Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Refinements of some classical inequalities via superquadraticity

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Abstract

Some new refined versions of the Jensen, Minkowski, and Hardy inequalities are stated and proved. In particular, these results both generalize and unify several results of this type. Some results are also new for the classical situation.

MSC: Primary 26D10; secondary 26D15; 39B62; 46E27; 26A51; 26B25

Keywords: Inequalities; Refinements; Superquadratic function; Jensen–; Minkowski-; Beckenbach–Dresher-; Hardy-type inequalities; Banach function spaces

1 Introduction

Classical inequalities are of great importance for the development of several areas both within the mathematical sciences and beyond. Hence, it is not surprising that the area "Inequalities" has been developed to an independent area of increasing interest. Several wonderful generalizations, sharpening, and applications have been presented. In particular, fairly lately even refinements of these inequalities have been derived. See e.g. [1–4, 12, 14, 20] and the references given there.

In this paper we derive new such refinements of some classical inequalities. More exactly, the main content of this paper is as follows:

In Sect. 2 some new refinements of Jensen's inequality can be found (see Theorems 2.3 and 2.4). In Sect. 3 we state, prove, and apply some new refinements of the Minkowski inequality and new Beckenbach–Dresher type inequality (see Theorems 3.3 and 3.6). In Sect. 4 we derive some corresponding refinements of Hardy's inequality even in a Banach function space setting. Finally, in Sect. 5 we give some concluding remarks and results, which in particular put our results to a more general context.

In all our results we use the concept of superquadratic function, so we finish this section with the following crucial information.

Definition 1.1 (See [2, Definition 2.1]) A function $\varphi : [0,\infty) \to \mathbf{R}$ is superquadratic provided that for all $x \geq 0$ there exists a constant $C_x \in \mathbf{R}$ such that

$$\varphi(y) - \varphi(x) - \varphi\big(|y-x|\big) \geq C_x(y-x)$$

for all $y \geq 0$.

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We say that f is subquadratic if $-f$ is superquadratic.

We cite the following result, which is very useful in the proofs of our main results (see [2, 3], and [21] for further details).

Theorem 1.2 (See [2, Theorem 2.3]) *Let (Ω, μ) be a probability measure space. The inequality*

$$\begin{aligned}&\varphi\left(\int_\Omega f(s)\,\mathrm{d}\mu(s)\right)\\&\quad\le \int_\Omega \varphi\big(f(s)\big)\,\mathrm{d}\mu(s) - \int_\Omega \varphi\left(\left|f(s) - \int_\Omega f(s)\,\mathrm{d}\mu(s)\right|\right)\mathrm{d}\mu(s)\end{aligned} \tag{1}$$

holds for all probability measures μ and all nonnegative μ-integrable functions f if and only if φ is superquadratic. Moreover, (1) holds in the reversed direction if and only if φ is subquadratic.

If φ is a nonnegative superquadratic function, then φ is convex (see [2, Lemma 2.2]) and inequality (1) is a refinement of the Jensen inequality for a convex function which states

$$\varphi\left(\int_\Omega f(s)\,\mathrm{d}\mu(s)\right) \le \int_\Omega \varphi\big(f(s)\big)\,\mathrm{d}\mu(s).$$

Convention. Throughout this paper we assume that f is a measurable function on the considered measure space.

2 Refinements of Jensen's inequality

We need the following useful special case of Theorem 1.2.

Lemma 2.1 *Let φ be a superquadratic function, and let t be a nonnegative measurable function such that $T = \int_\Omega t(s)\,\mathrm{d}s$. The inequality*

$$\varphi(\overline{f}) \le \frac{1}{T}\int_\Omega t(s)\varphi\big(f(s)\big)\,\mathrm{d}s - \frac{1}{T}\int_\Omega t(s)\varphi\big(\big|f(s) - \overline{f}\big|\big)\,\mathrm{d}s \tag{2}$$

holds for all nonnegative functions f, where $\overline{f} = \frac{1}{T}\int_\Omega t(s)f(s)\,\mathrm{d}s$. Moreover, (2) holds in the reversed direction if φ is subquadratic.

Proof Set $\mathrm{d}\mu(s) = \frac{t(s)}{T}\,\mathrm{d}s$. Then (2) follows from (1). The proof is complete. □

Example 2.2 Let φ be a superquadratic function, let x_1, x_2 be two nonnegative real numbers and $\lambda \in [0,1]$. Then

$$\begin{aligned}&\varphi\big(\lambda x_1 + (1-\lambda)x_2\big)\\&\quad\le \lambda\varphi(x_1) + (1-\lambda)\varphi(x_2) - \lambda\varphi\big((1-\lambda)|x_1 - x_2|\big) - (1-\lambda)\varphi\big(\lambda|x_1 - x_2|\big).\end{aligned} \tag{3}$$

Moreover, (3) holds in the reversed direction if φ is subquadratic.

In fact, by taking $\Omega = [0,1]$, $t(s) = 1$, and

$$f(s) = \begin{cases} x_1 : s \in [0,\lambda], \\ x_2 : s \in (\lambda,1], \end{cases}$$

we see that (3) follows from (2).

Consider the nonnegative measurable functions α and β satisfying

$$\alpha(s) + \beta(s) = 1 \quad \text{for all } s \in \Omega.$$

Denote

$$T = \int_\Omega t(s)\,\mathrm{d}s, \qquad Q = \int_\Omega \alpha(s)t(s)\,\mathrm{d}s, \qquad R = \int_\Omega \beta(s)t(s)\,\mathrm{d}s.$$

Our first main result in this section reads as follows.

Theorem 2.3 *Let $\varphi : [0,\infty) \to \mathbf{R}$ be a superquadratic function, and let f be a nonnegative and measurable function. Then the following refined variant of Jensen type inequality*

$$\begin{aligned} \varphi(\overline{f}) &\le \frac{Q}{T}\varphi(\overline{f}_Q) + \frac{R}{T}\varphi(\overline{f}_R) - \frac{Q}{T}\varphi\left(\frac{R}{T}|\overline{f}_Q - \overline{f}_R|\right) - \frac{R}{T}\varphi\left(\frac{Q}{T}|\overline{f}_Q - \overline{f}_R|\right) \\ &= \frac{Q}{T}\varphi(\overline{f}_Q) + \frac{R}{T}\varphi(\overline{f}_R) - \frac{Q}{T}\varphi\big(|\overline{f}_Q - \overline{f}|\big) - \frac{R}{T}\varphi\big(|\overline{f}_R - \overline{f}|\big), \end{aligned} \tag{4}$$

holds, where

$$\overline{f} = \frac{1}{T}\int_\Omega t(s)f(s)\,\mathrm{d}s, \qquad \overline{f}_Q = \frac{1}{Q}\int_\Omega \alpha(s)t(s)f(s)\,\mathrm{d}s, \qquad \overline{f}_R = \frac{1}{R}\int_\Omega \beta(s)t(s)f(s)\,\mathrm{d}s.$$

Moreover, (4) *holds in the reversed direction if φ is subquadratic.*

Proof Set $x_1 = \overline{f}_Q$, $x_2 = \overline{f}_R$, and $\lambda = \frac{Q}{T}$. It is clear that

$$1 - \lambda = \frac{R}{T} \quad \text{and} \quad \lambda x_1 + (1-\lambda)x_2 = \overline{f}.$$

Then from Example 2.2 it follows that

$$\varphi(\overline{f}) \le \frac{Q}{T}\varphi(\overline{f}_Q) + \frac{R}{T}\varphi(\overline{f}_R) - \frac{Q}{T}\varphi\left(\frac{R}{T}|\overline{f}_Q - \overline{f}_R|\right) - \frac{R}{T}\varphi\left(\frac{Q}{T}|\overline{f}_Q - \overline{f}_R|\right).$$

Moreover, using the fact that

$$\frac{R}{T}|\overline{f}_Q - \overline{f}_R| = |\overline{f}_Q - \overline{f}| \quad \text{and} \quad \frac{Q}{T}|\overline{f}_Q - \overline{f}_R| = |\overline{f}_R - \overline{f}|,$$

we obtain inequality (4). The proof is complete since the proof of the reversed inequality is similar to the proof above, and we can omit the details. □

By making a further restriction of φ, we can also state the following version of Theorem 2.3.

Theorem 2.4 *Let $\varphi : [0,\infty) \to \mathbf{R}$ be a nondecreasing and superquadratic function such that*

$$\varphi(a+b) \le c\big(\varphi(a)+\varphi(b)\big) \quad \text{for some } c>0. \tag{5}$$

Then the following refined variant of Jensen-type inequality

$$\varphi(\overline{f}) \le I \le \frac{1}{T}\int_\Omega t(s)\varphi\big(f(s)\big)\,\mathrm{d}s - \frac{1}{cT}\int_\Omega t(s)\varphi\big(|f(s)-\overline{f}|\big)\,\mathrm{d}s$$

holds for all nonnegative measurable functions f, where

$$I = \frac{Q}{T}\varphi(\overline{f}_Q) + \frac{R}{T}\varphi(\overline{f}_R) - \frac{Q}{T}\varphi\big(|\overline{f}_Q-\overline{f}|\big) - \frac{R}{T}\varphi\big(|\overline{f}_R-\overline{f}|\big).$$

Proof We proved the first inequality $\varphi(\overline{f}) \le I$ in Theorem 2.3, so we only need to prove the second inequality.

By applying Lemma 2.1 in the first two terms of I, we get that

$$I \le \frac{1}{T}\int_\Omega t(s)\varphi\big(f(s)\big)\,\mathrm{d}s - A_1 - B_1,$$

where

$$A_1 := \frac{Q}{T}\varphi\big(|\overline{f}_Q-\overline{f}|\big) + \frac{R}{T}\varphi\big(|\overline{f}_R-\overline{f}|\big)$$

and

$$B_1 := \frac{1}{T}\int_\Omega \alpha(s)t(s)\varphi\big(|f(s)-\overline{f}_Q|\big)\,\mathrm{d}s + \frac{1}{T}\int_\Omega \beta(s)t(s)\varphi\big(|f(s)-\overline{f}_R|\big)\,\mathrm{d}s.$$

To finish the proof, it is enough to prove that

$$\frac{1}{cT}\int_\Omega t(s)\varphi\big(|f(s)-\overline{f}|\big)\,\mathrm{d}s \le A_1 + B_1. \tag{6}$$

Now, by using the triangle inequality, the nondecreasing property of φ, and (5), we obtain that

$$\begin{aligned}&\frac{1}{T}\int_\Omega \alpha(s)t(s)\varphi\big(|f(s)-\overline{f}|\big)\,\mathrm{d}s\\ &\quad\le c\left(\frac{1}{T}\int_\Omega \alpha(s)t(s)\varphi\big(|f(s)-\overline{f}_Q|\big)\,\mathrm{d}s + \frac{Q}{T}\varphi\big(|\overline{f}_Q-\overline{f}|\big)\right)\end{aligned}$$

and

$$\frac{1}{T}\int_\Omega \beta(s)t(s)\varphi\big(|f(s)-\overline{f}|\big)\,\mathrm{d}s$$

$$\leq c\left(\frac{1}{T}\int_{\Omega}\beta(s)t(s)\varphi\left(|f(s)-\overline{f}_R|\right)\mathrm{d}s+\frac{R}{T}\varphi\left(|\overline{f}_R-\overline{f}|\right)\right).$$

Hence (6) follows as a sum of the above two inequalities. The proof is complete. □

3 Refinements of continuous Minkowski inequality

In the following discussion we consider the measurable spaces (X,μ), (X,λ), and (Y,ν). Moreover, $\mathrm{d}\mu$, $\mathrm{d}\lambda$, and $\mathrm{d}\nu$ are notations for $\mathrm{d}\mu(x)$, $\mathrm{d}\lambda(x)$, and $\mathrm{d}\nu(y)$, respectively. First, we remind about the following interesting refinement of the Hölder inequality by G. Sinnamon [21, Theorem 1.1].

Lemma 3.1 *Let $p\geq 2$ and $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\int_{\Omega}fg\,\mathrm{d}\nu\leq\left(\int_{\Omega}\left(f^p-h^p\right)\mathrm{d}\nu\right)^{\frac{1}{p}}\left(\int_{\Omega}g^q\,\mathrm{d}\nu\right)^{\frac{1}{q}}\tag{7}$$

holds for any two nonnegative ν-measurable functions f and g, where

$$h=\left|f-\frac{g^{q-1}\int_{\Omega}fg\,\mathrm{d}\nu}{\int_{\Omega}g^q\,\mathrm{d}\nu}\right|.$$

Moreover, (7) *holds in the reversed direction if $1<p\leq 2$.*

The continuous Minkowski inequality reads as follows (see [17, p. 41]).

Theorem 3.2 *Let f be a nonnegative measurable function on $X\times Y$ with respect to the measure $\mu\times\nu$, and let $p\geq 1$. Then*

$$\left(\int_X\left(\int_Y f\,\mathrm{d}\nu\right)^p\mathrm{d}\mu\right)^{\frac{1}{p}}\leq\int_Y\left(\int_X f^p\,\mathrm{d}\mu\right)^{\frac{1}{p}}\mathrm{d}\nu.$$

Our first main result in this section is the following refinement of the Minkowski inequality.

Theorem 3.3 *Let f be a nonnegative measurable function on $X\times Y$ with respect to the measure $\mu\times\nu$, and let $p\geq 2$. Then*

$$\left(\int_X\left(\int_Y f\,\mathrm{d}\nu\right)^p\mathrm{d}\mu\right)^{\frac{1}{p}}\leq\int_Y\left(\int_X\left(f^p-h^p\right)\mathrm{d}\mu\right)^{\frac{1}{p}}\mathrm{d}\nu,\tag{8}$$

where

$$h=\left|f-\frac{H\int_X fH^{p-1}\,\mathrm{d}\mu}{\int_X H^p\,\mathrm{d}\mu}\right|,\qquad H(x)=\int_Y f(x,y)\,\mathrm{d}\nu.$$

If $1<p\leq 2$, then (8) *holds in the reversed direction.*

Proof Let $H(x)=\int_Y f(x,y)\,\mathrm{d}\nu$. Let $p\geq 2$ and $\frac{1}{p}+\frac{1}{q}=1$. Using Lemma 3.1, by replacing $f(x)$ and $g(x)$ with $f(x,y)$ and $H^{p-1}(x)$, respectively, we get that

$$\int_X fH^{p-1}\,\mathrm{d}\mu\leq\left(\int_X\left(f^p-h^p\right)\mathrm{d}\mu\right)^{\frac{1}{p}}\left(\int_X H^p\,\mathrm{d}\mu\right)^{\frac{1}{q}},\tag{9}$$

where

$$h(x,y) = \left| f - \frac{H \int_X f H^{p-1} \,\mathrm{d}\mu}{\int_X H^p \,\mathrm{d}\mu} \right|.$$

We integrate inequality (9) over Y, apply Fubini's theorem on the left side of the inequality to find that

$$\left(\int_X H^p \,\mathrm{d}\mu \right) \leq \left(\int_X H^p \,\mathrm{d}\mu \right)^{\frac{1}{q}} \int_Y \left(\int_X \left(f^p - h^p \right) \mathrm{d}\mu \right)^{\frac{1}{p}} \mathrm{d}\nu. \tag{10}$$

Since $H(x) = \int_Y f(x,y) \,\mathrm{d}\nu$ and $1 - \frac{1}{q} = \frac{1}{p}$, we deduce that

$$\left(\int_X \left(\int_Y f \,\mathrm{d}\nu \right)^p \mathrm{d}\mu \right)^{\frac{1}{p}} \leq \int_Y \left(\int_X \left(f^p - h^p \right) \mathrm{d}\mu \right)^{\frac{1}{p}} \mathrm{d}\nu.$$

The proof of the case $1 < p \leq 2$ is similar so we omit the details and the proof is complete. □

Next, we point out the following discrete version of the above theorem.

Corollary 3.4 *Let $p \geq 2$ and let $f_1, f_2, \ldots, f_n$ be nonnegative μ-measurable functions. Then*

$$\left(\int_X \left(\sum_{i=1}^n f_i \right)^p \mathrm{d}\mu \right)^{\frac{1}{p}} \leq \sum_{i=1}^n \left(\int_X \left(f_i^p - h_i^p \right) \mathrm{d}\mu \right)^{\frac{1}{p}}, \tag{11}$$

where

$$h_i = \left| f_i - \frac{H \int_X f_i H^{p-1} \,\mathrm{d}\mu}{\int_X H^p \,\mathrm{d}\mu} \right|, \quad \forall i = 1, \ldots, n, \quad \text{and} \quad H(x) = \sum_{i=1}^n f_i(x).$$

If $1 < p \leq 2$, then (11) holds in the reversed direction.

Proof Let $Y = \bigcup_{i=1}^n Y_i$, where $Y_i = [i-1, i)$, for all $i = 1, \ldots, n$, and let $\mathrm{d}\nu = \mathrm{d}y$ be the Lebesgue measure.

Define $f(x,y) = \sum_{i=1}^n f_i(x) \chi_{Y_i}(y)$. Then

$$H(x) = \int_Y f(x,y) \,\mathrm{d}\nu = \sum_{i=1}^n f_i(x)$$

and

$$\begin{aligned} h(x,y) &= \left| f - \frac{H \int_X f H^{p-1} \,\mathrm{d}\mu}{\int_X H^p \,\mathrm{d}\mu} \right| = \sum_{i=1}^n \chi_{Y_i}(y) \left| f_i - \frac{H \int_X f_i H^{p-1} \,\mathrm{d}\mu}{\int_X H^p \,\mathrm{d}\mu} \right| \\ &= \sum_{i=1}^n \chi_{Y_i}(y) h_i(x), \end{aligned}$$

where

$$h_i = \left| f_i - \frac{H \int_X f_i H^{p-1} \,\mathrm{d}\mu}{\int_X H^p \,\mathrm{d}\mu} \right|.$$

Therefore, by applying Theorem 3.3, one can complete the proof. □

The continuous form of the Beckenbach–Dresher inequality was first derived in [9, Theorem 3.1] (see also [22]). It has the following form.

Theorem 3.5 *Let f and u be nonnegative measurable functions on $X \times Y$ with respect to the measures $\mu \times \nu$ and $\lambda \times \nu$, respectively, and let*

(i) $s \geq 1$, $q \leq 1 \leq p$, $(q \neq 0)$ *or*

(ii) $s \leq 0$, $p \leq 1 \leq q$, $(p \neq 0)$.

Then

$$\frac{(\int_X (\int_Y f \,\mathrm{d}\nu)^p \,\mathrm{d}\mu)^{\frac{s}{p}}}{(\int_X (\int_Y u \,\mathrm{d}\nu)^q \,\mathrm{d}\lambda)^{\frac{s-1}{q}}} \leq \int_Y \frac{(\int_X f^p \,\mathrm{d}\mu)^{\frac{s}{p}}}{(\int_X u^q \,\mathrm{d}\lambda)^{\frac{s-1}{q}}} \,\mathrm{d}\nu,$$

provided all occurring integrals exist.

If $0 < s \leq 1$, $p \leq 1$, and $q \leq 1$ $(p, q \neq 0)$, then inequality (12) is reversed.

Our new result related to the continuous Beckenbach–Dresher inequality reads as follows.

Theorem 3.6 *Let f and u be nonnegative measurable functions on $X \times Y$ with respect to the measures $\mu \times \nu$ and $\lambda \times \nu$, respectively, and let $1 < q \leq 2 \leq p$, $s \geq 1$. Then*

$$\frac{(\int_X (\int_Y f \,\mathrm{d}\nu)^p \,\mathrm{d}\mu)^{\frac{s}{p}}}{(\int_X (\int_Y u \,\mathrm{d}\nu)^q \,\mathrm{d}\lambda)^{\frac{s-1}{q}}} \leq \int_Y \frac{(\int_X (f^p - h^p) \,\mathrm{d}\mu)^{\frac{s}{p}}}{(\int_X (u^q - r^q) \,\mathrm{d}\lambda)^{\frac{s-1}{q}}} \,\mathrm{d}\nu,$$

where

$$h = \left| f - \frac{H \int_X f H^{p-1} \,\mathrm{d}\mu}{\int_X H^p \,\mathrm{d}\mu} \right|, \qquad H(x) = \int_Y f(x,y) \,\mathrm{d}\nu,$$

$$r = \left| u - \frac{\hat{H} \int_X u \hat{H}^{q-1} \,\mathrm{d}\mu}{\int_X \hat{H}^q \,\mathrm{d}\mu} \right|, \qquad \hat{H}(x) = \int_Y u(x,y) \,\mathrm{d}\nu.$$

Proof Let $1 < q \leq 2 \leq p$. Then, in view of Theorem 3.3, for $p \geq 2$ and $1 < q \leq 2$, we have that

$$\begin{aligned}
\frac{(\int_X (\int_Y f \,\mathrm{d}\nu)^p \,\mathrm{d}\mu)^{\frac{s}{p}}}{(\int_X (\int_Y u \,\mathrm{d}\nu)^q \,\mathrm{d}\lambda)^{\frac{s-1}{q}}} &\leq \frac{(\int_Y (\int_X (f^p - h^p) \,\mathrm{d}\mu)^{\frac{1}{p}} \,\mathrm{d}\nu)^s}{(\int_Y (\int_X (u^q - r^q) \,\mathrm{d}\lambda)^{\frac{1}{q}} \,\mathrm{d}\nu)^{s-1}} \\
&= \left(\int_Y a^{\frac{1}{s}} \,\mathrm{d}\nu \right)^s \left(\int_Y b^{\frac{1}{1-s}} \,\mathrm{d}\nu \right)^{1-s} \\
&\leq \int_Y ab \,\mathrm{d}\nu,
\end{aligned}$$

where $a^{\frac{1}{s}} = (\int_X (f^p - h^p)\,\mathrm{d}\mu)^{\frac{1}{p}}$ and $b^{\frac{1}{1-s}} = (\int_X (u^q - r^q)\,\mathrm{d}\lambda)^{\frac{1}{q}}$. In the last inequality we used the reverse Hölder inequality for two functions a and b when one exponent $(1-s)$ is negative and the other exponent s is positive. The proof is complete. □

By using Theorem 3.6 and similar arguments as those in the proof of Corollary 3.4, we can also derive the following discrete version.

Corollary 3.7 *Let* $1 < q \le 2 \le p$, $s \ge 1$, $f_i, u_i : X \to [0,\infty)$, $f_i^p, u_i^q \in L^1$ *for all* $i = 1,\dots,n$. *Then*

$$\frac{(\int_X (\sum_{i=1}^n f_i)^p\,\mathrm{d}\mu)^{\frac{s}{p}}}{(\int_X (\sum_{i=1}^n u_i)^q\,\mathrm{d}\lambda)^{\frac{s-1}{q}}} \le \sum_{i=1}^n \frac{(\int_X (f_i^p - h_i^p)\,\mathrm{d}\mu)^{\frac{s}{p}}}{(\int_X (u_i^q - r_i^q)\,\mathrm{d}\lambda)^{\frac{s-1}{q}}},$$

where

$$h_i = \left| f_i - \frac{H \int_X f_i H^{p-1}\,\mathrm{d}\mu}{\int_X H^p\,\mathrm{d}\mu} \right|, \qquad H(x) = \sum_{i=1}^n f_i(x),$$

$$r_i = \left| u_i - \frac{\hat{H} \int_X u_i \hat{H}^{q-1}\,\mathrm{d}\mu}{\int_X \hat{H}^q\,\mathrm{d}\mu} \right|, \qquad \hat{H}(x) = \sum_{i=1}^n u_i(x).$$

As an application of Corollary 3.7, by making the substitution $s = \frac{p}{p-q}$, $p \ne q$, we obtain the following Beckenbach–Dresher type inequality.

Example 3.8 Let $1 < q \le 2 \le p$, $q \ne p$, $f_i, u_i : X \to [0,\infty)$, $f_i^p, u_i^q \in L^1$ for all $i = 1,\dots,n$. Then

$$\left(\frac{\int_X (\sum_{i=1}^n f_i)^p\,\mathrm{d}\mu}{\int_X (\sum_{i=1}^n u_i)^q\,\mathrm{d}\lambda} \right)^{\frac{1}{p-q}} \le \sum_{i=1}^n \left(\frac{\int_X (f_i^p - h_i^p)\,\mathrm{d}\mu}{\int_X (u_i^q - r_i^q)\,\mathrm{d}\lambda} \right)^{\frac{1}{p-q}},$$

where h_i and r_i are as in Corollary 3.7.

4 Refinements of Hardy's inequality

The results in this section may be seen as complements and further generalizations of some results in [14] and [19]. In [6, Theorem 2.1] the following Hardy-type inequality was given.

Theorem 4.1 *Let* $0 < b \le \infty$, $-\infty \le a < c \le \infty$, *let* φ *be a positive convex function on* (a,c) *and* E *be a Banach function space on* $[0,b)$. *If* E *has the Fatou property and* $a < f(x) < c$, *then*

$$\left\| \varphi\left(\frac{1}{x} \int_0^x f(t)\,\mathrm{d}t \right) \right\|_E \le \int_0^b \varphi\big(f(t)\big) \left\| \frac{1}{x} \chi_{[t,b)}(x) \right\|_E \mathrm{d}t,$$

provided that both sides have sense.

To prove our main results, we need the following lemma (see [13, 18]).

Lemma 4.2 (See [18]) *Assume that the Banach function space E has the Fatou property. Let $f(x,t) \geq 0$ on $\Omega \times T$ and let for almost every $t \in T$, $f(x,t) \in E$. If the function $\|f^r(x,t)\|_E^{\frac{1}{r}}$ is integrable on T, then, for $r \geq 1$,*

$$\left\| \left(\int_T f(x,t)\,\mathrm{d}t \right)^r \right\|_E^{\frac{1}{r}} \leq \int_T \left\| f^r(x,t) \right\|_E^{\frac{1}{r}} \mathrm{d}t.$$

Our first main result in this section reads as follows.

Theorem 4.3 *Let $0 < b \leq \infty$, $-\infty \leq a < c \leq \infty$, let φ be a positive and superquadratic function on (a,c) and E be a Banach function space on $[0,b)$. If E has the Fatou property and $a < f(x) < c$, then*

$$\begin{aligned} &\left\| \varphi\left(\frac{1}{x} \int_0^x f(t)\,\mathrm{d}t \right) \right\|_E \\ &\quad \leq \int_0^b \varphi\big(f(t)\big) \left\| \left(1 - \frac{\varphi(|f(t) - \frac{1}{x}\int_0^x f(s)\,\mathrm{d}s|)}{\varphi(f(t))} \right) \frac{1}{x} \chi_{[t,b)}(x) \right\|_E \mathrm{d}t, \end{aligned} \tag{12}$$

provided that both sides have sense.

Proof Let $D = \{(x,t) : 0 \leq x \leq b, 0 \leq t \leq x\}$. Then

$$\chi_D(x,t) = \chi_{[0,x]}(t) = \chi_{[t,b]}(x). \tag{13}$$

By using Theorem 1.2, the lattice property of E, Lemma 4.2 with $r = 1$, and (13), we find that

$$\begin{aligned} &\left\| \varphi\left(\frac{1}{x} \int_0^x f(t)\,\mathrm{d}t \right) \right\|_E \\ &\quad \leq \left\| \int_0^x \frac{\varphi(f(t)) - \varphi(|f(t) - \frac{1}{x}\int_0^x f(s)\,\mathrm{d}s|)}{x}\,\mathrm{d}t \right\|_E \\ &\quad = \left\| \int_0^b \frac{\varphi(f(t)) - \varphi(|f(t) - \frac{1}{x}\int_0^x f(s)\,\mathrm{d}s|)}{x} \chi_{[0,x]}(t)\,\mathrm{d}t \right\|_E \\ &\quad = \left\| \int_0^b \frac{\varphi(f(t)) - \varphi(|f(t) - \frac{1}{x}\int_0^x f(s)\,\mathrm{d}s|)}{x} \chi_D(x,t)\,\mathrm{d}t \right\|_E \\ &\quad \leq \int_0^b \left\| \frac{\varphi(f(t)) - \varphi(|f(t) - \frac{1}{x}\int_0^x f(s)\,\mathrm{d}s|)}{x} \chi_D(x,t) \right\|_E \mathrm{d}t \\ &\quad = \int_0^b \left\| \frac{\varphi(f(t)) - \varphi(|f(t) - \frac{1}{x}\int_0^x f(s)\,\mathrm{d}s|)}{x} \chi_{[t,b]}(x) \right\|_E \mathrm{d}t \\ &\quad = \int_0^b \varphi\big(f(t)\big) \left\| \left(1 - \frac{\varphi(|f(t) - \frac{1}{x}\int_0^x f(s)\,\mathrm{d}s|)}{\varphi(f(t))} \right) \frac{1}{x} \chi_{[t,b]}(x) \right\|_E \mathrm{d}t. \end{aligned}$$

The proof is complete. □

Here we just give one example of application of Theorem 4.3 (cf. [19, Proposition 2.1] and [14, Theorem 2.3]).

Corollary 4.4 *Let $0 < b \le \infty$, $u : (0,b) \to \mathbf{R}$ be a nonnegative weight function such that the function $x \mapsto \frac{u(x)}{x^2}$ is locally integrable on $(0,b)$, and define the weight function v by*

$$v(t) = t \int_t^b \frac{u(x)}{x^2}\,\mathrm{d}x, \quad t \in (0,b).$$

If the real-valued function φ is positive and superquadratic on (a,c), $0 \le a < c \le \infty$, then the inequality

$$\begin{aligned} &\int_0^b u(x)\varphi\left(\frac{1}{x}\int_0^x f(t)\,\mathrm{d}t\right)\frac{\mathrm{d}x}{x} \\ &\quad \le \int_0^b v(t)\varphi\big(f(t)\big)\frac{\mathrm{d}t}{t} - \int_0^b \int_t^b \varphi\left(\left|f(t) - \frac{1}{x}\int_0^x f(s)\,\mathrm{d}s\right|\right)\frac{u(x)}{x^2}\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{14}$$

holds for all f with $a < f(x) < c$, $0 < x \le b$.

Proof It is known that $E = L^1((0,b), \frac{u(x)}{x}\,\mathrm{d}x)$ satisfy the Fatou property (see e.g. [7]). Moreover,

$$\begin{aligned} &\left\| \left(1 - \frac{\varphi(|f(t) - \frac{1}{x}\int_0^x f(s)\,\mathrm{d}s|)}{\varphi(f(t))}\right)\frac{1}{x}\chi_{[t,b]}(x)\right\|_E \\ &\quad = \int_t^b \frac{u(x)}{x^2}\,\mathrm{d}x - \frac{1}{\varphi(f(t))}\int_t^b \varphi\left(\left|f(t) - \frac{1}{x}\int_0^x f(s)\,\mathrm{d}s\right|\right)\frac{u(x)}{x^2}\,\mathrm{d}x \\ &\quad = \frac{v(t)}{t} - \frac{1}{\varphi(f(t))}\int_t^b \varphi\left(\left|f(t) - \frac{1}{x}\int_0^x f(s)\,\mathrm{d}s\right|\right)\frac{u(x)}{x^2}\,\mathrm{d}x. \end{aligned} \tag{15}$$

Therefore, (14) follows from (15) and Theorem 4.3. The proof is complete. □

Next we state a "dual" version of Theorem 4.3. Note that the natural dual operator of the Hardy operator $H : H(f)(x) = \frac{1}{x}\int_0^x f(t)\,\mathrm{d}t$ is $\hat{H} : \hat{H}(f)(x) = \int_x^\infty \frac{f(t)}{t}\,\mathrm{d}t$, but here we use its alternative $H^* : H^*(f)(x) = x\int_x^\infty \frac{f(t)}{t^2}\,\mathrm{d}t$.

Theorem 4.5 *Let $-\infty \le a < c \le \infty$, let φ be a positive and superquadratic function on (a,c) and E be a Banach function space on $[b,\infty)$, $b \ge 0$, with the Fatou property. Then, whenever $a < f(x) < c$,*

$$\begin{aligned} &\left\|\varphi\left(x\int_x^\infty \frac{f(t)}{t^2}\,\mathrm{d}t\right)\right\|_E \\ &\quad \le \int_b^\infty \varphi\big(f(t)\big)\left\|\left(1 - \frac{\varphi(|f(t) - x\int_x^\infty f(s)\frac{\mathrm{d}s}{s^2}|)}{\varphi(f(t))}\right)x\chi_{[b,t]}(x)\right\|_E \frac{\mathrm{d}t}{t^2}. \end{aligned}$$

Proof Let $D = \{(x,t) : b \le x, x \le t < \infty\}$. Then

$$\chi_D(x,t) = \chi_{[x,\infty)}(t) = \chi_{[b,t]}(x). \tag{16}$$

By using (16) and the same arguments as in the proof of Theorem 4.3 we obtain that

$$\begin{aligned}
&\left\|\varphi\left(x\int_x^\infty \frac{f(t)}{t^2}\,\mathrm{d}t\right)\right\|_E \\
&\quad\le \left\|\int_x^\infty \left(\varphi\big(f(t)\big)-\varphi\left(\left|f(t)-x\int_x^\infty f(s)\frac{\mathrm{d}s}{s^2}\right|\right)\right)x\frac{\mathrm{d}t}{t^2}\right\|_E \\
&\quad= \left\|\int_b^\infty \left(\varphi\big(f(t)\big)-\varphi\left(\left|f(t)-x\int_x^\infty f(s)\frac{\mathrm{d}s}{s^2}\right|\right)\right)x\chi_{[x,\infty)}(t)\frac{\mathrm{d}t}{t^2}\right\|_E \\
&\quad= \left\|\int_b^\infty \left(\varphi\big(f(t)\big)-\varphi\left(\left|f(t)-x\int_x^\infty f(s)\frac{\mathrm{d}s}{s^2}\right|\right)\right)x\chi_D(x,t)\frac{\mathrm{d}t}{t^2}\right\|_E \\
&\quad\le \int_b^\infty \left\|\left(\varphi\big(f(t)\big)-\varphi\left(\left|f(t)-x\int_x^\infty f(s)\frac{\mathrm{d}s}{s^2}\right|\right)\right)x\chi_D(x,t)\right\|_E\frac{\mathrm{d}t}{t^2} \\
&\quad= \int_b^\infty \left\|\left(\varphi\big(f(t)\big)-\varphi\left(\left|f(t)-x\int_x^\infty f(s)\frac{\mathrm{d}s}{s^2}\right|\right)\right)x\chi_{[b,t]}(x)\right\|_E\frac{\mathrm{d}t}{t^2} \\
&\quad= \int_b^\infty \varphi\big(f(t)\big)\left\|\left(1-\frac{\varphi(|f(t)-x\int_x^\infty f(s)\frac{\mathrm{d}s}{s^2}|)}{\varphi(f(t))}\right)x\chi_{[b,t]}(x)\right\|_E\frac{\mathrm{d}t}{t^2}.
\end{aligned}$$

The proof is complete. □

We give the following example of application of Theorem 4.5 (cf. [19, Proposition 2.2]).

Corollary 4.6 *Let $0\le b<\infty$, $u:(b,\infty)\to\mathbf{R}$ be a nonnegative locally integrable function on (b,∞), and define the function v by*

$$v(t)=\frac{1}{t}\int_b^t u(x)\,\mathrm{d}x,\quad t\in(b,\infty).$$

If the real-valued function φ is positive and superquadratic on (a,c), $0\le a<c\le\infty$, then the inequality

$$\begin{aligned}
&\int_b^\infty u(x)\varphi\left(x\int_x^\infty f(t)\frac{\mathrm{d}t}{t^2}\right)\frac{\mathrm{d}x}{x} \\
&\quad\le \int_b^\infty v(t)\varphi\big(f(t)\big)\frac{\mathrm{d}t}{t}-\int_b^\infty\int_b^t \varphi\left(\left|f(t)-x\int_x^\infty f(s)\frac{\mathrm{d}s}{s^2}\right|\right)u(x)\,\mathrm{d}x\frac{\mathrm{d}t}{t^2}
\end{aligned}\tag{17}$$

holds for all f with $a<f(x)<c$, $x\ge b$.

Proof It is known that $E=L^1([b,\infty),\frac{u}{x}\,\mathrm{d}x)$ satisfy the Fatou property (see e.g. [7]). Moreover,

$$\begin{aligned}
&\left\|\left(1-\frac{\varphi(|f(t)-x\int_x^\infty f(s)\frac{\mathrm{d}s}{s^2}|)}{\varphi(f(t))}\right)x\chi_{[b,t]}(x)\right\|_E \\
&\quad= \int_b^t u(x)\,\mathrm{d}x-\frac{1}{\varphi(f(t))}\int_b^t \varphi\left(\left|f(t)-x\int_x^\infty f(s)\frac{\mathrm{d}s}{s^2}\right|\right)u(x)\,\mathrm{d}x \\
&\quad= tv(t)-\frac{1}{\varphi(f(t))}\int_t^b \varphi\left(\left|f(t)-x\int_x^\infty f(s)\frac{\mathrm{d}s}{s^2}\right|\right)u(x)\,\mathrm{d}x.
\end{aligned}\tag{18}$$

Therefore, (17) follows from (18) and Theorem 4.5, so the proof is complete. □

5 Concluding remarks and results

Remark 5.1 The natural "turning point" in Minkowski and Beckenbach–Dresher type inequalities is 1, but in our versions of these inequalities, we have proved the first inequalities of this type with turning point 2 (see Theorem 3.3 and 3.6).

Our first new result of this type in this section is the following improved version of the inequality in [23, Theorem 1.2].

Proposition 5.2 *Let p, s, and t be different real numbers such that $s \geq 2$, $t \geq 2$ and $(s-t)/(p-t) > 1$. Then, for any positive μ-measurable functions $f_1, \dots, f_n$,*

$$\left(\int_X \left(\sum_{i=1}^n f_i \right)^p \mathrm{d}\mu \right)^{s-t} \leq \frac{(\sum_{i=1}^n (\int_X (f_i^s - h_i^s)\,\mathrm{d}\mu)^{\frac{1}{s}})^{s(p-t)}}{(\sum_{i=1}^n (\int_X (f_i^t - r_i^t)\,\mathrm{d}\mu)^{\frac{1}{t}})^{t(p-s)}}, \tag{19}$$

where

$$h_i = \left| f_i - \frac{H \int_X f_i H^{s-1}\,\mathrm{d}\mu}{\int_X H^s\,\mathrm{d}\nu} \right|, \qquad H = \sum_{i=1}^n f_i, \quad \text{and} \quad r_i = \left| f_i - \frac{H \int_X f_i H^{t-1}\,\mathrm{d}\mu}{\int_X H^t\,\mathrm{d}\mu} \right|.$$

Moreover, if $p \neq 0$, $1 < t < 2$, $1 < s < 2$, and $(s-t)/(p-t) < 1$, then (19) holds in the reversed direction.

Proof Let $s \geq 2$, $t \geq 2$ such that $\frac{s-t}{p-t} > 1$. Then, by Hölder's inequality,

$$\begin{aligned} \int_X \left(\sum_{i=1}^n f_i \right)^p \mathrm{d}\mu &= \int_X \left[(f_1 + \cdots + f_n)^s \right]^{\frac{p-t}{s-t}} \left[(f_1 + \cdots + f_n)^t \right]^{\frac{s-p}{s-t}} \mathrm{d}\mu \\ &\leq \left(\int_X (f_1 + \cdots + f_n)^s\,\mathrm{d}\mu \right)^{\frac{p-t}{s-t}} \left(\int_X (f_1 + \cdots + f_n)^t\,\mathrm{d}\mu \right)^{\frac{s-p}{s-t}}. \end{aligned}$$

In view of Corollary 3.4, the above inequality becomes

$$\int_X \left(\sum_{i=1}^n f_i \right)^p \mathrm{d}\mu \leq \left(\sum_{i=1}^n \left(\int_X (f_i^s - h_i^s)\,\mathrm{d}\mu \right)^{\frac{1}{s}} \right)^{s\frac{p-t}{s-t}} \left(\sum_{i=1}^n \left(\int_X (f_i^t - r_i^t)\,\mathrm{d}\mu \right)^{\frac{1}{t}} \right)^{t\frac{s-p}{s-t}},$$

where

$$h_i = \left| f_i - \frac{H \int_X f_i H^{s-1}\,\mathrm{d}\mu}{\int_X H^s\,\mathrm{d}\mu} \right|, \qquad H = \sum_{i=1}^n f_i, r_i = \left| f_i - \frac{H \int_X f_i H^{t-1}\,\mathrm{d}\mu}{\int_X H^t\,\mathrm{d}\mu} \right|.$$

The proof of the other case is similar, so we omit the details and the proof is complete. □

Remark 5.3 In [23, Theorem 1.2] only the case $n = 2$ was considered, so Proposition 5.2 is both a generalization and refinement of this result.

Proposition 5.4 *Suppose that ν is a measure, and f_1 and f_2 are nonnegative measurable functions such that f_i^p are ν-integrable for $i = 1, 2$.*

(a) *If* $p > 0$, *then*

$$\left\| \sum_{i=1}^{2} f_i \right\|_{L^p(\nu)}^p \leq \left(\sum_{i=1}^{2} \|f_i\|_{L^p(\nu)} \right)^p + \frac{1}{2} \sum_{i=1}^{2} \|f_i + h_i\|_{L^p(\nu)}^p.$$

(b) *If* $p \geq \frac{1}{2}$, *then*

$$\left\| \sum_{i=1}^{2} f_i \right\|_{L^p(\nu)}^p \leq \left(\sum_{i=1}^{2} \|f_i\|_{L^p(\nu)} \right)^p + \frac{1}{2} \sum_{i=1}^{2} \left(\|f_i + h_i\|_{L^p(\nu)}^p - \|f_i\|_{L^p(\nu)}^p \right),$$

where

$$h_i = \left| g_i^p - \frac{f_i^p \|g_i\|_{L^p(\nu)}^p}{\|f_i\|_{L^p(\nu)}^p} \right|^{\frac{1}{p}}, \quad g_i = (f_1 + f_2) - f_i, i = 1, 2.$$

Proof In view of [2, Theorem 4.3], we have

$$-\left(1 + \left(\int F \, d\mu\right)^{\frac{1}{p}}\right)^p \leq -\int \left(1 + F^{\frac{1}{p}}\right)^p d\mu + \int \left(1 + \left|F - \int F \, d\mu\right|^{\frac{1}{p}}\right)^p d\mu \tag{20}$$

for $p > 0$ and

$$-\left(1 + \left(\int F \, d\mu\right)^{\frac{1}{p}}\right)^p + 1 \leq -\int \left(1 + F^{\frac{1}{p}}\right)^p d\mu + \int \left(1 + \left|F - \int F \, d\mu\right|^{\frac{1}{p}}\right)^p d\mu \tag{21}$$

for $p \geq \frac{1}{2}$.

By substituting $F = \frac{f_2^p}{f_1^p}$ and $d\mu = \frac{f_1^p}{\int f_1^p \, d\nu} \, d\nu$ in both of inequalities (20) and (21), we obtain that

$$\|f_1 + f_2\|_{L^p(\nu)}^p \leq \left(\|f_1\|_{L^p(\nu)} + \|f_2\|_{L^p(\nu)} \right)^p + \|f_1 + h_1\|_{L^p(\nu)}^p \tag{22}$$

for $p > 0$ and

$$\|f_1 + f_2\|_{L^p(\nu)}^p \leq \left(\sum_{i=1}^{2} \|f_i\|_{L^p(\nu)} \right)^p + \|f_1 + h_1\|_{L^p(\nu)}^p - \|f_1\|_{L^p(\nu)}^p \tag{23}$$

for $p \geq \frac{1}{2}$, where

$$h_1 = \left| f_2^p - \frac{f_1^p \|f_2\|_{L^p(\nu)}^p}{\|f_1\|_{L^p(\nu)}^p} \right|^{\frac{1}{p}}.$$

By interchanging the role of f_1 and f_2 in the above discussion, we have that

$$\|f_1 + f_2\|_{L^p(\nu)}^p \leq \left(\|f_1\|_{L^p(\nu)} + \|f_2\|_{L^p(\nu)} \right)^p + \|f_2 + h_2\|_{L^p(\nu)}^p \tag{24}$$

for $p > 0$ and

$$\|f_1 + f_2\|_{L^p(v)}^p \le \left(\sum_{i=1}^{2} \|f_i\|_{L^p(v)} \right)^p + \|f_2 + h_2\|_{L^p(v)}^p - \|f_2\|_{L^p(v)}^p \tag{25}$$

for $p \ge \frac{1}{2}$, where

$$h_2 = \left| f_1^p - \frac{f_2^p \|f_1\|_{L^p(v)}^p}{\|f_2\|_{L^p(v)}^p} \right|^{\frac{1}{p}}.$$

Consequently, by taking the sum of inequalities (22) with (24) and (23) with (25), we get the results in (a) and (b). The proof is complete. □

Remark 5.5 The concept of superquadratic function was formally introduced in [2, 3] but this idea seems to be known even before (see e.g. [21] and the references therein).

Remark 5.6 The first important book in the area of inequalities was (the bible) [11], but after that more than 30 books or monographs in this area have been published.

Remark 5.7 Concerning Hardy-type inequalities in Sect. 4, the first result was proved in 1925 (see [10]). The dramatic history and prehistory up to 2007 is described in the book [15] (see also [20]). The corresponding history up to 2017 is given in detail in the book [16]. Our results in Sect. 4 are just one example of the fact that the development of the theory of this fascinating inequality still continues. For the development of classical inequalities in a continuous and/or Banach function space setting, we refer to [18] and the references given there, see also [5, 6], and [8].

Acknowledgements
The forth named author thanks International Science Program (ISP) in Uppsala, Sweden, for financial support. We thank both referees for good suggestions, which have improved the final version of this paper.

Funding
The publication charges for this manuscript are supported by a grant from the publication fund of UiT The Arctic University of Norway. Open access funding provided by UiT The Arctic University of Norway (incl University Hospital of North Norway).

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors contributed equally and significantly in writing this paper. LEP analyzed and interpreted the results regarding the position in recent research, and he and MFY are the main authors concerning the results from Sects. 3, 4, and 5. LN's and SV's main contributions are in Sects. 2 and 3. MFY typed the manuscript. All the authors read and approved the final manuscript.

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Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 27 December 2021 Accepted: 2 June 2022 Published online: 21 June 2022

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