix} 2\frac{(n_1-n_{-1})^2}{1-n_o} - 2n_1 - 2n_{-1} \text{ where } n_1, n_{-1}, n_o \in \mathbb{Z}_+, \\ n_1 + n_{-1} + n_o = n \text{ with } n_o = 0 \text{ or } \left|\frac{n_1-n_{-1}}{n_o-1}\right| < 1 \end{matrix}\right\}.$$

In particular,

$$\inf_{x\in\mathbb{R}^n} \mathcal{F}_{\mathsf{D}_n}(x) = \begin{cases} -2n, & n \text{ even}, \\ -2(n-1), & n \text{ odd}, \end{cases}$$

and $\chi(\mathsf{D}_n) \geqslant n$ for even n, and $\chi(\mathsf{D}_n) \geqslant n+1$ for odd n.

Proof. By the symmetries of the function, we can restrict ourselves to $0 \leqslant x_i \leqslant 1/2$. We consider the sum

$$\begin{aligned} S &= \sum_{u\in R(\mathsf{D}_n)} e^{2\pi i u\cdot x} \\ &= 2\sum_{1\leqslant r<s\leqslant n} (\cos(2\pi x_r + 2\pi x_s) + \cos(2\pi x_r - 2\pi x_s)) \\ &= 4\sum_{1\leqslant r<s\leqslant n} \cos(2\pi x_r)\cos(2\pi x_s) \\ &= 2\left(W^2 - \sum_{r=1}^{n}(\cos(2\pi x_r))^2\right). \end{aligned}$$

where $W = \sum_{r=1}^n \cos(2\pi x_r)$. To find the critical values we compute the gradient of S (as a function of the x_r), which is

$$\frac{\partial S}{\partial x_r} = -8\pi\sin(2\pi x_r)(W - \cos(2\pi x_r)), \quad r = 1, \ldots, n.$$

Let x be a critical point. Define S_1, S_{-1}, and S_o the set of $i \in \{1,\ldots,n\}$ such that $x_i = 0$, $x_i = 1/2$ and $0 < x_i < 1/2$ so that $\cos(2\pi x_i) = 1$ or $\cos(2\pi x_i) = -1$ or $\cos(2\pi x_i) \in (-1,1)$.

For $i \in S_o$ we have $W = \cos(2\pi x_i)$. Thus, $|W| < 1$ if $S_o \neq \emptyset$.

We define $n_1 = |S_1|$, $n_{-1} = |S_{-1}|$ and $n_o = |S_o|$ and we have $n_1 + n_{-1} + n_o = n$, and

$$W = \cos(0)n_1 + \cos(\pi)n_{-1} + n_o W. \tag{5}$$

If $n_o = 1$ we have

$$S = 2\left(W^2 - W^2 - \sum_{r\in S_{-1}\cup S_1} (\cos(2\pi x_r))^2\right) = -2(n-1).$$

If $n_o \neq 1$, then the equation for W (5) gives

$$W = \frac{n_1 - n_{-1}}{1 - n_o}.$$

The value of the function is then

$$\begin{aligned} S &= 2\big(W^2 - (n_1 + n_{-1} + n_o W^2)\big) \\ &= 2\tfrac{(n_1-n_{-1})^2}{1-n_o} - 2n_1 - 2n_{-1}. \end{aligned}$$

The lower bound on $\chi(\mathsf{D}_n)$ then follows from Corollary 5.2. □

5.2.3. E_7, E_8, *and sums of squares.* We start by giving an alternative construction of the E_8 lattice which is based on lifting the (extended) *Hamming code* $\mathcal{H}_8$, which is the vector space over the finite field $\mathbb{F}_2$ (consisting of the elements 0 and 1) spanned by the rows of the matrix

$$G = \left(\begin{array}{c|c} 1000 & 0111 \\ 0100 & 1011 \\ 0010 & 1101 \\ 0001 & 1110 \end{array}\right) \in \mathbb{F}_2^{4\times 8},$$

It consists of $2^4 = 16$ code words:

$$\begin{array}{ccccc} 0000 \mid 0000 & 1000 \mid 0111 & 1100 \mid 1100 & 0111 \mid 1000 & 1111 \mid 1111 \\ & 0100 \mid 1011 & 1010 \mid 1010 & 1011 \mid 0100 & \\ & 0010 \mid 1101 & 1001 \mid 1001 & 1101 \mid 0010 & \\ & 0001 \mid 1110 & 0110 \mid 0110 & 1110 \mid 0001 & \\ & & 0101 \mid 0101 & & \\ & & 0011 \mid 0011 & & \end{array}$$

We can define the lattice E_8 by the following lifting construction (which is usually called *Construction A*):

$$\mathsf{E}_8 = \left\{ \frac{1}{\sqrt{2}} x : x \in \mathbb{Z}^8,\ x \bmod 2 \in \mathcal{H}_8 \right\}.$$

Now it is immediate to see that E_8 has 240 shortest (nonzero) vectors:

$$\begin{array}{ll} 16 = 2^4 \text{ vectors:} & \pm\sqrt{2}e_i,\ i = 1,\dots,8 \\ 224 = 2^4 \cdot 14 \text{ vectors:} & \frac{1}{\sqrt{2}} \sum_{j=1}^{8} (\pm c_j) e_j,\ c \in \mathcal{H}_8 \text{ and } \operatorname{wt}(c) = 4, \end{array}$$

where $e_1, \dots, e_8$ are the standard basis vectors of $\mathbb{R}^8$ and where $\operatorname{wt}(c) = |\{i : c_i \neq 0\}|$ denotes the Hamming weight of a code word c.

Observe that the lower bound $\chi(\mathsf{E}_8) \geqslant 16$ is implied through the spectral bound by the following inequality (Theorem B.3 gives a stronger result by providing a complete list of all critical values):

$$S(x) = \sum_{i=1}^{8} 2\cos(2\pi\sqrt{2}x_i) + \sum_{c\in\mathcal{H}_8, \operatorname{wt}(c)=4} \sum_{\pm} \cos\left(2\pi \frac{1}{\sqrt{2}} \sum_{j=1}^{8} (\pm c_j) x_j\right) + 16 \geqslant 0.$$

for all $x \in \mathbb{R}^8$.

To simplify the formula we apply a change of variables by setting $T(x) = S(\frac{\sqrt{2}}{2\pi}x)$. Then,

$$T(x) = \sum_{i=1}^{8} 2\cos(2x_i) + \sum_{c\in\mathcal{H}_8, \operatorname{wt}(c)=4} \sum_{\pm} \cos\left(\sum_{j=1}^{8} (\pm c_j) x_j\right) + 16.$$

Applying the cosine addition formula multiple times, we get

$$T(x) = 4\sum_{i=1}^{8} \cos(x_i)^2 + \sum_{c\in\mathcal{H}_8, \operatorname{wt}(c)=4} 16 \prod_{j=1}^{8} \cos(c_j x_j).$$

Function T is globally nonnegative if and only if the polynomial

$$p(t) = \sum_{i=1}^{8} t_i^2 + 4 \sum_{c\in\mathcal{H}_8, \operatorname{wt}(c)=4, \operatorname{supp} c = \{i,j,k,l\}} t_i t_j t_k t_l$$

is nonnegative on the cube $t \in [-1,+1]^8$. This nonnegativity can be verified algorithmically. By

$$\Sigma_{n,d} = \{p \in \mathbb{R}[x_1,\dots,x_n] : \deg p \leqslant d, \text{ there are } p_1,\dots,p_m \in \mathbb{R}[x_1,\dots,x_n] : \\ p = p_1^2 + \cdots + p_m^2\}$$

denote the cone of real polynomials in n indeterminates of degree at most d. One can verify membership in this cone by using semidefinite optimization, see, for example, [**37**]. We checked numerically (up to machine precision) that there are polynomials

$$q \in \Sigma_{8,8}, q_1,\dots,q_8 \in \Sigma_{8,6}$$

so that the following identity holds true:

$$p(t) = q(t) + \sum_{i=1}^{8}(1-t_i^2)q_i(t).$$

It is interesting to observe that using smaller degree did not work.

One can easily modify this proof technique for the case E_7 to show $\chi(\mathsf{E}_7) \geqslant 10$. To define E_7 we apply Construction A on the [7,3,4] code $\mathcal{H}_7^*$, which one obtains from $\mathcal{H}_8$ by deleting its first coordinate. Here one shows that the polynomial

$$\sum_{i=1}^{7} t_i^2 + 4 \sum_{c\in\mathcal{H}_7^*, \mathrm{wt}(c)=4, \mathrm{supp}\, c=\{i,j,k,l\}} t_i t_j t_k t_l$$

is nonnegative on the cube $t \in [-1,+1]^7$ again using sum of squares.

5.2.4. E_6. The E_6 lattice we can handle without computer as follows: In the proof of Theorem 6.3 we shall color E_6 with nine colors. On the other hand, the Schläfli polytope is a Delaunay polytope of E_6 whose vertex–edge graph — the Schläfli graph on 27 vertices having 216 edges — is a finite subgraph of $\mathrm{Cayley}(\mathsf{E}_6, \mathrm{Vor}(\mathsf{E}_6))$. It is known, see, for example, [**5**, Chapter 8, page 55] that the Hoffman bound of the Schläfli graph equals nine. It is also known, see [**3**, Section 10.1], that the Hoffman bound of an infinite edge transitive graph is at least the Hoffman bound of any of its finite subgraphs. Hence, the spectral bound of E_6 equals nine.

6. *The chromatic number of irreducible root lattices and their duals*

In this section, we complete our study of the chromatic number of irreducible root lattices and their duals. The knowledge that we use about Delaunay polytopes of root lattices can be found in [**17**, Section 14.3]. Our claims regarding the sublattices that we consider and the colorings of certain small graphs can be conveniently checked with computer assistance, for example, by using `Magma` [**7**] or `Polyhedral` [**19**].

When we cannot directly compute the chromatic number of a graph, we apply other methods, computationally easier, in order to get lower and upper bounds. A lower bound for the chromatic number of a graph $G = (V,E)$ is given by its *fractional chromatic number*: Denote by $\mathcal{I}_G$ the set of all independent sets of G. The fractional chromatic number of G is the solution of the following linear program:

$$\min\left\{\sum_{I\in\mathcal{I}_G}\lambda_I : \lambda_I \in \mathbb{R}_{\geqslant 0} \text{ for } I \in \mathcal{I}_G, \sum_{I\in\mathcal{I}_G \text{ with } v\in I} \lambda_I \geqslant 1 \text{ for } v \in V\right\}.$$

If G affords symmetries, one can use them to reduce the number of variables of this linear program. Regarding upper bounds, given a number k of colors and a graph G, proving the existence of a k-coloring of G can be turned into a satisfiability problem, that can be solved, for instance, by using `Minisat` [**23**].

6.1. D_n *and its dual*

The half cube $\frac{1}{2}H_n$, sometimes also called the parity polytope, is defined as

$$\frac{1}{2}H_n = \mathrm{conv}\left\{x \in \{0,1\}^n : \sum_i x_i = 0 \bmod 2\right\}.$$

It is one of the two Delaunay polytopes of the root lattice D_n.

THEOREM 6.1. *For all $n \geqslant 4$ we have $\chi(\mathsf{D}_n) = \chi(\frac{1}{2}H_n)$, where we consider the vertex–edge graph of the half cube.*

Proof. The inequality $\chi(\mathsf{D}_n) \geqslant \chi(\frac{1}{2}H_n)$ comes from the fact that $\frac{1}{2}H_n$ is a Delaunay polytope of $\chi(\mathsf{D}_n)$.

Let c be a proper coloring of $\frac{1}{2}H_n$. We extend it to D_n by giving to any $x \in \mathsf{D}_n$ the color $c(x \bmod 2)$. Assume that two vectors x_1 and x_2 are adjacent in $\mathrm{Cayley}(\mathsf{D}_n, \mathrm{Vor}(\mathsf{D}_n))$. Since the relevant vectors of D_n are of the form $\pm e_i \pm e_j$, the difference $x_1 - x_2 \bmod 2$ is also such a vector, so that $x_1 \bmod 2$ and $x_2 \bmod 2$ are adjacent in $\frac{1}{2}H_n$, and $c(x_1 \bmod 2) \neq c(x_2 \bmod 2)$. Hence, $\chi(\mathsf{D}_n) \leqslant \chi(\frac{1}{2}H_n)$. □

THEOREM 6.2. *For every $n \geqslant 4$, the chromatic number of D_n^* is 4.*

Proof. The relevant vectors of the lattice $\mathsf{D}_n^* = \mathbb{Z}^n \cup ((1/2, \ldots, 1/2) + \mathbb{Z}^n)$ are the $2n$ vectors $\pm e_i$ and the 2^n vectors of the form $(\pm 1/2, \ldots, \pm 1/2)$. The four vectors 0, $(1, 0, \ldots, 0)$, $(1/2, \ldots, 1/2)$, and $(1/2, \ldots, 1/2, -1/2)$ define a clique in $\mathrm{Cayley}(\mathsf{D}_n^*, \mathrm{Vor}(\mathsf{D}_n^*))$; and the unique way to color D_n^* with four colors is by coloring each copy of $\mathbb{Z}^n$ with two different colors. □

6.2. $\mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$, *and their duals*

THEOREM 6.3. *We have $\chi(\mathsf{E}_6) = 9$, $\chi(\mathsf{E}_7) = 14$ and $\chi(\mathsf{E}_8) = 16$.*

Proof. By Theorem 6.1 we know that $\chi(\mathsf{D}_8) = 8$. So one can color the root lattice $\mathsf{E}_8 = \mathsf{D}_8 \cup ((1/2, \ldots, 1/2)) + \mathsf{D}_8)$ with 16 colors. The lower bound from Theorem B.3 concludes the case of E_8.

For E_7 there are two orbits of Delaunay polytopes. One of them is the Gosset polytope with 56 vertices, whose vertex–edge graph has chromatic number 14, which shows that $\chi(\mathsf{E}_7) \geqslant 14$. Moreover, we have a lamination of E_7 over the lattice A_6:

$$\mathsf{E}_7 = \bigcup_{n \in \mathbb{Z}} (nu + \mathsf{A}_6) \text{ for some } u \in \mathsf{E}_7.$$

If v and w belong to two layers which differ by an even index, then $v - w$ is not a relevant vector. Following Section 3, we know that $\chi(\mathsf{A}_6) = 7$. Thus we can color the even layers by colors in $\{1, \ldots 7\}$ and the odd layers by colors in $\{8, \ldots, 14\}$. This proves that $\chi(\mathsf{E}_7) = 14$.

The unique Delaunay polytope of E_6 is the Schläfli polytope whose vertex-edge graph is the Schläfli graph with 27 vertices. It is well known that its chromatic number is 9, so that $\chi(\mathsf{E}_6) \geqslant 9$. There is just one orbit of independent triples of vertices. Moreover, there are just two orbits of colorings with nine colors of the Schäfli graph: one orbit of size 160 and another of size 40. Let us take a coloring from the orbit of size 40. It is composed of nine different triples of elements. For each such triple $\{v_1, v_2, v_3\}$ we consider the set of vectors $\{v_1 - v_2, v_2 - v_3, v_3 - v_1\}$. Since we have nine triples, this gives in total 27 vectors. Those vectors span a sublattice L of E_6 of index 9. It turns out that none of the relevant vectors of E_6 belongs to L. Thus following Lemma 2.1, $\chi(\mathsf{E}_6) \leqslant 9$. □

THEOREM 6.4. *The chromatic number of* E_n^* *is* 16 *for* $n = 6, 7$.

Proof. Upon rescaling to an integral lattice, the norms of the vectors of E_6^* are 4, 6, 10, and so on. The norms of the relevant vectors are 4 and 6. We consider the 432 vectors of norm 10 and enumerate the sublattices of E_6^* of dimension 6 spanned by those vectors that do not contain any relevant vector. We found 1393 orbits of such lattices. Exactly one of them is of index 16 which proves that $\chi(\mathsf{E}_6^*) \leqslant 16$.

The lower bound is obtained in the following way. We consider the graph formed by the origin 0 and the 126 relevant vectors with two vectors adjacent if their difference is a relevant vector. The fractional chromatic number of this graph is 77/5. Thus $\chi(\mathsf{E}_6^*) \geqslant \lceil 77/5 \rceil = 16$.

The lower bound on the chromatic number of E_7^* is obtained by the same technique as for E_6^*. The upper bound is obtained in the following way: Consider the quotient $\mathsf{E}_7^*/4\mathsf{E}_7^*$ with 16 384 elements. One coloring with 16 colors can be obtained by solving the satisfiability problem using `Minisat`. □

Appendix A. *Recollections on compact Lie groups*

In this section, we collect, for the benefit of the unfamiliar reader, without proof but with references, a few facts about semisimple compact Lie groups and representation theory. All of the following results are standard, although it is not easy to find them conveniently stated in a single place, so we hope that this compendium will be helpful.

The main reason for this appendix insofar as the present paper is concerned is to explain the reason behind the reformulation which we give in Theorem B.1 of Serre's [**49**, Theorem 3'], namely, the connection between the Fourier transform of a root system Φ and the character of the adjoint representation of the Lie groups associated with Φ: this is provided by A.13. We have, however, stated a few additional results which are not strictly necessary toward that goal but which, we hope, help give a clearer overall picture. The secondary reason for this appendix is to provide the necessary framework for Appendix B (although the latter could, in principle, be reworded so as to eliminate all mentions of Lie groups just like we did for Theorem B.1, we believe that this would be unnecessarily contrived).

REMARK A.1. We have chosen to focus these recollections on semisimple compact real Lie groups, but the classification and representation theory of semisimple *complex* Lie groups is identical (Weyl's "unitarian trick", cf. [**32**, § 6.2] and [**26**, § 26]): we simply mention that the role of the tangent Lie algebra $\mathfrak{t}$ to a maximal torus in what follows is played, in the complex setting, by Cartan subalgebras $\mathfrak{h}$ of $\mathfrak{g}$ ([**32**, Definition 6.32] and [**26**, § 14.1 and Appendix *D*]).

A.2. A *compact* (*real*) *Lie group* is a compact connected real smooth manifold G, together with a group structure on G such that the multiplication and inverse maps are smooth (C^∞). The tangent space $\mathfrak{g}$ at the identity 1 of G is then endowed with a linear action Ad of G, called the *adjoint representation* of G, defined by letting $\mathrm{Ad}(g)\colon \mathfrak{g} \to \mathfrak{g}$ (for $g \in G$) be the differential at 1 of $u \mapsto gug^{-1}$; this in turn defines a linear map $\mathrm{ad}(x)\colon \mathfrak{g} \to \mathfrak{g}$ for $x \in \mathfrak{g}$ by letting $\mathrm{ad}\colon \mathfrak{g} \to L(\mathfrak{g}, \mathfrak{g})$ (where $L(U, V)$ stands for the vector space of linear maps between two vector spaces U and V) be the differential at 1 of $\mathrm{Ad}\colon G \to L(\mathfrak{g}, \mathfrak{g})$ itself (see [**26**, § 8.1]): writing $[x, y]$ for $\mathrm{ad}(x)(y)$, this gives $\mathfrak{g}$ the structure of a (real) *Lie algebra* (simply known as the Lie algebra of G).

We also recall the exponential map $\exp\colon \mathfrak{g} \to G$, which takes $x \in \mathfrak{g}$ to the value at 1 of the unique smooth group homomorphism $\mathbb{R} \to G$ (a.k.a. "1-parameter subgroup") whose differential at the origin is x. While exp is not a group homomorphism in general, it is one whenever G is abelian (whenever the Lie bracket of $\mathfrak{g}$ vanishes; otherwise, the so-called Baker–Campbell–Hausdorff formula expresses the relation of $\exp(x)\,\exp(y)$ to an exponential). Rather than using the exponential, we will find it more convenient to use the function $\mathbf{e}\colon x \mapsto \exp(2\pi x)$; just as exp itself, the function $\mathbf{e}$ in question is surjective (cf. A.5 below).

The *Killing form* $B\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is the bilinear form $B(x, y) = \mathrm{tr}(\mathrm{ad}(x) \circ \mathrm{ad}(y))$, which is G-invariant; in the real compact case in which we placed ourselves, this form is negative semidefinite ([**32**, Theorem 6.10] or [**1**, Theorem 2.13]), and we say that G or $\mathfrak{g}$ is *semisimple* when B is nondegenerate ([**26**, Proposition C10], [**32**, Theorem 5.53]), that is, negative definite in the real compact case.

As an example, the group SO_{2n} of $(2n) \times (2n)$ real orthogonal matrices with determinant $+1$ is a compact real Lie group, whose Lie algebra $\mathfrak{so}_{2n}$ consists of antisymmetric $(2n) \times (2n)$ matrices, the Lie bracket $[x, y]$ being the usual $xy - yx$, and the Killing form on $\mathfrak{so}_{2n}$ is given by $B(x, y) = 2(n - 1)\,\mathrm{tr}(xy)$, so that $\mathfrak{so}_{2n}$ is semisimple if and only if $n \geqslant 2$.

A.3. If G is a compact Lie group with Lie algebra $\mathfrak{g}$, the map taking a connected closed subgroup H of G to its Lie algebra seen as a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ (that is, a vector subspace closed under the Lie bracket) is a bijection ([**32**, Theorem 3.40]).

Two *simply connected* compact Lie groups are isomorphic if and only if their Lie algebras are isomorphic ([**32**, Theorem 3.43]); thus, two compact Lie groups with isomorphic Lie algebras have isomorphic universal coverings: they are then said to be *isogenous*. (We note that, for the purposes of this paper, isogenous Lie groups are an irrelevant complication.) Beware, however, that the universal covering of a compact Lie group need not be compact as the case of tori shows.

Conversely, any real Lie algebra with a negative semidefinite Killing form is the Lie algebra of some compact Lie group (unique up to isogeny, by the previous paragraph). We return in A.14 to the question of which Lie groups are possible in the semisimple case.

A.4. If G is a compact Lie group, a *torus* in G is an abelian connected closed subgroup of G, or equivalently, one whose Lie algebra is abelian (meaning that its Lie bracket is trivial). A *maximal torus*, of course, is a torus that is maximal for inclusion; by A.3, maximal tori of G are in bijection with maximal abelian Lie subalgebras of the Lie algebra $\mathfrak{g}$ of G.

As an example, a maximal torus in SO_{2n} is given by the block diagonal matrices whose diagonal blocks are 2×2 rotation matrices.

Crucial results by Cartan concerning maximal tori of compact Lie groups are ([**11**, Theorems 16.4 and 16.5] or [**1**, Theorem 2.15] or [**33**, Corollaries 4.35 and 4.46]): **(a)** every element of G belongs to some maximal torus and **(b)** all maximal tori of G are conjugate; in particular, each element of G is conjugate to some element of any fixed maximal torus of G.

The dimension of some (any) maximal torus T in G (or equivalently, of its Lie algebra) is known as the *rank* of G. The quotient $N_G(T)/T$ of the normalizer of T (in G) by T itself is known as the *Weyl group* W of G: so a W-orbit in T is precisely a full set of G-conjugate elements of T, and the set of conjugacy classes in G can be identified (as a set) with T/W. We note that W acts as a group of automorphisms of T, so it also acts (linearly) on the Lie algebra $\mathfrak{t}$ of T and (by inverse transpose) on the dual $\mathfrak{t}^*$ of $\mathfrak{t}$.

A.5. If T is an abstract torus, that is, an abelian compact Lie group, and $\mathfrak{t}$ is its Lie algebra, the map $x \mapsto \mathbf{e}(x) := \exp(2\pi x)$ (in other words the differentiable group homomorphism $\mathfrak{t} \to T$ whose differential at 0 is 2π times the identity) defines a surjective homomorphism $\mathfrak{t} \to T$, whose kernel is a discrete subgroup Γ of $\mathfrak{t}$. Thus, we can identify T with $\mathfrak{t}/\Gamma$ (as a differentiable group), and functions on T with Γ-periodic functions on $\mathfrak{t}$. We will call Γ the *period lattice* of the torus T.

A.6. If G is a compact Lie group, a (finite-dimensional) *representation* of G on a finite-dimensional complex vector space V is a differentiable linear action of G on V, that is, a differentiable group homomorphism $\rho\colon G \to GL(V)$. The *character* of said representation is the map $g \mapsto \operatorname{tr}\rho(g)$ (a differentiable function on G, invariant under conjugation). The representation is said to be *irreducible* when the only G-invariant subspaces of V are 0 and V. It turns out that every representation of G is a direct sum of irreducible representations ([**32**, Theorem 4.40]); and a representation is characterized (up to isomorphism) by its character ([**11**, Theorem 2.5] or [**32**, Theorem 4.46]).

Furthermore, although we will not use this, it might be worth pointing out the Peter–Weyl theorem: the characters of the irreducible representations of G form a Hilbert orthonormal basis for the closed subspace consisting of conjugation-invariant functions in the Hilbert space $L^2(G)$ of square-integrable functions on G ([**32**, Theorem 4.50]; incidentally, these functions are also eigenvalues of the Laplace–Beltrami operator on G seen as a Riemannian manifold).

Among the representations of G, the adjoint representation (defined in A.2 above as a map $\mathrm{Ad}\colon G \to GL(\mathfrak{g})$, which we see as an action on the complexified vector space $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$) is of particular importance; its character $g \mapsto \operatorname{tr}\mathrm{Ad}(g)$ is called the character of the adjoint representation, or simply the adjoint character, of G; the adjoint representation is irreducible if and only if G is simple (this can be taken as a definition[†]).

A.7. Representation theory on a torus T is well known: writing $T = \mathfrak{t}/\Gamma$ through $x \mapsto \mathbf{e}(x) := \exp(2\pi x)$ as in A.5, the irreducible characters of T are of the form $\mathbf{e}(\lambda) : \mathbf{e}(x) \mapsto \exp(2\pi i\lambda(x))$ with λ ranging over the lattice Γ^* dual to Γ (in the vector space $\mathfrak{t}^*$ dual to $\mathfrak{t}$), which we call the *character lattice* of T. The corresponding representations are all 1-dimensional (acting by multiplication by the character just defined). In other words, the irreducible representations of T are indexed by Γ^*, and the orthonormal basis of characters of T predicted by the Peter–Weyl theorem is the usual Fourier basis on $T = \mathfrak{t}/\Gamma$. (As mentioned in A.5, we write $\mathbf{e}(\lambda)$ both for the function $T \to \mathbb{C}$ defined earlier, and for the Γ-periodic function $\mathfrak{t} \to \mathbb{C}$ given by $x \mapsto \exp(2\pi i\lambda(x))$.)

We note that the ring of linear combinations (over $\mathbb{Z}$, respectvely, $\mathbb{C}$) of the $\mathbf{e}(\lambda)$, or *character ring* (respectively, character $\mathbb{C}$-algebra) of T (cf. A.19 below), is the group ring (respectively, group $\mathbb{C}$-algebra) of the lattice Γ^*. (Choosing a basis for Γ^* shows that this is a ring of Laurent polynomials.)

A.8. If now G is a compact Lie group and T is a maximal torus in G with corresponding Lie algebras $\mathfrak{t} \subseteq \mathfrak{g}$, given a representation V of G having character χ, we can restrict them to T and consider the Fourier decomposition of $\chi|_T$, that is, its decomposition $\chi|_T = \sum_{\lambda\in\Gamma^*} m_\lambda \mathbf{e}(\lambda)$ in terms of the characters $\mathbf{e}(\lambda)$ (for $\lambda \in \Gamma^*$) defined in the previous paragraph: clearly m_λ is the dimension of the subspace V^λ of V consisting of those $z \in V$ such that $\rho(u)(z) = \mathbf{e}(\lambda)(u)$ for each $u \in T$. In particular, $m_\lambda \in \mathbb{N}$. The λ such that $m_\lambda > 0$ are known as the *weights* of the representation V (or of the character χ), and we emphasize that they belong to Γ^*; the value m_λ is known as the *multiplicity* of the weight λ (in V or in χ), and the subspace V^λ on which T acts through $\mathbf{e}(\lambda)$ is known as the *weight* (*eigen*)*space*; we note that the weight space can be defined at the Lie algebra level as the set of z such that $d\rho_1(x)(z) = i\lambda(x)\,z$ for all $x \in \mathfrak{t}$ (where $d\rho_1$ is the differential of ρ at 1; compare [**32**, Definition 8.1]: we have added a factor i here for convenience in the compact case, but it is a matter of convention).

[†]A Lie group G is called "simple" when it does not have nontrivial *connected* normal subgroups: this allows for a finite center (the term "quasisimple" might be more appropriate).

Since, as explained in A.4 above, all maximal tori of G are conjugate, the weights and multiplicities do not depend on the choice of T; furthermore, they are invariant under the action of the Weyl group W. Seen as a function on $\mathfrak{t}$, the character values are both Γ-periodic and W-invariant, so they are $\Gamma \rtimes W$-invariant. (Let us mention here the paper [**46**], which can serve as link between the "Fourier" and "Lie group characters" points of view.)

A.9. We temporarily leave aside Lie groups to recall the following definitions and facts in relation with abstract root systems (see [**29**, § 9.2], [**32**, § 7.1], and [**8**, Chapter VI]). A (reduced, crystallographic) *root system* is a set Φ of vectors in a finite-dimensional real vector space E such that (1) Φ is finite, does not contain 0, and spans E, (2) for every $\alpha \in \Phi$, there exists $\alpha^\vee$ in the dual space E^* of E such that $\alpha^\vee(\alpha) = 2$ and such that the (symmetry) map $s_\alpha\colon x \mapsto x - \alpha^\vee(x)\,\alpha$ leaves Φ stable (it is easy to see that $\alpha^\vee$ is uniquely defined, cf. [**8**, Chapter VI, § 1, n°1, Lemma 1], so that the notation is legitimate), (3) for every $\alpha, \beta \in \Phi$ we have $\alpha^\vee(\beta) \in \mathbb{Z}$, and (4) if $\alpha \in \Phi$ and $c\alpha \in \Phi$ then $c \in \{\pm 1\}$. The elements of Φ are called *roots*, and the $\alpha^\vee$ are the *coroots*. The set $\Phi^\vee := \{\alpha^\vee : \alpha \in \Phi\}$ of coroots is itself a root system (with $(\alpha^\vee)^\vee = \alpha$), known as the *dual* root system to Φ. The group generated by the s_α is known as the *Weyl group* of Φ, and it is finite.

Two root systems $\Phi \subseteq E$ and $\Phi' \subseteq E'$ are said to be *isomorphic* when there is a linear isomorphism between E and E' taking Φ to Φ'. In this case, they have isomorphic dual systems and isomorphic Weyl groups.

The root system $\Phi \subseteq E$ is said to be *reducible* when it is the union ("sum") of root systems Φ_1, Φ_2 in E_1, E_2 with $E = E_1 \oplus E_2$, respectively, *irreducible* otherwise. Every root system can be written in a unique way as the sum of irreducible root systems, and any sum of root systems is a root systems.

Given a root system Φ in E, there exists a Euclidean structure on E such that every s_α (and consequently, every element of the Weyl group) is orthogonal; equivalently, a Euclidean structure which identifies E with its dual E^* so that each coroot $\alpha^\vee$ is proportional to the corresponding root α. Such a Euclidean structure is said to be *compatible* with Φ. (The definition of root systems is often written in a manner that preassumes the Euclidean structure: in this case, the coroot $\alpha^\vee$ associated to α is defined as $2\alpha/\|\alpha\|^2$.) For Φ irreducible, this Euclidean structure is unique up to a multiplicative constant, that is, up to the definition of the lengths of the roots; in the case of the simply laced root system (those in which every root has the same length), which concerns us in the present paper, the constant is generally chosen such that the squared root length is 2, so that Φ and $\Phi^\vee$ can be identified. Nevertheless, it might be useful for expositional clarity to keep the distinction between Φ and $\Phi^\vee$ and the greater generality afforded by the not necessarily simply laced root system, so we do not perform this identification (but the reader may choose to do so).

Associated with a root system Φ as above are four lattices: the lattice $Q := \mathbb{Z}\Phi \subseteq E$ generated by Φ is known as the *root lattice*, the lattice $Q^\vee := \mathbb{Z}\Phi^\vee \subseteq E^*$ generated by $\Phi^\vee$ is known as the *coroot lattice*; the lattice $P := (Q^\vee)^*$ (in E) dual to the coroot lattice is known as the *weight lattice* and contains the root lattice; and the lattice $P^\vee := Q^*$ (in E^*) dual to the root lattice is known as the *coweight lattice* and contains the coroot lattice. The quotient of the weight lattice by the root lattice, or equivalently of the coweight lattice by the coroot lattice, is sometimes known as the *fundamental group* of Φ for reasons that will be clarified in A.14.

A.10. Continuing the exposition of root systems started in A.9, if h is a linear form on E such that $h(\alpha) \neq 0$ for each $\alpha \in \Phi$, the roots such that $h(\alpha) > 0$ are then known as the *positive roots*, and those such that $h(\alpha) < 0$ as the *negative roots* relative to h: a subset $\Phi_+ := \{\alpha \in \Phi : h(\alpha) > 0\}$ which can be obtained in this manner is known as a *choice of positive roots* for Φ. The positive roots which cannot be written as sums of other positive roots are known as *simple roots* (for this choice of positive roots): it is then a fact that the simple roots form

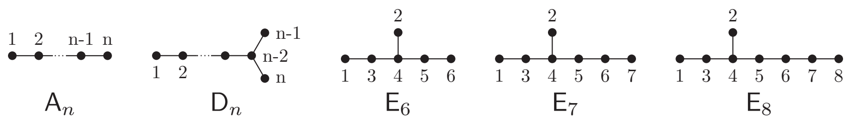


FIGURE A.1. *Simply laced Dynkin diagrams with the Bourbaki numbering of their nodes.*

a basis of E, and that every positive root is a linear combination of the simple roots with nonnegative integer coefficients (not all zero); so the choice of positive roots can be defined equivalently by the set of simple roots. The Weyl group acts simply transitively on the set of all choices of positive roots (or the set of all choices of simple roots).

The choice of positive roots also gives a choice of positive coroots (defined from h as above by identifying E with E^*, or by saying that the positive coroots are the coroots associated with positive roots). The dual basis $\varpi_1, \ldots, \varpi_n$ to the set $\alpha_1^\vee, \ldots, \alpha_n^\vee$ of simple coroots is known as the set of *fundamental weights* (for the choice of positive roots); symmetrically, the dual basis to the set of simple roots is known as the set of *fundamental coweights*.

The convex cone in E generated by the fundamental weights is known as the *closed Weyl chamber* in E corresponding to the choice of positive roots: it is the dual cone to the positive coroot cone, in other words, it is defined by the inequalities $\alpha^\vee(x) \geqslant 0$ for all positive coroots (or equivalently, for all simple coroots) $\alpha^\vee$; the open Weyl chamber, defined by the inequalities $\alpha^\vee(x) > 0$, is the interior of the closed Weyl chamber (and the closed Weyl chamber is its closure). Dually, the cone in E^* generated by the fundamental coweights, which is the dual cone to the positive root cone, is also known as the closed Weyl chamber (in E^*). The choice of a Weyl chamber is equivalent to a choice of positive roots: the Weyl group acts simply transitively on the set of Weyl chambers.

Given a choice of positive roots $\Phi_+ \subseteq \Phi$, there exists a unique $\beta \in \Phi_+$ such that $\beta + \alpha \notin \Phi_+$ for all $\alpha \in \Phi_+$: this is known as the *highest root* of Φ (relative to this choice of positive roots), and it belongs to the open Weyl chamber. The (integer) coefficients m_i of β on the basis of simple roots, that is, the m_i such that $\beta = \sum_{i=1}^n m_i \alpha_i$ where $\alpha_1, \ldots, \alpha_n$ are the simple roots, often come up in formulae involving G or its root system. It is often more convenient to define $\alpha_0 = -\beta$ (the *lowest root*) and $m_0 = 1$ so that $\sum_{i=0}^n m_i \alpha_i = 0$.

A.11. Given a root system Φ and a choice of positive roots, we define the *Dynkin diagram* of Φ as the graph whose vertices ("nodes") are the simple roots, two nodes α, β being connected by a single, double or triple edge, or by no edge at all, according as the angle between them is $2\pi/3$, $3\pi/4$ or $5\pi/6$, or $\pi/2$ for no edge at all, these being the only possible values; in the case of a double or triple edge, it is oriented by pointing from the simple root with the larger norm to that with the smaller norm. (These constructions rely on a Euclidean structure compatible with Φ, but are independent of the choice of such a structure.)

The Dynkin diagram of a root system Φ determines the latter up to isomorphism. Furthermore, all possible irreducible root systems can be classified, with the help of Dynkin diagrams. The simply laced Dynkin diagrams are shown on Figure A.1.

A.12. We now return to the setup of a compact Lie group G as in A.8, and we furthermore assume G to be semisimple.

The nonzero weights of the adjoint representation of G are known as the *roots* of G (or of $\mathfrak{g}$), and each one occurs with multiplicity 1; as for the zero weight space, it is the complexification of $\mathfrak{t}$ itself (in other words, its multiplicity is the rank of G). So, writing $\mathfrak{g}_{\mathbb{C}}^{\alpha} := \{z \in \mathfrak{g}_{\mathbb{C}} : [x, z] = i\alpha(x)\, z \text{ for all } x \in \mathfrak{t}\}$ for the weight space of $\alpha \in \Phi$ acting on $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, we have the weight space decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_{\mathbb{C}}^{\alpha}$ (compare [**33**, formula (2.16)] and [**32**, Theorem 6.38]).

The set Φ of these roots is an abstract (reduced, crystallographic) root system ([**11**, Theorem 19.2] or [**32**, Theorem 6.44]), whose Weyl group is that which we have already associated to G (cf. A.4); it is irreducible if and only if G is simple. Furthermore, this induces a bijection between the isomorphism classes of semisimple compact Lie algebras and root systems ([**32**, Corollary 7.55] or [**33**, Corollary 7.3]), or equivalently, isogeny classes of semisimple compact Lie groups or isomorphism classes of semisimple *simply connected* compact Lie groups ([**26**, § 7.3]); we clarify in A.14 below the classification of groups inside an isogeny class.

Note that, as a set, Φ depends not only on G but also on the choice of the maximal torus T used to define the weights (cf. A.8), or equivalently, not only on $\mathfrak{g}$ but also on $\mathfrak{t}$: in fact, Φ is a subset of the dual $\mathfrak{t}^*$ of $\mathfrak{t}$; however, as an abstract root system, it does not depend on this choice.

The following proposition follows immediately from what has already been said.

Proposition A.13. *If G is a semisimple compact Lie group with rank n, then the range of values taken by the adjoint character* $\mathrm{ch}_{\mathrm{ad}}$ *of G is precisely the range of the function $n + \mathscr{F}_\Phi$ where $\mathscr{F}_\Phi : x \mapsto \sum_{\alpha\in\Phi} e^{2\pi i\alpha(x)}$ is the Fourier transform of Φ.*

More precisely, if T is a maximal torus in G with Lie algebra $\mathfrak{t}$, and $u \in T$ is written $\exp(2\pi x)$ *for $x \in \mathfrak{t}$, then* $\mathrm{ch}_{\mathrm{ad}}(u) = n + \mathscr{F}_\Phi(x)$ *(and we have pointed out in A.4 that each element g of G is conjugate to an element u of T, which then obviously has* $\mathrm{ch}_{\mathrm{ad}}(g) = \mathrm{ch}_{\mathrm{ad}}(u)$*).*

Proof. As explained in A.12, the weights of the adjoint representation are the elements of Φ each with multiplicity 1, and 0 with multiplicity n, that is, $\mathrm{ch}_{\mathrm{ad}}|_T = n \cdot \mathbf{e}(0) + \sum_{\alpha\in\Phi} \mathbf{e}(\alpha)$, which is precisely the statement of the second paragraph. □

A.14. We briefly clarify the relation between isogenous (cf. A.3) compact Lie groups in the semisimple case (this subsection is required for completeness, but for the purposes of this paper, we care only about the simply connected groups):

If G is a semisimple compact Lie group, we have noted that its root system Φ can be defined directly from its Lie algebra $\mathfrak{g}$ and that $\mathfrak{t}$ of a maximal torus T of G (which is the same as a maximal abelian subalgebra of $\mathfrak{g}$, cf. A.4), namely, as the set of nonzero $\alpha \in \mathfrak{t}^*$ (where $\mathfrak{t}^*$ is the dual vector space to $\mathfrak{t}$) such that $\mathfrak{g}^\alpha_{\mathbb{C}} := \{z \in \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} : [x, z] = i\alpha(x)\, z \text{ for all } x \in \mathfrak{t}\}$ is nontrivial (cf. A.8). So the root lattice $Q := \mathbb{Z}\Phi$ and weight lattice $P := (\mathbb{Z}\Phi^\vee)^*$ defined in A.9, inside $\mathfrak{t}^*$, are defined at the Lie algebra level: they depend only on the isogeny class of G. We have $Q \subseteq \Gamma^* \subseteq P$ or equivalently $Q^\vee \subseteq \Gamma \subseteq P^\vee$ where $Q^\vee$ is the coroot and $P^\vee$ the coweight lattice (the inclusion $Q \subseteq \Gamma^*$ follows from the fact that the weights of any representation of G, as defined in A.8, belong to Γ^*, and in particular the roots belong to Γ^*; the inclusion $Q^\vee \subseteq \Gamma$ follows from the fact that the coroots can also be defined at the Lie algebra level).

The classification of compact Lie groups having Lie algebra $\mathfrak{g}$ is then as follows: G is uniquely defined by giving the lattice Γ satisfying $Q \subseteq \Gamma^* \subseteq P$, or equivalently $Q^\vee \subseteq \Gamma \subseteq P^\vee$; and conversely, for any such Γ, there exists a unique corresponding Lie group G; furthermore, the fundamental group of G is Abelian, finite, and canonically isomorphic to $\Gamma/Q^\vee$, and the center $Z(G)$ of G is finite and canonically isomorphic to $P^\vee/\Gamma$. ([**44**, Chapter 4, § 3, 6°, Theorems 9 and 10]; see also [**11**, Theorem 23.1] and [**33**, Corollary 5.109].)

(In particular, the universal covering of a semisimple compact Lie group G is still compact, and corresponds to taking Γ equal to the coroot lattice $Q^\vee$. At the other extreme, the centerless group $G/Z(G)$ corresponds to taking Γ equal to the coweight lattice $P^\vee$; this is also known as the "adjoint" form, because it is the image of the adjoint representation $G \to GL(\mathfrak{g})$. The fundamental group of the adjoint form, or equivalently the center of the universal covering, is defined at the Lie algebra level, and is $P^\vee/Q^\vee$.)

REMARK A.15. If G is a semisimple compact Lie group with maximal torus $T = \mathfrak{t}/\Gamma$, we have already pointed out in A.4 that the set of conjugacy classes of G can be identified (as a set) with T/W, where W is the Weyl group. Lifting to the Lie algebra $\mathfrak{t}$ of T, it can be identified with $\mathfrak{t}/(\Gamma \rtimes W)$. This point of view is particularly important when G is simply connected (Γ is the coroot lattice) because then it can be shown that the "affine Weyl group" $\Gamma \rtimes W$ is an affine Coxeter group, having a fundamental domain, known as the *Weyl alcove*, which is the simplex whose vertices are 0 and the $\varpi_i^\vee/m_i$, where the $\varpi_i^\vee$ are the fundamental coweights (cf. A.10) and m_i are the coefficients of the highest root (cf. A.10). (See [**31**, Chapter 11].)

A.16. We now briefly review the classification of irreducible representations of a semisimple compact Lie group as provided by "highest weight theory."

As explained in A.8, the weights of a (finite-dimensional) representation V of a semisimple compact Lie group G (relative to the choice of a maximal torus $T \subseteq G$) are the $\lambda \in \Gamma^*$ such that $V^\lambda := \{z \in V : (\forall u \in T)\, u \cdot z = \mathbf{e}(\lambda)(u)\}$ is nonzero, the multiplicity m_λ being $\dim V^\lambda$. Now fix a choice of positive roots of G (cf. A.10): a *highest weight* of V (or of its character, χ) is a weight λ such that $\lambda + \alpha$ is not a weight for any positive root α; a *dominant integral weight* (for G) is a $\lambda \in \Gamma^*$ belonging to the closed Weyl chamber, that is, such that $\lambda(\alpha^\vee) \geqslant 0$ for each simple (or equivalently, positive) coroot $\alpha^\vee$.

Highest weight theory tells us that ([**33**, Theorem 5.5], [**32**, § 8.3] or [**44**, Chapter 4, § 3, 7°, Theorem 11] or [**9**, VI.1.7]):

- every irreducible representation of G has a unique highest weight, which is a dominant integral weight, and its multiplicity is 1;
- if V is an irreducible representation of G with highest weight λ, then the set of weights of V is the intersection of $\lambda + Q$, where Q is the root lattice, and of the convex hull of the orbit of λ under the Weyl group;
- two irreducible representations of G are isomorphic if and only if they have the same highest weight (thus, we can speak of "the" irreducible representation with highest weight λ);
- every dominant integral weight is the highest weight of an irreducible representation of G (it is unique by the previous point);
- if V and V' are irreducible representations of G with the highest weights λ and λ', respectively, then $V \otimes V'$ has the highest weight $\lambda + \lambda'$ and has a unique irreducible factor with that weight ([**9**, VI.2.8]).

The highest weight of the adjoint representation is the highest root ($-\alpha_0$).

A.17. If G is a semisimple compact Lie group and T a maximal torus of G, then the *Weyl character formula* ([**32**, § 8.5], [**33**, Theorems 5.75–5.77] or [**9**, VI.1.7]) expresses the value of the irreducible character χ_λ with the highest weight λ (that is, the character of the irreducible representation with highest weight λ) as the ratio of two skew-W-invariant polynomials on T, namely,

$$\mathrm{ch}_\lambda = \frac{\sum_{w \in W} \mathrm{sgn}(w)\, \mathbf{e}(w(\lambda + \rho))}{\sum_{w \in W} \mathrm{sgn}(w)\, \mathbf{e}(w(\rho))},$$

where W is the Weyl group and $\mathrm{sgn}\colon W \to \{\pm 1\}$ the group homomorphism taking the value -1 on each reflection s_α (that is, $\mathrm{sgn}(w)$ is the determinant of w acting on the Lie algebra $\mathfrak{t}$ of T); and the *Weyl vector*$\rho := \frac{1}{2} \sum_{\alpha \in \Phi_+} \alpha$ is half the sum of the positive roots, which is also the sum $\sum_{i=1}^{n} \varpi_i$ of the fundamental weights. The denominator of the above expression can be factored using the *Weyl denominator formula*:

$$\sum_{w\in W} \operatorname{sgn}(w)\,\mathbf{e}(w(\rho)) = \prod_{\alpha\in\Phi_+} (\mathbf{e}(\alpha/2) - \mathbf{e}(-\alpha/2)).$$

A.18. Assuming that G (still a semisimple compact Lie group) is simply connected (so that Γ^* is the weight lattice, cf. A.14), the irreducible representations having the fundamental weights (cf. A.10) as the highest weights are known as *fundamental representations*, and their characters as the *fundamental characters* of G.

A.19. The *character ring* of a compact Lie group G is the ring generated by the characters of G (that is, the set of differences $\chi_1 - \chi_2$ between two characters of G) for pointwise sum and product. Equivalently, if we define a *virtual representation* of G to be the formal difference $V_1 \ominus V_2$ of two (finite dimensional) representations, identifying $V_1 \ominus V_2$ with $V_1' \ominus V_2'$ whenever $V_1 \oplus V_2'$ and $V_1' \oplus V_2$ are isomorphic ("Grothendieck ring" construction), and if we define the (virtual) character of $V_1 \ominus V_2$ to be $\chi_1 - \chi_2$ where χ_i is the character of V_i, the character ring can be defined as the set of virtual representations of G with addition and multiplication being defined as the direct sum and tensor product (extended in the obvious fashion to virtual representations).

To put it differently, the character ring of G consists of $\mathbb{Z}$-linear combinations of the irreducible characters (or representations) of G, the product being defined by the decomposition into irreducibles of a product of characters (tensor product of representations). One can similarly define the character $\mathbb{C}$-algebra as the set of $\mathbb{C}$-linear combinations of the irreducible characters (or representations) of G.

Highest weight theory implies that: (1) for a semisimple compact Lie group G with maximal torus T, the character ring of G is simply the invariant part under the Weyl group of the character ring of T (the latter being the group ring of Γ^*, cf. A.7), and (2) when G is, additionally, simply connected, the character ring is isomorphic to the polynomial algebra, with coefficients in $\mathbb{Z}$, over indeterminates corresponding to the fundamental representations ([**9**, VI.2.1 and VI.2.11]). The corresponding statements also hold with complex coefficients instead of integers.

Remark A.20. If G is a semisimple simply connected compact Lie group with maximal torus $T = \mathfrak{t}/\Gamma$, then the character $\mathbb{C}$-algebra of G can be identified (via restriction to T) with the set of W-invariant trigonometric polynomials on T (with complex coefficients), where W is the Weyl group, or, lifting to $\mathfrak{t}$, of W-invariant (hence $\Gamma \rtimes W$-invariant) combinations of the $\mathbf{e}(\lambda)$ for $\lambda \in \Gamma^*$. Also note that such functions are entirely defined by their values on the Weyl alcove (cf. A.15).

If we are mostly interested in the character values on T (they determine those on G by A.4), and in this paper we are, the irreducible representations of G are something of a needless complication: the character ring of a semisimple simply connected compact Lie group has a $\mathbb{Z}$-basis consisting of the sums $\sum_{\lambda\in W\lambda_0} \mathbf{e}(\lambda)$ for λ ranging over an orbit of the Weyl group W acting on Γ^*.

A.21. We have recalled in A.4 that every element g of a compact Lie group G belongs to a maximal torus T; when the torus in question is *unique*, the element g is said to be *regular*. Assuming that G is semisimple, this is equivalent ([**11**, Theorem 22.3(ii)]) to saying that g is not in the kernel of any $\mathbf{e}(\alpha)$ for root $\alpha \in \Phi$ (cf. A.7). Correspondingly, we say that an element x of the Lie algebra $\mathfrak{g}$ of G is regular when $\mathbf{e}(x)$ is regular, that is, when $\alpha(x) \notin \mathbb{Z}$ for all $\alpha \in \Phi$ (this means that x is represented by an element in the interior of the Weyl alcove, cf. A.15; also compare [**9**, V.7.8]).

The following fact is crucial to the proof given in Appendix B:

PROPOSITION A.22. *Let G be a semisimple simply connected compact Lie group. If $g \in G$ and T is a maximal torus containing g, then the following are equivalent:*

- *the element g is regular,*
- *the differentials $d\,\mathrm{ch}_1, \ldots, d\,\mathrm{ch}_n$ of the fundamental characters of G (cf. A.18) are independent at g,*
- *the differentials $d\,\mathrm{ch}_1|_T, \ldots, d\,\mathrm{ch}_n|_T$ of the fundamental characters of G restricted to T are independent at g.*

In a more general context, the equivalence of the first two statements is due to Kostant ([**36**, Theorem 0.1]) and Steinberg ([**51**, Theorem 8.1]); however, since we are only considering compact Lie group, every element g belongs to a maximal torus (that is, is "semisimple"), making the proof of the equivalence considerably easier and giving the third statement as a byproduct (as detailed in [**51**, § 8.2–8.6]).

A.23. We now briefly discuss how a subset of the nodes of the Dynkin diagram of a semisimple compact Lie group defines a Lie subgroup with the Dynkin diagram defined by the subset in question (that is, the induced subgraph).

So let G be a semisimple compact Lie group, fix a maximal torus T in G, and let $\mathfrak{g}, \mathfrak{t}$ be the corresponding Lie algebras and Φ the root system of G (cf. A.12); choose a system of simple roots $\alpha_1, \ldots, \alpha_n \in \Phi$ (where n is the rank of G). Now if $I \subseteq \{1, \ldots, n\}$, this defines a root system $\Phi_I \subseteq \Phi$, sometimes known as the *parabolic* subsystem associated to I, namely, the set $\Phi_I := \Phi \cap \bigoplus_{i \in I} \mathbb{Z}\alpha_i = \Phi \cap \bigoplus_{i \in I} \mathbb{R}\alpha_i$ "generated by" the α_i for $i \in I$ (see [**31**, § 5.1] for a discussion, or [**40**, § 12.1]), so that its Dynkin diagram consists of the nodes of that of Φ labeled by elements of I.

We now fix such an I and explain how to define a Lie subgroup of G with root system $\Psi := \Phi_I$. See also [**1**, § 7.3–7.4] for a more detailed and pedagogical account of this construction.

For $\alpha \in \Phi$, let $\mathfrak{g}_{\mathbb{C}}^{\alpha} := \{z \in \mathfrak{g}_{\mathbb{C}} : [x, z] = i\alpha(x)\, z \text{ for all } x \in \mathfrak{t}\}$ inside $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ be the corresponding weight space. Then (see [**33**, Corollary 5.94], or [**40**, Proposition 12.6] in a different context) $\mathfrak{l}_{\mathbb{C}} := \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Psi} \mathfrak{g}_{\mathbb{C}}^{\alpha}$ is a (complex) Lie subalgebra of $\mathfrak{g}_{\mathbb{C}}$ (sometimes known as a "parabolic Levi factor"), which can be further factored as a Lie algebra direct sum (that is, with trivial bracket between the summands) of its center $\mathfrak{z}(\mathfrak{l}_{\mathbb{C}}) = \bigcap_{\alpha \in \Psi} \ker \alpha \subseteq \mathfrak{t}_{\mathbb{C}}$ and its semisimple subalgebra $\mathfrak{l}'_{\mathbb{C}} = [\mathfrak{l}_{\mathbb{C}}, \mathfrak{l}_{\mathbb{C}}] = \mathfrak{t}'_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Psi} \mathfrak{g}_{\mathbb{C}}^{\alpha}$ where $\mathfrak{t}'_{\mathbb{C}}$ is the complex subspace spanned by the coroots $\alpha^{\vee}$ for $\alpha \in \Psi$ (seen as elements of $\mathfrak{t}_{\mathbb{C}}$). Since $\mathfrak{l}_{\mathbb{C}}$ and $\mathfrak{l}'_{\mathbb{C}}$ are stable under complex conjugation, they define (real!) Lie subalgebras $\mathfrak{l} := \mathfrak{l}_{\mathbb{C}} \cap \mathfrak{g}$ and $\mathfrak{l}' := \mathfrak{l}'_{\mathbb{C}} \cap \mathfrak{g}$ of $\mathfrak{g}$, hence compact Lie subgroups L, L' of G having these Lie algebras (see [**33**, Theorem 5.114]). Then L' is a semisimple Lie subgroup of G having root system Ψ and maximal torus T' with Lie algebra $\mathfrak{t}'$ (as for L, it is isogenous to the product of L' with a torus of rank $n - \#I$, so that it has the same rank n as G).

(In fact, the only properties of Ψ used above are that it is a "closed" subsystem of Φ: see [**40**, Definition 13.2 and Theorem 13.6].)

In Appendix B, we will use the above construction in the case where I is the complement of a single node $\{i\}$ in the Dynkin diagram.

Appendix B. *Serre's result on characters of compact Lie groups*

In this section, we prove the E_n cases of [**49**, Theorem 3']. The proof in the E_7 and E_8 cases has been kindly communicated to us by J.-P. Serre and only slightly adapted for symbolic computation with Sage (J.-P. Serre was able to perform the entire computation by hand and we have not attempted to reproduce this feat; any errors in the following expositions are, of course, entirely our own) and straightforwardly extended to compute all critical values of the adjoint character; in the E_6 case, J.-P. Serre referred us to a proof devised by A. Connes, which we do not follow here, preferring instead a straightforward analog of the E_7 and E_8 cases, at the cost of considerably more computing power (the E_6 case does not seem doable by hand with the technique presented below).

THEOREM B.1. *If G is a semisimple compact Lie group of type E_6, E_7, or E_8, respectively, and* $\mathrm{ch}_{\mathrm{ad}}$ *its adjoint character (cf. A.6); then* $\inf_{g\in G}\mathrm{ch}_{\mathrm{ad}}(g)$ *is equal to* -3, -7, *or* -8, *respectively.*

(The proof for A_n and D_n has been given in Theorems 5.3 and 5.4, respectively.)

In each case, we divide the proof in two steps: the *reduction step* and the *computation step*, the second being itself subdivided into an *elimination substep* and a *ruling-out substep*.

We will call G the simply connected semisimple compact Lie group of type E_6, E_7, or E_8 as the case may be, T its maximal torus (cf. A.4) and $\mathfrak{t}$ the Lie algebra of the latter.

The reduction step uses the trick (†) explained below (and based essentially on A.22) to reduce the number of variables by observing that any critical point of $\mathrm{ch}_{\mathrm{ad}}$ must lie on certain linear subspaces of $\mathfrak{t}$. The *computation step* then finds the critical values of some polynomial function h of several variables $x_1,\dots,x_r$ by elimination theory: there are slight variations in each of the cases below, but broadly speaking, consider the ideal of $\mathbb{C}[x_1,\dots,x_r,y]$ generated by $\frac{\partial h}{\partial x_i}$ and $y-h$ (defining the — often 0-dimensional — algebraic variety of critical points of h) and use a Gröbner basis for some elimination order (that is, a monomial order such that $y < x_1^{i_1}\cdots x_r^{i_r}$ for all $i_1,\dots,i_r$ not all zero) to find the projection of this variety on the y coordinate (represented by a polynomial in y which one factors to find the actual values, which for some currently mysterious reason happen to be always rational); unfortunately, elimination theory considers all *complex* values of $x_1,\dots,x_r$, so there are many spurious values, and one must then consider each computed value, or at least those that are smaller than the actual minimum, and rule them out by showing that, for some reason, they cannot be realized for real values $x_1,\dots,x_r$ (generally by noticing that some other element in the Gröbner basis does not have roots in the domain considered).

Let us now explain the idea of the reduction step in more detail. We generally follow Appendix A for notation: for example, we call Φ the root system of G.

The reduction trick (†) is as follows. Suppose that $z\in\mathfrak{t}$ is a critical value of the adjoint character (which is a fundamental character in each of E_6, E_7, and E_8), or more generally that it is a critical value of any polynomial in the fundamental characters in which no fundamental character appears more than once; then the result of Kostant and Steinberg A.22 implies that z is on a root hyperplane $\mathfrak{t}'=\{\alpha=m\}$ (where $\alpha\in\Phi$ and $m\in\mathbb{Z}$); now the affine Weyl group $\Gamma\rtimes W$ (cf. A.15) acts transitively on the set of such root hyperplanes and preserves all character values, so we can assume that the z lies on the root hyperplane $\{\alpha_0=0\}$, where $-\alpha_0$ is the highest root, which (for D_n, E_6, E_7, E_8) is also one of the fundamental weight, say ϖ_i. (This i can be read by taking the *extended* Dynkin diagram for Φ: it is the node to which the extender node attaches.) In other words, the coordinates of z on the basis of coroots $\alpha_1^\vee,\dots,\alpha_n^\vee$ have a zero at coordinate i (the one which is measured by ϖ_i): so z lives, in fact, in the hyperplane generated by $\alpha_j^\vee$ for $j\neq i$. Now, as explained in A.23, the linear subspace $\mathfrak{t}'$ of $\mathfrak{t}$ generated by $\alpha_j^\vee$ for $j\in I$ (here $I:=\{1,\dots,n\}\setminus\{i\}$) is, in a natural way, the Lie algebra of the maximal torus T' of a semisimple Lie subgroup G' of G (denoted as L' in A.23), whose root system Φ' has a Dynkin diagram obtained from that of Φ by keeping only the roots labeled by an element of I, so, in our case, by deleting node i. So by restricting the character (initially the adjoint character of G) to G', we are left with a character on a Lie group G' having a rank smaller by 1. Of course, the character restricted to the subgroup G' in question is no longer the adjoint character, but it can be expressed in terms of the fundamental characters of G' using standard tables of "branching rules" or a computer program like Sage[†] (in fact, branching are typically given first for the restriction from G to an intermediate subgroup $G \supseteq G_1 \supseteq G'$, maximal in G, and described through the removal of the node i from the *extended* Dynkin diagram of G by means of so-called "Borel-de Siebenthal theory: see [**56**, § 8.10] or [**31**, chapter 12]; the restriction from G_1 to G is then straightforward as G is a factor of G_1, and we will give both in what follows).

In what follows, we number the nodes of the Dynkin diagrams as in Bourbaki (cf. Figure A.1). We write ch_i^G for the ith fundamental representation of the simple group G and $\mathrm{ch}_{\mathrm{ad}}$ for the adjoint representation: thus, $\mathrm{ch}_{\mathrm{ad}}^{\mathsf{E}_6} = \mathrm{ch}_2^{\mathsf{E}_6}$ and $\mathrm{ch}_{\mathrm{ad}}^{\mathsf{E}_7} = \mathrm{ch}_1^{\mathsf{E}_7}$ and $\mathrm{ch}_{\mathrm{ad}}^{\mathsf{E}_8} = \mathrm{ch}_8^{\mathsf{E}_8}$. More generally, we write ch_λ for the character with highest weight λ (written as a combination of fundamental weights ϖ_i), so that ch_i is an abbreviation for ch_{ϖ_i}.

Theorem B.2. *The set of critical values of* $\mathrm{ch}_{\mathrm{ad}}^{\mathsf{E}_6}$ *is* -3, -2, 6, 14, *and* 78. *We therefore have* $\inf_x \mathcal{F}_{\mathsf{E}_6} = -9$ *and* $\chi(\mathsf{E}_6) \geqslant 9$.

Proof. If z is a critical point of $\mathrm{ch}_{\mathrm{ad}}^{\mathsf{E}_6} = \mathrm{ch}_2^{\mathsf{E}_6}$, that is, if the differential $d\,\mathrm{ch}_2^{\mathsf{E}_6}$ vanishes there, then by (†) we can assume that z belongs to (the Lie algebra $\mathfrak{t}$ of the maximal torus of) the Lie subgroup A_5 defined by removing node 2 from the Dynkin diagram of E_6.

Now $\mathrm{ch}_2^{\mathsf{E}_6}|_{\mathsf{A}_5} = 2\,\mathrm{ch}_3^{\mathsf{A}_5} + \mathrm{ch}_1^{\mathsf{A}_5}\,\mathrm{ch}_5^{\mathsf{A}_5} + 2$. In more details, the branching rule for the maximal subgroup $\mathsf{A}_1 \times \mathsf{A}_5$ of E_6 gives: $\mathrm{ch}_2^{\mathsf{E}_6}|_{\mathsf{A}_1\times\mathsf{A}_5} = \mathrm{ch}_{\varpi_1}^{\mathsf{A}_1}\,\mathrm{ch}_{\varpi_3}^{\mathsf{A}_5} + \mathrm{ch}_{\varpi_1+\varpi_5}^{\mathsf{A}_5} + \mathrm{ch}_{2\varpi_1}^{\mathsf{A}_1}$ as witnessed by Sage:

```
sage: E6 = WeylCharacterRing("E6", style="coroots")
sage: br = branching_rule(E6, "A1xA5","extended")
sage: br.branch(E6(E6.fundamental_weights()[2]))
A1xA5(1,0,0,1,0,0) + A1xA5(2,0,0,0,0,0) + A1xA5(0,1,0,0,0,1)
```

We then observe that

$$\mathrm{ch}_{\varpi_1+\varpi_5}^{\mathsf{A}_5} = \mathrm{ch}_1^{\mathsf{A}_5}\,\mathrm{ch}_5^{\mathsf{A}_5} - 1$$

and that

$$\mathrm{ch}_{2\varpi_1}^{\mathsf{A}_1} = (\mathrm{ch}_1^{\mathsf{A}_1})^2 - 1,$$

so that

$$\mathrm{ch}_2^{\mathsf{E}_6}|_{\mathsf{A}_1\times\mathsf{A}_5} = \mathrm{ch}_1^{\mathsf{A}_1}\,\mathrm{ch}_3^{\mathsf{A}_5} + (\mathrm{ch}_1^{\mathsf{A}_5}\,\mathrm{ch}_5^{\mathsf{A}_5} - 1) + ((\mathrm{ch}_1^{\mathsf{A}_1})^2 - 1).$$

Evaluating at the identity of A_1 (where $\mathrm{ch}_1^{\mathsf{A}_1} = 2$), we get

$$\mathrm{ch}_2^{\mathsf{E}_6}|_{\mathsf{A}_5} = 2\,\mathrm{ch}_3^{\mathsf{A}_5} + \mathrm{ch}_1^{\mathsf{A}_5}\,\mathrm{ch}_5^{\mathsf{A}_5} + 2$$

as announced.

Now the fundamental characters ch_i of A_5 are the elementary symmetric functions σ_i of six variables $u_0, \dots, u_5$ ranging over the unit circle $\mathbb{U} := \{u \in \mathbb{C} : |u| = 1\}$ and constrained by $u_0u_1u_2u_3u_4u_5 = 1$ (the eigenvalues of the element of SU_6). This means that we are to compute the critical values of $h := 2\sigma_3 + \sigma_1\sigma_5 + 2$ over $\{(u_0, \dots, u_5) \in \mathbb{U}^6 : u_0u_1u_2u_3u_4u_5 = 1\}$ (which is the maximal torus of A_5).

This concludes the reduction step (which is simpler in the case of E_6 than for $\mathsf{E}_7, \mathsf{E}_8$), and we now proceed to the computation step (which, compared to $\mathsf{E}_7, \mathsf{E}_8$, has fewer cases to consider, but is computationally more challenging).

By elimination theory, we can compute the critical values for $u_0, \dots, u_5$ ranging over $\mathbb{C}^6$ subject to $u_0u_1u_2u_3u_4u_5 = 1$: consider the ideal of $\mathbb{C}[u_0, \dots, u_5, y]$ generated by $u_i\frac{\partial h}{\partial u_i} - u_0\frac{\partial h}{\partial u_0}$

[†] http://doc.sagemath.org/html/en/thematic_tutorials/lie/branching_rules.html

for $i = 1, \dots, 5$ (because saying that dh is proportional to $d(u_0 \cdots u_5)$ means $u_i \frac{\partial h}{\partial u_i} = u_0 \frac{\partial h}{\partial u_0}$) and also $y - h$; and perform elimination of the variables $u_0, \dots, u_5$ (by computing a Gröbner basis for a monomial order for which $y < u_0^{i_0} \cdots u_5^{i_5}$ for any $i_0, \dots, i_5$ not all zero) in this ideal to obtain the projection on the y coordinate of the critical points.

By computing the Gröbner basis of the corresponding ideal, we obtain that the resulting set of possible critical values is -66, -3, -2, 6, 14, and 78.

Now the critical value -66 cannot be attained on

$$\{(u_0, \dots, u_5) \in \mathbb{U}^6 : u_0 u_1 u_2 u_3 u_4 u_5 = 1\}.$$

We add the inequality $y + 66$ to the ideal and recompute a Gröbner basis. In this basis we have $\sigma_5^3 = -6^3$.

And this is impossible because σ_5 is the sum of the u_i^{-1}, which can only take the value 6 in absolute value, provided that all the u_i are equal to 1 and the same sixth root of unity ζ, in which case $\sigma_1 = 6\zeta$ and $\sigma_5 = 6\zeta^{-1}$ and $\sigma_3 = 20\zeta^3$ and by checking the possible ζ, one notices that $h = 2\sigma_3 + \sigma_1 \sigma_5 + 2$ does not, in fact, take the value -66.

The value -3, on the other hand, is attained, namely, when three of the u_i are equal to one primitive cube root of unity and the other three are equal to the other. So it is its minimum and so a critical value.

By further Gröbner basis computation we obtain, the 78 is attained only at $u_i = 1$. We also obtained that the critical value 14 is attained only with four u_i set at -1 and two u_i set at 1. The critical value 6 is attained only by fixing two u_i at 1, two at $e^{i2\pi/3}$, and two at $e^{-2i\pi/3}$. The critical value -2 corresponds to a manifold of dimension 2. This manifold is the point with four u_i set at 1 and two u_i set at -1.

From the formula $\mathcal{F}_{\mathsf{E}_6}(x) = \mathrm{ch}_{\mathrm{ad}}^{\mathsf{E}_6}(x) - 6$, we get $\inf_x \mathcal{F}_{\mathsf{E}_6}(x) = -9$ and then using Corollary 5.2$\chi(\mathsf{E}_6) \geqslant 1 - (-9/72)^{-1} = 9$. $\square$

THEOREM B.3. *The set of critical values of* $\mathrm{ch}_{\mathrm{ad}}^{\mathsf{E}_8}$ *is* -8, -4, $-\frac{104}{27}$, $-\frac{57}{16}$, -3, -2, 0, 5, 24, 248. *We therefore have* $\inf_x \mathcal{F}_{\mathsf{E}_8} = -16$ *and* $\chi(\mathsf{E}_8) \geqslant 16$.

Proof. If z is a critical point of $\mathrm{ch}_{\mathrm{ad}}^{\mathsf{E}_8} = \mathrm{ch}_8^{\mathsf{E}_8}$, that is, if the differential $d\,\mathrm{ch}_8^{\mathsf{E}_8}$ vanishes there, then by (†) we can assume that z belongs to (the Lie algebra $\mathfrak{t}$ of the maximal torus of) the Lie subgroup E_7 defined by removing node 8 from the Dynkin diagram of E_8.

Now

$$\mathrm{ch}_8^{\mathsf{E}_8}|_{\mathsf{E}_7} = \mathrm{ch}_1^{\mathsf{E}_7} + 2\,\mathrm{ch}_7^{\mathsf{E}_7} + 3 \text{ (from } \mathrm{ch}_8^{\mathsf{E}_8}|_{\mathsf{A}_1 \times \mathsf{E}_7} = \mathrm{ch}_1^{\mathsf{E}_7} + \mathrm{ch}_1^{\mathsf{A}_1}\ \mathrm{ch}_7^{\mathsf{E}_7} + ((\mathrm{ch}_1^{\mathsf{A}_1})^2 - 1)).$$

Now since neither $\mathrm{ch}_1^{\mathsf{E}_7}$ nor $\mathrm{ch}_7^{\mathsf{E}_7}$ appear more than once (or with any exponent) in $\mathrm{ch}_1^{\mathsf{E}_7} + 2\,\mathrm{ch}_7^{\mathsf{E}_7} + 3$, we can apply (†) again: at a point z where the differential of this expression vanishes, the differentials of the fundamental characters are not independent, so z belongs to (the Lie algebra $\mathfrak{t}$ of the maximal torus of) the Lie subgroup D_6 defined by removing node 1 from the Dynkin diagram of E_7.

We have

$$\mathrm{ch}_1^{\mathsf{E}_7}|_{\mathsf{D}_6} = \mathrm{ch}_2^{\mathsf{D}_6} + 2\,\mathrm{ch}_5^{\mathsf{D}_6} + 3 \text{ (from } \mathrm{ch}_1^{\mathsf{E}_7}|_{\mathsf{A}_1 \times \mathsf{D}_6} = \mathrm{ch}_2^{\mathsf{D}_6} + \mathrm{ch}_1^{\mathsf{A}_1}\ \mathrm{ch}_5^{\mathsf{D}_6} + ((\mathrm{ch}_1^{\mathsf{A}_1})^2 - 1))$$

and

$$\mathrm{ch}_7^{\mathsf{E}_7}|_{\mathsf{D}_6} = \mathrm{ch}_6^{\mathsf{D}_6} + 2\,\mathrm{ch}_1^{\mathsf{D}_6} \text{ (from } \mathrm{ch}_7^{\mathsf{E}_7}|_{\mathsf{A}_1 \times \mathsf{D}_6} = \mathrm{ch}_6^{\mathsf{D}_6} + \mathrm{ch}_1^{\mathsf{A}_1}\ \mathrm{ch}_1^{\mathsf{D}_6}),$$

giving:

$$\mathrm{ch}_{\mathrm{ad}}^{\mathsf{E}_8}|_{\mathsf{D}_6} = \mathrm{ch}_2^{\mathsf{D}_6} + 2\,\mathrm{ch}_5^{\mathsf{D}_6} + 2\,\mathrm{ch}_6^{\mathsf{D}_6} + 4\,\mathrm{ch}_1^{\mathsf{D}_6} + 6.$$

Again, there are no multiple occurrences of the various $\mathrm{ch}_i^{\mathsf{D}_6}$, so we can apply (†) one more time: at a point z where the differential of this expression vanishes, the differentials of the fundamental characters are not independent, so z belongs to (the Lie algebra $\mathfrak{t}$ of the maximal torus of) the Lie subgroup $\mathsf{D}_4 \times \mathsf{A}_1$ defined by removing node 2 from the Dynkin diagram of D_6.

Here the branching gets more complicated: We use

```
sage: WeylCharacterRing("D6").maximal_subgroups()
```

within Sage to find the correct rule for branching to $\mathsf{A}_1 \times \mathsf{A}_1 \times \mathsf{D}_4$, and in principle the two A_1 factors are not symmetric (although in the end it turns out that they are, up to a symmetry of D_4), so one must use `br.describe()` to chop off the correct A_1 factor (call it A_1° in what follows). We find:

- $\mathrm{ch}_2^{\mathsf{D}_6}|_{\mathsf{A}_1\times\mathsf{D}_4} = \mathrm{ch}_2^{\mathsf{D}_4} + 2\,\mathrm{ch}_1^{\mathsf{A}_1}\ \mathrm{ch}_1^{\mathsf{D}_4} + ((\mathrm{ch}_1^{\mathsf{A}_1})^2 - 1) + 3$ (from $\mathrm{ch}_2^{\mathsf{D}_6}|_{\mathsf{A}_1^\circ\times\mathsf{A}_1\times\mathsf{D}_4} = \mathrm{ch}_2^{\mathsf{D}_4} + \mathrm{ch}_1^{\mathsf{A}_1^\circ}\ \mathrm{ch}_1^{\mathsf{A}_1}\ \mathrm{ch}_1^{\mathsf{D}_4} + ((\mathrm{ch}_1^{\mathsf{A}_1})^2 - 1) + ((\mathrm{ch}_1^{\mathsf{A}_1^\circ})^2 - 1)$).
- $\mathrm{ch}_5^{\mathsf{D}_6}|_{\mathsf{A}_1\times\mathsf{D}_4} = \mathrm{ch}_1^{\mathsf{A}_1}\ \mathrm{ch}_4^{\mathsf{D}_4} + 2\,\mathrm{ch}_3^{\mathsf{D}_4}$ (from $\mathrm{ch}_5^{\mathsf{D}_6}|_{\mathsf{A}_1^\circ\times\mathsf{A}_1\times\mathsf{D}_4} = \mathrm{ch}_1^{\mathsf{A}_1}\ \mathrm{ch}_4^{\mathsf{D}_4} + \mathrm{ch}_1^{\mathsf{A}_1^\circ}\ \mathrm{ch}_3^{\mathsf{D}_4}$).
- $\mathrm{ch}_6^{\mathsf{D}_6}|_{\mathsf{A}_1\times\mathsf{D}_4} = \mathrm{ch}_1^{\mathsf{A}_1}\ \mathrm{ch}_3^{\mathsf{D}_4} + 2\,\mathrm{ch}_4^{\mathsf{D}_4}$ (from $\mathrm{ch}_5^{\mathsf{D}_6}|_{\mathsf{A}_1^\circ\times\mathsf{A}_1\times\mathsf{D}_4} = \mathrm{ch}_1^{\mathsf{A}_1}\ \mathrm{ch}_3^{\mathsf{D}_4} + \mathrm{ch}_1^{\mathsf{A}_1^\circ}\ \mathrm{ch}_4^{\mathsf{D}_4}$).
- $\mathrm{ch}_1^{\mathsf{D}_6}|_{\mathsf{A}_1\times\mathsf{D}_4} = \mathrm{ch}_1^{\mathsf{D}_4} + 2\,\mathrm{ch}_1^{\mathsf{A}_1}$ (from $\mathrm{ch}_1^{\mathsf{D}_6}|_{\mathsf{A}_1^\circ\times\mathsf{A}_1\times\mathsf{D}_4} = \mathrm{ch}_1^{\mathsf{D}_4} + \mathrm{ch}_1^{\mathsf{A}_1^\circ}\ \mathrm{ch}_1^{\mathsf{A}_1}$).

Finally, we get:

$$\begin{aligned}\mathrm{ch}_{\mathrm{ad}}^{\mathsf{E}_8}|_{\mathsf{A}_1\times\mathsf{D}_4} ={}& \mathrm{ch}_2^{\mathsf{D}_4} + 2\,\mathrm{ch}_1^{\mathsf{A}_1}\ \mathrm{ch}_1^{\mathsf{D}_4} + 2\,\mathrm{ch}_1^{\mathsf{A}_1}\ \mathrm{ch}_4^{\mathsf{D}_4} + 2\,\mathrm{ch}_1^{\mathsf{A}_1}\ \mathrm{ch}_3^{\mathsf{D}_4} + 4\,\mathrm{ch}_1^{\mathsf{D}_4} + 4\,\mathrm{ch}_4^{\mathsf{D}_4} + 4\,\mathrm{ch}_3^{\mathsf{D}_4} \\ &+ (\mathrm{ch}_1^{\mathsf{A}_1})^2 + 8(\mathrm{ch}_1^{\mathsf{A}_1}) + 8.\end{aligned}$$

We can now apply the reduction trick (†) one last time, for the D_4 factor: since $\mathrm{ch}_1^{\mathsf{A}_1}$ is obviously independent from the $\mathrm{ch}_i^{\mathsf{D}_4}$, at a point z where the differential of the expression $\mathrm{ch}_{\mathrm{ad}}^{\mathsf{E}_8}|_{\mathsf{A}_1\times\mathsf{D}_4}$ above vanishes, the differentials of the fundamental characters $\mathrm{ch}_i^{\mathsf{D}_4}$ are not independent, so z belongs to (the Lie algebra $\mathfrak{t}$ of the maximal torus of) the Lie subgroup $\mathsf{A}_1 \times (\mathsf{A}_1)^3$ defined by removing node 2 from the Dynkin diagram of D_4.

To compute this restriction, first examine the restriction of D_4 to the maximal subgroup $\mathsf{A}_1 \times \mathsf{A}_1 \times \mathsf{A}_1 \times \mathsf{A}_1$ of D_4 (seen by extending the Dynkin diagram of D_4 and removing the node connected to the four others): if we call $t_1, \ldots, t_4$, the (single) fundamental characters of the various A_1 factors, numbered in the same way as the nodes of the extended diagram of D_4 from which they come (except that t_2 comes from the extending node), then $\mathrm{ch}_1^{\mathsf{D}_4}|_{(\mathsf{A}_1)^4} = t_1t_2 + t_3t_4$ and $\mathrm{ch}_3^{\mathsf{D}_4}|_{(\mathsf{A}_1)^4} = t_1t_4 + t_2t_3$ and $\mathrm{ch}_4^{\mathsf{D}_4}|_{(\mathsf{A}_1)^4} = t_1t_3 + t_2t_4$ and finally

$$\mathrm{ch}_2^{\mathsf{D}_4}|_{(\mathsf{A}_1)^4} = t_1t_2t_3t_4 + t_1^2 + t_2^2 + t_3^2 + t_4^2 - 4.$$

Finally, restricting $\mathrm{ch}_{\mathrm{ad}}^{\mathsf{E}_8}$ to $\mathsf{A}_1 \times (\mathsf{A}_1)^3$ (the first A_1 factor being the factor A_1 in $\mathsf{A}_1 \times \mathsf{D}_4$ earlier and the other three coming from nodes 1,3,4 of D_4 as described in the previous paragraphs), we have

$$\mathrm{ch}_{\mathrm{ad}}^{\mathsf{E}_8}|_{\mathsf{A}_1\times(\mathsf{A}_1)^3} = 2\sigma_3 + \sigma_1^2 + 2(s+1)\sigma_2 + 4(s+2)\sigma_1 + s^2 + 8s + 4,$$

where s is the fundamental character from the first A_1 factor and σ_i are the elementary symmetric functions in the fundamental characters t_1, t_3, t_4 of the three other A_1 factors.

We now need to find the critical values of this function

$$h = 2\sigma_3 + \sigma_1^2 + 2(s+1)\sigma_2 + 4(s+2)\sigma_1 + s^2 + 8s + 4.$$

There is one subtlety, however: "critical" means that for each t_i, as well as for s, we either have $\frac{\partial h}{\partial t_i} = 0$ (respectively, $\frac{\partial h}{\partial s} = 0$) *or* $t_i = \pm 2$ (respectively, $s = \pm 2$). Indeed, each t_i (as well as s) is a character of A_1, so it is $u + u^{-1}$ for the two eigenvalues u, u^{-1} of the element of SU_2 in question, so the critical values of t_i itself are ± 2.

This concludes the *reduction step* for E_8. The *computation step* is then to use elimination theory, in each possible case depending on how many of the t_i satisfy $\frac{\partial h}{\partial t_i} = 0$ and how many satisfy $t_i = 2$ and $t_i = -2$, and similarly for s, to find the corresponding values of $h = 2\sigma_3 + \sigma_1^2 + 2(s+1)\sigma_2 + 4(s+2)\sigma_1 + s^2 + 8s + 4$.

For this, we must consider $10 \times 3 = 30$ cases according to constraints placed on the t_i (which can be set equal to $+2$ or to -2 or to satisfy $\frac{\partial h}{\partial t_i} = 0$, which we denote as "$t_i = \partial$" for short) and on s (similarly $s = +2$ or $s = -2$ or $s = \partial$). In each case, we consider the ideal of $\mathbb{C}[s, t_1, t_3, t_4, y]$ generated by the $t_i - 2$ or $t_i + 2$ or $\frac{\partial h}{\partial t_i}$ as the case may be, and similarly for s, and also $y - h$; and perform elimination of the variables s, t_1, t_3, t_4 (by computing a Gröbner basis for a monomial order for which $y < s^j t_1^{i_1} t_3^{i_3} t_4^{i_4}$ for any j, i_1, i_3, i_4 not all zero) in this ideal to obtain the projection on the y coordinate of the critical points.

By considering all cases we find that the set of possible critical values is -652, -27, -12, $-\frac{64}{7}$, -8, -4, $-\frac{104}{27}$, $-\frac{57}{16}$, -3, -2, 0, 5, 24, and 248. For each such value and each possible critical value, we compute the manifold which turns out to be always 0-dimensional. The points of those manifolds can be enumerated and we obtain the list of critical values by keeping only the values for which at least one of the point has $|t_1|$, $|t_3|$, $|t_4|$, $|s| \leqslant 2$.

From the formula $\mathcal{F}_{\mathsf{E}_8}(x) = \mathrm{ch}_{\mathrm{ad}}^{\mathsf{E}_8}(x) - 8$, we get $\inf_x \mathcal{F}_{\mathsf{E}_8}(x) = -16$ and then using Corollary 5.2$\chi(\mathsf{E}_8) \geqslant 1 - (-16/240)^{-1} = 16$. $\square$

Theorem B.4. *The set of critical values of* $\mathrm{ch}_{\mathrm{ad}}^{\mathsf{E}_7}$ *is* -7, -3, -2, 1, $\frac{17}{5}$, 5, 25, 133. *We therefore have* $\inf_x \mathcal{F}_{\mathsf{E}_7} = -14$ *and* $\chi(\mathsf{E}_7) \geqslant 10$.

Proof. We have $\mathrm{ch}_{\mathrm{ad}}^{\mathsf{E}_7} = \mathrm{ch}_1^{\mathsf{E}_7}$ and all the reduction step has already been explained above in the E_8 case: we get

$$\mathrm{ch}_{\mathrm{ad}}^{\mathsf{E}_7}|_{\mathsf{A}_1 \times \mathsf{D}_4} = \mathrm{ch}_2^{\mathsf{D}_4} + 2\,\mathrm{ch}_1^{\mathsf{A}_1}\ \mathrm{ch}_1^{\mathsf{D}_4} + 2\,\mathrm{ch}_1^{\mathsf{A}_1}\ \mathrm{ch}_4^{\mathsf{D}_4} + 4\,\mathrm{ch}_3^{\mathsf{D}_4} + (\mathrm{ch}_1^{\mathsf{A}_1})^2 + 5,$$

so that

$$\mathrm{ch}_{\mathrm{ad}}^{\mathsf{E}_7}|_{\mathsf{A}_1 \times (\mathsf{A}_1)^4} = 2(\sigma_3 - \sigma_2) + \sigma_1^2 + 2s(t_1 + t_4)(t_3 + 2) + 4(t_1 t_4 + 2t_3) + s^2 + 5.$$

The computation step is then similar to E_8, except there is now less symmetry between the t_i (one can only exchange t_1 and t_4): one must therefore distinguish $3 \times 3 \times 6$ cases.

By computing a Gröbner basis we obtain that the set of possible critical values is -191, -11, $-35/4$, -7, -3, -2, 1, $\frac{17}{5}$, 5, 25, and 133. For each case and critical value we compute the corresponding manifold and its complex points. In each case except one, the manifold is 0-dimensional and for two 0-dimensional cases, the computation of the points does not finish. Those three problematic cases occur for the value -2.

For the other cases we compute the points and keep only the values for which one of the points has $|t_1|$, $|t_3|$, $|t_4|$, $|s| \leqslant 2$. It turns out that the value -2 is attained by one of those points and so the three problematic cases do not prevent us from concluding that -2 is a critical value.

From the formula $\mathcal{F}_{\mathsf{E}_7}(x) = \mathrm{ch}_{\mathrm{ad}}^{\mathsf{E}_7}(x) - 7$ we get $\inf_x \mathcal{F}_{\mathsf{E}_7}(x) = -14$ and then using Corollary 5.2$\chi(\mathsf{E}_7) \geqslant 1 - (-14/126)^{-1} = 10$. $\square$

One possible extension of this work could be to consider the nonsimply laced diagrams, that is, B_n, C_n, F_4, and G_2.

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