

Differential invariants of curves in G_2 flag varieties

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ABSTRACT

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We compute the algebra of differential invariants of unparametrized curves in the homogeneous G_2 flag varieties, namely in G_2/P . This gives a solution to the equivalence problem for such curves. We consider the cases of integral and generic curves and relate the equivalence problems for all three choices of the parabolic subgroup P .

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0. Introduction

Computation of differential invariants of (unparametrized) curves was an important topic in XIX-th century mathematics. Besides Frenet-Serret formulas for Euclidean spaces, curves were extensively studied in projective spaces. In particular, Klein and Lie derived special ODEs describing projective invariant classes of curves in the plane [11] and Halphen computed the invariants that govern projective equivalence [9], see [12] for a modern approach and review. Wilczynski [16] derived a complete set of invariants for such curves in any dimension, starting from the Laguerre-Forsyth normal form for linear ODEs.

Invariants of curves were derived in various non-metric geometries, for instance in conformal, Grassmannian and others [1,7,8]. Fundamental invariants for some classes of curves in generalized flag varieties were computed by Doubrov and Zelenko [5]. This approach was based on the moving frame method [4,15] and the theory of parabolic geometries [2]. In particular, it covered general curves in projective spaces and integral curves in the $G = G_2$ flag varieties for split algebraic $G_2 = G_2^*$, namely homogeneous space G/P with some parabolic subgroup P .

The goal of this paper is to revisit the case of curves in $M = G_2/P$ based on the theory of differential invariants for which we refer the reader to [14] (we assume the curves are regular, so that its velocity never vanishes). The algebra of such (absolute rational) invariants depends on the type of 1-jet of a curve, and we compute the Hilbert function counting the number of differential invariants for every type. Then we concentrate on the two poles, corresponding to the minimal and the maximal orbits of the action of G on

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$J^1(M, 1) = \mathbb{P}TM$. For integral curves, representing minimal orbits, we obtain the invariants differently and more explicitly than in [5]. For generic curves, representing maximal orbits, our results are apparently new.

We give more details to the case $M = G_2/P_1$ (labelling of the parabolic subgroups corresponds to the Bourbaki numeration), where our tool is the canonical conformal structure associated to the $(2, 3, 5)$ distribution by Nurowski. In the flat case that we discuss, it corresponds to another homogeneous representation $M = SO(3, 4)/P_1$ (here P_1 is the subgroup of B_3 naturally extending the previous P_1) induced by the inclusion $G_2 \subset SO(3, 4)$. We also discuss differential invariants of curves in other G_2 flag varieties with a different choice of the parabolic, namely G_2/P_{12} and G_2/P_2 , but since the formulae are large we do not provide full details of computations.

It turns out that there is a transformation (multiple-valued to one side) between generic curves in G_2/P_1 and G_2/P_2 based on the twistor correspondence via G_2/P_{12} and the geometry of those spaces. For minimal integral curves (no restriction for G_2/P_1 but corresponding to triple root Petrov type in the case G_2/P_2) such a transformation is known as the prolongation, but it does not exist for all curves. The transformation we propose for generic curves (it is $1 : 1$ to one side and $1 : 2$ to the other) allows to reduce computation of invariants to those of G_2/P_1 .

On historical side we note that two realisations of G_2 corresponding to homogeneous spaces G_2/P_1 and G_2/P_2 were obtained in 1893 by E. Cartan [3] and F. Engel [6]. An interplay between these models is a base for the above mentioned correspondence.

We also note that the method developed in this paper is applicable to obtain differential invariants of curves in curved geometries of type (G_2, P) , as we will briefly discuss in the conclusion, though we do not provide any explicit formulae for those. It would be interesting to extend the results to parabolic geometries of general type (G, P) .

Maple computations supporting the results of this paper, and containing some large formulae, can be found in a supplement to arXiv:2107.03664.

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1. Invariants of curves in $M^5 = G_2/P_1$

Associated with P_1 is the gradation of $\mathfrak{g} = \text{Lie}(G_2)$ of depth 3

$$\mathfrak{g} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3 \quad (1)$$

with $\dim \mathfrak{g}_{\pm 1} = \dim \mathfrak{g}_{\pm 3} = 2$, $\dim \mathfrak{g}_{\pm 2} = 1$ and $\mathfrak{g}_0 = \mathfrak{gl}_2$. The filtration $\mathfrak{g}^i = \bigoplus_{j \geq i} \mathfrak{g}_j$ is invariant with respect to $\mathfrak{p} = \mathfrak{g}^0$ and its Lie group $P_1 = GL_2 \times \exp(\mathfrak{p}_+)$, which is equal to $\text{Stab}_o(G_2)$, $o = [P_1]$, for the G_2 action on the 5-dimensional homogeneous space $M = G_2/P_1$.

We identify T_oM , as well as tangent spaces at other points of M , with $\mathfrak{m} = \mathfrak{g}/\mathfrak{p}$ and it will be convenient to interpret it as $\mathfrak{g}_- = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ though the gradation is not P_1 invariant. Furthermore $\mathfrak{g}^{-1} \bmod \mathfrak{p}$ defines a G_2 invariant rank 2 distribution Π on M with growth vector $(2, 3, 5)$, which exhibits ranks of the derived distributions $\Pi^2 = [\Pi, \Pi]$ and $\Pi^3 = [\Pi, \Pi^2] = TM$.

We use the coordinates (x, y, p, q, z) on M obtained from the model Monge equation $y'' = (z')^2$ with local G_2 symmetry. Namely, denoting $y' = p$ and $y'' = q$ the distribution Π is induced from the Cartan distribution in jets and has the following expression:

$$\Pi = \langle \partial_x + p\partial_y + q\partial_p + q^2\partial_z, \partial_q \rangle. \tag{2}$$

The G_2 invariant conformal structure $[g]$ is given by the representative

$$g = q^2 dx^2 - 2q dx dp + 6p dx dq - 3 dx dz - 6 dy dq + 4 dp^2. \tag{3}$$

Let $J^k(M, 1)$ denote the space of k -jets of unparametrized regular curves (for details on jet-spaces we refer to [13]). These will be represented as parametrized curves $\gamma : I \rightarrow M$ modulo the right action of the pseudogroup $\text{Diff}_{\text{loc}}(\mathbb{R})$ of reparametrizations. A 0-jet is just $\gamma(0) \in M$, while 1-jet is $\dot{\gamma}(0) \neq 0$ up to rescaling. Thus $J^0(M, 1) = M$ and $J^1(M, 1) = \mathbb{P}TM$. The fiber of the latter bundle over $o \in M$ will be identified with $(T_oM \setminus \{0\})/\mathbb{R}_\times \simeq \mathbb{P}\mathfrak{m}$, on which P_1 acts.

A general curve, transversal to the foliation $\{x = \text{const}\}$, can be parametrized as $y = y(x)$, $p = p(x)$, $q = q(x)$, $z = z(x)$. This introduces an affine chart in $J^k(M, 1)$ with coordinates x, y, p, q, z and y_i, p_i, q_i, z_i for $1 \leq i \leq k$. The action of G_2 will be expressed in these coordinates, as well as the invariants of the action. We note that the orbits of G_2 in $J^k(M, 1)$ are bijective with the orbits of P_1 on $J^k_o(M, 1)$, and we begin with a discussion of $k = 1$ case.

1.1. Action and orbits of P_1 on 1-jet

It is convenient to describe the action on the vector space \mathfrak{m} , and then pass to the corresponding projective space (here when describing the P action we use linear coordinates on \mathfrak{m} , and then return to the coordinates on M used before).

A basis e_1, e_2 of \mathfrak{g}_{-1} induces bases $e_3 = [e_1, e_2]$ of \mathfrak{g}_{-2} and $e_4 = [e_1, e_3], e_5 = [e_2, e_3]$ of \mathfrak{g}_{-3} . Thus we get a basis of $\mathfrak{m} \simeq \bigoplus_{i=-3}^{-1} \mathfrak{g}_i$. This introduces coordinates v_i on \mathfrak{m} considered as a graded nilpotent Lie algebra

$$v = v_1e_1 + v_2e_2 + v_3e_3 + v_4e_4 + v_5e_5 \in \mathfrak{m},$$

and hence also on $\exp(\mathfrak{m}) \simeq M$ via the exponential map. Note that the unity o corresponds to 0 and the inverse of $g \in \exp(\mathfrak{m})$ is $-g$. (The Lie bracket on \mathfrak{m} induces the group structure on $\exp(\mathfrak{m})$ by the Baker-Campbell-Hausdorff formula that is finite due to nilpotency of \mathfrak{m} .)

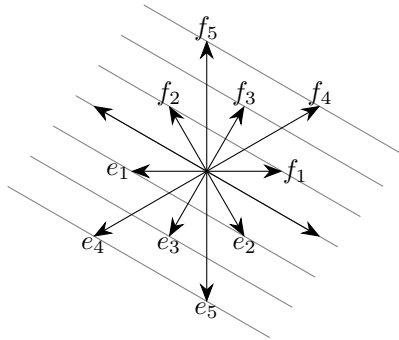
In these coordinates the canonical conformal structure (3) has constant coefficients:

$$\langle v, w \rangle = v_1w_5 + v_5w_1 - v_2w_4 - v_4w_2 + v_3w_3. \tag{4}$$

Indeed, this is a unique (up to scale) left-invariant metric on M that is also invariant with respect to semi-simple part of the reductive component GL_2 of P_1 (this preserves the grading on \mathfrak{m}) and has weight 4 with respect to its center:

$$\langle \text{Ad}_g(v), \text{Ad}_g(w) \rangle = \langle v, w \rangle \quad \forall g \in \exp(\mathfrak{m}) \quad \text{and} \quad \langle Av, Aw \rangle = (\det A)^2 \langle v, w \rangle \quad \forall A \in GL_2.$$

To obtain P_1 action on $\mathfrak{m} = \mathfrak{g}/\mathfrak{p}$ we derive first the action of \mathfrak{p}_+ using the root diagram of \mathfrak{g} . We choose the basis f_i of $\mathfrak{p}_+ \simeq \mathfrak{m}^*$ dual to e_i : these are the root vectors indicated near the corresponding roots on the diagram below (gradation is shown by a family of parallel lines). Their commutator relations are $[f_1, f_2] = f_3, [f_1, f_3] = f_4, [f_2, f_3] = f_5$.



Brackets between basis elements in \mathfrak{p}_+ and \mathfrak{m} correspond, up to scale, to vector summation of the corresponding root vectors in the root diagram: $[e_\alpha, e_\beta] = k_{\alpha\beta}e_{\alpha+\beta}$. The factors $k_{\alpha\beta}$ can be fixed by the condition that the metric is invariant under \mathfrak{p}_+ :

$$\langle \text{ad}_{f_i} v, w \rangle + \langle v, \text{ad}_{f_i} w \rangle = 0.$$

The diagram implies that $\mathfrak{g}_3 = \langle f_4, f_5 \rangle$ acts trivially on \mathfrak{m} and \mathfrak{p}_+ acts trivially on \mathfrak{g}_{-1} . The remaining commutation relations are as follows.

$$\begin{aligned} [f_1, e_3] &= e_2, & [f_2, e_3] &= -e_1, & [f_3, e_3] &= 0, \\ [f_1, e_4] &= e_3, & [f_2, e_4] &= 0, & [f_3, e_4] &= e_1, \\ [f_1, e_5] &= 0, & [f_2, e_5] &= e_3, & [f_3, e_5] &= e_2. \end{aligned}$$

Thus the action of $\exp(\mathfrak{p}_+)$ is encoded through the action of $\rho = \exp(s_1 f_1 + s_2 f_2 + s_3 f_3)$ with real parameters s_1, s_2, s_3 as follows:

$$\rho(v_1, v_2, v_3, v_4, v_5) = (v_1 - s_2 v_3 + (s_3 - s_1 s_2)v_4 - \frac{1}{2}s_2^2 v_5, v_2 + s_1 v_3 + \frac{1}{2}s_1^2 v_4 + (s_3 + s_1 s_2)v_5, v_3 + s_1 v_4 + s_2 v_5, v_4, v_5).$$

With these at hand we can now compute that, under the action of P_1 , a general point $v_1 e_1 + v_2 e_2 + v_3 e_3 + v_4 e_4 + v_5 e_5 \in \mathfrak{m} \setminus \{0\}$ can be mapped to one of the following 5 representatives:

- $e_5 \pm e_1$ if $(v_4, v_5) \neq (0, 0)$ and $2v_1 v_5 - 2v_2 v_4 + v_3^2 \geq 0$,
- e_5 if $(v_4, v_5) \neq (0, 0)$ and $2v_1 v_5 - 2v_2 v_4 + v_3^2 = 0$,
- e_3 if $v_4 = v_5 = 0$ and $v_3 \neq 0$,
- e_1 if $v_3 = v_4 = v_5 = 0$.

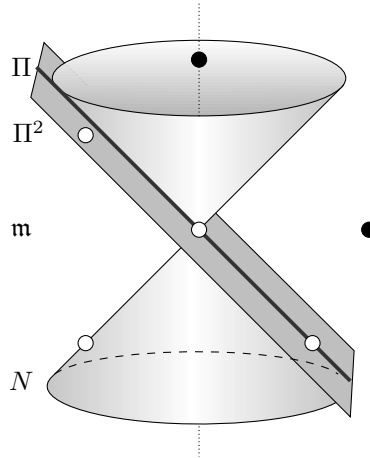
Associated to the conformal structure (4) is the null cone

$$N = \{v_1 e_1 + v_2 e_2 + v_3 e_3 + v_4 e_4 + v_5 e_5 \in \mathfrak{m} \mid 2v_1 v_5 - 2v_2 v_4 + v_3^2 = 0\}.$$

Its relation to the distributions Π and Π^2 is the following:

$$N \cap \Pi^2 = \Pi.$$

Each of the 5 points above and the singular orbit 0 represent orbits of the action of P_1 on \mathfrak{m} . Therefore there are 5 orbits of the action of P_1 on $\mathbb{P}\mathfrak{m}$.



The points $e_5 \pm e_1$ represent 2 open orbits, separated by the null cone (both orbits connected, the above 3D picture is just an analogy). The other orbits are (projectivizations of) $\Pi \setminus \{0\} \ni e_1$, $\Pi^2 \setminus \Pi \ni e_3$, $N \setminus \Pi \ni e_5$.

1.2. Number of invariants

Define s_k to be the transcendence degree of the field of rational differential invariants of the G_2 action on $J^k(M, 1)$. By [14] this value is equal to codimension of the regular orbit of the action, and so s_k can be interpreted as the number of invariants of order k . By the preceding computations, $s_0 = s_1 = 0$.

Hilbert function is defined as $h_k = s_k - s_{k-1}$ and it can be interpreted as the number of invariants of pure order k . We will consider curves of a fixed type \mathfrak{t} of their 1-jet. In other words, we assume that at any point 1-jet of the curve belongs to the same P_1 orbit in $\mathbb{P}\mathfrak{m}$.

If $d_{\mathfrak{t}}$ is dimension of the orbit, then such curves are given by $4 - d_{\mathfrak{t}}$ equations of the first order. Geometrically this specifies a submanifold $\mathcal{E}_{\mathfrak{t}} \subset J^1(M, 1)$ of codimension $4 - d_{\mathfrak{t}}$, and prolongations $\mathcal{E}_{\mathfrak{t}}^k := \mathcal{E}_{\mathfrak{t}}^{(k-1)} \subset J^k(M, 1)$ form a tower of bundles with the rank of $\pi_{k,k-1} : \mathcal{E}_{\mathfrak{t}}^k \rightarrow \mathcal{E}_{\mathfrak{t}}^{k-1}$ equal to $d_{\mathfrak{t}}$. Because G_2 is finite-dimensional, we will occasionally have $h_k = d_{\mathfrak{t}}$ for $k \gg 1$.

Let us consider the case of open orbits in $J^1(M, 1)$. Even though there are two types of such, the count is the same because under complexification these become one orbit and the action is algebraic. Due to transitivity we can fix 0-jet to be $a_0 = (0, 0, 0, 0, 0)$ in (x, y, p, q, z) coordinates. A generic 1-jet a_1 over a_0 can be represented by $(y_1, p_1, q_1, z_1) = (0, 0, 0, -1)$; note that $z_1 = +1$ for the other open orbit but we will focus on the first.

Denote by $\mathfrak{p}^{(k)}$ the prolongation of the isotropy \mathfrak{p} of the point $o = a_0$ in \mathfrak{g} . The action in the fibers of $\pi_{k,k-1}$ for $k > 1$ is affine, so we can restrict to usage of Lie algebras (not groups). Denote the isotropy subalgebra at the point a_1 by

$$\text{stab}_{a_1} = \{v \in \mathfrak{p}^{(1)} : v(a_1) = 0\}$$

and by $\text{stab}_{a_1}^k$ its prolongations to k -jets. As an abstract Lie algebra, $\text{stab}(a_1)$ is solvable and it is defined by the following structure relations

$$\begin{aligned} [s_1, s_2] &= s_2, & [s_1, s_3] &= s_3, & [s_1, s_4] &= s_4, \\ [s_1, s_5] &= 2s_5, & [s_2, s_4] &= 3s_5, & [s_3, s_4] &= 4s_5. \end{aligned}$$

The action of $\text{stab}_{a_1}^2$ on $\pi_{2,1}^{-1}(a_1) \simeq \mathbb{R}^4(y_2, p_2, q_2, z_2)$ is generated by the following vector fields

$$\partial_{p_2} - z_2\partial_{q_2} + 4y_2\partial_{z_2}, \quad \partial_{q_2}, \quad \partial_{z_2}, \quad p_2\partial_{p_2} + 2q_2\partial_{q_2} + z_2\partial_{z_2}, \quad 3y_2\partial_{p_2} + 4p_2\partial_{q_2}$$

and so the orbits have dimension 3. We have one invariant $I'_2 = y_2$ in $\pi_{2,1}^{-1}(a_1)$. Let $a_2 = (I'_2, 0, 0, 0) \in J^2(M, 1)$ be a point above a_1 (the expression for I'_2 is invariant only above a_1).

The isotropy algebra of a_2 is a 2-dimensional solvable subalgebra $\text{stab}_{a_2} \subset \mathfrak{p}^{(2)}$. Its prolongation to 3-jets, namely to $\pi_{3,2}^{-1}(a_2) \simeq \mathbb{R}(y_3, p_3, q_3, z_3)$, has generators

$$3y_3\partial_{p_3} + (3y_2z_3 + 4p_3)\partial_{q_3} - 12y_2y_3\partial_{z_3}, \quad y_3\partial_{y_3} + 2p_3\partial_{p_3} + 3q_3\partial_{q_3} + 2z_3\partial_{z_3},$$

and so we get 2 independent invariants in $\pi_{3,2}^{-1}(a_2)$. The isotropy algebra of a generic point in 3-jet is already trivial, so there will be 4 more independent invariants for every jet of order $k \geq 4$.

The count of invariants for other types \mathfrak{t} of 1-jet is performed similarly, so we omit the details. Summarizing, the Hilbert function h_k counting differential invariants is given in the table:

$\mathfrak{t} \setminus k$	0	1	2	3	4	5	6	7	8	9	10	...
$TM \setminus (N \cup \Pi^2)$	0	0	1	2	4	4	4	4	4	4	4	4
$N \setminus \Pi^2$	0	0	0	1	2	3	3	3	3	3	3	3
$\Pi^2 \setminus \Pi$	0	0	0	0	0	1	2	2	2	2	2	2
$\Pi \setminus \{0\}$	0	0	0	0	0	0	0	0	0	0	1	1

Let us observe from the first two rows that for the corresponding \mathfrak{t} -types there is dimensional freedom for the group to act freely on the level of 2- and 3-jets, respectively, yet there appear invariants. The situation with the last two rows is what could be expected in a general position.

In what follows we will focus on two particular cases: curves with minimal and maximal \mathfrak{t} -types of 1-jet (last and first rows) for which we describe the algebra of differential invariants explicitly. These algebras of differential invariants will be denoted by \mathcal{A}_i and \mathcal{A}_g respectively.

1.3. Invariants of integral curves

Consider curves in M tangent to Π . For such there are no absolute differential invariants up to jet-order 9, and the first invariant arises in order 10. There are however relative differential invariants: such functions R of order k satisfy $L_v R = \alpha(v)R$ for $v \in \mathfrak{g}^{(\infty)}$ and $\alpha \in C^\infty(J^k) \otimes \mathfrak{g}^*$. We find those by the method of Sophus Lie, namely via a computation of the loci where the rank of prolongations $e_j^{(k)} \in \mathfrak{X}(J^k)$ drop, for a basis $e_j \in \mathfrak{g}$.

The integral curves are subject to the constraints

$$y_1 = p, \quad p_1 = q, \quad z_1 = q^2, \tag{5}$$

their prolongation define the equation $\mathcal{E}_\Pi \subset J^\infty(M, 1)$ with coordinates $x, y, p, q, z, q_k, k \geq 1$.

The simplest relative differential invariant is q_2 . Next, such invariant arises in order 8, namely

$$R_8 = 196 q_2^5 q_8 - 2352 q_2^4 q_3 q_7 - 5040 q_2^4 q_4 q_6 - 3255 q_2^4 q_5^2 + 16632 q_2^3 q_3^2 q_6 + 59598 q_2^3 q_3 q_4 q_5 + 13772 q_2^3 q_4^3 - 83160 q_2^2 q_3^3 q_5 - 174735 q_2^2 q_3^2 q_4^2 + 297000 q_2 q_3^4 q_4 - 118800 q_3^6.$$

One more relative invariant of order 10 is given by the formula

$$R_{10} = 21q_2 R_8 \mathcal{D}_x (q_2 \mathcal{D}_x R_8) - \frac{91}{4} (q_2 \mathcal{D}_x R_8)^2 + 9R_8^2 (13q_3^2 - 19q_2 q_4).$$

The two latter relative invariants have proportional weights (that is, 1-forms α), which makes a combination of them an absolute differential invariant

$$I_{10} = \frac{R_{10}^3}{R_8^7}.$$

This is accompanied by an invariant derivation, i.e. a linear map $\square : C^\infty(J^k) \rightarrow C^\infty(J^{k+1})$ that satisfies the Leibniz rule and commutes with the action of G_2 . We search for it in the form $\square = h\mathcal{D}_x$, $h \in C^\infty(J^k)$, where

$$\mathcal{D}_x = \partial_x + p\partial_y + q\partial_p + q^2\partial_z + \sum_{i=0}^{\infty} q_{i+1}\partial_{q_i}$$

is the operator of total derivative on \mathcal{E}_Π . Then the invariance condition

$$[v^{(\infty)}, \square] = 0 \quad \forall v \in \mathfrak{g}$$

applied to x writes for a basis $e_j \in \mathfrak{g}$

$$L_{e_j^{(k)}}(h) = h\mathcal{D}_x(e_j(x)).$$

Finding h from this equation, we get the following invariant derivation (ι for ‘integral’):

$$\square_\iota = \frac{q_2}{R_8^{1/6}}\mathcal{D}_x.$$

The invariant derivation \square_ι produces the next differential invariant $I_{11} = \square_\iota(I_{10})$ of order 11, then $I_{12} = \square_\iota(I_{11})$ of order 12, and successively generates all the higher order invariants.

Theorem 1. *The algebra \mathcal{A}_ι of (micro-local) differential invariants of integral curves is generated in the Lie-Tresse sense by I_{10} and \square_ι .*

Remark 1. The invariant derivation \square_ι has non-rational coefficient. This is sufficient for micro-local invariants (defined in open non-invariant sets in jets), however is at odd with the claim that the global differential invariants are rational in jets [14]. To remedy this one passes to rational invariant derivation (Tresse derivative associated to I_{10})

$$\bar{\square}_\iota = \frac{d}{dI_{10}} := \frac{I_{10}}{\square_\iota(I_{10})} \cdot \square_\iota.$$

Then the algebra $\bar{\mathcal{A}}_\iota$ of global differential invariants for integral curves is generated by I_{10} , $\bar{I}_{11} = I_{11}^6$ and $\bar{\square}_\iota$. Note that $\bar{\square}_\iota(I_{10}) = 1$.

Proof. By the count of invariants there is precisely one independent differential invariant of pure order k for any $k \geq 10$. These are $\square_\iota^{k-10}I_{10}$. The micro-local claim follows.

To obtain the algebra of rational differential invariants let us note that the invariants $\bar{\square}_\iota^{k-11}\bar{I}_{11}$ are affine in jets of order $k > 11$. Invariants of order ≤ 11 are algebraically generated by I_{10} and \bar{I}_{11} since the ideal generated by them (in the ring of rational functions that are polynomial in jets of order > 9) is radical. This proves the claim. \square

1.4. *Invariants of generic curves*

Now we consider curves transversal to the distribution and not null with respect to conformal structure (3). Investigation of curves of both general type \mathfrak{t} of 1-jets goes parallel, so we may assume that the tangent $X = \dot{\gamma}$ to the curve satisfies $g(X, X) > 0$ (one has to take another normalization below for $g(X, X) < 0$).

When the curve is parametrized coordinately $\gamma(t) = (x(t), y(t), p(t), q(t), z(t))$, its tangent vector

$$X = \partial_x + y_1\partial_y + p_1\partial_p + q_1\partial_q + z_1\partial_z \tag{6}$$

is given by the truncated total derivative. Recall it is defined up to scale. We are going to exploit the change of scales in order to construct an invariant frame along the curve.

The metric g in (3) is defined up to rescaling. Another representative of $[g]$ is given by $\bar{g} = e^{2f}g$. Let $\nabla, \bar{\nabla}$ be the Levi-Civita connections of g, \bar{g} . They are related as follows:

$$\bar{\nabla}_X U = \nabla_X U + X(f)U + U(f)X - g(X, U)\nabla f \tag{7}$$

for $X, U \in \mathfrak{X}(M)$. Therefore $\nabla_X U$ is defined up to X, U and ∇f . The latter is difficult to control, so we will apply this formula only for $g(X, U) = 0$. Also, to obtain invariant quantities we can only differentiate in the direction of the curve, so X will be taken as in (6).

Since Π^2/Π has rank 1, there is a conformal identification $\Pi \simeq TM/\Pi^2$ based on $\mathfrak{g}_{-1} \simeq [\mathfrak{g}_{-1}, \mathfrak{g}_{-2}] = \mathfrak{g}_{-3}$ of (1). Since γ is a generic curve, its tangent $X \notin \Pi^2$ has a conformal dual $Y \in \Pi$. For instance, choosing the vector $e_3 = [\partial_q, \partial_x + p\partial_y + q\partial_p + q^2\partial_z] = \partial_p + 2q\partial_z \in \Pi^2$ we find a unique $Y \in \Pi$ from

$$[Y, e_3] = X \text{ mod } \Pi^2.$$

This Y is defined up to scale and satisfies $g(X, Y) = 0$. Therefore the covariant derivative $\nabla_X Y$ is defined up to X, Y and determines unambiguously the subspace

$$\Pi_X = \langle X, Y, \nabla_X Y \rangle \subset TM.$$

Generically Π_X has rank 3, $\Pi_X \text{ mod } \Pi$ rank 2, and $\Pi_X \text{ mod } \Pi^2$ rank 1. We change the generator $\nabla_X Y$ of Π_X to

$$Z \in \Pi_X \cap \Pi^2 \quad \text{such that} \quad g(X, Z - X) = 0.$$

This Z is defined up to Y and up to scale. However, due to the above relation, X and Z are subject to rescaling by the same factor. Therefore we have

$$\Pi_X = \langle X, Y, Z \rangle$$

and we use Z to fix the scale of Y : the conformal identification $[Y, Z] = X \text{ mod } \Pi^2$ determines Y uniquely.

The Gram matrix of Y, Z, X (in this order) is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \varkappa_2 & \varkappa_1 \\ 0 & \varkappa_1 & \varkappa_1 \end{pmatrix}$$

where $\varkappa_1 = 2R_1$, $\varkappa_2 = \frac{R_1^4}{162R_2^2}$ and

$$\begin{aligned}
 R_1 &= (q + 2p_1)^2 + 6(q_1(p - y_1) - qp_1) - 3z_1, \\
 R_2 &= -\frac{(q+2p_1)^3}{18} + (p - y_1)\left(qp_2 - \frac{z_2}{2}\right) + (q + y_2)\left(qp_1 - \frac{z_1}{2}\right) + p_1z_1 - q^2\frac{y_2}{2}
 \end{aligned}
 \tag{8}$$

are relative invariants of orders 1 and 2 respectively. Their ratio

$$I_2 = \frac{\varkappa_1}{324\varkappa_2} = \frac{R_2^2}{R_1^3}
 \tag{9}$$

is the first absolute differential invariant in $J^2(M, 1)$; note that this invariant, when restricted to $\pi_{2,1}^{-1}(a_1)$, differs from I'_2 of subsection 1.2 only by a power and a factor: $108I_2 = (I'_2)^2$.

As long as $\varkappa_1 \neq \varkappa_2$, which is generically true, the conformal metric has rank 2 on the 3-dimensional Π_X . Hence Π_X^\perp has rank 2 and $\Pi_X \cap \Pi_X^\perp = \langle Y \rangle$. Choose

$$V \in \Pi \quad \text{such that} \quad g(X, V - X) = 0.$$

This V is defined up to Y and up to scale with the same factor as X . Therefore we have

$$\Pi_X + \Pi_X^\perp = \langle V, Y, Z, X \rangle.$$

To complete to a (yet non-canonical) frame we add the vector

$$W = \nabla_X(Z - X).$$

Since $g(X, Z - X) = 0$, this W defined up to scale, and up to X and Z , which in turn is defined up to X and Y . The vectors X and W are independent mod Π^2 .

The Gram matrix of Y, V, Z, X, W (in this order) is

$$\begin{pmatrix}
 0 & 0 & 0 & 0 & \varkappa_3 \\
 0 & 0 & 0 & \varkappa_1 & k_4 \\
 0 & 0 & \varkappa_2 & \varkappa_1 & k_3 \\
 0 & \varkappa_1 & \varkappa_1 & \varkappa_1 & k_2 \\
 \varkappa_3 & k_4 & k_3 & k_2 & k_1
 \end{pmatrix}$$

where $\varkappa_3 = 2R_1\left(108I_2 - \frac{1}{3}\right)$ is a relative invariant.

Keeping track of the choices, we change our vectors and the Gram matrix changes accordingly. Precisely, the freedom we have in defining our vectors allows us to make the following transformations:

$$\begin{aligned}
 X &\mapsto c_1X, \\
 Y &\mapsto Y, \\
 Z &\mapsto c_1Z + c_2Y, \\
 V &\mapsto c_1V + c_3Y, \\
 W &\mapsto c_1^2W + c_4Z - (c_4 + c_1c_2k_5)X + c_5Y,
 \end{aligned}$$

where $k_5 = \frac{p_1 - q}{3} + \frac{3R_2}{R_1}$ and we have 5 degrees of freedom given by c_1, c_2, c_3, c_4, c_5 . We find c_2, c_3, c_4, c_5 such that $k_1 = k_2 = k_3 = k_4 = 0$. Under this transformation of our vectors, the Gram matrix of Y, V, Z, X, W takes the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & c_1^2 \varkappa_3 \\ 0 & 0 & 0 & c_1^2 \varkappa_1 & 0 \\ 0 & 0 & c_1^2 \varkappa_2 & c_1^2 \varkappa_1 & 0 \\ 0 & c_1^2 \varkappa_1 & c_1^2 \varkappa_1 & c_1^2 \varkappa_1 & 0 \\ c_1^2 \varkappa_3 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It only remains to fix the scale of X and the scale of the metric g .

We fix c_1 by the condition $\mathcal{L}_X I_2 = 1$, that is, by setting

$$c_1 = \frac{1}{\mathcal{D}_x I_2},$$

where \mathcal{D}_x is the operator of total derivative:

$$\mathcal{D}_x = \partial_x + \sum_{i=0}^{\infty} (y_{i+1} \partial_{y_i} + p_{i+1} \partial_{p_i} + q_{i+1} \partial_{q_i} + z_{i+1} \partial_{z_i}).$$

Then we fix the scale of the metric $\bar{g} = e^{2f} g$ by

$$\bar{g}(X, X) = 1. \tag{10}$$

This fixes an invariant frame adapted to the distribution and the conformal structure. With this the above Gram matrix becomes

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 108I_2 - \frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{324} I_2^{-1} & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 108I_2 - \frac{1}{3} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The next step is to generate differential invariants of orders 3 and 4 in $J^4(M, 1)$. In order to do so first note that the new metric given by (10) involves rescaling depending on 3-jet, and so its Levi-Civita connection is uncomputable (we can only differentiate along the curve), yet formula (7) applied to $U \in \langle X \rangle^\perp$ has a well-defined output $\bar{\nabla}_X U \bmod X$.

Let w_i denote the basis $Y, Z - X, V - X, W$ of $\langle X \rangle^\perp$. Then we decompose

$$\bar{\nabla}_X w_i = \sum_{j=1}^4 a_{ij} w_j \bmod X, \quad 1 \leq i \leq 4.$$

The coefficients a_{ij} are differential invariants. Some of them are constants or expressed through I_2 , some are related due to the fact that $\bar{\nabla}_X \bar{g} = 0$, but the others will determine 2 differential invariants I_{3a}, I_{3b} of order 3 and 4 differential invariants $I_{4a}, I_{4b}, I_{4c}, I_{4d}$ of order 4 (all rational and independent). The formulae are too large to be given explicitly, but in $\pi_{4,2}^{-1}(a_2)$ the invariants I_{3a}, I_{3b} are equal to $\frac{4I_2 p_3 + z_3}{y_3^2}, \frac{16p_3^2 - 24y_3 q_3 - 3z_3^2}{y_3^4}$, while I_{4j} are affine in y_4, p_4, q_4, z_4 .

Finally the normalized X yields the invariant derivation (Tresse derivative associated to I_2)

$$\square_g = \frac{d}{dI_2} := \frac{1}{\mathcal{D}_x I_2} \cdot \mathcal{D}_x$$

which on scalars coincides with $L_X = \bar{\nabla}_X$.

Theorem 2. *The algebra \mathcal{A}_g of differential invariants of generic curves is generated in the Lie-Tresse sense by seven differential invariants I_2, I_{3i}, I_{4j} and one invariant derivation \square_g .*

Proof. By construction the seven invariants generate all differential invariants of order ≤ 4 . Invariant derivation provides independent invariants affine in jets of order $k > 4$ in the totality equal to the number of those jets. Thus any differential invariant from \mathcal{A}_g can be rationally expressed through the given generators. \square

2. Invariants of curves in $\hat{M}^6 = G_2/P_{12}$

Associated with P_{12} is the gradation of $\mathfrak{g} = \text{Lie}(G_2)$ of depth 5

$$\mathfrak{g} = \mathfrak{g}_{-5} \oplus \mathfrak{g}_{-4} \oplus \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3 \oplus \mathfrak{g}_4 \oplus \mathfrak{g}_5 \tag{11}$$

with $\dim \mathfrak{g}_{\pm 1} = \dim \mathfrak{g}_0 = 2$ and $\dim \mathfrak{g}_i = 1$ otherwise. The filtration \mathfrak{g}^i is introduced as in Section 1 and it defines the distribution Δ of growth $(2, 3, 4, 5, 6)$ invariant with respect to G_2 ; the stabilizer of $o \in \hat{M}$ is $P_{12} = (\mathbb{R}^1_{\times} \times \mathbb{R}^1_{\times}) \ltimes \exp(\mathfrak{p}_+)$.

To introduce coordinates on \hat{M} it is convenient to identify it with the prolongation of (M, Π) , namely the \mathbb{P}^1 bundle $\mathbb{P}\Pi$ over M : its points are $\hat{a} = (a, \ell)$, where $a \in M$ is a point and $\ell \subset \Pi_x$ is a line. Thus we can use the coordinates (x, y, p, q, z, r) for an open chart in \hat{M} , where the first 5-tuple gives a chart in M as in Section 1 and the line ℓ has coordinates $[1 : r]$ in the basis (2) of Π . This gives the representation

$$\Delta = \hat{\Pi} = \langle \partial_x + p\partial_y + q\partial_p + q^2\partial_z + r\partial_q, \partial_r \rangle \tag{12}$$

where both generators are distinguished: the first by the prolongation procedure described above and the second as the kernel of the differential of the projection $\pi_l : \hat{M} \rightarrow M$.

This projection relates the derived distributions as follows: $\pi_l^{-1}(\Pi) = \Delta^2$ and $\pi_l^{-1}(\Pi^2) = \Delta^3$. In addition, the pullback of the conformal structure on M gives a degenerate conformal structure on \hat{M} with the null cone $\hat{N} = \pi_l^{-1}(N) \simeq N \times \mathbb{R}^1$ in $T\hat{M}$.

We parametrize the curves again by x , so the jet-coordinates on $J^\infty(\hat{M}, 1)$ are (x, y, p, q, z, r) and $(y_k, p_k, q_k, z_k, r_k)$. The null cone is given by the condition $R_1 = 0$ of the G_2/P_1 case (8). The equation for derived flag is given by the following conditions

$$\begin{aligned} \mathcal{E}_\Delta &= \{q_1 = r, p_1 = q, y_1 = p, z_1 = q^2\} \\ \mathcal{E}_{\Delta^2} &= \{p_1 = q, y_1 = p, z_1 = q^2\} \\ \mathcal{E}_{\Delta^3} &= \{y_1 = p, z_1 = 2qp_1 - q^2\} \\ \mathcal{E}_{\Delta^4} &= \{z_1 = 2pr - 2ry_1 + 2qp_1 - q^2\} \end{aligned}$$

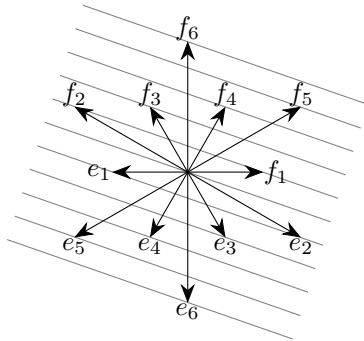
that determine some types of 1-jets; curves with the given fixed type are solutions to the corresponding prolonged first order systems $\mathcal{E}_{\Delta^s} \subset J^\infty(\hat{M}, 1)$. There are however more types of 1-jets of curves in \hat{M} as we will describe next.

2.1. Action and orbits of P_{12} on 1-jets

The generators of (12) correspond to a basis e_1, e_2 of \mathfrak{g}_{-1} in (11), which generates a basis $\{e_i\}$ of \mathfrak{m} via commutation:

$$\begin{aligned} [e_1, e_2] &= e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_3, e_4] = e_6, [e_2, e_5] = -e_6, \\ [f_1, f_2] &= f_3, [f_1, f_3] = f_4, [f_1, f_4] = f_5, [f_3, f_4] = f_6, [f_2, f_5] = -f_6. \end{aligned}$$

This and its dual basis $\{f_i\}$ of \mathfrak{p}_+ are indicated on the root diagram as before.



From Serre’s relations we find brackets involving \mathfrak{g}_0 :

$$\begin{aligned}
 [h_1, e_1] &= 2e_1 & [h_1, e_2] &= -3e_2 & [h_2, e_1] &= -e_1 & [h_2, e_2] &= 2e_2 & [e_1, f_1] &= h_1 \\
 [h_1, f_1] &= -2f_1 & [h_1, f_2] &= 3f_2 & [h_2, f_1] &= f_1 & [h_2, f_2] &= -2f_2 & [e_2, f_2] &= h_2
 \end{aligned}$$

The remaining structure relations are written basing on the root arithmetic with unknown coefficients, which are then uniquely determined from the Jacobi identity.

With this knowledge we compute the action of \mathfrak{p}_+ on \mathfrak{m} . This in turn determines the action of $\rho = \exp\left(\sum_{k=1}^5 s_k f_k\right) \in \exp(\mathfrak{p}_+)$ on $v = \sum_{k=1}^6 v_k e_k$ (f_6 acts trivially) as follows:

$$\begin{aligned}
 &\rho(v_1, v_2, v_3, v_4, v_5, v_6) = \\
 &\left(v_1 - s_2 v_3 + (4s_3 - 2s_1)v_4 + (6s_1 s_3 - 2s_1^2)v_5 + \left(\frac{1}{2}s_1^2 s_2^2 - 6s_3^2 + 12s_2 s_4\right)v_6, \right. \\
 &\quad v_2 + 3s_1 v_3 + 6s_1^2 v_4 + 6s_1^3 v_5 - (6s_1^2 s_3 + \frac{3}{2}s_1^3 s_2 + 18s_1 s_4)v_6, \\
 &\quad \left. v_3 + 4s_1 v_4 + 6s_1^2 v_5 - (2s_2 + 6s_3 + \frac{1}{3}s_4)v_6, v_4 + 3s_1 v_5 - (3s_3 + \frac{3}{2}s_1 s_2)v_6, v_5 - s_2 v_6, v_6 \right).
 \end{aligned}$$

The group $G_0 = \mathbb{R}_\times \times \mathbb{R}_\times$ action on \mathfrak{g}_{-1} extends to an automorphism of \mathfrak{m} . So we derive the action of $P_{12} = G_0 \ltimes \exp(\mathfrak{p}_+)$ on $\mathfrak{m} \setminus \{0\}$ and this yields the decomposition into orbits as follows:

- 3 orbits in $TM \setminus \Delta^4$: one closed in \hat{N} and two open separated by \hat{N} ,
- ∞ orbits in $\Delta^4 \setminus (\Delta^3 \cup H_3)$: there is an absolute invariant in $\Delta^4 \setminus \hat{N}$,
- 3 orbits in $H_3 \setminus \Delta^3$: one closed in \hat{N} and two open separated by \hat{N} ,
- 2 orbits in $\Delta^3 \setminus \Delta^2$: one closed in H_2 and one open in the complement,
- 1 orbit in $\Delta^2 \setminus \Delta$,
- 3 orbits in $\Delta \setminus \{0\}$: two lines and the complement.

Here $H_2 = \{a_1 \in \Delta^3 : h_2(a_1) = 0\}$ and $H_3 = \{a_1 \in \Delta^4 : h_3(a_1) = 0\}$ have the defining equations:

$$\begin{aligned}
 h_2 &= 8p_1 r_1 - 8q r_1 - 3q_1^2 + 6r q_1 - 3r^2, \\
 h_3 &= 9p^2 r_1 + 9p q_1 p_1 - 9p r p_1 - 9p q q_1 + 9p q r - 18p r_1 y_1 + 4p_1^3 - 12q p_1^2 \\
 &\quad + 12q^2 p_1 - 9p_1 q_1 y_1 + 9r p_1 y_1 - 4q^3 + 9q q_1 y_1 - 9q r y_1 + 9r_1 y_1^2.
 \end{aligned}$$

We note that h_2 is a relative invariant in Δ^3 and h_3 is a relative invariant in Δ^4 . Moreover, $h_3 = R_2|_{\Delta^4}$, where R_2 is the same relative invariant as in G_2/P_1 (note that the order of R_2 drops to 1 when we restrict to

the prolongation of the equation for Δ^4). Actually, the restriction of I_2 to $\Delta^4 \setminus \hat{N}$ is an absolute differential invariant, where I_2 is the second order differential invariant of generic curves in G_2/P_1 pulled back to G_2/P_{12} .

The relative invariant h_2 comes from a relative invariant for integral curves in G_2/P_2 .

2.2. Number of invariants

Similar to what is done in Section 1, we compute the Hilbert function h_k counting the number of differential invariants of pure order k , depending on the type \mathfrak{t} of the orbit of 1-jet, and tabulate it as follows.

$\mathfrak{t} \setminus k$	0	1	2	3	4	5	6	7	8	9	...
$T\hat{M} \setminus (\Delta^4 \cup \hat{N})$	0	0	2	5	5	5	5	5	5	5	5
$\hat{N} \setminus \Delta^4$	0	0	1	3	4	4	4	4	4	4	4
$\Delta^4 \setminus (\Delta^3 \cup \hat{N} \cup H_3)$	0	1	0	3	4	4	4	4	4	4	4
$H_3 \setminus (\Delta^3 \cup \hat{N})$	0	0	0	1	3	3	3	3	3	3	3
$(\Delta^4 \cap \hat{N}) \setminus (\Delta^3 \cup H_3)$	0	0	0	1	3	3	3	3	3	3	3
$(\hat{N} \cap H_3) \setminus \Delta^3$	0	0	0	0	0	2	2	2	2	2	2
$\Delta^3 \setminus (\Delta^2 \cup H_2)$	0	0	0	1	3	3	3	3	3	3	3
$H_2 \setminus \Delta^2$	0	0	0	0	0	2	2	2	2	2	2
$\Delta^2 \setminus \Delta$	0	0	0	0	0	2	2	2	2	2	2
$\Delta \setminus \{0\}$	0	0	0	0	0	0	0	0	0	1	1

Again we consider in more details only differential invariants of curves of constant type with either minimal (actually next to it: integral) or maximal type \mathfrak{t} of the orbit in 1-jets.

2.3. Invariants of integral curves

The first absolute differential invariant \hat{I}_9 of curves tangent to Δ in $\hat{M} = G_2/P_{12}$ occurs in order 9. It coincides with the invariant I_{10} for curves tangent to Π in G_2/P_1 after the change of coordinates $r_i = q_{i+1} \forall i$. (Recall that the differential equation for Δ is obtained from that of Π by intersecting with $q_1 = r$ and its prolongation.)

An invariant derivation is

$$\hat{\square}_i = \frac{r_1}{\hat{R}_7^{1/6}} \hat{\mathcal{D}}_x,$$

where \hat{R}_7 coincides with R_8 from Section 1.3 after the same change of coordinates, and

$$\hat{\mathcal{D}}_x = \partial_x + p\partial_y + q\partial_p + q^2\partial_z + r\partial_q + \sum_{i=0}^{\infty} r_{i+1}\partial_{r_i}$$

is the operator of total derivative on \mathcal{E}_Δ .

Thus the algebra of differential invariants $\hat{\mathcal{A}}_i$ is generated similarly to Section 1.3, and this is not surprising: there is a bijection between integral curves of Π and those of $\Delta = \hat{\Pi}$. Indeed, the prolongation $\gamma(t) \mapsto (\gamma(t), \dot{\gamma}(t))$ lifts the integral curves of Π to the integral curves of Δ , and the projection gives the inverse map.

2.4. Invariants of generic curves

We have the same relative invariants R_1 and R_2 as in G_2/P_1 . In addition, we have the first order relative invariant

$$R_3 = 2pr + 2qp_1 - 2ry_1 - q^2 - z_1,$$

that is the pull-back of the contact condition (integral curves) in G_2/P_2 .

Some other differential invariants have been computed, like relative invariant R_4 of order 2 (in Maple), but their formulae are long and we describe the algebra $\hat{\mathcal{A}}_g$ of invariants differently.

For a generic curve $\hat{\gamma}(t) \subset \hat{M}$ its projection $\gamma(t) = \pi_l \circ \hat{\gamma}(t) \subset M$ is also generic, and hence by the results of Section 1.4 possesses a frame Y, V, Z, X, W along it. A point $\hat{\gamma}(t)$ over $\gamma(t)$ can be interpreted as a line $\ell_{\gamma(t)} \subset \Pi_{\gamma(t)}$. There exists a unique $\varrho = \varrho(t) \in \bar{\mathbb{R}} = \mathbb{R} \cup \infty$ such that $Y + \varrho V \in \ell_{\gamma(t)}$. This ϱ is a function on the curve, and it defines a rational function on the space of jets of generic curves in \hat{M} , denoted by the same symbol.

Let us also note that the differential parameter along the curve $\hat{\gamma}$ can be induced from the differential parameter along its projection γ . In other words, the invariant derivation \square_g from Section 1.4 induces the following invariant derivation in $J^\infty(\hat{M}, 1)$:

$$\hat{\square}_g = \frac{1}{\mathcal{D}_x I_2} \cdot \hat{\mathcal{D}}_x,$$

where we use the operator of total derivative

$$\hat{\mathcal{D}}_x = \partial_x + \sum_{i=0}^{\infty} \left(y_{i+1} \partial_{y_i} + p_{i+1} \partial_{p_i} + q_{i+1} \partial_{q_i} + z_{i+1} \partial_{z_i} + r_{i+1} \partial_{r_i} \right).$$

Theorem 3. *The algebra $\hat{\mathcal{A}}_g$ of differential invariants of generic curves in \hat{M} is generated by the differential invariants from Theorem 2, pulled back from $J^\infty(M, 1)$ to $J^\infty(\hat{M}, 1)$, the invariant ϱ and the derivation $\hat{\square}_g$.*

Proof. A curve $\hat{\gamma} \subset \hat{M}$ is uniquely encoded by its projection $\gamma \subset M$ and its enhancement ℓ_γ , equivalently represented by the function ϱ . Hence it suffices to add this invariant to the generating set for \mathcal{A}_g to generate $\hat{\mathcal{A}}_g$. \square

3. Invariants of curves in $K^5 = G_2/P_2$

Associated with P_2 is the contact gradation of $\mathfrak{g} = \text{Lie}(G_2)$

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \tag{13}$$

with $\dim \mathfrak{g}_{\pm 1} = 4$, $\dim \mathfrak{g}_{\pm 2} = 1$ and $\mathfrak{g}_0 = \mathfrak{gl}_2$. The manifold $K = G_2/P_2$ possesses G_2 -invariant contact structure $D \subset TK$ and a field of rational normal curves (RNC) in $\mathbb{P}D$, corresponding to the minimal orbit of (the reductive part of) the structure group $G_0 = GL_2$, also identified with its cone field $\Gamma \subset D$ (in projectivization we will write $[\Gamma]$).

In coordinates (x, y, p, q, z) on K the contact structure is the annihilator of $\alpha = dz - p dx - q dy$ and the rational normal cone (also abbreviated RNC) is given by the following ideal in $S^\bullet D^*$:

$$\langle 3 dx dp - dy dq, \sqrt{3} dx dy - dq^2, \sqrt{3} dp dq - dy^2 \rangle.$$

In other words, RNC is given by the union of 1-parametric family of lines

$$\Gamma = \bigcup_{r \in \bar{\mathbb{R}}} \xi_r \subset D, \quad \text{where} \quad \xi_r = \langle (\partial_x + p \partial_z) + r \sqrt{3} \partial_q + r^2 \sqrt{3} (\partial_y + q \partial_z) + r^3 \partial_p \rangle. \tag{14}$$

The tangent to the RNC is the hypersurface in D of degree 4 given by

$$T\Gamma = \{a(\partial_x + p\partial_z) + b\partial_q + c(\partial_y + q\partial_z) + d\partial_p : 4(ac^3 + b^3d) = \sqrt{3}(b^2c^2 - 3a^2d^2 + 6abcd)\}.$$

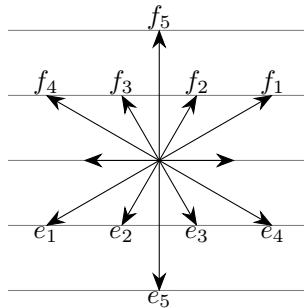
The curves tangent to those varieties are given by the following equations respectively:

$$\begin{aligned} \mathcal{E}_D &= \{z_1 = p + qy_1\}, \\ \mathcal{E}_{T\Gamma} &= \{4(y_1^3 + p_1q_1^3) = \sqrt{3}(y_1^2q_1^2 - 3p_1^2 + 6y_1p_1q_1), z_1 = p + qy_1\}, \\ \mathcal{E}_\Gamma &= \left\{y_1 = \frac{q_1^2}{\sqrt{3}}, p_1 = \frac{q_1^3}{3\sqrt{3}}, z_1 = p + \frac{qq_1^2}{\sqrt{3}}\right\}. \end{aligned} \tag{15}$$

Note that in our coordinates (x, y, p, q, z) the invariant conformally symplectic structure has the canonical form $\omega = d\alpha = dx \wedge dp + dy \wedge dq$, while the RNC has coefficients involving $\sqrt{3}$ (if we normalize RNC standardly, then the symplectic structure has a coefficient 3).

3.1. Action and orbits of P_2 on 1-jets

The action of \mathfrak{p}_+ on $\mathfrak{m} = \mathfrak{g}/\mathfrak{p}$ for $\mathfrak{p} = \mathfrak{p}_2$ is nontrivial only on \mathfrak{g}_{-2} . Moreover \mathfrak{g}_2 acts trivially and parametrizing \mathfrak{g}_1 by the coefficients s_1, s_2, s_3, s_4 in the basis f_1, f_2, f_3, f_4 of \mathfrak{g}_1 dual to the basis e_1, e_2, e_3, e_4 of \mathfrak{g}_{-1} as described on the picture



we encode the action as follows:

$$(v_1, v_2, v_3, v_4, v_5) \mapsto (v_1 + s_1v_5, v_2 + s_2v_5, v_3 + s_3v_5, v_4 + s_4v_5, v_5),$$

where v_i are coordinates on $\mathfrak{m} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}$ associated to the basis e_1, \dots, e_5 .

The action of $G_0 = GL_2$ on \mathfrak{g}_{-1} is given by the matrix

$$\begin{pmatrix} a^3 & \sqrt{3}a^2c & \sqrt{3}ac^2 & c^3 \\ \sqrt{3}a^2b & a^2d + 2abc & 2acd + bc^2 & \sqrt{3}c^2d \\ \sqrt{3}ab^2 & 2abd + b^2c & ad^2 + 2bcd & \sqrt{3}cd^2 \\ b^3 & \sqrt{3}b^2d & \sqrt{3}bd^2 & d^3 \end{pmatrix}$$

in coordinates (v_1, v_2, v_3, v_4) and it extends to \mathfrak{g}_{-2} by $v_5 \mapsto (ad - bc)^3v_5$.

Hence the action of P_2 decomposes \mathfrak{m} into the following orbits

- One orbit in $TM \setminus D$,
- Two orbits in $D \setminus T\Gamma$,

- One orbit in $T\Gamma \setminus \Gamma$,
- One orbit in $\Gamma \setminus \{0\}$.

The curves of fixed type \mathfrak{t} of their 1-jet according to the orbit type as above, are given by the equations $\mathcal{E}_{\mathfrak{t}}$ described in (15).

3.2. Number of invariants

Similar to what is done in Section 1, we compute the Hilbert function h_k counting the number of differential invariants of pure order k , depending on the type \mathfrak{t} of the orbit of 1-jet, and tabulate it as follows.

$\mathfrak{t} \setminus k$	0	1	2	3	4	5	6	7	8	9	10	...
$TM \setminus D$	0	0	0	3	4	4	4	4	4	4	4	4
$D \setminus T\Gamma$	0	0	0	1	2	3	3	3	3	3	3	3
$T\Gamma \setminus \Gamma$	0	0	0	0	0	1	2	2	2	2	2	2
$\Gamma \setminus \{0\}$	0	0	0	0	0	0	0	0	0	0	1	1

Again we consider in more details only differential invariants of curves of constant type with either minimal or maximal type \mathfrak{t} of the orbit in 1-jets.

3.3. Invariants of minimal integral curves

There are several types \mathfrak{t} of integral curves for G_2/P_2 , we consider those that are tangent to RNC Γ . The algebra of differential invariants $\mathcal{I}_{\mathfrak{t}}$ of these curves is generated by

$$\tilde{I}_{10} = \frac{\tilde{R}_{10}}{\tilde{R}_8^{7/3}} \quad \text{and} \quad \tilde{\square} = \frac{q_2}{\tilde{R}_8^{1/6}} \mathcal{D}_x,$$

where \mathcal{D}_x is the operator of total derivative restricted to \mathcal{E}_{Γ} ,

$$\begin{aligned} \tilde{R}_8 = & 196 q_8 q_2^5 - 3136 q_2^4 q_3 q_7 - 5936 q_2^4 q_4 q_6 - 3605 q_2^4 q_5^2 + 26208 q_2^3 q_3^2 q_6 + 83538 q_2^3 q_3 q_4 q_5 \\ & + 18252 q_2^3 q_4^3 - 144144 q_2^2 q_3^3 q_5 - 281853 q_2^2 q_3^2 q_4^2 + 555984 q_2^4 q_4 q_2 - 247104 q_3^6 \end{aligned}$$

corresponds to the invariant R_8 from the case G_2/P_1 , and similar for \tilde{R}_{10} (see Theorem 1 and Remark 1). Explicitly this correspondence will be explained in the next section.

3.4. Invariants of generic curves

For generic curves $\gamma \subset K$ transversal to the distribution D , the first relative invariant appears in order 1 and it corresponds to tangency with D :

$$\tilde{R}_1 = z_1 - p - qy_1.$$

The second order relative invariant \tilde{R}_2 has more complicated formula, but the most difficult are absolute differential invariants $\tilde{I}_{3a}, \tilde{I}_{3b}, \tilde{I}_{3c}, \tilde{I}_{4d}$ of orders $3(\times 3)$ and 4, which together with the invariant derivation $\tilde{\square}_g$ generate the algebra of differential invariants \mathcal{I}_g . Below we explain how to obtain an invariant frame that, in principle, determines all basic invariants.

It is known [2] that for every contact parabolic geometry, in particular for $K = G_2/P_2$, there is a unique (up to projective reparametrization) canonical (distinguished) curve through any point $a \in K$ in any direction $v \in T_aK \setminus D_a$. For $\gamma(0) = a$ choose $X = \dot{\gamma}(0)$ and denote this curve by δ_X . We treat both curves γ and δ_X as unparametrized. They have the same 1-jet, and therefore their difference canonically determines 2-jet $\zeta_X \in S^2T_a^*\gamma \otimes \nu_a$, where $\nu_a = T_aK/\langle X \rangle$ is the normal to γ at a . The image $\zeta_X(X, X) \in T_aK \text{ mod } X$ defines uniquely a 2-plane in T_aK , containing X , the intersection of which with D_a is a line denoted Υ_X .

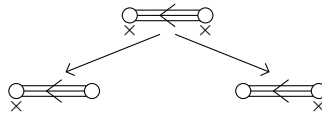
Next we use the projective geometry of $\mathbb{P}D$ equipped with RNC $[\Gamma]$. The above construction gives a point $[\Upsilon_X] \in \mathbb{P}D$. We assume the genericity condition $[\Upsilon_X] \notin T[\Gamma]$. In this case there exists a unique bisection L of $[\Gamma]$ containing $[\Upsilon_X]$, intersecting the RNC in two points λ_X^\pm (there is no canonical way to distinguish between \pm , so these points enter non-numerated; in the case $[\Upsilon_X] \in T[\Gamma] \setminus [\Gamma]$ they coincide and in the case $[\Upsilon_X] \in [\Gamma]$ there are infinitely many lines L).

Moreover we can introduce two more points $\mu_X^\pm = T_{\lambda_X^\pm}[\Gamma] \cap T_{\lambda_X^\mp}^2[\Gamma]$ on the (unique) intersection of the first and second tangents at the points λ_X^+ and λ_X^- (or interchange). The corresponding lines in D can be normalized so that to form a conformally symplectic basis with respect to ω . Only one overall scale is missing to obtain the frame from those vectors jointly with X and to fix the contact form. This can be normalized via a differential invariant.

The formulae are rather complicated, so instead we show in the next section how to relate the equivalence problem for generic curves in $K = G_2/P_2$ to those in $M = G_2/P_1$.

4. Twistor correspondence

The three realizations of G_2 , acting on various G_2/P as discussed above, are conveniently related by the following double fibration (parabolic subgroups P correspond to crosses on the Dynkin-Satake diagrams).



The arrows are projections corresponding to the inclusions $P_1 \leftarrow P_{12} \hookrightarrow P_2$. Below we explain how this correspondence relates three equivalence problems studied in this paper.

4.1. Correspondence for points

For G_2/P_{12} we used the nomenclature \hat{M} because it was the geometric prolongation of $M = G_2/P_1$. In coordinate language the affine chart $\mathbb{R}^5(x, y, p, q, z)$ of M is covered by the affine chart $\mathbb{R}^6(x, y, p, q, z, r)$, where r is such that the point of \hat{M} is represented by the line $\ell = \langle \partial_x + p\partial_y + q\partial_p + q^2\partial_z + r\partial_q \rangle$ in the distribution (2). The rank 2 distribution Δ of G_2/P_{12} with the canonical split into $1 + 1$ line subbundles is given by (12).

We can also represent G_2/P_{12} as the geometric prolongation of $K = G_2/P_2$, so that its points $\hat{b} = (b, \xi_r)$ are lines in RNC $\xi_r \subset D_b$, $b \in K$, see (14). In coordinates this gives $G_2/P_{12} = \hat{K}$ with affine chart $\mathbb{R}^6(x, y, p, q, z, r)$, and this is naturally equipped with the rank 2 distribution of growth (2, 3, 4, 5, 6) that is canonically split into $1 + 1$ lines subbundles as follows:

$$\tilde{\Delta} = \langle (\partial_x + p\partial_z) + r\sqrt{3}\partial_q + r^2\sqrt{3}(\partial_y + q\partial_z) + r^3\partial_p, \partial_r \rangle. \tag{16}$$

There is a diffeomorphism $\varphi : (\hat{M}, \Delta) \rightarrow (\hat{K}, \tilde{\Delta})$ that interchanges the first and the second generators of the distributions. In other words, the vertical line (fiber to π_l) in Δ is mapped to the horizontal line in $\tilde{\Delta}$ and the horizontal line in Δ is mapped to the vertical line (fiber to π_r) in $\tilde{\Delta}$. This fits the following diagram

$$\begin{array}{ccc}
 G_2/P_{12} : (\hat{M}, \Delta) & \xrightarrow{\varphi} & (\hat{K}, \tilde{\Delta}) \\
 \swarrow \pi_l & & \searrow \pi_r \\
 G_2/P_1 : (M, \Delta) & & G_2/P_2 : (K, D, \Gamma)
 \end{array}$$

where $\pi_l(x, y, p, q, z, r) = (x, y, p, q, z)$ and $\pi_r(x, y, p, q, z, r) = (x, y, p, q, z)$ in the corresponding coordinates. The required transformation is given by formula

$$\varphi(x, y, p, q, z, r) = \left(-\frac{1}{r}, \sqrt{3}\left(2p - \frac{q^2}{r}\right), 3z - \frac{q^3}{r}, \sqrt{3}\left(x - \frac{q}{r}\right), 6(xp - y) - \frac{3}{r}(z + xq^2) + \frac{2q^3}{r^2}, q \right).$$

4.2. Correspondence for integral curves

Integral curves for (M, Π) are given by equation (5) and their prolongations to (\hat{M}, Δ) are determined by the additional constraint $r = q_1$. Thus there is a 1:1 correspondence between integral curves of (M, Π) and integral curves of (\hat{M}, Δ) . The invariant constraint $r_1 = 0$ (or $r = \text{const}$) determines a 1-parametric family of integral curves through any point called abnormal extremals for Π . They are projections of the integral curves for the horizontal line distribution given by the first generator of (12).

Minimal integral curves for (K, D, Γ) are given by equation \mathcal{E}_Γ of (15) and their prolongation to $(\hat{K}, \tilde{\Delta})$ are determined by the additional constraint $r = q_1/\sqrt{3}$. Thus there is a 1:1 correspondence between integral curves of (K, D, Γ) and integral curves of $(\hat{K}, \tilde{\Delta})$. The invariant constraint $r_1 = 0$ (or $r = \text{const}$) determines straight line generators of the RNC through any point of K . They are projections of the integral curves for the horizontal line distribution given by the first generator of (16).

This correspondence on the level of jets is summarized in the following diagram, where we denote by $j_{l,r}$ the lifts defined above, they are right inverse to the projections $\pi_{l,r}$.

$$\begin{array}{ccc}
 & \mathcal{E}_\Delta & \\
 \swarrow \pi_l & & \searrow \pi_r \\
 \mathcal{E}_\Pi & & \mathcal{E}_\Gamma
 \end{array}
 \begin{array}{ccc}
 & \mathcal{E}_\Delta & \\
 \swarrow j_l & & \searrow j_r \\
 \mathcal{E}_\Pi & & \mathcal{E}_\Gamma
 \end{array}$$

Note that $\mathcal{E}_\Pi \times \mathbb{P}^1 \simeq \mathcal{E}_\Delta \simeq \mathcal{E}_\Gamma \times \mathbb{P}^1$ (since any integral curve is uniquely lifted given a point in the fiber) and $\mathcal{E}_\Pi \simeq \mathcal{E}_\Gamma$, which explains isomorphism of the algebras of differential invariants.

4.3. Correspondence for generic curves

The above correspondence cannot be extended to all curves, however we can produce lifts for generic curves.

On the left side of the double (twistor) fibration the lift is determined from the observation of Section 1.4 that X given by (6) determines $Y \in \Pi$ up to scale by the condition $g(X, Y) = 0$. Setting $Y = \partial_x + p\partial_y + q\partial_p + q^2\partial_z + r\partial_q$ this and (3) gives the formula for r , from which we conclude that the lift is given by the following formula (and its prolongations):

$$j_l(x, y, p, q, z) = \left(x, y, p, q, z, \frac{2qp_1 - z_1 - q^2}{2(y_1 - p)} \right).$$

On the right side of the double fibration the computation is a bit more involved. First we derive the formula for distinguished curves in direction \mathfrak{g}_{-2} . By the mentioned general result [2, §5.3.7], there is a unique unparametrized distinguished curve of that type in any non-contact direction on TK . This gives an invariant section $J^1(K, 1) \dashrightarrow J^2(K, 1)$ defined on a Zariski open set. (An alternative way to check it: the stabilizer of a generic $b_1 \in J^1$ is $GL_2 \times \mathbb{R}_\times$ that acts on $\pi_{2,1}^{-1}(b_1) \simeq \mathbb{R}^4$ via an irreducible representation of GL_2 that has one fixed point.)

The explicit formula involves matrix realization $G_2 \subset SO(3, 4)$ described in [10] on the level of Lie algebras; the reference specifies the \mathfrak{p}_1 grading but one can also identify \mathfrak{p}_2 . The corresponding parabolic subgroup P_2 can be coordinized via $GL_2 \times \exp(\mathfrak{p}_+)$ and the action of this on $\mathfrak{m} = \mathfrak{g}_-$ is then explicitly derived. The formulae for the distinguished curves (omitted here, see Maple’s supplement) imply the formula for the above section, or equivalently for a point $[\Upsilon_X] \in \mathbb{P}D$ as defined in Section 3.4:

$$\begin{aligned} \Upsilon_X = & \frac{2\sqrt{3}q_1^3 - 9qy_2 - 18y_1q_1 + 9z_2}{9(qy_1 + p - z_1)} (\partial_x + p\partial_z) + \left(q_2 - \frac{2\sqrt{3}y_1^2 + 3qq_1y_2 + 2y_1q_1^2 - 3q_1z_2}{3(qy_1 + p - z_1)} \right) \partial_q \\ & + \left(y_2 + \frac{2\sqrt{3}p_1q_1^2 - 3qy_1y_2 - 4y_1^2q_1 - 6y_1p_1 + 3y_1z_2}{3(qy_1 + p - z_1)} \right) (\partial_y + q\partial_z) \\ & + \left(p_2 - \frac{2\sqrt{3}y_1^3 + 9qp_1y_2 + 18p_1^2 - 9p_1z_2}{9(qy_1 + p - z_1)} \right) \partial_p. \end{aligned}$$

Next, given a point $[a : b : c : d] \in \mathbb{P}D$ the RNC secant line through it intersects $[\Gamma]$ at the points corresponding to the parameter r from (14) so:

$$\lambda_X^\pm = \frac{3ad - bc \pm \sqrt{(3ad - bc)^2 - 4(\sqrt{3}ac - b^2)(\sqrt{3}bd - c^2)}}{2(\sqrt{3}ac - b^2)}.$$

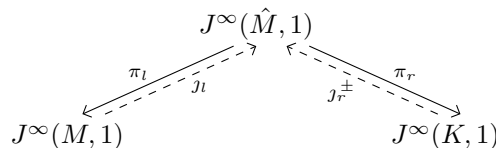
Equivalently, the points $r = \lambda_X^\pm$ are the solutions of the quadratic equation

$$(\sqrt{3}ac - b^2)r^2 + (bc - 3ad)r + \sqrt{3}bd - c^2 = 0.$$

This formula for r composed with the formula for Υ_X (the coefficients a, b, c, d are extracted in the order of appearance) defines two lifts of generic curves from K to $\hat{K} \simeq \hat{M}$:

$$j_r^\pm(x, y, p, q, z) = (x, y, p, q, z, \lambda_X^\pm \circ [\Upsilon_X]).$$

This correspondence on the level of jets is summarized in the following diagram:



The dashed arrows are defined on open dense subsets of their domains, are right inverse to the corresponding projections, and in addition, j_r^\pm is 1:2 map. This can be seen as an analog of the Bäcklund transformation, so that for one (jet of) curve in K we obtain two such in M . This allows to derive the algebra of differential invariants \mathcal{I}_g of curves in G_2/P_2 from the results of Section 1 by averaging the invariants thereof on the two branches $\pi_l \circ j_r^\pm$.

5. Concluding remarks

The computations in this paper demonstrate the method of differential invariants for G_2 action on curves in generalized flag varieties. The group is more complicated than the projective group PSL_{n+1} mentioned in the introduction, and we address the corresponding challenges.

For (minimal) integral curves the approach is very effective and provides a complete description of the algebra. This has to be compared with the method of moving frame [4]; a moving frame for this problem was constructed in [5] but the algebra of invariants was not derived.

A modification of this method, the equivariant moving frame [15], is not applicable as it relies on an explicit Lie group parametrization, which is non-trivial for G_2 (one has to resolve the quadratic and cubic equations defining the group). We worked mainly with the Lie algebra. Even in this case for generic curves the direct computations fail, and we had to evoke geometric arguments to arrive to the basic invariants, in particular exploiting the ideas of moving frames.

The results of this paper concern only curves in homogeneous flag varieties G_2/P , but they can be extended to more general case of curves in curved M^5 of type (G_2, P_1) etc. Indeed in this more general case the symmetry algebra of such M^5 is smaller than G_2 yet the invariants can be found by the same method. In particular, the stratification of 1-jets makes a perfect sense in the curved case and one can derive relative invariants similar to R_1, R_2 in Section 1 (for R_1 this is straightforward) leading to absolute differential invariants.

The invariance is meant here in the following sense: If $\phi : M_1 \rightarrow M_2$ is an equivalence between two different spaces with their $(2, 3, 5)$ distributions, sending one curve $\gamma_1 \subset M_1$ to another $\gamma_2 \subset M_2$ then the invariants are superposed. Since the structural group P_1 for the Cartan bundle associated to this normal parabolic geometry [2] was central in our computations, the basic invariants are expected to generalize.

For the geometry of type (G_2, P_{12}) the situation is completely similar because it is functorially equivalent to the geometry of type (G_2, P_1) . However in the curves case (G_2, P_2) type geometry fails the twistor correspondence, so this would require a separate consideration.

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