Weighted Boundedness of Certain Sublinear Operators in Generalized Morrey Spaces on Quasi-Metric Measure Spaces Under the Growth Condition

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Abstract

We prove weighted boundedness of Calderón–Zygmund and maximal singular operators in generalized Morrey spaces on quasi-metric measure spaces, in general non-homogeneous, only under the growth condition on the measure, for a certain class of weights. Weights and characteristic of the spaces are independent of each other. Weighted boundedness of the maximal operator is also proved in the case when lower and upper Ahlfors exponents coincide with each other. Our approach is based on two important steps. The first is a certain transference theorem, where without use homogeneity of the space, we provide a condition which insures that every sublinear operator with the size condition, bounded in Lebesgue space, is also bounded in generalized Morrey space. The second is a reduction theorem which reduces weighted boundedness of the considered sublinear operators to that of weighted Hardy operators and non-weighted boundedness of some special operators.

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1 Introduction

We study weighted boundedness of certain sublinear operators in generalized Morrey spaces $\mathcal{L}_{\Pi}^{p,\varphi}(X)$ defined on quasi-metric measure spaces (X, d, μ). We do not suppose that (X, d, μ) is homogeneous, i.e. we assume that it satisfies the growth condition

$$\mu B(x, r) \leq cr^{\nu}, \ 0 < r < \ \mathrm{diam} \ X \leq \infty, \nu > 0. \tag{1.1}$$

The study includes Calderón–Zygmund singular operators with standard kernel, the corresponding maximal singular operator and the standard maximal operator. In the case of the singular and maximal singular operators we obtain results on weighted boundedness in the generalized Morrey spaces, under the only assumption that the measure satisfies the growth condition (1.1).

Sublinear operators under consideration are supposed to satisfy the following two conditions:

(1) they are bounded in $\mathcal{L}^p(X)$,

(2) they satisfy a certain size condition, related to the exponent of the growth condition.

For the study of sublinear operators of singular type in the space $L^p(X)$ under the growth condition (2.2) we refer to [25]. There are known results on the boundedness of such operators in $L^p(X)$ under the growth condition more general than (2.2), where r^{ν} is replaced by a given dominant $\lambda(x, r)$, see [14] and [15].

We consider the generalized Morrey spaces defined by the norm

$$\|f\|_{\mathcal{L}_{\Pi}^{p,\varphi}(X)} = \sup_{x \in \Pi, r > 0} \left(\frac{1}{\varphi(x, r)} \int\limits_{B(x,r)} |f(y)|^p \, d\mu(y) \right)^{\frac{1}{p}}, \tag{1.2}$$

where Π is any subset in X. Introduction of Π helps to unite local and global Morrey spaces.

For a sublinear operator T satisfying the conditions (1) and (2), we study the boundedness of the weighted operators

$$wT\frac{1}{w}$$

in the spaces $\mathcal{L}_{\Pi}^{p,\varphi}(X)$ under the growth condition (2.2). In other words, we study the operators T themselves in the weighted space $\mathcal{L}_{\Pi}^{p,\varphi}(X, w)$ defined by the norm

$$\|f\|_{\mathcal{L}_{\Pi}^{p,\varphi}(X,w)} = \sup_{x \in \Pi, r > 0} \left(\frac{1}{\varphi(x, r)} \int\limits_{B(x,r)} |f(y)|^p \, w(y)^p d\mu(y) \right)^{\frac{1}{p}},$$

where the weight w and the function $\varphi(x, r)$ are completely independent of each other.

For classical and generalized Morrey spaces and their applications we refer, for instance, to the books [8, 19, 27, 36, 38, 39] and the overview paper [28].

The boundedness of operators of harmonic analysis in Morrey spaces was studied in a variety of papers, see for instance, [4–6, 9, 11, 12, 20, 24, 33, 35] for the Euclidean case, and [10, 17, 18, 21–23, 34, 37, 40] for the general setting of quasi-metric measure spaces, see also references therein. In particular, in [34] there was studied the boundedness of singular-type operators in classical Morrey spaces under the growth condition (2.2).

Singular-type operators under more general growth condition (in the sense of [14] and [15]) were studied in [21] and [40]. In [21] the operators T were studied in the non-weighted case, while in [40] they were considered in the weighted space $L^{p,k}(X, w)$ of specific form, which goes back to [20], namely in the case

$$\|f\|_{L^{p,k}(X,w)} = \sup_{x,r} \left(\frac{1}{w[B(x,r)]^k} \int\limits_{B(x,r)} |f(y)|^p \, w(y) d\mu(y) \right)^{\frac{1}{p}} .$$

Note that in this case, the Morrey space $L^{p,k}(X, w)$ is in fact the non-weighted classical Morrey space $L^{p,k}(X)$ with respect to the measure $\mu_w(E) = \int\limits_E w(y) d\mu(y)$, $E \subset X$.

In this paper we study sublinear operators satisfying the properties (1) and (2) in weighted generalized Morrey spaces on quasi-metric measure space (X, d, μ), under the "classical" growth condition (2.2). We consider "radial" weights $w(x) = v[d(x, x_0)]$, $x_0 \in X$ and the function v belongs to some class $V_+ \bigcup V_-$, see its definition in Sect. 2.3.

Our main results are as follows.

First we show that the known way of transference of L^p-boundedness to Morrey-boundedness under the size condition, may be proved without using homogeneity of the space, see Transference Theorem in Sect. 3.1. More precisely, we show that the condition

$$\sup_{x \in \Pi, r>0} \frac{r^{\frac{v}{p}}}{\varphi(x,r)^{\frac{1}{p}}} \int_r^\ell \frac{\varphi(x,t)^{\frac{1}{p}}}{t^{1+\frac{v}{p}}} \, dt < \infty, \ \ell = \text{ diam } X,$$

with v from (1.1), imposed on the function $\varphi(x, r)$ defining the Morrey space, guarantees that any sublinear operator with the size condition, bounded in $L^p(X)$, is also bounded in the Morrey space $\mathcal{L}^{p,\varphi}_\Pi(X)$.

Moreover, under the only growth condition, we are able to efficiently estimate the Morrey modular of Tf via that of f, see Theorem 3.8, which leads to the boundedness result in Morrey spaces in Theorem 3.9.

Further, we provide a certain pointwise estimate for weighted singular, maximal singular and maximal operators, with above mentioned radial weights, via non-weighted such operators plus the following operators: *weighted Hardy operators*, certain non-weighted operators which may be considered as a kind of *hybrids of Hardy operators* and *potential operators*, see Reduction Theorem 3.11.

Since the estimate in this theorem is pointwise, it reduces the weighted boundedness of the weighted singular, maximal singular and maximal operators in any Banach functions spaces with lattice properties to the boundedness of non-weighted operators, weighted Hardy operators with the same weight and some specific "hybrids". In this paper we use this estimate for the case of the generalized Morrey space $\mathcal{L}_{\Pi}^{p,\varphi}(X)$.

As a separate result of interest we show that some of those hybrids are dominated by the modified maximal operator (modification concerns the use of the growth condition), see Theorem 3.5.

This reduction and the above mentioned Transference Theorem together with the L^p results [25], allow us to obtain a result on the weighted boundedness of the weighted singular, maximal singular and maximal operators in the spaces $\mathcal{L}_{\Pi}^{p,\varphi}(X)$ as given in Theorem 3.20. To this end, we obtain conditions for the weighted boundedness of Hardy operators in the spaces $\mathcal{L}_{\Pi}^{p,\varphi}(X)$ under the only growth condition for (X, d, μ).

The paper is organized as follows. In Sect. 2 we provide necessary information on quasi-metric measure spaces (X, d, μ) together with definition of the space $\mathcal{L}_{\Pi}^{p,\varphi}(X)$ and define the class of weights. Sect. 3 contains our main results. In Sect. 3.1 we prove the above mentioned Transference Theorem for an arbitrary sublinear operator with the size condition. In Sect. 3.2 we pass to weighted operators and prove the above mentioned Reduction Theorem containing the pointwise estimate of weighted operators. Section 3.3 starts with a result of weighted boundedness of Hardy operators in generalized Morrey spaces $\mathcal{L}_{\Pi}^{p,\varphi}(X)$. This allows us to apply Transference and Reduction Theorems to obtain conditions on the weight and the function $\varphi(x, r)$, insuring the weighted boundedness of singular, maximal singular and maximal operators in the spaces $\mathcal{L}_{\Pi}^{p,\varphi}(X)$. In Corollary 3.21, where we take $\varphi(x, r) = r^\lambda$ for simplicity, we give sufficient conditions for the validity of those conditions in terms of Matuszewska-Orlicz indices of the weight. Finally, in Sect. 1 (Appendix), for reader's convenience, we provide necessary information for Matuszewska-Orlicz indices.

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2 Preliminaries

2.1 Preliminaries on Quasi-Metric Measure Spaces

Basics on quasi-metric measure spaces may be found e.g. in [7] and [13]. Below we provide necessary definitions which we use in the paper.

Let (X, d, μ) be a quasi-metric measure space with measure μ and quasi-distance d:

$$d(x, y) \leq k[d(x, z) + d(y, z)], \quad k \geq 1 \tag{2.1}$$

$d(x, y) = 0 \iff x = y$, $d(x, y) = d(y, x)$ and $\ell = diam\, X$, $0 < \ell \leq \infty$, $B(x, r) = \{y \in X : d(x, y) < r\}$. Everywhere in the sequel we suppose that the following properties of (X, d, μ) hold:

(1) all balls are open sets;

(2) the spheres $S(x, r) := \{y \in X : d(y, x) = r\}$ have zero measure for all x and r;

(3) $\mu B(x, r)$ is continuous in $r \in [0, \ell)$ for every $x \in X$.

The set (X, d, μ) is said to satisfy the *growth condition* if there exist a constant $A > 0$ and exponent $\nu > 0$, which is fractional in general, such that

$$\mu B(x, r) \le A r^\nu, \tag{2.2}$$

where $x \in X$ and $r \in (0, \ell)$. For more general notion of the growth condition, i.e. with a given dominant of measure of balls we refer to [14] and [15]. In this paper we use the growth condition of the form (2.2).

We say that (X, d, μ) is *regular*, if the measure satisfies the lower and upper Ahlfors conditions with coinciding exponents, i.e.

$$C_1 r^\nu \le \mu B(x, r) \le C_2 r^\nu, \quad x \in X, r > 0, \; \nu > 0. \tag{2.3}$$

Estimates of the type provided by the lemma below are known but we give its short proof for completeness of presentation.

Lemma 2.1 *Let (X, d, μ) satisfy the growth condition (2.2) and $\gamma > 0$. Then*

$$\int_{B(x,r)} \frac{d\mu(y)}{d(x, y)^{\nu-\gamma}} \le C \, r^\gamma, \tag{2.4}$$

where $C = A \max\{1, 2^{\gamma-\nu}\} \frac{2^\nu}{2^\gamma - 1}$ does not depend on $x \in X$ and $r \in (0, \ell)$.

Proof We have

$$\int_{B(x,r)} \frac{d\mu(y)}{d(x, y)^{\nu-\gamma}} = \sum_{k=0}^\infty \int_{2^{-k-1}r < d(x,y) < 2^{-k}r} \frac{d\mu(y)}{d(x, y)^{\nu-\gamma}}$$

$$\le \max\{1, 2^{\gamma-\nu}\} \sum_{k=0}^\infty \frac{1}{(2^{-k-1}r)^{\nu-\gamma}} \int_{d(x,y) < 2^{-k}r} d\mu(y)$$

$$= A \max\{1, 2^{\gamma-\nu}\} \sum_{k=0}^\infty \frac{(2^{-k}r)^\nu}{(2^{-k-1}r)^{\nu-\gamma}}$$

$$= A \max\{1, 2^{\gamma-\nu}\} \frac{2^\nu}{2^\gamma - 1} r^\gamma.$$

$\square$

Lemma 2.2 below provides a certain replacement of the formula of passage to polar coordinates used in the case $X = \mathbb{R}^n$. This lemma is a simplified version of more general estimates proved in [32].

Let Π be an arbitrary set of points in X. We use the uniform doubling condition

$$L(\xi, 2t) \le CL(\xi, t), \tag{2.5}$$

on a function $L(\xi, t)$, where $C > 0$ does not depend on $(\xi, t) \in \Pi \times \mathbb{R}_+$.

In the sequel we use the abbreviations: *a.i. = almost increasing* and *a.d. = almost decreasing*

Lemma 2.2 [32,Lemmas 2.5 and 2.8] *Let* (X, d, μ) *satisfy the growth condition* (2.2), $L(\xi, t)$ *be a non-negative function on* $\Pi \times (0, \ell)$, $0 < \ell \le \infty$, *a.i. in t uniformly in* $x \in \Pi$ *and the doubling condition* (2.5) *be satisfied. Then*

$$\int_{B(x,r)} \frac{L[\xi, d(x, y)]}{d(x, y)^a} d\mu(y) \le C \int_0^r \frac{t^{\nu-1} L(\xi, t)}{t^a} dt \tag{2.6}$$

and

$$\int_{X \backslash B(x,r)} \frac{L[\xi, d(x, y)]}{d(x, y)^a} d\mu(y) \le C \int_r^\ell \frac{t^{\nu-1} L(\xi, t)}{t^a} dt, \tag{2.7}$$

where $\xi \in \Pi$, $x \in (0, \ell)$, $a \in \mathbb{R}$ *and* $0 < r < \ell \le \infty$, *whenever the right hand side of these estimates exists or not.*

2.2 Generalized Morrey Spaces $\mathcal{L}^{p,\varphi}(X)$

Let $1 \le p < \infty$.

The generalized Morrey spaces are defined by the norm:

$$\|f\|_{\mathcal{L}^{p,\varphi}(X)} = \sup_{x \in X, r > 0} \left(\frac{1}{\varphi(x, r)} \int_{B(x,r)} |f(y)|^p d\mu(y) \right)^{\frac{1}{p}}, \tag{2.8}$$

where it is assumed that $\varphi(x, r)$ is a positive measurable function on $X \times (0, \ell)$, $\ell = \text{diam } X$, positive for all $(x, t) \in X \times (0, \ell]$.

The spaces defined by the norm

$$\sup_{r > 0} \left(\frac{1}{\varphi(x_0, r)} \int_{B(x_0,r)} |f(y)|^p d\mu(y) \right)^{\frac{1}{p}}, \quad x_0 \in X, \tag{2.9}$$

are often called generalized local Morrey spaces. The spaces defined by the norm (2.8) are correspondingly called generalized global Morrey spaces. Both may be united in a

single approach by the localization applied with respect to an arbitrary set $\Pi \subseteq X$, not just with respect to the case $\Pi = \{x_0\}$ of an isolated point. That is, one can estimate the Morrey-regularity of functions f at an arbitrary given subset of X, with admission of the extremal cases $\Pi = X$ and $\Pi = \{x_0\}$, $x_0 \in X$. The corresponding space defined by the norm

$$\|f\|_{\mathcal{L}_\Pi^{p,\varphi}(X)} := \sup_{x \in \Pi, r > 0} \left(\frac{1}{\varphi(x,r)} \int\limits_{B(x,r)} |f(y)|^p \, d\mu(y) \right)^{\frac{1}{p}} \tag{2.10}$$

will be denoted by $\mathcal{L}_\Pi^{p,\varphi}(X)$. The principal estimates on which the proofs in this paper are based, are pointwise, see Sect. 3.2.

Everywhere in the sequel we suppose that $\varphi(x,r)$ is a positive measurable function on $\Pi \times (0, \ell)$, $\ell = \operatorname{diam} X$, $0 < \ell \leq \infty$, and the following *à priori* assumptions hold:

(1) $\varphi(x,r)$ is a.i. in r uniformly in $x \in \Pi$:

$$\varphi(x, \varrho) \leq c\varphi(x,r), \ 0 < \varrho < r < \ell. \tag{2.11}$$

(2) $\frac{\varphi(x,r)}{r^\nu}$ is a.d. in r uniformly in $x \in \Pi$:

$$\frac{\varphi(x,r)}{\varphi(x,\varrho)} \leq c \left(\frac{r}{\varrho} \right)^\nu, \ 0 < \varrho < r < \ell. \tag{2.12}$$

Note that $\mathcal{L}_\Pi^{p,\varphi}(X) = L^p(X)$, if $\inf\limits_{(x,r) \in \Pi \times (0,\ell)} \varphi(x,r) > 0$.

In the sequel we use the notation

$$\mathfrak{M}(f; x, r) := \frac{1}{\varphi(x,r)} \int\limits_{B(x,r)} |f(y)|^p \, d\mu(y). \tag{2.13}$$

For classical Morrey spaces $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$, as is known, $|x|^{\frac{\lambda-n}{p}} \in \mathcal{L}^{p,\lambda}(\mathbb{R}^n)$, if $0 < \lambda \leq n$ or $\lambda > 0$, when $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ is global or local centered at the origin, respectively. We shall deal with the corresponding "model" function

$$\Phi_{x_0}(x) = \left(\frac{\varphi(x, d(x, x_0))}{d(x, x_0)^\nu} \right)^{\frac{1}{p}}$$

in the general setting of quasi-metric measure spaces with growth condition.

To this end we introduce the assumption that:

Uniform Zygmund conditions hold:

$$\int\limits_0^r \frac{\varphi(x,t)}{t} dt \leq c\varphi(x,r) \tag{2.14}$$

where $0 < r < \ell$, $x \in \Pi$ and c does not depend on x and r.

Theorem 2.3 *Let (X, d, μ) satisfy the growth condition and $\varphi(x, r)$ satisfy the Zygmund condition* (2.14). *Then*

$$\Phi_{x_0} \in \mathcal{L}_{\Pi}^{p,\varphi}(X).$$

Proof Let $x \in \Pi$. For the modular $\mathfrak{M}(\Phi_{x_0}; x, r)$ we have

$$\mathfrak{M}(\Phi_{x_0}; x, r) = \frac{1}{\varphi(x, r)} \int\limits_{B(x,r)} \frac{\varphi(x, d(y, x_0))}{d(y, x_0)^\nu} d\mu(y).$$

We distinguish the cases $d(x, x_0) \leq 2kr$ and $d(x, x_0) \geq 2kr$. Let first $d(x, x_0) \leq 2kr$. Then $d(y, x_0) \leq kd(y, x) + kd(x, x_0) \leq kr + 2k^2r = k(1 + 2k)r$, i.e. $B(x, r) \subset B(x_0, k(1 + 2k)r)$. Then

$$\mathfrak{M}(\Phi_{x_0}; x, r) \leq \frac{1}{\varphi(x, r)} \int\limits_{B(x_0, k(1+2k)r)} \frac{\varphi(y, d(y, x_0))}{d(y, x_0)^\nu} d\mu(y).$$

On the right hand side we can apply the inequality (2.6) with $L(x, r) = \varphi(x, r)$. Note that the condition (2.5) of Lemma 2.2 is satisfied, being easily derived from (2.12). By (2.6) we obtain

$$\mathfrak{M}(\Phi_{x_0}; x, r) \leq \frac{C}{\varphi(x, r)} \int\limits_0^{k(1+2k)r} \frac{\varphi(x, t)}{t} dt \leq c < \infty,$$

due to (2.14).

Let $d(x, x_0) > 2kr$.

By the triangle inequality we have $d(y, x_0) \geq \frac{1}{k}d(x, x_0) - d(y, x) \geq 2r - r \geq d(y, x)$. Since $\frac{\varphi(x,s)}{s^\nu}$ is a.d. in s, we obtain

$$\mathfrak{M}(\Phi_{x_0}; x, r) \leq C \int\limits_{B(x,r)} \frac{\varphi(x, d(y, x))}{d(y, x)^\nu} d\mu(y),$$

where the inequality (2.6) is applicable and we can proceed as in the previous case. □

Let $w(y)$ be an arbitrary weight on (X, d, μ), i.e. μ-a.e. positive function in $L^1_{loc}(X)$. We define the weighted generalized Morrey space $\mathcal{L}_{\Pi,w}^{p,\varphi}(X)$ as the space of functions with the finite norm

$$\|f\|_{\mathcal{L}_{\Pi,w}^{p,\varphi}(X)} := \|wf\|_{\mathcal{L}_{\Pi}^{p,\varphi}(X)} = \sup_{x \in X, r > 0} \left(\frac{1}{\varphi(x, r)} \int\limits_{B(x,r)} |wf(y)|^p \, d\mu(y) \right)^{\frac{1}{p}} < \infty. \tag{2.15}$$

2.3 Classes V_+ and V_- of Radial Weights

The following classes of weight functions were introduced in [30], see also [26].

Definition 2.4 By $\mathbf{V}_\pm$, we denote the classes of functions v positive on $(0, \ell)$, $0 < \ell \leq \infty$, defined by the conditions:

$$\mathbf{V}_+ : \quad \frac{|v(t) - v(\tau)|}{|t - \tau|} \leq C \frac{v(t_+)}{t_+}, \tag{2.16}$$

$$\mathbf{V}_- : \quad \frac{|v(t) - v(\tau)|}{|t - \tau|} \leq C \frac{v(t_-)}{t_+}, \tag{2.17}$$

where $t, \tau \in (0, \ell), t \neq \tau$, and $t_+ = \max(t, \tau), \ t_- = \min(t, \tau)$.

Lemma 2.5 [30] *Functions $v \in \mathbf{V}_+$ are a.i. and functions $v \in \mathbf{V}_-$ are a.d..*

Note that for power weights we have

$$t^\gamma \in \mathbf{V}_+ \iff \gamma \geq 0, \quad t^\gamma \in \mathbf{V}_- \iff \gamma \leq 0.$$

The following lemma provides sufficient conditions for functions to belong to the classes $\mathbf{V}_+$ and $\mathbf{V}_-$.

Lemma 2.6 [30,Lemma 2.11 and Example 2.12] *Let v be a function positive and differentiable on $(0, \ell)$. If*

$$0 \leq v'(t) \leq c \frac{v(t)}{t}, \quad 0 < t < \ell,$$

for some $c > 0$, then $v \in \mathbf{V}_+$. If

$$-c \frac{v(t)}{t} \leq v'(t) \leq 0, \ 0 < t < \ell,$$

for some $c > 0$, then $v \in \mathbf{V}_-$.
  In particular,

$$t^\alpha \left(\ln \frac{A}{t} \right)^\beta \in \begin{cases} \mathbf{V}_+, & \text{if } \alpha > 0, \ \beta \in \mathbb{R} \text{ or } \alpha = 0 \text{ and } \beta \leq 0 \\ \mathbf{V}_-, & \text{if } \alpha < 0, \ \beta \in \mathbb{R} \text{ or } \alpha = 0 \text{ and } \beta \geq 0, \end{cases}$$

if $A > \ell$, where it is assumed that $\ell < \infty$. In the case $\ell = \infty$, the statement holds with $\log \frac{A}{t}$ replaced by $\log e \max\{t, \frac{1}{t}\}$.

3 Main Results

3.1 Lebesgue to Morrey Transference Theorems for *p*-Admissible Operators of Singular-Type

Let T be a sublinear operator on (X, d, μ), i.e. $|T(f + g)| \leq |Tf| + |Tg|$, $f, g : X \to \mathbb{R}$.

Definition 3.1 Let $1 < p < \infty$. A sublinear operator T will be called *p-admissible singular-type operator*, if:

(1) T satisfies the size condition of the form

$$\chi_{B(x,r)}(z)|T(f\chi_{X\setminus B(x,2kr)})(z)| \leq C\chi_{B(x,r)}(z) \int_{X\setminus B(x,2kr)} \frac{|f(y)|}{d(y,z)^\nu} d\mu(y), \; x \in X, \; r > 0,$$

(3.1)

where ν comes from the growth condition (2.2) ;

(2) T is bounded in $L^p(X, d, \mu)$.

Remark 3.2 Usually the size condition is defined in the form

$$|Tf(x)| \leq C \int_X \frac{|f(y)|}{d(x,y)^\nu} d\mu(y), \; x \notin \text{supp } f,$$

(3.2)

which insures (3.1). In the main theorem of this section, i.e. in the transference of L^p-boundedness to Morrey-boundedness, the form (3.1) of the size condition is sufficient for our goals.

First of all we keep in mind singular-type operators as p-admissible operators in view of Theorem 3.3. To be precise we define the *singular operator T* as

$$Tf(x) = \lim_{\varepsilon > 0} \int_{d(x,y) > \varepsilon} K(x,y)f(y)d\mu(y),$$

(3.3)

where the kernel $K(x, y)$ satisfies the *conditions*

$$|K(x,y) - K(x,z)| \leq C\frac{d(y,z)^\sigma}{d(x,y)^{\nu+\sigma}}, \; \text{if} \; d(x,y) > 2d(y,z),$$

(3.4)

$$|K(x,y) - K(\xi,y)| \leq C\frac{d(x,\xi)^\sigma}{d(x,y)^{\nu+\sigma}}, \; \text{if} \; d(x,y) > 2d(x,\xi),$$

(3.5)

for some $\sigma > 0$, and

$$\left|\int_X K(x,y)f(y)d\mu(y)\right| \leq C \int_X \frac{|f(y)|}{d(x,y)^\nu} d\mu(y), \; x \notin supp f,$$

(3.6)

where ν comes from the growth condition (2.2). Kernels satisfying the conditions (3.4), (3.5) and (3.6) or some similar versions of these conditions, are known as *standard singular kernels*. For this notion we refer e.g. to [1], [3,p.99] and [25,p.4]. Note that the conditions (3.4) and (3.5) go back to such Lipschitz-type condition on the kernel introduced in integral form in [2,p.75].

Besides the operator (3.3), we consider the maximal singular operator

$$T^{\sharp} f(x) = \sup_{\varepsilon > 0} T_{\varepsilon} f(x), \ T_{\varepsilon} f(x) = \int_{X \backslash B(x,\varepsilon)} K(x, y) f(y) d\mu(y), \qquad (3.7)$$

assuming, as usual (see, for instance, [25]) that the operator $T^{\sharp}$ is bounded in $L^2(X)$.

The following is known.

Theorem 3.3 ([25,Theorem 1.1], see also [14] and [15] in a more general setting) *Let* (X, d, μ) *satisfy the growth condition* (2.2), $1 < p < \infty$. *The singular operator* T *and the maximal singular operator* $T^{\sharp}$ *with a standard kernel, if bounded in* $L^2(X)$, *are bounded in* $L^p(X)$.

As other examples we mention the Hardy-type operators

$$Hf(x) = \frac{1}{d(x, x_0)^{\nu}} \int_{B(x_0, d(x, x_0))} f(y) d\mu(y) \quad \text{and} \quad \mathcal{H}f(x)$$

$$= \int_{X \backslash B(x_0, d(x, x_0))} \frac{f(y)}{d(y, x_0)^{\nu}} d\mu(y) \qquad (3.8)$$

and the following "hybrids"

$$K_{\gamma,\nu} f(x) = \frac{1}{d(x, x_0)^{\gamma}} \int_{B(x_0, d(x, x_0))} \frac{f(y) d\mu(y)}{d(x, y)^{\nu - \gamma}} \text{ and } \mathcal{K}_{\gamma,\nu} f(x)$$

$$= \int_{X \backslash B(x_0, d(x, x_0))} \frac{f(y) d\mu(y)}{d(y, x_0)^{\gamma} d(x, y)^{\nu - \gamma}} \qquad (3.9)$$

of Hardy and potential operators, where $0 < \gamma \leq \nu$.

Note that $K_{\gamma,\nu}\big|_{\gamma=\nu} = H$ and $\mathcal{K}_{\gamma,\nu}\big|_{\gamma=\nu} = \mathcal{H}$.

Operators (3.9) arise in the sequel in the reduction of weighted boundedness of weighted singular operators in Morrey spaces to the boundedness of non-weighted singular operators, see Sect. 3.2. The operators $K_{\gamma,\nu}$ and $\mathcal{K}_{\gamma,\nu}$ are p-admissible operators as follows from Lemmas 3.4 and Theorem 3.5, taking Remark 3.6 into account.

Lemma 3.4 *Let* $x \in X \backslash \operatorname{supp} f$ *and* $0 < \gamma \leq \nu$. *Then the operators* $K_{\gamma,\nu}$ *and* $\mathcal{K}_{\gamma,\nu}$ *satisfy the size condition:*

$$|K_{\gamma,v}f(x)| \le (2k)^{\gamma} \int_X \frac{|f(y)|}{d(x,y)^v} d\mu(y) \text{ and } |\mathcal{K}_{\gamma,v}f(x)| \le (2k)^{\gamma} \int_X \frac{|f(y)|}{d(x,y)^v} d\mu(y)$$

$$(3.10)$$

Proof To prove (3.10) it suffices to note that $d(x,x_0) \ge \frac{1}{2k}d(x,y)$ in $K_{\gamma,v}f$ and $d(y,x_0) \ge \frac{1}{2k}d(x,y)$ in $\mathcal{K}_{\gamma,v}f$.

□

In the theorem below we use the modified maximal operator

$$M_N f(x) = \sup_{r \in (0,\frac{\ell}{N})} \frac{1}{\mu B(x,Nr)} \int_{B(x,r)} |f(y)|d\mu(y), \ x \in X, \text{ with every } N \ge 1.$$

$$(3.11)$$

We write $M := M_N\big|_{N=1}$.

The operator $K_{\gamma,v}$ is dominated by the operator M_N as shown in the next theorem.

Theorem 3.5 *Let (X,d,μ) satisfy the growth condition (2.2) and $0 < \gamma \le v$. Then*

$$|K_{\gamma,v}f(x)| \le CM_N f(x), \ x \in X, \ C = Ak^{\gamma}\frac{2^{v+\gamma}}{2^{\gamma}-1}N^v,$$

$$(3.12)$$

where A is the constant from the growth condition (2.2).

Proof By the triangle inequality we have $B(x_0, d(x,x_0)) \subset B(x, 2kd(x,x_0))$. Therefore

$$K_{\gamma,v}f(x) \le \frac{1}{d(x,x_0)^{\gamma}} \sum_{j=0}^{\infty} \int_{2^{-j}kd(x,x_0)<d(y,x)<2^{1-j}kd(x,x_0)} \frac{|f(y)|d\mu(y)}{d(y,x)^{v-\gamma}}$$

$$\le \frac{1}{d(x,x_0)^{\gamma}} \sum_{j=0}^{\infty} [2^{-j}kd(x,x_0)]^{\gamma-v} \int_{B(x,2^{1-j}kd(x,x_0))} |f(y)|d\mu(y)$$

$$\le \frac{1}{d(x,x_0)^{\gamma}} \sum_{j=0}^{\infty} [2^{-j}kd(x,x_0)]^{\gamma-v}[\mu B(x,2^{1-j}Nkd(x,x_0))]M_N f(x)$$

$$\le A(2N)^v k^{\gamma} \sum_{j=0}^{\infty} 2^{-j\gamma} M_N f(x) = CM_N f(x).$$

□

Remark 3.6 The operators $K_{\gamma,v}$ and $\mathcal{K}_{\gamma,v}$, $0 < \gamma \le v$ are p-admissible in view of Lemma 3.4 and Theorem 3.5, since

(1) the maximal operator M_N is bounded in $L^p(X)$, $1 < p \leq \infty$, if $N \geq 3k$, see [7,Proposition 6.1.1], and then the Hardy operator H and the operator $K_{\gamma,v}$, $\gamma > 0$, are bounded in $L^p(X)$ by Theorem 3.5;

(2) then the operators $\mathcal{H}$ and $\mathcal{K}_{\gamma,v}$, $\gamma > 0$, are bounded in $L^p(X)$, $1 \leq p < \infty$, by duality arguments.

Proof of Theorem 3.8 is based on the following crucial lemma.

Lemma 3.7 *Let (X, d, μ) satisfy the growth condition* (2.2), *$1 \leq p \leq \infty$, and $\in \mathbb{R}$. Then*

$$\int_{X \setminus B(x,r)} \frac{|f(y)|}{d(x, y)^v} d\mu(y) \leq C \int_r^{\ell} \frac{\|f\|_{L^p(B(x,s))} \, ds}{s^{\frac{v}{p}+1}}, \qquad (3.13)$$

where C does not depend on f, $x \in X$ and $r \in \left(0, \frac{\ell}{2}\right)$.

Proof The inequality (3.13), is proved by the known trick. We have

$$\int_{X \setminus B(x,r)} \frac{|f(y)| \, d\mu(y)}{d(x, y)^v} = \int_{X \setminus B(x,r)} \frac{|f(y)|}{d(x, y)^{v-\beta}} \frac{d\mu(y)}{d(x, y)^{\beta}},$$

where we choose $\beta > \max\left\{0, \frac{v}{p}\right\}$. It is easy to check that

$$\frac{1}{r^{\beta}} \leq c\beta \int_r^{\ell} \frac{dt}{t^{1+\beta}}, \text{ with } c = \frac{2^{\beta}}{2^{\beta}-1}$$

when $0 < r < \frac{\ell}{2}$ and $\ell < \infty$; in the case $\ell = \infty$ this holds with $c = 1$ and $''\leq''$ replaced by $''=''$. Then

$$\int_{X \setminus B(x,r)} \frac{|f(y)| \, d\mu(y)}{d(x, y)^v} \leq c\beta \int_{X \setminus B(x,r)} \frac{|f(y)|}{d(x, y)^{v-\beta}} \left(\int_{d(x,y)}^{\ell} \frac{ds}{s^{\beta+1}}\right) d\mu(y)$$

$$= c\beta \int_r^{\ell} \frac{1}{s^{\beta+1}} \left(\int_{B(x,s) \setminus B(x,r)\}} \frac{|f(y)| \, d\mu(y)}{d(x, y)^{v-\beta}}\right) ds$$

$$\leq C \int_r^{\ell} s^{-\beta-1} \|f\|_{L^p(B(x,s))} \left(\int_{B(x,s)} \frac{d\mu(y)}{d(x, y)^{(v-\beta)p'}}\right)^{\frac{1}{p'}} ds.$$

Since $(v - \beta)p' < v$, by Lemma 2.1 we then obtain

$$\int_{X \setminus B(x,r)} \frac{|f(y)| \, d\mu(y)}{d(x, y)^v} \leq C \int_r^{\infty} \frac{\|f\|_{L^p(B(x,s))} ds}{s^{1+\frac{v}{p}}}.$$

$\square$

Theorem 3.8 *Let (X, d, μ) satisfy the growth condition* (2.2) *and $\ell = diam\, X \leq \infty$, let $1 < p < \infty$ and T be a p-admissible sublinear operator of singular-type. Then*

$$\mathfrak{M}(Tf; x, r) \leq \frac{Cr^{\nu}}{\varphi(x, r)} \left(\int_r^{\ell} \frac{\varphi^{\frac{1}{p}}(x, t)}{t^{1+\frac{\nu}{p}}} \left(\mathfrak{M}(f; x, t) \right)^{\frac{1}{p}} dt \right)^p, \quad 0 < r < \frac{\ell}{2}, (3.14)$$

for every $f \in L_{loc}^p(X)$, where C does not depend on $x \in X$, $r \in (0, \ell)$ and f.

Proof We split the function f into the parts supported in a neighbourhood of the point x and outside it, in the usual way:

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{B(x,2kr)}(y), \quad f_2(y) = f(y)\chi_{X \setminus B(x,2kr)}(y) (3.15)$$

where $r > 0$, and by the sublinearity of the operator T we have

$$\|Tf\|_{L^p(B(x,r))} \leq \|Tf_1\|_{L^p(B(x,r))} + \|Tf_2\|_{L^p(B(x,r))}.$$

By the assumption 2) in Definition 3.1, we obtain

$$\|Tf_1\|_{L^p(B(x,r))} \leq \|Tf_1\|_{L^p(X)} \leq C\|f_1\|_{L^p(X)} = C\|f\|_{L^p(B(x,2kr))}. \quad (3.16)$$

To estimate Tf_2, we make use of the assumption 1) from Definition 3.1:

$$|Tf_2(z)| \leq C \int_{X \setminus B(x,2kr)} \frac{|f(y)|\, d\mu(y)}{d(y, z)^{\nu}}, \quad z \in B(x, r).$$

By the triangle inequality (2.1) it is easy to check that the conditions $z \in B(x, r)$ and $y \in X \setminus B(x, 2kr)$ imply that

$$\left(k + \frac{1}{2} \right)^{-1} d(y, z) \leq d(x, y) \leq 2kd(y, z).$$

Therefore,

$$|Tf_2(z)| \leq C \int_{X \setminus B(x,2kr)} \frac{|f(y)|\, d\mu(y)}{d(x, y)^{\nu}},$$

where the right-hand side does not depend on z, so that

$$\|Tf_2\|_{L^p(B(x,r))} \leq Cr^{\frac{\nu}{p}} \int_{X \setminus B(x,r)} \frac{|f(y)|\, d\mu(y)}{d(x, y)^{\nu}}$$

and then applying Lemma 3.7, we get

$$\|Tf_2\|_{L^p(B(x,r))} \leq Cr^{\frac{\nu}{p}} \int_r^\ell \frac{\|f\|_{L^p(B(x,s))} \, ds}{s^{1+\frac{\nu}{p}}}. \tag{3.17}$$

The simpler direct estimate (3.16) for $\|Tf_1\|_{L^p(B(x,r))}$, as can be easily seen, is dominated by the estimate of similar form:

$$\|Tf_1\|_{L^p(B(x,r))} \leq Cr^{\frac{\nu}{p}} \int_r^\ell t^{-\frac{\nu}{p}-1} \|f\|_{L^p(B(x,t))} dt \tag{3.18}$$

which yields (3.14).   □

Theorem 3.9 (Transference Theorem). *Let (X, d, μ) satisfy the growth condition (2.2) and $1 < p < \infty$. Let also*

$$C_{p,\nu}(\varphi) := \sup_{x \in \Pi, r > 0} \frac{r^{\frac{\nu}{p}}}{\varphi(x,r)^{\frac{1}{p}}} \int_r^\ell \frac{\varphi(x,t)^{\frac{1}{p}}}{t^{1+\frac{\nu}{p}}} \, dt < \infty. \tag{3.19}$$

Then any sublinear operator T satisfying the size condition (3.1), bounded in $L^p(X)$, is also bounded in $\mathcal{L}_\Pi^{p,\varphi}(X)$, and $\|Tf\|_{\mathcal{L}_\Pi^{p,\varphi}(X)} \leq c\, C_{p,\nu}(\varphi)\|f\|_{\mathcal{L}_\Pi^{p,\varphi}(X)}$, where c does not depend on φ.

Proof The proof of this theorem is prepared by the pointwise estimate of Theorem 3.8: since

$$\|f\|_{\mathcal{L}_\Pi^{p,\varphi}(X)} = \sup_{x \in \Pi, r > 0} \mathfrak{M}(f; x, t)^{\frac{1}{p}},$$

it remains to pass to supremum in (3.14).   □

Note that in [22] there was studied the boundedness of the Hardy operator H in local Morrey spaces $\mathcal{L}_{x_0}^{p,\varphi}(X)$ and local vanishing Morrey spaces under other assumptions on the function φ and the triplet (X, d, μ).

Remark 3.10 The condition (3.19) is not needed in the case where $\varphi(x, r)$ does not depend on x or if Π contains a finite number of points.

3.2 Reduction of Boundedness of Weighted Singular Integral Operators with Size Condition and the Weighted Maximal Operator to the Weighted Boundeness of Hardy Operators

In this section we consider integral operators, in general of singular type:

$$Tf(x) = \int_X K(x, y)f(y)d\mu(y) = \lim_{\varepsilon \to 0} \int_{X \setminus B(x,\varepsilon)} K(x, y)f(y)d\mu(y) \tag{3.20}$$

under the only assumption that its kernel $K(x, y)$ satisfies the size condition

$$|K(x, y)| \le c d(x, y)^{-\nu}, \tag{3.21}$$

where ν comes from the growth condition. Note that in this section in fact we even do not need to know that ν comes from the growth condition, since in the proof of the pointwise estimate in the theorem below we use only properties of weights of the classes $V_\pm$, the fact that the operator T has the size condition with some $\nu > 0$ and do not use at all any information about (X, d, μ).

Our goal is to study the boundedness of such operators in weighted Morrey spaces $\mathcal{L}^{p,\varphi}_{\Pi,w}(X)$. It is clear that the boundedness of the operator T in the weighted space $\mathcal{L}^{p,\varphi}_{\Pi,w}(X)$ is identical to the boundedness of the weighted operator

$$wT\frac{1}{w}$$

in the non-weighted space $\mathcal{L}^{p,\varphi}_{\Pi}(X)$. We shall study the operator $wT\frac{1}{w}$ with "radial" weights

$$w(y) = v[d(y, x_0)], \text{ where } v \in V_+ \text{ or } v \in V_-, \; x_0 \in X. \tag{3.22}$$

The pointwise estimate of Theorem 3.11 shows that for any Banach function space with lattice property over an arbitrary quasi-metric measure space (X, d, μ), the boundedness of the weighted operator $wT\frac{1}{w}$ with $w(y) = v[d(y, x_0)]$, $v \in V_+ \bigcup V_-$, $x_0 \in X$, is reduced to the non-weighted boundedness of the operator T, boundedness of the weighted Hardy operators $wH\frac{1}{w}$, $w\mathcal{H}\frac{1}{w}$ and non-weighted boundedness of simple operators $K_{\gamma,\nu}$ and $\mathcal{K}_{\gamma,\nu}$.

We provide also a similar reduction for the weighted maximal operator.

Theorem 3.11 (Reduction Theorem) *Let (X, d, μ) be an arbitrary quasi-metric measure space, T, $T^\sharp$ be the operators* (3.20) *and* (3.7), M_N *be the operator* (3.11) *and $w(y) = v[d(y, x_0)]$, where $v \in V_+ \bigcup V_-$, $x_0 \in X$. Then*

$$\left| wT\frac{1}{w}f(x) \right| \le |Tf(x)| + C \left(wH\frac{1}{w}(|f|)(x) + \mathcal{K}_{1,\nu}(|f|)(x) \right.$$
$$\left. + \sum_{m=1}^{\bar\nu-1} K_{m,\nu}(|f|)(x) \right), \quad v \in \mathbf{V}_+ \tag{3.23}$$

and

$$\left| wT\frac{1}{w}f(x) \right| \le |Tf(x)| + C \left(w\mathcal{H}\frac{1}{w}(|f|)(x) + K_{1,\nu}(|f|)(x) \right.$$
$$\left. + \sum_{m=1}^{\bar\nu-1} \mathcal{K}_{m,\nu}(|f|)(x) \right), \quad v \in \mathbf{V}_-; \tag{3.24}$$

$$\left| wT^{\sharp} \frac{1}{w} f(x) \right| \le |T^{\sharp} f(x)| + C \left(wH \frac{1}{w}(|f|)(x) + \mathcal{K}_{1,\nu}(|f|)(x) \right.$$

$$\left. + \sum_{m=1}^{\bar{\nu}-1} K_{m,\nu}(|f|)(x) \right), \quad \nu \in \mathbf{V}_{+} \tag{3.25}$$

and

$$\left| wT^{\sharp} \frac{1}{w} f(x) \right| \le |T^{\sharp} f(x)| + C \left(w\mathcal{H} \frac{1}{w}(|f|)(x) + K_{1,\nu}(|f|)(x) \right.$$

$$\left. + \sum_{m=1}^{\bar{\nu}-1} \mathcal{K}_{m,\nu}(|f|)(x) \right), \quad \nu \in \mathbf{V}_{-}, \tag{3.26}$$

where $\bar{\nu}$ is the least integer greater or equal to ν and the sum $\sum_{m=1}^{\bar{\nu}-1}$ should be omitted in the case $\bar{\nu} = 1$.

If (X, d, μ) satisfies the condition

$$\mu B(x, r) \ge C r^{\alpha} \tag{3.27}$$

for some $C > 0$ and $\alpha > 0$, then

$$\left| wM_N \frac{1}{w} f(x) \right| \le |M_N f(x)| + C \left(wH \frac{1}{w}(|f|)(x) + \mathcal{K}_{1,\nu}(|f|)(x) \right.$$

$$\left. + \sum_{m=1}^{\bar{\alpha}-1} K_{m,\alpha}(|f|)(x) \right), \tag{3.28}$$

when $\nu \in \mathbf{V}_{+}$ and

$$\left| wM_N \frac{1}{w} f(x) \right| \le |M_N f(x)| + C \left(w\mathcal{H} \frac{1}{w}(|f|)(x) + K_{1,\nu}(|f|)(x) \right.$$

$$\left. + \sum_{m=1}^{\bar{\alpha}-1} \mathcal{K}_{m,\alpha}(|f|)(x) \right), \tag{3.29}$$

when $\nu \in \mathbf{V}_{-}$, where $\bar{\alpha}$ is the least integer greater or equal to α and the sum $\sum_{m=1}^{\bar{\alpha}-1}$ should be omitted in the case $\bar{\alpha} = 1$.

Proof We assume that $f(x) > 0$, $x \in X$, without loss of generality. By the size condition we have

$$\left| wT \frac{1}{w} f(x) - Tf(x) \right| \le C \int_{X} \frac{|w(x) - w(y)|}{w(y)} \frac{f(y)}{d(x, y)^{\nu}} d\mu(y). \tag{3.30}$$

We shall prove the estimate (3.23), the proof of (3.24) being similar.
For brevity we denote

$$d_x = d(x, x_0) \text{ and } d_y = d(y, x_0).$$

Let $X_+ = \{y \in X : d_y \leq d_x\}$ and $X_- = \{y \in X : d_y \geq d_x\}$. By the definition of the class V_+ and the triangle inequality we have

$$\left| wT\frac{1}{w}f(x) - Tf(x) \right| \leq C \left(\frac{v(d_x)}{d_x} \int_{X_+} \frac{f(y)}{v(d_y)} \frac{d\mu(y)}{d(x,y)^{\nu-1}} + \int_{X_-} \frac{f(y)}{d_y} \frac{d\mu(y)}{d(x,y)^{\nu-1}} \right).$$
(3.31)

Let first $\nu \leq 1$. Then

$$d(x,y)^{1-\nu} \leq (2kd_x)^{1-\nu} \text{ in the first integral and } d(x,y)^{1-\nu} \leq (2kd_y)^{1-\nu}$$
in the second integral. (3.32)

Consequently,

$$\left| wT\frac{1}{w}f(x) - Tf(x) \right| \leq C \left(wH\frac{1}{w}f(x) + \mathcal{K}_{1,\nu}f(x) \right),$$

which provides (3.23).

Let $\nu > 1$. From (3.31) we have

$$\left| wT\frac{1}{w}f(x) - Tf(x) \right| \leq C \left(\frac{1}{d_x} \int_{X_+} \frac{v(d_x) - v(d_y)}{v(d_y)} \frac{f(y)d\mu(y)}{d(x,y)^{\nu-1}} \right.$$
$$\left. + K_{1,\nu}f(x) + \mathcal{K}_{1,\nu}f(x) \right).$$

$$\left| wT\frac{1}{w}f(x) - Tf(x) \right| \leq C \left(\frac{v(d_x)}{d_x^2} \int_{X_+} \frac{1}{v(d_y)} \frac{f(y)d\mu(y)}{d(x,y)^{\nu-2}} \right.$$
$$\left. + K_{1,\nu}f(x) + \mathcal{K}_{1,\nu}f(x) \right).$$

If $\nu \leq 2$, we estimate $d(x,y)^{2-\nu}$ similarly to (3.32) and get

$$\left| wT\frac{1}{w}f(x) - Tf(x) \right| \leq C \left(wH\frac{1}{w}f(x) + K_{2,\nu}f(x) + K_{1,\nu}f(x) + \mathcal{K}_{1,\nu}f(x) \right).$$

Iterating this procedure, we arrive at (3.23) with $\sum_{m=1}^{\bar{v}} K_{m,v}f$, but the last term $K_{m,v}f\big|_{m=\bar{v}}$ can be eliminated, since $K_{\bar{v}}f = H$ and

$$Hf(x) \le C \, wH\frac{1}{w}f(x) \text{ when } v \in V_+,$$

because $v \in V_+$ is a.i. by Lemma 2.5.

For the maximal singular operator $T^\sharp$, using the size condition (3.21), we have

$$\left| wT^\sharp\frac{1}{w}f(x) - T^\sharp f(x) \right| = \sup_{\varepsilon>0} \left| \int_{X\setminus B(x,\varepsilon)} \frac{w(x)-w(y)}{w(y)} K(x,y)f(y)d\mu(y) \right|$$
$$\le \int_X \frac{|w(x)-w(y)|}{w(y)} \frac{|f(y)|}{d(x,y)^v} d\mu(y),$$

after which the estimation is the same as after (3.30). Note that this estimation uses only properties of the distance but not of the measure.

For the maximal operator M_N we have

$$|wM_N\frac{1}{w}f(x) - M_N f(x)|$$

$$= \left| \sup_{r>0} \frac{1}{\mu(B(x,Nr))} \int_{B(x,r)} \frac{w(x)}{w(y)}|f(y)|d\mu(y) - \sup_{r>0} \frac{1}{\mu(B(x,Nr))} \int_{B(x,r)} |f(y)|d\mu(y) \right|$$

$$\le \sup_{r>0} \frac{1}{\mu(B(x,Nr))} \int_{B(x,r)} \frac{|w(x)-w(y)|}{w(y)}|f(y)|d\mu(y).$$

By (3.27) we then have

$$|wM_N\frac{1}{w}f(x) - M_N f(x)| \le C \int_X \frac{|w(x)-w(y)|}{w(y)d(x,y)^\alpha}|f(y)|d\mu(y),$$

after which the estimation is also the same as after (3.30). □

3.3 Weighted Boundedness of Hardy, Singular and Maximal Operators in Generalized Morrey Spaces $\mathcal{L}_\Pi^{p,\varphi}(X)$

According to the reduction procedure of the Sect. 3.2, we have to study the boundedness of the operators $wH\frac{1}{w}$ and $w\mathcal{H}\frac{1}{w}$ which we do in the following section.

3.3.1 Hardy Operators with Quasi-monotone Weights in the Spaces $\mathcal{L}_{\Pi}^{p,\varphi}(X)$

In the following theorem we use the notation

$$\Psi_{x_0}(x) = \frac{v(d(x,x_0))}{d(x,x_0)^{\nu}} \int\limits_0^{d(x,x_0)} \frac{t^{\frac{\nu}{p'}-1}\varphi(x,t)^{\frac{1}{p}}}{v(t)} dt.$$

Theorem 3.12 *Let* (X,d,μ) *satisfy the growth condition* (2.2), $1 \le p < \infty$ *and* $w(x) = v[d(x,x_0)]$, $x_0 \in X$, *where* $v \in V_+$. *If the function* $\frac{t^{\frac{\nu}{p'}}\varphi(x,t)^{\frac{1}{p}}}{v(t)}$ *is a.i. in* $t \in (0,\ell)$ *uniformly in* $x \in \Pi$, *then there holds the pointwise estimate*

$$\left| wH\frac{f}{w}(x) \right| \le c_w \Psi_{x_0}(x) \|f\|_{\mathcal{L}_{\Pi}^{p,\varphi}(X)}, x \in X, \tag{3.33}$$

where c_w *does not depend on* x *and* f.

Proof We have

$$wH\frac{f}{w}(x) = \frac{v(d_x)}{d_x^{\nu}} \int\limits_{B(x_0,d_x)} \frac{f(y)d\mu(y)}{v(d_y)} = \frac{v(d_x)}{d_x^{\nu}} \sum_{k=0}^{\infty} \int\limits_{2^{-k-1}d_x < d_y < 2^{-k}d_x} \frac{f(y)d\mu(y)}{v(d_y)},$$

where the notation $d_x = d(x,x_0)$, $d_y = d(y,x_0)$ is used. The function v is a.i. by Lemma 2.5. Consequently,

$$\left| wH\frac{f}{w}(x) \right| \le c\frac{v(d_x)}{d_x^{\nu}} \sum_{k=0}^{\infty} \frac{1}{v(2^{-k-1}d_x)} \|f\|_{L^p(B(x_0,2^{-k}d_x))} \mu B(x_0,2^{-k}d_x))^{\frac{1}{p'}}.$$

In view of the growth condition we then get

$$\left| wH\frac{f}{w}(x) \right| \le c\frac{v(d_x)}{d_x^{\nu}} \sum_{k=0}^{\infty} \frac{(2^{-k}d_x)^{\frac{\nu}{p'}}\varphi(x,2^{-k}d_x)^{\frac{1}{p}}}{v(2^{-k-1}d_x)} \cdot \frac{1}{\varphi(x,2^{-k}d_x)^{\frac{1}{p}}} \|f\|_{L^p(B(x_0,2^{-k}d_x))}$$

$$\le c\frac{v(d_x)}{d_x^{\nu}} \|f\|_{\mathcal{L}_{\Pi}^{p,\varphi}(X)} \sum_{k=0}^{\infty} \frac{\varphi(x,2^{-k}d_x)^{\frac{1}{p}}}{v(2^{-k-1}d_x)} (2^{-k}d_x)^{\frac{\nu}{p'}}.$$

From (2.12) it follows that $\varphi(x,r)$ has doubling property in r uniformly in x, so that

$$\sum_{k=0}^{\infty} \frac{\varphi(x,2^{-k}d_x)^{\frac{1}{p}}}{v(2^{-k-1}d_x)} (2^{-k}d_x)^{\frac{\nu}{p'}} \le C\sum_{k=0}^{\infty} \frac{\varphi(x,2^{-k-1}d_x)^{\frac{1}{p}}}{v(2^{-k-1}d_x)} (2^{-k}d_x)^{\frac{\nu}{p'}}$$

$$\le c \int\limits_0^{d_x} \frac{t^{\frac{\nu}{p'}-1}\varphi(x,\frac{t}{2})^{\frac{1}{p}}}{v(\frac{t}{2})} dt \le c \int\limits_0^{d_x} \frac{t^{\frac{\nu}{p'}-1}\varphi(x,t)^{\frac{1}{p}}}{v(t)} dt.$$

Then

$$\left| wH\frac{f}{w}(x) \right| \le c\,\frac{v(d_x)}{d_x^v} \int\limits_0^{d_x} \frac{t^{\frac{v}{p'}-1}\varphi(x,t)^{\frac{1}{p}}}{v(t)}dt \cdot \|f\|_{\mathcal{L}_\Pi^{p,\varphi}(X)},$$

which proves (3.33). □

We introduce the following Zygmund-type condition

$$\int\limits_0^r \frac{t^{\frac{v}{p'}}\varphi(x,t)^{\frac{1}{p}}}{v(t)}\frac{dt}{t} \le C\frac{r^{\frac{v}{p'}}\varphi(x,r)^{\frac{1}{p}}}{v(r)},\ 0<r<\ell, \tag{3.34}$$

where C does not depend on $x \in X$ and $r \in (0,\ell)$.

Theorem 3.12 leads to the following result.

Theorem 3.13 *Let* (X,d,μ) *satisfy the growth condition* (2.2)*,* $1 \le p < \infty$ *and* $w(x) = v[d(x,x_0)]$*,* $x_0 \in X$*, where* $v \in V_+$*, and let the function* $\frac{t^{\frac{v}{p'}}\varphi(x,t)^{\frac{1}{p}}}{v(t)}$ *be a.i. in* $t \in (0,\ell)$ *uniformly in* $x \in \Pi$*. If the Zygmund conditions* (2.14) *and* (3.34) *hold, then the weighted Hardy operator* $wH\frac{1}{w}$ *is bounded in the space* $\mathcal{L}_\Pi^{p,\varphi}(X)$ *and*

$$\left\| wH\frac{f}{w} \right\|_{\mathcal{L}_\Pi^{p,\varphi}(X)} \le c\|\Phi_{x_0}\|_{\mathcal{L}_\Pi^{p,\varphi}(X)}\|f\|_{\mathcal{L}_\Pi^{p,\varphi}(X)}. \tag{3.35}$$

Proof The condition (3.34) says that $\Psi_{x_0}(x) \le C\ \Phi_{x_0}(x)$, so that $\left| wH\frac{f}{w} \right| \le c\ \Phi_{x_0}\|f\|_{\mathcal{L}_\Pi^{p,\varphi}(X)}$. The condition (2.14) and Theorem 2.3 imply that $\Phi \in \mathcal{L}_\Pi^{p,\varphi}(X)$, which proves the theorem. □

In the sequel we use the notion of Matuszewska-Orlicz indices (see [31,Appendix]). For reader's convenience we provide necessary definitions in Appendix.

Remark 3.14 In Theorem 3.13 two Zygmund conditions (2.14) and (3.34) are supposed to hold. It is natural to compare them. First, we note that (2.14) is equivalent to

$$\int\limits_0^r \frac{\varphi(x,t)^{\frac{1}{p}}}{t}dt \le C\varphi(x,r)^{\frac{1}{p}},$$

since (2.14) for a.i. functions φ holds if and only if its lower Matuszewska-Orlicz index $m(\varphi)$ in the variable t is positive (uniformly in x in our case), see [16], and $m(\varphi^{\frac{1}{p}}) = \frac{1}{p}m(\varphi)$. Then it is easy to see that (2.14) implies (3.34) when the function $\frac{r^{\frac{v}{p'}}}{v(r)}$ is a.i. and (3.34) implies (2.14) if this function is a.d.. In terms of Matuszewska-Orlicz indices of the weight $v(r)$, when v is quasi-monotone near the origin and at

infinity (the latter if $\ell = \infty$), in view of Lemma 4.1, the following is true:

$$(2.14) \implies (3.34) \quad \text{when} \quad \max\{M_0(v), \ M_\infty(v)\} < \frac{v}{p'}$$

$$(3.34) \implies (2.14) \quad \text{when} \quad \min\{m_0(v), \ m_\infty(v)\} > \frac{v}{p'}$$

From Theorems 3.12 and 3.13, by means of Lemma 4.2 we arrive at the following corollary in terms of Matuszewska-Orlicz indices. We take $\ell = \infty$ in this corollary; if $\ell < \infty$ the information about m_∞ should be omitted.

Corollary 3.15 *Let the assumptions of Theorem 3.12 be satisfied and let v be quasi-monotone near the origin and at infinity (the latter if $\ell = \infty$). If*

$$\inf_{x \in \Pi} \min \left\{ m_0 \left(\frac{\varphi(x, \cdot)^{\frac{1}{p}}}{v(\cdot)} \right), m_\infty \left(\frac{\varphi(x, \cdot)^{\frac{1}{p}}}{v(\cdot)} \right) \right\} > -\frac{v}{p'}, \tag{3.36}$$

then $\left| wH\frac{f}{w}(x) \right| \leq c \ \Phi_{x_0}(x) \|f\|_{\mathcal{L}_\Pi^{p,\varphi}(X)}$, $x \in X$, *holds. If also* $\inf_{x \in \Pi}$ $\min\{m_0(\varphi(\cdot)), m_\infty(\varphi(\cdot))\} > 0$, *then the operator* $wH\frac{1}{w}$ *is bounded in the space* $\mathcal{L}_\Pi^{p,\varphi}(X)$.

In particular, in the case $\varphi(x, r) = r^\lambda$ and $v(r) = r^\gamma$, the condition (3.36) takes the form $\gamma < \frac{\lambda}{p} + \frac{v}{p'}$.

Corollary 3.16 *The operator H is bounded in the Morrey space $\mathcal{L}_\Pi^{p,\varphi}(X)$ if (2.14) holds.*

Proof It suffices to observe that (2.14) implies (3.34), when $w(x) \equiv 1$. $\qquad\square$

For the operator $w\mathcal{H}\frac{1}{w}$ the following result holds.

Theorem 3.17 *Let (X, d, μ) satisfy the growth condition (2.2), $1 \leq p < \infty$ and $w(x) = v[d(x, x_0)]$, $x_0 \in X$, where $v \in V_-$, the function $\frac{\varphi(x,t)^{\frac{1}{p}}}{t^{\frac{v}{p}} v(t)}$ be a.d. in $t \in (0, \ell)$ uniformly in $x \in \Pi$, and (2.14) hold. If*

$$\int_r^\ell \frac{\varphi(x, t)^{\frac{1}{p}}}{t^{\frac{v}{p}+1} v(t)} dt \leq C \frac{\varphi(x, r)^{\frac{1}{p}}}{r^{\frac{v}{p}} v(r)}, \tag{3.37}$$

then

$$\left| w\mathcal{H}\frac{f}{w}(x) \right| \leq c \ \Phi_{x_0}(x) \|f\|_{\mathcal{L}_\Pi^{p,\varphi}(X)}, x \in X, \tag{3.38}$$

where c does not depend on x and f, and the operator $w\mathcal{H}\frac{1}{w}$ is bounded in the space $\mathcal{L}_\Pi^{p,\varphi}(X)$.

The proof of Theorem 3.17 follows the same lines as in the proof of Theorems 3.12 and 3.13 and is omitted.

From Theorem 3.17, by means of Lemma 4.2 we arrive at the following corollary in terms of Matuszewska-Orlicz indices. We take $\ell = \infty$ in this corollary; if $\ell < \infty$ the information about m_∞ should be omitted.

Corollary 3.18 *Let the assumptions of Theorem 3.17 be satisfied and let v be quasi-monotone near the origin and at infinity (the latter if $\ell = \infty$). If*

$$\sup_{x \in \Pi} \max \left\{ M_0 \left(\frac{\varphi(x, \cdot)^{\frac{1}{p}}}{v(\cdot)} \right), M_\infty \left(\frac{\varphi(x, \cdot)^{\frac{1}{p}}}{v(\cdot)} \right) \right\} < \frac{v}{p}, \qquad (3.39)$$

then $\left| w \mathcal{H} \frac{f}{w}(x) \right| \leq c \ \Phi_{x_0}(x) \| f \|_{\mathcal{L}_\Pi^{p,\varphi}(X)}, \ x \in X,$ *holds. If also* $\inf_{x \in \Pi} \min\{m_0(\varphi(\cdot)),$ $m_\infty(\varphi(\cdot))\} > 0,$ *then the operator* $w \mathcal{H} \frac{1}{w}$ *is bounded in the space* $\mathcal{L}_\Pi^{p,\varphi}(X).$

In particular, in the case $\varphi(x, r) = r^\lambda$ and $v(r) = r^\gamma$, the condition (3.39) takes the form $\gamma > \frac{\lambda - v}{p}$.

Corollary 3.19 *The operator $\mathcal{H}$ is bounded in the Morrey space $\mathcal{L}_\Pi^{p,\varphi}(X)$ if (2.14) holds and*

$$\int_r^\ell \frac{\varphi(x, t)^{\frac{1}{p}}}{t^{\frac{v}{p}+1}} dt \leq C \frac{\varphi(x, r)^{\frac{1}{p}}}{r^{\frac{v}{p}}},$$

where C does not depend on $(x, r) \in \Pi \times (0, \ell).$

3.3.2 Boundedness of Weighted Singular and Maximal Operators

In the following theorem T is the singular operator (3.3) with a standard kernel, bounded in $L^2(X)$, $T^\sharp$ is the maximal singular operator (3.7) and M_N is the modified maximal operator (3.11). We use the notation

$$\phi_v(x, r) := \frac{\varphi(x, r)^{\frac{1}{p}}}{v(r)}$$

Theorem 3.20 *Let (X, d, μ) satisfy the growth condition (2.2), $\ell = \operatorname{diam} X \leq \infty$, $1 < p < \infty$, and let $\varphi(x, t)$ satisfy the assumptions (2.14) and (3.19). Then the weighted singular operator $wT \frac{1}{w}$ and the weighted maximal singular operator $wT^\sharp \frac{1}{w}$ with the weight $w(x) = v([d(x, x_0)]), \ x_0 \in X$, where $v \in V_+ \bigcup V_-$, are bounded in the Morrey space $\mathcal{L}_\Pi^{p,\varphi}(X)$, if the following conditions on the weight hold:*
(1) $r^{\frac{v}{p'}} \phi_v(x, r)$ is a.i. in t uniformly in $x \in \Pi$ and

$$\int_0^r t^{\frac{v}{p'}-1} \phi_v(x, t) dt \leq cr^{\frac{v}{p'}} \phi_v(x, r), \ (x, r) \in \Pi \times (0, \ell), \qquad (3.40)$$

when $v \in V_+$, and

(2) $\frac{\phi_v(x,r)}{r^{\frac{v}{p}}}$ is a.d. in t uniformly in $x \in \Pi$ and

$$\int\limits_r^\ell \frac{\phi_v(x,t)}{t^{\frac{v}{p}+1}} dt \le c \frac{\phi_v(x,r)}{r^{\frac{v}{p}}}, \quad (x,r) \in \Pi \times (0,\ell), \tag{3.41}$$

when $v \in V_-$; where c does not depend on $(x,r) \in \Pi \times (0,\ell)$.

If (X,d,μ) is regular in the sense (2.3) then the maximal operator M is bounded in the Morrey space $\mathcal{L}_\Pi^{p,\varphi}(X)$ under the same conditions.

Proof We first consider the weighted operators $wT\frac{1}{w}$ and $wT^\sharp \frac{1}{w}$. According to the pointwise estimates (3.23)–(3.24) and (3.25)–(3.26), respectively, of Theorem 3.11 we have to insure the boundedness in the space $\mathcal{L}_\Pi^{p,\varphi}(X)$ of the operators T and $T^\sharp$ themselves, the operator $wH\frac{1}{w}$ when $w \in V_+$ and the operator $w\mathcal{H}\frac{1}{w}$ when $w \in V_-$ and also the operators $K_{m,v}$ and $\mathcal{K}_{m,v}$.

The non-weighted operators T and $T^\sharp$ are bounded in $L^p(X)$ by Theorem 3.3. For the boundedness of the operators $K_{m,v}$ and $\mathcal{K}_{m,v}$ see Remark 3.6. Then these operators are bounded in the Morrey space $\mathcal{L}_\Pi^{p,\varphi}(X)$ by Theorem 3.9 under the condition(3.19).

Finally, taking into account conditions of Theorems 3.13 and 3.17 for the boundedness of the weighted Hardy operators and gathering all the assumptions of Theorems 3.9, 3.13 and 3.17, we arrive at the statement of the theorem for the operators $wT\frac{1}{w}$ and $wT^\sharp \frac{1}{w}$.

When (X,d,μ) is regular, the arguments for the maximal operator M are the same in view of (3.28) and (3.29), since $\alpha = v$. We also take into account that M is p-admissible, being bounded in $L^p(X)$ in the case of regular metric space. $\quad\square$

For the case of classical Morrey spaces, i.e. $\varphi(r) \equiv r^\lambda$, $\lambda > 0$, and the weight $v \in V_+ \bigcup V_-$ is quasi-monotone near the origin and at infinity (the latter if $\ell = \infty$), we obtain the following corollary.

Corollary 3.21 *Let (X,d,μ) satisfy the growth condition (2.2), $\ell = $ diam $X \le \infty$, $1 < p < \infty$, and let T and $T^\sharp$ be the singular and maximal singular operators (3.20) and (3.7). The weighted operators $wT\frac{1}{w}$ and $wT^\sharp \frac{1}{w}$ are bounded in the Morrey space $\mathcal{L}_\Pi^{p,\varphi}\big|_{\varphi(r)\equiv r^\lambda}$, $0 < \lambda < v$, under the condition*

$$-\frac{v}{p} + \frac{\lambda}{p} < \min\{m_0(v), m_\infty(v)\} \le \max\{M_0(v), M_\infty(v)\} < \frac{v}{p'} + \frac{\lambda}{p}. \tag{3.42}$$

The same is true for the maximal operator M in the case when (X,d,μ) is regular.

Appendix: Matuszewska–Orlicz (MO)-Type Indices

We provide definitions of Matuszewska-Orlicz indices of positive quasi-monotone functions functions on $(0, \ell)$, $0 < \ell \leq \infty$, and some their properties. For more details we refer for instance to [16, 29] and [31, Appendix].

A function $v(t)$ positive on $(0, \ell)$ is called quasi-monotone near the origin if there exist numbers $\alpha, \beta \in \mathbb{R}$ such that $\frac{v(t)}{t^\alpha}$ is a.i. and $\frac{v(t)}{t^\beta}$ is a.d. in a neighborhood of the origin. In the case $\ell = \infty$ it is called quasi-monotone at infinity if there exist $a, b \in \mathbb{R}$ such that $\frac{v(t)}{t^a}$ is a.i. and $\frac{v(t)}{t^b}$ is a.d. in a neighborhood of infinity.

Functions quasi-monotone at the origin and infinity have finite Matuszewska-Orlicz indices at the origin and infinity, respectively. These indices are defined as follows:

$$m_0(v) = \sup_{x>1} \frac{\ln\left(\liminf_{h\to 0} \frac{v(hx)}{v(h)}\right)}{\ln x} = \sup_{0<x<1} \frac{\ln\left(\limsup_{h\to 0} \frac{v(hx)}{v(h)}\right)}{\ln x}$$

$$= \lim_{x\to 0} \frac{\ln\left(\limsup_{h\to 0} \frac{v(hx)}{v(h)}\right)}{\ln x}, \tag{4.1}$$

$$M_0(v) = \inf_{x>1} \frac{\ln\left(\limsup_{h\to 0} \frac{v(hx)}{v(h)}\right)}{\ln x} = \lim_{x\to\infty} \frac{\ln\left(\limsup_{h\to 0} \frac{v(hx)}{v(h)}\right)}{\ln x} \tag{4.2}$$

and

$$m_\infty(v) = \sup_{x>1} \frac{\ln\left[\liminf_{h\to\infty} \frac{v(xh)}{v(h)}\right]}{\ln x} , \quad M_\infty(v) = \inf_{x>1} \frac{\ln\left[\limsup_{h\to\infty} \frac{v(xh)}{v(h)}\right]}{\ln x}. \tag{4.3}$$

Some properties of the indices are given in the following lemmas.

Lemma 4.1 *If v is quasi-monotone near the origin, then*

$$m_0(v) = \sup\left\{\alpha > 0 : \frac{v(x)}{x^\alpha} \text{ is a.i.}\right\} \text{ and } M_0(v) = \inf\left\{\beta > 0 : \frac{v(x)}{x^\beta} \text{ is a.d.}\right\}.$$

If v is quasi-monotone at infinity, then

$$m_\infty(v) = \sup \left\{ a > 0 : \ \frac{v(x)}{x^a} \text{ is a.i.} \right\} \text{ and } M_\infty(v) = \inf \left\{ b > 0 : \ \frac{v(x)}{x^b} \text{ is a.d.} \right\}.$$

Lemma 4.2 *Let a function v positive on $(0, \ell)$ be quasi-monotone near the origin and infinity (the latter in the case $\ell = \infty$). The inequalities*

$$\int_0^x \frac{v(t)}{t^{1+\gamma}} dt \le c \frac{v(x)}{x^\gamma} \text{ and } \int_x^\ell \frac{v(t)}{t^{1+\delta}} dt \le c \frac{v(x)}{x^\delta}, \ \gamma, \delta \in \mathbb{R}$$

are equivalent to the conditions $m_0(v) > \gamma$ and $M_0(v) < \delta$, respectively, when $\ell < \infty$; and to the conditions $\min\{m_0(v), m_\infty(v)\} > \gamma$ and $\max\{M_0(v), M_\infty(v)\} < \delta$, when $\ell = \infty$.

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