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Vilenkin–Lebesgue Points and Almost Everywhere Convergence for Some Classical Summability Methods

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Abstract. The concept of Vilenkin–Lebesgue points was introduced in [12], where the almost everywhere convergence of Fejer means of Vilenkin–Fourier series was proved. In this paper, we present a different (and simpler) approach to prove a similar result, which can be used to prove that the corresponding result holds also in a more general context, namely for regular Norlund and T-means.

Mathematics Subject Classification. 42C10.

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1. Introduction

Concerning some definitions and notations used in this introduction, we refer to Sect. 2.

The fact that the Walsh system is the group of characters of a compact abelian group connects Walsh analysis with abstract harmonic analysis was discovered independently by Fine [7] and Vilenkin [40]. Later on, in 1947 Vilenkin [40–42] actually introduced a large class of compact groups (now called Vilenkin groups) and the corresponding characters which includes the dyadic group and the Walsh system as a special case. For general references to the haar measure and harmonic analysis on groups see Pontryagin [33], Rudin [34], and Hewitt and Ross [14]. In particular, Vilenkin investigated the group G_m, which is a direct product of the additive groups $Z_{m_k} =: \{0, 1, \ldots, m_k-1\}$ of integers modulo m_k, where $m =: (m_0, m_1, \ldots)$ are positive integers not less than 2, and introduced the Vilenkin systems $\{\psi_j\}_{j=0}^{\infty}$. These systems include

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as a special case the Walsh system and many of the proofs presented for the Walsh system can be generalized readily to the Vilenkin case.

Fejer's theorem shows that (see, e.g., [1,5,6,37]) if one replaces ordinary summation by Fejer means σ_n defined by

$$\sigma_n f := \frac{1}{n}\sum_{k=1}^{n} S_k f,$$

then, for any $1 \le p \le \infty$, there exists an absolute constant C_p, depending only on p such that

$$\|\sigma_n f\|_p \le C_p \|f\|_p .$$

If we define the maximal operator σ^* of Fejer means by

$$\sigma^* f := \sup_{n\in\mathbb{N}} |\sigma_n f| ,$$

then the weak type inequality

$$\mu(\sigma^* f > \lambda) \le \frac{c}{\lambda}\|f\|_1 , \qquad (\lambda > 0)$$

holds for any integrable function. For example, this result can be found in Zygmund [47] (see also [9,19]) for trigonometric series, in Schipp [35] for Walsh series and in Pál, Simon [28] (see also [30,44–46]) for bounded Vilenkin series. It follows that the Fejer means with respect to trigonometric and Vilenkin systems of any integrable function converges a.e to this function.

It is known that almost every point x is a Lebesgue point of a function $f \in L^1$ and the Fejer means $\sigma_n^T f$ of the trigonometric Fourier series of $f \in L^1$ converge to f at each Lebesgue point.

Weisz [43] introduced the Walsh–Lebesgue points and proved the analogue of the preceding result: almost every point is a Walsh–Lebesgue point of an integrable function $f \in L^1$ and the Walsh–Fejer means of f converge to f at each Walsh–Lebesgue point. Later, Goginava and Gogoladze [12] introduced the Vilenkin–Lebesgue points and proved similar result. They used methods of martingale Hardy spaces.

In this paper, we consider some more general summability methods, which are called Nörlund and T-means. In particular, the n-th Nörlund mean t_n and T-mean T_n of the Fourier series of f are, respectively, defined by

$$t_n f := \frac{1}{Q_n}\sum_{k=1}^{n} q_{n-k} S_k f \tag{1}$$

and

$$T_n f := \frac{1}{Q_n}\sum_{k=0}^{n-1} q_k S_k f, \tag{2}$$

where

$$Q_n := \sum_{k=0}^{n-1} q_k .$$

Here, $\{q_k : k \ge 0\}$ is a sequence of nonnegative numbers, where $q_0 > 0$ and

$$\lim_{n\to\infty} Q_n = \infty.$$

Then, the summability method (1) generated by $\{q_k : k \geq 0\}$ is regular if and only if (see [17])

$$\lim_{n\to\infty} \frac{q_{n-1}}{Q_n} = 0.$$

Moreover, the summability method (2) is regular if and only if

$$\lim_{n\to\infty} Q_n = \infty.$$

It is well known (for details see, e.g., [32]) that every Nörlund summability method generated by non-increasing sequence $(q_k, k \in \mathbb{N})$ is regular, but Nörlund means generated by non-decreasing sequence $(q_k, k \in \mathbb{N})$ is not always regular. On the other hand, every T-mean generated by non-decreasing sequence $(q_k, k \in \mathbb{N})$ is regular, but T-means generated by non-increasing sequence $(q_k, k \in \mathbb{N})$ is not always regular. In this paper, we investigate only regular Nörlund and T-means.

Almost everywhere convergence and summability of Nörlund and T-means were studied by several authors. We mentioned Bhahota, Persson and Tephnadze [3] (see also [2,4,16,31]), Tutberidze [38,39], Fridli, Manchanda, Siddiqi [8], Móricz and Siddiqi [18] Nagy [20–23] (see also [24–27]).

We also define the maximal operator t^* of Nörlund means by

$$t^* f := \sup_{n\in\mathbb{N}} |t_n f|.$$

If $\{q_k : k \in \mathbb{N}\}$ is non-increasing and satisfying the condition

$$\frac{1}{Q_n} = O\left(\frac{1}{n}\right), \quad \text{as} \quad n \to \infty, \tag{3}$$

then the weak-type inequality

$$y\mu\{t^* f > y\} \leq c\,\|f\|_1, \quad f \in L^1(G_m), \quad y > 0 \tag{4}$$

was proved in [30]. When the sequence $\{q_k : k \in \mathbb{N}\}$ is non-decreasing, then the weak-(1,1) type inequality (4) holds for every maximal operator of Nörlund means. It follows that for such Nörlund means of $f \in L^1(G_m)$, we have that

$$\lim_{n\to\infty} t_n f(x) = f(x), \quad \text{a.e. on} \quad G_m.$$

Define the maximal operator T^* of T-means by

$$T^* f := \sup_{n\in\mathbb{N}} |T_n f|.$$

It was proved in [38] that if $\{q_k : k \in \mathbb{N}\}$ is non-increasing or if $\{q_k : k \in \mathbb{N}\}$ is non-decreasing and satisfying the condition

$$\frac{q_{n-1}}{Q_n} = O\left(\frac{1}{n}\right), \quad \text{as} \quad n \to \infty, \tag{5}$$

then the following weak-type inequality holds:

$$y\mu\{T^* f > y\} \leq c\,\|f\|_1, \quad f \in L^1(G_m), \quad y > 0.$$

It follows that for such T-means and for $f \in L^1(G_m)$, we have that

$$\lim_{n\to\infty} T_n f(x) = f(x), \quad \text{a.e. on} \quad G_m.$$

The main aim of this paper is to find a different and simpler approach, with the help of which we can generalize the results in [12] and prove them for a more large class of regular Norlund and T-means.

The paper is organized as follows: the main results are presented, proved and discussed in Sect. 3. In particular, Theorems 1 and 2 are parts of this new approach. The announced results for Norlund and T-means can be found in Theorems 3 and 4, respectively. In order not to disturb the presentations in Sect. 3, we use Sect. 2 for some necessary preliminaries (e.g., definitions, notations, lemmas). In particular, Lemma 2 is new and of independent interest.

2. Preliminaries

Let $\mathbb{N}_+$ denote the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Let $m := (m_0, m_1, \dots)$ denote a sequence of the positive integers not less than 2. Denote by

$$Z_{m_k} := \{0, 1, \dots, m_k - 1\}$$

the additive group of integers modulo m_k.

Define the groups G_m as the complete direct product of the group Z_{m_j} with the product of the discrete topologies of Z_{m_j} ' s. The direct product μ of the measures

$$\mu_k(\{j\}) := 1/m_k \qquad (j \in Z_{m_k})$$

is the Haar measure on G_m with $\mu(G_m) = 1$. In this paper, we discuss bounded Vilenkin groups only, that is

$$\sup_{n \in \mathbb{N}} m_n < \infty.$$

The elements of G_m are represented by the sequences

$$x := (x_0, x_1, \dots, x_k, \dots) \qquad (x_k \in Z_{m_k}).$$

It is easy to give a base for the neighborhood of G_m, namely

$$I_0(x) := G_m,$$
$$I_n(x) := \{y \in G_m \mid y_0 = x_0, \dots, y_{n-1} = x_{n-1}\} \, (x \in G_m, n \in \mathbb{N}).$$

The intervals $I_n(x)$ $(n \in \mathbb{N}, x \in G_m)$ are called Vilenkin intervals. Denote $I_n := I_n(0)$ for $n \in \mathbb{N}$ and $\overline{I_n} := G_m \backslash I_n$. Let

$$e_n := (0, \dots, 0, x_n = 1, 0, \dots) \in G_m \;\; (n \in \mathbb{N}).$$

If we define the so-called generalized number system based on m in the following way:

$$M_0 := 1, \;\; M_{k+1} := m_k M_k, \;\; (k \in \mathbb{N}),$$

then every $n \in \mathbb{N}$ can be uniquely expressed as

$$n = \sum_{k=0}^{\infty} n_j M_j, \quad \text{where} \quad n_j \in Z_{m_j} \quad (j \in \mathbb{N})$$

and only a finite number of n_j's differ from zero. Let

$$|n| := \max\{j \in \mathbb{N}, \quad n_j \neq 0\}.$$

Defining $\overline{I_n} := G_m \backslash I_n$ and

$$I_N^{k,l} := \begin{cases} I_N(0,\ldots,0,x_k \neq 0,0,...,0,x_l \neq 0,x_{l+1},\ldots,x_{N-1},\ldots), \\ \text{for } 0 \leq k < l < N, \\ I_N(0,\ldots,0,x_k \neq 0,x_{k+1}=0,\ldots,x_{N-1}=0,x_N,\ldots), \\ \text{for } \ 0 \leq k < l = N, \end{cases}$$

we have

$$\overline{I_N} = \bigcup_{s=0}^{N-1} I_s \backslash I_{s+1} = \left(\bigcup_{k=0}^{N-2} \bigcup_{l=k+1}^{N-1} I_N^{k,l} \right) \bigcup \left(\bigcup_{k=0}^{N-1} I_N^{k,N} \right). \tag{6}$$

Next, we introduce on G_m an orthonormal system, which is called the Vilenkin system. First, define the complex valued function $r_k(x) : G_m \to \mathbb{C}$, the generalized Rademacher functions, as

$$r_k(x) := \exp(2\pi\imath x_k/m_k) \quad \left(\imath^2 = -1, \ x \in G_m, \ k \in \mathbb{N}\right).$$

We define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on G_m as

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \quad (n \in \mathbb{N}).$$

Especially, we call this system the Walsh–Paley one if $m \equiv 2$ (for details see [13,36]). The Vilenkin system is orthonormal and complete in $L^2(G_m)$ (for details see, e.g., [1,36,40]).

Next, we introduce analogues of the usual definitions in Fourier analysis. If $f \in L^1(G_m)$, we can define the Fourier coefficients, the partial sums of the Fourier series, the Fejer means, the Dirichlet and Fejer kernels with respect to the Vilenkin system ψ in the usual manner:

$$\begin{aligned} \widehat{f}(k) &:= \int_{G_m} f\overline{\psi}_k \mathrm{d}\mu, && (k \in \mathbb{N}), \\ S_n f &:= \sum_{k=0}^{n-1} \widehat{f}(k)\psi_k, && (n \in \mathbb{N}_+, \ S_0 f := 0), \\ \sigma_n f &:= \frac{1}{n}\sum_{k=0}^{n-1} S_k f, && (n \in \mathbb{N}_+), \\ D_n &:= \sum_{k=0}^{n-1} \psi_k, && (n \in \mathbb{N}_+), \\ K_n &:= \frac{1}{n}\sum_{k=0}^{n-1} D_k, && (n \in \mathbb{N}_+). \end{aligned}$$

Recall that (for details see, e.g., [1,10,11]),

$$D_{M_n}(x) = \begin{cases} M_n, & \text{if } x \in I_n, \\ 0, & \text{if } x \notin I_n, \end{cases} \tag{7}$$

$$n\left|K_n\right| \le c\sum_{l=0}^{|n|} M_l\left|K_{M_l}\right|, \tag{8}$$

and

$$\int_{G_m} K_n(x)\mathrm{d}\mu(x) = 1, \qquad \sup_{n\in\mathbb{N}}\int_{G_m}\left|K_n(x)\right|\mathrm{d}\mu(x) \le c < \infty. \tag{9}$$

Moreover, if $n > t$, $t, n \in \mathbb{N}$, then

$$K_{M_n}(x) = \begin{cases} \frac{M_t}{1-r_t(x)}, \ x \in I_t \backslash I_{t+1}, & x - x_t e_t \in I_n, \\ \frac{M_n+1}{2}, \quad x \in I_n, & \\ 0, \qquad\quad \text{otherwise}, & \end{cases}$$

and

$$\left|K_{M_n}(x)\right| \le c\sum_{s=0}^{n} M_s \sum_{r=1}^{m_s-1} 1_{I_n(x-re_s)}. \tag{10}$$

A point x is called a Lebesgue point of an integrable function f if

$$\lim_{h\to 0}\frac{1}{h}\int_x^{x+h}\left|f(t)-f(x)\right|\mathrm{d}\mu(t) = 0.$$

Weisz [43] introduced the concept of Walsh–Lebesgue points for the dyadic group with the help of the operator

$$W_A f(x) := \sum_{s=0}^{A} 2^s \int_{I_A(x-e_s)}\left|f(t)-f(x)\right|\mathrm{d}\mu(t).$$

Similarly to [12], now we generalize this by

$$W_A f(x) := \sum_{s=0}^{A} M_s \sum_{r=1}^{m_s-1}\int_{I_A(x-re_s)}\left|f(t)-f(x)\right|\mathrm{d}\mu(t).$$

A point $x \in G_m$ is called a Vilenkin–Lebesgue point of the function $f \in L^1(G_m)$, if

$$\lim_{A\to\infty} W_A f(x) = 0.$$

We also define the operator V_A by

$$V_A f(x) := \sum_{s=0}^{A} M_s \sum_{r=1}^{m_s-1}\int_{I_A(x-re_s)} f(t)\mathrm{d}\mu(t).$$

It is evident that

$$\begin{aligned} V_A f(x) &= \sum_{s=0}^{A}\frac{M_s}{M_A}\sum_{r=1}^{m_s-1}\int_{G_m} D_{M_A}(x-re_s-t)f(t)\mathrm{d}\mu(t) \\ &= \int_{G_m}\left(\sum_{s=0}^{A}\frac{M_s}{M_A}\sum_{r=1}^{m_s-1} D_{M_A}(x-re_s-t)\right)f(t)\mathrm{d}\mu(t) \\ &= \int_{G_m} Y_A(x-t)f(t)\mathrm{d}\mu(t), \end{aligned}$$

where

$$Y_A(x)=\sum_{s=0}^{A}\frac{M_s}{M_A}\sum_{r=1}^{m_s-1}D_{M_A}(x-re_s).$$

It is obvious that

$$\lim_{A\to\infty}W_Af(x)=0,$$

if and only if

$$\lim_{A\to\infty}V_A|f-f(x)|(x)=0.$$

Next, we state the following Lemma, which is very important to study almost everywhere convergence of Vilenkin–Fejer means (see, e.g., [44]).

Lemma 1. *Suppose that the sigma-sublinear operator V is bounded from L^{p_1} to L^{p_1} for some $1<p_1\le\infty$ and*

$$\int_{\overline{I}}|Vf|\,\mathrm{d}\mu\le C\,\|f\|_1$$

for $f\in L^1$ and Vilenkin interval I, which satisfies that

$$\mathit{supp}f\subset I\quad \mathit{and}\quad \int_{G_m}f\mathrm{d}\mu=0. \tag{11}$$

Then, the operator V is of weak type $(1,1)$, i.e.,

$$\sup_{y>0}y\mu\left(\{Vf>y\}\right)\le\|f\|_1\,.$$

We also need the following new Lemma of independent interest:

Lemma 2. *Let $N\in\mathbb{N}$. Then,*

$$\int_{G_m\backslash I_N}\sup_{A>N}|Y_A|\,\mathrm{d}\mu\le c<\infty,$$

where c is an absolute constant.

Proof. Let $A>N$ and $x\in I_N^{k,l}$, $k=0,\dots,N-2$ and $l=k+1,\dots,N-1$. Then it is easy to prove that $x-re_s\in G_m\backslash I_N$ for all $r=1,\dots,m_s-1$. Using (7), we get that

$$D_{M_A}(x-re_s)=0\quad\text{for}\quad A>N$$

so that

$$Y_A(x)=\left|\sum_{s=0}^{A}\frac{M_s}{M_A}\sum_{r=1}^{m_s-1}D_{M_A}(x-re_s)\right|=0\quad\text{for}\quad A>N. \tag{12}$$

Let $A>N$ and $x\in I_N^{k,N}$. Using again (7), we can conclude that $D_{M_A}(x-re_s)=0$ if $s\neq k$ and $D_{M_A}(x-re_k)=0$ if $r\neq x_k$. Moreover,

$$D_{M_A}(x-x_ke_k)=\begin{cases}M_A,\ x\in I_A(x_ke_k),\\ 0,\qquad x\in G_m\backslash I_A(x_ke_k).\end{cases}$$

Hence,

$$\left|\sum_{s=0}^{A}\frac{M_s}{M_A}\sum_{r=1}^{m_s-1}D_{M_A}(x-re_s)\right| = \frac{M_k}{M_A}\left|D_{M_A}(x-x_ke_k)\right| = \begin{cases} M_k, & x\in I_A(x_ke_k),\\ 0, & x\in G_m\backslash I_A(x_ke_k).\end{cases} \tag{13}$$

By combining (6), (12) and (13), we find that

$$\begin{aligned}
&\int_{G_m\backslash I_N}\sup_{A>N}|Y_A(x)|\,\mathrm{d}\mu(x)\\
&=\sum_{k=0}^{N-2}\sum_{l=k+1}^{N-1}\sum_{j=l+1}^{N-1}\sum_{x_j=0}^{m_{j-1}}\int_{I_N^{k,l}}\sup_{A>N}|Y_A(x)|\,\mathrm{d}\mu(x)\\
&\quad+\sum_{k=0}^{N-1}\int_{I_N^{k,N}}\sup_{A>N}|Y_A(x)|\,\mathrm{d}\mu(x)\\
&=\sum_{k=0}^{N-1}\int_{I_A(x_ke_k)}\sup_{A>N}\left|\sum_{s=0}^{A}\frac{M_s}{M_A}\sum_{r=1}^{m_s-1}D_{M_A}(x-re_s)\right|\mathrm{d}\mu(x)\\
&\le c\sum_{k=0}^{N-1}\frac{M_k}{M_N}<C<\infty.
\end{aligned}$$

The proof is complete. □

3. The Main Results with Applications

In our first main result, we consider the maximal operator V^* defined by

$$V^*f(x) := \sup_{A\in\mathbb{N}}|V_Af(x)|.$$

Theorem 1. *Let $f\in L^1(G_m)$. Then, the operator V^* is of weak type $(1,1)$, i.e.,*

$$\sup_{y>0} y\mu\left\{V^*f>y\right\}\le\|f\|_1.$$

Proof. Since

$$\|V^*f\|_\infty\le c\|f\|_\infty\sup_{A\in\mathbb{N}}\frac{1}{M_A}\sum_{s=0}^{A}M_s\le c\|f\|_\infty,$$

we obtain that V^* is bounded from $L^\infty(G_m)$ to $L^\infty(G_m)$. According to Lemma 1, the proof will be complete if we prove that

$$\int_{\overline{I}}|V^*f|\,\mathrm{d}\mu\le c\|f\|_1 \tag{14}$$

for every function f satisfying the conditions in (11), where I denotes the support of the function f. Without loss the generality, we may assume that f is a function with support I and $\mu(I)=M_N$. We may assume that $I=I_N$.

It is easy to see that $V_n f = 0$ when $n \le M_N$. Therefore, we can suppose that $n > M_N$. Hence,

$$|V^* f(x)| = \sup_{n>M_N} \left| \int_{I_N} Y_n(x-t) f(t) \mathrm{d}\mu(t) \right|.$$

Let $t \in I_N$ and $x \in \overline{I_N}$. Then $x - t \in \overline{I_N}$ and by applying Lemma 2, we get that

$$\begin{aligned}
\int_{\overline{I_N}} |V^* f(x)| \,\mathrm{d}\mu(x) &\le \int_{\overline{I_N}} \int_{I_N} \sup_{n>M_N} |Y_n(x-t) f(t)| \,\mathrm{d}\mu(t) \,\mathrm{d}\mu(x) \\
&\le \int_{I_N} \int_{\overline{I_N}} \sup_{n>M_N} |Y_n(x-t) f(t)| \,\mathrm{d}\mu(x) \,\mathrm{d}\mu(t) \\
&\le \int_{I_N} \int_{\overline{I_N}} \sup_{n>M_N} |Y_n(x) f(t)| \,\mathrm{d}\mu(x) \,\mathrm{d}\mu(t) \\
&= \int_{I_N} |f(t)| \,\mathrm{d}\mu(t) \int_{\overline{I_N}} \sup_{n>M_N} |Y_n(x)| \,\mathrm{d}\mu(x) \\
&\le \|f\|_1 \int_{\overline{I_N}} \sup_{n>M_N} |Y_n(x)| \,\mathrm{d}\mu(x) \le c \|f\|_1 ,
\end{aligned}$$

which means that (14) holds so the proof is complete. □

Next, we state the following convergence result for the operator W_A :

Corollary 1. *Let $f \in L^1(G_m)$. Then*

$$\lim_{A\to\infty} W_A f(x) = 0 \qquad a.e. \qquad x \in G_m.$$

Proof. It is easy to see that

$$\lim_{A\to\infty} W_A f(x) = 0$$

for every Vilenkin polynomial. Hence, since the Vilenkin polynomials are dense in $L^1(G_m)$, the usual density argument (see Marcinkiewicz and Zygmund [15]) and Theorem 1 imply the proof. □

Our convergence result for the Fejer means reads:

Theorem 2. *Let $f \in L^1(G_m)$. Then,*

$$\lim_{n\to\infty} \sigma_n f(x) = f(x),$$

for all Vilenkin–Lebesgue points of f.

Proof. By combining (8), (9) and (10), we get that

$$|\sigma_n f(x) - f(x)| \le \frac{c}{n} \sum_{A=0}^{|n|} M_A \int_{G_m} |f(t) - f(x)| |K_{M_A}(x-t)| \mathrm{d}\mu(t)$$

$$\leq \frac{c}{n}\sum_{A=0}^{|n|} M_A \sum_{s=0}^{A} M_s \sum_{r=1}^{m_s-1} \int_{I_A(x-re_s)} |f(t)-f(x)| \,\mathrm{d}\mu(t)$$

$$\leq \frac{c}{n}\sum_{A=0}^{|n|} M_A W_A f(x) \to 0, \text{ as } n\to\infty.$$

The proof is complete. □

Corollary 2. *Let $f \in L^1(G_m)$. Then,*

$$\lim_{n\to\infty} \sigma_n f(x) = f(x) \quad a.e. \text{ on } \quad G_m.$$

Based on Theorem 2, we can prove our next main result.

Theorem 3. *Suppose that $f \in L^1(G_m)$ and for some $x \in G_m$,*

$$\lim_{n\to\infty} \sigma_n f(x) = f(x).$$

The following statements hold true:

a) *Let t_n be a regular Nörlund mean generated by non-decreasing sequence $\{q_k : k \in \mathbb{N}\}$. Then,*

$$\lim_{n\to\infty} t_n f(x) = f(x).$$

b) *Let t_n be a Nörlund mean generated by non-increasing sequence $\{q_k : k \in \mathbb{N}\}$ satisfying condition (3). Then*

$$\lim_{n\to\infty} t_n f(x) \to f(x).$$

Note that if $\{q_k : k \in \mathbb{N}\}$ is non-increasing, then the Norlund means are regular. If this sequence is non-decreasing, then (3) is obviously satisfied.

Proof. a) Suppose that

$$\lim_{n\to\infty} |\sigma_n f(x) - f(x)| = 0$$

for some $x \in G_m$. If we invoke Abel transformation we get the following identities:

$$Q_n := \sum_{j=0}^{n-1} q_j = \sum_{j=1}^{n} q_{n-j}\cdot 1 = \sum_{j=1}^{n-1} (q_{n-j}-q_{n-j-1})\, j + q_0 n \tag{15}$$

and

$$t_n = \frac{1}{Q_n}\left(\sum_{j=1}^{n-1} (q_{n-j}-q_{n-j-1})\, j\sigma_j + q_0 n\sigma_n\right). \tag{16}$$

By combining (15) and (16), we can conclude that

$$|t_n f(x) - f(x)|$$
$$\leq \frac{1}{Q_n}\left(\sum_{j=1}^{n-1} (q_{n-j}-q_{n-j-1})\, j|\sigma_j f(x)-f(x)| + q_0 n|\sigma_n f(x)-f(x)|\right)$$

$$\leq \frac{1}{Q_n}\sum_{j=0}^{n-1}\left(q_{n-j}-q_{n-j-1}\right)j\alpha_j+\frac{q_0 n\alpha_n}{Q_n}$$
$$:= I + II,$$

where

$$\alpha_n := |\sigma_n f(x) - f(x)| \to 0, \quad \text{as} \quad n \to \infty.$$

Since t_n are regular Nörlund means, generated by sequence of non-decreasing numbers $\{q_k : k \in \mathbb{N}\}$ we obtain that

$$II \leq \frac{q_0 n\alpha_n}{Q_n} \leq C\alpha_n \to 0, \quad \text{as} \quad n \to \infty.$$

Moreover, since α_n converges to 0, we get that there exists an absolute constant A, such that $\alpha_n \leq A$ for any $n \in \mathbb{N}$ and for any $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$, such that $\alpha_n < \varepsilon$ when $n > N_0$. Hence,

$$I = \frac{1}{Q_n}\sum_{j=1}^{N_0}\left(q_{n-j}-q_{n-j-1}\right)j\alpha_j+\frac{1}{Q_n}\sum_{j=N_0+1}^{n-1}\left(q_{n-j}-q_{n-j-1}\right)j\alpha_j := I_1 + I_2.$$

Since $\alpha_n < A$, we obtain that

$$I_1 = \frac{1}{Q_n}\sum_{j=1}^{N_0}\left(q_{n-j}-q_{n-j-1}\right)j\alpha_j \leq \frac{AN_0 q_{n-1}}{Q_n} \to 0, \quad \text{as} \quad n \to \infty.$$

Moreover, by (15),

$$\begin{aligned} I_2 &= \frac{1}{Q_n}\sum_{j=N_0+1}^{n-1}\left(q_{n-j}-q_{n-j-1}\right)j\alpha_j \\ &\leq \frac{\varepsilon}{Q_n}\sum_{j=N_0+1}^{n-1}\left(q_{n-j}-q_{n-j-1}\right)j \\ &\leq \frac{\varepsilon}{Q_n}\sum_{j=0}^{n-1}\left(q_{n-j}-q_{n-j-1}\right)j < \varepsilon. \end{aligned}$$

We conclude that also $I_2 \to 0$, so the proof of a) is complete.

b) In view of condition (3), the proof of part b) is step-by-step analogous to that of part a) so we omit the details. The proof is complete. □

Corollary 3. a) *Let Let t_n be a regular Nörlund mean generated by non-decreasing sequence $\{q_k : k \in \mathbb{N}\}$. Then, for all Vilenkin–Lebesgue points of $f \in L^1(G_m)$,*

$$\lim_{n\to\infty} t_n f(x) = f(x).$$

b) *Let t_n be a Nörlund mean generated by non-increasing sequence $\{q_k : k \in \mathbb{N}\}$ satisfying condition (3). Then, for all Vilenkin–Lebesgue points of $f \in L^1(G_m)$,*

$$\lim_{n\to\infty} t_n f(x) = f(x).$$

Analogously, we can state the following results for T-means with respect to Vilenkin systems.

Theorem 4. *Suppose that $f \in L^1(G_m)$ and, for some $x \in G_m$,*

$$\lim_{n\to\infty} \sigma_n f(x) = f(x).$$

The following statements hold true:

a) Let T_n be a regular T-mean generated by non-increasing sequence $\{q_k : k \in \mathbb{N}\}$. Then,

$$\lim_{n\to\infty} T_n f(x) = f(x).$$

b) Let T_n be a T-mean generated by non-decreasing sequence $\{q_k : k \in \mathbb{N}\}$ satisfying condition (5). Then,

$$\lim_{n\to\infty} T_n(x) = f(x).$$

Note that if $\{q_k : k \in \mathbb{N}\}$ is non-decreasing, then the T-means are regular and if it is non-increasing then (5) holds automatically.

Proof. The proof is step by step analogous to that of Theorem 3, so we omit the details. We just need to replace condition (3) by condition (5) in the proof. □

Corollary 4. a) *Let T_n be a regular T-mean generated by non-increasing sequence $\{q_k : k \in \mathbb{N}\}$. Then, for all Vilenkin–Lebesgue points of $f \in L^1(G_m)$,*

$$\lim_{n\to\infty} T_n f(x) = f(x).$$

b) *Let T_n be a T-mean generated by non-decreasing sequence $\{q_k : k \in \mathbb{N}\}$ satisfying condition (5). Then, for any T-means and for all Vilenkin–Lebesgue points of $f \in L^1(G_m)$,*

$$\lim_{n\to\infty} T_n f(x) = f(x).$$

Final Remark: We refer to our new book [32] for some complementary new information and frame of this paper. For another (of Carleson–Hunt type) new convergence result, we refer to our recent paper [29].

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