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RESEARCH ARTICLE

Almost every path structure is not variational

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Abstract

Given a smooth family of unparameterized curves such that through every point in every direction there passes exactly one curve, does there exist a Lagrangian with extremals being precisely this family? It is known that in dimension 2 the answer is positive. In dimension 3, it follows from the work of Douglas that the answer is, in general, negative. We generalise this result to all higher dimensions and show that the answer is actually negative for almost every such a family of curves, also known as path structure or path geometry. On the other hand, we consider path geometries possessing infinitesimal symmetries and show that path and projective structures with submaximal symmetry dimensions are variational. Note that the projective structure with the submaximal symmetry algebra, the so-called Egorov structure, is not pseudo-Riemannian metrizable; we show that it is metrizable in the class of Kropina pseudo-metrics and explicitly construct the corresponding Kropina Lagrangian.

Keywords Path structure · Projective structure · Euler-Lagrange equation · Symmetry analysis · Geodesics · Inverse variational problem · Jet space · Metrization · Finsler metrics · Kropina metrics

Contents

This article belongs to a Topical Collection: Singularity theorems, causality, and all that (SCRI21).

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1 Introduction

1.1 Definitions and motivations

Consider the following system of second order ODEs on a space M of dimension $n+1$ with coordinates $\boldsymbol{x} = (x^0, \dots, x^n)$:

$$x^i_{tt} + h^i(\boldsymbol{x}, \boldsymbol{x}_t) = \nu x^i_t, \qquad 0 \le i \le n. \tag{1.1}$$

Here functions $h^i(\boldsymbol{x}, \boldsymbol{x}_t)$ are assumed to be positively homogeneous of the second degree in $\boldsymbol{x}_t$, i.e., $\boldsymbol{h}(\boldsymbol{x}, \lambda\boldsymbol{x}_t) = \lambda^2\boldsymbol{h}(\boldsymbol{x}, \boldsymbol{x}_t)$ for every $\lambda > 0$, and ν is an arbitrary functional parameter to be eliminated. That is, a solution of the system is a vector-function $\boldsymbol{x}(t)$ such that there exists a function $\nu(t)$ for which (1.1) holds; $\boldsymbol{x}_t$ and $\boldsymbol{x}_{tt}$ denote the first and second derivatives of the vector-function $\boldsymbol{x}(t)$ in t. This system is clearly underdetermined and effectively consists of n equations on $n+1$ unknown functions. From the physical viewpoint it can be interpreted as the condition that at every point the acceleration $\boldsymbol{x}_{tt} + \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{x}_t)$ is linearly dependent with the velocity $\boldsymbol{x}_t$.

Since $\boldsymbol{h}$ is positively homogeneous of the second degree in $\boldsymbol{x}_t$, for every solution $\boldsymbol{x}(t)$ of system (1.1) and for any local diffeomorphismn $t \mapsto \tau(t)$ of $\mathbb{R}$ with $\tau'(t) > 0$ the reparameterized curve $\boldsymbol{x}(\tau(t))$ is also a solution. Therefore, solutions of (1.1) are arbitrary orientation-preserving reparameterizations of solutions of the system

$$x^i_{tt} + h^i(\boldsymbol{x}, \boldsymbol{x}_t) = 0, \qquad 0 \le i \le n. \tag{1.2}$$

For any point and any oriented direction there exists exactly one solution with these initial data.

A *path structure* (also known as *path geometry*) is the solution space of an equation of the form (1.1) or equivalently of (1.2) where we forget parametrization of solution-curves (henceforth called paths). Geometrically, it is defined as a smooth family of *unparameterized* curves such that there exists precisely one curve from the family through every point in every oriented direction.

The simplest example of a path structure is the *flat* structure on an affine space, where all the curves of the family are straight lines. (A locally equivalent path structure is given by the geodesic family on a Riemannian space of constant sectional curvature.)

We say that a path structure is *reversible*, if for every point and any oriented direction the path passing through this point in this direction geometrically coincides with the path passing in the reversed direction. For example, the flat structure is reversible. Clearly, reversibility is equivalent to the property that for every $(\boldsymbol{x}, \boldsymbol{x}_t)$ the difference $\boldsymbol{h}(\boldsymbol{x}, \boldsymbol{x}_t) - \boldsymbol{h}(\boldsymbol{x}, -\boldsymbol{x}_t)$ is proportional to $\boldsymbol{x}_t$.

Path structures naturally appear in differential geometry and in mathematical relativity. Indeed, for a Lagrangian[1] $\hat{L}(\boldsymbol{x}, \boldsymbol{x}_t)$ positively homogeneous of degree one in velocities (that is $\hat{L}(\boldsymbol{x}, \lambda \boldsymbol{x}_t) = \lambda \hat{L}(\boldsymbol{x}, \boldsymbol{x}_t)$ for $\lambda > 0$) and such that for the "energy function" $\hat{E} := \frac{1}{2}\hat{L}^2$ the Hessian $\left(\frac{\partial^2 \hat{E}}{\partial x_t^i \partial x_t^j}\right)$ with respect to $\boldsymbol{x}_t$ is nondegenerate, the Euler-Lagrange equation is algebraically-equivalent to a system of the form (1.1). Since unparameterized geodesics of Riemannian and pseudo-Riemannian metrics are extremals of the Langrangian $\hat{L}$ equal to the square root of the kinetic energy, system (1.1) contains the equation of geodesics as a special case. The same is true in Finsler geometry (and pseudo-Finsler generalisations), where geodesics are extremals of the Lagrangian $\hat{L}$ equal to the Finsler norm; if the Finsler norm is only positively homogeneous the corresponding path structure can be irreversible.

Investigation of path structures, as differential equations, and in particular their symmetries, goes back to the works of Lie [38] and his student Tresse [51]. For a scalar ODE of the form

$$y_{xx} = f(x, y, y_x), \tag{1.3}$$

they considered the path structure on $\mathbb{R}^2(x, y)$ whose paths are given by $x \to (x, y(x))$, where $y(x)$ is a solution of (1.3). This path structure is singular in the sense that in the vertical direction the paths are not defined. Symmetries of this path structure are called *point transformations* of the ODE; they correspond to changes of variables mixing dependent and independent variables.

In the context of metric geometry, path structures were studied by H. Busemann [11]; one of the question he considered is whether for a given path structure there exists a Finsler metric whose unparameterized geodesics are paths.

A *projective structure* is a path structure given by equation (1.1) with the functions h^i of the form

$$h^i(\boldsymbol{x}, \boldsymbol{x}_t) = \sum_{j,k=0}^{n} \Gamma^i_{jk}(\boldsymbol{x}) x_t^j x_t^k.$$

The corresponding paths are unparameterized geodesics of the affine connection (Γ^i_{jk}). Clearly, it is a reversible path structure. Projective equivalence of affine connections is their equivalence as path structures, and was studied since H. Weyl [53, 54] who in particular proved that in dimension $n + 1 \geq 3$ the Weyl projective curvature tensor W^ℓ_{ijk} vanishes if and only if the projective structure is flat. See also E. Cartan [15], who

[1] We use hat on autonomous Lagrangians (which for most part of the paper can be assumed to be homogeneous in $\boldsymbol{x}_t$) to distinguish them from nonautonomous Lagrangians in a space of one dimension less used later on.

constructed the fundamental systems of differential invariants for projective structures (in dimension $n + 1 > 2$; the case $n = 1$ is due to [51]).

A closely related classical problem is when two different metrics have the same geodesics viewed as unparameterized curves. First nontrivial results in this direction are due to E. Beltrami [4] who proved that in dimension two a Riemannian metric generating a flat projective structure has constant curvature, and to U. Dini [20] who gave a local description of pairs of 2-dimensional Riemannian metrics sharing the same (unparameterized) geodesics. Results of Beltrami and Dini were generalised to all dimensions by F. Schur [46] and T. Levi-Civita [37].

In the framework of mathematical relativity, projective structures were studied since H. Weyl [53, 54]. He proposed to base the geometric framework of gravity theory on the observable structures of particle trajectories and light propagation, i.e., on unparameterized geodesics and the conformal structure, see also O. Veblen and T. Thomas [52]. In a fundamental and widely read paper [27] J. Ehlers, F. Pirani and A. Schild claimed that a projective structure and a conformal structure on a differentiable manifold M determine a Weylian metric (Weyl structure), if and only if the *light-like geodesics* of the conformal structure are paths of the projective structure. This claim has been recently proven in [43]; see also [39], [40] and [9, §12].

Path structures which are not projective structures also naturally appear within mathematical relativity, see the survey by Ch. Pfeifer [44]. In particular, according to the Fermat principle, projection of null geodesics of a stationary spacetime to a Cauchy hypersurface are geodesics of a Randers (Finsler) metric, see e.g. E. Caponio et al [13]. These geodesics come without preferred parameterization, since a parameterization depends on the choice of a Cauchy hypersurface. Note that path structures coming from most Randers metrics are not reversible; moreover, if a path structure coming from a Randers metric is not reversible, then one can uniquely reconstruct this metric up to a trivial projective change by [42].

In our paper we discuss the question whether a given path structure is variational, that is whether there exists a Lagrangian function $\hat{L}(x, x_t, x_{tt}, ...)$ whose extremals are precisely the paths of the structure. This question is important, because many physical systems can be described mathematically with the help of the Hamilton-Jacobi formalism and was considered already by H. Helmholz [29]. In differential geometry, this question was explicitly asked by H. Busemann [10].

In the calculus of variations, this question is one of the so-called *inverse problems*, and there is a vast literature on this topic, see e.g. books by I. Anderson and G. Thompson [3] and by J. Grifone and Z. Muzsnay [28] for two different approaches to this problem (note that the second reference treats mostly parameterized solution-curves of differential equations and is not directly applicable to our problem), as well as the recent surveys [21, 22] by T. Do and G. Prince.

1.2 Results

We consider a path structure in dimension $n + 1$ and ask whether there exists an autonomous Lagrangian such that every curve of our path structure (with any parameterization), is an extremal of the Lagrangian and vice versa. We will call such path structures *variational*.

We first observe that we can eliminate higher order derivatives in the Lagrangian $\hat{L}$:

Proposition 1 *Suppose a path structure is variational. Then it is variational in the class of Lagrangians of order one: there exists a positively homogeneous of degree one in velocities function* $\hat{L}(\boldsymbol{x}, \boldsymbol{x}_t)$ *whose extremals are precisely the curves of the path structure.*

Almost equivalent statements can be found in the literature, see e.g. [45, Theorem 1] or [2, Theorem 3.2], so we do not pretend that the result is new. We will give a short proof to be self-contained.

Next, we will reduce the problem to a similar one, but in dimension one less. In this reduced problem we will look for nonautonomous Lagrangians (such a reduction was used in e.g. [4, 38], see also [24, §3]). In order to do this, we parametrize the curves of our path structure by the first coordinate $x^0 = x$ (this is possible locally for almost all solutions). In the notations $\boldsymbol{y} = (y^1, \dots, y^n)$, $y^j = x^j$ for $1 \le j \le n$, the curves are given by $x \mapsto (x, \boldsymbol{y}(x))$. Thus a path structure on a manifold M is given by a system of second order ODEs, which in local coordinates can be written as follows (dot means the derivative by x):

$$\ddot{y}^i = f^i(x, \boldsymbol{y}, \dot{\boldsymbol{y}}), \qquad 1 \le i \le n. \tag{1.4}$$

Paths of the path structure are the curves of the form $x \mapsto (x, \boldsymbol{y}(x))$, where $\boldsymbol{y}(x)$ are solutions of (1.4). On the language of geometric theory of ODEs, local diffeomorphisms of the space $(x, \boldsymbol{y})$ preserving the path structure are called *point transformations*.

We will recall in §3 relations between systems (1.1) and (1.4), and explain that the inverse variational problem for both systems is essentially the same. We treat it in the second (reduced) version. The corresponding Lagrangian L is a function on the ray-projectivized (or spherical) tangent bundle STM.

Recall that the space $J^k := J^k(\mathbb{R}^{2n+1}, \mathbb{R}^n)$ of k-jets of vector-functions $\boldsymbol{f} = (f^i(z))_{i=1}^n$ of the argument $z = (x, \boldsymbol{y}, \dot{\boldsymbol{y}})$ consists of the values of independent and dependent variables and their derivatives up to order k. The jet-lift of $\boldsymbol{f}$ is the map $j^k\boldsymbol{f} : \mathbb{R}^{2n+1} \to J^k, z \mapsto (z, \{\partial^j \boldsymbol{f}(z)\}_{j=0}^k)$.

Theorem 2 *Let* $\ell = 4$ *for* $n > 2$ *and* $\ell = 5$ *for* $n = 2$. *There exists an open dense set* $\mathcal{U} \subset J^\ell$ *such that if* $j^\ell \boldsymbol{f}(U) \cap \mathcal{U} \neq \emptyset$ *for* $U \subset STM$ *for the right-hand side of (1.4) then the path structure of (1.4) is not variational via a first-order Lagrangian* $L(x, \boldsymbol{y}, \dot{\boldsymbol{y}})$ *even microlocally on* U.

It is well-known that fibers of the bundle $J^k \to J^1$ carry a natural affine structure, while fibers of $J^1 \to J^0$ can be identified with (open charts in) Grassmanians, see e.g. [31]. Hence, fibers of $J^k \to J^0$ are algebraic, so we can use the Zariski topology. Recall that open sets in a Zariski topology are open dense in the standard topology, and the above set $\mathcal{U}$ can be taken Zariski open. This straightforwardly implies the following statement:

Corollary 3 *In dimension $n+1$, a generic smooth path structure in C^4 topology for $n \geq 3$ and in C^5 topology for $n = 2$ is not variational (hence not Finsler).*

In other words, in proper topology, every path structure $\mathcal{P}$ can be deformed by an arbitrary small deformation to a nonvariational path structure $\tilde{\mathcal{P}}$ and any sufficiently small deformation of $\tilde{\mathcal{P}}$ remains nonvariational.

Let us now discuss the dimension $n+1 = 2$. It is known since 1886, see N. Sonin [47] and G. Darboux [18], that in this case every equation (1.4) is (equivalent to) the Euler-Lagrange equation of an nonautonomous Lagrangian. This result was improved in [1] where it was shown that for every reversible path structure $\mathcal{P}$ there exists a reversible Finsler metric whose geodesics are paths of the structure. The irreversible case is still open, see e.g. [49] where the case when all paths are circles was investigated in details.

The case $n+1 = 3$ was considered by Douglas [23], who in particular constructed the first example of a nonvariational projective structure. He also discussed the PDE system for the inverse variational problem in the case of general n, but did not investigate it in detail. We recall this fundamental system in §3 and in §4 we show how to exploit it for specific path structures and for all dimensions.

Let us now discuss the question whether all the curves of a given path structure are geodesics of some pseudo-Riemannian metric. In the literature, this problem is known as "metrizability". Of course, in this case we may assume that the path structure is actually projective.

Our way to prove Theorem 2 easily implies:

Corollary 4 *In dimension $n+1$, a generic smooth projective structure in C^4 topology for $n \geq 3$ and in C^5 topology for $n = 2$ is not variational, hence not metrizable.*

The last portion of our results concerns path/projective structures with large Lie algebras of symmetries. Recall that *symmetry* of a path or projective structure is a local diffeomorphism that sends paths to paths. It is known that the flat structure in dimension $n+1$ has maximal symmetry dimension (i.e., dimension of the symmetry algebra) equal to n^2+4n+3. Of course, this path structure is variational since geodesics of the Lagrangian $\sqrt{(x_t^0)^2 + \ldots + (x_t^n)^2}$ are straight lines.

The next possible symmetry dimensions are n^2+5 (for general path structures) and n^2+4 (for projective structures), see [34]. In §5.1–5.2 we will demonstrate that these structures are variational by exhibiting Lagrangians (of Kropina type). However they are not metrizable: for the submaximally symmetric projective structure, called Egorov structure, this follows from [32]; the submaximally symmetric path structure is not a projective structure hence can not be metrizable by any pseudo-Riemannian metric. This implies the following result:

Corollary 5 *In dimension $n+1 > 1$ there exists a projective structure that is variational, but not metrizable.*

Note that §5.1 implies this results for $n \geq 2$. For $n = 1$, the result is known and follows from e.g. R. Bryant et al [6, 7]. Note also that $(n+1 = 2)$-dimensional projective structures admitting infinitesimal symmetries and the metrization problem for them was solved completely in [6, 41]. As mentioned above, 2-dimensional projective structures are always variational. J. Lang in [36] constructed Lagrangians for 2-dimensional path and projective structures with the submaximal symmetry algebra (of dimension 3), see also [30, 50].

We will also show that the Egorov projective structure is not (regular) Finsler metrizable. We expect, in the spirit of our results above, that generic variational projective structures are not metrizable (neither via pseudo-Riemannian nor via Finsler metrics). We briefly discuss other examples in §5.3 in relation to the inverse variational problem.

Our main result on non-variationality should be expected by experts. Indeed, the freedom of choosing system (1.4) is n functions of $2n + 1$ variables, while the Lagrangian is just one such function. This indicates that the system of PDEs expressing the existence of a Lagrangian is overdetermined, so one expects that for generic $\boldsymbol{f} = (f^i)$ it is not solvable. This arguments however requires high regularity (smoothness) of $\boldsymbol{f}$. See also [8, 48] for a treatment of this system using the machinery of Finsler geometry.

2 Proof of Proposition 1

By the Vainberg-Tonti formula [35], if the second order ODE system (1.1) is variational, then without loss of generality we may assume that the Lagrangian has the form $\hat{L} = \hat{L}(\boldsymbol{x}, \boldsymbol{x}_t, \boldsymbol{x}_{tt})$. The corresponding Euler-Lagrange equation then reads:

$$\frac{d^2}{dt^2}\frac{\partial \hat{L}}{\partial x^i_{tt}} - \frac{d}{dt}\frac{\partial \hat{L}}{\partial x^i_t} + \frac{\partial \hat{L}}{\partial x^i} = 0. \tag{2.1}$$

In this formula the possible highest t-derivative of $\boldsymbol{x}$ has order 4 and can come from the terms $\frac{d^2}{dt^2}\frac{\partial \hat{L}}{\partial x^i_{tt}}$ only. Since (1.1) does not have terms involving x^i_{tttt}, $\hat{L}$ must have the following form:

$$\hat{L}(\boldsymbol{x}, \boldsymbol{x}_t, \boldsymbol{x}_{tt}) = F(\boldsymbol{x}, \boldsymbol{x}_t) + \sum_s x^s_{tt}\lambda_s(\boldsymbol{x}, \boldsymbol{x}_t). \tag{2.2}$$

Let is now look on the third t-derivatives of $\boldsymbol{x}$: since the terms with x^i_{ttt} in the equation (2.1) with $\hat{L}$ given by (2.1) must cancel, we obtain:

$$\sum_s \left(\frac{\partial \lambda_s}{\partial x^i_t} - \frac{\partial \lambda_i}{\partial x^s_t}\right) x^s_{tt} = 0. \tag{2.3}$$

Then there exists a function $\Lambda(\boldsymbol{x}, \boldsymbol{x}_t)$ such that $\lambda_s = \frac{\partial \Lambda}{\partial x^s_t}$ implying $\sum_s x^s_{tt}\frac{\partial \Lambda}{\partial x^s_t} = \frac{d}{dt}\Lambda - \sum_s x^s_t \frac{\partial \Lambda}{\partial x^s}$.

Since the addition of the total derivative $-\frac{d}{dt}\Lambda$ to a Lagrangian does not change the complete variation, the Euler-Lagrange equation with Lagrangian (2.2) coincides with that substituted by $\tilde{L} = F(\boldsymbol{x}, \boldsymbol{x}_t) - \sum_s x^s_t \frac{\partial \Lambda}{\partial x^s}$. We see that $\tilde{L}$ is independent of

$\boldsymbol{x}_{tt}$ implying the first claim of Proposition 1. Next, since by our assumptions for a solution $\boldsymbol{x}(t)$ any of its reparameterization $\boldsymbol{x}(\tau(t))$ is also a solution, the Lagrangian $\tilde{L}$ is necessary homogeneous in t of degree 1.

3 PDE setup for the inverse problem

Here we work with inhomogeneous ODE (1.4) and the corresponding Lagrangian L, which now can be assumed of the first order. The variational problem

$$\int L(x, \boldsymbol{y}, \dot{\boldsymbol{y}})\, dx \to \min \tag{3.1}$$

leads to the Euler-Lagrange equations

$$\frac{\partial L}{\partial y^j} - \frac{d}{dx}\frac{\partial L}{\partial \dot{y}^j} = 0, \quad 1 \le j \le n, \tag{3.2}$$

where $\frac{d}{dx} = \partial_x + \dot{y}^j \partial_{y^j} + f^j \partial_{\dot{y}^j}$ is the operator of total derivative.

This is an overdetermined 2nd order PDE system on a scalar function $L = L(x, \boldsymbol{y}, \dot{\boldsymbol{y}})$ and it is equivalent to (1.4) if and only if: Euler-Lagrange system (3.2) vanishes modulo ODE system (1.4) and the Hessian matrix of L is nondegenerate (to be able to express the ODE from the EL)

$$\det\left[\frac{\partial^2 L}{\partial \dot{y}^i \partial \dot{y}^j}\right]_{i,j=1}^n \neq 0.$$

Note that system (3.2) is not of finite type, i.e. its solution is non-unique (modulo divergences and rescalings) and may be even not finitely parametric but contain arbitrary functions. Indeed, when $f^i = 0$ the problem (3.1) with straight lines as extremals has infinite-dimensional space of solutions. These are the so-called Minkowski Finsler metrics, given by translationally invariant Lagrangians $L = L(\dot{\boldsymbol{y}})$. Clearly there is a functional freedom in choosing such a Finsler metric.

3.1 The fundamental system

In [19] Davis and in [23] Douglas derived the following fundamental overdetermined system on the symmetric nondegenerate matrix $\phi_{ij} = \frac{\partial^2 L}{\partial \dot{y}^i \partial \dot{y}^j} = \phi_{ji}$:

$$\frac{\partial \phi_{ik}}{\partial \dot{y}^j} = \frac{\partial \phi_{jk}}{\partial \dot{y}^i}, \tag{3.3}$$

$$\frac{d}{dx}\phi_{ij} + \frac{1}{2}\frac{\partial f^k}{\partial \dot{y}^i}\phi_{kj} + \frac{1}{2}\frac{\partial f^k}{\partial \dot{y}^j}\phi_{ki} = 0, \tag{3.4}$$

$$A_i^k \phi_{kj} = A_j^k \phi_{ki}, \tag{3.5}$$

where

$$A_i^j = \frac{d}{dx}\frac{\partial f^j}{\partial \dot{y}^i} - 2\frac{\partial f^j}{\partial y^i} - \frac{1}{2}\frac{\partial f^k}{\partial \dot{y}^i}\frac{\partial f^j}{\partial \dot{y}^k}.$$

Note that A is a (1,1)-tensor (or a field of operators, which is more obvious than in [23], when written in proper indices), so condition (3.5) means that A is symmetric with respect to metric ϕ: $\phi(A\xi, \eta) = \phi(\xi, A\eta)$. However A is a given field and the unknown in this equation is ϕ. Yet, there are many solutions (depending on arbitrary functions).

To restrict those solutions further note that A, in general, is not integrable (its Nijenhuis tensor does not vanish), and there are more constraints coming from (3.4), and also (3.3). Namely, passing from A to $A' = \frac{d}{dx}A - \frac{1}{2}AJ - \frac{1}{2}J^*A$, where $J_j^k = \frac{\partial f^k}{\partial \dot{y}^j}$ and $*$ is conjugation with respect to ϕ we get the equation $\phi(A'\xi, \eta) = \phi(\xi, A'\eta)$. One can further iterate this recursive generation of constraints, and this is what is done in [23] for $n = 2$. However, as we will see, for $n > 2$ already the first iteration is generically sufficient.

3.2 On reparametrizations

If $\hat{L} = \hat{L}(\boldsymbol{x}, \boldsymbol{x}_t)$ is 1-homogeneous in velocity $\boldsymbol{x}_t$, then the functional on curves in M

$$\boldsymbol{x}(t) \mapsto \int \hat{L}(\boldsymbol{x}(t), \boldsymbol{x}_t(t))\, dt$$

is reparameterization invariant. In particular for a path $x^i = x^i(t)$ choosing $x^0 = x$ instead of parameter t we obtain the integral in (3.1): indeed when $x^0 = t$ we get $x_t^0 = 1$ and

$$L(x, y^j, \dot{y}^j) = \hat{L}(x, y^j, 1, \dot{y}^j).$$

Conversely, given $L(x, y^j, \dot{y}^j)$ can be extended to a function 1-homogeneous in velocities on TM (we view TM as a cone over STM) as follows (for nonsymmetric L, i.e. if $L(-v) \neq L(v)$, $v \in ST_xM$, one has to distinguish between $x_t^0 < 0$ and $x_t^0 > 0$ that may be not possible locally over domains in M, but only microlocally on small domains $U \subset STM$):

$$\hat{L}(x^0, x^1, \dots, x^n, x_t^0, x_t^1, \dots, x_t^n) = L\left(x^0, x^1, \dots, x^n, \frac{x_t^1}{x_t^0}, \dots, \frac{x_t^n}{x_t^0}\right) \cdot x_t^0.$$

Recall that the condition for $\hat{L}$ to define a Finsler metric is the subadditivity in velocities, which is equivalent (provided $\hat{L}$ smooth on $TM\backslash 0_M$) to the strong convexity condition: for any $x \in M$ and $0 \neq v \in T_xM$ the Hessian of $\hat{L}^2|_{T_xM}$ is positive definite at v.

Note that $\hat{L}^2$ is nondegenerate, i.e. $\det \mathrm{Hess}\big(\hat{L}^2|_{CU}\big) \neq 0$, for the cone $CU \subset T_xM$ over an open dense subset $U \subset ST_xM$, if L is nonvanishing and nondegenerate:

$$\det \mathrm{Hess}\big(\hat{L}^2|_{CU}\big) = 2^{n+1} L^{n+2} \det \mathrm{Hess}\big(L|_U\big)$$

(in general there are no relations between nondegeneracy of L and L^2. Note also that, due to 1-homogeneity, $\det \mathrm{Hess}\big(\hat{L}|_{T_xM}\big) \equiv 0$). We will call such $\hat{L}$ a pseudo-Finsler metric (an example is $\sqrt{|g(v,v)|}$ for a Lorentzian metric g on M). In this case equation $\{\hat{L} = 1\}$ in T_xM does not necessary define a convex but a nondegenerate (almost everywhere) hypersurface.

4 Proof of Theorem 2

In the case $f^i = f^i(x, \boldsymbol{y})$ we have $A_i^j = -2\frac{\partial f^j}{\partial y^i}$ and the fundamental system and its prolongation contian the following algebraic subsystem

$$A_i^k \phi_{kj} = A_j^k \phi_{ki}, \quad \big(\tfrac{d}{dx} A_i^k\big)\phi_{kj} = \big(\tfrac{d}{dx} A_j^k\big)\phi_{ki}. \tag{4.1}$$

This linear homogeneous system consists of $n(n-1)$ equations on $\frac{1}{2}n(n+1)$ unknowns, and so is determined for $n = 3$ and overdetermined for $n > 3$. We claim that generically it attains the maximal rank, and hence the only solution is $\phi_{ij} = 0$.

Note that for $n = 2$ the system is underdetermined, hence as in [23] one should add one more linear equation,[2] namely $\big(\frac{d^2}{dx^2} A_i^k\big)\phi_{kj} = \big(\frac{d^2}{dx^2} A_j^k\big)\phi_{ki}$. Then we get the 3×3 matrix of the system, which is generically nondegenerate, whence the same conclusion.

4.1 Nonexistence of solutions to the inverse problem

To prove the above claim for $n > 2$ we first exhibit a system for which the maximal rank is attained. This is given by

$$f^1 = \sum_{k=1}^{n} (y^k)^2,\ f^2 = (y^1)^2,\ f^3 = (y^2)^2,\ \dots,\ f^{n-1} = (y^{n-2})^2,\ f^n = y^{n-1}. \tag{4.2}$$

The $n(n-1) \times \binom{n+1}{2}$ matrix $\mathbb{A}$ of system (4.1) (to obtain it write ϕ_{ki} into a column Φ and write the system in matrix form $\mathbb{A} \cdot \Phi = 0$) depends on $(x, \boldsymbol{y}, \dot{\boldsymbol{y}})$ and has maximal rank $\binom{n+1}{2}$, for instance, at the point $x = 0$, $y^j = \delta_n^j$, $\dot{y}^j = 1$. (We omit this tedious verification.) Since the rank is generally maximal and the data are algebraic in 4-jets, the rank is generically maximal.

Moreover, when we perturb the condition $\frac{\partial f^i}{\partial \dot{y}^j} = 0$ the matrix $\mathbb{A}$ changes but still is of maximal rank at generic points, and this persists for generic 4-jets of the vector

[2] In the general case $\boldsymbol{f} = \boldsymbol{f}(x, \boldsymbol{y}, \dot{\boldsymbol{y}})$ the operator $A \mapsto A'$ of §3.1 should be used instead of $\frac{d}{dx}$.

function $\boldsymbol{f}$. Indeed, the second set of equations in (4.1) will express (ij)-symmetry of $\left(\frac{d}{dx}A_i^k\right)\phi_{kj} - \frac{1}{2}A_i^k\frac{\partial f^s}{\partial \dot{y}^k}\phi_{sj} - \frac{1}{2}A_i^k\frac{\partial f^s}{\partial \dot{y}^j}\phi_{ks}$ due to equation (3.4), and the matrix of this system is a perturbation of $\mathbb{A}$ provided that the values of derivatives $\frac{\partial f^s}{\partial \dot{y}^k}$ are small at the reference point. This implies the claim for $n > 2$.

Let us give an alternative geometric, less computational, agrument for computing the rank of $\mathbb{A}$. The matrices involved in (4.1) have the form (note that $\mathbb{A}$ is not this matrix below, but is easily derived from it):

$$\left(A\,\middle|\,\frac{d}{dx}A\right)^t = -4\cdot\left(\begin{array}{ccccccc|ccccccc}
y^1 & y^1 & 0 & 0 & \dots & 0 & 0 & \dot{y}^1 & \dot{y}^1 & 0 & 0 & \dots & 0 & 0\\
y^2 & 0 & y^2 & 0 & \dots & 0 & 0 & \dot{y}^2 & 0 & \dot{y}^2 & 0 & \dots & 0 & 0\\
y^3 & 0 & 0 & y^3 & \dots & 0 & 0 & \dot{y}^3 & 0 & 0 & \dot{y}^3 & \dots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
y^{n-2} & 0 & 0 & 0 & \dots & y^{n-2} & 0 & \dot{y}^{n-2} & 0 & 0 & 0 & \dots & \dot{y}^{n-2} & 0\\
y^{n-1} & 0 & 0 & 0 & \dots & 0 & \frac{1}{2} & \dot{y}^{n-1} & 0 & 0 & 0 & \dots & 0 & 0\\
y^n & 0 & 0 & 0 & \dots & 0 & 0 & \dot{y}^n & 0 & 0 & 0 & \dots & 0 & 0
\end{array}\right)$$

For generic entries the blocks have different eigenvalues each and are mutually independent. The quadraric form ϕ has eigenvectors of each block as orthogonal basis, but in general two bases cannot be simultenously orthogonal for one metric (any signature). This finished the proof.

For $n = 2$ the same argument works with the same ODE system (4.2). In fact, this system for $n = 2$ was already indicated by Douglas, and in [23, formula (3.1)] the 3×3 matrix Δ is nondegenerate, implying $\phi_{ij} = 0$ as the only solution. Our observation extends his result without going into detailed analysis of solvability of the fundamental system.

4.2 Other approaches

Let us consider one more example of nonexistence, namely a higher-dimensional version of another system from [23]:

$$f^1 = \sum_{k=1}^n (y^k)^2,\ \ f^2 = 0,\ \dots,\ f^{n-1} = 0,\ \ f^n = 0. \tag{4.3}$$

Then for the matrix $\mathbb{A}$ of system (4.1) its $n \times n$ minor consisting of rows with numbers $\left(1, \dots, n-1, \binom{n+1}{2} - 1\right)$ and columns $(1, \dots, n)$ is equal to $(-2)\times$ the matrix

$$\begin{pmatrix}
y^2 & y^1 & 0 & 0 & \cdots & 0 & 0\\
y^3 & 0 & y^1 & 0 & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
y^{n-1} & 0 & 0 & 0 & \cdots & y^1 & 0\\
y^n & 0 & 0 & 0 & \cdots & 0 & y^1\\
\dot{y}^n & 0 & 0 & 0 & \cdots & 0 & \dot{y}^1
\end{pmatrix} \quad \text{with} \quad \det = (y^1)^{n-2}(y^1\dot{y}^n - y^n\dot{y}^1) \not\equiv 0,$$

while the columns $\left(n+1,\ldots,\binom{n+1}{2}\right)$ vanish identically. This implies that $\phi_{1i}=0$ and hence $\det(\phi_{ij})=0$. Therefore ODE system (4.3) is not variational.

However this argument does not survive perturbation, as it belongs to a lower strata in branching the compatibility analysis of system (3.3)-(3.5). In particular, it can not be used as a substitute for (4.2) in our proof of Theorem 2. Complete analysis depending on ranks of the arising matrices was performed for $n=2$ in [23]. However the number of branches grows rapidly with $n>2$ and it would be unreasonable to expect a complete answer due to complexity.

One can exploit the idea of [33] to find the number of independent solutions (dimension) when system (3.3)-(3.5) is of finite type. Namely, its prolongation, obtained by differentiation of all equations of the system to a sufficiently large order N at a particular point $z\in U$, stabilizes the solution space, given by $(N+1)$-jet of ϕ_{ij} at z. In practice this procedure allows to effectively decide solvability of the system.

5 Submaximally symmetric structures are variational

In this section we discuss several examples, where we can resolve the fundamental system for the inverse problem. Namely we consider path structures admitting infinitesimal symmetries, i.e., local diffeomorphisms preserving the structure. A flat structure on a manifold M of dimension $n+1$ has maximal possible symmetry dimension n^2+4n+3 and is variational.

The next, so-called submaximal symmetry dimension is equal to n^2+5 for $n>1$; let us specify submaximal symmetry depending on the type of (nonzero) harmonic curvature, namely Fels torsion T or Fels curvature S, see [31, §5.3−5.4]. In the zero curvature module ($S=0$) we get projective geometry, and in the torsion-free module ($T=0$) we het general path geometry (Segré branch; non-projective). We consider those in turn.

5.1 The Egorov projective structure

This structure is originally [26] given by the nonzero Christoffel coefficients $\Gamma^0_{12}=\Gamma^0_{21}=x^1$ on $M=\mathbb{R}^{n+1}(\boldsymbol{x})$, $\boldsymbol{x}=(x^0,\ldots,x^n)$. The corresponding inhomogeneous system (1.4) has the following form[3]:

$$\ddot{y}^j=2y^1\dot{y}^1\dot{y}^2\dot{y}^j,\qquad 1\le j\le n. \tag{5.1}$$

This structure has maximal symmetry dimension n^2+4 among all nonflat projective structures [26, 34] and up to local diffeomorphism it is unique such [50]; it is non-metrizable by [32], i.e. there is no Levi-Civita connection in its projective class.

Surprisingly, the structure is variational, at least micro-locally:

[3] Projective structure with a representative connection Γ^c_{ab} $(0\le a,b,c\le n)$ can be encoded as the ODE system $\ddot{y}^j=\Gamma^0_{ik}\dot{y}^i\dot{y}^j\dot{y}^k+2\Gamma^0_{0i}\dot{y}^i\dot{y}^j-\Gamma^j_{ik}\dot{y}^i\dot{y}^k+\Gamma^0_{00}\dot{y}^j-2\Gamma^j_{0i}\dot{y}^i-\Gamma^j_{00}$ with summation over $1\le i,k\le n$.

Proposition 6 *There exists a Lagrangian function L defined for almost all velocities, which generates the Egorov projective structure.*

To see this note first following [5, remark after Theorem 3] that equation (5.1) is linearizable, namely the point transformation $(x, y^1, \dots, y^n) \mapsto (y^1 - \frac{x^2}{2}y^2, x, y^2, \dots, y^n)$ maps it to the ODE

$$\ddot{y}^1 = y^2,\ \ddot{y}^2 = 0,\ \dots,\ \ddot{y}^n = 0. \tag{5.2}$$

We will treat therefore this system. It is precisely of the kind considered at the beginning of this section. Thus considering system (3.3)-(3.5) for this choice of f^i we find a Lagrangian

$$L = (\dot{y}^1 - x\, y^2)\dot{y}^2 + \sum_3^n (\dot{y}^i)^2$$

with extremals given by (5.2). The corresponding 1-homogeneous Lagrangian is

$$\hat{L} = \left(\frac{x_t^1}{x_t^0} - x^0 x^2\right) x_t^2 + \frac{(x_t^3)^2 + \cdots + (x_t^n)}{x_t^0}.$$

Its extremal curves satisfy the (underdetermined) ODE with the same paths as (5.2):

$$x_t^0 x_{tt}^1 - x_t^1 x_{tt}^0 = (x_t^0)^3 x^2,\ x_t^0 x_{tt}^j - x_t^j x_{tt}^0 = 0 \quad (1 < j \le n).$$

Thus the Egorov structure is variational.

Remark 7 Lagrangians of the form $\hat{L} = \frac{g(\boldsymbol{x}_t, \boldsymbol{x}_t)}{\alpha(\boldsymbol{x}_t)}$ for a Riemannian or pseudo-Riemannian metric g and for a 1-form α are called Kropina (pseudo-Finsler) metrics. Kropina metrics were also considered in the framework of mathematical relativity, see e.g. E. Caponio et al [14]. Kropina metrics are not defined on the vectors $\boldsymbol{x}_t$ lying in the kernel of α. Note that in our case the form α is closed so its extremals define a projective structure by [17, Corollary 3.6].

In this context, the following question is natural: does there exist a strictly convex Finsler metric (without singularities and defined on the whole slit tangent bundle) whose geodesics are curves of the Egorov projective structure? The next proposition answers this question negatively:

Proposition 8 *The path structure given by (5.2), and hence by (5.1), is not Finsler metrizable.*

Indeed, in this case we can obtain the general solution of the fundamental system, which due to a very simple form $A_i^j = -2\delta_1^j \delta_i^2$, is as follows:

$$\phi_{11} = \phi_{13} = \cdots = \phi_{1n} = 0,\ \phi_{12} \neq 0,\ \tfrac{d}{dx}\phi_{ij} = 0,$$

where $\frac{d}{dx} = \partial_x + \dot{y}^i \partial_{y^i} + y^2 \partial_{\dot{y}^1}$. This implies the form (we omit dependence of ψ_0, ψ_1 on $\boldsymbol{x}$, indicating only velocity $\boldsymbol{x}_t$) of the homogeneous Lagrangian:

$$\hat{L} = \psi_0\left(\frac{x_t^2}{x_t^0}, \dots, \frac{x_t^n}{x_t^0}\right) x_t^0 + \psi_1\left(\frac{x_t^2}{x_t^0}, \dots, \frac{x_t^n}{x_t^0}\right) x_t^1.$$

One can easily see that for any choice of ψ_0, ψ_1 the function $\hat{L}^2$ is not convex.

5.2 Submaximally symmetric path structure

The maximally symmetric nonflat path structure has dimension of the symmetry algebra $n^2 + 5$, see [34]. Uniqueness of such a structure has been recently established in [50]. For $n + 1 = 3$ this structure was discussed in [16] in relation to self-dual gravity, the corresponding spacetime[4] is Ricci flat of Petrov type N. The ODE system generating this metric via the twistor correspondence is

$$\ddot{y}^1 = (\dot{y}^2)^3, \ \ddot{y}^2 = 0, \ \dots, \ \ddot{y}^n = 0. \tag{5.3}$$

The fundamental system for the inverse problem is solvable; one solution is given by

$$L = \left(\sqrt{\pi}\, \dot{y}^1 \operatorname{erf}\left(\frac{\dot{y}^1}{(\dot{y}^2)^{3/2}}\right) + (\dot{y}^2)^{3/2} \exp\left(-\frac{(\dot{y}^1)^2}{(\dot{y}^2)^3}\right) + \dot{y}^1\right) e^{2y^1} + \sum_3^n (\dot{y}^i)^2.$$

Thus the path structure (5.3) is variational. The corresponding 1-homogeneous Lagrangian $\hat{L}$ can be derived straightforwardly.

5.3 More examples

Another notable path structure is given by a family of distinguished curves of the trivial scalar ODE, encoded as the flat $A_2/P_{1,2}$ homogeneous geometry [12]. The distinguished curves transversal to the contact structure on $J^1(\mathbb{R}^1)$ are given by a pair of differential equations on unknowns $y^1(x)$, $y^2(x)$ (cf. an equivalent form in [16, §7.2]):

$$\ddot{y}^1 = \frac{2(\dot{y}^1)^2}{y^1 - \dot{y}^2}, \ \ddot{y}^2 = 0. \tag{5.4}$$

This ODEs system is also related to anti self-dual conformal metrics, namely it generates via the twistor correspondence an Einstein metric of constant negative scalar curvature [16].

The fundamental system passes the compatibility test (as discussed at the end of §4.2), so from the Cartan-Kähler theorem it follows that it possesses solutions with

[4] This Plebanski type metric has coordinate expression $g = dx\,dw + dy\,dz - y^2 dw^2$.

any admissible Cauchy initial data; in particular, we conclude that system (5.4) is variational. Indeed, for

$$L = \frac{\dot{y}^1}{\dot{y}^2 - y^1}$$

extremals are exactly the paths given by (5.4). The corresponding 1-homogeneous Lagrangian $\hat{L}$ can be derived straightforwardly.

Remark 9 An elliptic version of this example consist of chains in (not necessary flat) CR geometry. It was proven in [17] that in any dimension the path geometry of chains is variational, with the Lagrangian being a Kropina metric.

6 Conclusion

The inverse variational problem for nonautonomous ODE systems (1.4) has attracted a lot of interest in the literature; several criteria for variationability were obtained. We have shown that a generic path structure in dimension $n+1 \geq 3$ is not variational. The proof is done in terms of jets. Our methods allow to derive a proper subanalytic subset $\Sigma \subset J^k$ such that (regular) variational structures given as (1.4) are subject to the constraints $j^\ell \boldsymbol{f}(U) \subset \Sigma, U \subset STM$.

In particular if a path structure comes from experimental observations and should be variational by physical reasons, our methods may help to confirm correctness of the experiment; and also find a variational structure that (in some sense) is closest to the experimental data. We leave aside a related question on optimal regularity C^k, where our results hold.

A corollary of our main theorem implies that a generic projective structure is not metrizable in the class of Riemannian or pseudo-Riemannian metrics. This result was expected: indeed, the metrization problem can be reduced to an overdetermined system of PDEs of finite type (see e.g. M. Eastwood et al [25]). Nevertheless, this result was formally established only in dimension 2 (R. Bryant et al [7]) and in dimension 3 (M. Dunajski et al [24]); we proved it in any dimension.

We also demonstrated that the Egorov projective structure is variational in any dimension $n+1 \geq 3$ by exhibiting a Kropina type pseudo-Finsler metric. For $n=2$ this could be obtained from the results of Douglas [23]; in [3] another Lagrangian was derived for ODE (5.2) though without any relation to the Egorov structure. By our previous work [32] it is not metrizable in the pseudi-Riemannian setting. In this work we proved it is not metrizable in the Finsler setting. We also demonstrated variationability of some other notable path geometries with many infinitesimal symmetries.

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