UiT The Arctic University of Norway

Faculty of Engineering Science and Technology
Department of Computer Science and Computational Engineering

Some new mathematical and engineering results connected to structural problems

Harpal Singh

A dissertation for the degree of Philosophiae Doctor December 2021

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Dedicated to my late grandparents Sukhdev Singh and Seetal Kaur.

Abstract

The research in this PhD thesis is focused on some problems of general interest in both applied mathematics and engineering sciences. It contains new results, which can be directly used for solving some important structural problems in engineering sciences, but which are also of interest in pure mathematics. The main body of this PhD thesis consists of six papers (Papers A – F).

In Paper A we present and discuss some recent developments concerning operational modal analysis (OMA) techniques, and also give a concrete example where the most popular OMA techniques have been implemented and applied on a steel truss bridge located over the Åby river in Sweden.

In Paper B a basic mathematical model of vibrating structure is presented. We also review and compare some signal processing techniques that are of great importance for OMA and structural health monitoring (SHM) of civil engineering structures. A new application of OMA on a high rise building in Sweden, where some of the most popular OMA techniques are applied is included in this paper.

In Paper C we prove and discuss some new Fourier inequalities in the generalized Lorentz type spaces, and in the important case with unbounded orthogonal systems. The derived results generalize, complement and unify several results in the literature for this general case.

In Paper D we further compliment and develop the results in Paper C. In this paper we also prove and discuss the corresponding Jackson-Nikol'skii type inequalities, still in the important case with unbounded orthonormal systems.

In Paper E we discuss some signal processing problems in a Bayesian framework and semi-group theory in the general case with non-separable function spaces. In particular, this is done for the case of an abstract Cauchy problem, with initial data in a non-separable Morrey space.

In Paper F we present a brief description of the Hålogaland suspension bridge, along with some challenges which have already appeared. Moreover, the problems and challenges which have appeared in a number of bridges of this type in the past are also reported on and discussed. The aim of this Paper is that it can serve as a basis for our planned future research concerning SHM of this bridge.

These new results are put into a more general frame in an introduction, where, in particular, some important information and challenges connected to the Hålogaland bridge in Narvik are discussed in the light of this frame.

Preface

This PhD thesis in Engineering Science is submitted in fulfillment of the requirements for the degree Doctor of Philosophy at UiT The Arctic University of Norway. The research presented here is conducted under the supervision of Professor Per Johan Nicklasson and co-supervision of Professor Lars-Erik Persson and Professor Dag Lukkassen.

The thesis is composed of six Papers A – F, and a matching introduction. In the introduction, the papers are discussed and put into a more general frame. The introduction is also of independent interest, since it contains a brief discussion on the important interplay between applied mathematics and engineering applications, illustrated by comparing with some relevant international research presented in this light.

A very brief presentation of the main content of the six papers can be found in the abstract above, and a more complete description at the end of the Introduction.

List of Papers

Paper A: Harpal Singh and Niklas Grip. "Recent trends in operation modal analysis techniques and its application on a steel truss bridge". *Journal of Nonlinear Studies,* Vol. 26, Nr. 4, 911–927, 2019.

Paper B: Harpal Singh, Niklas Grip and Per Johan Nicklasson. "A comprehensive study of signal processing techniques of importance for operation modal analysis (OMA) and its application to a high-rise building.". *Journal of Nonlinear Studies,* Vol. 28, Nr. 2, 389–412, 2021.

Paper C: Gabdolla Akishev, Lars-Erik Persson and Harpal Singh. "Inequalities for the Fourier coefficients in unbounded orthogonal systems in generalized Lorentz spaces." *Journal of Nonlinear Studies,* Vol. 27, Nr. 4, 1137–1155, 2020.

Paper D: Gabdolla Akishev, Lars-Erik Persson and Harpal Singh. "Some new Fourier and Jackson–Nikol'skii type inequalities in unbounded orthonormal systems." *Constructive Mathematical Analysis.* Vol. 4, Nr. 3, 291–394, 2021.

Paper E: Natasha Samko and Harpal Singh. "A note on contributions concerning non-separable spaces with respect to signal processing within Bayesian frameworks". *Technical report, UiT The Arctic University of Norway, 2021.* 11 pages. *Submitted for publication in journal.*

Paper F: Harpal Singh. "The Hålogaland bridge - descriptions, challenges and related research under arctic conditions". *Technical report, UiT The Arctic University of Norway, 2021.* 16 pages. *Submitted for publication in conference.*

In addition to the papers above, the following paper is related to this PhD thesis:

[*] D. Baramidze, L.E. Persson, H. Singh and G. Tephnadze. "Some new results and inequalities for subsequences of Nörlund logarithmic means of Walsh-Fourier series". 17 Pages. 2022, Submitted for journal publication.

Also this paper is focused on Fourier analysis, but of a completely different kind than in Papers C and D. Hence, we have chosen not to include it in the thesis.

Acknowledgements

First of all I would like to express my deepest gratitude to my supervisors Professor Per Johan Nicklasson, Professor Lars-Erik Persson and Professor Dag Lukassen for introducing me to the topics covered in this PhD thesis and for their invaluable support, advice, help, encouragement and care during all of this work.

Secondly, I am indebted to the Faculty of Engineering Science and Technology at UiT The Arctic University of Norway, Campus Narvik, for providing me the economic possibility to work with the questions studied and presented in this PhD thesis.

I also want to thank my other co-authors, Professor Gabdolla Akishev, Professor Natasha Samko and Associate Professor Niklas Grip for the successful collaborations, which have increased the quality of this PhD thesis.

I am also grateful to Assad Jamal (Project Director COWI Denmark), Issac Farreras Alcover (Senior Specialist Project Manager, Bridges International, COWI Denmark) and Roy Eivind Antonsen (Project Manager, Statens Veivesen, Narvik), for several fruitful discussions and generous advises.

Furthermore, I want to thank my friends who have been there for me always in the most stressful periods, and helped me to explore the amazing nature in Northern Norway.

Last, but not least, my deepest thanks go to my parents Jaswinder Kaur and Kultar Singh, and my sisters and relatives for the love and support during this work.

I would like to dedicate my PhD thesis to my late grandparents Sukhdev Singh and Seetal Kaur. They passed away while I was working with my PhD.

Harpal Singh
Narvik, December 2021

Chapter 1

Introduction

Civil engineering infrastructure around the world is in need of rehabilitation and repair. Ageing processes and damages resulting from wind, traffic and the surrounding environment, cannot be avoided. This is causing many serious problems, and new challenging research techniques are needed to solve them.

Governments and municipalities have to devote more time and budget for maintenance and repairs of damaged structures. In some cases it is a need for making new structures in place of deteriorated ones, to provide decent services. For example, in Germany the value of the constructed infrastructures is about 20 trillion Euro. If the life of infrastructures is assumed to be 100 years, then the replacement rate is approximately 200 billion Euro per year (see [225]).

A similar study was carried out by the American Society of Civil Engineers (ASCE) regarding the infrastructure in the USA, and it was concluded that an investment of over 2 trillion USD is needed over the next 10 years to reduce the risks of ageing infrastructures (see [155]). It was also discovered that more than 27 percent of the national bridges in the USA are either structural deficient or functionally obsolete, and the average age of the bridges in the USA is about 43 years (see [1]). The ASCE estimated that an investment of 930 billion USD is needed to bring the bridge infrastructure up to code.

Analysis of such problems is important e.g. in Scandinavia, as the impact of the extreme arctic conditions is quite intense. In 2017, the newspaper *"Verdens Gang"* (VG) got access to a report published by the Norwegian Public Roads Administration (Statens Vegvesen). According to this report, there are approximately 16971 bridges in Norway, and Norwegian Public Roads Administration has been violating inspection rules for many of them (see [247]).

It was revealed that for one in every two bridges, proper inspection of bridges was lacking. Moreover, 1087 bridges have damages that are described as critical to the load bearing capacity, or traffic safety by the bridge inspectors (see [247]). The newspaper VG has created a map of all the bridges from this report, and has marked the bridges as seriously injured, delaying action and lacking inspection (see [248]). The Norwegian Public Roads Administration confirmed this report.

The research in this PhD thesis is to a great extent inspired by the challenges described above, and aim at showing how engineering mathematics can contribute to tackling these serious problem. Here, in particular, we also have the focus on some challenges connected to the fairly new Hålogaland suspension bridge quite close to the UiT The Arctic University of Norway, Campus Narvik. Such challenges are also common to many types of infrastructure.

The main part of the thesis is focused on some problems of general interest in engineering sciences. It contains a broad spectrum from contributions which can be directly used for solving some important structural problems, to contributions which are of interest also in pure mathematics.

Such a broad view of science, and in particular the interplay between mathematics, applied mathematics and engineering sciences, are increasingly important for several reasons, e.g. for the development of research in various areas of engineering sciences. Correspondingly, nowadays there exists even some international journals which invites papers on such a broad scale of science. As an important example, let us here mention the Journal of "Nonlinear Studies" with P.L. Lions (Fields medalist) and S. Sivasundaram as Editors-in Chiefs (see [143], www.nonlinearstudies.com).

The December 2019 issue of this journal was devoted to the 75th anniversary of one of my supervisors, and in the preface the Editors P.L. Lions, N. Samko and S. Sivasundaram wrote some motivation in this spirit, (see [144] and also [149]). One of my articles, "Recent trends in operation modal analysis techniques and its application on a steel truss bridge" (see [225]) has been published in this issue of the journal.

Papers A and B form the basis of this PhD thesis. In Paper A we review and discuss some important operational modal analysis (OMA) techniques used for civil engineering constructions. As a concrete application, where such techniques are used, we present an example, where a steel truss bridge in Sweden is analysed in detail. Paper B complements Paper A in various ways, where some signal processing techniques of importance for OMA and structural health monitoring (SHM) are reviewed and discussed. As a concrete application, where such techniques are used, we present the details concerning a high rise building, namely the fire station in Luleå, Sweden.

The research discussed in Papers A and B is also related to some recent research in more theoretical Fourier analysis (see e.g. the PhD thesis of A. Seger [216] and the paper [9]). Fourier analysis form the basis of signal processing techniques that are used in vibration analysis of the structures. In this PhD thesis, we further develop the investigations in [8] and [9]. A lot of theoretical research has been done for Fourier coefficients in bounded systems, but it has its limitations with respect to some applications. This motivated the researchers to further explore and analyse inequalities for Fourier coefficients in unbounded orthogonal systems.

In Paper C some new results concerning Fourier inequalities in unbounded orthogonal systems are presented and discussed. In Paper D these results are complemented, and also some new Jackson-Nikol'skii type inequalities in unbounded orthonormal systems are proved. It is important to work with unbounded orthogonal systems, since some of the most powerful such systems used in practice have this property (e.g. wavelets system). Actual interest in Papers C and D is equipped with some applications related to structural problems in complex civil engineering structures, that focus about the development of signal processing techniques which can be used for detecting damages in bridges and high rise buildings see e.g. [216].

Vibration based damage detection (VBDD) techniques contain a lot of uncertainties that could lead to inaccurate damage detection of structures. Bayesian methods have proven to be the most rigorous probabilistic framework for identifying target variables, and evaluate corresponding uncertainties using the available information (see [252]). Therefore, further investigations in this area was done, when studying problems tackled in a Bayesian framework. Techniques for non-separable function spaces were suggested. This made the background for Paper E, where some new contributions have been stated and proved. In this paper we initiate the study of solving some signal processing problems within the Bayesian framework and non-separable spaces. For applications such an approach is important in situations where classical approximations cannot be done.

Summing up, Papers A and B form a basis for this PhD thesis. Papers C, D and E gives some other new contributions of importance for future research in this area. All these contributions contain results of interest also in pure mathematics. Papers A and B are intended to be able to be used for direct applications. All Papers A, B, C, D and E are typical papers in what we call “Engineering Mathematics”, which means that they contain results of interest for concrete applications in engineering sciences, but also in pure mathematics.

One main idea is that the research described in these papers can be of great importance for the challenges related to the new Hålogaland bridge in Narvik. Hence, in Paper F of this PhD thesis, we have reported investigations related to the challenges in the suspensions bridges around the world. Moreover, the challenges that already have appeared in Hålogaland bridge is described, along with the measures taken to overcome them. We hope that this can be the starting point for our planned future research where Statens Vegvesen (the Norwegian Public Roads Administration) and COWI can be our main collaboration partners. We have already initiated very promising preliminary discussions with them.

1.1 A short description of the results in papers A - F

1.1.1 Paper A

In Paper A we review some OMA methods that are used to find modal parameters of complex civil engineering structures (e.g. bridges, dams, high-rise buildings, etc.), that are difficult to excite with artificial excitation forces. Such methods use ambient vibrations from wind and traffic as unknown inputs, and output-only analysis is done to determine the resulting vibration modes.

There are mainly three approaches that are used to do an assessment of the conditions of the structures which are well acknowledged: dynamic analysis of the finite element model (FEM), finite element model updating and OMA. Dynamic analysis of the FEM is used to calculate mode shapes and mode frequencies of structures. In this approach bending and strains of the structures are compared against bending and strains of FEM. FEM is the most appropriate tool for modelling the structure, and is based of the following second order differential equation:

$$M\frac{d^2u(t)}{dt^2} + C_2\frac{du}{dt} + Ku(t) = B_2 f(t), \tag{1.1}$$

where M is the mass matrix, C_2 is the damping matrix, K is the stiffness matrix and B_2 is the selection matrix (input matrix), $f(t)$ is a vector with nodal forces and $u(t)$ is a vector with nodal displacements (see [199]). Mathematical modelling to compute the modal parameters is explained in detail in Paper B.

High accuracy is needed in the FEM for implementing structural control and SHM strategies. This accuracy depends on the type of FEM used to represent the structural elements, as well as the properties assigned to these elements. FEM has a lot of uncertainties in deciding the boundary conditions, geometry or material properties that change when the material deteriorates. Thus, there is a need to calibrate the FEM based on information from the real structure.

A numerical optimization technique, known as FEM updating, is used to calibrate the key parameters in the FEM of the structure. This minimizes the smallest possible difference between the measured vibrations and the simulated vibrations. FEM updating techniques can be classified as: updating using modal data, updating using frequency response functions, and updating using gradient and gradient free methods.

Damages in a structure affects its dynamic properties. In order to do damage detection by use of SHM, one of the most important parameters that needs a good estimation is the modal damping, since it is more sensitive to damages in comparison to mode frequencies (see [58]). OMA is a tool which can do good estimation of modal parameters. OMA methods can be classified as time domain methods, frequency domain methods and time-frequency methods. Two of the most popular time domain methods, namely the auto regressive moving average (ARMA), and stochastic subspace identification (SSI) methods, are described in this paper.

Time domain methods deal with the free responses that are present over the entire time span. However in the frequency domain, each mode has a small frequency band where the mode dominates. Hence in the frequency domain we have an advantage of natural modal decomposition by just considering the different frequency bands where the mode dominates (see [33]). This is the major advantage of this approach. Some of the most popular frequency domain methods, basic frequency domain (BFD), frequency domain decomposition (FDD), poly-reference least square complex frequency method (p-LSCF) are discussed in Section 4 of this paper.

An application of OMA on a steel truss bridge is also presented in Paper B. This steel truss bridge is located over the Åby river, about 45 Km west of Piteå in northern Sweden. The bridge was to be replaced by a new bridge in 2012. Vibration measurements were performed on the bridge while the old bridge was still in use. In addition to ambient vibrations (i.e. excitation's by the wind and river), a train was running over the bridge before each measurement. Modal analysis was performed on the data with the ARTemis software (see [90]). The software was able to identify 9 modes. For more details of the experiment, see Section 5 of this paper and the research report [93].

OMA techniques have been used for applications in many areas especially where the structures are difficult to excite. Research groups around the globe are working on projects in the area of OMA, that have applications for historical structures, lighthouses, dam structures, stadium buildings, onshore wind turbines and bridges. OMA has a much wider scope than the other two methods (FEM and FEM updating), and research groups around the world are exploring areas of civil engineering, mechanical engineering, aerospace engineering and offshore engineering with these techniques under different conditions.

The FDD and SSI methods have become the most popular techniques for OMA in the past decade, but in order to do damage detection, these techniques have some limitations. Therefore in Paper B we do a comprehensive study of signal processing techniques that are important for SHM and OMA.

The results in Paper A are related to the following publications: [17], [18], [19], [22], [23], [24], [29], [33], [34], [35], [36], [39], [40], [41], [43], [45], [46], [47], [50], [57], [58], [63], [75], [76], [80], [85], [90], [93], [94], [134], [137], [140], [147], [153], [155], [160], [163], [167], [172], [176], [178], [179], [180], [181], [182], [183], [196], [199], [203], [206], [207], [208], [245], [251], [253], [256], [266], [271] and [273].

1.1.2 Paper B

Ageing of civil engineering infrastructures such as dams, bridges, tunnels and buildings, causes many problems with great consequences, both from practical and economical points of view. The main aim of this paper, together with [225], is to describe the state of the art of known methods, and how to tackle this important problem using mathematical methods. A brief description of the methods, as well as comparison between the methods is provided in this Paper. Moreover, in this article we also include a new application of OMA, on a high-rise building in Luleå, Sweden, where some of the most popular OMA techniques have been applied.

As a continuation of Paper A, we present mathematical modelling of vibrating structures and explain how the eigen frequencies and modal damping factors can be computed. In general, civil engineering structures such as bridges, dams, high rise buildings, are made from materials such as concrete, glass, steel, etc. that do not have ideal linear behaviour and exhibit non-linear stress-strain characteristics (see [116]). Moreover, due to ageing and degradation, such structures deviate from their ideal expected behaviour. In order to study such a non-linear behaviour, more advanced signal processing techniques are needed to do vibration analysis. A detailed mathematical framework of signal processing techniques used in the study of OMA and SHM can be found in [24], [32], [33], [47], [75], [196] and [253].

During the last decade, research in the area of OMA and its applications in SHM have seen a tremendous growth. The development of automated OMA with high accuracy in the estimation of modal parameters from structural response, opened up new prospects for structural damage detection. More recent developments in the field of OMA make modal based SHM a fairly mature technology (see [197]). Signal processing methods like Fourier based transforms such as the short-time Fourier transform (STFT), the fast Fourier transform (FFT), the wavelet transforms, quadratic time frequency methods, multiple signal classification (also called the Music algorithm), the S-transform and the Hilbert-Huang transforms (HHT) are described and discussed in this paper. Some details concerning these methods can be found in [100] and [131].

Furthermore, some concrete results of OMA of a high-rise building are presented and illustrated in this paper. The building under investigation was Luleå fire tower, located in northern Sweden. The structure of this fire station is special due to symmetry reasons, as it can have multiple modes with the same, or almost the same frequency. The tower has three concrete walls covered with bricks, and a front wall consisting of glass windows with steel beams, see Figure 1. Here and below all references of figures and tables are described in Paper B. The sensor placement and floor design is also illustrated and described in Figure 1. Table 1 of this paper shows the sensor location, and the letters corresponds to the locations as described in the measurement plan.

The modal analysis software ARTeMIS was used to find different modes of the structure from the recorded data. Figure 2 shows the mode shapes that were found with the FDD method, and Table 2 represents the frequencies at which the mode shapes were found. Moreover, Figure 3 shows the mode shapes found with the EFDD method, and Table 3 represents the frequencies at which the mode shapes were found, along with standing frequency, damping ratio as well as standing damping ratio.

In this Paper B we also briefly present some recently developed signal processing methods and algorithms which have a potential for OMA and SHM, and should be analysed in more detail. In particular, Section 5 of Paper B focus in this direction, and new methods like the fast S-transform, the synchrosqueezed wavelet transform, the emperical wavelet transform, the adaptive optimal kernel time-frequency analysis and the enhanced fast Fourier transform are described.

Summing up, I believe that Papers A and B can be very useful as a basis when investigating the problems related to signal processing techniques for detecting damages in engineering structures.

The results in Paper B are related to the following publications: [1], [5], [13], [14], [15], [16], [24], [25], [27], [28], [31], [32], [33], [35], [37], [38], [43], [46], [47], [49], [51], [52], [53], [54], [55], [60], [61], [62], [64], [68], [69], [75], [77], [79], [82], [83], [84], [87], [89], [91], [93], [99], [100], [101], [104], [106], [107], [108], [109], [110], [112], [116], [117], [118], [122], [123], [124], [131], [135], [138], [141], [142], [145], [146], [150], [151], [157], [158], [159], [160], [165], [172], [176], [180], [192], [193], [195], [196], [197], [199], [205], [216], [217], [221], [223], [225], [227], [236], [237], [238], [239], [243], [244], [249], [250], [253], [256], [257], [258], [259], [260], [261], [265], [267], [269], [272], [274], [275], [276], [277], [278].

1.1.3 Papers C and D

In Papers C and D we prove some new Fourier inequalities for unbounded systems. To shortly describe the background of such research, the following result derived by Lars-Erik Persson in his PhD thesis from 1974 (see [186] and also [184]) is our starting point.

Theorem A. Let $0 < p < \infty$ and $\Phi = \{e^{2\pi ikt}\}_{k=-\infty}^{+\infty}$ be the trigonometrical system.
a) If there exists a positive number $\delta > 0$ so that $\psi(t)t^{-\delta}$ is an increasing function of t and $\psi(t)t^{-(\frac{1}{2}-\delta)}$ is a decreasing function of t, then

$$\left(\sum_{k=1}^{\infty}\left(a_k^*\psi(k)\right)^p\frac{1}{k}\right)^{\frac{1}{p}} \leqslant C\|f\|_{L_{\psi,p}}.$$

b) If there exists a positive number $\delta > 0$ such that $\psi(t)t^{-\frac{1}{2}-\delta}$ is an increasing function of t and $\psi(t)t^{-1+\delta}$ is a decreasing function of t, then

$$\|f\|_{L_{\psi,p}} \leqslant C\left(\sum_{k=1}^{\infty}\left(a_k^*\psi(k)\right)^p\frac{1}{k}\right)^{\frac{1}{p}},$$

where $\{a_k^*\}_{k=0}^{\infty}$ is the non-increasing rearrangement of the sequence $\{|a_k|\}_{k=-\infty}^{\infty}$ of Fourier coefficients of f with respect to the system Φ.

Here, as usual, the generalized Lorentz space $L_{\psi,p}$ consists of the functions f on $[0,1]$ such that $\|f\|_{L_{\psi,p}} < \infty$, where

$$\|f\|_{L_{\psi,p}} := \begin{cases} \left(\int\limits_0^1 (f^*(t)\psi(t))^p \frac{dt}{t}\right)^{\frac{1}{p}}, & \text{for } 0 < p < \infty, \\ \sup\limits_{0\leqslant t\leqslant 1} f^*(t)\psi(t), & \text{for } p = \infty, \end{cases} \tag{1.2}$$

where f^* denotes the non-increasing rearrangement of the function $|f|$ (see e.g. [220]).

Remark 1.1.1. *Theorem A may be regarded as a unification and generalization of several classical results e.g. those by Marcinkiewicz, Zygmund, Hausdorff, Young, Paley, Riesz, Pitt and Stein. A very good description of this prehistory of Theorem A is given in the PhD thesis of Kopezhanova from 2017 (see [129]).*

Theorem A was generalized to the case with a general bounded orthogonal system (this means that $|a_n| \leqslant C < \infty, \forall n$) in [129] and [130]. However, it is not known whether or not Theorem A can by generalized to the case with unbounded orthogonal systems. However, some results are known also for this case but in more restrictive cases e.g. for Lebesgue spaces, see for example various contributions by Marcinkiewicz and Zygmund [152], Kolyada [127], Kirillov [125], Flett [78], Maslov [154], Stein [233], Bochkarev [30], Nursultanov [169] and Tleukhanova and Mussabaeva [242].

For several applications e.g. some considered in this PhD thesis, it is important to generalize such results to the case with unbounded orthogonal systems. For example many systems related to wavelet theory are of this type. Here we only give the following example:

Example 1.1.2. (a) $\{\chi_n\}$–orthonormal system of Haar functions (see e.g. [129]). The functions $\chi_n(t)$ are defined as follows: $\chi_1(t) := 1$ for $t \in [0,1]$ and for $n = 2^m + k$, $k = 1, \ldots, m$ and $m = 0, 1, \ldots$ put

$$\chi_n(t) = \begin{cases} \sqrt{2^m}, & t \in (\frac{2k-2}{2^{m+1}}, \frac{2k-1}{2^{m+1}}), \\ -\sqrt{2^m}, & t \in (\frac{2k-1}{2^{m+1}}, \frac{2k}{2^{m+1}}), \\ 0, & t \bar{\in} \left[\frac{r}{m_k}, \frac{r+1}{m_k}\right]. \end{cases}$$

The value of $\chi_n(t)$ in a discontinuity point t is defined as

$$\chi_n(t) = \frac{1}{2} \lim_{\varepsilon \to 0} [\chi_n(t+\varepsilon) + \chi_n(t-\varepsilon)].$$

For more such important examples of unbounded orthogonal systems see Remark 4.11 in [10]
By an unbounded orthogonal system $\{\varphi_n\}$ in $L_2[0,1]$ we mean that (see [5])

$$\|\varphi_n\|_s \leq M_n, \quad n \in \mathbb{N}, \tag{1.3}$$

Moreover, let

$$\mu_n = \mu_n^{(s)} := \sup\{\| \sum_{k=1}^{n} c_k \varphi_k \|_s : \sum_{k=1}^{n} |c_k|^2 = 1\}, \quad \rho_n := \left(\sum_{k=n}^{\infty} |a_k|^2 \right)^{\frac{1}{2}}, \tag{1.4}$$

for some $s \in (2, +\infty)$. Here $M_n \uparrow$ and $M_n \geq 1$ (see e.g. [152],).

The two main theorems from paper C generalize and unify some other recent results, e.g. by Akishev, Lukassen and Persson (see [8]), in this general context. The main results in this paper read as follows:

Theorem 1.1.3. *Let $\{\varphi_n\}_{n=1}^{\infty}$ be an orthogonal system, which satisfies the condition* (1.3)*, $s \in (2, +\infty]$, $0 < \theta \leq 2$, and the function ψ satisfy the conditions $1 < \alpha_\psi = \beta_\psi = 2^{1/2}$, $\frac{\psi(t)}{t^{1/2}} \in SVL$ and*

$$\sup_{x \in (0,1]} \frac{t^{1/2}}{\psi(t)} < \infty.$$

If $\{a_n\} \in l_2$ and

$$\Lambda_{\psi,\theta}(a) = \sum_{n=1}^{\infty} \left(\frac{\psi((1+\mu_n)^{-1})}{\sqrt{(1+\mu_n)^{-1}}} \right)^{\theta} (\log(1+\mu_n))^{(\frac{1}{\theta}-\frac{1}{2})\theta} \left(\rho_n^{\theta} - \rho_{n+1}^{\theta}\right) < +\infty,$$

where μ_n and ρ_n are defined by (1.4)*, then the series* $\sum\limits_{n=1}^{\infty} a_n\varphi_n(x)$ *converges in the space* $L_{\psi,\theta}$ *to some function* $f \in L_{\psi,\theta}$ *and* $\|f\|_{L_{\psi,\theta}} \leqslant C(\Lambda_{\psi,\theta})^{1/\theta}$.

Theorem 1.1.4. *Let $\{\varphi_n\}_{n=1}^{\infty}$ be an orthogonal system, which satisfies the condition (1.2), $s \in (2, +\infty]$, $2 < \theta < +\infty$ and $\alpha < 0$ and the function ψ satisfy the conditions $1 < \alpha_\psi = \beta_\psi = 2^{1/2}$, $\frac{t^{1/2}}{\psi(t)} \in SVL$ and*

$$\sup_{x\in(0,1]} \frac{\psi(t)}{t^{1/2}} < \infty.$$

If the function $f \in L_{\psi,\theta}$, then $A_{\psi,\theta}(f) \leqslant C\,\|f\|_{\psi,\theta}$, where

$$A_{\psi,\theta}(f) :=$$

$$\left\{ \sum_{n=1}^{\infty} \left(\frac{\sqrt{\left(1+\mu_{\nu_{n+1}-1}\right)^{-1}}}{\psi((1+\mu_{\nu_{n+1}-1})^{-1})} \right)^{\theta} \left(\log\left(1+\mu_{\nu_{n+1}-1}\right)\right)^{1-\frac{\theta}{2}} \left(\sum_{k=\nu_n}^{\nu_{n+1}-1} a_k^2(f) \right)^{\frac{\theta}{2}} \right\}^{\frac{1}{\theta}},$$

μ_j are defined by (1.4) *and $a_k(f)$, $k \in \mathbb{N}$, as usual denote the Fourier coefficients with respect to the system $\{\varphi_n\}_{n=1}^{\infty}$.*

Here, as usual, we denote by SVL the set of all non-negative functions on $[0,1]$ of $\psi(t)$ for which $(\log 2/t)^{\varepsilon}\psi(t) \uparrow +\infty$ and $(\log 2/t)^{-\varepsilon}\psi(t) \downarrow 0$ for $t \downarrow 0$ (see [224] and [240]). Moreover, for the function ψ we define the "indices"

$$\alpha_\psi := \underline{\lim}_{t\to 0} \frac{\psi(2t)}{\psi(t)}, \qquad \beta_\psi := \overline{\lim}_{t\to 0} \frac{\psi(2t)}{\psi(t)}.$$

It is known that $1 \leq \alpha_\psi \leqslant \beta_\psi \leqslant 2$ (see e.g. [218]).

In paper D we do some further generalizations and unifications (also of the related Jakson-Nikolśkii type inequalities) of several classical and new results (eg. by J. Marcinkiewiez, A. Zygmund, S.V. Bochkarev, V.I. Ovchinnikov, V.A. Raspopova, V.A. Rodin, L.R.Ya Doktoroski, N. Fleukhanova, G. Mussabaeva and others) in this general context. The main theorems in this paper reads:

Theorem 1.1.5. *Let ψ be a function satisfying the conditions $1 < \alpha_\psi = \beta_\psi = 2^{1/2}$, $\frac{t^{1/2}}{\psi(t)} \in SVL$,*

$$\sup_{t\in(0,1]} \frac{\psi(t)}{t^{1/2}} < \infty,$$

and assume that the orthonormal system $\{\varphi_n\}_{n\in\mathbb{N}}$ for some $r \in (2, +\infty]$ satisfies the condition (1). Then, for any function $f \in L_{\psi,q}$, $2 < q \leqslant \infty$, the following inequality holds:

$$\Big[\sum_{k\in A} |\hat{f}(k)|^2\Big]^{\frac{1}{2}} \leqslant C\|f\|_{\psi,q}\Big[\ln(1+\sum_{j\in A} M_j^2)\Big]^{\frac{1}{2}-\frac{1}{q}} \frac{\sqrt{(1+\sum_{j\in A} M_j^2)^{-1}}}{\psi((1+\sum_{j\in A} M_j^2)^{-1})},$$

where A is a nonempty set in $\mathbb{N}$ and C is positive constant which depends only on q and r.

Theorem 1.1.6. *Let the function ψ satisfy the conditions $1 < \alpha_\psi = \beta_\psi = 2^{1/2}$, $\frac{\psi(t)}{t^{1/2}} \in SVL$,*

$$\sup_{t\in(0,1]} \frac{t^{1/2}}{\psi(t)} < \infty, \tag{6}$$

let the orthonormal system $\{\varphi_n\}_{n\in\mathbb{N}}$ for some $r \in (2, +\infty]$ satisfy the condition (1) and $f_n(x) = \sum_{k=1}^n c_k\varphi_k(x)$.

1) If $1 < q < 2$, then

$$\|f_n\|_{\psi,q} \leqslant C \left(\frac{\sqrt{\left(1+\sum_{j=1}^n M_j^2\right)^{-1}}}{\psi((1+\sum_{j=1}^n M_j^2)^{-1})} \right)^{-1} \left(\log\left(1+\sum_{k=1}^n M_k^2\right)\right)^{\frac{1}{q}-\frac{1}{2}} \|f_n\|_2$$

for some constant C depending only on q.

2) If $1 < p < 2 < q < +\infty$, then

$$\|f_n\|_{\psi,p} \leqslant C(p,q)\|f_n\|_{\psi,q} \left(\log(1+\sum_{k=1}^n M_k^2)\right)^{\frac{1}{p}-\frac{1}{q}}$$

for some constant C depending only on p and q.

3) If $2 < p < q < +\infty$, then

$$\|f_n\|_{\psi,p} \leqslant C(p,q)\|f_n\|_{\psi,q} \left(\log(1+\sum_{k=1}^n M_k^2)\right)^{\frac{1}{p}-\frac{1}{q}}$$

for some constant C depending only on p and q.

The results in Paper C are related to the following publications: [8], [9], [30], [71], [78], [96], [125], [127], [129], [130], [132], [136], [152], [154], [164], [169], [174], [184], [185], [186], [187], [216], [218], [220], [224], [225], [233], [240], [242].

The results in Paper D are related to the following publications: [2], [3], [4], [5], [6], [7], [8], [9], [12], [21], [30], [65], [66], [67], [88], [113], [114], [119], [130], [152], [161], [162], [166], [168], [170], [171], [174], [184], [201], [218], [219], [220], [235], [241], [242].

1.1.4 Paper E

In this paper we discuss the study of some signal processing problems within the Bayesian frameworks and semigroups theory, in the case where the function space under consideration may be non-separable. For applications, the suggested approach is of special interest in situations where approximation in the norm of the space is not possible.

This paper was inspired by the survey paper [97] on the Bayesian approach to signal processing problems, in which the signal is a solution of a stochastic partial differential equation (SPDE). The goal of the approach discussed in the mentioned survey, was to find the signal as a solution to the SPDE, taking into account the noisy observations. The Bayesian interface is a probabilistic method of interference, that allows to form probabilistic estimates of certain parameters from a given series of observations.

This approach helps to avoid dealing with the algorithms to which the use of known methods based on application of the standard Markov chain Monte Carlo (MCMC) method after a discretization leads. One of the disadvantages of such algorithms, is that they perform poorly under refinement of the discretization (see [97] and the references given therein).

The use of a proper mathematical formulation of the problems on domain space while working with probability measures on function spaces, leads to efficient sampling techniques, defined on the path-space as the domain space, and therefore is robust under the introduction of discretization. A wide variety of signal processing problems is overviewed in [97]. These problems lead to a posterior probability measure on a separable Banach space.

The theory of parabolic equations has deep connections with functional analysis, especially with the theory of evolution equations with unbounded operators in Banach spaces, and the theory of semigroups. The theory of stochastic processes, especially the theory of Markov processes and stochastic differential equations, very closely interacts with the theory of parabolic equations (see [103]).

In this paper, we suggest an approach for the study of properties of solutions to such equations in the case of non-separable function spaces. Moreover, in the case of problems formulated as filtering problems in addition to smoothing problems, an approach based on study in the framework of weighted nonstandard function spaces could also be useful.

To prove our main results (Theorems 3.1 and 3.3), we use the following well known theorem (see [188] , Theorem 3.1):

Theorem B Let E be a Banach space. Consider a closed linear operator A with the domain $D(A)$ dense in E and non-empty resolvent set. Suppose that the Cauchy problem

$$\begin{cases} u_t = Au, \\ u|_{t=0} = u_0. \end{cases}, \tag{1.5}$$

is uniquely solvable for every $u_0 \in D(A)$. Then there exists a semigroup T_t of the class C_0 which solves the Cauchy problem (1.5).

Here, as usual, that the semi-group T_t is of the class C_0 means that

$$\lim_{t\to 0} \|T_t f - f\|_E = 0.$$

By using Theorem B a solution of the Cauchy problem via semigroup theory is derived and the two main results (Theorems 3.1 and 3.3) are proved in Section 3:

Theorem 1.1.7. *Let the Banach space E and the operator A satisfy all the assumptions of Theorem B and let X be an arbitrary Banach space, non-necessarily separable, such that $X \hookrightarrow E$. Then $u(t) \in X,\ t > 0$ for all $u_0 \in D_X(A)$, if the operator $T_t,\ t > 0$ is bounded in X. Suppose that the Cauchy problem* (1.5) *is uniquely solvable for every $u_0 \in D(A)$. Then there exists a semigroup T_t of the class C_0 which solves the Cauchy problem* (1.5).

Theorem 1.1.8. *Let $u_0 \in \mathcal{L}^{p,\lambda}(\mathbb{R}^n), 1 < p < \infty,\ 0 < \lambda < n$. Then the unique solution $u(x,t)$ of the problem*

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = k(\Delta u)(x,t),\ x \in \mathbb{R}^n \\ u(x,0) = u_0(x),\ x \in \mathbb{R}^n, \end{cases} \tag{1.6}$$

has the property $u(\cdot,t) \in \mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ uniformly in $t \in \mathbb{R}_+$. The convergence

$$\|T_t u_0 - u_0\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)} \to 0 \text{ as } t \to 0 \tag{1.7}$$

holds if u_0 is in the Zorko subspace $Z^{p,\lambda}(\mathbb{R}^n)$ of $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$.

Here, as usual, $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ denotes the Morrey space defined by the norm

$$\|f\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n} \sup_{r>0} \left(\frac{1}{r^\lambda} \int_{B(x,r)} |f(y)|^p \, dy \right)^{\frac{1}{p}}.$$

Moreover, $Z^{p,\lambda}(\mathbb{R}^n)$ is the subspace of $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ such that

$$\lim_{h\to 0} \|f(\cdot + h) - f(\cdot)\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)} = 0.$$

This paper is the first where non-separable function spaces have been used to give a new possibility to avoid approximation problems for example in signal processing related to damage detection in bridges.

Summing up, we think that our Paper E can be very useful as a basis when investigating such types of problems in engineering sciences.

The results in Paper E are related to the following publications: [20], [42], [48], [59], [70], [72], [73], [86], [97], [103], [121], [126], [128], [133], [148], [156], [177], [188], [194], [198], [200], [204], [210], [211] [213], [214], [234], [268], [279].

1.1.5 Paper F

In Paper F, a very general description of the Hålogaland bridge is presented, along with its basic design. The Hålogaland bridge is a part of European Route E6. The bridge has been featured on the Science Channel show "Building Giants", and the episode was titled "Arctic Mega Structure." The book titled "Hålogalandsbrua: Superbru i verdensklasse" (in Norwegian), is an important and very well written book. Along with the emphasis on development and construction, the book provides an insight knowledge to explain the fact that why it took so long to put infrastructure in place in northern Norway. The book provides lots of useful information and details the entire project (see [115]).

Challenges connected to the Hålogaland bridge are also discussed in this paper, along with the precautionary measures that have been taken to overcome these challenges. A brief discussion is made about how Gulf stream and wind under arctic conditions can have impact on the Hålogaland bridge.

Particularly, in case of suspension bridges, main suspension cables and hangers can suffer from severe corrosion and fatigue damage, as discussed in the research report [102]. Such damages can have a critical effect on the lifetime of the bridges.

Especially in northern Norway and the countries inside the arctic circle, where the weather conditions are extreme and due to the climate changes storms are getting bigger and sea waves are becoming more violent. The problems arising due to extreme weather and corrosion are also more sensible as salting of roads in winter further speeds up the process.

Therefore, structures need more routine checkups to carry out operation and maintenance activities to get a reduced downtime. The Hålogaland bridge is of great importance, as the cargo related to fishing is expected to quadruple in the next 10 years, and its downtime can have a huge economic impact.

Moreover, in Paper F various challenges that have appeared in the past in the suspension bridges have been discussed with concrete examples. Various vibration based damage detection (VBDD) techniques that can be used to detect such damages are also mentioned. It is clear that there are many factors that influence the deterioration of bridges such as environmental factors like wind, rain, storms, earthquakes and overloading of structures.

Localization is an issue with VBDD techniques due to the presence of uncertainties. As a result different approaches can lead to inaccurate damage detection results. Among various approaches Bayesian methods have proven to be the most rigorous probabilistic framework to identify target variables and evaluate corresponding uncertainties using the available information.

In order to have accurate vibration analysis, it is important to have accurate signals/data from the industry standard vibration transducers such as accelerometers. In SHM system, data retrieved from the sensors might have some deviation or unwanted characteristics that needs to be resolved. Therefore, preprocessing of data is required before the data can be prepared for analysis and interpretation.

For a bridge of the scale of Hålogaland bridge in arctic environment where weather changes fast and conditions are extreme, SHM in real-time will generate a lot of data. Therefore, a good data management system with the use of artificial intelligence can make the process of of damage detection and maintenance operations more efficient.

Finally, in Paper F we also include some suggestions for future research connected to Hålogaland bridge, where the new findings in Papers A-E can be used.

The results in Paper F are related to the following publications: [5], [8], [10], [11], [26], [44], [56], [63], [231], [74], [75], [79], [81], [92], [95], [97], [98], [102], [105], [111], [115], [120], [139], [173], [175], [189], [190], [191], [202], [209], [212], [215], [216], [222], [225], [226], [228], [229], [230], [231], [232], [246], [250], [252], [254], [255], [262], [263], [264] and [270].

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Papers

Paper A

Recent trends in operation modal analysis techniques and its application on a steel truss bridge

Harpal Singh, Niklas Grip

Published in *Journal of Nonlinear Studies*, Volume 26, Number 4, Pages 911-927, 2019.

Paper B

A comprehensive study of signal processing techniques of importance for operation modal analysis (OMA) and its application to a high-rise building

Harpal Singh, Niklas Grip, Per Johan Nicklasson

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B

Paper C

Inequalities for the Fourier coefficients in unbounded orthogonal systems in generalized Lorentz spaces

Gabdolla Akishev, Lars-Erik Persson, Harpal Singh

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C

Paper D

Some New Fourier and Jackson–Nikol'skii Type Inequalities in Unbounded Orthonormal Systems

Gabdolla Akishev, Lars-Erik Persson, Harpal Singh

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Research Article

Some New Fourier and Jackson–Nikol'skii Type Inequalities in Unbounded Orthonormal Systems

GABDOLLA AKISHEV, LARS ERIK PERSSON*, AND HARPAL SINGH

ABSTRACT. We consider the generalized Lorentz space $L_{\psi,q}$ defined via a continuous and concave function ψ and the Fourier series of a function with respect to an unbounded orthonormal system. Some new Fourier and Jackson-Nikol'skii type inequalities in this frame are stated, proved and discussed. In particular, the derived results generalize and unify several well-known results but also some new applications are pointed out.

Keywords: Inequalities, generalized Lorentz spaces, unbounded orthonormal system, Fourier inequalities, Jackson–Nikol'skii inequality.

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1. INTRODUCTION

Let the function ψ be continuous and concave by $[0,1]$, $\psi(0)=0$ and $0<q\leqslant\infty$. Such functions are called Φ functions. The generalized Lorentz space $L_{\psi,q}$ is the set of measurable functions f on $[0,1]$ for which

$$\|f\|_{\psi,q} := \left(\int_0^1 f^{*^q}(t)\psi^q(t)\frac{dt}{t}\right)^{1/q} < \infty,$$

where f^* is the non-increasing rearrangement of the function $|f|$ (see e.g. [36]).
For a given function $\psi(t), t\in[0,1]$, we define

$$\alpha_\psi := \underline{\lim}_{t\to 0}\frac{\psi(2t)}{\psi(t)}, \quad \beta_\psi := \overline{\lim}_{t\to 0}\frac{\psi(2t)}{\psi(t)}.$$

It is known that $1\leqslant\alpha_\psi\leqslant\beta_\psi\leqslant 2$ (see e.g. [35]) .
Note that for $\psi(t)=t^{1/p}$, the space $L_{\psi,q}$ coincides with the Lorentz space $L_{p,q}$, $0<q,p<\infty$, which consists of all functions f such that (see e.g. [38, p. 228])

$$\|f\|_{p,q} := \left(\int_0^1 f^{*^q}(t)t^{\frac{q}{p}-1}dt\right)^{1/q}.$$

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*Corresponding author: L. E. Persson; larserik6.pers@gmail.com
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In particular, for the case $p = q$, we have the usual Lebesgue space with the norm (quasi-norm if $0 < q < 1$)

$$\|f\|_q := \left(\int_0^1 |f(x)|^q dx \right)^{1/q}, \quad 0 < q < \infty.$$

Let $q, p \in (0, +\infty)$ and $\alpha \in R = (-\infty, +\infty)$. The Lorentz-Zygmund space $L_{p,q}(\log L)^\alpha$ is the set of all functions f measurable on $[0, 1]$ for which (see e.g. [37])

$$\|f\|_{p,q,\alpha} := \left\{ \int_0^1 (f^*(t))^q (1 + |\log t|)^{\alpha q} t^{\frac{q}{p} - 1} dt \right\}^{\frac{1}{q}} < +\infty.$$

For A, B the notation $A \asymp B$ means that there exits positive constants C_1, C_2 such that $C_1 A \leqslant B \leqslant C_2 A$.

We consider the orthonormal system $\{\varphi_n\}_{n\in\mathbb{N}} \subset L_2[0, 1]$ (see [22, p. 58]) satisfying the condition

$$\|\varphi_n\|_r := \left(\int_0^1 |\varphi_n(x)|^r dx \right)^{\frac{1}{r}} \leqslant M_n, \quad n \in \mathbb{N} \tag{1}$$

for some $r \in (2, +\infty]$. Here, we assume that $\{M_n\}$ is a non-decreasing sequence.

Let $\hat{f}(n)$ be the Fourier coefficients of the function f with respect to the orthonormal system $\{\varphi_n\}_{n\in\mathbb{N}}$.

J. Marcinkiewicz and A. Zygmund [22] proved some inequalities for the sums of the Fourier coefficients of the orthogonal system $\{\varphi_n\}_{n\in\mathbb{N}}$ satisfying condition (1) and norms of the function $f \in L_p$, $1 < p < \infty$. Later, many authors investigated this problem in other functional spaces (for example, see [3], [6], [7], [8], [11], [13], [21], [30], [32], [33], [42] and bibliographic references in them).

In particular, the following statement is known (see S.V. Bochkarev [11]):

Theorem 1.1. *Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be an orthonormal system of complex-valued functions*

$$\|\varphi_n\|_\infty \leqslant M, \ n = 1, 2, ... \tag{2}$$

for some $M < \infty$. Then, for any $2 < q \leqslant \infty$ and $n = 2, 3, ...$, the following inequality holds:

$$\left[\sum_{k=1}^n (\hat{f}^*(k))^2 \right]^{\frac{1}{2}} \leqslant CM \|f\|_{2,q} (\log n)^{\frac{1}{2} - \frac{1}{q}}.$$

In the case $q = \infty$, Theorem 1.1 was previously proved by V.I. Ovchinnikov, V.D. Raspopova and V.A. Rodin [32].

In the case when $\{\varphi_n\}_{n\in\mathbb{N}}$ is a trigonometric system, in the Lorentz-Zygmund space $L_{2,q,}(\log L)^\alpha$ H. Oba , E. Sato and Y. Sato [30] stated and proved the following:

Theorem 1.2. *Let $2 < q \leqslant \infty$, $n \geqslant 3$ and $\alpha \in \mathbb{R}$. Then the following inequality holds:*

$$\left[\sum_{k=1}^n (\hat{f}^*(k))^2 \right]^{\frac{1}{2}} \leqslant CA_n \|f\|_{2,q,\alpha}$$

for some constant C which is independent of n and f, and A_n is as follows:

$$A_n = \begin{cases} (\log n)^{\frac{1}{2} - \frac{1}{q} - \alpha}, & \text{if } \alpha < \frac{1}{2} - \frac{1}{q}, \\ (\log(\log n))^\alpha, & \text{if } \alpha = \frac{1}{2} - \frac{1}{q}, \\ 1, & \text{if } \alpha > \frac{1}{2} - \frac{1}{q}. \end{cases}$$

A generalization of this theorem for the orthonormal system $\{\varphi_n\}_{n\in\mathbb{N}}$ satisfying condition (2) was proved by L.R.Ya. Doktorski (see [13]). Moreover, N. Tleukhanova and G. Mussabaeva [42] for the orthonormal system $\{\varphi_n\}_{n\in\mathbb{N}}$ satisfying condition (2) proved the inequality

$$\sup_{n\in\mathbb{N}} \frac{1}{n^{1/2}(\log(n+1))^{\frac{1}{2}-\frac{1}{q}}} \sum_{k=1}^{n} \hat{f}^*(k) \leqslant C\|f\|_{2,q} \tag{3}$$

for any function $f \in L_{2,q}, 2 < q \leqslant \infty$.
Most results concerning Fourier inequalities are derived for bounded orthonormal systems. However, for several applications it is also important to derive such results for unbounded orthonormal systems like those described in our final Remark 4.11. One aim of this paper is to further complement our recent research in this direction (see [6], [7] and [8]) and also prove and discuss some new related Nikol'skii type inequalities of this type. Let us first mention that in [3] for an unbounded orthonormal system $\{\varphi_n\}_{n\in\mathbb{N}}$, the following statement was proved (for the case $\alpha = 0$, see [2]).

Theorem 1.3. *Let the orthonormal system $\{\varphi_n\}_{n\in\mathbb{N}}$ for some $r \in (2, +\infty]$ satisfy the condition (1). Then, for any function $L_{2,q}(\log L)^{\alpha}, 2 < q \leqslant \infty$, $\alpha < \frac{1}{2} - \frac{1}{q}$, $n \in \mathbb{N}$, the following inequality holds:*

$$\Big[\sum_{k=1}^{n} |\hat{f}(k)|^2\Big]^{\frac{1}{2}} \leqslant C\|f\|_{2,q,\alpha}\Big[\ln(1+\sum_{j=1}^{n} M_j^2)\Big]^{\frac{1}{2}-\frac{1}{q}-\alpha}.$$

For a trigonometric polynomial

$$T_n(x) = \sum_{k=-n}^{n} a_k e^{ikx}, n \in \mathbb{N}$$

the following Jackson–Nikol'skii inequality is well known (see [17], [27])

$$\|T_n\|_q \leqslant 2n^{1/p}\|T_n\|_p \tag{4}$$

for $1 \leqslant p < q \leqslant \infty$. This inequality is also called the inequality of different metrics for a trigonometric polynomial.
For case $0 < p < q \leqslant \infty$, inequality (4) was proved in [16] and [10]. Moreover, for $p = 0 < q < \infty$, it was proved by V.V. Arestov [10].
Nowadays, there are various generalizations of the Jackson-Nikol'skii inequality (see [5], [12], [29] and the bibliography therein). One of the generalizations is its extension to polynomials in orthonormal systems of functions. In particular, M.F. Timan [40] proved the following statement:

Theorem 1.4. *Let $1 \leqslant p \leqslant 2$, $p < q \leqslant \infty$ and $\{\varphi_n\}_{n=1}^{\infty}$ be a uniformly bounded sequence of orthonormal systems of functions. Then for the polynomial*

$$f_n(x) = \sum_{k=1}^{n} c_k\varphi_k(x), n \in \mathbb{N},$$

holds the following inequality:

$$\|f_n\|_q \leqslant Cn^{1/p-1/q}\|f_n\|_p. \tag{5}$$

A multidimensional version of inequality (5) in the spaces L_p was established by R.J. Nessel and G. Wilmes [25], [26]. The Jackson-Nikol'skii inequality for polynomials in a uniformly bounded system of functions in some symmetric spaces was proved by V.A. Rodin [34]. Moreover, L.R.Ya. Doktorski and D.Gendler [14] proved the inequality of different metrics for polynomials in a uniformly bounded orthonormal system of functions in the Lorentz–Zygmund

space. Jackson–Nikol'skii inequality is also known for polynomials in an unbounded orthonormal system of functions (see, for example, [19], [20], [23], [24]).
In this paper, we complement the results above by proving some new Fourier and Jackson-Nikol'skii type inequalities in the generalized Lorentz space $L_{\psi,q}$ and in unbounded systems satisfying (1).
In Section 2, we present and discuss our main results. The announced generalizations and unifications of Fourier type inequalities can be found in Theorem 2.1 while the corresponding results concerning Jackson-Nikol'skii type inequalities are given in Theorem 2.2. These detailed proofs are presented; in Section 3 and Section 4 is reserved for some concluding remarks and result (see Proposition 4.1).

2. The main results

We denote by SVL (slowly varing) the set of all non-negative functions on $[0,1]$ of $\psi(t)$ for which $(\log 2/t)^{\varepsilon}\psi(t)\uparrow +\infty$ and $(\log 2/t)^{-\varepsilon}\psi(t)\downarrow 0$ for $t\downarrow 0$ (see e.g. [8]).
First, we formulate the following generalization and unification of Theorem 1.1, Theorem 1.2 for the case $\alpha<\frac{1}{2}-\frac{1}{q}$, assertion 1) of Theorem 1.3 and inequality (3):

Theorem 2.1. *Let ψ a function satisfying the conditions $1<\alpha_\psi=\beta_\psi=2^{1/2}$, $\frac{t^{1/2}}{\psi(t)}\in SVL$,*

$$\sup_{t\in(0,1]}\frac{\psi(t)}{t^{1/2}}<\infty,$$

and assume that the orthonormal system $\{\varphi_n\}_{n\in\mathbb{N}}$ for some $r\in(2,+\infty]$ satisfies the condition (1). Then, for any function $f\in L_{\psi,q}$, $2<q\leqslant\infty$, the following inequality holds:

$$\Big[\sum_{k\in A}|\hat{f}(k)|^2\Big]^{\frac{1}{2}}\leqslant C\|f\|_{\psi,q}\Big[\ln(1+\sum_{j\in A}M_j^2)\Big]^{\frac{1}{2}-\frac{1}{q}}\frac{\sqrt{(1+\sum_{j\in A}M_j^2)^{-1}}}{\psi((1+\sum_{j\in A}M_j^2)^{-1})},$$

where A is a non-empty set in $\mathbb{N}$ and C is positive constant which depends only on q and r.

Corollary 2.1. *Let ψ be a function satisfying the conditions of Theorem 2.1 and the orthonormal system $\{\varphi_n\}_{n\in\mathbb{N}}$ for some $r\in(2,+\infty]$ satisfying the condition (2). Then, for any function $f\in L_{\psi,q}, 2<q\leqslant\infty$, we have the inequality*

$$\Big[\sum_{k=1}^{|A|}(\hat{f}^*(k))^2\Big]^{\frac{1}{2}}\leqslant C\|f\|_{\psi,q}\Big[\log(1+|A|M^2)\Big]^{\frac{1}{2}-\frac{1}{q}}\frac{\sqrt{(1+|A|M^2)^{-1}}}{\psi((1+|A|M^2)^{-1})},$$

where $|A|$ is the number of elements in the set $A\subset\mathbb{N}$.

Corollary 2.2. *Let ψ be a function satisfying the conditions of Theorem 2.1 and let the orthonormal system $\{\varphi_n\}_{n\in\mathbb{N}}$ for some $r\in(2,+\infty]$ satisfying the condition (2). Then, for any function $f\in L_{\psi,q}, 2<q\leqslant\infty$, the following inequality holds:*

$$\sup_{n\in\mathbb{N}}n^{-1/2}\Big[\log(1+nM^2)\Big]^{\frac{1}{q}-\frac{1}{2}}\left(\frac{\sqrt{(1+nM^2)^{-1}}}{\psi((1+nM^2)^{-1})}\right)^{-1}\sum_{k=1}^{n}\hat{f}^*(k)\leqslant C\|f\|_{\psi,q}.$$

Remark 2.1. *In the case $\psi(t)=t^{1/2}$ from Corollary 2.1 and Corollary 2.2, we accordingly obtain the statement of Theorem 1.1 and inequality (3).*

Remark 2.2. *In the case $\psi(t)=t^{1/2}(1+|\log t|)^{\alpha}$ and $\{\varphi_n\}$ the trigonometric system from Corollary 2.2, we obtain the statement in Theorem 1.2 for $\alpha<\frac{1}{2}-\frac{1}{q}$.*

Remark 2.3. *If $\psi(t) = t^{1/2}(1 + |\log t|)^{\alpha}$ and the orthonormal system $\{\varphi_n\}_{n\in\mathbb{N}}$ for some $r \in (2, +\infty]$ satisfies condition (2), then from Corollary 2.2, we obtain assertion 1) of Theorem 1.3.*

Remark 2.4. *In the case $\psi(t) = t^{1/2}$ and $A = \{1, ..., n\}$, it was proved in [11] that the inequality in Corollary 2.1 is exact for the multiplicative Crestenson–Levy system . This fact for a trigonometric system in the Lorentz–Zygmund space $L_{2,q,}(\log L)^{\alpha}$ was proved in [30]. By also using Theorem 2 in [5], we obtain the following statement:*

Corollary 2.3. *Let ψ be a function satisfying the conditions of Theorem 2.1, $2 < q < \infty$ and $\{e^{inx}\}_{n\in\mathbb{Z}}$ be the trigonometric system. Then*

$$\sup_{f\neq 0} \frac{\left(\sum_{k=1}^{2n+1}(\hat{f}^*(k))^2\right)^{1/2}}{\|f\|_{\psi,q}} \asymp \frac{\sqrt{(1+n)^{-1}}}{\psi((1+n)^{-1})}\left[\log(1+n)\right]^{\frac{1}{2}-\frac{1}{q}}.$$

Next, we state a Jackson–Nikol'skii type inequality which generalizes some results for the trigonometric system in [17] and [27], [28] (for a complementary bibliography see also [4], [5]).

Theorem 2.2. *Let the function ψ satisfy the conditions $1 < \alpha_{\psi} = \beta_{\psi} = 2^{1/2}$, $\frac{\psi(t)}{t^{1/2}} \in SVL$,*

$$\sup_{t\in(0,1]} \frac{t^{1/2}}{\psi(t)} < \infty, \tag{6}$$

let the orthonormal system $\{\varphi_n\}_{n\in\mathbb{N}}$ for some $r \in (2, +\infty]$ satisfy the condition (1) and $f_n(x) = \sum_{k=1}^{n} c_k\varphi_k(x)$.
1) If $1 < q < 2$, then

$$\|f_n\|_{\psi,q} \leqslant C\left(\frac{\sqrt{\left(1+\sum_{j=1}^{n} M_j^2\right)^{-1}}}{\psi((1+\sum_{j=1}^{n} M_j^2)^{-1})}\right)^{-1}\left(\log\left(1+\sum_{k=1}^{n} M_k^2\right)\right)^{\frac{1}{q}-\frac{1}{2}}\|f_n\|_2$$

for some constant C depending only on q.
2) If $1 < p < 2 < q < +\infty$, then

$$\|f_n\|_{\psi,p} \leqslant C(p,q)\|f_n\|_{\psi,q}\left(\log(1+\sum_{k=1}^{n} M_k^2)\right)^{\frac{1}{p}-\frac{1}{q}}$$

for some constant C depending only on p and q.
3) If $2 < p < q < +\infty$, then

$$\|f_n\|_{\psi,p} \leqslant C(p,q)\|f_n\|_{\psi,q}\left(\log(1+\sum_{k=1}^{n} M_k^2)\right)^{\frac{1}{p}-\frac{1}{q}}$$

for some constant C depending only on p and q.

3. Proofs

Proof of Theorem 2.1. Let $f \in L_{\psi,q}$. This function can be represented as $f(x) = f_1(x) + f_2(x)$, where

$$f_1(x) = \begin{cases} f(x), & \text{when } |f(x)| \leqslant f^*(\tau), \\ 0, & \text{when } |f(x)| > f^*(\tau), \end{cases}$$
$$f_2(x) = f(x) - f_1(x), \quad 0 < \tau < 1.$$

Then, by the Minkowski inequality, we have that

$$\Big[\sum_{k\in A}|\hat{f}(k)|^2\Big]^{1/2}\leqslant\Big[\sum_{k\in A}|\hat{f}_1(k)|^2\Big]^{1/2}+\Big[\sum_{k\in A}|\hat{f}_2(k)|^2\Big]^{1/2}. \tag{7}$$

Now, we prove that each of the functions $f_i, i=1,2$, satisfies the inequality

$$\Big[\sum_{k\in A}|\hat{f}_i(k)|^2\Big]^{1/2}\leqslant C(q,r)\Big(\ln(1+\sum_{k\in A}M_k^2)\Big)^{\frac{1}{2}-\frac{1}{q}}\frac{\sqrt{(1+\sum_{j\in A}M_j^2)^{-\frac{r}{2(r-2)}}}}{\psi((1+\sum_{j\in A}M_j^2)^{-\frac{r}{r-2}})}\|f\|_{\psi,q}. \tag{8}$$

According to the Parseval equality for an orthonormal system and Hölder's inequality for $\theta=\frac{q}{2}>1$, $\frac{1}{\theta}+\frac{1}{\theta'}=1$ for the function f_1, we find that

$$\sum_{k\in A}|\hat{f}_1(k)|^2\leqslant\|f_1\|_2^2\leqslant\int_\tau^1 {f^*}^2(t)dt\leqslant\|f\|_{\psi,q}^2\Big[\int_\tau^1\Big(\frac{t^{1/2}}{\psi(t)}\Big)^{2\theta'}t^{-1}dt\Big]^{\frac{1}{\theta'}}. \tag{9}$$

Since $\frac{t^{1/2}}{\psi(t)}\in SVL$, then $\frac{t^{1/2}}{\psi(t)}\log^\varepsilon 2/t\leqslant\frac{\tau^{1/2}}{\psi(\tau)}\log^\varepsilon 2/\tau$ for $t\in[\tau,1]$, $\forall\varepsilon>0$. Therefore

$$\Big[\int_\tau^1\Big(\frac{t^{1/2}}{\psi(t)}\Big)^{2\theta'}t^{-1}dt\Big]^{\frac{1}{\theta'}}\leqslant\Big(\frac{\tau^{1/2}}{\psi(\tau)}\Big)^2\log^{2\varepsilon}2/\tau\Big[\int_\tau^1(\log 2/t)^{-2\varepsilon\theta'}t^{-1}dt\Big]^{\frac{1}{\theta'}}. \tag{10}$$

Choose the number $\varepsilon\in(0,\frac{1}{2}-\frac{1}{q})$. Then, $1-2\varepsilon\theta'>0$ so that

$$\int_\tau^1(\log 2/t)^{-2\varepsilon\theta'}t^{-1}dt=\frac{1}{1-2\varepsilon\theta'}\Big[(\log 2/t)^{1-2\varepsilon\theta'}-1\Big].$$

Therefore, from inequality (10), it follows that

$$\Big[\int_\tau^1\Big(\frac{t^{1/2}}{\psi(t)}\Big)^{2\theta'}t^{-1}dt\Big]^{\frac{1}{\theta'}}\leqslant\frac{1}{1-2\varepsilon\theta'}\Big(\frac{\tau^{1/2}}{\psi(\tau)}\Big)^2(\log 2/t)^{\frac{1}{\theta'}}. \tag{11}$$

Now by using inequalities (9) and (11), we obtain that

$$\Big(\sum_{k\in A}|\hat{f}_1(k)|^2\Big)^{\frac{1}{2}}\leqslant\frac{1}{1-2\varepsilon\theta'}\frac{\tau^{1/2}}{\psi(\tau)}(\log 2/\tau)^{\frac{1}{2}-\frac{1}{q}}\|f\|_{\psi,q}. \tag{12}$$

In this formula, we put $\tau=(1+\sum_{j\in A}M_j^2)^{-\frac{r}{r-2}}$. Then, for the function f_1 from (12), we can conclude that

$$\Big(\sum_{k\in A}|\hat{f}_1(k)|^2\Big)^{\frac{1}{2}}$$
$$\leqslant C\Big(\ln(1+\sum_{k\in A}M_k^2)\Big)^{\frac{1}{2}-\frac{1}{q}}\frac{\sqrt{(1+\sum_{j\in A}M_j^2)^{-\frac{r}{2(r-2)}}}}{\psi((1+\sum_{j\in A}M_j^2)^{-\frac{r}{r-2}})}\Big(\ln(1+\sum_{k\in A}M_k^2)\Big)^{\frac{1}{2}-\frac{1}{q}}\|f\|_{\psi,q},$$

so (8) holds with $i=1$. For the function $f_2\in L_{r'}$ by the definition of the coefficient expansions and Hölder's inequality ($2<r<+\infty, r'=\frac{r}{r-1}$), we have that

$$|\hat{f}_2(k)|=\Big|\int_0^1 f_2(x)\varphi_k(x)dx\Big|\leqslant\|f_2\|_{r'}\|\varphi_k\|_r\leqslant M_k\|f\|_{r'}.$$

Hence,

$$\sum_{k\in A}|\hat{f}_2(k)|^2\leqslant\|f_2\|_{r'}^2\sum_{k\in A}M_k^2=\Big(\int_0^\tau {f^*}^{r'}(t)dt\Big)^{2/r'}\sum_{k\in A}M_k^2. \tag{13}$$

Since the function f^* is non-increasing and ψ is non-decreasing, then

$$\begin{aligned}\|f\|_{\psi,q} &\geqslant \Big(\int\limits_{x/2}^{x} {f^*}^{q}(t)\psi^{q}(t)\frac{dt}{t}\Big)^{1/q}\\ &\geqslant f^*(x)\psi(x/2)\Big(\int\limits_{x/2}^{x}\frac{dt}{t}\Big)^{1/q} = f^*(x)\psi(x/2)(\ln 2)^{1/q},\ x\in(0,1].\end{aligned}$$

Therefore, from inequality (13), it follows that

$$\sum_{k\in A}|\hat{f}_2(k)|^2 \leqslant \|f\|^2_{\psi,q}\Big(\int\limits_0^{\tau}\psi^{-r'}(t/2)dt\Big)^{2/r'}\sum_{k\in A}M_k^2. \tag{14}$$

Since $\frac{t^{1/2}}{\psi(t)}\in SVL$, then

$$\begin{aligned}\Big(\int\limits_0^{\tau}\psi^{-r'}(t/2)dt\Big)^{2/r'} &= \Big(\int\limits_0^{\tau}\Big(\frac{\sqrt{t/2}}{\psi(t/2)}\Big)^{r'}(t/2)^{-r'/2}dt\Big)^{2/r'}\\ &\leqslant \Big(\frac{\sqrt{\tau/2}}{\psi(\tau/2)}\log^{\varepsilon}\frac{2}{\tau/2}\Big)^2\Big(\int\limits_0^{\tau}(\log\frac{2}{t/2})^{-\varepsilon r'}(t/2)^{-r'/2}dt\Big)^{2/r'}. \qquad (15)\end{aligned}$$

If $0<t<\tau$, then $(\log\frac{2}{t/2})^{-\varepsilon} < (\log\frac{2}{\tau/2})^{-\varepsilon}$, for $\varepsilon>0$. Therefore, by using (15), we obtain that

$$\begin{aligned}\Big(\int\limits_0^{\tau}\psi^{-r'}(t/2)dt\Big)^{2/r'} &\leqslant \Big(\frac{\sqrt{\tau/2}}{\psi(\tau/2)}\Big)^2\Big(\int\limits_0^{\tau}(t/2)^{-r'/2}dt\Big)^{2/r'}\\ &= (\frac{2}{2-r'})^{2/r'}\Big(\frac{\sqrt{\tau/2}}{\psi(\tau/2)}\Big)^2(\tau/2)^{\frac{2}{r'}-1} = (\frac{2}{2-r'})^{2/r'}\Big(\frac{1}{\psi(\tau/2)}\Big)^2 2^{-\frac{2}{r'}}\tau^{\frac{2}{r'}}. (16)\end{aligned}$$

Now, it follows from inequalities (14) and (16) that

$$\Big(\sum_{k\in A}|\hat{f}_2(k)|^2\Big)^{1/2} \leqslant C\|f\|_{\psi,q}\frac{1}{\psi(\tau)}\tau^{\frac{1}{r'}}\Big(\sum_{k\in A}M_k^2\Big)^{1/2}.$$

In this formula, we put $\tau = (1+\sum_{j\in A}M_j^2)^{-\frac{r}{r-2}}$. Then

$$\begin{aligned}\Big(\sum_{k=1}^{n}|\hat{f}_2(k)|^2\Big)^{1‘/2} &\leqslant C\|f\|_{\psi,q}\frac{1}{\psi((1+\sum_{j=1}^{n}M_j^2)^{-\frac{r}{r-2}})}\Big(1+\sum_{j=1}^{n}M_j^2\Big)^{-\frac{r}{r'(r-2)}}\Big(\sum_{k=1}^{n}M_k^2\Big)^{1/2}\\ &= C\frac{1}{\psi((1+\sum_{j=1}^{n}M_j^2)^{-\frac{r}{r-2}})}\Big(1+\sum_{j=1}^{n}M_j^2\Big)^{-\frac{r}{2(r-2)}}\|f\|_{\psi,q}.\end{aligned}$$

Now, taking into account that $1/2-1/q>0$, we get from here that

$$\begin{aligned}&\Big(\sum_{k\in A}|\hat{f}_2(k)|^2\Big)^{1‘/2}\\ \leqslant & C\frac{1}{\psi((1+\sum_{j\in A}M_j^2)^{-\frac{r}{r-2}})}\Big(1+\sum_{j\in A}M_j^2\Big)^{-\frac{r}{2(r-2)}}\Big(\log\Big(1+\sum_{j\in A}M_j^2\Big)\Big)^{1/2-1/q}\|f\|_{\psi,q},\end{aligned}$$

so (8) holds also for $i = 2$. From inequalities (7) and (8), it follows that

$$\Big(\sum_{k\in A}|\hat{f}(k)|^2\Big)^{1`/2}$$
$$\leqslant C\frac{1}{\psi((1+\sum_{j\in A}M_j^2)^{-\frac{r}{r-2}})}\Big(1+\sum_{j\in A}M_j^2\Big)^{-\frac{r}{2(r-2)}}\Big(\log\Big(1+\sum_{j\in A}M_j^2\Big)\Big)^{1/2-1/q}\|f\|_{\psi,q}. \tag{17}$$

Since $\frac{t^{1/2}}{\psi(t)} \in SVL$ and $\Big(1+\sum_{j\in A}M_j^2\Big)^{-\frac{r}{2(r-2)}} < \Big(1+\sum_{j\in A}M_j^2\Big)^{-1}$, then

$$\frac{\sqrt{\Big(1+\sum_{j\in A}M_j^2\Big)^{-\frac{r}{(r-2)}}}}{\psi((1+\sum_{j\in A}M_j^2)^{-\frac{r}{r-2}})}$$
$$\leqslant\frac{\sqrt{\Big(1+\sum_{j\in A}M_j^2\Big)^{-1}}}{\psi((1+\sum_{j\in A}M_j^2)^{-1})}\Big(\log\frac{2}{\Big(1+\sum_{j\in A}M_j^2\Big)^{-1}}\Big)^{-\varepsilon}\Big(\log\frac{2}{\Big(1+\sum_{j\in A}M_j^2\Big)^{-\frac{r}{(r-2)}}}\Big)^{\varepsilon}$$
$$\leqslant\frac{\sqrt{\Big(1+\sum_{j\in A}M_j^2\Big)^{-1}}}{\psi((1+\sum_{j\in A}M_j^2)^{-1})}\Big(\log 2\Big(1+\sum_{j\in A}M_j^2\Big)\Big)^{-\varepsilon}\Big(\frac{r}{r-2}\log 2\Big(1+\sum_{j\in A}M_j^2\Big)\Big)^{\varepsilon}$$
$$=\frac{r}{r-2}\frac{\sqrt{\Big(1+\sum_{j\in A}M_j^2\Big)^{-1}}}{\psi((1+\sum_{j\in A}M_j^2)^{-1})}. \tag{18}$$

It follows from inequalities (17) and (18) that

$$\Big(\sum_{k\in A}|\hat{f}(k)|^2\Big)^{1`/2}\leqslant\frac{r}{r-2}\frac{\sqrt{\Big(1+\sum_{j\in A}M_j^2\Big)^{-1}}}{\psi((1+\sum_{j\in A}M_j^2)^{-1})}\Big(\log\Big(1+\sum_{j\in A}M_j^2\Big)\Big)^{1/2-1/q}\|f\|_{\psi,q}.$$

The proof is complete. □

Proof of Corollary 2.1. In view of the fact that $M_j = M$, $j = 1, 2, \ldots$ and the property of non-increasing rearrangement of numbers, it yields that

$$\sum_{k\in A}|\hat{f}(k)|^2=\sum_{k=1}^{|A|}(\hat{f}^*(k))^2,$$

so the proof follows by just applying Theorem 2.1. □

Proof of Corollary 2.2. According to Hölder's inequality, we have that

$$\sum_{k=1}^{n}\hat{f}^*(k)\leqslant n^{1/2}\Big(\sum_{k=1}^{n}(\hat{f}^*(k))^2\Big)^{1/2}.$$

Therefore, the assertion of Corollary 2.2 follows by applying Corollary 2.1 with $A = \{1, 2, \ldots, n\}$. □

Proof of Corollary 2.3. For the set $A = \{-n, ..., -1, 0, 1, ..., n\}$ from Corollary 2.1, we get

$$\sup_{f \neq 0} \frac{\left(\sum_{k=1}^{2n+1} (\hat{f}^*(k))^2\right)^{1/2}}{\|f\|_{\psi,q}} \leqslant C \frac{\sqrt{(1+n)^{-1}}}{\psi((1+n)^{-1})} \Big[\log(1+n)\Big]^{\frac{1}{2}-\frac{1}{q}}.$$

To prove the reversed inequality, we consider the trigonometric polynomial

$$f_n(x) = \sum_{k=-n}^{n} a_k e^{ikx}.$$

Then, by using Theorem 2 in [5] for $\psi_1(t) = t^{1/2}, \tau_1 = 2$, $\psi_2(t) = \psi(t), \tau_2 = q$, we have that

$$\sup_{f_n \neq 0} \frac{\|f_n\|_2}{\|f_n\|_{\psi,q}} \geqslant C \frac{\sqrt{(1+n)^{-1}}}{\psi((1+n)^{-1})} \Big[\log(1+n)\Big]^{\frac{1}{2}-\frac{1}{q}}.$$

Therefore

$$\sup_{f \neq 0} \frac{\left(\sum_{k=1}^{2n+1} (\hat{f}^*(k))^2\right)^{1/2}}{\|f\|_{\psi,q}} \geqslant \sup_{f_n \neq 0} \frac{\|f_n\|_2}{\|f_n\|_{\psi,q}} \geqslant C \frac{\sqrt{(1+n)^{-1}}}{\psi((1+n)^{-1})} \Big[\log(1+n)\Big]^{\frac{1}{2}-\frac{1}{q}}.$$

The proof is complete. □

Proof of Theorem 2.2. For the generalized Lorentz space $L_{\psi,q}$, we have the relation (see [2])

$$\|f\|_{\psi,q} \asymp \sup_{\|f\|_{\bar{\psi},q'} \leqslant 1} \left| \int_0^1 f(x)g(x)dx \right|, \tag{19}$$

where $\bar{\psi}(t) = \frac{t}{\psi(t)}$, $t \in (0,1]$, $1 < q < \infty$, $q' = \frac{q}{q-1}$. Since the system $\{\varphi_n\}$ is orthonormal, then

$$\int_0^1 f_n(x)g(x)dx = \sum_{k=1}^{n} c_k \hat{g}(k), \;\; g \in L_{\bar{\psi},q'}$$

for any $n \in \mathbb{N}$.
Note that condition (6) implies that

$$\sup_{t \in (0,1]} \frac{\bar{\psi}(t)}{t^{1/2}} < \infty.$$

By applying Hölder's inequality, Theorem 2.1, and Parseval's equality, we obtain that

$$\begin{aligned} \left| \int_0^1 f_n(x)g(x)dx \right| &\leqslant \left(\sum_{k=1}^{n} |c_k|^2\right)^{1/2} \left(\sum_{k=1}^{n} |\hat{g}(k)|^2\right)^{1/2} \\ &\leqslant C \frac{\sqrt{\left(1+\sum_{j=1}^n M_j^2\right)^{-1}}}{\bar{\psi}((1+\sum_{j=1}^n M_j^2)^{-1})} \left(\log\left(1+\sum_{j=1}^n M_j^2\right)\right)^{1/2-1/q'} \|g\|_{\bar{\psi},q'} \|f_n\|_2. \end{aligned}$$

Therefore, in virtue of relation (19), we have that

$$\|f_n\|_{\psi,q} \leqslant C \frac{\psi((1+\sum_{j=1}^n M_j^2)^{-1})}{\sqrt{\left(1+\sum_{j=1}^n M_j^2\right)^{-1}}} \left(\log\left(1+\sum_{j=1}^n M_j^2\right)\right)^{1/q-1/2} \|f_n\|_2$$

and 1) is proved.

We will now prove the second statement. Since $1 < p < 2$, according to item 1), it yields that

$$\|f_n\|_{\psi,p} \leqslant C\left(\frac{\sqrt{\left(1+\sum_{j=1}^{n} M_j^2\right)^{-1}}}{\psi((1+\sum_{j=1}^{n} M_j^2)^{-1})}\right)^{-1}\left(\log\left(1+\sum_{k=1}^{n} M_k^2\right)\right)^{\frac{1}{p}-\frac{1}{2}}\|f_n\|_2. \tag{20}$$

Moreover, since $2 < q < \infty$, by Theorem 2.1 and Parseval's equality, we find that

$$\|f_n\|_2 \leqslant \frac{\sqrt{\left(1+\sum_{j=1}^{n} M_j^2\right)^{-1}}}{\psi((1+\sum_{j=1}^{n} M_j^2)^{-1})}\left(\log\left(1+\sum_{j=1}^{n} M_j^2\right)\right)^{1/2-1/q}\|f\|_{\psi,q}. \tag{21}$$

Now from inequalities (20) and (21), it follows that

$$\|f_n\|_{\psi,p} \leqslant C\left(\log\left(1+\sum_{j=1}^{n} M_j^2\right)\right)^{1/p-1/q}\|f\|_{\psi,q}$$

and 2) is proved.
Finally, let $2 < p < q < +\infty$. In the generalized Lorentz space $L_{\psi,q}$, the following inequality hold (see [36], p. 491):

$$\|g\|_{\psi,p} \leqslant \|g\|_{\psi,q}^{\frac{\frac{1}{\tau}-\frac{1}{p}}{\frac{1}{\tau}-\frac{1}{q}}}\|g\|_{\psi,\tau}^{\frac{\frac{1}{p}-\frac{1}{q}}{\frac{1}{\tau}-\frac{1}{q}}} \tag{22}$$

for $1 < \tau < p < q < +\infty$. Choose the number $\tau \in (1, 2)$. Then, according to the second statement, we have that

$$\|f_n\|_{\psi,\tau} \leqslant C\left(\log\left(1+\sum_{j=1}^{n} M_j^2\right)\right)^{1/\tau-1/q}\|f\|_{\psi,q}. \tag{23}$$

Now by in equality (22) setting $g = f_n$ and taking into account (23), we obtain that

$$\|f_n\|_{\psi,p} \leqslant \|f_n\|_{\psi,q}^{\frac{\frac{1}{\tau}-\frac{1}{p}}{\frac{1}{\tau}-\frac{1}{q}}}\left\{C\left(\log\left(1+\sum_{j=1}^{n} M_j^2\right)\right)^{1/\tau-1/q}\|f\|_{\psi,q}\right\}^{\frac{\frac{1}{p}-\frac{1}{q}}{\frac{1}{\tau}-\frac{1}{q}}}$$
$$= C\left(\log\left(1+\sum_{j=1}^{n} M_j^2\right)\right)^{1/p-1/q}\|f\|_{\psi,q}$$

and also 3) is proved. The proof is complete. □

4. Concluding remarks result

Remark 4.5. *In the case $\psi(t) = t^{1/p}(1+|\log t|)^{\alpha}$, $1 < p < \infty$, Theorem 2.2 was previously proved in* [3]. *For the case $\alpha = 0$ see also* [2].

Remark 4.6. *In the case $\psi(t) = t^{1/p}(1+|\log t|)^{\alpha}$, $0 < p < 2$, Theorem 2.2 for polynomials in a uniformly bounded system was proved in* [14], *Theorem 3 i).*

Remark 4.7. *A similar statement as that in Theorem 2.1 was recently proved and discussed in* [8].

Remark 4.8. *It is well-known that each concave function $\psi = \psi(t)$ has the quasi-monotonicity properties that $\frac{\psi(t)}{t}$ is non-increasing and $\psi(t)$ is non-decreasing. Moreover, the definition of the SVL clam means that the functions satisfy two quasi-monotonicity conditions but now on a logarithmic scale.*

These facts opens the possibility that some of the results in this paper can be further generalized in this direction.

From Theorem 2.1 and Theorem 2.2, we can also derive the following generalization of a result in [5]:

Proposition 4.1. *Let the functions ψ_1 and ψ_2 satisfy the conditions $1 < \alpha_{\psi_1} = \beta_{\psi_2} = 2^{1/2}$, $\frac{t^{1/2}}{\psi_1(t)} \in SVL$, $\frac{t^{1/2}}{\psi_2(t)} \in SVL$,*

$$\sup_{t\in(0,1]} \frac{\psi_2(t)}{\psi_1(t)} < \infty \tag{24}$$

and assume that the orthonormal system $\{\varphi_n\}_{n\in\mathbb{N}}$ for some $r \in (2, +\infty]$ satisfies condition (1). If $1 < p \leqslant 2 < q < \infty$, then for any polynomial

$$f_n(x) = \sum_{k=1}^{n} c_k \varphi_k(x),$$

the following inequality holds:

$$\|f_n\|_{\psi_1,p} \leqslant C \frac{\psi_1((1+\sum_{j=1}^n M_j^2))^{-1}}{\psi_2((1+\sum_{j=1}^n M_j^2)^{-1})} \left(\log\left(1+\sum_{k=1}^n M_k^2\right)\right)^{\frac{1}{p}-\frac{1}{q}} \|f_n\|_{\psi_2,q}.$$

Proof. Since $\frac{t^{1/2}}{\psi_1(t)} \in SVL$ and $1 < p \leqslant 2$, according to the first statement of Theorem 2.2, the following inequality holds:

$$\|f_n\|_{\psi_1,p} \leqslant C \left(\frac{\sqrt{\left(1+\sum_{j=1}^n M_j^2\right)^{-1}}}{\psi_1((1+\sum_{j=1}^n M_j^2)^{-1})} \right)^{-1} \left(\log\left(1+\sum_{k=1}^n M_k^2\right)\right)^{\frac{1}{p}-\frac{1}{2}} \|f_n\|_2.$$

Taking into account that $\frac{t^{1/2}}{\psi_2(t)} \in SVL$ and $2 < q < \infty$ by Theorem 2.1, we have that

$$\|f_n\|_2 \leqslant C \left(\frac{\sqrt{\left(1+\sum_{j=1}^n M_j^2\right)^{-1}}}{\psi_2((1+\sum_{j=1}^n M_j^2)^{-1})} \right) \left(\log\left(1+\sum_{k=1}^n M_k^2\right)\right)^{\frac{1}{2}-\frac{1}{q}} \|f_n\|_{\psi_2,q}.$$

From these inequalities, it follows that

$$\begin{aligned}
\|f_n\|_{\psi_1,p} &\leqslant C \left(\frac{\sqrt{\left(1+\sum_{j=1}^n M_j^2\right)^{-1}}}{\psi_1((1+\sum_{j=1}^n M_j^2)^{-1})} \right)^{-1} \left(\log\left(1+\sum_{k=1}^n M_k^2\right)\right)^{\frac{1}{p}-\frac{1}{2}} \\
&\quad \times \frac{\sqrt{\left(1+\sum_{j=1}^n M_j^2\right)^{-1}}}{\psi_2((1+\sum_{j=1}^n M_j^2)^{-1})} \left(\log\left(1+\sum_{k=1}^n M_k^2\right)\right)^{\frac{1}{2}-\frac{1}{q}} \|f_n\|_{\psi_2,q} \\
&= \frac{\psi_1((1+\sum_{j=1}^n M_j^2)^{-1})}{\psi_2((1+\sum_{j=1}^n M_j^2)^{-1})} \left(\log\left(1+\sum_{k=1}^n M_k^2\right)\right)^{\frac{1}{p}-\frac{1}{q}} \|f_n\|_{\psi_2,q}
\end{aligned}$$

for $1 < p \leqslant 2 < q < \infty$. The proof is complete. □

Remark 4.9. *To investigate a statement as that Proposition 4.1 in the case of $1 < p < q \leqslant 2$ is an interesting open question. This case for polynomials in the trigonometric system was investigated in [5]. Furthermore, it seems to be possible to consider Proposition 4.1 also in the more general case $1 \leqslant \beta_{\psi_2} < \alpha_{\psi_1} \leqslant 2$.*

Remark 4.10. *In [4], it was proved that condition (24) implies that $L_{\psi_1,p} \subset L_{\psi_2,q}$, $1 < p < q < \infty$, in the case $\psi_1 = \psi_2$ see [36].*

Remark 4.11 (Final Remark)**.** *Most results concerning Fourier and Jackson–Nikol'skii type inequalities are derived for the case with bounded orthonormal systems. But since there are many important unbounded orthonormal systems, it is of importance to develop the theory to cover such cases too. Examples of such unbounded systems are the following:*
(a) $\{\chi_n\}$–orthonormal system of Haar functions (see e.g. [9]). The functions $\chi_n(t)$ are defined as follows: $\chi_1(t) := 1$ for $t \in [0,1]$ and for $n = 2^m + k$, $k = 1, \dots, m$ and $m = 0, 1, \dots$ put

$$\chi_n(t) = \begin{cases} \sqrt{2^m}, & t \in (\frac{2k-2}{2^{m+1}}, \frac{2k-1}{2^{m+1}}), \\ -\sqrt{2^m}, & t \in (\frac{2k-1}{2^{m+1}}, \frac{2k}{2^{m+1}}), \\ 0, & t \bar{\in} \left[\frac{r}{m_k}, \frac{r+1}{m_k}\right]. \end{cases}$$

The value of $\chi_n(t)$ in a discontinuity point t is defined as

$$\chi_n(t) = \frac{1}{2} \lim_{\varepsilon \to 0} [\chi_n(t+\varepsilon) + \chi_n(t-\varepsilon)].$$

(b) Let there be given an infinite sequence of integers $\{p_n\}$ such that $p_n \geqslant 2 \quad (n = 1, 2, \ldots)$. We put $m_n = p_1 \ldots p_n$, $n \geqslant 1$. Then for any point $t \in [0,1] \setminus A$, there exists the unique expansion

$$t = \sum_{k=1}^{\infty} \frac{\alpha_k(t)}{m_k}, \quad \alpha_k(t) = 0, 1, \ldots, p_k - 1,$$

where $A = \{\frac{l}{m_k}\}$, $l = 0, 1, \dots, m_k$. The generalized Haar system $\chi\{p_k\} := \{\chi_n(t)\}$ on $[0,1]$ is defined as follows (see [15]):
$\chi_1(t) = 1$ for $t \in [0,1]$ and if $n \geqslant 2$, then $n = m_k + r(p_{k+1} - 1) + s$, where $m_0 = 1$ and $m_k = p_1 p_2 \ldots p_k$; $k = 1, \ldots$; $r = 0, 1, \ldots, m_k - 1$; $s = 1, 2, \ldots, p_{k+1} - 1$.
We put

$$\chi_n(t) := \chi_{k,r}^{(s)}(t) := \begin{cases} \sqrt{m_k} exp^{\frac{2\pi i s \alpha_{k+1}(t)}{p_{k+1}}} & , t \in \left(\frac{r}{m_k}, \frac{r+1}{m_k}\right) \cap B, \\ 0 & , t \bar{\in} \left[\frac{r}{m_k}, \frac{r+1}{m_k}\right], \end{cases}$$

where $B := [0,1] \setminus A$. At the remaining points of the interval $(0,1)$, $\chi_n(t)$ is equal to the half-sum of its right-hand and left-hand limits on the set $[0,1] \setminus A$, and at the endpoints of $[0,1]$, to the limits from within the interval.
(c) Other generalizations of the Haar system were defined by A.M. Olevskii [31] and A. Kamont [18]. Jackson–Nikol'skii inequalities for polynomials in the $\chi\{p_n\}$ system in the Lebesgue spaces L_p and Lorentz spaces $L_{p,\tau}$ were proved in [1], [19], [39] and [41].

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GABDOLLA AKISHEV
LOMONOSOV MOSCOW STATE UNIVERSITY
DEPARTMENT OF MATHEMATICS AND INFORMATICS
KAZAKHSTAN BRANCH OF MSU,
NUR–SULTAN, REPUBLIC KAZAKHSTAN
INSTITUTE OF NATURAL SCIENCES AND MATHEMATICS,
URAL FEDERAL UNIVERSITY,
YEKATERINBURG, RUSSIA
ORCID: 0000-0002-8340-9771
E-mail address: akishev_g@mail.ru

LARS ERIK PERSSON
UIT THE ARCTIC UNIVERSITY OF NORWAY
DEPARTMENT OF COMPUTER SCIENCE AND COMPUTATIONAL ENGINEERING
NARVIK, NORWAY
KARLSTAD UNIVERSITY
DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
SWEDEN
ORCID: 0000-0001-9140-6724
E-mail address: larserik6.pers@gmail.com

HARPAL SINGH
UIT THE ARCTIC UNIVERSITY OF NORWAY
DEPARTMENT OF COMPUTER SCIENCE AND COMPUTATIONAL ENGINEERING
NARVIK, NORWAY
ORCID: 0000-0002-8340-9772
E-mail address: harpal.singh@uit.no

Paper E

A note on contributions concerning non-separable spaces with respect to signal processing within Bayesian frameworks

Natasha Samko, Harpal Singh

Submitted.

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A note on contributions concerning non-separable spaces with respect to signal processing within Bayesian frameworks

N. Samko and H. Singh
UiT The Arctic University of Norway, Norway
email: *Natasha.G.Samko@uit.no* and *Harpal.Singh.@uit.no*

Abstract

In this paper we discuss the study of some signal processing problems within Bayesian frameworks and semigroups theory, in the case where the Banach space under consideration may be non-separable. For applications the suggested approach may be of interest in situations where approximation in the norm of the space is not possible. We describe the idea for the case of the abstract Cauchy problem for the evolution equation and provide more detailed example of the diffusion equation with the initial data in the non-separable Morrey space.

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1 Introduction

Bayesian methods have proven to be the most rigorous probabilistic framework to identify target variables and evaluate the corresponding uncertainties using the available information. As known, the Bayesian interface is a probabilistic method of interference that allows to form probabilistic estimates of certain parameters from a given series of observations. We refer, for instanse, to the survey paper [9] for the Bayesian approach to signal processing problems in which the signal is a solution of a stochastic partial differential equation (SPDE).

In the survey [9], a focus there was to show that the idea to work with probability measures on a suitable function space is a key idea, which leads to the notion of a well-posed signal-processing problem. This approach helps to avoid dealing with the algorithms to which the use of known methods based on application of the standard Markov chain Monte Carlo (MCMC) method after a discretization leads. One of the disadvantages of such algorithms is that they perform poorly under refinement of the discretization.

The use of a proper mathematical formulation of the problems on domain space while working with probability measures on function spaces, leads to efficient sampling techniques, defined on path-space as the domain space, and therefore robust under the introduction of discretization. A wide variety of signal processing problems is overviewed in [9]. These problems lead to a posterior probability measure on a separable Banach space. The separability of a Banach space provides a possibility for approximation in the space. However, when looking at the problems tackled in Bayesian frameworks, it is also of interest to explore what potential "gains" and/or "losses" might be in the case where the space is not necessarily separable. Such kind of questions sometimes are discussed also at some on-line forums. In this note we touch some questions concerning such issues.

As known, the theory of Markov stochastic processes is a natural source of interesting parabolic equations. When passing to a macro description of such processes with respect to the densities of transition probabilities, under special conditions there appear partial differential equations. It is worth mentioning that A.P. Kolmogoroff yet in 1934 (see [12]), proceeding from the problems of the theory of probability, defined and studied an interesting equation, which he called the equation of diffusion with inertia. The properties of solutions of such equations and the methods of their study, for natural reasons, are very closely related to the properties of solutions of the heat equation and diffusion equations. Parabolic equations of diffusion-type and their various generalizations are known to be widely studied and may be found in various books, in particular in the Hörmander books [10].

Study of such equations under special conditions and in various function spaces could make possible a more precise observing the results and the possibilities of their further applications.

The theory of parabolic equations has deep connections with functional analysis, especially with the theory of evolution equations with unbounded operators in Banach spaces and the theory of semigroups.

The theory of stochastic processes, especially the theory of Markov processes and stochastic differential equations, very closely interacts with the theory of parabolic equations.

In this paper, we suggest an approach for the study of properties of solutions to such equations in the case of non-separable function spaces. In case of problems formulated as filtering problems in addition to smoothing problems, an approach based on study in the frameworks of weighted non-standard function spaces could also be useful.

One of the main approaches to evolution equations relates to the theory of semigroups of operators. This approach is under discussion in our paper with respect to the Cauchy problem for an abstract evolution equation.

2 Solution of Cauchy problem via semigroups

One of the approaches to the study of evolution equations is based on the use of semigroups of operators. This approach has a long history for about 80 years. Already in the paper [18] of 1954, the reader can find a comparison of different settings of the Cauchy problem for an abstract evolution equation, for which the operator semigroups approach is applicable. The presentation of this approach may be found in a big variety of books and surveys. We refer, for instance, to [1], [2], [6], [7], [13], [15], [20], [21], [22] and [28] on semigroups of operators and their applications to partial differential equations.

Below we provide necessary standard definitions concerning operator semigroups.

A family of continuous linear operators $T_t, t > 0$, in a Banach space E is called a semigroup of operators if

$$T_t T_s = T_{t+s},\ t > 0, s > 0 \ \text{ and } \ T_0 = I.$$

A semigroup of operators is called strongly continuous if

$$T_t f \to T_s f \text{ in } E \text{ as } t \to s,\ s > 0, \quad \text{for all } f \in E.$$

A semigroup T_t is said to be of class C_0 if

$$\lim_{t\to 0} \|T_t f - f\|_E = 0.$$

The operator

$$Af := \lim_{t\to 0} \frac{1}{t} \left(T_t f - f\right),$$

defined for $f \in E$ whenever this limit exists, is called the infinitesimal operator of the semigroup T_t. If the operator A admits an extension to a closed operator $\bar{A}$, then the operator $\bar{A}$ is referred to as the generator of the semigroup T_t.

More information on semigroups of operators may be found e.g. in the books [2], [16], [17] and [28].

To describe the principal idea of the operator semigroup approach, we consider the standard Cauchy problem

$$\begin{cases} u_t = Au, \\ u|_{t=0} = u_0 \end{cases}, \tag{2.1}$$

where u is the unknown function and A is a linear operator. It may be a function $u(x,t)$, where $x \in \Omega \subseteq \mathbb{R}^n, t > 0$ and $A = \Delta = \frac{\partial^2}{\partial x_1^2} + \cdots \frac{\partial^2}{\partial x_n^2}$ being the classical Laplace operator corresponding to the heat or diffusion processes. More generally, it may be in abstract form, i.e. u is an element of a Banach space E, depending on parameter $t > 0$, with the derivative $u_t = \frac{\partial u}{\partial t}$ interpreted in a proper sense and A is a linear operator with the domain $D(A) \subset E$ (now we present a formal procedure, precise assumptions and formulations being considered later).

Formally solving the equation $\frac{\partial u}{\partial t} = Au$ as an ordinary differential equation and taking the initial condition into account, we get

$$u = e^{tA}u_0,$$

assuming that e^{tA} is well defined. Formally again $\{e^{tA}\}_{t>0}$ is a semigroup. Thus, we say that the semigroup $T_t = e^{tA}$ solves the Cauchy problem (2.1). Certainly, this formal procedure needs justification. Such a justification needs certain assumptions on the space E and the operator A and is well known in the literature. These assumptions usually include the notion of non-positivity or resolvent set of the operator A and the condition

$$D(A) = E. \tag{2.2}$$

For our goals in the sequel, we underline that in applications the condition (2.2) is in fact the assumption that the space E is separable. In this paper we discuss a possibility to adjust the above operator semigroup procedure for non-necessarily separable spaces.

The similar procedure is also known to be applied to non-linear partial differential equations, for instance, in the case when the operator A in (2.1) is non-linear. However, then the construction of the corresponding semigroup T_t is more complicated.

Concretely for the abstract Cauchy problem (2.1) the semigroup approach under various assumptions is dispersed in the literature and goes back to the paper [18].

The proof of the following theorem was given in [18, Theorem 3.1], see also [21].

Theorem 2.1. *Let E be a Banach space. Consider a closed linear operator A with the domain $D(A)$ dense in E and non-empty resolvent set. Suppose that the Cauchy problem* (2.1) *is uniquely solvable for every $u_0 \in D(A)$. Then there exists a semigroup T_t of the class C_0 which solves the Cauchy problem* (2.1).

Remark 2.2. There are known sufficient conditions for the unique solvability of the Cauchy problem supposed in Theorem 2.1 given in terms of the generator of the semigroup T_t, see e.g. [18, Theorem 3.3].

3 What can be saved if the Banach space is non-separable?

Note that the assumption that the domain of the operator is dense in the considered Banach space in general reassumes non-separable spaces. In this section we provide some arguments which allow us to include non-separable spaces into the operator semigroup approach. We explain these arguments for the model case of the Cauchy problem (2.1). Let X be a non-separable space. Consider any Banach space E presumably separable, such that $X \hookrightarrow E$ and the operator A obeys Theorem 2.1 in the space E. By $D_X(A)$ and $D_E(A)$ we denote the domains of the operator A in the spaces X and E, respectively. Then clearly for $f \in D_X(A) \subseteq D_E(A)$ by Theorem 2.1 we have the solution $u(t) = T_t u_0 \in E$ for all $t > 0$. However, our interest is to know that $u(t) \in X,\ t > 0$ and $u(t) \to u_0$ in X as $t \to 0$. The requirement that $u(t) \in X,\ t > 0$, is easily covered by the assumption that the operators T_t are bounded in X for $t > 0$. However, we cannot assume that T_t is of class C_0 in X, since the latter in general does not hold in non-separable spaces. Anyway, we have a weaker convergence $\|u(t) - u_0\|_E \to 0$ as $t \to 0$. Thus, dealing with a smaller space $X \hookrightarrow E$, from the known solvability in the space E we can gain the information that $u(t) \in X,\ t > 0$, but have to keep a weaker E-convergence of $u(t)$ to u_0. We summarize this in the form of the following theorem:

Theorem 3.1. *Let the Banach space E and the operator A satisfy all the assumptions of Theorem 2.1, and let X be an arbitrary Banach space, non-necessarily separable, such that $X \hookrightarrow E$. Then $u(t) \in X,\ t > 0$ for all $u_0 \in D_X(A)$, if the operator $T_t,\ t > 0$ is bounded in X. Suppose that the Cauchy problem* (2.1) *is uniquely solvable for every $u_0 \in D(A)$. Then there exists a semigroup T_t of the class C_0 which solves the Cauchy problem* (2.1)*.*

In the above scheme X could be an arbitrary space. There is known a variety of non-separable function spaces. Apart of the well known space L^∞ of essentially bounded functions in Analysis and PDEs there are also known such non-separable spaces as Hölder spaces, Morrey spaces and recently developed grand Lebesgue spaces. Observe that grand Lebesgue spaces proved to be very important in some applications to PDEs, see for instance the papers [4] and [26]. In the example below we take as X the Morrey space $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$. Note that Morrey spces are very popular both in analysis and applications to PDEs, see for instance the books [8] and [25].

Let us consider an example of the diffusion equation

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = k(\Delta u)(x,t),\ x \in \mathbb{R}^n \\ u(x,0) = u_0(x),\ x \in \mathbb{R}^n, \end{cases} \tag{3.1}$$

where $k > 0$. We take $E = L^q_\beta(\mathbb{R}^n),\ 1 < q < \infty,\ \beta \in \mathbb{R}$, where $L^q_\beta(\mathbb{R}^n)$ is the weighted Lebesgue space defined by the norm

$$\|f\|_{L^q_\beta(\mathbb{R}^n)} = \left[\int\limits_{\mathbb{R}^n} \left| \frac{f(x)}{(1+|x|)^\beta} \right|^q dx \right]^{\frac{1}{q}},$$

and choose $X = \mathcal{L}^{p,\lambda}(\mathbb{R}^n),\ 0 < p < \infty,\ 0 < \lambda < n$, where $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ is the Morrey space defined by the norm

$$\|f\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n} \sup_{r>0} \left(\frac{1}{r^\lambda} \int\limits_{B(x,r)} |f(y)|^p \, dy \right)^{\frac{1}{p}}.$$

It is a non-separable space. For more details on Morrey spaces we refer to the book [14]. The subspace of functions $f \in \mathcal{L}^{p,\lambda}(\mathbb{R}^n)$, such that

$$\lim_{h\to 0} \|f(\cdot + h) - f(\cdot)\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)} = 0$$

is referred to as the Zorko space, see [29]. The Zorko space is a proper subspace of the Morrey space and it is known that this is the maximal subspace of the Morrey space where the heat semigroup is of the class C_0, see [11, Lemma 3.1].

The embedding $X \hookrightarrow E$ in this case holds under the conditions

$$q < p \text{ and } \beta > \frac{\lambda}{p} + n\left(\frac{1}{q} - \frac{1}{p}\right), \tag{3.2}$$

as derived from [24, Corollary 3.14].

As is well known, the solution of the problem (3.1) is given by the semigroup

$$u(x,t) = T_t u_0(x),\ T_t u_0(x) := \int\limits_{\mathbb{R}^n} \mathcal{K}_t(x-y)u_0(y)dy, \tag{3.3}$$

where

$$\mathcal{K}_t(x) = \frac{1}{(4\pi kt)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4kt}}.$$

The fact that this semigroup is of class C_0 in the space $L^q(R^n)$ is well known. In the lemma below we justify that the same holds for the weighted Lebesgue space $L^q_\beta(\mathbb{R}^n)$.

Lemma 3.2. *Let $q > 1$, and $\frac{1}{q} + \frac{1}{q'} = 1$. Then the semigroup* (3.3) *is of class C_0 in the space $L^q_\beta(\mathbb{R}^n)$ if $-\frac{n}{q'} < \beta < \frac{n}{q}$.*

Proof. The kernel $\mathcal{K}_t(x)$ of the convolution operator (3.3) has the form

$$\mathcal{K}_t(x) = \frac{1}{(\sqrt{t})^n}\mathcal{K}_1(x)\left(\frac{x}{\sqrt{t}}\right).$$

Convolution operators with such a delation kernel are uniformly in t dominated by the maximal operator, provided that $\mathcal{K}_1(x)$ is radial, integrable on $\mathbb{R}^n$ and as a radial function is increasing on $\mathbb{R}_+$, see [27]. Hence

$$|T_t u_0(x)| \le cMu_0(x),\ Mf(x) = \sup_{r>0} \frac{1}{r^n} \int\limits_{|y-x|<r} |u_0(y)|dy. \tag{3.4}$$

The maximal operator M is bounded in the weighted space $L^q_\beta(\mathbb{R}^n)$ if $-\beta$ lies in the "Muckenhoupt interval" $\left(-\frac{n}{q}, \frac{n}{q'}\right)$, see [5]. Thus, the operators $T_t,\ t \in \mathbb{R}_+$ are even uniformly in t bounded in $L^q_\beta(\mathbb{R}^n)$.

It remains to verify that

$$\lim_{t\to 0} \|T_t u_0 - u_0\|_{L^q_\beta(\mathbb{R}^n)} = 0.$$

This can be checked by the standard procedure via approximation of $u_0 \in L^q_\beta(\mathbb{R}^n)$ by C_0^∞-functions in $L^q_\beta(\mathbb{R}^n)$ and using the uniform boundedness of T_t. The proof is complete. □

Finally, we present the following statement for the non-separable Morrey space $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$:

Theorem 3.3. *Let $u_0 \in \mathcal{L}^{p,\lambda}(\mathbb{R}^n), 1 < p < \infty,\ 0 < \lambda < n$. Then the unique solution $u(x,t)$ of the problem* (3.1) *has the property $u(\cdot,t) \in \mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ uniformly in $t \in \mathbb{R}_+$. The convergence*

$$\|T_t u_0 - u_0\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)} \to 0 \text{ as } t \to 0 \tag{3.5}$$

holds if u_0 is in the Zorko subspace $Z^{p,\lambda}(\mathbb{R}^n)$ of $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$.

Proof. We embed the space $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ into the space L^q_β with some q and β, which is possible under the conditions (3.2). Choose also $\beta < \frac{n}{q}$ and observe that the interval $\left(\frac{\lambda}{p} + \frac{n}{q} - \frac{n}{p}, \frac{n}{q}\right)$ is non-empty. Then in view of this embedding and Lemma 3.2 for the solution $u(x,t)$ we have the representation (3.3). Then the uniformness

$$\sup_{t>0} \|u(\cdot,t)\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)} = \sup_{t>0} \|T_t u_0\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)} \leq \infty$$

of the inclusion $u(t) \in \mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ follows from the uniform point-wise domination of the semigroup T_t by the maximal operator, see (3.4), and the boundedness of the maximal operator in Morrey spaces. The latter was proved in [3], se also the proof for more general class of sublinear operators in [23].

As for the convergence $\|T_t u_0 - u_0\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)} \to 0$ as $t \to 0$, it certainly does not hold for all $u_0 \in \mathcal{L}^{p,\lambda}(\mathbb{R}^n)$, see counterexample in [11, Example 3.4].

The convergence (3.5) with $u_0 \in Z^{p,\lambda}(\mathbb{R}^n)$ for the heat semigroup was proved in [11, Lemma 3.1].

Observe that in any case we have such a convergence for all $u_0 \in \mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ in a weaker norm $\|\cdot\|_{L^q_\beta(\mathbb{R}^n)}$, for $\beta > \frac{\lambda}{p}$ and arbitrarily close to $\frac{\lambda}{p}$, since q in (3.2) may be chosen arbitrarily close to p. Convergence in $L^q_\beta(\mathbb{R}^n)$-norm becomes "less weak" when $\beta \to \frac{\lambda}{p}$).

□

Remark 3.4. Theorem 3.3 may be extended to the so called generalized Morrey spaces (see their definition e.g. in [19]) via the use of embedding between Morrey and Lebesgue spaces obtained in [24].

Declarations

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Conflict of interest

• The authors declare that they have no conflict of interest.

The corresponding author is N. Samko.

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Paper F

The Hålogaland bridge - descriptions, challenges and related research under arctic conditions

Harpal Singh

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The Hålogaland bridge - descriptions, challenges and related research under arctic conditions

Harpal Singh*
*UiT – The Arctic University of Norway

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Abstract

In this article a brief description of the Hålogaland suspension bridge is presented, along with some of the challenges which already have appeared. Moreover, the problems and challenges which have appeared in a number of bridges of this type in the past are reported. Further, some vibration analysis techniques and concrete examples to detect damages in suspension bridges are discussed. New research can be needed to overcome some of the not solved research challenges. In this article, we propose some new methods which can be useful in this connection, namely to develop the Fourier theory and proved the corresponding inequalities in unbounded systems and to develop signal processing techniques in non-separable function spaces within Bayesian framework.

Keywords: Fourier analysis, Function spaces, Bayesian methods, Operation modal analysis, Structural health monitoring, Damage detection, Suspension bridges, High-rise building, Off-shore Structures, Global warming, Arctic conditions

AMS Classification (2020): 42A16, 46E30, 65T50, 65T60

1 Introduction

Bridges are the major transportation infrastructure assets of a region, which propel regional cooperation, social development and stimulate economic growth. Suspension bridges are very expensive civil engineering structures where the deck is hung below the suspension cables on vertical suspenders. Suspension bridges are increasingly used in the creation of new civil infrastructure as they are very flexible structures and have an ability in bridging larger spans. Moreover, suspension bridges are rich in architectural features and once the construction is finished, few materials are required for maintenance in comparison to other types of bridge constructions.

The effects of traffic and environmental loading are carefully accounted in the design of the bridges and included in the life cycle assessment and bridges are also subjected to periodic inspection, and maintenance. However, the aging process and the bridge performance from the wind, traffic and surrounding environment are interesting to monitor to understand the overall aging process and support the life cycle cost assessment.

Especially in northern Norway extreme arctic conditions further speed up the aging process. Moreover, due to global warming storms are getting bigger and waves in the sea are becoming more violent. Proper monitoring and maintenance of bridges is performed to identify and correct in a proactive manner any unexpected deterioration and avoid significant effect on economic and social development.

Structural health monitoring (SHM) covers, among others, the process of implementing a damage detection and characterization procedure for engineering structures. Here damage is defined as a change in the geometric properties of the structural system or a change in the boundary conditions of structural system, a change in the boundary conditions, or a change in system connectivity, that adversely affects the performance of the structural system (see [10]). SHM can help to timely carry out the maintenance operations that could help mitigate the problem of reduced performance or failure of the structure. Thus, SHM is one significant focus area to ensure the safety and serviceability of bridges, and further reducing the downtime as a result of long-term degradation and extreme loading.

In order for SHM to be successful, monitoring objectives should be defined together with bridge owners to ensure data acquired can be used to answer specific and relevant questions for the operation and maintenance of bridges. Moreover, the full potential of SHM is only unlocked when a holistic approach is taken combining all sources of information, e.g. inspection, tests and monitoring data in assessment processes.

The research in this paper is inspired by some challenges related to the new Hålogaland suspension bridge in Narvik. Moreover, the challenges which have appeared in a number of bridges of this type in the past are reported. Furthermore, we describe some vibration analysis techniques and concrete examples to detect damages in suspension bridges. We suggest that new research can be needed to overcome some of the not solved challenges. In this article, we propose some new methods which can be useful in this connection, namely to develop the Fourier theory and prove the corresponding inequalities in unbounded systems and to develop signal processing techniques in non-separable function spaces within Bayesian framework. We also include a description of the special fact that the challenges occurring in the Hålogaland bridge are strongly influenced by Gulf stream and arctic conditions. Finally, some suggestions of supporting research for the future is proposed.

The paper is organized as follows: In Section 2 the announced description of the Hålogaland suspension bridge is presented. Some challenges that have appeared in the the Hålogaland suspension bridge are described along with the procedures that were done to overcome these challenges is discussed in Section 3. The Hålogaland suspension bridge is located inside the arctic circle. Therefore, in this section also some special challenges naturally occur, which are also briefly mentioned. Furthermore, several problems, challenges and accidents that appeared in the history all over the world are described and discussed in Section 4. Some research based on vibration based damage detection for suspension bridges is presented and analyzed in Section 5. Section 6 is reserved to present some more challenges that appear in signal processing of data. Pre-processing of data is required as it can appear some errors in synchronization of sensors as well as some errors can appear while recording the data. In Section 7 the new supporting results in the area of non-separable function spaces within statistical Bayesian framework are described related to approximation problems in signal processing (for more details see [30]). Some related research concerning Fourier analysis especially concerning inequalities appear in Section 8. In general, Fourier inequalities are proved for bounded systems, but some practically used Fourier methods (e.g. the Haar wavelets system) are unbounded, therefore we propose to develop the theory for such unbounded orthonormal and orthogonal systems for future use (for more details see [2] and [3]). Finally, some concluding remarks are presented in Section 9.

2 The Hålogaland suspension bridge

The Hålogaland bridge is the longest suspension bridge within the arctic circle area at the time of construction. It is the second longest suspension bridge in Norway. Figure 1 shows the actual picture of the Hålogaland bridge, this picture is taken by Rune Dahl. It is a part of the European Route E6. Construction work of the bridge was featured on the Science Channel show Building Gaints, titled Arctic Mega Structure. The bridge is 1533 m long and has 3 spans. The longest suspension span has a length of 1145 m that is connected by 2 viaducts, one from Karistrand and the other from Øyjord. The viaduct from Karistrand has a length of 250 m and while the viaduct from Øyjord has a length of 148m. The structure is supported by 2 suspension cables with a diameter of 47 cm and a length of 1621 meters that weights about 2000 tonnes each.

The "A shaped" towers are build in reinforced concrete, where steel is embedded inside concrete. The Karistranda tower has a height of 179.1 m while the Øyjord tower has a height of 173.5 m. A geometrical drawing of the bridge is shown in Figure 2 [37]. The steel box girder constitutes the actual bridge deck in the suspension span. It is constructed as a trapeze-shaped closed steel box as shown in Figure 3 [37]. There are a total of 30 steel box girders of length 40 meters that weight about 7000 tonnes of steel. Figure 2 and Figure 3 are taken from the technical brochure available on the website of Statens Veivesen (see [37]). Detailed design and technical information is available in the technical report [7] and the book [20]. The construction work began on February 2013 and the project was completed in 2018 at a cost of approximately NOK 4 billions. The Prime minister of Norway Erna Solberg inaugurated the bridge on December 9, 2018, and the bridge was opened to traffic the same day.

A detailed design of the Hålogaland bridge and its various construction phases can be found in [7]. The first 35 mode shapes of the bridge are shown in the Appendix F of this report. Furthermore, the eigenfrequencies are determined for the finished bridge, free system construction stage, and for the system with the buffer fixed at Karistranda. Figure 4(taken from the report [7]) shows the first 5 mode shapes (isometric view) and mode frequencies of the Hålogaland bridge with finished system. These modes are as per the updated system of Hålogaland bridge with all the assumptions. But, in reality the original system may vary, as it is difficult to determine exact boundary conditions. It may be interesting to do OMA of the Hålogaland Bridge and get the final system be updated to real case scenario, that can provide interesting information about the Hålogaland bridge and further confirm the detailed analysis carried out during the design. It shall be noted that the complex and detailed analysis carried out during design and the safety parameter provided by the code system provide the structure with proper safety and integrity. An OMA of the bridge can further confirm the analyses and find areas of focus and optimization.

3 Challenges connected to the Hålogaland Bridge

3.1 Appeared challenges

On January 14, 2019 some fractures in one of the anchor bolts holding the main cables in place was found to have cracked. The Norwegian Public Roads Administration inspected the support system of the cables to find the cause, but the bridge was kept open to road users after a safety assessment while the investigations were carried out. Review of the documentation about production and the tests were carried out, and it was found that material failure or incorrect assembly are possible explanations [36].

It was decided to inspect and monitor all the 344 bolts that are fastened to support the cables. The total number of strands is 44 and the cable strand that was anchored by these two rods has a very small load only, i.e. the 43 other cable strands has to carry the total load. This is shared evenly, meaning that each of the 43 strands has now higher load than before. In addition, there is a small reserve in the design, so even with 43 strands, the total capacity is still meeting the full demands of the project. Therefore, the loss of two rods (effectively one cable strand) does not by itself lead to any concern in regards to general safety or functionality of the bridge, even when the bridge will be subjected to design loads from traffic and wind.

The probable cause of this fracture in the bolt, is called "hydrogen-induced stress crack growth" [40]. On December 20, 2019, the expert group appointed by the Norwegian Public Road Administration made a final statement on this case. According to this group, the type of material used in the bolts gives a very high strength and hardness, but at the same time it

Figure 1: The Hålogaland Bridge, the longest suspension bridge in the arctic region.

is sensitive to brittle fractures if not produced with careful workmanship. The most probable reason that triggered the fracture in the bolt was being exposed to the weather and strong wind for a certain period of time during the construction period and the water accumulated around them. The weather is a trigger but not a root cause for the fractured bolts and similar high strength bolts are used on other bridges. The bolts were regularly monitored and inspected since the fracture was discovered. No more cracks in rest of the bolts have been reported. The load bearing capacity of the bridge is stable. However, still there is uncertainty in the bolts that has weakness. So a decision to replace the bolts with weakness was made, to remove this uncertainty. The situation was classified as not critical, so the maintenance work can be carried out over a period of time [39].

Quite recently in February 2021 northern Norway was hit by the storm ”Frank.” The storm had a significant impact on the Hålogaland bridge. Several of the vibration dampers on the bridge were completely or partially destroyed, while some parts fell into the sea when the storm was ranging. This damage has no effect on the strength and load-bearing capacity of the bridge.

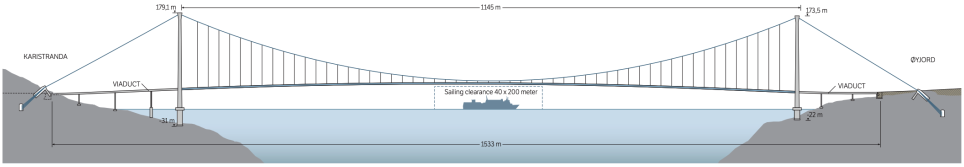


Figure 2: Specifications of the Hålogaland bridge .

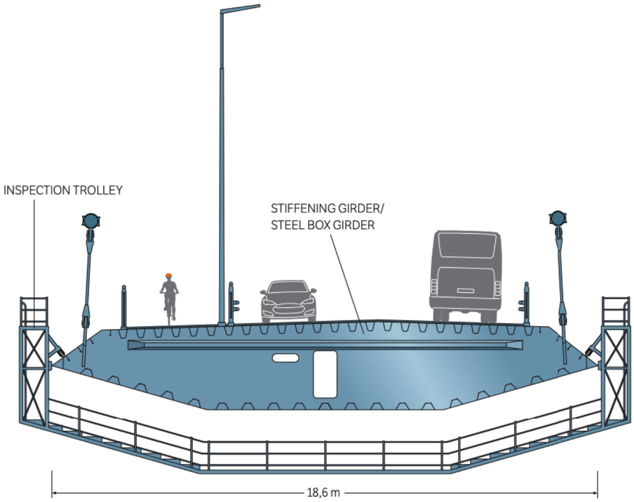


Figure 3: Cross-section view of the stiffening girder of the Hålogaland bridge.

According to the project manager Hans Jack Arntzen all the vibration dampers have been inspected after the storm and some of the damaged dampers have been sent to the manufacturer to find the cause of the damage. This vibration damping system is well tested and has been mounted on many bridges across the globe and according to supplier no such damage has been reported earlier. Now, The Norwegian Public Roads Administration is waiting for the results and tests reports from the manufacturer while monitoring the vibration dampers that are still mounted [38].

Some problems from the icing on the suspension cables of the Hålogaland Bridge had also been reported. Most of the suspension bridges can really withstand the additional load of snow or ice mass, but the real problem lies in the melting phase when the large chunk of ice are prone to drop from suspension cables and can damage cars or pedestrian. According to the COWI, similar problem appeared in the Øresund bridge that links Sweden and Denmark (see [28]). But the recommended solution was to install the heating cables at the locations where most icing occurred or just shut the bridge and manually remove the ice.

3.2 Effects of gulf stream and wind under arctic conditions

Reinforced concrete structures (RCS) is not so vulnerable to damages if the temperature is continuously positive or continuously negative. RCS is more exposed to damages when there is a continuous variation between positive and negative temperatures. Especially, in Narvik region, there is a lot of fluctuation in temperature due to the gulf stream in winter. Gulf stream

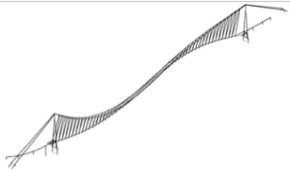
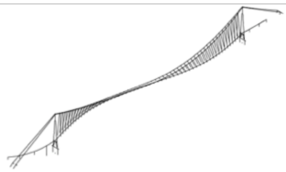
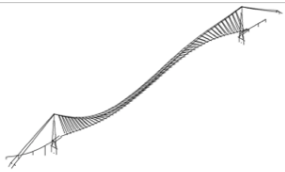
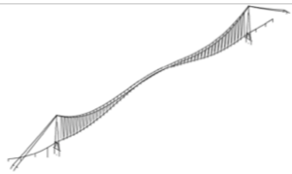
Mode Shape of Finished System	Mode Frequency (Hz)
	0.05305 Hz
	0.08617 Hz
	0.11671 Hz
	0.13719 Hz
	0.14295 Hz

Figure 4: The first 5 mode shapes (isometric view) and mode frequencies of Hålogaland bridge.

is the reason that the sea port in Narvik is ice free in the winter. This continuous variation of temperature is very harsh for RCS. Further, the Hålogaland bridge is exposed to saline conditions and even roads in the arctic are salted to remove icing.

Frost damage in concrete occurs due to differing thermal expansion of ice and concrete and by the volume expansion of water that freeze in the concrete pores (see [6]). The process accelerates in presence of saline conditions. Due to the difference in the thermal expansion of ice and concrete, stress arises, which further leaves the ice in tension if the temperature drops [16]. This leads to a phenomenon called surface scaling, where a crack in the brine ice penetrates in the substrate. This superficial damage results in spalling of concrete surface, while the concrete beneath is not so much unaffected.

Due to the volume expansion of the freezing water restrained by the surrounding concrete can lead to the formation of micro and macro cracks in the concrete body. This can cause severe known as internal frost damage [16], which affects the compressive strength, tensile strength, elastic modulus, fracture energy and the bond between the reinforcing bar and surrounding concrete in the damaged regions of the structure (see [26], [33] and [9]). Therefore, this continuous variation of temperature resulting from gulf stream in winter and salting of roads are bringing additional challenges for the Hålogaland bridge. Moreover, the Hålogaland bridge is further exposed to the winds coming from the sea towards the Ofoten fjord. This also bring variation in temperatures. The most crucial factor with this phenomenon is that the bridge is exposed to more vulnerable conditions on one side (i.e. direction of wind from sea to fjord) than the other.

3.3 Global warming and future analysis

Due to climate warming sea ice extent is reducing, sea levels are rising and increased storminess make low-lying arctic coastal regions susceptible to storm surges [41]. Climate warming together with reduced sea ice extent, result in strong wave action and the wave height integrated over the open water seasons has increased dramatically (see [24]). Further analyses indicate that this surge in storms will make arctic coastal regions increasingly hostile environment. The arctic ocean and the arctic coast are projected to harbour more intense summer storms, with an increase of 1-2 arctic cyclone per summer [23]. The warming Atlantic water layers has a noticeable impact on the sea ice in the Nordic seas and Barents sea. More and more open water combine with the prevailing atmospheric pattern of airflow and leads to strong storms (e.g. Storm Frank in December 2015 that leads to high energy transport to the high arctic [4]. Storm Frank caused strong winds in Europe. The authors of the article [4] found two other stronger storms in the past one in December 1986 and the other in December 1999 that had an stronger intensity and the authors also compiled a list of the strongest North Atlantic cyclones in the table S1 of supplementary section.

Structures in the arctic environment are exposed to extreme weather conditions that degrade the performance and leads to faster wear and tear. As we have seen from the trends that the storms in the arctic region are not just increasing, but also getting more intense. This will have more impact on the structures. In future, civil engineering structures will need more routine checkups to carry out operation and maintenance activities to have a reduced downtime. Especially in arctic regions a lot of logistics are needed to carry out such an activity and it gets very expensive. A good solution of this problem could be a hybrid system for the SHM of civil engineering structures that can analyse the condition of the structure after severe storms. To maximize the benefit of such an approach, clear objectives and performance indicators should be defined prior to the implementation of a SHMS. In fact, this could contribute reducing the downtime of structure or recommending timely maintenance so the structure can be saved from deterioration.

4 Suspension bridge challenges in general

There are many factors that influence the structural deterioration of bridges and hence cause damage to their components. Changes in the load patterns on bridges due to an increase in traffic, results in exceeding the load carrying capacity of structural components (see [44]). Moreover, different environmental conditions such as floods, hurricanes, storms, tornadoes and earthquakes accelerate the structural deterioration of bridges. According to [31] overloading, environmental deterioration and lateral excitation caused by vehicles have crucial effects on bridge deterioration.

Life expectancy of a suspension bridge is directly correlated to the conditions of the cable system, and the major problems that arise in suspension bridges are the corrosion in the cable system and wire breaks [44]. Corrosion is an electrochemical process, in which the corroding metal is oxidised in the presence of humid environment and presence of humidity further accelerate this process. According to [17], corrosion in the cable system begins when moist air and water get through the coating system due to imperfections in coating. If the water and moist air get inside the cable system, then its evaporation is impossible. This is a consequence of the fact that the wires in the cable system get exposed to the moist environment for a long time. Examples of corrosion in the suspension bridge cables can be found in [44] and the report [17]. In the earlier times it has been a negligence in maintenance of the suspension bridges in USA for many years, as a result many of them became structurally deficient due to corrosion. In a similar study conducted in Japan, it was found that the newest suspension bridges cables have corrosion issues (see [12]). On Hålogaland bridge, de-humidification system of the main cables are installed to protect the main cables from corrosion.

Hangers are very crucial components of the suspension bridges, that are exposed to severe corrosion. Due to corrosion, structural capacity of the hangers decreases and this can lead to sudden failure. This sudden failure of hangers causes strong vibrations and as a result there could be huge changes in the internal forces of the suspension bridge. It is noted that design of modern bridges account for hanger rupture load cases. Furthermore, hangers are detailed to prevent water ingress and thereby corrosion. Hangers from the Hålogaland bridge have high-density polyethylene (HPDE) sheathing. Therefore, corrosion in hangers have detrimental influences on the capacity and serviceability of the structure and in the extreme case it results in bridge failure [44].

In 2001, Yibin Southgate bridge in China collapsed and fell down in the river after 8 vertical cables broke. In an another incidence Kutai Kartanegara Bridge in Indonesia suffered a progressive collapse resulting from one broken hanger (see [27]). The incidence on the Kutai Kartanegara bridge took place when the workers were performing maintenance on the hangers. It led to mass casualties, 11 people were killed and more than 30 were missing.

In hanger breakage event, there is a great bending moment in the girders. The bending moment is not just confined to the girders that are adjacent to the broken hanger near the tower but also in the section of girders that are far away from the broken hanger. Moreover, it also causes large dynamic responses in the stiffening girders even in the sections that are far away from the broken hanger. Due to mechanical degradation of the hangers resulting from corrosion, there is a slight increase in the structural dynamic responses of the girders(see [45]).

Progressive collapse is evaluated with respect to demand capacity ratio (DCR), which is defined as the ratio of the force in the structural member after the instantaneous removal of a column for each scenario to the member capacity [25]. A structural member is classified as failed if DCR value exceeds more than 2 for a typical structural configuration. The hangers that are adjacent to the broken hanger receives most of the redistributed load and becomes the critical member. DCR of a hanger progressively decreases with the growth of corrosion. Corroded hangers should be replaced in time in order to improve the structural robustness after sudden breakage of a hanger.

Suspension bridges are susceptible to long term fatigue damage as a result of many service years and increased loading. This leads to weakening of some components or parts of the structure. The main cables are protected with corrosion protection systems, that shields the load carrying wires from visual inspections. These factors further reveals the importance of SHM in informing operation and maintenance strategies for suspension bridges for their safer performance. Some examples where SHM systems have been installed in long span bridges are the Skarnsundet bridge in Norway, the new HaengJu Bridge in Korea, The Storck's bridge in Switzerland and the New Carquinez bridge in California [44]. In the following Section we define structural damage and various vibration based damage detection methods.

5 Vibration based damage detection

Structural damage is defined as the weakening of a structure that leads to reduced performance as well as a deviation in the structure's original geometric or material properties, that causes undesirable stress, displacement or vibrations. A damage in a structure can be linear if the structure remains linearly elastic after damage or else it is classified as non-linear damage. Main objectives of the damage detection methods can be classified at 5 levels: determination of the existence of damage, determination of the location of damage, determination of the type of damage, determination of the extent of damage and lastly the prediction of the remaining service life after the damage [10]. First 4 levels are classified as damage diagnosis state, while the last level is damage prognosis state. Damage diagnosis and prognosis are very challenging for complex civil structures like suspension bridges, thus this research focuses on damage detection for suspension bridges using vibration response.

In the article [35], we have discussed the mathematical modelling of the vibrating structure and various techniques for the dynamic analysis of structures are described in [34]. Vibration based damage detection is based on the fact that their is change in the structural properties i.e. change in mass, damping and stiffness due to damage. This change in structural properties arising due to damage, causes change in modal properties i.e., mode shape, natural frequency and modal damping. Since dynamic characteristics of a structure are functions of its stiffness and mass, the variation in modal properties can be an effective indicator of structural deterioration.

Models that are based on vibration based damage detection (VBDD) method uses analytical model to identify damage. These analytical models are finite element (FE) model or updated FE model. Thus there is an interest for model updating of analytical FE model in order to detect damage (see [34]). This model updating based approach is expensive and time consuming but it has been used in large civil engineering structures like bridges successfully. On the other hand non-model based damage detection methods are not so expensive and quite straightforward since they don't require an analytical model. They are also referred to as damage index (DI) methods. In these methods, the changes in modal parameters between the intact and damaged states are used to develop the DI for the damage detection of the structure under consideration.

DI's computed using primary vibration properties are classified as: natural frequency based methods, damping based methods, and direct mode shape methods. DI's computed using secondary vibration properties can be classified as curvature mode shaped based methods, flexibility based methods and modal strain energy based methods. These methods have been discussed in detail in the PhD thesis of Wickramasinghe (see [44]) along with its advantages and disadvantages.

VBDD contains many uncertainties along with measurement noises, modelling errors and methodology errors. Neglecting these uncertainties could lead to inaccurate damage detection results. Researchers have proposed statistical damage identification based approaches such as statistical pattern recognition [14] , perturbation techniques [18] and Bayesian methods [5] and [19]. In these approaches uncertainties in vibration data are described as random variables. OMA has gained huge popularity in the last decade in theoretical development as well as practical applications. In the articles [34] and [35] we have discussed experimental results of OMA for a steel truss bridge and a high-rise building, respectively. It is very challenging and difficult to control the test environment for OMA. Thus uncertainty plays a crucial role in OMA and needed to be backed by a statistical interference problem to provide robustness of the mode identification results [48]. (Italy and portugal oma based damage detection add material, rune bricnker and carlos vantura)

Various statistical approaches have been developed to quantify uncertainty over the decades, but two main approaches are non-Bayesian approach and Bayesian approach. Examples of non-Bayesian approaches are frequency domain maximum likelihood (ML) technique and stochastic subspace identification (SSI) based methods (see [34]). Significant development has been achieved in structural dynamics that is based on Bayesian statistical framework. Bayesian fast Fourier transform (BFFT) (see [49]) and Bayesian spectral density approach (BSDA) ([21]) are computationally demanding and mathematically rigorous. A two stage fast Bayesian spectral density approach (FBSDA) based on the framework of BFFT and BSDA was developed (see [46] and [47]), where a variable separation techniques was presented to separate the spectrum variables (such as: frequency, damping ratio, PSD of modal excitation and prediction error) and spatial variables (such as: mode shape). This resulted in the reduction of dimension involved for computation and have ease of implementation. A successful implementation of a two stage FBSDA has been reported on a long-span suspension bridge [22]. In the next section new results in the non-separable function space, with in the Bayesian framework are discussed.

6 Pre-processing of data and data management

In order to do effective vibrational analysis, one of the most important requirements is to have as accurate signals as possible from the industry standard vibration transducers such as accelerometers. Depending upon the need of the application, the signal can be processed directly or be improvised by using mathematical integrators to other units of vibration measurements. The signal can be conditioned using high pass or low pass filters with respect to the frequency of interest. In order to have the required results, the signal might be averaged and sampled multiple times. The other important thing required is to decide on the number of samples and the sampling rate.

In a SHM system, the acceleration data retrieved from the site might require some preprocessing to prepare data for analysis and interpretation. However, it can be some unwanted characteristics that needs to be resolved (see [8]) e.g.:

- Lack of synchronization in the system clocks of various data-loggers in the distributed monitoring system.
- Due to high-speed data acquisition, it is a possibility that there could be some gaps in the data files due to missed samples.
- There could be a duplication of data samples.
- It is a possibility that there could be a slow drift of the voltage measurement baseline over time that could lead to constant baseline offset in acceleration time history signals.
- There could be a disturbance form electrical noise at harmonics of electrical frequency that can lead to high-frequency noise in the data.
- Data from a dynamic event could be partitioned in multiple separate files due to an interruption in the communication system or a conflict originating during data collection and transmission.
- There might be a possibility that a data file can contain data from multiple triggered data events.

Due to all these factors, file sorting and processing of raw data is performed in order to fix the problems mentioned above and generate suitable data that could be used for analysis and visualization. Data processing, data visualization and data analysis techniques used in the Confederation bridge, in Canada are explained in detail in [8]. In order to have high precision in the measurements, accelerometers are needed to be calibrated between different measurement occasions. A simple calibration method has been proposed in [13]. This technique is non-iterative. Therefore there are no complicated convergence issues in relation to input parameters and round-off errors. The technique requires fewer arithmetic operations in comparison to other iterative approaches. An algorithmic framework for the reconstruction of time-delayed and incomplete binary signals from the energy-lean SHM system has been discussed in [29].

For a bridge of the size of the Hålogaland Bridge in Arctic environment where the weather changes too fast and conditions are extreme, SHM in real-time will generate a lot of data. For a project of such a scale it will be good to have data management with the use of artificial intelligence(AI) where the useful data and information can be extracted and can help to further automate the process of damage detection. Department of Computer Science and Computational Engineering at UiT-The Arctic University of Norway, Campus Narvik has already invested in AI based server that can be used, if there comes a project for the SHM of Hålogaland Bridge with UiT Campus Narvik as a partner.

7 Signal processing problems in non-separable function spaces

Vibration based damage detection techniques contains lots of uncertainties that leads to inaccurate damage detection results. Researchers all over the world have proposed and worked to develop statistical damage detection approaches based on vibration data, where uncertainties are described as random variables. Various approaches include perturbation techniques, Monte Carlo simulations, statistical pattern recognition and Bayesian methods. Bayesian methods have proven to be the most rigorous probabilistic framework to identify target variables and evaluate corresponding uncertainties using the available information (see [43]). In the preliminary stage of structural failure, damage usually occur at limited locations in the structure. Sparse Bayesian learning has been widely applied to sparse signal reconstruction and compressed sensing for the development of structural damage identification. Bayesian interface is a probabilistic method of interference that allow to form probabilistic estimates of certain parameters from a given series of observations. This method can be used in a couple of different ways in SHM including model updating, monitoring by inferring structural parameters over time, as well as in determining the optimal placement of sensors.

In reference [15] to the review paper on a Bayesian approach to signal processing problems, signal is a solution of a stochastic partial differential equation (SPDE). Main objective of this approach is to find the signal as a solution of the SPDE, taking into account the noisy observations of its solution. In the overview, a focus there is on showing that the idea to work with probability measures on function space, that lead to the formulation of a well-posed signal-processing problem. With this approach one can avoid dealing with the algorithms where methods based on application of the standard Markov chain Monte Carlo (MCMC) method after a discretization leads. A major disadvantages of such algorithms is that they function badly under refinement of discretization.

Providing a proper mathematical formulation of the problems on domain space while working with probability measures on a function space, leads to efficient sampling techniques, defined on path-space as the domain space, and therefore it is robust under the introduction of discretization [30]. In article [15], a wide variety of signal processing problems is over-viewed which leads to a posterior probability measure on a separable Banach space. In our submitted article [30] main aim was to go a step further ahead and investigate cases, when studying problems tackled in a Bayesian framework, is in non-separable space. Therefore, techniques for non-separable function spaces are suggested. Further, in cases where problems are formulated as filtering problems in addition to smoothing problems, development of an approach based on the framework of weighted function spaces is very useful.

In the new article [30], some new contributions in this connection have been stated, proved and discussed. This article is the first where non-separable function spaces has been used to give a new possibility to avoid approximation problems in signal processing e.g. related to damage detection in bridges.

For new contributions in this see [30].

8 Some new Fourier based inequalities in unbounded orthogonal and orthonormal systems

Fourier series as defined in the books [11] and [42] is an expansion of a periodic function, in terms of sinusoids, combined by a weighted summation. Fourier series uses the orthogonality relationship of the sine and cosine functions. The computation and study of Fourier series is known as an important part of harmonic analysis. In engineering sciences, the process of decomposing a function into oscillatory components is called Fourier analysis, while the process of reconstructing the function from these pieces is known as Fourier synthesis. Fourier transform and inverse Fourier transform as defined in [35] is used to transform a time domain signal with given properties to frequency domain. In terms of mathematics, Fourier transform and inverse Fourier transform are analogies of the Fourier analysis and Fourier synthesis for functions on unbounded intervals. Discrete version of the Fourier transform can be computed much faster and more efficiently by fast Fourier transform (FFT) algorithm.

In the papers [34] and [35] we have discussed various Fourier based methods that are used in OMA and SHM when the task of signal processing is required. We presented how frequency domain decomposition (FDD) was used together with fast Fourier transform (FFT) to find the modal parameters of a steel truss bridge and Luleå fire house tower. For the applications of FFT the system to be analysed should be linear and the data that has to be processed should be strictly periodic or stationary, else it can lead to misleading results [35]. Short time Fourier transform (STFT) as suggested by Dennis Gabor uses a window function and can overcome some of the limitations of FFT.

As for now, a lot of theoretical research has been carried out for Fourier coefficients in bounded orthogonal and orthonormal systems, but it has lots of limitations. Moreover some practically used Fourier methods, e.g. Haar wavelets are unbounded. This motivated us to further explore and analyse inequalities for Fourier coefficients in unbounded orthogonal systems based on [1] and new results are presented in [2]. In this work a number of classical Fourier inequalities related to Fourier coefficients in unbounded orthogonal systems are generalized and complemented in generalized Lorentz spaces. Some new Fourier and Jackson-Nikol´skii inequalities in unbounded orthonormal systems have been stated, proved and discussed in [3]. In this article, we consider the Fourier series of the function with respect to unbounded orthonormal system and the generalized Lorentz space is defined via a continuous and concave function. The derived results generalize and unify several well known results. Our actual interest is equipped with some applications related to structural problems in engineering, see e.g. PhD thesis of Seger [32] and the papers [34] and [35]. These structural problems focus about the safety of structure especially detecting damages in dams, bridges, high rise buildings and tunnels. Authors of [1], [2], [3], [34] and [35] believe that further development of this theory in unbounded systems can help to overcome some of the shortcoming of the Fourier methods already in use for SHM and OMA.

9 Concluding Remarks

Remark 9.1 *A limited research has been conducted for the damage detection in suspension bridges using VBDD methods to detect damages in main girder, towers, bearings. Damage detection in the main cables and hangers has not been fully examined for the case of single and multiple damages under moderate damage conditions. Hence, more research and use of such methods can contribute to solve the mentioned problems. In particular,it is worth developing a VBDD method considering different vibration modes (vertical, lateral, torsional and coupled modes) when damage occurs in the main suspension cable and hangers considering their dominant modes of vibration. Sensitivity analysis assessing the deterioration extent needed for detection as well as an assessment of the effect of ambient and operational conditions on vibrational responses are considered key elements of future research initiatives.*

Remark 9.2 *Several minor damages and challenges on the newly constructed Hålogaland bridge has been reported by Statens Veivesen. As seen in the literature various similar bridges across the world has problems related to corrosion and damage in main suspension cables and hangers. Implementation of a SHM system, with clear objectives agreed with relevant stakeholders prior to its installation, for the Hålogaland bridge can help in informing proactive operation and maintenance strategies. This will enhance the safety of the users and the maintenance workers. Moreover, the bridge is newly build, it will be of interest how the crucial parameters changes in arctic conditions, that will provide valuable information for the future construction of infrastructure in arctic regions.*

Remark 9.3 *Furthermore, development of Fourier analysis and inequalities also in unbounded orthogonal and orthonormal system can provide or help in the improvement of the signal processing techniques used for the damage detection in suspension bridges and related structures.*

Remark 9.4 *Data management system along with the use of AI can help in the automation and improving the efficiency of SHM for complex civil engineering structures like the Hålogaland bridge.*

Remark 9.5 *BFFT, BSDA and two stage FBSDA based approaches has shown significant improvements in the OMA of civil engineering structures. However, in the theory some approximation problems appear. Hence, in order to overcome these problem we suggest to use non-separable spaces with applications on signal processing techniques with in Bayesian framework can provide more insights on Bayesian state-approach for damage detection for suspension bridges and other civil engineering structures such as dams, tunnels and high-rise buildings.*

Remark 9.6 *From the literature on global warming and the Arctic environment it can be concluded that the storms in the Arctic are not just increasing, rather they are getting bigger and stronger. Thus the behaviour of the Hålogaland bridge in the changing arctic conditions will be very useful information that can further help in the improvements of future constructions in the extreme Arctic environment.*

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