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Cartan-Geometric Approaches to Submaximally Symmetric Ordinary Differential Equations

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ABSTRACT.

This thesis is concerned with a symmetry classification problem for ordinary differential equations (ODEs) that dates back to Sophus Lie. We focus on *higher order* ODEs, i.e. scalar ODEs of order ≥ 4 or vector ODEs of order ≥ 3, up to contact transformations. The *maximal* contact symmetry algebra dimensions for these ODEs are known. We determine for all higher order ODEs the *submaximal* (i.e. next largest realizable) contact symmetry dimensions $\mathfrak{S}$. Using the known contact fundamental (generalized Wilczynski or C-class) invariants for higher order ODEs, we also determine submaximal symmetry dimensions for several classes of the ODEs that are contact invariant. Moreover, we give a complete local classification of all submaximally symmetric vector ODEs of *C-class*, i.e. ODEs with symmetry dimensions realizing $\mathfrak{S}$ that are characterized by the vanishing of all generalized Wilczynski invariants. Our results refine the classical results for scalar ODEs, and also provide generalizations of those results to vector ODEs.

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Contents

CHAPTER 1

Overview of the Thesis

Ordinary differential equations (ODEs) are fundamental mathematical objects that are used in a wide variety of disciplines, from physics, chemistry and biology to economics and engineering. Solving explicitly given ODEs in quadratures is an important theoretical problem, and the classical approach to this (due to Sophus Lie) is by using symmetries (see for example [43]). This leads to the natural question: which ODEs have sufficiently many symmetries and this question attracted considerable attention (see [2, 31, 33, 36, 43, 45] and references therein).

In this thesis, we consider (systems of) $(n+1)$-st order ODEs $\mathcal{E}$

$$\mathbf{u}^{(n+1)} = \mathbf{f}(t, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(n)}). \tag{1.1}$$

Here $\mathbf{u}$ is an $\mathbb{R}^m$-valued function of t where $m \geq 1$, and $\mathbf{u}^{(k)}$ is its k-th derivative. We recall that a k-jet of $\mathbf{u}$ is the collection of data invariantly encoding the Taylor expansion of $\mathbf{u}(t)$ up to order k, which at a point t_0 can be identified with the values of all derivatives of $\mathbf{u}$ at t_0 up to order k. With this approach, an ODE of order k can be considered as a submanifold $\mathcal{E}$ in the space J^k of k-jets of functions $\mathbf{u} : \mathbb{R} \to \mathbb{R}^m$ (see for example [29] and references therein). Below we fix the order $k = n + 1$.

Symmetries of an ODE (1.1) of order $n+1$ are contact transformations of J^{n+1} that preserve the equation submanifold $\mathcal{E} \subset J^{n+1}$. By the Lie–Bäcklund theorem, such transformations are prolongations of

- $m = 1$: contact transformations of J^1, which are the most general invertible local diffeomorphisms that preserve the contact structure;
- $m \geq 2$: point transformations, i.e. invertible local diffeomorphisms of $J^0 \cong \mathbb{R} \times \mathbb{R}^m$.

Infinitesimally, symmetries of such an ODE are contact vector fields on J^{n+1}, i.e. vector fields whose local flow are contact transformations, that are tangent to $\mathcal{E}$ [29].

It is known that for $n \geq 1$, an ODE (1.1) admits a finite-dimensional contact symmetry Lie algebra, with the exception of a scalar second order ODE (i.e. $m = 1, n = 1$). We remark that the largest realizable *(maximal)* symmetry dimensions $\mathfrak{M}$ for the scalar cases are classical results due to Lie [39] from 1893, whereas the vector cases $\mathfrak{M}$ were established much later. In fact, for the 2nd order vector ODEs this was done in 1983 by González-Gascón and González-López [23] (see also Fels

[22] where the result was reproved using Cartan's equivalence method), and for vector ODEs of order ≥ 3 this can be deduced (see below) from the main result of Doubrov–Komrakov–Morimoto [16] that appeared in 1999.

We will focus on *higher order* ODEs (1.1), i.e. scalar ODEs of order ≥ 4 ($m = 1, n \geq 3$) or vector ODEs of order ≥ 3 ($m \geq 2, n \geq 2$). We consider and resolve the *symmetry gap problem* for scalar and vector ODEs, which concerns determining *submaximal* (i.e. the next largest realizable) symmetry dimensions $\mathfrak{S}$. We then give a complete local classification of submaximally symmetric vector ODEs of C-class [6, 10] (see below).

This is an article-based thesis, and the chapters consist of my joint articles with Dennis The:

(1) J.A. Kessy and D. The, *Symmetry gaps for higher order ordinary differential equations*, J. Math. Anal. Appl. **516** (2022), 126475, 1–23.
(2) J.A. Kessy and D. The, *On uniqueness of submaximally symmetric vector ordinary differential equations of C-class*, arXiv: 2301.09364 (2023).

We note that the classification of submaximally symmetric higher order scalar ODEs is due to Lie [40]. He obtained those using his complete classification of Lie algebras of contact vector fields on the complex plane. This involves classifying invariant ODEs under the prolonged action of a Lie algebra of contact vector fields, which amounts to classifying all relative and absolute differential invariants. Lie's approach certainly generalizes to higher order vector ODEs, but it is not feasible, since complete classifications of Lie algebras of contact vector fields on $\mathbb{C}^n$ or $\mathbb{R}^n$ for $n \geq 3$ are not available [15, 44]. We remark that even if the classifications were available, the computations associated with this approach would be tedious and difficult since the classification of relative and absolute invariants is in general not enough to find the invariant vector ODEs (see for example [35]). So, different approaches are needed to:

(1) compute the submaximal symmetry dimensions for vector ODEs;
(2) classify (up to point-equivalence) submaximally symmetric vector ODEs.

Our approach is based on the fact that all higher order ODEs (1.1) modulo contact transformations can be equivalently reformulated as *(regular, normal) Cartan geometries* $(\mathcal{G} \to \mathcal{E}, \omega)$ of type (G, P) for an appropriate Lie group G and a closed subgroup $P \subset G$ [6, 16]. Here, a Cartan geometry consists of:

- $\mathcal{G} \to \mathcal{E}$ is a (right) principal P-bundle, and
- ω is a *Cartan connection*, i.e. a $\mathfrak{g}$-valued 1-form on $\mathcal{G}$, where $\mathfrak{g}$ is the Lie algebra of G, such that:
  (i) For any $u \in \mathcal{G}$, $\omega_u : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism;
  (ii) $R_g^*\omega = \mathrm{Ad}_{g^{-1}} \circ \omega$ for any $g \in P$, i.e. ω is P-equivariant;
  (iii) $\omega(\zeta_A) = A$, where $A \in \mathfrak{p}$, where ζ_A is the fundamental vertical vector field defined by $\zeta_A(u) := \frac{d}{dt}\big|_{t=0} u \cdot \exp(tA)$.

The *trivial* ODE $\mathbf{u}_{n+1} = \mathbf{0}$ is associated to the Klein geometry $(G \to G/P, \omega_G)$, where ω_G is the Maurer–Cartan form of G, which is called the *flat model* for all Cartan geometries of type (G, P). A relevant immediate consequence is that existence of this Cartan-geometric description establishes the finite dimensionality for the contact symmetry algebra for all higher ODEs with $\mathfrak{M} = \dim G$. Moreover, $\mathfrak{M}$ is locally uniquely realized by the trivial ODE.

Given a Cartan geometry $(\mathcal{G} \to \mathcal{E}, \omega)$ of type (G, P), its curvature is the 2-form $K = d\omega + \frac{1}{2}[\omega, \omega]$, which due to its P-equivariancy, can be identified with a (horizontal) function κ on $\mathcal{G}$ taking values in $\bigwedge^2 \mathfrak{g}^* \otimes \mathfrak{g}$. This is a fundamental relative invariant, and all other differential invariants of the Cartan geometry can be derived from it. For parabolic geometries [8], i.e. (regular, normal) Cartan geometries modelled on a *semisimple* Lie group G and a parabolic subgroup $P \subset G$, there is a fundamental quantity called the *harmonic curvature* κ_H that is a complete obstruction to local flatness, i.e. $\kappa_H \equiv 0$ iff the geometry is locally equivalent to the flat model (this is characterized by $\kappa = 0$).

Geometries associated to higher order ODEs are Cartan, but *not* parabolic, namely the corresponding group G is non-semisimple. However, due to the works by Doubrov and Medvedev [13, 17, 41, 42], they can be treated on the same footing as parabolic Cartan geometries, namely they have harmonic curvatures. In particular, κ_H is comprised of the fundamental invariants for the ODE. Consequently, an ODE (1.1) is contact-equivalent to the trivial ODE iff all fundamental invariants vanish identically. The fundamental invariants for these ODEs (1.1) were computed by Doubrov [13] for scalar ODEs of order ≥ 4, Medvedev [41, 42] for 3rd order vector ODEs, and Doubrov–Medvedev [17] for vector ODEs of order ≥ 4. These invariants are valued in a certain G_0-submodule $\mathbb{E} \subsetneq H^2(\mathfrak{g}_-, \mathfrak{g})$ of a Lie algebra cohomology group called the *effective part*. Here $G_0 \subset P$ is the reductive part (the groups G, G_0 and P will be described in detail in the main part of the text). The submodule $\mathbb{E}$ has been computed in the aforementioned works of Doubrov and Medvedev, and is an important ingredient for our approach. The fundamental invariants consist of generalized Wilczynski invariants [14] and C-class invariants, and these are valued in G_0-irreducible submodules $\mathbb{U} \subset \mathbb{E}$ that we shall refer to as Wilczynski modules and C-class modules respectively.

Let us now briefly describe the projects comprising this thesis.

1.1 Project 1: Symmetry gaps for higher order ODEs

The goal of this project is to resolve the symmetry gap problem for higher order ODEs. We note that the symmetry gap problem for geometric structures is a classical problem in differential geometry (see [9, 28, 48, 49] and references therein).

For geometric structures that can be reformulated as parabolic geometries, substantial recent progress in resolving the symmetry gap problem was made following Kruglikov–The [36] (see for example [19, 32, 34, 38] and references therein). Examples of parabolic (ODE) geometries include 2nd order scalar ODEs mod point transformations, 3rd order scalar ODEs mod contact transformations, and 2nd order vector ODEs mod point transformations.

We observed that some key features underlying the parabolic study [36] including the harmonic curvature κ_H and *Tanaka prolongation algebra* (see Definition 2.3.8) have parallels to (non-parabolic) Cartan geometries associated to higher order ODEs. We then adapted the Kruglikov–The approach to the ODE setting and resolved the symmetry gap problem. Briefly, we first established an algebraic upper bound $\mathfrak{U}$ on the submaximal symmetry dimension $\mathfrak{S}$, and gave ODE models of order $n+1$ with contact symmetry algebra dimensions realizing

$$\begin{aligned} \mathfrak{U} = \mathfrak{S} &= \begin{cases} \mathfrak{M}-1, & \text{if } m=1,\ n\in\{4,6\}; \\ \mathfrak{M}-2, & \text{otherwise}; \end{cases} \\ \text{where} \quad \mathfrak{M} &= \dim G = m^2+(n+1)m+3. \end{aligned} \tag{1.2}$$

Our result for $\mathfrak{S}$ was not known for the vector cases. For the scalar cases we recovered the classical result for due to Lie [40] (see [43, p.205]).

We remark that our approach allowed us to compute submaximal symmetry dimensions for several classes of higher order ODEs that are invariant under contact transformations as we next describe. Fix a G_0-irrep $\mathbb{U}\subset\mathbb{E}$ and its corresponding fundamental invariant $\mathcal{U}$, and let $C_{\mathcal{U}}$ denotes the set of all ODEs (1.1) with $\mathcal{U}\not\equiv 0$ and *all* remaining fundamental invariants vanishing identically. We defined $\mathfrak{S}_{\mathbb{U}}$ and $\mathfrak{U}_{\mathbb{U}}$ analogously to $\mathfrak{S}$ and $\mathfrak{U}$ by restricting to ODEs in $C_{\mathcal{U}}$. We then established that $\mathfrak{S}_{\mathbb{U}}\leq\mathfrak{U}_{\mathbb{U}}$. We proved that for all vector cases and most of the scalar cases we have $\mathfrak{S}_{\mathbb{U}}=\mathfrak{U}_{\mathbb{U}}$. Scalar exceptions where $\mathfrak{S}_{\mathbb{U}}<\mathfrak{U}_{\mathbb{U}}$ were also addressed. Note that our result $\mathfrak{S}=\mathfrak{U}$ stated above is immediate from the aforementioned results. We remark that even for the scalar cases, these finer results were missing in the literature, and hence is our new contribution.

1.2 Project 2: On uniqueness of submaximally symmetric vector ODEs of C-class

Having computed the submaximal symmetry dimensions in project 1, we are naturally led to considering the local classification problem (up to contact-equivalence) for submaximally symmetric ODEs. In this project, we consider and completely resolve the problem for vector ODEs of C-class. This is an important class of ODEs, taking its origin in the work by É. Cartan [10], for which the explicit integration of a generic ODE of the class is entirely an algebraic / differential problem. According to [6], an ODE of C-class can be characterized by the vanishing of all generalized Wilczynski invariants.

Fix an irreducible C-class module $\mathbb{U}\subset\mathbb{E}$ and its corresponding C-class invariant, and recall $C_{\mathcal{U}}$ and $\mathfrak{S}_{\mathbb{U}}$ from above. The goal of the project is to classify (up to point-equivalence) all submaximally symmetric vector ODEs (1.1) of C-class of order ≥ 3 in $C_{\mathcal{U}}$, i.e. ODEs of C-class with

(1) $\mathcal{U} \not\equiv 0$ and *all* remaining C-class invariants vanishing identically, and
(2) symmetry dimension equal to $\mathfrak{S}_{\mathbb{U}}$.

A key fact that we proved in project 1 is that the regular, normal Cartan geometries $(\mathcal{G} \to \mathcal{E}, \omega)$ of type (G, P) corresponding to submaximally symmetric vector ODEs are locally *homogeneous*, i.e. there exists a (left) action by a local Lie group F on $\mathcal{G} \to \mathcal{E}$ by principal bundle morphisms preserving ω that projects onto a transitive action on $\mathcal{E}$. We remark that the *Cartan reduction* method [9, 18, 43] can potentially be used to classify such (homogeneous) geometric structures. However, implementation of the method is extremely challenging since setting up the correct structure equations is a difficult task, and normalizations will involve cumbersome computations.

Our approach is motivated by that of The [46, 47] for parabolic geometries, and is based on the well known result that regular, normal homogeneous Cartan geometries can be encoded by algebraic data [8, Prop 1.5.15] (see also [25]). Our classification (up to point-equivalence) for vector ODEs at hand then translates into a classification (up to an appropriate notion of equivalence) for the so-called *algebraic models of ODE type* (see Definition 2.6.2), which are defined in terms of algebraic data. The computations associated with our approach will be efficiently done using representation theory.

For each such $\mathbb{U} \subset \mathbb{E}$, we gave a complete local classification over $\mathbb{C}$ or $\mathbb{R}$ of all vector ODEs of C-class in $C_{\mathcal{U}}$. Our results are entirely new, and provide generalizations of classical results for submaximally symmetric scalar ODEs due to Lie [40] (see the introduction to Chapter 3). A new ingredient underlying these results is a key advance concerning the harmonic theory associated with the structure of vector ODEs of C-class. Namely, for each irreducible C-class module, we provide an explicit identification of a lowest weight vector as a harmonic 2-cochain.

To conclude this introduction, we remark that in this work we left aside the questions of maximal and submaximal symmetry for systems of ODEs of *mixed* orders [20]. This is an interesting subject on its own with various approaches to symmetry, and several important classes of differential equations. We hope that the methods developed in our work will find further applications in these and related problems.

CHAPTER 2

Symmetry gaps for higher order ordinary differential equations

This chapter consists of contents from my joint article [26] with Dennis The.

2.1 Abstract

The maximal contact symmetry dimensions for scalar ODEs of order ≥ 4 and vector ODEs of order ≥ 3 are well known. Using a Cartan-geometric approach, we determine for these ODEs the next largest realizable (*submaximal*) symmetry dimension. Moreover, finer curvature-constrained submaximal symmetry dimensions are also classified.

2.2 Introduction

Consider a system of $m \geq 1$ ordinary differential equations (ODEs) of order $n + 1 \geq 2$ given by

$$\mathbf{u}^{(n+1)} = \mathbf{f}(t, \mathbf{u}, \dot{\mathbf{u}}, \dots, \mathbf{u}^{(n)}), \tag{2.1}$$

where $\mathbf{u}$ is an $\mathbb{R}^m$-valued function of t, and $\mathbf{u}^{(k)}$ is its k-th derivative. We will focus on the geometry of such ODEs under local *contact* transformations, which by the Lie–Bäcklund theorem agrees with the geometry under local *point* transformations when $m \geq 2$ (vector ODEs).

Except when $n = m = 1$ (scalar 2nd order), the ODE (2.1) admits a finite-dimensional contact symmetry algebra and the largest realizable (*maximal*) symmetry dimension $\mathfrak{M}$ is known – see for example [3, §1] for a historical survey. Indeed, the trivial ODE $\mathbf{u}^{(n+1)} = 0$ is uniquely (up to contact equivalence) maximally symmetric among (2.1), cf. Corollary 2.3.7 below, and the dimension of its Lie algebra of (infinitesimal) contact symmetries is given by

$$\mathfrak{M} = \begin{cases} 10, & \text{if } m = 1,\ n = 2; \\ (m+2)^2 - 1, & \text{if } m \geq 2,\ n = 1; \\ m^2 + (n+1)m + 3, & \text{if } m = 1,\ n \geq 3 \ \text{ or } \ m, n \geq 2. \end{cases} \tag{2.2}$$

In contrast, all scalar 2nd order ODEs are locally contact equivalent to the trivial ODE $\ddot{u} = 0$, which admits an *infinite*-dimensional contact symmetry algebra. Under point transformations, $\ddot{u} = 0$ has point symmetry algebra of dimension $\mathfrak{M} = 8$ and is maximally symmetric.

In all cases with a finite maximal symmetry dimension, a natural classification problem is to *determine the next largest realizable (submaximal) symmetry dimension* $\mathfrak{S}$. There is often a sizable gap between $\mathfrak{M}$ and $\mathfrak{S}$, so this is referred to as the *symmetry gap* problem. For ODEs, examples of this are given in Table 1. See [36] for details on these cases where the underlying geometric structure is a *parabolic geometry* (see below).

Geometry	$\mathfrak{S}$	Sample ODE	Reference
Scalar 2nd order ODEs mod point transformations	3	$\ddot{u} = \exp(\dot{u})$	(1896) [48]
Scalar 3rd order ODEs mod contact transformations	5	$\dddot{u} = b\dot{u} + u$	(2002) [45]
Vector 2nd order ODEs mod point transformations	$m^2 + 5$	$\ddot{u}^a = (\dot{u}^1)^3 \delta^a_m$ $(1 \leq a \leq m)$	$m = 2$: (2013) [11] $m \geq 3$: (2017) [36]

TABLE 1. Submaximal symmetry dimensions $\mathfrak{S}$ for ODEs among parabolic geometries

We consider the symmetry gap problem for *higher order* ODEs (scalar ODEs of order ≥ 4 or vector ODEs of order ≥ 3) and prove that:

THEOREM 2.2.1. *Fix (n, m) with $m = 1, n \geq 3$ or $m, n \geq 2$. Among the ODEs (2.1) of order $n + 1$, the submaximal contact symmetry dimension is*

$$\mathfrak{S} = \begin{cases} \mathfrak{M} - 1, & \textit{if } m = 1,\ n \in \{4, 6\}; \\ \mathfrak{M} - 2, & \textit{otherwise}. \end{cases} \tag{2.3}$$

This corrects a recent conjecture [3, §10] for $\mathfrak{S}$ when $m, n \geq 2$, stated as

$$\begin{cases} \mathfrak{M} - 2m + 2, & \text{if } m \in \{2, 3\}; \\ \mathfrak{M} - 2m + 1, & \text{if } m \geq 4. \end{cases} \tag{2.4}$$

The results for scalar ODEs recover Lie's [40] (see [43, p.205] for a brief summary), which he obtained based on [43, Thm. 6.36] and the complete classification of Lie algebras of contact vector fields on the (complex) plane. This requires classifying the fundamental differential invariants for each such Lie algebra of vector fields as well as investigating their Lie determinants (see [43, Table 5]). However, attempting to apply Lie's approach to vector ODEs in order to prove Theorem 2.2.1 is not feasible: this would require as a first step classifying Lie algebras of vector fields in general dimension. This is far out of reach, as evidenced by the fact that even the classification in dimension three remains incomplete (although large branches have been settled), see [15, 44] for recent progress and references therein. Moreover, even if such classifications were available, the computations involved with the approach would be extremely tedious, and establishing refinements as in Theorem 2.2.2 below would be even more difficult. Different techniques are required to address the vector cases.

Our approach is based on a categorically equivalent reformulation of ODEs $\mathcal{E}$ given by (2.1) (mod contact) as *regular, normal Cartan geometries* $(\mathcal{G} \to \mathcal{E}, \omega)$ of type (G, P), for some appropriate Lie group G and closed subgroup $P \subset G$ (see §2.3.1.2). The construction of such canonical Cartan connections ω for ODEs was discussed in [6, 13, 16, 17]. The trivial ODE corresponds to the flat model $(G \to {}^{G}/_{P}, \omega_G)$, which has symmetry dimension $\dim G$, and more generally $\dim G$ bounds the symmetry dimension of any Cartan geometry of type (G, P), so $\mathfrak{M} = \dim G$.

Parabolic geometries are Cartan geometries modelled on the quotient of a semisimple Lie group by a parabolic subgroup. For this diverse class of geometric structures (whose underlying structures includes those ODEs from Table 1), significant progress on the symmetry gap problem was made in [36]. In particular, a universal algebraic upper bound $\mathfrak{U}$ on $\mathfrak{S}$ was established, effective methods for the computation of $\mathfrak{U}$ were given in the complex or split-real settings, and in almost all of these cases it was shown that $\mathfrak{S} = \mathfrak{U}$ by presenting (abstract) models.

All higher order ODEs ($m = 1$, $n \geq 3$ or $m, n \geq 2$) admit equivalent descriptions as *non-parabolic* Cartan geometries. For these ODEs, we adapt certain key features from the parabolic study to our specific non-parabolic setting. The main ingredients for establishing $\mathfrak{S} \leq \mathfrak{U}$ are *harmonic curvature* κ_H, which is a complete obstruction to local flatness, and *Tanaka prolongation*, both of which have parallels in the ODE setting. The key technical fact underpinning our $\mathfrak{S} \leq \mathfrak{U}$ proof is that $\kappa_H \not\equiv 0$ is valued in a certain completely reducible P-module, which was established in [6, Cor.3.8], so only the action of the reductive part $G_0 \subset P$ is relevant. (In fact, the strategy of our proof is a simplified version of that given in [37], which yields a stronger statement than the approach from [36] – see Remark 2.) Our upper bound result is formulated in Theorem 2.3.10.

By complete reducibility, the codomain of κ_H can be identified with a certain proper G_0-submodule $\mathbb{E} \subsetneq H^2_+(\mathfrak{g}_-, \mathfrak{g})$ of a Lie algebra cohomology group. This *effective part* $\mathbb{E}$ has already been computed in the literature by Doubrov [12, 13] for scalar ODEs, Medvedev [41] for vector 3rd order ODEs, and by Doubrov–Medvedev [17] for vector higher order ODEs. In §2.4, we summarize their classifications in Tables 5 and 6, organized as irreducible G_0-submodules $\mathbb{U} \subset \mathbb{E}$, and use these to efficiently compute the corresponding restricted quantities $\mathfrak{U}_{\mathbb{U}}$, from which $\mathfrak{U}$ can be obtained via (2.34).

We note that the aforementioned upper bound proof also yields the finer results $\mathfrak{S}_{\mathbb{U}} \leq \mathfrak{U}_{\mathbb{U}}$, where $\mathfrak{S}_{\mathbb{U}}$ is analogous to $\mathfrak{S}$ but with the additional constraint that $\kappa_H \not\equiv 0$ is valued in $\mathbb{U} \subset \mathbb{E}$. Thus, we can consider the finer symmetry gap problem of determining $\mathfrak{S}_{\mathbb{U}}$ for a fixed $\mathbb{U}$. For ODEs that are parabolic geometries, such constrained problems were resolved in [36]. In our non-parabolic setting, using the known fundamental (relative) differential invariants for higher order ODEs derived in [1, 13, 17, 42, 50], we exhibit realizability of $\mathfrak{U}_{\mathbb{U}}$ in §2.5 by finding explicit ODEs realizing these symmetry dimensions and with $\kappa_H \neq 0$ concentrated in $\mathbb{U}$. In addition to proving Theorem 2.2.1, we obtain the following curvature-adapted result:

THEOREM 2.2.2. *Fix (n, m) with $m = 1, n \geq 3$ or $m, n \geq 2$, and consider ODEs (2.1) of order $n + 1$. Let $\mathbb{U}$ be a G_0-irrep contained in the effective part $\mathbb{E} \subsetneq H^2_+(\mathfrak{g}_-, \mathfrak{g})$. Then $\mathfrak{S}_{\mathbb{U}}$ is given in Table 2.*

n	m	G_0-irrep $\mathbb{U} \subset \mathbb{E}$	$\mathfrak{S}_{\mathbb{U}}$
≥ 3	1	$\mathbb{W}_r$ $(3 \leq r \leq n+1)$	$\mathfrak{M} - 2 = \mathfrak{U}_{\mathbb{W}_r}$
3	1	$\mathbb{B}_3$	$\mathfrak{M} - 3 = \mathfrak{U}_{\mathbb{B}_3} - 1$
3	1	$\mathbb{B}_4$	$\mathfrak{M} - 2 = \mathfrak{U}_{\mathbb{B}_4}$
4	1	$\mathbb{B}_6$	$\mathfrak{M} - 1 = \mathfrak{U}_{\mathbb{B}_6}$
≥ 4	1	$\mathbb{A}_2$	$\mathfrak{M} - 2 = \mathfrak{U}_{\mathbb{A}_2}$
5	1	$\mathbb{A}_3$	$\mathfrak{M} - 3 = \mathfrak{U}_{\mathbb{A}_3} - 1$
≥ 6	1	$\mathbb{A}_3$	$\leq \mathfrak{M} - 3 = \mathfrak{U}_{\mathbb{A}_3} - 1$
6	1	$\mathbb{A}_4$	$\mathfrak{M} - 1 = \mathfrak{U}_{\mathbb{A}_4}$
≥ 7	1	$\mathbb{A}_4$	$\mathfrak{M} - 3 = \mathfrak{U}_{\mathbb{A}_4} - 1$ or $\mathfrak{M} - 4$
≥ 2	≥ 2	$\mathbb{W}^{\mathrm{tf}}_r$ $(2 \leq r \leq n+1)$	$\mathfrak{M} - 2m + 1 = \mathfrak{U}_{\mathbb{W}^{\mathrm{tf}}_r}$
≥ 2	≥ 2	$\mathbb{W}^{\mathrm{tr}}_r$ $(3 \leq r \leq n+1)$	$\mathfrak{M} - 2 = \mathfrak{U}_{\mathbb{W}^{\mathrm{tr}}_r}$
2	≥ 2	$\mathbb{B}_4$	$\mathfrak{M} - m = \mathfrak{U}_{\mathbb{B}_4}$
2	≥ 2	$\mathbb{A}^{\mathrm{tf}}_2$	$\mathfrak{M} - 2m + 2 = \mathfrak{U}_{\mathbb{A}^{\mathrm{tf}}_2}$
≥ 2	≥ 2	$\mathbb{A}^{\mathrm{tf}}_2$	$\mathfrak{M} - 2m + 1 = \mathfrak{U}_{\mathbb{A}^{\mathrm{tf}}_2}$
≥ 3	≥ 2	$\mathbb{A}^{\mathrm{tr}}_2$	$\mathfrak{M} - m - 1 = \mathfrak{U}_{\mathbb{A}^{\mathrm{tr}}_2}$

(Recall $\mathfrak{M} = m^2 + (n+1)m + 3$ from (2.2).)

TABLE 2. Curvature-constrained submaximal symmetry dimensions for ODEs of order $n + 1$

We note that all vector cases and most scalar cases satisfy $\mathfrak{S}_{\mathbb{U}} = \mathfrak{U}_{\mathbb{U}}$. The exceptional scalar cases are: $(n, \mathbb{U}) = (3, \mathbb{B}_3), (\geq 5, \mathbb{A}_3)$ or $(\geq 7, \mathbb{A}_4)$. The assertions $\mathfrak{S}_{\mathbb{U}} < \mathfrak{U}_{\mathbb{U}}$ here can be deduced from the known classification of submaximally symmetric scalar ODEs (see [43, p. 206]). In Appendix 2.6, we outline an alternative algebraic method for establishing these $\mathfrak{S}_{\mathbb{U}} < \mathfrak{U}_{\mathbb{U}}$ exceptions.

We conclude this introduction with explicit examples of ODEs (in Tables 3 and 4) that realize $\mathfrak{S}_{\mathbb{U}}$ from Table 2 (aside from the above exceptions). We use the notation $\mathbf{u}^{(k)} := (u^1_k, \dots, u^m_k)$ for the k-th derivative of $\mathbf{u} := (u^1, \dots, u^m)$ with respect to t. The assertions about the given ODEs can be directly verified using the relative invariants summarized in §2.5 and explicit infinitesimal symmetries given in Tables 8, 9, and 10.

n	G_0-irrep $\mathbb{U} \subset \mathbb{E}$	Example ODE with $\mathrm{im}(\kappa_H) \subset \mathbb{U}$
≥ 3	$\mathbb{W}_r$ $(3 \leq r \leq n+1)$	$u_{n+1} = u_{n+1-r}$
3	$\mathbb{B}_4$	$n u_{n-1} u_{n+1} - (n+1)(u_n)^2 = 0$
≥ 4	$\mathbb{A}_2$	
4	$\mathbb{B}_6$	$9(u_2)^2 u_5 - 45 u_2 u_3 u_4 + 40 (u_3)^3 = 0$
6	$\mathbb{A}_4$	$10(u_3)^3 u_7 - 70(u_3)^2 u_4 u_6 - 49(u_3)^2 (u_5)^2 + 280 u_3 (u_4)^2 u_5 - 175 (u_4)^4 = 0$

TABLE 3. Scalar ODEs of order $n+1 \geq 4$ realizing $\mathfrak{S}_{\mathbb{U}}$

n	G_0-irrep $\mathbb{U} \subset \mathbb{E}$	Example ODE with $\mathrm{im}(\kappa_H) \subset \mathbb{U}$
≥ 2	$\mathbb{W}^{\mathrm{tr}}_r$ $(3 \leq r \leq n+1)$	$u^a_{n+1} = u^a_{n+1-r}$ $(1 \leq a \leq m)$
≥ 2	$\mathbb{W}^{\mathrm{tf}}_r$ $(2 \leq r \leq n+1)$	$u^a_{n+1} = u^2_{n+1-r} \delta^a_1$ $(1 \leq a \leq m)$
2	$\mathbb{B}_4$	$u^a_{n+1} = \dfrac{(n+1) u^1_n u^a_n}{n u^1_{n-1}}$ $(1 \leq a \leq m)$
≥ 3	$\mathbb{A}^{\mathrm{tr}}_2$	
≥ 2	$\mathbb{A}^{\mathrm{tf}}_2$	$u^a_{n+1} = (u^2_n)^2 \delta^a_1$ $(1 \leq a \leq m)$

TABLE 4. Vector ODEs of order $n+1 \geq 3$ (for $m \geq 2$ functions) realizing $\mathfrak{S}_{\mathbb{U}}$

2.3 An upper bound on submaximal symmetry dimensions

We begin by reviewing the Cartan-geometric perspective on ODEs, and then use it to prove an upper bound formula for submaximal symmetry dimensions (Theorem 2.3.10).

2.3.1 Canonical Cartan connections

2.3.1.1 ODEs as filtered G_0-structures Consider the space $J^{n+1}(\mathbb{R},\mathbb{R}^m)$ of $(n+1)$-jets of smooth maps from $\mathbb{R}$ into $\mathbb{R}^m$, with the natural projection $\pi^{n+1}_n : J^{n+1}(\mathbb{R},\mathbb{R}^m) \to J^n(\mathbb{R},\mathbb{R}^m)$ and denote by C the *Cartan distribution* on it. Denoting $\mathbf{u}_r = (u^1_r, \ldots, u^m_r)$, we let $(t, \mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_{n+1})$ be standard (bundle-adapted) local coordinates on $J^{n+1}(\mathbb{R},\mathbb{R}^m)$, for which the Cartan distribution C is given by

$$C = \langle \partial_t + \mathbf{u}_1 \partial_{\mathbf{u}_0} + \ldots + \mathbf{u}_{n+1} \partial_{\mathbf{u}_n}, \partial_{\mathbf{u}_{n+1}} \rangle. \tag{2.5}$$

(Here, $\mathbf{u}_1 \partial_{\mathbf{u}_0}$ is our compact notation for $\sum_{a=1}^m u^a_1 \partial_{u^a_0}$, etc. and $\partial_{\mathbf{u}_{n+1}}$ refers to $\partial_{u^1_{n+1}}, \ldots, \partial_{u^m_{n+1}}$.)

We will consider (2.1) up to *contact transformations*. These are diffeomorphisms ϕ of $J^{n+1}(\mathbb{R},\mathbb{R}^m)$ that preserve the distribution C, i.e. $\phi_*(C) = C$. By the Lie–Bäcklund theorem, such transformations are the *prolongations* [43] of contact transformations on $J^1(\mathbb{R},\mathbb{R}^m)$. Moreover, for $m \geq 2$ they are the prolongations of diffeomorphisms on $J^0(\mathbb{R},\mathbb{R}^m) \cong \mathbb{R} \times \mathbb{R}^m$ (*point transformations*). At the infinitesimal level, a *contact vector field* ξ is a vector field whose flow is a (local) contact transformation. Equivalently, $\mathcal{L}_\xi C \subset C$, where $\mathcal{L}_\xi$ is the Lie derivative with respect to ξ.

Rephrased geometrically, the $(n+1)$-st order ODE (2.1) is a codimension m submanifold $\mathcal{E} = \{\mathbf{u}_{n+1} = \mathbf{f}\}$ in $J^{n+1}(\mathbb{R},\mathbb{R}^m)$ transverse to the projection map π^{n+1}_n. So, $\mathcal{E}$ can be (locally) identified with its diffeomorphic image in $J^n(\mathbb{R},\mathbb{R}^m)$.

DEFINITION 2.3.1. *A contact symmetry of the ODE $\mathcal{E} \subset J^{n+1}(\mathbb{R},\mathbb{R}^m)$ is a contact vector field ξ on $J^{n+1}(\mathbb{R},\mathbb{R}^m)$ that is tangent to $\mathcal{E}$.*

We associate $\mathcal{E}$ with a pair (E, V) of subdistributions of C described below:

- the line bundle E over $\mathcal{E}$ whose integral curves are lifts of solution curves to (2.1);
- the rank m Frobenius-integrable distribution $V := \ker(d\pi^{n+1}_n|_{\mathcal{E}})$.

As proven in [16, Thm 1], the pair (E, V) encodes $\mathcal{E}$ up to the contact transformations and therefore defines a geometric structure associated to (2.1).

Equivalently, a contact symmetry of the ODE $\mathcal{E} \subset J^{n+1}(\mathbb{R},\mathbb{R}^m)$ is a vector field ξ on $\mathcal{E}$ such that $\mathcal{L}_\xi E \subset E$ and $\mathcal{L}_\xi V \subset V$. In standard local coordinates,

$$E = \left\langle \frac{d}{dt} := \partial_t + \mathbf{u}_1 \partial_{\mathbf{u}_0} + \cdots + \mathbf{u}_n \partial_{\mathbf{u}_{n-1}} + \mathbf{f} \partial_{\mathbf{u}_n} \right\rangle, \quad V = \langle \partial_{\mathbf{u}_n} \rangle. \tag{2.6}$$

In the sequel, we shall refer to $\frac{d}{dt}$ as the *total derivative*.

The distribution $D := E \oplus V \subset T\mathcal{E}$ is bracket-generating and its weak-derived flag defines a filtration on the tangent bundle $T\mathcal{E}$:

$$T\mathcal{E} = D^{-n-1} \supset \cdots \supset D^{-2} \supset D^{-1}, \tag{2.7}$$

where $D^{-1} := D$ and $D^{-j-1} := D^{-j} + [D^{-j}, D^{-1}]$ for $j > 0$. Then $(\mathcal{E}, \{D^j\})$ becomes a *filtered manifold*, since the Lie bracket of vector fields on $\mathcal{E}$ is compatible with the tangential filtration $\{D^j\}$, i.e

$$[\Gamma(D^i), \Gamma(D^j)] \subset \Gamma(D^{i+j}). \tag{2.8}$$

From (2.6), we can moreover verify that

$$[\Gamma(D^i), \Gamma(D^j)] \subset \Gamma(D^{\min(i,j)-1}), \tag{2.9}$$

which is a stronger condition if $i, j \leq -2$.

Furthermore, (2.1) admits an equivalent description as a *filtered G_0-structure* described below. The *associated graded* to the filtration (2.7) is given by

$$\operatorname{gr}(T\mathcal{E}) := \bigoplus_{j=-n-1}^{-1} \operatorname{gr}_j(T\mathcal{E}), \quad \text{where} \quad \operatorname{gr}_j(T\mathcal{E}) := D^j\mathcal{E}/D^{j+1}\mathcal{E}.$$

For $x \in \mathcal{E}$, the Lie bracket of vector fields induces a (Levi) bracket on $\mathfrak{m}(x) := \operatorname{gr}(T_x\mathcal{E})$ turning it into a nilpotent graded Lie algebra (NGLA) with $\mathfrak{m}_j(x) := \operatorname{gr}_j(T_x\mathcal{E})$. It is called the *symbol algebra* at x. For distinct points $x, y \in \mathcal{E}$, $\mathfrak{m}(x)$ and $\mathfrak{m}(y)$ belong to the same NGLA isomorphism class. Let $\mathfrak{m}$ be a fixed NGLA with $\mathfrak{m} \cong \mathfrak{m}(x), \forall x \in \mathcal{E}$. Since D is bracket-generating, then $\mathfrak{m}$ is generated by $\mathfrak{m}_{-1}$.

For $x \in \mathcal{E}$, denote by $F_{\mathrm{gr}}(x)$ the set of all NGLA isomorphisms from $\mathfrak{m}$ to $\mathfrak{m}(x)$ and $F_{\mathrm{gr}}(\mathcal{E}) := \bigcup_{x \in \mathcal{E}} F_{\mathrm{gr}}(x)$. Then $F_{\mathrm{gr}}(\mathcal{E}) \to \mathcal{E}$ is a principal fiber bundle with structure group $\mathrm{Aut}_{\mathrm{gr}}(\mathfrak{m})$ consisting of all graded automorphisms of $\mathfrak{m}$. In fact, $\mathrm{Aut}_{\mathrm{gr}}(\mathfrak{m}) \hookrightarrow \mathrm{GL}(\mathfrak{m}_{-1})$, since $\mathfrak{m}$ is generated by $\mathfrak{m}_{-1}$.

The splitting of D implies a splitting of $\mathfrak{m}_{-1}$. Let $G_0 \leq \mathrm{Aut}_{\mathrm{gr}}(\mathfrak{m})$ be the subgroup preserving this splitting of $\mathfrak{m}_{-1}$. There is a corresponding proper subbundle $\mathcal{G}_0 \to \mathcal{E}$, which is a principal fiber bundle with reduced structure group $G_0 \cong \mathbb{R}^\times \times \mathrm{GL}_m$. This realizes the ODE as a so-called *filtered G_0-structure* [5, Defn 2.2]. We immediately caution that not all filtered G_0-structures arise from ODEs (see Remark 1).

2.3.1.2 The trivial ODE Consider the trivial system of $m \geq 1$ ODEs $\mathbf{u}_{n+1} = 0$ of order $n+1$. Throughout, we will restrict to the *higher order* cases $m = 1, n \geq 3$ and $m, n \geq 2$. The contact symmetry vector fields for the trivial ODE were given in [6, Section 2.2]. Abstractly, the contact symmetry algebra $\mathfrak{g}$ has the structure

$$\mathfrak{g} := \mathfrak{q} \ltimes V, \quad \text{where} \quad \mathfrak{q} := \mathfrak{sl}_2 \times \mathfrak{gl}_m, \quad V := \mathbb{V}_n \otimes W. \tag{2.10}$$

Here, $\mathbb{V}_n$ is the unique (up to isomorphism) $\mathfrak{sl}_2$-irrep of dimension $n+1$ and $W = \mathbb{R}^m$ is the standard representation of $\mathfrak{gl}_m$. The trivial ODE admits the maximal symmetry dimension among (2.1) for fixed (n, m), c.f. Corollary 2.3.7. Consequently, we denote:

$$\mathfrak{M} := \dim \mathfrak{g} = m^2 + (n+1)m + 3. \tag{2.11}$$

We work with the following basis for $\mathfrak{g}$. Let $\{w_a\}$ be the standard basis for $W = \mathbb{R}^m$, let $\mathfrak{gl}_m \cong \mathfrak{gl}(W)$ be spanned by $\{e^a_b\}$, where $e^a_b w_c = \delta^a_c w_b$, and let $\mathrm{id}_m := \sum_{a=1}^m e^a_a$. Letting $\{x, y\}$ be the standard basis for $\mathbb{R}^2$, consider the standard $\mathfrak{sl}_2$-triple

$$\mathsf{X} = x\partial_y, \quad \mathsf{H} = x\partial_x - y\partial_y, \quad \mathsf{Y} = y\partial_x, \tag{2.12}$$

and consider the weight vectors for $\mathbb{V}_n$ given by

$$E_i = \frac{1}{i!} x^{n-i} y^i, \quad i = 0, \dots, n. \tag{2.13}$$

Following [13, 17], we give $\mathfrak{g}$ the structure of a $\mathbb{Z}$-graded Lie algebra $\mathfrak{g} = \mathfrak{g}_{-n-1} \oplus \dots \oplus \mathfrak{g}_1$, where

$$\begin{aligned} &\mathfrak{g}_1 = \mathbb{R}\mathsf{Y}, \quad \mathfrak{g}_0 = \mathbb{R}\mathsf{H} \oplus \mathfrak{gl}_m, \quad \mathfrak{g}_{-1} = \mathbb{R}\mathsf{X} \oplus (\mathbb{R}E_n \otimes W), \\ &\mathfrak{g}_i = \mathbb{R}E_{n+1+i} \otimes W, \quad i = -2, \dots, -n-1. \end{aligned} \tag{2.14}$$

We note that $\mathfrak{g}_- \cong \mathfrak{m}$, the symbol algebra defined in §2.3.1.1.

The splitting on $\mathfrak{g}_{-1}$ reflects the splitting on the distribution $D = E \oplus V$ from §2.3.1.1. Note that $\mathfrak{g}_0$ is reductive and $\mathfrak{g}_-$ is generated by $\mathfrak{g}_{-1}$. Alternatively, introducing the *grading element*

$$\mathsf{Z} := -\frac{1}{2}\left(\mathsf{H} + (n+2)\,\mathrm{id}_m\right), \tag{2.15}$$

the eigenspaces of $\mathrm{ad}_{\mathsf{Z}} \in \mathfrak{gl}(\mathfrak{g})$ are precisely $\mathfrak{g}_i = \{x \in \mathfrak{g} : [\mathsf{Z}, x] = ix\}$ for all $i \in \mathbb{Z}$. We visualize this as in Figure 1.

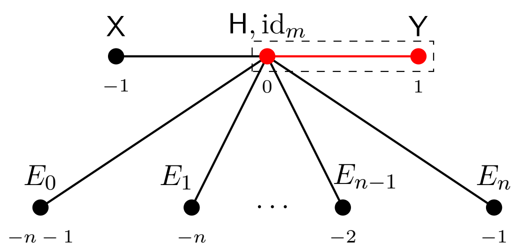

FIGURE 1. Grading on $\mathfrak{g}$, with basis specified in the scalar case

We also endow $\mathfrak{g}$ with the corresponding filtration $\mathfrak{g}^i := \sum_{j \geq i} \mathfrak{g}_j$, and let

$$\mathfrak{p} := \mathfrak{g}^0 = \langle \mathsf{H}, e^a_b, \mathsf{Y} \rangle, \quad \mathfrak{p}_+ := \mathfrak{g}^1 = \langle \mathsf{Y} \rangle. \tag{2.16}$$

Let $\mathrm{gr}_i : \mathfrak{g}^i \to \mathfrak{g}^i/\mathfrak{g}^{i+1}$ denote the natural quotient and let $\mathrm{gr}(\mathfrak{g}) := \bigoplus_i \mathrm{gr}_i(\mathfrak{g})$ denote the associated graded, which is isomorphic as a $\mathfrak{g}_0 \cong \mathrm{gr}_0(\mathfrak{g})$ module to $\mathfrak{g}$ as a graded Lie algebra.

At the group level, let

- $m = 1$: $G = \mathrm{GL}_2 \ltimes \mathbb{V}_n$ and $P = \mathrm{ST}_2 \subset \mathrm{GL}_2$, the subgroup of lower triangular matrices;
- $m \geq 2$: $G = (\mathrm{SL}_2 \times \mathrm{GL}_m) \ltimes V$ and $P = \mathrm{ST}_2 \times \mathrm{GL}_m$.

In either case, let $G_0 := \{g \in P : \mathrm{Ad}_g(\mathfrak{g}_0) \subset \mathfrak{g}_0\}$. We note that the filtration on $\mathfrak{g}$ is P-invariant.

2.3.1.3 Cartan geometries All ODEs (2.1) are filtered G_0-structures, and these admit an equivalent description as (normalized) Cartan geometries of type (G, P). We describe the precise setup in this section.

DEFINITION 2.3.2. *A* Cartan geometry *$(\mathcal{G} \to M, \omega)$ of type (G, P) consists of a (right) principal P-bundle $\mathcal{G} \to M$ endowed with a $\mathfrak{g}$-valued one-form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, called a* Cartan connection*, such that:*

(i) For any $u \in \mathcal{G}$, $\omega_u : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism;
(ii) ω is P-equivariant, i.e. $R_g^*\omega = Ad_{g^{-1}} \circ \omega$ for any $g \in P$;
(iii) $\omega(\zeta_A) = A$, where $A \in \mathfrak{p}$, where ζ_A is the fundamental vertical vector field *defined by $\zeta_A(u) := \frac{d}{dt}\big|_{t=0} u \cdot \exp(tA)$.*

Because of (i), the tangent bundle of $\mathcal{G}$ is trivialized, i.e. $T\mathcal{G} \cong \mathcal{G} \times \mathfrak{g}$, and the P-invariant filtration on $\mathfrak{g}$ induces a corresponding filtration of $T\mathcal{G}$:

$$T^{-n-1}\mathcal{G} \supset \ldots \supset T^{-1}\mathcal{G} \supset T^0\mathcal{G} \supset T^1\mathcal{G}. \tag{2.17}$$

Let us also note the following consequence of (ii). Fixing $u \in \mathcal{G}$, consider a P-invariant vector field $\eta \in \Gamma(T\mathcal{G})^P$ with $A := \omega(\eta_u) \in \mathfrak{p}$, and let f be a P-equivariant function on $\mathcal{G}$. Then:

$$(\eta \cdot f)(u) = \left.\frac{d}{dt}\right|_{t=0} f\left(u \cdot \exp(At)\right) = \left.\frac{d}{dt}\right|_{t=0} \exp(-At) \cdot f(u) = -A \cdot f(u). \tag{2.18}$$

The *Klein geometry* $(G \to {}^G/_P, \omega_G)$, where ω_G is the *Maurer–Cartan form* on G, is called the *flat model* for Cartan geometries of type (G, P). Given a Cartan geometry, its *curvature form* $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ is given by

$$K(\xi, \eta) = d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)], \tag{2.19}$$

which is P-equivariant and horizontal, i.e. $K(\zeta_A, \cdot) = 0$, $A \in \mathfrak{p}$. By horizontality, it is determined by the P-equivariant *curvature function* $\kappa : \mathcal{G} \to \bigwedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$, defined by

$$\kappa(A, B) = K(\omega^{-1}(A), \omega^{-1}(B)), \quad A, B \in \mathfrak{g}. \tag{2.20}$$

For (G, P) from §2.3.1.2, and the filtration $\{\mathfrak{g}^i\}$ introduced there, we say that a Cartan connection ω is *regular* if $\kappa(\mathfrak{g}^i, \mathfrak{g}^j) \subset \mathfrak{g}^{i+j+1}$ for all i, j. Equivalently, κ has image in the subspace of $\bigwedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ on which the grading element Z acts with positive eigenvalues (*degrees*).

For normality of ω, we follow the description in [6, §3]. Let us denote by $C^k(\mathfrak{g}, \mathfrak{g}) := \bigwedge^2 \mathfrak{g}^* \otimes \mathfrak{g}$, and consider the P-invariant subspace

$$C^k_{\mathrm{hor}}(\mathfrak{g}, \mathfrak{g}) := \{\psi \in C^k(\mathfrak{g}, \mathfrak{g}) : \iota_A \psi = 0, \forall A \in \mathfrak{p}\} \cong \bigwedge\nolimits^k(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}. \tag{2.21}$$

Both of these inherit filtrations from the filtration on $\mathfrak{g}$. Their associated graded can be identified with $C^k(\mathfrak{g}_-, \mathfrak{g})$, i.e. the cochain spaces for a complex $C^\bullet(\mathfrak{g}_-, \mathfrak{g})$ with the standard differential ∂ for computing Lie algebra cohomology groups $H^k(\mathfrak{g}_-, \mathfrak{g})$. There is an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ whose extension to $C^k(\mathfrak{g}, \mathfrak{g})$ is such that the adjoint ∂^* of the standard differential $\partial_\mathfrak{g}$ on $C^\bullet(\mathfrak{g}, \mathfrak{g})$ (with respect to $\langle \cdot, \cdot \rangle$) restricts to a P-equivariant map $\partial^* : \bigwedge^k(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g} \to \bigwedge^{k-1}(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$. (See [6, Lemma 3.2] for details.) In terms of this map ∂^*, we say that ω is *normal* if $\partial^* \kappa = 0$. From [6, Thm.2.2] (see also [13, 16, 17]), we have the following important starting point:

THEOREM 2.3.3. *Fix (G, P) as above. There is an equivalence of categories between filtered G_0-structures and regular, normal Cartan geometries of type (G, P).*

REMARK 1. A regular, normal Cartan connection *associated to an ODE* (2.1) satisfies the *strong regularity* condition $\kappa(\mathfrak{g}^i, \mathfrak{g}^j) \subset \mathfrak{g}^{i+j+1} \cap \mathfrak{g}^{\min(i,j)-1}$, $\forall i, j$ [6, Rem 2.3]. Consequently, not all filtered G_0-structures arise from ODE. For example, in [6, §3.5] there is a G_2-invariant filtered G_0-structure with the same symbol as that of an 11th order scalar ODE, but it is not realizable by any such ODE.

Since $(\partial^*)^2 = 0$, then for regular, normal Cartan geometries one obtains the (P-equivariant) *harmonic curvature* function

$$\kappa_H : \mathcal{G} \to \frac{\ker \partial^*}{\operatorname{im} \partial^*}, \tag{2.22}$$

which is valued in the filtrand of positive degree (by regularity). It is a fundamental fact that κ_H completely obstructs local flatness [5], i.e $\kappa_H \equiv 0$ if and only if the geometry is locally equivalent to the flat model, which corresponds to the trivial ODE. Furthermore,

LEMMA 2.3.4. *The P-module $\frac{\ker \partial^*}{\operatorname{im} \partial^*}$ is completely reducible, i.e. $\mathfrak{g}^1$ acts trivially.*

PROOF. See [6, Corollary 3.8]. ■

The above complete reducibility property will be important in subsequent sections. Consequently, only the G_0-action on $\frac{\ker \partial^*}{\operatorname{im} \partial^*}$ is relevant. Identifying

$\bigwedge^2 (\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g} \cong \bigwedge^2 \mathfrak{g}_-^* \otimes \mathfrak{g}$ as G_0-modules, and defining the Laplacian operator $\Box := \partial \circ \partial^* + \partial^* \circ \partial$ on $\bigwedge^2 \mathfrak{g}_-^* \otimes \mathfrak{g}$, we have a Hodge decomposition and the following G_0 isomorphisms:

$$\bigwedge^2 \mathfrak{g}_-^* \otimes \mathfrak{g} \cong \overbrace{\operatorname{im}\partial^* \oplus \underbrace{\ker\Box}_{}}^{\ker\partial^*} \!\!\!\underbrace{\oplus \operatorname{im}\partial}_{\ker\partial}, \tag{2.23}$$

$$\ker\Box \cong \frac{\ker\partial^*}{\operatorname{im}\partial^*} \cong \frac{\ker\partial}{\operatorname{im}\partial} =: H^2(\mathfrak{g}_-, \mathfrak{g}). \tag{2.24}$$

Regularity of ω and complete reducibility imply that the codomain of κ_H can be identified with the subspace $H^2_+(\mathfrak{g}_-, \mathfrak{g}) \subset H^2(\mathfrak{g}_-, \mathfrak{g})$ on which Z acts with positive eigenvalues.

Not all filtered G_0-structures are realizable by ODE, so some of $H^2_+(\mathfrak{g}_-, \mathfrak{g})$ is extraneous for ODE.

DEFINITION 2.3.5. *Let $\mathbb{E} \subset H^2_+(\mathfrak{g}_-, \mathfrak{g})$ denote the effective part, i.e. the minimal G_0-submodule in which κ_H is valued, for any regular, normal Cartan geometry of type (G, P) associated to an ODE (for fixed n, m).*

This important submodule has already been computed in the literature [12, 13, 17, 41]. All irreducible components are summarized in Tables 5 and 6.

2.3.2 ODE symmetries viewed Cartan-geometrically

Given a Cartan geometry $(\mathcal{G} \to M, \omega)$ of type (G, P), an *(infinitesimal) symmetry* is a P-invariant vector field on $\mathcal{G}$ that preserves ω under Lie differentiation. The collection of all such symmetries forms a Lie algebra, which we denote by

$$\mathfrak{inf}(\mathcal{G}, \omega) := \left\{ \xi \in \Gamma(\mathcal{G})^P : \mathcal{L}_\xi \omega = 0 \right\}. \tag{2.25}$$

PROPOSITION 2.3.6. *Let $(\mathcal{G} \to M, \omega)$ be a Cartan geometry of type (G, P) and fix $u \in \mathcal{G}$ arbitrary. Then:*

(i) The map $\xi \mapsto \omega(\xi_u)$ is a linear injection from $\mathfrak{inf}(\mathcal{G}, \omega)$ into $\mathfrak{g}$. Let $\mathfrak{f}(u)$ denote the image subspace.

(ii) Equipping $\mathfrak{f}(u)$ with the inherited filtration $\mathfrak{f}(u)^k := \mathfrak{f}(u) \cap \mathfrak{g}^k$ and bracket

$$[X, Y]_{\mathfrak{f}(u)} := [X, Y] - \kappa(u)(X, Y), \quad \forall X, Y \in \mathfrak{f}(u), \tag{2.26}$$

we have that $(\mathfrak{f}(u), [\cdot,\cdot]_{\mathfrak{f}(u)})$ is a filtered Lie algebra isomorphic to $\mathfrak{inf}(\mathcal{G}, \omega)$.

(iii) The associated graded Lie algebra $\mathfrak{s}(u) := \operatorname{gr}(\mathfrak{f}(u))$ is a graded Lie subalgebra of $\mathfrak{g}$.

(iv) $\mathfrak{s}_0(u) \subseteq \mathfrak{ann}(\kappa_H(u)) \subseteq \mathfrak{g}_0$.

PROOF. The statements (i)–(iii) were proved in [7, Thm.4] for bracket-generating distributions that lead to parabolic geometries of type (G, P). Although (G, P)

there refers to the parabolic setting, the same proof works for our (G,P) considered here. For (iv), let $A\in\mathfrak{p}$ with $A\in\mathfrak{f}^0(u)$, and let η be a symmetry with $\omega(\eta_u)=A$. Use (2.18) with $f=\kappa_H$ to obtain $A\cdot\kappa_H(u)=0$. Since $\frac{\ker\partial^*}{\operatorname{im}\partial^*}$ is completely reducible, this statement only depends on $A \bmod \mathfrak{f}^1\in\mathfrak{s}_0(u)$, so (iv) follows. ■

Using Cartan-geometric methods, we have:

COROLLARY 2.3.7. *Let $(n,m)\neq(1,1)$. Up to (local) contact transformations, the trivial ODE $\mathbf{u}^{(n+1)}=0$ of order $n+1\geq 2$ with $m\geq 1$ dependent variables is uniquely maximally symmetric among (2.1).*

PROOF. The scalar 3rd order ($n=2, m=1$) and vector 2nd order ($n=1, m\geq 2$) cases correspond to parabolic geometries – see [36, Prop.2.3.2] for a uniqueness statement. The proof for higher order ODE cases is analogous and we give this here. Given an ODE (2.1), let $(\mathcal{G}\to M,\omega)$ be the corresponding regular, normal Cartan geometry of type (G,P). Fix any $u\in\mathcal{G}$. By Proposition 2.3.6 (iii), $\mathfrak{s}(u)\subset\mathfrak{g}$, so $\dim\inf(\mathcal{G},\omega)=\dim\mathfrak{s}(u)\leq\dim\mathfrak{g}$. The trivial ODE in particular has symmetry dimension $\mathfrak{M}=\dim\mathfrak{g}$, so this is indeed maximal. Now supposing $\dim\inf(\mathcal{G},\omega)=\dim\mathfrak{g}$, we must have $\mathfrak{s}(u)=\mathfrak{g}$, so $\mathfrak{g}_0=\mathfrak{s}_0(u)=\mathfrak{ann}(\kappa_H(u))$ follows from Proposition 2.3.6 (iv). In particular, the grading element satisfies $\mathsf{Z}\in\mathfrak{s}_0(u)$. Since $\kappa_H(u)\in H^2_+(\mathfrak{g}_-,\mathfrak{g})$, then $\kappa_H(u)=0$, so $\kappa_H\equiv 0$ and the geometry is flat. Thus, the ODE is locally equivalent to the trivial one. ■

We note that the results for the scalar case are due to Lie [39], while Fels [21] established uniqueness for the case of second and third order systems using Cartan's method of equivalence.

2.3.3 An algebraic bound on submaximal symmetry dimensions

Fix (G,P) as above. We define the submaximal symmetry dimension $\mathfrak{S}$ by:

$$\begin{aligned}\mathfrak{S}:=\max\{\dim\mathfrak{inf}(\mathcal{G},\omega):(\mathcal{G}\to M,\omega)\ &\text{regular, normal of type }(G,P)\\ &\text{associated to an ODE, with }\kappa_H\not\equiv 0\}.\end{aligned}\tag{2.27}$$

Following [36], we define:

DEFINITION 2.3.8. *Let $\mathfrak{g}$ be a graded Lie algebra with $\mathfrak{g}_-$ generated by $\mathfrak{g}_{-1}$. For $\mathfrak{a}_0\subset\mathfrak{g}_0$, the* Tanaka prolongation algebra *is the graded subalgebra $\mathfrak{a}:=pr(\mathfrak{g}_-,\mathfrak{a}_0)$ of $\mathfrak{g}$ with $\mathfrak{a}_-:=\mathfrak{g}_-$ and $\mathfrak{a}_k$ defined iteratively for $k>0$ by $\mathfrak{a}_k:=\{X\in\mathfrak{g}_k:[X,\mathfrak{g}_{-1}]\subset\mathfrak{a}_{k-1}\}$. Given ϕ in some $\mathfrak{g}_0$-module, let $\mathfrak{ann}(\phi)\subset\mathfrak{g}_0$ be its annihilator and define $\mathfrak{a}^\phi:=\mathrm{pr}(\mathfrak{g}_-,\mathfrak{ann}(\phi))$.*

In terms of the effective part $\mathbb{E}\subset H^2_+(\mathfrak{g}_-,\mathfrak{g})$, we define

$$\mathfrak{U}:=\max\left\{\dim\mathfrak{a}^\phi:0\neq\phi\in\mathbb{E}\right\}.\tag{2.28}$$

Clearly $\mathfrak{U} < \dim \mathfrak{g}$. (Otherwise $\mathfrak{a}^{\phi} = \mathfrak{g}$ for some $0 \neq \phi \in \mathbb{E}$, and so $\mathsf{Z} \in \mathfrak{ann}(\phi)$. But necessarily Z acts non-trivially since $\phi \in H^2_+(\mathfrak{g}_-, \mathfrak{g})$, which is a contradiction.) We will show that $\mathfrak{S} \leq \mathfrak{U}$.

LEMMA 2.3.9. *Let $(\mathcal{G} \to M, \omega)$ be a regular, normal Cartan geometry of type (G, P). Let $u \in \mathcal{G}$ be arbitrary. Let $\xi \in \mathfrak{inf}(\mathcal{G}, \omega)$ with $\omega(\xi_u) \in \mathfrak{g}^1 \subset \mathfrak{p}$ and $\eta \in \Gamma(T^{-1}\mathcal{G})^P$. Then:*

$$[\omega(\xi_u), \omega(\eta_u)] \cdot \kappa_H(u) = 0. \tag{2.29}$$

PROOF. Fix $u \in \mathcal{G}$ as above with $A := \omega(\xi_u) \in \mathfrak{g}^1$ and $B := \omega(\eta_u) \in \mathfrak{g}^{-1}$. Since ξ is a symmetry, then $0 = (\mathcal{L}_\xi \omega)(\eta) = d\omega(\xi, \eta) + \eta \cdot \omega(\xi) = \xi \cdot \omega(\eta) - \omega([\xi, \eta])$. Evaluation at u now yields

$$\omega([\xi, \eta])(u) = (\xi \cdot \omega(\eta))(u) = -[A, B] \in \mathfrak{p}, \tag{2.30}$$

using P-equivariancy of $\omega(\eta)$ and (2.18).

Since ξ is a symmetry, then $\xi \cdot \kappa = 0$ and $\xi \cdot \kappa_H = 0$. We get the prolonged equation

$$0 = \eta \cdot (\xi \cdot \kappa_H) = \xi \cdot (\eta \cdot \kappa_H) + [\eta, \xi] \cdot \kappa_H. \tag{2.31}$$

Now evaluate at u:

- Since η is P-invariant and κ_H is P-equivariant, then $\eta \cdot \kappa_H : \mathcal{G} \to \frac{\ker \partial^*}{\operatorname{im} \partial^*}$ is P-equivariant. Thus, $(\xi \cdot (\eta \cdot \kappa_H))(u) = -A \cdot (\eta \cdot \kappa_H)(u) = 0$ using (2.18) and Lemma 2.3.4 (since $A \in \mathfrak{g}^1$).
- Since $[\xi, \eta]$ is P-invariant with $\omega([\xi, \eta])(u) \in \mathfrak{p}$, then

$$0 \overset{(2.31)}{=} ([\eta, \xi] \cdot \kappa_H)(u) \overset{(2.18)}{=} \omega([\xi, \eta])(u) \cdot \kappa_H(u) \overset{(2.30)}{=} -[A, B] \cdot \kappa_H(u). \tag{2.32}$$

■

THEOREM 2.3.10. *Let $(\pi : \mathcal{G} \to M, \omega)$ be a regular, normal Cartan geometry of type (G, P) associated to an ODE. For any $u \in \mathcal{G}$, we have $\mathfrak{s}(u) \subseteq \mathfrak{a}^{\kappa_H(u)}$. Moreover, $\mathfrak{S} \leq \mathfrak{U} < \dim \mathfrak{g}$.*

PROOF. Fix any $u \in \mathcal{G}$. We have $\mathfrak{s}_0(u) \subseteq \mathfrak{ann}(\kappa_H(u))$ from Proposition 2.3.6 (iv), so for the first claim it suffices to prove that $\mathfrak{s}_1(u) \subseteq \mathfrak{a}_1^{\kappa_H(u)}$. Suppose $\mathfrak{s}_1(u) \neq 0$, then we must have $\mathfrak{s}_1(u) = \mathbb{R}\mathsf{Y}$. Pick any $B \in \mathfrak{g}_{-1}$. Let $\xi \in \mathfrak{inf}(\mathcal{G}, \omega)$ and $\eta \in \Gamma(T^{-1}\mathcal{G})^P$ with $\omega(\xi_u) = \mathsf{Y}$ and $\omega(\eta_u) = B$. Then (2.29) with $A := \mathsf{Y}$ implies that $[\mathsf{Y}, B] \cdot \kappa_H(u) = 0$, hence $\mathsf{Y} \in \mathfrak{a}_1^{\kappa_H(u)}$ and the first claim follows. We deduce that $\dim \mathfrak{inf}(\mathcal{G}, \omega) = \dim \mathfrak{s}(u) \leq \dim \mathfrak{a}^{\kappa_H(u)} \leq \mathfrak{U}$, since κ_H is valued in the effective part $\mathbb{E}$. We conclude that $\mathfrak{S} \leq \mathfrak{U} < \dim \mathfrak{g}$. ■

REMARK 2. In the parabolic setting, the analogous statement $\mathfrak{s}(u) \subseteq \mathfrak{a}^{\kappa_H(u)}$ was proved in [36, §3] on an open dense set of so-called *regular points* (using a

Frobenius integrability argument). This was strengthened to all points in [37] using the fundamental derivative and calculus on the adjoint tractor bundle. Our proof in this section is adapted from the latter, but can be formulated and proven more simply since the positive part $\mathfrak{g}_+ = \mathfrak{g}_1$ consists of only a single grading level (with dimension one).

Let $\mathcal{O} \subset \mathbb{E}$ be a G_0-invariant subset. We define $\mathfrak{S}_{\mathcal{O}}$ analogously to $\mathfrak{S}$ from (2.27), but with the additional constraint that κ_H is valued in $\mathcal{O}$. We also set $\mathfrak{U}_{\mathcal{O}} := \max\{\dim \mathfrak{a}^\phi : 0 \neq \phi \in \mathcal{O}\}$. The same argument as in Theorem 2.3.10 allows us to conclude:

$$\mathfrak{S}_{\mathcal{O}} \leq \mathfrak{U}_{\mathcal{O}}. \tag{2.33}$$

Of particular interest to us will be the case where $\mathcal{O} \subset \mathbb{E}$ is a G_0-irrep $\mathbb{U}$, so that $\mathfrak{S}_{\mathbb{U}} \leq \mathfrak{U}_{\mathbb{U}}$.

Suppose that $\mathbb{E} = \bigoplus_i \mathbb{U}_i$ is the decomposition into G_0-irreps $\mathbb{U}_i$, which exists since G_0 is reductive. From the definition of $\mathfrak{U}$ and $\mathfrak{U}_{\mathbb{U}_i}$, we remark that the following equality is immediate:

$$\mathfrak{U} = \max_i \mathfrak{U}_{\mathbb{U}_i}. \tag{2.34}$$

A priori, the corresponding statement $\mathfrak{S} = \max_i \mathfrak{S}_{\mathbb{U}_i}$ may not hold, in particular when $\mathfrak{S}_{\mathbb{U}_i} \neq \mathfrak{U}_{\mathbb{U}_i}$. Furthermore, submaximally symmetric models may exist with κ_H not concentrated along a single irreducible component.

2.4 Computation of upper bounds

In this entirely algebraic section, we compute $\mathfrak{U}$ and $\mathfrak{U}_{\mathbb{U}}$ for each $\mathfrak{g}_0$-irrep $\mathbb{U} \subset \mathbb{E} \subset H^2_+(\mathfrak{g}_-, \mathfrak{g})$. In view of Theorem 2.3.10, these provide upper bounds on the respective submaximal symmetry dimensions $\mathfrak{S}$ and $\mathfrak{S}_{\mathbb{U}}$.

2.4.1 Bi-gradings

In (2.14), we introduced a $\mathfrak{g}_0$-invariant splitting on $\mathfrak{g}_{-1}$. Such splittings similarly arise for parabolic geometries (with respect to non-maximal parabolic subgroups). Analogously as in that setting [36], we refine the grading to a *bi-grading*. Define $\mathsf{Z}_1, \mathsf{Z}_2 \in \mathfrak{z}(\mathfrak{g}_0)$ with $\mathsf{Z} = \mathsf{Z}_1 + \mathsf{Z}_2$ (see (2.15)) by

$$\mathsf{Z}_1 = -\frac{1}{2}(\mathsf{H} + n\,\mathrm{id}_m), \quad \mathsf{Z}_2 = -\,\mathrm{id}_m\,. \tag{2.35}$$

We refer to the ordered pair $(\mathsf{Z}_1, \mathsf{Z}_2)$ as the *bi-grading element*, and then the joint eigenspaces $\mathfrak{g}_{a,b} := \{x \in \mathfrak{g} : [\mathsf{Z}_1, x] = ax,\ [\mathsf{Z}_2, x] = bx\}$ define the bi-grading $\mathfrak{g} = \bigoplus_{(a,b)\in\mathbb{Z}^2} \mathfrak{g}_{a,b}$. Note that $\mathfrak{g}_0 = \mathfrak{g}_{0,0}$ and $\mathfrak{g}_{-1} = \mathfrak{g}_{-1,0} \oplus \mathfrak{g}_{0,-1}$, and we visualize the bi-grading as in Figure 2.

The bi-grading on $\mathfrak{g}$ induces a bi-grading on cochains and cohomology (since ∂ is $\mathfrak{g}_0$-equivariant), in particular on the effective part $\mathbb{E} \subset H^2_+(\mathfrak{g}_-, \mathfrak{g})$. Given $(a, b) \in \mathbb{Z}^2$, let $\mathbb{E}_{a,b} = \{\phi \in \mathbb{E} : \mathsf{Z}_1 \cdot \phi = a\phi,\ \mathsf{Z}_2 \cdot \phi = b\phi\}$ be the corresponding joint eigenspace.

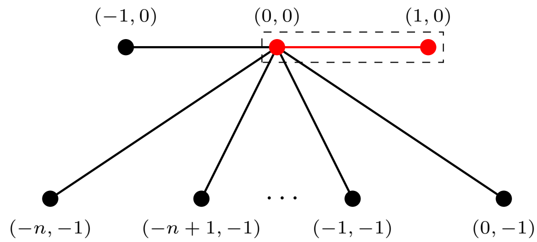

FIGURE 2. Bi-grading on $\mathfrak{g}$

We note that Z_2 acts on $\bigwedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ with eigenvalues (Z_2-degrees) $0, 1$ or 2. We will refer to the G_0-submodules in $\mathbb{E}$ of *positive* Z_2-degree as *C-class modules* and those with *zero* Z_2-degree as *Wilczynski modules* (see §2.5 for this terminology).

DEFINITION 2.4.1. *Let $\mathbb{E}_C \subsetneq \mathbb{E}$ denote the direct sum of* all *C-class modules and $\mathbb{W} \subsetneq \mathbb{E}$ the direct sum of* all *Wilczynski modules in $\mathbb{E}$, i.e. $\mathbb{E} = \mathbb{W} \oplus \mathbb{E}_C$.*

REMARK 3. In the articles [12, 13, 17, 41] computing the effective part $\mathbb{E}$, the gradings on $\mathfrak{g}_0$-submodules of $\mathbb{E}$ were explicitly stated, but bi-gradings were not used. However, these can be easily deduced from the cohomology results there (in particular, their realizations as (harmonic) 2-cochains) using the fact that V and $\mathfrak{q}$ have Z_2-degrees -1 and 0 respectively.

2.4.2 Prolongation-rigidity

In view of §2.3.3, it is important to understand when the Tanaka prolongation algebra $\mathfrak{a}^\phi$ has non-trivial prolongation in degree +1.

LEMMA 2.4.2. *Let $0 \neq \phi \in \mathbb{E}$. Then $\mathfrak{a}_1^\phi \neq 0$ if and only if ϕ lies in the direct sum of all $\mathbb{E}_{a,b}$ for (a, b) that is a multiple of $(n, 2)$.*

PROOF. Note that $\mathfrak{a}_1^\phi \neq 0$ if and only if $\mathfrak{a}_1^\phi = \mathfrak{g}_1 = \mathbb{R}\mathsf{Y}$. Since $[\mathsf{Y}, \mathfrak{g}_{0,-1}] = 0$, then this occurs if and only if $[\mathsf{Y}, \mathsf{X}] = -\mathsf{H} \in \mathfrak{a}_0^\phi := \mathfrak{ann}(\phi)$. From (2.35), we have $\mathsf{H} = -2\mathsf{Z}_1 + n\mathsf{Z}_2$, so $\mathsf{H} \in \mathfrak{ann}(\phi)$ if and only if ϕ lies in the direct sum of the claimed modules. ■

DEFINITION 2.4.3. *We say that a $\mathfrak{g}_0$-submodule $\mathcal{O} \subseteq \mathbb{E}$ is prolongation-rigid (PR) if $\mathfrak{a}_1^\phi = 0$ for any $0 \neq \phi \in \mathcal{O}$.*

2.4.3 Scalar case

For scalar ODEs, the effective part $\mathbb{E} \subset H^2_+(\mathfrak{g}_-, \mathfrak{g})$ (Table 5) was computed by Doubrov – see [13, Prop.4] for a summary and [12] for details. (Bi-gradings are asserted using Remark 3.) Since $\mathfrak{g}_0$ is spanned by Z_1 and Z_2, then all $\mathfrak{g}_0$-irreps

$\mathbb{U} \subset \mathbb{E}$ are 1-dimensional.

Type	n	$\mathfrak{g}_0$-irrep $\mathbb{U} \subset \mathbb{E}$	Bi-grade
Wilczynski	≥ 3	$\mathbb{W}_r$ $(3 \leq r \leq n+1)$	$(r, 0)$
C-class	3	$\mathbb{B}_3$	$(1, 2)$
	3	$\mathbb{B}_4$	$(2, 2)$
	4	$\mathbb{B}_6$	$(4, 2)$
	≥ 4	$\mathbb{A}_2$	$(1, 1)$
	≥ 5	$\mathbb{A}_3$	$(2, 1)$
	≥ 6	$\mathbb{A}_4$	$(3, 1)$

TABLE 5. Effective part $\mathbb{E} \subsetneq H^2_+(\mathfrak{g}_-, \mathfrak{g})$ for scalar ODEs of order $n + 1 \geq 4$

LEMMA 2.4.4. *Consider the effective part $\mathbb{E}$ for scalar ODEs of order $n + 1 \geq 4$. Then:*

(a) $\mathbb{E}$ is not PR if and only if $n = 4$ or 6. In particular, $(n, \mathbb{U}) = (4, \mathbb{B}_6)$ and $(6, \mathbb{A}_4)$ are not PR.

(b) If $\mathbb{U} \subset \mathbb{E}$ is a $\mathfrak{g}_0$-irrep, then $\mathfrak{U}_{\mathbb{U}} = \begin{cases} n+4, & \textit{if } (n, \mathbb{U}) = (4, \mathbb{B}_6) \textit{ or } (6, \mathbb{A}_4); \\ n+3, & \textit{otherwise}. \end{cases}$

(c) $\mathfrak{U} = \begin{cases} \mathfrak{M} - 1 = n+4, & \textit{if } n = 4, 6; \\ \mathfrak{M} - 2 = n+3, & \textit{otherwise}. \end{cases}$

PROOF. Part (a) directly follows from Lemma 2.4.2 and Table 5. For part (b), recall that $\dim \mathfrak{g}_- = n + 2$ and $\dim \mathfrak{ann}(\phi) = 1$ for $0 \neq \phi \in \mathbb{U}$ since $\mathbb{U}$ is irreducible and $\mathsf{Z} \notin \mathfrak{ann}(\phi)$ (by regularity). Thus, $\dim \mathfrak{a}^\phi_{\leq 0} = n+3$, so $\mathfrak{U}_{\mathbb{U}} = n+3$ when $\mathbb{U}$ is PR and $\mathfrak{U}_{\mathbb{U}} = n+4$ when $\mathbb{U}$ is not PR (when $(n, \mathbb{U}) = (4, \mathbb{B}_6)$ or $(6, \mathbb{A}_4)$). Part (c) now follows by using (2.34). ∎

LEMMA 2.4.5. *Consider the effective part $\mathbb{E}$ for scalar ODEs (2.1) of order $n + 1 \geq 4$ and $\mathbb{E}_C = \bigoplus_i \mathbb{U}_i \subset \mathbb{E}$, the direct sum of all irreducible C-class modules $\mathbb{U}_i$. Then, for $0 \neq \phi \in \mathbb{E}_C$ such that $\dim \mathfrak{a}^\phi \geq n + 3$, we have $\phi \in \mathbb{U}_i \subset \mathbb{E}_C$ for some i.*

PROOF. Suppose that for $0 \neq \phi \in \mathbb{E}_C$, $\dim \mathfrak{a}^\phi \geq n + 3$. Since $\dim \mathfrak{g}_- = \dim \mathfrak{a}^\phi_- = n + 2$, then $\mathfrak{a}^\phi_0 = \mathfrak{ann}(\phi)$ is a non-trivial proper subspace of $\mathfrak{g}_0$. Since $\dim \mathfrak{g}_0 = 2$, then $\dim \mathfrak{a}^\phi_0 = 1$. None of the bi-grades for the C-class modules in Table 5 is a multiple of any other, so $\dim \mathfrak{a}^\phi_0 = 1$ forces $\phi \in \mathbb{U}_i \subset \mathbb{E}_C$ for some i. ∎

2.4.4 Vector case

For vector ODEs, the effective part $\mathbb{E} \subset H^2_+(\mathfrak{g}_-, \mathfrak{g})$ (Table 6) was computed by Medvedev [42] for the 3rd order case, and Doubrov–Medvedev [17] for the higher order cases. (Bi-gradings are asserted using Remark 3.) We have $\mathfrak{g}_0 = \operatorname{span}\{\mathsf{Z}_1, \mathsf{Z}_2\} \oplus \mathfrak{sl}(W)$, so any $\mathfrak{g}_0$-irrep $\mathbb{U} \subset \mathbb{E}$ is completely determined by its bi-grading and highest weight λ with respect to $\mathfrak{sl}(W) \cong \mathfrak{sl}_m$. The latter can be expressed in terms of the fundamental weights $\lambda_1, \dots, \lambda_{m-1}$ of $\mathfrak{sl}_m$ with respect to the standard choice of Cartan subalgebra and simple roots. We note that some of the modules appearing in [17, 42] are not $\mathfrak{g}_0$-irreducible, so we have decomposed them here into their trace-free and trace parts. We also define $\mathbb{W}_r := \mathbb{W}_r^{\mathrm{tf}} + \mathbb{W}_r^{\mathrm{tr}}$ and $\mathbb{A}_2 := \mathbb{A}_2^{\mathrm{tf}} + \mathbb{A}_2^{\mathrm{tr}}$.

Type	n	$\mathfrak{g}_0$-irrep $\mathbb{U}$	Bi-grade	$\mathfrak{sl}(W)$-module $\mathbb{U}$	$\mathfrak{sl}(W)$ h.w. λ
Wilczynski	≥ 2	$\mathbb{W}_r^{\mathrm{tf}}$ $(2 \leq r \leq n+1)$	$(r, 0)$	$\mathfrak{sl}(W)$	$\lambda_1 + \lambda_{m-1}$
	≥ 2	$\mathbb{W}_r^{\mathrm{tr}}$ $(3 \leq r \leq n+1)$	$(r, 0)$	$\mathbb{R}\,\mathrm{id}_m$	0
C-class	2	$\mathbb{B}_4$	$(2, 2)$	$S^2 W^*$	$2\lambda_{m-1}$
	≥ 2	$\mathbb{A}_2^{\mathrm{tf}}$	$(1, 1)$	$(S^2 W^* \otimes W)_0$	$\lambda_1 + 2\lambda_{m-1}$
	≥ 3	$\mathbb{A}_2^{\mathrm{tr}}$	$(1, 1)$	W^*	λ_{m-1}

TABLE 6. Effective part $\mathbb{E} \subsetneq H^2_+(\mathfrak{g}_-, \mathfrak{g})$ for vector ODEs of order $n + 1 \geq 3$ with $m \geq 2$

Type	n	$\mathfrak{g}_0$-irrep $\mathbb{U} \subset \mathbb{E}$	$\max\limits_{0 \neq \phi \in \mathbb{U}} \dim \mathfrak{ann}(\phi)$	$\mathbb{U}$ PR?	$\mathfrak{U}_{\mathbb{U}}$
Wilczynski	≥ 2	$\mathbb{W}_r^{\mathrm{tf}}$ $(2 \leq r \leq n+1)$	$m^2 - 2m + 3$	✓	$\mathfrak{M} - 2m + 1$
	≥ 2	$\mathbb{W}_r^{\mathrm{tr}}$ $(3 \leq r \leq n+1)$	m^2	✓	$\mathfrak{M} - 2$
C-class	2	$\mathbb{B}_4$	$m^2 - m + 1$	×	$\mathfrak{M} - m$
	2	$\mathbb{A}_2^{\mathrm{tf}}$	$m^2 - 2m + 3$	×	$\mathfrak{M} - 2m + 2$
	≥ 3	$\mathbb{A}_2^{\mathrm{tf}}$	$m^2 - 2m + 3$	✓	$\mathfrak{M} - 2m + 1$
	≥ 3	$\mathbb{A}_2^{\mathrm{tr}}$	$m^2 - m + 1$	✓	$\mathfrak{M} - m - 1$

(The contact symmetry dimension of the trivial ODE is $\mathfrak{M} = m^2 + (n+1)m + 3$.)

TABLE 7. Upper bounds $\mathfrak{U}_{\mathbb{U}}$ for vector ODEs of order $n+1 \geq 3$ with $m \geq 2$

LEMMA 2.4.6. *Consider the effective part $\mathbb{E}$ for vector ODEs of order $n + 1 \geq 3$ with $m \geq 2$. Then:*

(a) $\mathbb{E}$ is not PR if and only if $n = 2$. When $n = 2$, $\mathbb{A}_2^{\mathrm{tf}}$ and $\mathbb{B}_4$ are not PR, while $\mathbb{W}_r^{\mathrm{tf}}$ and $\mathbb{W}_r^{\mathrm{tr}}$ are PR.

(b) If $\mathbb{U} \subset \mathbb{E}$ is a $\mathfrak{g}_0$-irrep, then $\mathfrak{U}_\mathbb{U}$ is given in Table 7.
(c) $\mathfrak{U} = \mathfrak{M} - 2 = m^2 + (n+1)m + 1$.

PROOF. Part (a) directly follows from Lemma 2.4.2 and Table 6. Let us prove part (b). In order to compute $\mathfrak{U}_\mathbb{U}$, it suffices to maximize $\dim \mathfrak{ann}(\phi)$ among $0 \neq \phi \in \mathbb{U}$. (If $\mathbb{U}$ is not PR, then $\mathfrak{a}_1^\phi = \mathbb{R}\mathsf{Y}$ for all $0 \neq \phi \in \mathbb{U}$.) Since $\mathbb{U}$ is $\mathfrak{g}_0$-irreducible, the maximum is achieved on any highest weight vector ϕ_0 (and indeed, along the SL_m-orbit through ϕ_0). Let $\mathfrak{u} \subset \mathfrak{sl}(W) \cong \mathfrak{sl}_m$ be the parabolic subalgebra preserving ϕ_0 up to a scaling factor. Since Z_1 and Z_2 also preserve ϕ_0 up to scale, then we obtain

$$\dim \mathfrak{ann}(\phi_0) = 1 + \dim \mathfrak{u}. \tag{2.36}$$

For each $\mathfrak{g}_0$-irrep $\mathbb{U} \subset \mathbb{E}$, the highest $\mathfrak{sl}_m$-weight λ and parabolic $\mathfrak{u} \subset \mathfrak{sl}_m$ is given below.

$\mathbb{U}$	$\mathbb{W}_r^{\mathrm{tf}}$	$\mathbb{W}_r^{\mathrm{tr}}$	$\mathbb{B}_4$	$\mathbb{A}_2^{\mathrm{tf}}$	$\mathbb{A}_2^{\mathrm{tr}}$
λ	$\lambda_1 + \lambda_{m-1}$	0	$2\lambda_{m-1}$	$\lambda_1 + 2\lambda_{m-1}$	λ_{m-1}
$\mathfrak{u}$	$\mathfrak{p}_{1,m-1}$	$\mathfrak{sl}_m$	$\mathfrak{p}_{m-1}$	$\mathfrak{p}_{1,m-1}$	$\mathfrak{p}_{m-1}$

(2.37)

The subscript notation for parabolics is the same as that used in [36]. (We caution that $\mathfrak{p}$ ornamented with subscripts here is not related to P for the trivial ODE.) Concretely, each such $\mathfrak{u}$ is a block upper triangular, trace-free $m \times m$ matrix with diagonal blocks of size:

- $1, m-2, 1$ for $\mathfrak{p}_{1,m-1}$, so $\dim \mathfrak{u} = m^2 - 1 - 2(m-2) - 1 = m^2 - 2m + 2$;
- $m-1, 1$ for $\mathfrak{p}_{m-1}$, so $\dim \mathfrak{u} = m^2 - 1 - (m-1) = m^2 - m$.

Using $\dim \mathfrak{g}_- = 1 + (n+1)m$ and (2.36), we obtain $\dim \mathfrak{a}_{\leq 0}^{\phi_0}$. When $\mathbb{U}$ is PR, this equals $\mathfrak{U}_\mathbb{U}$. When $\mathbb{U}$ is not PR, we must augment it by one. Part (c) now follows by using (2.34). ■

2.5 Submaximal symmetry dimensions

For higher order ODEs, we review the known local expressions for κ_H, labelled here by:

- $\mathcal{W}_r$: *Generalized Wilczynski* invariants (with Z_2-degree 0);
- $\mathcal{A}_r, \mathcal{B}_r$: *C-class* invariants (with Z_2-degrees 1 and 2 respectively).

These correspond to the $\mathfrak{g}_0$-irreps $\mathbb{W}_r, \mathbb{A}_r, \mathbb{B}_r \subset \mathbb{E}$ introduced earlier in §2.4.3 and §2.4.4. (The expressions for these invariants were computed with respect to some adapted coframing. If a different adapted coframing is used, these expressions would transform tensorially according to the structure of the indicated modules.) For each irreducible $\mathfrak{g}_0$-submodule $\mathbb{U} \subset \mathbb{E}$, we use these differential invariants to exhibit explicit ODE models with abundant symmetries having κ_H non-zero and concentrated in $\mathbb{U} \subset \mathbb{E}$.

For *all* vector cases and most scalar cases, these exhibited models realize $\mathfrak{S}_\mathbb{U} = \mathfrak{U}_\mathbb{U}$, cf. Tables 8, 9 and 10. The contact symmetries of the given ODE models are stated in terms of their projections to $(t, \mathbf{u})$-space, i.e. $J^0(\mathbb{R}, \mathbb{R}^m)$, in the case of point symmetries, or in terms of their projections to $(t, \mathbf{u}, \mathbf{u}_1)$-space, i.e. $J^1(\mathbb{R}, \mathbb{R}^m)$, in the case of genuine contact symmetries. In §2.5.3, exceptional cases (where $\mathfrak{S}_{\mathbb{U}} < \mathfrak{U}_{\mathbb{U}}$) are discussed and we conclude the proofs of Theorems 2.2.1 and 2.2.2.

2.5.1 Generalized Wilczynski invariants

Consider the class of linear ODEs of order $n+1$:

$$\mathbf{u}_{n+1} + R_n(t)\mathbf{u}_n + \ldots + R_1(t)\mathbf{u}_1 + R_0(t)\mathbf{u} = 0, \tag{2.38}$$

where $R_j(t)$ is an $\mathrm{End}(\mathbb{R}^m)$-valued function. The invertible transformations

$$(t, \mathbf{u}) \mapsto (\lambda(t), \mu(t)\mathbf{u}), \quad \text{where} \quad \lambda : \mathbb{R} \to \mathbb{R}^\times, \quad \mu : \mathbb{R} \to \mathrm{GL}(m), \tag{2.39}$$

constitute the most general Lie pseudogroup preserving the class (2.38). Using (2.39), any equation (2.38) can be brought into *canonical Laguerre–Forsyth form* defined by $R_n = 0$ and $\mathrm{tr}(R_{n-1}) = 0$.

As proved by Wilczynski [50] for $m = 1$ and Se-ashi [1] for $m \geq 2$, the following expressions

$$\Theta_r = \sum_{k=1}^{r-1} (-1)^{k+1} \frac{(2r-k-1)!(n-r+k)!}{(r-k)!(k-1)!} R_{n-r+k}^{(k-1)}, \quad r = 2, \ldots, n+1, \tag{2.40}$$

are fundamental (relative) invariants with respect to those transformations (2.39) preserving the Laguerre–Forsyth form. These invariants are called the *Se-ashi–Wilczynski invariants* and r is the degree of the invariant. We remark that:

- If all R_j are independent of t, then all Θ_r are constant multiples of R_{n+1-r}.
- For $m = 1$ (scalar ODEs), we have $R_{n-1}(t) = 0$ and this forces $\Theta_2 \equiv 0$.

The generalized Wilczynski invariants $\mathcal{W}_r$ directly generalize the Se-ashi–Wilczynski invariants to non-linear ODEs. We refer to the corresponding modules $\mathbb{W}_r$ as being of *Wilczynski-type*. (Similarly for trace or trace-free parts.)

DEFINITION 2.5.1. *For (2.1), $\mathcal{W}_r$ are defined as Θ_r evaluated at its linearization along a solution* $\mathbf{u}$. *Formally, $\mathcal{W}_r$ are obtained from (2.38) by substituting $R_r(t)$ by the matrices $\left(-\frac{\partial f^a}{\partial u_r^b}\right)$ and the usual derivative by the total derivative.*

It was proved by Doubrov [14] that $\mathcal{W}_r$ do not depend on the choice of solution $\mathbf{u}$ and are indeed (relative) contact invariants of (2.1). Table 8 exhibits *constant coefficient* linear ODEs with $\kappa_H \not\equiv 0$, $\mathrm{im}(\kappa_H) \subset \mathbb{U}$ and contact symmetry dimension realizing $\mathfrak{U}_{\mathbb{U}}$, so $\mathfrak{S}_{\mathbb{U}} = \mathfrak{U}_{\mathbb{U}}$ for modules $\mathbb{U}$ of Wilczynski type.

n	m	$\mathbb{U}$	ODE with $\mathrm{im}(\kappa_H) \subset \mathbb{U}$	Sym dim	Contact symmetries
≥ 3	1	$\underset{(3 \leq r \leq n+1)}{\mathbb{W}_r}$	$u_{n+1} = u_{n+1-r}$	$\mathfrak{M} - 2$	$\partial_t,\ u\partial_u,\ s_k\partial_u,$ $\{s_k\}_{k=1}^{n+1}$ solns of $u_{n+1} = u_{n+1-r}$
≥ 2	≥ 2	$\underset{(3 \leq r \leq n+1)}{\mathbb{W}_r^{\mathrm{tr}}}$	$\underset{(1 \leq a \leq m)}{u^a_{n+1} = u^a_{n+1-r}}$	$\mathfrak{M} - 2$	$\partial_t,\ u^a\partial_{u^b},\ s_k\partial_{u^a},$ $1 \leq a, b \leq m,$ $\{s_k\}_{k=1}^{n+1}$ solns of $u_{n+1} = u_{n+1-r}$
≥ 2	≥ 2	$\underset{(2 \leq r \leq n+1)}{\mathbb{W}_r^{\mathrm{tf}}}$	$\underset{(1 \leq a \leq m)}{u^a_{n+1} = u^2_{n+1-r}\delta^a_1}$	$\mathfrak{M} - 2m + 1$	$\partial_t,\ \partial_{u^a},\ t^i\partial_{u^a},$ $u^b\partial_{u^a},$ $1 \leq a, b \leq m,$ $a \neq 2,\ b \neq 1,$ $1 \leq i \leq n,$ $t\partial_t + ru^1\partial_{u^1},$ $u^1\partial_{u^1} + u^2\partial_{u^2},$ $\frac{t^k}{k!}\partial_{u^1} + \frac{t^{k-r}}{(k-r)!}\partial_{u^2},$ $n+1 \leq k \leq n+r,$ for $2 \leq r \leq n$ in addition: $t^\ell\partial_{u^2},$ $0 \leq \ell \leq n-r$

(The contact symmetry dimension of the trivial ODE is $\mathfrak{M} = m^2 + (n+1)m + 3$.)

TABLE 8. Constant coefficient linear ODEs realizing $\mathfrak{S}_{\mathbb{U}} = \mathfrak{U}_{\mathbb{U}}$ for $\mathbb{U}$ of Wilczynski type

2.5.2 C-class invariants

As formulated in [6], an ODE (2.1) is of *C-class* if the curvature of the corresponding canonical Cartan geometry satisfies $\kappa(\mathsf{X}, \cdot) = 0$. This can be characterized at the harmonic level in terms of the generalized Wilczynski invariants $\mathcal{W}_r$. Necessity of all $\mathcal{W}_r \equiv 0$ follows from [6, Thm.4.1], while sufficiency is established in [6, Thm.4.2]. Here, we abuse the terminology and refer to the modules $\mathbb{A}_r, \mathbb{B}_r$ and corresponding invariants $\mathcal{A}_r, \mathcal{B}_r$ as being of *C-class* type (despite the fact that they are defined in general, even for ODEs that are not of C-class).

Below are the C-class invariants of (2.1):

- *Scalar case*: The C-class invariants of $u_{n+1} = f(t, u, u_1, ..., u_n)$ were computed by Doubrov [13] (see also [17, Example 6]):

$$
\begin{aligned}
&n = 3: \quad \mathcal{B}_3 = f_{333},\\
&n = 3: \quad \mathcal{B}_4 = f_{233} + \frac{1}{6}(f_{33})^2 + \frac{9}{8} f_3 f_{333} + \frac{3}{4}\frac{d}{dt} f_{333},\\
&n = 4: \quad \mathcal{B}_6 = f_{234} - \frac{2}{3} f_{333} - \frac{1}{2}(f_{34})^2 \mod \langle \mathcal{A}_2, \mathcal{W}_3 \rangle,\\
&n \geq 4: \quad \mathcal{A}_2 = f_{nn},\\
&n \geq 5: \quad \mathcal{A}_3 = f_{n,n-1} + \frac{n(n-1)}{(n+1)(n-2)} f_n f_{nn} + \frac{n}{n-2}\frac{d}{dt} f_{nn},\\
&n \geq 6: \quad \mathcal{A}_4 = f_{n-1,n-1} \mod \langle \mathcal{A}_2, \mathcal{A}_3, \mathcal{W}_3 \rangle.
\end{aligned}
\tag{2.41}
$$

Here, $f_i := \frac{\partial f}{\partial u_i}$, see (2.6) for $\frac{d}{dt}$, and $\langle \mathcal{I} \rangle$ denotes the differential ideal generated by an invariant $\mathcal{I}$.

- *Vector case*: For $m \geq 2$, the C-class invariants were computed by Medvedev [42] for $n = 2$ and by Doubrov–Medvedev [17] for $n \geq 3$. Letting tf refer to the trace-free part, we have:

$$
\begin{aligned}
&n \geq 2: \quad (\mathcal{A}_2^{\mathrm{tf}})^a_{bc} = \mathrm{tf}\left(\frac{\partial^2 f^a}{\partial u_n^b \, \partial u_n^c} \right),\\
&n \geq 3: \quad (\mathcal{A}_2^{\mathrm{tr}})^a_{bc} = \mathrm{tr}\left(\frac{\partial^2 f^a}{\partial u_n^b \, \partial u_n^c} \right),\\
&n = 2: \quad (\mathcal{B}_4)_{bc} = -\frac{\partial H_c^{-1}}{\partial u_1^b} + \frac{\partial}{\partial u_2^b}\frac{\partial}{\partial u_2^c} H^t - \frac{\partial}{\partial u_2^c}\frac{d}{dt} H_b^{-1}\\
&\qquad\qquad - \frac{\partial}{\partial u_2^c}\left(\sum_{a=1}^m H_a^{-1} \frac{\partial f^a}{\partial u_2^b} \right) + 2 H_b^{-1} H_c^{-1},
\end{aligned}
\tag{2.42}
$$

where

$$
\begin{aligned}
H_b^{-1} &= \frac{1}{6(m+1)} \sum_{a=1}^m \frac{\partial^2 f^a}{\partial u_2^a \, \partial u_2^b},\\
H^t &= -\frac{1}{4m} \sum_{a=1}^m \left(\frac{\partial f^a}{\partial u_1^a} - \frac{d}{dt}\frac{\partial f^a}{\partial u_2^a} + \frac{1}{3}\sum_{c=1}^m \frac{\partial f^a}{\partial u_2^c}\frac{\partial f^c}{\partial u_2^a} \right).
\end{aligned}
\tag{2.43}
$$

Tables 9 and 10 respectively exhibit scalar ODEs and vector ODEs with $\kappa_H \not\equiv 0$, $\mathrm{im}(\kappa_H) \subset \mathbb{U}$ and contact symmetry dimension realizing $\mathfrak{S}_{\mathbb{U}} = \mathfrak{U}_{\mathbb{U}}$ for modules $\mathbb{U}$ of C-class type. These ODEs are examples of C-class equations since all $\mathcal{W}_r \equiv 0$. These scalar ODEs are well-known and stated for example in [43, pp. 205-206], but their harmonic curvature classification was not given there. We remark that for the ODE in the first row of Table 9, the κ_H-classification is deduced from the invariants when $n = 3$. For $n \geq 4$ however, $\mathrm{im}(\kappa_H) \subset \mathbb{A}_2$ cannot be asserted by using the invariants alone since $\mathcal{B}_6$ and $\mathcal{A}_4$ were computed only up to a differential ideal containing $\mathcal{A}_2$, and we have $\mathcal{A}_2 \neq 0$ for this ODE (and $\mathcal{A}_3 \equiv 0$ for $n \geq 5$). However, since the ODE admits an $(n+3)$-dimensional contact symmetry algebra, then by Lemma 2.4.5 the conclusion $\mathrm{im}(\kappa_H) \subset \mathbb{A}_2$ follows.

n	$\mathbb{U}$	ODE with $\mathrm{im}(\kappa_H) \subset \mathbb{U}$	Sym dim	Contact symmetries
3 ≥ 4	$\mathbb{B}_4$ $\mathbb{A}_2$	$nu_{n-1}u_{n+1} - (n+1)(u_n)^2 = 0$	$\mathfrak{M} - 2 = n+3$	$\partial_t,\ \partial_u,\ t\partial_t,\ u\partial_u,\ t^2\partial_t + (n-2)tu\partial_u,\ t\partial_u,\ \dots,\ t^{n-2}\partial_u$
4	$\mathbb{B}_6$	$9(u_2)^2u_5 - 45u_2u_3u_4 + 40(u_3)^3 = 0$	$\mathfrak{M} - 1 = 8$	$\partial_t,\ \partial_u,\ t\partial_t,\ u\partial_t,\ t\partial_u,\ u\partial_u,\ tu\partial_t + u^2\partial_u,\ t^2\partial_t + (n-3)tu\partial_u$
6	$\mathbb{A}_4$	$10(u_3)^3u_7 - 70(u_3)^2u_4u_6 - 49(u_3)^2(u_5)^2 + 280u_3(u_4)^2u_5 - 175(u_4)^4 = 0$	$\mathfrak{M} - 1 = 10$	$\partial_t,\ \partial_u,\ t\partial_t - u_1\partial_{u_1},\ t\partial_u + \partial_{u_1},\ t^2\partial_u + 2t\partial_{u_1},\ u\partial_u + u_1\partial_{u_1},\ 2u_1\partial_t + u_1^2\partial_u,\ t^2\partial_t + 2tu\partial_u + 2u\partial_{u_1},\ (2tu_1 - 2u)\partial_t + tu_1^2\partial_u + u_1^2\partial_{u_1},\ (2t^2u_1 - 4tu)\partial_t + (t^2u_1^2 - 4u^2)\partial_u + (2tu_1^2 - 4uu_1)\partial_{u_1}$

TABLE 9. Scalar ODEs realizing $\mathfrak{S}_{\mathbb{U}} = \mathfrak{U}_{\mathbb{U}}$ for $\mathbb{U}$ of C-class type

2.5.3 Exceptional scalar cases and conclusion

By Theorem 2.3.10, we have $\mathfrak{S}_{\mathbb{U}} \leq \mathfrak{U}_{\mathbb{U}}$ and $\mathfrak{S} \leq \mathfrak{U}$. The upper bounds were computed in Lemmas 2.4.4 and 2.4.6, from which we obtain (using (2.34)):

$$\mathfrak{U} = \begin{cases} \mathfrak{M} - 1, & \text{if } m = 1,\ n \in \{4, 6\}; \\ \mathfrak{M} - 2, & \text{otherwise.} \end{cases} \tag{2.44}$$

These are realized by ODEs in Tables 8 and 9, so $\mathfrak{S} = \mathfrak{U}$ and Theorem 2.2.1 is proved.

Let us now turn to completing the proof of Theorem 2.2.2. The equality $\mathfrak{S}_{\mathbb{U}} = \mathfrak{U}_{\mathbb{U}}$ has already been established for all vector cases and most scalar cases. The following scalar cases remain:

$$(n, \mathbb{U}) = (3, \mathbb{B}_3), \quad (\geq 5, \mathbb{A}_3), \quad (\geq 7, \mathbb{A}_4), \tag{2.45}$$

n	$\mathbb{U}$	ODE with $\operatorname{im}(\kappa_H) \subset \mathbb{U}$	Sym dim	Contact (point) symmetries
2 ≥ 3	$\mathbb{B}_4$ $\mathbb{A}_2^{\mathrm{tr}}$	$u^a_{n+1} = \dfrac{(n+1)u^1_n u^a_n}{n u^1_{n-1}}$ $\scriptstyle (1 \leq a \leq m)$	$\mathfrak{M} - m - 1 + \delta^n_2$	$\partial_t,\ t\partial_t,\ u^1\partial_{u^1},$ $\partial_{u^a},\ u^a\partial_{u^b},$ $tu^1\partial_{u^b},\ t^j\partial_{u^1},$ $t^i\partial_{u^b},\ 1 \leq a,$ $b \leq m,\ b \neq 1,$ $1 \leq i,\ j \leq n-1,$ $j \neq n-1,$ $t^2\partial_t +$ $(n-2)tu^1\partial_{u^1} +$ $(n-1)t\sum_{a=2}^m u^a\partial_{u^a},$ for $n = 2$ in addition: $u^1\sum_{a=1}^m u^a\partial_{u^a}$
≥ 2	$\mathbb{A}_2^{\mathrm{tf}}$	$u^a_{n+1} = (u^2_n)^2\delta^a_1$ $\scriptstyle (1 \leq a \leq m)$	$\mathfrak{M} - 2m + 1 + \delta^n_2$	$\partial_t,\ \partial_{u^c},\ t^i\partial_{u^2},$ $t^j\partial_{u^a},\ u^b\partial_{u^a},$ $1 \leq a,\ b,\ c \leq m,$ $a \neq 2,\ b \neq 1,$ $1 \leq i,\ j \leq n,\ i \neq n,$ $t\partial_t - (n-1)u^1\partial_{u^1},$ $2u^1\partial_{u^1} + u^2\partial_{u^2},$ $2tu^2\partial_{u^1} + \frac{t^n(n+1)}{n!}\partial_{u^2},$ for $n = 2$ in addition: $3t^2\partial_t + 2(u^2)^2\partial_{u^1}$ $+6t\sum_{a=1}^m u^a\partial_{u^a}$

(The contact symmetry dimension of the trivial ODE is $\mathfrak{M} = m^2 + (n+1)m + 3$.)

TABLE 10. Vector ODEs realizing $\mathfrak{S}_{\mathbb{U}} = \mathfrak{U}_{\mathbb{U}}$ for $\mathbb{U}$ of C-class-type

for which $\mathfrak{U}_{\mathbb{U}} = \mathfrak{M} - 2 = n + 3$. (The $(6, \mathbb{A}_4)$ case was treated in Table 9.) Excluding $n = 4$ (for which $\mathfrak{S} = 8$) and $n = 6$ (for which $\mathfrak{S} = 10$), we already have $\mathfrak{S} = n + 3$ for scalar ODEs of order $n + 1 \geq 4$. From [43, p.206], which relies on results of Lie [40], all submaximally symmetric ODEs are either linear (but inequivalent to the trivial ODE $u_{n+1} = 0$) or equivalent to either:

$$n u_{n-1} u_{n+1} - (n+1)(u_n)^2 = 0, \quad \text{or} \quad 3u_2 u_4 - 5(u_3)^2 = 0. \tag{2.46}$$

We exclude the linear cases, for which all C-class invariants vanish. The first ODE in (2.46) has already appeared in Table 9 (associated to $(3, \mathbb{B}_4)$ or $(\geq 4, \mathbb{A}_2)$). The second ODE in (2.46) has κ_H concentrated in $\mathbb{B}_4$ (using the known relative invariants in §2.5.2). We conclude that

$$\mathfrak{S}_{\mathbb{U}} \leq n+2 < \mathfrak{U}_{\mathbb{U}} = n+3 \tag{2.47}$$

for all cases in (2.45) except possibly the $(6, \mathbb{A}_3)$ case. The latter case is resolved in §2.6.2.1 (Theorem 2.6.6) and indeed (2.47) also holds in this case.

Let us now exhibit model ODEs with $\kappa_H \not\equiv 0$, $\mathrm{im}(\kappa_H) \subset \mathbb{U}$ and all $\mathcal{W}_r \equiv 0$ (ODEs of C-class type). The assertions $\mathcal{W}_r \equiv 0$ and $\mathrm{im}(\kappa_H) \subset \mathbb{U}$ are established using Definition 2.5.1 and the differential invariants from §2.5.

- $(3, \mathbb{B}_3)$: The ODE $u_4 = (u_3)^k$ for $k \neq 0, 1$ has the 5-dimensional contact symmetry algebra:

$$\partial_t, \quad \partial_u, \quad t\partial_u, \quad t^2\partial_u, \quad (k-1)t\partial_t + (3k-4)u\partial_u. \tag{2.48}$$

  Generally, both $\mathcal{B}_3$ and $\mathcal{B}_4$ are nonzero. Requiring $\mathcal{B}_4 = 0$, i.e $\mathrm{im}(\kappa_H) \subset \mathbb{B}_3$ forces $k = \frac{74+2\sqrt{46}}{49}$. Thus, $\mathfrak{S}_{\mathbb{B}_3} = 5 < \mathfrak{U}_{\mathbb{B}_3} = 6$.
- $(\geq 5, \mathbb{A}_3)$: Consider the following ODE (obtained as $S_{n+1} = 0$ from [43, p. 475]):

$$\begin{aligned}(n-1)^2(u_{n-2})^2u_{n+1} - 3(n-1)(n+1)u_{n-2}u_{n-1}u_n+\\ 2n(n+1)(u_{n-1})^3 = 0,\end{aligned} \tag{2.49}$$

  which has the following $n+2$ contact symmetries when $n \geq 5$:

$$\begin{aligned}&\partial_t, \quad \partial_u, \quad t\partial_t, \quad u\partial_u, \quad t\partial_u, \quad \dots, \quad t^{n-3}\partial_u,\\ &t^2\partial_t + (n-3)tu\partial_u.\end{aligned} \tag{2.50}$$

  (Sidenote: when $n = 4$ the ODE (2.49) recovers the submaximally symmetric model from Table 9 in the $(4, \mathbb{B}_6)$ case, which admits eight symmetries: those in (2.50) and additionally $u\partial_t$ and $tu\partial_t + u^2\partial_u$.)

  We have $\mathcal{A}_2 \equiv 0$ (and $\mathcal{W}_r \equiv 0$), but $\mathcal{A}_3 \neq 0$. When $n = 5$, the invariant $\mathcal{A}_4$ does not arise, so in this case we can assert that $\mathrm{im}(\kappa_H) \subset \mathbb{A}_3$ and $\mathfrak{S}_{\mathbb{A}_3} = 7 < \mathfrak{U}_{\mathbb{A}_3} = 8$ (using (2.47)). For $n \geq 6$, since $\mathcal{A}_4$ was computed only up to the differential ideal $\langle \mathcal{A}_2, \mathcal{A}_3, \mathcal{W}_3 \rangle$, then the formula given in (2.41) for $\mathcal{A}_4$ is ambiguous, and so we cannot directly use it on (2.49). From (2.47), we can only assert $\mathfrak{S}_{\mathbb{A}_3} \leq n+2 < \mathfrak{U}_{\mathbb{A}_3} = n+3$ for $n \geq 6$.
- $(\geq 7, \mathbb{A}_4)$: The ODE $u_{n+1} = (u_{n-1})^2$ admits the following $n+1$ contact symmetries:

$$\partial_t, \quad \partial_u, \quad t\partial_u, \quad \dots, \quad t^{n-2}\partial_u, \quad t\partial_t + (n-3)u\partial_u. \tag{2.51}$$

  We confirm that it has vanishing $\mathcal{A}_2, \mathcal{A}_3, \mathcal{W}_3$, so the formula for $\mathcal{A}_4$ is unambiguous and $\mathcal{A}_4 \neq 0$, i.e. $\kappa_H \not\equiv 0$ and $\mathrm{im}(\kappa_H) \subset \mathbb{A}_4$. Hence, $n+1 \leq \mathfrak{S}_{\mathbb{A}_4} \leq n+2 < \mathfrak{U}_{\mathbb{A}_4} = n+3$.

This completes the proof of Theorem 2.2.2. We remark that for $(n, \mathbb{U}) = (\geq 6, \mathbb{A}_3)$ or $(\geq 7, \mathbb{A}_4)$, we currently do not know of any ODE (2.1) of order $n+1$

with $\kappa_H \not\equiv 0$ and $\mathrm{im}(\kappa_H) \subset \mathbb{U}$ that has contact symmetry dimension $n+2$. (See Remark 5 for further discussion.) Determining $\mathfrak{S}_{\mathbb{U}}$ for these cases remains open.

2.6 Appendix: Exceptional scalar cases

Fix (G, P) as in §2.3.1.2, and the effective part $\mathbb{E} \subset H^2_+(\mathfrak{g}_-, \mathfrak{g})$ as given in §2.4.3 and §2.4.4. Let $\mathbb{U} \subset \mathbb{E}$ be a $\mathfrak{g}_0$-irrep. Recall from §2.5 that for ODEs (2.1) of order $n+1$ with $\kappa_H \not\equiv 0$ and $\mathrm{im}(\kappa_H) \subset \mathbb{U}$, an algebraic upper bound $\mathfrak{U}_{\mathbb{U}}$ on the submaximal symmetry dimension $\mathfrak{S}_{\mathbb{U}}$ is realizable for all vector cases and the majority of scalar cases. Among the remaining scalar cases $(n, \mathbb{U}) = (3, \mathbb{B}_3), (\geq 5, \mathbb{A}_3)$ or $(\geq 7, \mathbb{A}_4)$, we asserted that $\mathfrak{S}_{\mathbb{U}} < \mathfrak{U}_{\mathbb{U}}$ for all of these in §2.5.3, except for $(6, \mathbb{A}_3)$, based on the known classification of submaximally symmetric scalar ODEs as described in [43, p. 206]. In this section, we outline a Cartan-geometric method for establishing $\mathfrak{S}_{\mathbb{U}} < \mathfrak{U}_{\mathbb{U}}$ for the exceptional scalar cases, and in particular establish $\mathfrak{S}_{\mathbb{A}_3} < \mathfrak{U}_{\mathbb{A}_3}$ for $n = 6$ (Theorem 2.6.6).

2.6.1 Local homogeneity and algebraic models

LEMMA 2.6.1. *For regular, normal Cartan geometries of type (G, P) and a $\mathfrak{g}_0$-irrep $\mathbb{U} \subset \mathbb{E}$, suppose that $\mathfrak{S}_{\mathbb{U}} = \mathfrak{U}_{\mathbb{U}}$. Then any geometry $(\mathcal{G} \to M, \omega)$ with κ_H valued in $\mathbb{U}$ and $\dim \mathfrak{inf}(\mathcal{G}, \omega) = \mathfrak{U}_{\mathbb{U}}$ is locally homogeneous near any $u \in \mathcal{G}$ with $\kappa_H(u) \neq 0$.*

PROOF. Fix $u \in \mathcal{G}$. By Theorem 2.3.10, $\mathfrak{s}(u) \subset \mathfrak{a}^{\kappa_H(u)}$. Then by definition of $\mathfrak{U}_{\mathbb{U}}$,

$$\mathfrak{S}_{\mathbb{U}} := \dim \mathfrak{inf}(\mathcal{G}, \omega) = \dim \mathfrak{s}(u) \leq \dim \mathfrak{a}^{\kappa_H(u)} \leq \mathfrak{U}_{\mathbb{U}}. \tag{2.52}$$

So, $\mathfrak{S}_{\mathbb{U}} = \mathfrak{U}_{\mathbb{U}}$ implies $\mathfrak{s}(u) = \mathfrak{a}^{\kappa_H(u)} \supset \mathfrak{g}_-$. The result then follows by Lie's third theorem. ■

It is well known that a homogeneous Cartan geometry $(\pi : \mathcal{G} \to M, \omega)$ of fixed type (G, P) can be encoded by algebraic data [8, Prop 1.5.15]. Fix $u \in \mathcal{G}$ and let $F^0 \subset F$ denotes stabilizer of a point $\pi(u) \in M$ and let $\mathfrak{f}^0$ and $\mathfrak{f}$ denote the Lie algebras of F^0 and F respectively. Then the induced F-action on M is transitive. Any F-invariant Cartan connection is completely determined by some distinguished linear map $\varpi : \mathfrak{f} \to \mathfrak{g}$ (*an algebraic Cartan connection* of type $(\mathfrak{g}, P)$). In particular, $\varpi|_{\mathfrak{f}^0}$ is a Lie algebra homomorphism, so $\ker(\varpi) \subset \mathfrak{f}^0$ is an ideal in $\mathfrak{f}$. Since the action of F on F/F^0 can be assumed to be *infinitesimally effective* (i.e. $\mathfrak{f}^0$ does not contain any non-trivial ideals of $\mathfrak{f}$), then without loss of generality we can restrict to injective maps ϖ. Consequently, we can identify $\mathfrak{f}$ with its image $\varpi(\mathfrak{f})$ in $\mathfrak{g}$. Analogous to [46, Defn 2.5] and in light of the fact that canonical Cartan connections for ODEs satisfy the strong regularity condition (Remark 1), any homogeneous Cartan geometry arising from an ODE can be encoded as:

DEFINITION 2.6.2. *An* algebraic model $(\mathfrak{f};\mathfrak{g},\mathfrak{p})$ *of ODE type is a Lie algebra* $(\mathfrak{f},[\cdot,\cdot]_{\mathfrak{f}})$ *satisfying:*

(A1) $\mathfrak{f}\subset\mathfrak{g}$ *is a filtered linear subspace such that* $\mathfrak{f}^i=\mathfrak{g}^i\cap\mathfrak{f}$ *and* $\mathfrak{s}:=\mathrm{gr}(\mathfrak{f})$ *with* $\mathfrak{s}_-=\mathfrak{g}_-$*;*

(A2) $\mathfrak{f}^0$ *inserts trivially into* $\kappa(X,Y):=[X,Y]-[X,Y]_{\mathfrak{f}}$*, i.e.* $\kappa(Z,\cdot)=0\quad\forall Z\in\mathfrak{f}^0$

(A3) $\partial^*\kappa=0$ *and* $\kappa(\mathfrak{g}^i,\mathfrak{g}^j)\subset\mathfrak{g}^{i+j+1}\cap\mathfrak{g}^{\min(i,j)-1}\quad\forall i,j.$

Recall from §2.3.1.3 that $\kappa_H:=\kappa\mod\mathrm{im}\,\partial^*$, where ∂^* is the adjoint of the Lie algebra cohomology differential with respect to a natural inner product on $\mathfrak{g}$.

PROPOSITION 2.6.3. *Let* $(\mathfrak{f};\mathfrak{g},\mathfrak{p})$ *be an algebraic model of ODE type. Then*

(a) $(\mathfrak{f},[\cdot,\cdot]_{\mathfrak{f}})$ *is a filtered Lie algebra.*

(b) $\mathfrak{f}^0\cdot\kappa=0$*, i.e.* $[Z,\kappa(X,Y)]_{\mathfrak{f}}=\kappa([Z,X]_{\mathfrak{f}},Y)+\kappa(X,[Z,Y]_{\mathfrak{f}})$*,* $\forall X,Y\in\mathfrak{f}$ *and* $\forall Z\in\mathfrak{f}^0$.

(c) $\mathfrak{s}\subset\mathfrak{a}^{\kappa_H}$.

PROOF. This is the same as for corresponding statements in the parabolic geometry setting [46, Prop 2.6]. ■

Fix (G,P) and denote by $\mathcal{N}$ the set of all algebraic models $(\mathfrak{f};\mathfrak{g},\mathfrak{p})$ of ODE type. Then $\mathcal{N}$:

(1) admits P-action: for $p\in P$ and $\mathfrak{f}\in\mathcal{N}$, $p\cdot\mathfrak{f}:=\mathrm{Ad}_p(\mathfrak{f})$. All algebraic models belonging to the same P-orbit are considered to be equivalent.

(2) a partially ordered set with relation $\leq$ defined as follows: for $\mathfrak{f},\widetilde{\mathfrak{f}}\in\mathcal{N}$ regard $\mathfrak{f}\leq\widetilde{\mathfrak{f}}$ if there exists an injection $\mathfrak{f}\hookrightarrow\widetilde{\mathfrak{f}}$ of Lie algebras. We will focus on maximal elements $\mathfrak{f}$ (for this partial order).

REMARK 4. By [36, Lemma 4.1.4], to each algebraic model $(\mathfrak{f};\mathfrak{g},\mathfrak{p})$ of ODE type, there exists a locally homogeneous geometry $(\mathcal{G}\to\mathcal{E},\omega)$ of type (G,P) with $\mathfrak{inf}(\mathcal{G},\omega)$ containing a subalgebra isomorphic to $\mathfrak{f}$. Moreover, if $\mathfrak{f}$ is maximal, then it is isomorphic to $\mathfrak{inf}(\mathcal{G},\omega)$.

At the level of vector spaces, $\mathfrak{f}\subset\mathfrak{g}$ can be understood as the graph of a linear map on $\mathfrak{s}$ into some subspace $\mathfrak{s}^\perp\subset\mathfrak{p}$ with $\mathfrak{g}=\mathfrak{s}\oplus\mathfrak{s}^\perp$ as follows. Choosing such a graded subspace $\mathfrak{s}^\perp$, we can write

$$\mathfrak{f}:=\bigoplus_i\langle x+\mathfrak{D}(x):x\in\mathfrak{s}_i\rangle\,,\tag{2.53}$$

for some *unique* linear (*deformation*) map $\mathfrak{D}:\mathfrak{s}\to\mathfrak{s}^\perp$ satisfying $\mathfrak{D}(x)\in\mathfrak{s}^\perp\cap\mathfrak{g}^{i+1}$ for $x\in\mathfrak{s}_i$. For $\widehat{x}:=x+\mathfrak{D}(x)\in\mathfrak{f}$, we will refer to $x\in\mathfrak{s}$ as the *leading part* and $\mathfrak{D}(x)$ as the *tail*.

LEMMA 2.6.4. *Let* $T\in\mathfrak{f}^0$ *and suppose that* $\mathfrak{s}$ *and* $\mathfrak{s}^\perp$ *are* ad_T*-invariant subspaces. Then* $T\cdot\mathfrak{D}=0$*, i.e* $\mathrm{ad}_T\circ\mathfrak{D}=\mathfrak{D}\circ\mathrm{ad}_T$.

PROOF. Recall $\mathfrak{s}, \mathfrak{s}^\perp \subset \mathfrak{g}$ are graded. Given $x \in \mathfrak{s}_i$, we have $x + \mathfrak{D}(x) \in \mathfrak{f}$ and $[T, x + \mathfrak{D}(x)]_\mathfrak{f} \in \mathfrak{f}$. Since $T \in \mathfrak{f}^0$, then $\kappa(T, \cdot) = 0$ and therefore $[T, x + \mathfrak{D}(x)]_\mathfrak{f} = [T, x + \mathfrak{D}(x)] = [T, x] + [T, \mathfrak{D}(x)]$. By ad_T-invariancy of $\mathfrak{s}$ and $\mathfrak{s}^\perp$, we have $[T, x] \in \mathfrak{s}$ and $[T, \mathfrak{D}(x)] \in \mathfrak{s}^\perp \cap \mathfrak{g}^{i+1}$. The uniqueness of $\mathfrak{D}$ then implies $[T, \mathfrak{D}(x)] = \mathfrak{D}([T, x])$. ■

2.6.2 Realizability of a curvature-constrained upper bound

From §2.6.1, $\mathfrak{S}_\mathbb{U} = \mathfrak{U}_\mathbb{U}$ implies local homogeneity (Lemma 2.6.1) and then the problem of realizability of $\mathfrak{U}_\mathbb{U}$ reduces to that of existence of an algebraic model $(\mathfrak{f}; \mathfrak{g}, \mathfrak{p})$ of ODE type with $\kappa_H \not\equiv 0$, $\mathrm{im}(\kappa_H) \subset \mathbb{U}$ and $\dim \mathfrak{f} = \mathfrak{U}_\mathbb{U}$. Recall from §2.3.1.2, §2.4.1 and §2.4.3:

- basis for $\mathfrak{g} \cong (\mathfrak{sl}_2 \times \mathfrak{gl}_1) \ltimes (\mathbb{V}_n \otimes \mathbb{R})$: $\mathsf{X}, \mathsf{H}, \mathsf{Y}$ (standard $\mathfrak{sl}_2$-triple), $E_0, \dots, E_n$ (for $\mathfrak{sl}_2$-irrep module $\mathbb{V}_n$) and id_1. And $\mathsf{Z}_1, \mathsf{Z}_2$ are the bi-grading elements.
- $\mathbb{U} \subset \mathbb{E}$ is one-dimensional with bi-grade $(a, b) = (1, 2), (2, 1), (3, 1)$ for $\mathbb{B}_3, \mathbb{A}_3, \mathbb{A}_4$ respectively. Thus, for any $0 \neq \phi \in \mathbb{U}$, we have $\mathfrak{ann}(\phi) = \langle T := b\mathsf{Z}_1 - a\mathsf{Z}_2 \rangle$. Since $\mathbb{U}$ is prolongation rigid (Lemma 2.4.4), then $\mathfrak{a}_1^\phi = 0$ for any $0 \neq \phi \in \mathbb{U}$. So, $\mathfrak{a} := \mathfrak{a}^\phi = \mathfrak{g}_- \oplus \mathfrak{ann}(\phi) \subset \mathfrak{g}$, is a graded subalgebra of dimension $n + 3$.

PROPOSITION 2.6.5. *Fix* $(n, \mathbb{U}) = (3, \mathbb{B}_3), (\geq 5, \mathbb{A}_3)$ *or* $(\geq 7, \mathbb{A}_4)$. *If there exists an algebraic model* $(\mathfrak{f}; \mathfrak{g}, \mathfrak{p})$ *of ODE type with* $\kappa_H \not\equiv 0$, $\mathrm{im}(\kappa_H) \subset \mathbb{U}$ *and* $\dim \mathfrak{f} = \mathfrak{U}_\mathbb{U} = n + 3 = \mathfrak{M} - 2$, *then fixing* $0 \neq \phi \in \mathbb{U}$ *and using the* P*-action* $\mathfrak{f} \mapsto \mathrm{Ad}_p \mathfrak{f}$, *we may normalize to* $\mathfrak{f} = \mathfrak{a}^\phi$ *as filtered vector spaces.*

PROOF. Suppose such an algebraic model with $\mathfrak{s} := \mathrm{gr}(\mathfrak{f}) = \mathfrak{a}^\phi$ exists. Let $\widehat{T} \in \mathfrak{f}^0$ with leading part T, so $\widehat{T} := b\mathsf{Z}_1 - a\mathsf{Z}_2 + \lambda\mathsf{Y}$. We use the P_+- action to normalize $\lambda = 0$:

$$\begin{aligned}\mathrm{Ad}_{\exp(t\mathsf{Y})}(\widehat{T}) &= \exp(\mathrm{ad}_{t\mathsf{Y}})(\widehat{T}) = \widehat{T} + [t\mathsf{Y}, \widehat{T}] + \frac{1}{2!}[t\mathsf{Y}, [t\mathsf{Y}, \widehat{T}]] + \cdots \\ &= b\mathsf{Z}_1 - a\mathsf{Z}_2 + (\lambda - bt)\mathsf{Y}.\end{aligned} \tag{2.54}$$

For our cases of interest, $(a, b) \in \{(1, 2), (2, 1), (3, 1)\}$, so $b \neq 0$ and choosing $t = \frac{\lambda}{b}$ normalizes $\widehat{T} = T$. So, $T = b\mathsf{Z}_1 - a\mathsf{Z}_2 \in \mathfrak{f}^0$ and by property (A2) of Definition 2.6.2, we have $[T, \cdot]_\mathfrak{f} := [T, \cdot]$. Consequently, $\mathfrak{s}$ and $\mathfrak{s}^\perp := \langle \mathsf{Z}_1, \mathsf{Y} \rangle$ are ad_T-invariant graded subspaces of $\mathfrak{g}$, so by Lemma 2.6.4, the deformation map $\mathfrak{D} : \mathfrak{s} \to \mathfrak{s}^\perp$ satisfies $T \cdot \mathfrak{D} = 0$.

We claim that $\mathfrak{D} = 0$. Equivalently, for $\widehat{\mathsf{X}}, \widehat{E}_i \in \mathfrak{f}$ with leading parts X, E_i respectively, we claim that $\widehat{\mathsf{X}} = \mathsf{X}$ and $\widehat{E}_i = E_i$. First focus on $\widehat{\mathsf{X}}$ and $\widehat{E}_n$, whose tails are valued in $\mathfrak{s}^\perp = \langle \mathsf{Z}_1, \mathsf{Y} \rangle$. Recall from Figure 2 that $\mathsf{X}, E_n, \mathsf{Z}_1, \mathsf{Y}$ are of bi-grades $(-1, 0), (0, -1), (0, 0), (1, 0)$. Letting ω^n and ω^X denote dual basis elements to E_n

and X respectively, the eigenvalues of $T = b\mathsf{Z}_1 - a\mathsf{Z}_2$ acting on

$$\omega^n \otimes \mathsf{Z}_1, \quad \omega^n \otimes \mathsf{Y}, \quad \omega^{\mathsf{X}} \otimes \mathsf{Z}_1, \quad \omega^{\mathsf{X}} \otimes \mathsf{Y} \tag{2.55}$$

are $-a, -a+b, b, 2b$. None of these are zero, so the condition $T \cdot \mathfrak{D} = 0$ forces $\mathfrak{D}(\mathsf{X}) = 0 = \mathfrak{D}(E_n)$ and hence $\widehat{\mathsf{X}} = \mathsf{X}$ and $\widehat{E}_n = E_n$. Any ODE with κ_H concentrated in any of the C-class modules $\mathbb{B}_3, \mathbb{A}_3, \mathbb{A}_4$ is of C-class, so $\kappa(\mathsf{X}, \cdot) = 0$ (see discussion in §2.5.2) and $[\mathsf{X}, \cdot]_{\mathfrak{f}} = [\mathsf{X}, \cdot]$. Since $\mathsf{X}, E_n \in \mathfrak{f}$, then $\mathfrak{f} \ni [\mathsf{X}, E_i]_{\mathfrak{f}} = [\mathsf{X}, E_i] = E_{i-1}$ inductively from $i = n$ to $i = 1$. Thus, $\widehat{E}_i = E_i \ \forall i$, so $\mathfrak{D} = 0$ and $\mathfrak{f} = \mathfrak{s} = \mathfrak{a}^\phi$. ■

2.6.2.1 Non-existence of algebraic models for the exceptional scalar cases We prove that for $(n, \mathbb{U}) = (6, \mathbb{A}_3)$, there are no algebraic models $(\mathfrak{f}; \mathfrak{p}, \mathfrak{g})$ with $\kappa_H \not\equiv 0$, $\mathrm{im}(\kappa_H) \subset \mathbb{A}_3$, and $\dim \mathfrak{f} = \mathfrak{U}_{\mathbb{A}_3}$. Thus, $\mathfrak{U}_{\mathbb{A}_3}$ is not realizable, i.e. $\mathfrak{S}_{\mathbb{A}_3} < \mathfrak{U}_{\mathbb{A}_3}$ (Theorem 2.6.6). From §2.3.1.3, $\kappa_H := \kappa \mod \mathrm{im}\, \partial^*$ with $\partial^* \kappa = 0$, but the determination of ∂^* is rather tedious, requiring specific information about the inner product on $\mathfrak{g}$. We have not provided details of this in our article since for our purposes here they can be completely circumvented. Namely in the proof of Theorem 2.6.6, instead of showing that "normal filtered deformations" provided by κ do not exist, we show that arbitrary "filtered deformations" do not exist. In a similar manner, $\mathfrak{S}_{\mathbb{U}} < \mathfrak{U}_{\mathbb{U}}$ can be established for $(n, \mathbb{U}) = (3, \mathbb{B}_3), (5, \mathbb{A}_3), (\geq 7, \mathbb{A}_3)$ or $(\geq 7, \mathbb{A}_4)$.

THEOREM 2.6.6. *There are no algebraic models $(\mathfrak{f}; \mathfrak{g}, \mathfrak{p})$ for seventh order ODEs (2.1) with $\kappa_H \not\equiv 0$, $\mathrm{im}(\kappa_H) \subset \mathbb{A}_3$ and $\dim \mathfrak{f} = \mathfrak{U}_{\mathbb{A}_3} = 9$. Thus, $\mathfrak{S}_{\mathbb{A}_3} \leq 8$.*

PROOF. Note that $n = 6$. Fix $0 \neq \phi \in \mathbb{A}_3$ (bi-grade $(2,1)$), $\mathfrak{a} := \mathfrak{a}^\phi$, and $\mathfrak{a}_0 = \langle T \rangle$, where $T = \mathsf{Z}_1 - 2\mathsf{Z}_2$. Assume there is an algebraic model $(\mathfrak{f}; \mathfrak{g}, \mathfrak{p})$ of ODE type with $\mathfrak{s} := \mathrm{gr}(\mathfrak{f}) = \mathfrak{a}$. By Proposition 2.6.5, we may assume that $\mathfrak{f} = \mathfrak{a}$. Let $\{\omega^0, \ldots, \omega^n, \omega^{\mathsf{X}}\}$ denote the dual basis to $\{E_0, \ldots, E_n, \mathsf{X}\}$. We note that any $\beta \in \wedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ has Z_2-degree at most 2. Since $\mathfrak{f}^0 \cdot \kappa = 0$ (Proposition 2.6.3 (b)) and $\kappa(\mathsf{X}, \cdot) = 0$ (the ODE is of C-class), then κ is a linear combination of the 2-cochains below:

Bi-grade	2-cochains
$(2,1)$	$\omega^0 \wedge \omega^4 \otimes E_0$, $\omega^1 \wedge \omega^3 \otimes E_0$, $\omega^0 \wedge \omega^5 \otimes E_1$, $\omega^1 \wedge \omega^4 \otimes E_1$, $\omega^2 \wedge \omega^3 \otimes E_1$, $\omega^0 \wedge \omega^6 \otimes E_2$, $\omega^1 \wedge \omega^5 \otimes E_2$, $\omega^2 \wedge \omega^4 \otimes E_2$, $\omega^1 \wedge \omega^6 \otimes E_3$, $\omega^2 \wedge \omega^5 \otimes E_3$, $\omega^3 \wedge \omega^4 \otimes E_3$, $\omega^2 \wedge \omega^6 \otimes E_4$, $\omega^3 \wedge \omega^5 \otimes E_4$, $\omega^3 \wedge \omega^6 \otimes E_5$, $\omega^4 \wedge \omega^5 \otimes E_5$, $\omega^4 \wedge \omega^6 \otimes E_6$
$(4,2)$	$\omega^1 \wedge \omega^6 \otimes \mathsf{X}$, $\omega^2 \wedge \omega^5 \otimes \mathsf{X}$, $\omega^3 \wedge \omega^4 \otimes \mathsf{X}$, $\omega^2 \wedge \omega^6 \otimes T$, $\omega^3 \wedge \omega^5 \otimes T$

We observe that all such 2-cochains are regular and satisfy the strong regularity condition, i.e. $\kappa(\mathfrak{g}^i, \mathfrak{g}^j) \subset \mathfrak{g}^{i+j+1} \cap \mathfrak{g}^{\min(i,j)-1} \quad \forall i, j$.

Next, we show that the Jacobi identity for $(\mathfrak{f}, [\cdot,\cdot]_{\mathfrak{f}})$ forces $\kappa \equiv 0$. For all $x, y, z \in \mathfrak{f}$, define

$$\mathrm{Jac}^{\mathfrak{f}}(x,y,z) := [x,[y,z]_{\mathfrak{f}}]_{\mathfrak{f}} - [[x,y]_{\mathfrak{f}}, z]_{\mathfrak{f}} - [y,[x,z]_{\mathfrak{f}}]_{\mathfrak{f}}. \tag{2.56}$$

For any $y, z \in \mathfrak{f}$, a direct computation shows that $0 = \mathrm{Jac}^{\mathfrak{f}}(\mathsf{X}, y, z) = (\mathsf{X}\cdot\kappa)(y,z)$. Expanding this gives many conditions (see the Maple file accompanying the arXiv submission of this article) and this leads to:

$$\begin{aligned}\kappa = \lambda\Big[&(\omega^0\wedge\omega^4 - \omega^1\wedge\omega^3)\otimes E_0 + (\omega^0\wedge\omega^5 - \omega^2\wedge\omega^3)\otimes E_1\\ &+ (\omega^0\wedge\omega^6 + \omega^1\wedge\omega^5 - \omega^2\wedge\omega^4)\otimes E_2 + (2\omega^1\wedge\omega^6 - \omega^3\wedge\omega^4)\otimes E_3\\ &+ (2\omega^2\wedge\omega^6 - \omega^3\wedge\omega^5)\otimes E_4 + (\omega^3\wedge\omega^6 - \omega^4\wedge\omega^5)\otimes E_5\Big]\\ &+ \mu(\omega^1\wedge\omega^6 - \omega^2\wedge\omega^5 + \omega^3\wedge\omega^4)\otimes\mathsf{X}.\end{aligned} \tag{2.57}$$

Then $\mathrm{Jac}^{\mathfrak{f}}(E_2, E_4, E_6) = 0$ implies $\lambda = 0$, while $\mathrm{Jac}^{\mathfrak{f}}(E_1, E_2, E_5) = 0$ then forces $\mu = 0$, and hence $\kappa \equiv 0$. Thus, an algebraic model with $0 \neq \kappa_H \subset \mathbb{A}_3$ with $\dim \mathfrak{f} = \mathfrak{U}_{\mathbb{A}_3}$ does not exist. ■

REMARK 5. Fix $(n, \mathbb{U}) = (\geq 6, \mathbb{A}_3)$ or $(\geq 7, \mathbb{A}_4)$ and recall from Table 5 that the bi-grades (a, b) for the C-class modules $\mathbb{A}_3$ and $\mathbb{A}_4$ are $(2, 1)$ and $(3, 1)$, respectively. Then from §2.5.3, we have

$$\mathfrak{S}_{\mathbb{U}} \leq n+2 < \mathfrak{U}_{\mathbb{U}} = n+3 = \mathfrak{M} - 2. \tag{2.58}$$

For $0 \neq \phi \in \mathbb{U}$, we have $\mathfrak{a} := \mathfrak{a}^{\phi} = \mathfrak{g}_- \oplus \mathfrak{ann}(\phi) = \mathfrak{g}_- \oplus \langle T := b\mathsf{Z}_1 - a\mathsf{Z}_2\rangle \subset \mathfrak{g}$, which is a graded subalgebra of dimension $n+3$. If there exists an ODE whose associated Cartan geometry $(\mathcal{G} \to M, \omega)$ satisfies $0 \neq \kappa_H(u) \in \mathbb{U}$, $\forall u \in \mathcal{G}$, then from Theorem 2.3.10 we have the graded Lie algebra inclusion

$$\mathfrak{s}(u) \subset \mathfrak{a}^{\kappa_H(u)} = \mathfrak{a}, \quad \forall u \in \mathcal{G}. \tag{2.59}$$

By (2.58), this inclusion is proper and a priori we do not need to have $\mathfrak{g}_- \subset \mathfrak{s}(u)$. If the contact symmetry dimension is $\mathfrak{U}_{\mathbb{U}} - 1 = n+2$, then there are three possibilities to investigate:

(i) inhomogeneous case: $\mathfrak{s}(u) = \langle E_0, \ldots, E_n, T\rangle$;
(ii) inhomogeneous case: $\mathfrak{s}(u) = \langle E_0, \ldots, E_{n-1}, X, T\rangle$;
(iii) homogeneous case: $\mathfrak{s}(u) = \langle E_0, \ldots, E_n, X\rangle = \mathfrak{g}_-$.

Thus, identifying $\mathfrak{S}_{\mathbb{U}}$ is more difficult and at this point we can only assert that:

$$\mathfrak{S}_{\mathbb{A}_3} \leq n+2, \qquad n+1 \leq \mathfrak{S}_{\mathbb{A}_4} \leq n+2. \tag{2.60}$$

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CHAPTER 3

On uniqueness of submaximally symmetric vector ordinary differential equations of C-class

This chapter consists of contents from my joint article [27] with Dennis The.

3.1 Abstract

The fundamental invariants for vector ODEs of order ≥ 3 considered up to point transformations consist of generalized Wilczynski invariants and C-class invariants. An ODE of C-class is characterized by the vanishing of the former. For any fixed C-class invariant $\mathcal{U}$, we give a local (point) classification for all submaximally symmetric ODEs of C-class with $\mathcal{U} \not\equiv 0$ and all remaining C-class invariants vanishing identically. Our results yield generalizations of a well-known classical result for scalar ODEs due to Sophus Lie.

Fundamental invariants correspond to the harmonic curvature of the associated Cartan geometry. A key new ingredient underlying our classification results is an advance concerning the harmonic theory associated with the structure of vector ODEs of C-class. Namely, for each irreducible C-class module, we provide an explicit identification of a lowest weight vector as a harmonic 2-cochain.

3.2 Introduction

Finite dimensionality of the contact symmetry algebra for scalar ODEs $u_{n+1} = f(t, u, u_1, \ldots, u_n)$ of order $n + 1 \geq 4$ is a classical result due to Sophus Lie [39] (see also [43, Thm 6.44]). (We use jet notation u_k instead of the more standard notation $u^{(k)}$ to denote the k-th derivative of u with respect to t.) The maximal symmetry dimension and the submaximal (i.e. next largest realizable) symmetry dimension are respectively:

$$\mathfrak{M} := n + 5 \qquad \text{and} \qquad \mathfrak{S} := \begin{cases} \mathfrak{M} - 1, & \text{for} \quad n = 4 \quad \text{or} \quad 6; \\ \mathfrak{M} - 2, & \text{otherwise.} \end{cases} \tag{3.1}$$

The former is realized locally uniquely by the trivial ODE $u_{n+1} = 0$. For ODEs realizing $\mathfrak{S}$, we have the following result (over $\mathbb{C}$) due to Lie [40] (see also [43, pp. 205-206]): *Any submaximally symmetric scalar ODE of order $n + 1 \geq 4$ is locally contact-equivalent to:*

(a) a linear equation, or

(b) exactly one of:[1]

(i) $n = 4$*:* $9(u_2)^2u_5 - 45u_2u_3u_4 + 40(u_3)^3 = 0$.

(ii) $n = 6$*:* $10(u_3)^3u_7 - 70(u_3)^2u_4u_6 - 49(u_3)^2(u_5)^2 + 280u_3(u_4)^2u_5 - 175(u_4)^4 = 0$.

(iii) $n \neq 4, 6$*:* $nu_{n-1}u_{n+1} - (n+1)(u_n)^2 = 0$.

The aim of our article is to establish analogous results for *vector* ODEs $\mathcal{E}$ of order $n + 1 \geq 3$:

$$\mathbf{u}_{n+1} = \mathbf{f}(t, \mathbf{u}, \mathbf{u}_1, \ldots, \mathbf{u}_n), \tag{3.2}$$

where $\mathbf{u}$ is an $\mathbb{R}^m$-valued function of t (for $m \geq 2$), and $\mathbf{u}_k$ is its k-th derivative. More precisely, we consider and completely resolve the classification problem (up to local contact equivalence) for submaximally symmetric vector ODEs (3.2) of order ≥ 3 of the *C-class* [6, 10]. Note that by the Lie–Bäcklund theorem, contact-equivalence agrees with point-equivalence for vector ODEs.

For vector ODEs (3.2) of order $n + 1 \geq 3$, the maximal and submaximal symmetry dimensions are:

$$\mathfrak{M} = m^2 + (n+1)m + 3 \qquad \text{and} \qquad \mathfrak{S} = \mathfrak{M} - 2, \tag{3.3}$$

with the latter established in our earlier work [26], along with numerous other symmetry gap results. The trivial vector ODE $\mathbf{u}_{n+1} = \mathbf{0}$ is locally uniquely maximally symmetric, cf. [26, Cor 2.8]. Examples of some submaximally symmetric vector ODEs were given in [26, Table 8], but no definitive classification lists for the submaximal strata were asserted. This is a focus of our current article.

Following Cartan [10] (see also [4, 6, 24]), a class of vector ODE (3.2) of order ≥ 3 is said to be a *C-class* if it is invariant under all contact transformations, and all (contact) differential invariants of any ODE in this class are *first integrals* of that ODE. Hence, generic C-class equations (having sufficiently many functionally independent first integrals) can be solved using these invariants. In [6, Thms 4.1 & 4.2], the C-class was characterized by the vanishing of the *generalized Wilczynski invariants*. These are a subset of the fundamental (relative) invariants, which additionally consist of *C-class invariants* (in the terminology of [26]).

We note from [26, Tables 8 & 10] that a vector ODE realizing $\mathfrak{S}$ given in (3.3) is either a 3rd order ODE pair of C-class (i.e. $(n, m) = (2, 2)$) or it is of *Wilczynski type* (i.e. an ODE with all C-class invariants vanishing identically). We will prove the following generalization of Lie's result above for vector ODEs:

THEOREM 3.2.1. *Any submaximally symmetric vector ODE (3.2) of order $n+1 \geq 3$ is either:*

(a) of Wilczynski type, or

[1] In [43, p. 206], the ODE $3u_2u_4 - 5(u_3)^2 = 0$ is also listed as a possibility, but this is in fact contact-equivalent to $nu_{n-1}u_{n+1} - (n+1)(u_n)^2 = 0$ when $n = 3$. We have verified this using Cartan-geometric techniques – details will be given elsewhere.

(b) *locally equivalent*[2] *over $\mathbb{R}$ to exactly one of the three 3rd order ODE pairs in Table 1. Over $\mathbb{C}$, the two 3rd order ODE pairs in the second row of Table 1 are locally equivalent.*

Lie obtained his result for submaximally symmetric scalar ODEs using his complete classification of Lie algebras of contact vector fields on the (complex) plane and classified invariant ODEs having sufficiently many symmetries. Certainly, this approach generalizes to vector ODEs, but it is not feasible: complete classifications for Lie algebras of (point) vector fields on $\mathbb{C}^n$ or $\mathbb{R}^n$ for $n \geq 3$ are known to be very difficult to establish [15, 44]. So, different techniques are needed to establish analogous results for submaximally symmetric vector ODEs.

Our approach to classifying all submaximally symmetric vector ODEs (3.2) of C-class of order ≥ 3 is motivated by that of [46, 47] in the setting of parabolic geometries [8], and is based on a categorically equivalent reformulation of vector ODEs (3.2) as *(strongly) regular, normal Cartan geometries* $(\mathcal{G} \to \mathcal{E}, \omega)$ of type (G, P) for a certain Lie group G and closed subgroup $P \subset G$ [6, 16] (see §3.3.3 below).

For such a (non-parabolic) Cartan geometry, the *harmonic curvature* κ_H, which corresponds to the fundamental invariants, is valued in a certain P-module that is completely reducible [6, Cor 3.8], so only the action of the reductive part $G_0 \subset P$ is relevant. Via a known algebraic Hodge theory associated with G_0, the codomain of κ_H can be identified with a certain G_0-submodule $\mathbb{E} \subsetneq H^2(\mathfrak{g}_-, \mathfrak{g})$ of a Lie algebra cohomology group called the *effective part* (see Definition 3.3.3). This has been already computed for ODEs (3.2) of order 3 in [41, 42] and of order ≥ 4 in [17]. The aforementioned fundamental invariants are valued in corresponding G_0-irreducible submodules $\mathbb{U} \subset \mathbb{E}$; see [26, Table 6] for a summary. The irreducible C-class modules are listed in Table 2.

We next formulate our second main result, which concerns the classification of vector ODEs (3.2) of C-class realizing the so-called constrained submaximal symmetry dimensions $\mathfrak{S}_{\mathbb{U}}$ identified in [26, Table 2]. Fix an irreducible C-class module $\mathbb{U} = \mathbb{B}_4, \mathbb{A}_2^{\mathrm{tr}}, \mathbb{A}_2^{\mathrm{tf}} \subset \mathbb{E}$ (see §3.3.4) and its corresponding C-class invariant $\mathcal{U} = \mathcal{B}_4, \mathcal{A}_2^{\mathrm{tr}}, \mathcal{A}_2^{\mathrm{tf}}$ (see §3.3.5). Let $C_{\mathcal{U}}$ denote the set of all ODEs (3.2) *with $\mathcal{U} \not\equiv 0$ and all remaining C-class invariants vanishing identically* (equivalently, $0 \not\equiv \mathrm{im}(\kappa_H) \subset \mathbb{U}$), and let $\mathfrak{S}_{\mathbb{U}}$ denote the largest realizable symmetry dimension among ODEs in $C_{\mathcal{U}}$. We will prove the following classification result:

THEOREM 3.2.2. Any *vector ODE (3.2) $\mathcal{E}$ of C-class of order $n+1 \geq 3$ in $C_{\mathcal{U}}$ realizing $\mathfrak{S}_{\mathbb{U}}$, near any point $x \in \mathcal{E}$ with $\mathcal{U}(x) \neq 0$, is* locally *(point) equivalent over $\mathbb{R}$ to exactly one of the ODEs given in Table 1. Over $\mathbb{C}$, the indicated 3rd order ODEs for $\mathbb{U} = \mathbb{B}_4$ are locally equivalent.*

[2] More precisely, "local equivalence" here is meant in a neighbourhood of a point in $\mathcal{E}$ where at least one of the C-class invariants is non-zero.

n	Irreducible C-class module $\mathbb{U} \subset \mathbb{E}$	$\mathfrak{S}_{\mathbb{U}}$	ODE with $0 \not\equiv \mathrm{im}(\kappa_H) \subset \mathbb{U}$ with symmetry dimension realizing $\mathfrak{S}_{\mathbb{U}}$
2	$\mathbb{B}_4$	$\mathfrak{M} - m$	$\underset{(1 \le a \le m)}{u_3^a = \dfrac{3u_2^1 u_2^a}{2u_1^1}}$ or $\underset{(1 \le a \le m)}{u_3^a = \dfrac{3u_1^1 u_2^1 u_2^a}{1 + (u_1^1)^2}}$
≥ 3	$\mathbb{A}_2^{\mathrm{tr}}$	$\mathfrak{M} - m - 1$	$\underset{(1 \le a \le m)}{u_{n+1}^a = \dfrac{(n+1)u_n^1 u_n^a}{n u_{n-1}^1}}$
≥ 2	$\mathbb{A}_2^{\mathrm{tf}}$	$\mathfrak{M} - 2m + 1 + \delta_2^n$	$\underset{(1 \le a \le m)}{u_{n+1}^a = (u_n^2)^2 \delta_1^a}$

(Recall $\mathfrak{M} = m^2 + (n+1)m + 3$ from (3.3).)

TABLE 1. Classification over $\mathbb{R}$ of submaximally symmetric vector ODEs of C-class of order $n + 1 \geq 3$

Our method for proving Theorems 3.2.1 and 3.2.2 will rely on the Cartan-geometric viewpoint for vector ODEs, and the associated computations will be efficiently done using representation theory. This will require important refinements to the existing structural results for vector ODEs of C-class stated in Table 2. Such refinements constitute our final main result, which we now briefly describe. In our non-parabolic ODE setting, the aforementioned algebraic Hodge theory establishes a G_0-equivariant identification of $H^2(\mathfrak{g}_-, \mathfrak{g})$ with the subspace $\ker \Box \subset \bigwedge^2 \mathfrak{g}_-^* \otimes \mathfrak{g}$ of harmonic 2-cochains (see §3.3.3). Analogous to Kostant's theorem [30], which is fundamental in the study of parabolic geometries, we may seek harmonic realizations of lowest weight vectors $\Phi_{\mathbb{U}} \in \mathbb{U}$ for each irreducible C-class submodule $\mathbb{U} \subset \mathbb{E} \subsetneq H^2(\mathfrak{g}_-, \mathfrak{g})$. Our Theorem 3.4.1 establishes such realizations (see Table 4). We anticipate that these structural results will be important for future geometric studies of the C-class and vector ODEs in general.

3.3 Cartan geometries and vector ODEs of C-class

In this section, we briefly review the Cartan-geometric reformulation for vector ODEs (3.2) of order ≥ 3 modulo point transformations, and summarize all relevant facts about vector ODEs of C-class.

3.3.1 ODE geometry and symmetry

We begin by summarizing [26, §2.1], which is based on [16], and refer the reader to these articles for more details. The $(n+1)$-st order ODE (3.2) defines a submanifold $\mathcal{E} = \{\mathbf{u}_{n+1} = \mathbf{f}\}$ of co-dimension $m \geq 2$ in the space of $(n+1)$-jets of functions $J^{n+1}(\mathbb{R}, \mathbb{R}^m)$ that is transverse to the projection $\pi^{n+1}_n : J^{n+1}(\mathbb{R}, \mathbb{R}^m) \to J^n(\mathbb{R}, \mathbb{R}^m)$. Let C denote the *Cartan distribution* on $J^{n+1}(\mathbb{R}, \mathbb{R}^m)$ with standard local coordinates $(t, \mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_{n+1})$, where $\mathbf{u}_r = (u^1_r, \ldots, u^m_r)$. Then C is given by

$$C = \operatorname{span}\left\{\partial_t + \mathbf{u}_1\partial_{\mathbf{u}_0} + \ldots + \mathbf{u}_{n+1}\partial_{\mathbf{u}_n}, \quad \partial_{\mathbf{u}_{n+1}}\right\}, \tag{3.4}$$

where $\mathbf{u}_i\partial_{\mathbf{u}_j} := \sum_{a=1}^m u^a_i \partial_{u^a_j}$ and $\partial_{\mathbf{u}_r}$ refers to $\partial_{u^1_r}, \ldots, \partial_{u^m_r}$. We also consider the restriction of C to $\mathcal{E}$ and abuse notation by also referring to this distribution as C.

We will consider ODEs (3.2) up to *point transformations*. These are diffeomorphisms $\Phi : J^{n+1}(\mathbb{R}, \mathbb{R}^m) \to J^{n+1}(\mathbb{R}, \mathbb{R}^m)$ that preserve C, i.e. $d\Phi(C) = C$. By the Lie–Bäcklund theorem, since $m \geq 2$, such transformations are the prolongations of diffeomorphisms on $J^0(\mathbb{R}, \mathbb{R}^m) \cong \mathbb{R} \times \mathbb{R}^m$. Infinitesimally, a *point vector field* is a vector field $\xi \in \mathfrak{X}(J^{n+1}(\mathbb{R}, \mathbb{R}^m))$ whose flow is a point transformation. Equivalently, $\mathcal{L}_\xi C \subset C$, where $\mathcal{L}_\xi$ is the Lie derivative with respect to ξ. A *point symmetry* of (3.2) is a point vector field that is tangent to $\mathcal{E}$.

The (point) geometry of $\mathcal{E}$ is encoded by a pair (E, V) of completely integrable sub-distributions of C on $\mathcal{E}$:

$$\begin{aligned} E &= \operatorname{span}\left\{\frac{d}{dt} := \partial_t + \mathbf{u}_1\partial_{\mathbf{u}_0} + \cdots + \mathbf{u}_n\partial_{\mathbf{u}_{n-1}} + \mathbf{f}\partial_{\mathbf{u}_n}\right\}, \\ V &= \operatorname{span}\left\{\partial_{\mathbf{u}_n}\right\}. \end{aligned} \tag{3.5}$$

(Note that integral curves of E are lifts of solution curves to (3.2).) Moreover, the distribution $D := E \oplus V \subset T\mathcal{E}$ is bracket-generating, and its weak-derived flag defines the following filtration on $T\mathcal{E}$:

$$T\mathcal{E} = D^{-n-1} \supset \cdots \supset D^{-2} \supset D^{-1} := D, \tag{3.6}$$

where $D^{-i-1} := D^{-i} + [D^{-i}, D^{-1}]$ for $i > 0$. Since $[\Gamma(D^j), \Gamma(D^k)] \subset \Gamma(D^{j+k})$, then the pair $(\mathcal{E}, \{D^i\})$ forms a *filtered manifold*. As we will describe below, this leads to the formulation of an ODE (3.2) as a filtered G_0-structure [5, §2.1].

Letting $T^i\mathcal{E} := D^i \subset T\mathcal{E}$ for $-n-1 \leq i \leq -1$ and $T^0\mathcal{E} := 0$, we define $\operatorname{gr}(T\mathcal{E}) := \bigoplus_{i=-n-1}^{-1} \operatorname{gr}_i(T\mathcal{E})$ where $\operatorname{gr}_i(T\mathcal{E}) := T^i\mathcal{E}/T^{i+1}\mathcal{E}$. Let $\operatorname{gr}_i(T_x\mathcal{E})$ denote the fiber of $\operatorname{gr}_i(T\mathcal{E})$ at $x \in \mathcal{E}$, i.e. $\mathfrak{m}_i(x) := \operatorname{gr}_i(T_x\mathcal{E}) = T^i_x\mathcal{E}/T^{i+1}_x\mathcal{E}$. Then $\mathfrak{m}(x) := \operatorname{gr}(T_x\mathcal{E}) = \bigoplus_{i=-n-1}^{-1} \mathfrak{m}_i(x)$ is a nilpotent graded Lie algebra (NGLA) under the (Levi) bracket induced by Lie bracket of vector fields. It is called the *symbol algebra* at x. Since the symbol algebras at all points are isomorphic, then we let $\mathfrak{m}$ denote a fixed NGLA with $\mathfrak{m} \cong \mathfrak{m}(x)$, $\forall x \in \mathcal{E}$, and we say that $(\mathcal{E}, \{D^i\})$ is *regular of type* $\mathfrak{m}$.

Let $\mathrm{Aut}_{\mathrm{gr}}(\mathfrak{m}) \leq \mathrm{GL}(\mathfrak{m})$ be the subgroup that preserves the grading of $\mathfrak{m}$. Since $\mathfrak{m}$ is generated by $\mathfrak{m}_{-1}$, then we have $\mathrm{Aut}_{\mathrm{gr}}(\mathfrak{m}) \hookrightarrow \mathrm{GL}(\mathfrak{m}_{-1})$. For $x \in \mathcal{E}$, we let $F_{\mathrm{gr}}(x)$ denote the set of all NGLA isomorphisms $\mathfrak{m} \to \mathfrak{m}(x)$. Then $F_{\mathrm{gr}}(\mathcal{E}) := \bigcup_{x \in \mathcal{E}} F_{\mathrm{gr}}(x)$ defines a principal fiber bundle $F_{\mathrm{gr}}(\mathcal{E}) \to \mathcal{E}$ with structure group $\mathrm{Aut}_{\mathrm{gr}}(\mathfrak{m})$, cf. [5, Prop 2.1]. The splitting of D implies a splitting of $\mathfrak{m}_{-1}$, and restricting to the subgroup $G_0 \leq \mathrm{Aut}_{\mathrm{gr}}(\mathfrak{m})$ that preserves the splitting yields a principal subbundle $\mathcal{G}_0 \to \mathcal{E}$ with reduced structure group $G_0 \cong \mathbb{R}^\times \times \mathrm{GL}_m$. By [5, Defn 2.2], an ODE (3.2) defines a *filtered G_0-structure.*

3.3.2 Structure underlying the trivial ODE

Let $n, m \geq 2$. The trivial ODE $\mathbf{u}_{n+1} = \mathbf{0}$ has point symmetry Lie algebra $\mathfrak{g}$ (see for example [6, §2.2] for explicit symmetry vector fields) with abstract structure given by

$$\mathfrak{g} \cong \mathfrak{q} \ltimes V, \qquad \mathfrak{q} := \mathfrak{sl}_2 \times \mathfrak{gl}(W), \qquad V := \mathbb{V}_n \otimes W, \qquad W := \mathbb{R}^m, \quad (3.7)$$

where $\mathbb{V}_n$ is the unique (up to isomorphism) $\mathfrak{sl}_2$-irrep of dimension $n+1$, and W is the standard rep of $\mathfrak{gl}(W)$. Here, V is taken to be an abelian subalgebra.

We now fix a basis for $\mathfrak{g}$. Let $\{w_a\}_{a=1}^m$ be the standard basis for W, and let $e_a{}^b$ be the $m \times m$ matrix such that $e_a{}^b w_c = \delta_c{}^b w_a$, so that $\{e_a{}^b\}_{a,b=1}^m$ spans $\mathfrak{gl}(W)$. Letting $\{x, y\}$ be the standard basis for $\mathbb{R}^2$, we identify $\mathbb{V}_n \cong S^n\mathbb{R}^2$. We obtain bases $\{E_i\}_{i=0}^n$ on $\mathbb{V}_n$ and $\{E_{i,a} : 0 \leq i \leq n,\ 1 \leq a \leq m\}$ on V via

$$E_i := \frac{x^i y^{n-i}}{(n-i)!}, \qquad E_{i,a} := E_i \otimes w_a. \quad (3.8)$$

(*For convenience, we define $E_i = 0$ for $i < 0$ or $i > n$. We also caution that our E_i corresponds to E_{n-i} in [26, §2.1.2].*) We complete our bases of V and $\mathfrak{gl}(W)$ to a basis of $\mathfrak{g}$ by introducing the standard $\mathfrak{sl}_2$-triple

$$\mathsf{X} := x\partial_y, \qquad \mathsf{H} := x\partial_x - y\partial_y, \qquad \mathsf{Y} := y\partial_x. \quad (3.9)$$

Note that $\mathfrak{sl}_2$ commutes with $\mathfrak{gl}(W)$, and the $\mathfrak{sl}_2$-actions on $\mathbb{V}_n$ and V are naturally induced, e.g.

$$\begin{aligned} &[\mathsf{X}, E_i] = E_{i+1}, \qquad [\mathsf{H}, E_i] = (2i-n)E_i, \\ &[\mathsf{Y}, E_i] = i(n+1-i)E_{i-1}. \end{aligned} \quad (3.10)$$

In particular, E_i and $E_{i,a}$ are weight vectors for the $\mathfrak{sl}_2$-action, i.e. eigenvectors with respect to H.

Now endow $\mathfrak{g}$ with a bi-grading as in [26, §3.1]. Letting $\mathrm{id}_m := \sum_{a=1}^m e_a{}^a$, define $\mathsf{Z}_1, \mathsf{Z}_2 \in \mathfrak{g}$ by

$$\mathsf{Z}_1 := -\frac{1}{2}(\mathsf{H} + n\,\mathrm{id}_m), \qquad \mathsf{Z}_2 := -\mathrm{id}_m\,. \quad (3.11)$$

Then $\mathfrak{g}$ decomposes into the joint eigenspaces of $\mathrm{ad}_{\mathsf{Z}_1}$ and $\mathrm{ad}_{\mathsf{Z}_2}$. We write

$$\mathfrak{g}_{s,t} := \{x \in \mathfrak{g} \,:\, \mathsf{Z}_1 \cdot x = sx,\ \mathsf{Z}_2 \cdot x = tx\}, \quad (3.12)$$

and refer to s and t as the Z_1-*degree* and Z_2-*degree* of x, respectively. The ordered pair $(s,t) \in \mathbb{Z} \times \mathbb{Z}$ is the *bi-grade* of x. It is helpful to picture $\mathfrak{g}$ as in Figure 1.

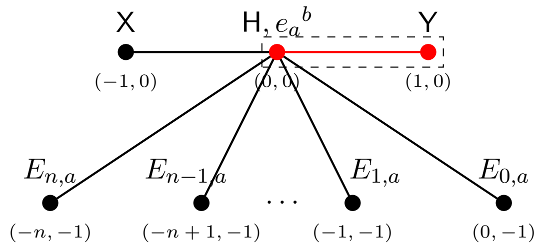

FIGURE 1. Bi-grading on $\mathfrak{g}$

Defining the *grading element* $\mathsf{Z} \in \mathfrak{z}(\mathfrak{g}_{0,0})$, we similarly induce the structure of a $\mathbb{Z}$-grading on $\mathfrak{g}$ via

$$\mathsf{Z} := \mathsf{Z}_1 + \mathsf{Z}_2 = -\frac{1}{2}\left(\mathsf{H} + (n+2)\,\mathrm{id}_m\right). \tag{3.13}$$

Then we have the decomposition $\mathfrak{g} = \mathfrak{g}_{-n-1} \oplus \ldots \oplus \mathfrak{g}_1$, where

$$\begin{aligned} \mathfrak{g}_1 &:= \mathfrak{g}_{1,0} = \mathbb{R}\mathsf{Y}, \\ \mathfrak{g}_0 &:= \mathfrak{g}_{0,0} = \mathbb{R}\mathsf{H} \oplus \mathfrak{gl}_m, \\ \mathfrak{g}_{-1} &:= \mathfrak{g}_{-1,0} \oplus \mathfrak{g}_{0,-1} = \mathbb{R}\mathsf{X} \oplus (\mathbb{R}E_0 \otimes W), \\ \mathfrak{g}_i &:= \mathfrak{g}_{i+1,-1} = \mathbb{R}E_{i+1} \otimes W, \qquad i = -2, \ldots, -n-1. \end{aligned} \tag{3.14}$$

We note that $\mathfrak{g}_- := \mathfrak{g}_{-n-1} \oplus \ldots \oplus \mathfrak{g}_{-1} \subset \mathfrak{g}$ is generated by $\mathfrak{g}_{-1}$.

We also endow $\mathfrak{g}$ with the *canonical filtration* $\mathfrak{g}^i := \sum_{j \geq i} \mathfrak{g}_j$, which turns $\mathfrak{g}$ into a filtered Lie algebra. Its associated graded $\mathrm{gr}(\mathfrak{g}) := \bigoplus_{k \in \mathbb{Z}} \mathrm{gr}_k(\mathfrak{g})$, where $\mathrm{gr}_k(\mathfrak{g}) = \mathfrak{g}^k/\mathfrak{g}^{k+1}$, is isomorphic to $\mathfrak{g}$ as graded Lie algebras. Using the isomorphism, we let $\mathrm{gr}_k : \mathfrak{g}^k \to \mathfrak{g}_k$ denote the *leading part*. Explicitly, if $x \in \mathfrak{g}^k$ with $x = x_k + x_{k+1} + \ldots$, where $x_j \in \mathfrak{g}_j$, then $\mathrm{gr}_k(x) := x_k$. The following notations will be convenient:

$$\mathfrak{p} := \mathfrak{g}^0 = \mathfrak{g}_0 \oplus \mathfrak{g}_1, \qquad \mathfrak{p}_+ := \mathfrak{g}^1 = \mathfrak{g}_1. \tag{3.15}$$

At the group level, let

$$\begin{aligned} &G := (\mathrm{SL}_2 \times \mathrm{GL}_m) \ltimes V, \qquad P := \mathrm{ST}_2 \times \mathrm{GL}_m, \\ &G_0 := \{g \in P : \mathrm{Ad}_g(\mathfrak{g}_0) \subset \mathfrak{g}_0\}, \end{aligned} \tag{3.16}$$

where $\mathrm{ST}_2 \subset \mathrm{SL}_2$ is the subgroup of lower triangular matrices. (Note that G_0 is isomorphic to that given in §3.3.1.) We also let $P_+ \subset P$ denote the connected Lie subgroup corresponding to $\mathfrak{p}_+ \subset \mathfrak{p}$. We remark that the canonical filtration on $\mathfrak{g}$ is P-invariant.

3.3.3 Cartan geometries associated to ODE

Fix G, P and G_0 as above. Recall also from §3.3.1 that all vector ODEs (3.2) can be formulated as filtered G_0-structures. Importantly, there is an equivalence of categories between filtered G_0-structures on $\mathcal{E}$ (which is a wider category than that arising from ODE – see below) and regular, normal Cartan geometries $(\mathcal{G} \to \mathcal{E}, \omega)$ of type (G, P) [6, 16]. A Cartan geometry consists of a (right) principal P-bundle $\mathcal{G} \to \mathcal{E}$ endowed with a *Cartan connection* ω, i.e. $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ is a $\mathfrak{g}$-valued 1-form on $\mathcal{G}$ such that:

(a) For any $u \in \mathcal{G}$, $\omega_u : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism;
(b) $R_g^*\omega = \mathrm{Ad}_{g^{-1}} \circ \omega$ for any $g \in P$, i.e. ω is P-equivariant;
(c) $\omega(\zeta_A) = A$, where $A \in \mathfrak{p}$, where ζ_A is the fundamental vertical vector field defined by $\zeta_A(u) := \frac{d}{dt}\big|_{t=0} u \cdot \exp(tA)$.

The *curvature* $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ of the geometry is given by $K(\xi, \eta) = d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)]$, which is P-equivariant and horizontal, i.e. $K(\zeta_A, \cdot) = 0$, $\forall A \in \mathfrak{p}$. Consequently, K is determined by the P-equivariant *curvature function* $\kappa : \mathcal{G} \to \bigwedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$, defined by $\kappa(u)(A, B) = K(\omega^{-1}(A), \omega^{-1}(B))(u),\ A, B \in \mathfrak{g}$. Letting ω_G be the Maurer–Cartan form on G, the *Klein geometry* $(G \to G/P, \omega_G)$ satisfies $K \equiv 0$ (Maurer–Cartan equation), and is the *flat model* for all Cartan geometries of type (G, P).

In terms of the canonical filtration $\{\mathfrak{g}^i\}$ on $\mathfrak{g}$ from §3.3.2, ω is said to be *regular* if $\kappa(\mathfrak{g}^i, \mathfrak{g}^j) \subset \mathfrak{g}^{i+j+1}$ $\forall i, j$. Importantly, it is known that for all filtered G_0-structures *arising from ODE*, the corresponding Cartan geometry has κ satisfying the *strong regularity* condition [6, Rem 2.3]:

$$\kappa(\mathfrak{g}^i, \mathfrak{g}^j) \subset \mathfrak{g}^{i+j+1} \cap \mathfrak{g}^{\min(i,j)-1}, \quad \forall i, j. \tag{3.17}$$

To define normality, we first fix an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ in terms of the basis introduced in §3.3.2:

DEFINITION 3.3.1. *Let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathfrak{g}$ such that $\{\mathsf{X}, \mathsf{H}, \mathsf{Y}, e_a{}^b, E_{i,a}\}$ is an orthogonal basis for $\mathfrak{g}$ with squared lengths of basis elements given below:*

$$\langle \mathsf{X}, \mathsf{X} \rangle = \langle \mathsf{Y}, \mathsf{Y} \rangle = 1, \quad \langle \mathsf{H}, \mathsf{H} \rangle = 2, \quad \langle e_a{}^b, e_a{}^b \rangle = 1, \quad \langle E_{i,a}, E_{i,a} \rangle = \frac{i!}{(n-i)!}. \tag{3.18}$$

Then $\forall A, B \in \mathfrak{q} = \mathfrak{sl}_2 \times \mathfrak{gl}_m$ and $\forall u, v \in V$, we have $\langle A, B \rangle = \mathrm{tr}(A^T B)$ and $\langle Au, v \rangle = \langle u, A^T v \rangle$.

Consider $C^k(\mathfrak{g}, \mathfrak{g}) := \bigwedge^k \mathfrak{g}^* \otimes \mathfrak{g}$ equipped with the induced canonical filtration from $\mathfrak{g}$ and let $\partial_\mathfrak{g}$ be the standard differential of the complex for computing Lie algebra cohomology groups $H^k(\mathfrak{g}, \mathfrak{g})$. Then, define the *codifferential* $\partial^* : C^k(\mathfrak{g}, \mathfrak{g}) \to C^{k-1}(\mathfrak{g}, \mathfrak{g})$ to be the adjoint of $\partial_\mathfrak{g}$ with respect to the induced inner product from $\mathfrak{g}$, i.e. for each k we have $\langle \partial_\mathfrak{g}\phi, \psi \rangle = \langle \phi, \partial^*\psi \rangle$ for all

$\phi \in C^{k-1}(\mathfrak{g}, \mathfrak{g})$ and $\psi \in C^k(\mathfrak{g}, \mathfrak{g})$. By [6, Lemma 3.2], the codifferential descends to a P-equivariant map $\partial^* : \bigwedge^k (\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g} \to \bigwedge^{k-1}(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$. A Cartan connection ω has curvature function κ valued in $\bigwedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$, and ω is said to be *normal* if $\partial^* \kappa = 0$. *In this article, we will always work with Cartan geometries of type (G, P) that are normal and strongly regular.*

Since $(\partial^*)^2 = 0$, then the (normal) curvature κ quotients to a P-equivariant function $\kappa_H : \mathcal{G} \to \frac{\ker \partial^*}{\operatorname{im} \partial^*}$ called the *harmonic curvature*. By regularity, κ_H is valued in the filtrand of positive degree of the P-module $\frac{\ker \partial^*}{\operatorname{im} \partial^*}$, which by [6, Corollary 3.8] is completely reducible, i.e. P_+ acts on it trivially, and therefore only the G_0-action is relevant. It is well-known (see Theorem 3.8.3 and references therein) that κ_H completely obstructs local flatness, i.e. $\kappa_H \equiv 0 \iff \kappa = 0$.

Identify $\bigwedge^k(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g} \cong \bigwedge^k \mathfrak{g}_-^* \otimes \mathfrak{g}$ as G_0-modules, and recall from §3.3.2 that $\mathfrak{g} \cong \mathfrak{q} \ltimes V$. Given $\phi \in C^k(\mathfrak{g}_-, \mathfrak{g}) := \bigwedge^k \mathfrak{g}_-^* \otimes \mathfrak{g}$, then we have $\phi = \mathsf{X}^* \wedge \phi_1 + \phi_2$, for $\phi_1 \in C^{k-1}(V, \mathfrak{g})$ and $\phi_2 \in C^k(V, \mathfrak{g})$, and where X^* is dual to X. Denoting $\phi := \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$, then $\partial\phi$ is given by [6, Lem 3.4]:

$$\partial \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} -\partial_V \phi_1 + \mathsf{X} \cdot \phi_2 \\ \partial_V \phi_2 \end{pmatrix}, \tag{3.19}$$

where

$$\partial_V \phi_2(x_0, \dots, x_k) = \sum_{i=0}^{k} (-1)^i x_i \cdot \phi_2(x_0, \dots, \widehat{x}_i, \dots, x_k), \tag{3.20}$$

for $x_0, \dots, x_k \in V$, and letting $\widehat{x}_i$ denote omission of x_i. A direct consequence of (3.19) is:

LEMMA 3.3.2. *Let $\phi \in \bigwedge^k V^* \otimes \mathfrak{g}$. Then $\partial\phi = 0$ if and only if $\mathsf{X} \cdot \phi = 0$ and $\partial_V \phi = 0$. Moreover, if in fact $\phi \in \bigwedge^k V^* \otimes V$, then $\partial\phi = 0$ if and only if $\mathsf{X} \cdot \phi = 0$.*

Defining $\Box := \partial \circ \partial^* + \partial^* \circ \partial : \bigwedge^k \mathfrak{g}_-^* \otimes \mathfrak{g} \to \bigwedge^k \mathfrak{g}_-^* \otimes \mathfrak{g}$, we then have the following G_0-isomorphisms,

$$\begin{aligned} &\bigwedge\nolimits^k \mathfrak{g}_-^* \otimes \mathfrak{g} \cong \overbrace{\operatorname{im} \partial^* \oplus \underbrace{\ker \Box}_{} }^{\ker \partial^*} \!\!\!\!\!\!\!\!\!\! \underbrace{ \oplus \operatorname{im} \partial}_{\ker \partial}, \\ &\ker \Box \cong \frac{\ker \partial^*}{\operatorname{im} \partial^*} \cong \frac{\ker \partial}{\operatorname{im} \partial} =: H^k(\mathfrak{g}_-, \mathfrak{g}). \end{aligned} \tag{3.21}$$

Consequently, for a regular, normal Cartan geometry, the codomain of κ_H can be identified with the subspace $H^2_+(\mathfrak{g}_-, \mathfrak{g}) \subset H^2(\mathfrak{g}_-, \mathfrak{g})$ on which the grading element $\mathsf{Z} = \mathsf{Z}_1 + \mathsf{Z}_2$ acts with positive eigenvalues. However, it should be emphasized that only part of $H^2_+(\mathfrak{g}_-, \mathfrak{g})$ is in fact realizable for geometries associated to ODE [17, 41]. Correspondingly, we define:

DEFINITION 3.3.3. *The effective part* $\mathbb{E} \subsetneq H^2_+(\mathfrak{g}_-, \mathfrak{g})$ *is the minimal* G_0*-module in which* κ_H *is valued, for any (strongly) regular, normal Cartan geometry of type* (G, P) *associated to an ODE (3.2) (for fixed* (n, m)*).*

3.3.4 Vector ODEs of C-class

We will focus on ODEs (3.2) of C-class, which have been characterized in [6] using curvatures κ of corresponding canonical Cartan connections ω described above. We define [6, Defn 2.4]:

DEFINITION 3.3.4. *An ODE (3.2) is said to be of C-class if the curvature* κ *of the corresponding strongly regular, normal Cartan geometry satisfies* $\kappa(\mathsf{X}, \cdot) = 0$*, where* $\mathsf{X} \in \mathfrak{g}_{-1}$ *was defined in §3.3.2.*

REMARK 6. Recall from §3.3.2 that $\mathfrak{g} \cong \mathfrak{q} \ltimes V$. We remark that for a Cartan geometry corresponding to an ODE of C-class, we can identify $\kappa \in \bigwedge^2(\mathfrak{g}/\mathfrak{q})^* \otimes \mathfrak{g} \cong \bigwedge^2 V^* \otimes \mathfrak{g}$.

As shown in [6], the notion of C-class can be concretely reformulated in terms of fundamental invariants for vector ODEs (3.2) of order ≥ 3 described below, which comprise the harmonic curvature of the geometry. We then have the following characterization of the C-class given in [6, Thms 4.1 & 4.2]:

THEOREM 3.3.5. *A vector ODE (3.2) of order* ≥ 3 *is of C-class if and only if all of its generalized Wilczynski invariants vanish.*

For concreteness, we now explicitly describe the fundamental invariants for vector ODEs (3.2) of order $n + 1 \geq 3$ consisting of *generalized Wilczynski invariants* $\mathcal{W}_r$ [14] and *C-class invariants* [17, 41, 42]:

- Consider a linear vector ODE of order $n + 1$:

  $$\mathbf{u}_{n+1} + P_n(t)\mathbf{u}_n + \ldots + P_1(t)\mathbf{u}_1 + P_0(t)\mathbf{u} = 0, \tag{3.22}$$

  where $P_j(t)$ is an $\mathrm{End}(\mathbb{R}^m)$-valued function. Using the invertible transformations $(t, \mathbf{u}) \mapsto (f(t), h(t)\mathbf{u})$ where $f : \mathbb{R} \to \mathbb{R}^\times$ and $h : \mathbb{R} \to \mathrm{GL}(m)$, which preserve the form of equation (3.22), we may normalize to $P_n = 0$ and $\mathrm{tr}(P_{n-1}) = 0$, i.e. *Laguerre–Forsyth canonical form*. Then

  $$\Theta_r = \sum_{k=1}^{r-1} (-1)^{k+1} \frac{(2r - k - 1)!(n - r + k)!}{(r - k)!(k - 1)!} P^{(k-1)}_{n-r+k}, \quad r = 2, \ldots, n + 1, \tag{3.23}$$

  are fundamental invariants found by Se-ashi [1], and r is the degree of the invariant. For (3.2), the generalized Wilczynski invariants $\mathcal{W}_r$ (for $r = 2, \ldots, n + 1$) are defined as Θ_r above evaluated at its linearization along a solution $\mathbf{u}$. Formally, $\mathcal{W}_r$ are obtained from (3.22) by replacing $P_r(t)$

by the matrices $-\left(\frac{\partial f^a}{\partial u^b_r}\right)$ and the usual derivative by the total derivative $\frac{d}{dt}$ given in (3.5). Moreover, $\mathcal{W}_r$ do not depend on the choice of solution $\mathbf{u}$, and are therefore contact invariants.

- *C-class invariants* are the following:

$$
\begin{aligned}
n \geq 2: \quad & (\mathcal{A}_2^{\mathrm{tf}})^a_{bc} = \mathrm{tf}\left(\frac{\partial^2 f^a}{\partial u^b_n\, \partial u^c_n}\right), \\
n \geq 3: \quad & (\mathcal{A}_2^{\mathrm{tr}})^a_{bc} = \mathrm{tr}\left(\frac{\partial^2 f^a}{\partial u^b_n\, \partial u^c_n}\right), \\
n = 2: \quad & (\mathcal{B}_4)_{bc} = -\frac{\partial H_c^{-1}}{\partial u_1^b} + \frac{\partial}{\partial u_2^b}\frac{\partial}{\partial u_2^c} H^t - \frac{\partial}{\partial u_2^c}\frac{d}{dt} H_b^{-1} \\
& \qquad - \frac{\partial}{\partial u_2^c}\left(\sum_{a=1}^m H_a^{-1}\frac{\partial f^a}{\partial u_2^b}\right) + 2H_b^{-1}H_c^{-1},
\end{aligned}
\tag{3.24}
$$

where

$$
\begin{aligned}
H_b^{-1} &= \frac{1}{6(m+1)}\sum_{a=1}^m \frac{\partial^2 f^a}{\partial u_2^a\, \partial u_2^b}, \\
H^t &= -\frac{1}{4m}\sum_{a=1}^m \left(\frac{\partial f^a}{\partial u_1^a} - \frac{d}{dt}\frac{\partial f^a}{\partial u_2^a} + \frac{1}{3}\sum_{c=1}^m \frac{\partial f^a}{\partial u_2^c}\frac{\partial f^c}{\partial u_2^a}\right).
\end{aligned}
\tag{3.25}
$$

3.3.5 C-class modules

The above fundamental invariants correspond to G_0-irreducible submodules in the effective part $\mathbb{E} \subsetneq H^2_+(\mathfrak{g}_-, \mathfrak{g})$ (Definition 3.3.3), which we now describe. Recall that $\mathfrak{g}_0 \cong \mathrm{span}\{\mathsf{Z}_1, \mathsf{Z}_2\} \oplus \mathfrak{sl}(W)$, and we have the induced action of Z_1 and Z_2 from (3.11) on $H^2_+(\mathfrak{g}_-, \mathfrak{g})$, and therefore on $\mathbb{E}$. Note that Z_2 acts with degrees $0, 1$ or 2. We define:

DEFINITION 3.3.6. *A G_0-submodule $\mathbb{U} \subset \mathbb{E} \subsetneq H^2_+(\mathfrak{g}_-, \mathfrak{g})$ on which Z_2 acts with positive degree(s) is called a* C-class module*, and we let $\mathbb{E}_C \subsetneq \mathbb{E}$ denote the direct sum of all irreducible C-class modules. On the other hand, if Z_2 acts on $\mathbb{U}$ with zero degree, we refer to $\mathbb{U}$ as a* Wilczynski module*.*

Any $\mathfrak{g}_0$-irrep $\mathbb{U} \subset \mathbb{E}$ is determined by its bi-grade and its lowest weight λ with respect to $\mathfrak{sl}(W) \cong \mathfrak{sl}_m$. Such λ can be expressed in terms of the fundamental weights $\lambda_1, \ldots, \lambda_{m-1}$ of $\mathfrak{sl}_m$ with respect to the Cartan subalgebra $\mathfrak{h}$ consisting of diagonal matrices in $\mathfrak{sl}_m$, and the standard choice of $m-1$ simple roots. Letting $h = \mathrm{diag}(h_1, \ldots, h_m) \in \mathfrak{h}$ and $\epsilon_a : \mathfrak{h} \to \mathbb{R}$ the linear functional $\epsilon_a(h) = h_a$, we then have $\epsilon_1 + ... + \epsilon_m = 0$ and $\lambda_i = \epsilon_1 + \ldots + \epsilon_i$ for $1 \leq i \leq m-1$.

Table 2 contains a summary of results for $\mathbb{E}_C \subsetneq \mathbb{E}$ for ODEs (3.2), due to Medvedev [41, 42] for order 3 and Doubrov–Medvedev [17] for order ≥ 4. Using the G_0-isomorphisms (3.21), we identify each irreducible C-class module $\mathbb{U} \subset \mathbb{E}_C$

from Table 2 with the corresponding module in $\ker \Box \subset C^2(\mathfrak{g}_-, \mathfrak{g})$ consisting of harmonic 2-cochains satisfying the strong regularity condition (3.17). (A further condition is formulated in §3.3.6 below.) Adopting the same notation from [26, Table 6], we let $\mathbb{A}_2$ and $\mathbb{B}_4$ denote the C-class submodules with bi-grades $(1,1)$ and $(2,2)$ respectively. From the respective Z_2-degrees, and since $\kappa \in \bigwedge^2 V^* \otimes \mathfrak{g}$ for C-class ODE, then we deduce that we may identify

$$\mathbb{A}_2 \subset \bigwedge^2 V^* \otimes V, \qquad \mathbb{B}_4 \subset \bigwedge^2 V^* \otimes \mathfrak{q}. \tag{3.26}$$

Since $\mathbb{A}_2$ is not irreducible, we decompose it into (irreducible) trace and trace-free parts: $\mathbb{A}_2 = \mathbb{A}_2^{\rm tr} \oplus \mathbb{A}_2^{\rm tf}$. (The C-class invariants $\mathcal{B}_4, \mathcal{A}_2^{\rm tr}, \mathcal{A}_2^{\rm tf}$ from §3.3.4 are valued in the corresponding irreducible C-class modules $\mathbb{B}_4, \mathbb{A}_2^{\rm tr}, \mathbb{A}_2^{\rm tf}$ respectively.)

n	Irreducible C-class module $\mathbb{U} \subset \mathbb{E}_C$	Bi-grade	$\mathfrak{sl}(W)$-module structure	$\mathfrak{sl}(W)$-lowest weight λ
2	$\mathbb{B}_4$	$(2,2)$	S^2W^*	$-2\epsilon_1$
≥ 3	$\mathbb{A}_2^{\rm tr}$	$(1,1)$	W^*	$-\epsilon_1$
≥ 2	$\mathbb{A}_2^{\rm tf}$	$(1,1)$	$(S^2W^* \otimes W)_0$	$\epsilon_m - 2\epsilon_1$

TABLE 2. C-class modules in $\mathbb{E}_C \subsetneq \mathbb{E} \subsetneq H^2_+(\mathfrak{g}_-, \mathfrak{g})$ for vector ODEs of order $n+1 \geq 3$

Since each $\mathbb{U}$ is a $\mathfrak{g}_0$-irrep, then up to scale $\mathbb{U}$ contains a unique *lowest weight vector* $\Phi_{\mathbb{U}}$. Since $\mathfrak{g}_0 \cong \mathrm{span}\{\mathsf{Z}_1, \mathsf{Z}_2\} \oplus \mathfrak{sl}(W)$, then being "lowest" means that $\Phi_{\mathbb{U}}$ is annihilated by all lowering operators, i.e. strictly lower triangular matrices, in $\mathfrak{sl}(W) \cong \mathfrak{sl}_m$. From Table 2, we can give an explicit description of the annihilators $\mathfrak{ann}(\Phi_{\mathbb{U}})$, which will be needed later. Namely, if $\tilde{\mathfrak{p}} \subset \mathfrak{sl}_m$ is the parabolic subalgebra preserving $\Phi_{\mathbb{U}}$ up to scale, then $\mathfrak{ann}(\Phi_{\mathbb{U}}) \subset \mathrm{span}\{\mathsf{Z}_1, \mathsf{Z}_2\} \oplus \tilde{\mathfrak{p}}$. For $a \neq c$, if ${e_a}^c \in \tilde{\mathfrak{p}}$, then ${e_a}^c \in \mathfrak{ann}(\Phi_{\mathbb{U}})$. It suffices to consider linear combinations of Z_1, Z_2, and diagonal elements $\mathfrak{h} \subset \tilde{\mathfrak{p}}$. If $\Phi_{\mathbb{U}}$ has $\mathfrak{sl}_m$-weight λ and Z_2-degree t, then we conclude that $\mathfrak{ann}(\Phi_{\mathbb{U}})$ is spanned by

$$\mathsf{Z}_1 - \mathsf{Z}_2, \qquad h - \frac{\lambda(h)}{t}\mathsf{Z}_2 \ \ (h \in \mathfrak{h}), \qquad {e_a}^c \in \tilde{\mathfrak{p}} \ \ (a \neq c), \tag{3.27}$$

where $\mathsf{Z}_1 - \mathsf{Z}_2 \in \mathfrak{ann}(\Phi_{\mathbb{U}})$ because of the bi-grading of $\mathbb{U}$. Applying (3.27) to (λ, t) from Table 2, we obtain Table 3. Here, $\tilde{\mathfrak{p}}_1$, $\tilde{\mathfrak{p}}_{1,m-1}$ are the parabolic subalgebras in $\mathfrak{sl}_m$ consisting of block *lower* triangular matrices with diagonal blocks of sizes $1, m-1$ and $1, m-2, 1$ respectively.

3.3.6 The Doubrov–Medvedev condition

We will be able to precisely identify $\mathbb{A}_2$ with the help of an additional linear condition formulated in [17, §3.1, Prop.4], and which we now summarize. Consider the $\mathfrak{p}$-invariant subspace $F = \mathrm{span}\{E_0, \ldots, E_{n-1}\} \otimes W \subset V$, and define

n	$\mathbb{U}$	$\dim \mathfrak{ann}(\Phi_{\mathbb{U}})$	Generators for $\mathfrak{ann}(\Phi_{\mathbb{U}}) \subset \mathfrak{g}_0$
2 ≥ 3	$\mathbb{B}_4$ $\mathbb{A}_2^{\rm tr}$	$m^2 - m + 1$	$\mathsf{Z}_1 - \mathsf{Z}_2,\ e_a{}^c \in \tilde{\mathfrak{p}}_1\ (a \neq c),$ $e_b{}^b - e_{b+1}{}^{b+1} + \delta_1{}^b \mathsf{Z}_2\ (1 \leq b \leq m-1)$
≥ 2	$\mathbb{A}_2^{\rm tf}$	$m^2 - 2m + 3$	$\mathsf{Z}_1 - \mathsf{Z}_2,\ e_a{}^c \in \tilde{\mathfrak{p}}_{1,m-1}\ (a \neq c),$ $e_b{}^b - e_{b+1}{}^{b+1} + (2\delta_1{}^b + \delta_{m-1}{}^b)\mathsf{Z}_2,$ $(1 \leq b \leq m-1)$

TABLE 3. $\mathfrak{ann}(\Phi_{\mathbb{U}}) \subset \mathfrak{g}_0$ for irreducible C-class modules $\mathbb{U} \subset \mathbb{E}$

$\delta : \mathrm{Hom}(F, \mathbb{R}\mathsf{X}) \to \mathrm{Hom}\left(\bigwedge^2 F, V/F\right)$ by

$$(\delta B)(x,y) = (B(x)\cdot y - B(y)\cdot x) \mod F, \quad \forall B \in \mathrm{Hom}(F, \mathbb{R}\mathsf{X}). \tag{3.28}$$

We have the inclusion $\iota_F : F \to V$, which induces $V/F \cong W$ (as $\mathfrak{p}$-modules) and natural quotient $\pi_W : V \to V/F$. Also induced is the inclusion $\iota_{\bigwedge^2 F} : \bigwedge^2 F \to \bigwedge^2 V$, from which we define $\vartheta : \mathrm{Hom}\left(\bigwedge^2 V, V\right) \to \mathrm{Hom}\left(\bigwedge^2 F, V/F\right)$ by $\vartheta = \pi_W \circ \iota^*_{\bigwedge^2 F}$, i.e.

$$\vartheta(A) = A|_{\bigwedge^2 F} \mod F. \tag{3.29}$$

From [17, §3.1, Prop 4] and Remark 6, we deduce that for a C-class ODE of order ≥ 4, the $\mathbb{A}_2$-component $\mathcal{A}_2$ of its harmonic curvature κ_H satisfies $\vartheta(\mathcal{A}_2) \in \mathrm{im}(\delta)$, which we refer to as the *Doubrov–Medvedev condition*. Correspondingly, for $n \geq 3$ we formulate the algebraic condition

$$\vartheta(A) \in \mathrm{im}(\delta), \quad \forall A \in \mathbb{A}_2, \tag{3.30}$$

which we refer to as the *DM condition*. (This condition is not present for 3rd order ODE.)

3.4 Lowest weight vectors for irreducible C-class modules

The $\mathfrak{g}_0$-module structure for irreducible C-class modules $\mathbb{U} \subset \mathbb{E}$ was stated in Table 2. While this abstract structural information proved useful in our previous study of symmetry gaps [26], more precise information is needed in our current study. Namely, viewing $\mathbb{U}$ as harmonic 2-cochains via the G_0-equivariant identification (3.21), we may ask for concrete realizations of lowest weight vectors $\Phi_{\mathbb{U}} \in \mathbb{U}$ (from which a full basis of $\mathbb{U}$ may be obtained by applying raising operators). These realizations are not found in the existing literature, and our main goal in this section is to provide them. This information will provide the starting point in subsequent sections for our classification of submaximally symmetric structures.

Given the notation introduced in §3.3.2, and letting $E^{i,a}$ denote the dual basis elements to $E_{i,a}$, we have:

THEOREM 3.4.1. *Fix $n, m \geq 2$ and an irreducible C-class module $\mathbb{U} \subset \mathbb{E}$, viewed as a G_0-submodule of $\ker \Box \subset C^2(\mathfrak{g}_-, \mathfrak{g})$ via (3.21). Then the unique lowest weight vector $\Phi_{\mathbb{U}} \in \mathbb{U}$, up to a scaling, is given in Table 4.*

n	$\mathbb{U}$	Lowest weight vector $\Phi_{\mathbb{U}} \in \mathbb{U}$
2	$\mathbb{B}_4$	$E^{2,1} \wedge E^{1,1} \otimes \mathsf{X} - \frac{1}{2} E^{2,1} \wedge E^{0,1} \otimes \mathsf{H} - \frac{1}{2} E^{1,1} \wedge E^{0,1} \otimes \mathsf{Y}$ $+ \sum_{a=1}^{m} \left(E^{2,1} \wedge E^{0,a} - E^{1,1} \wedge E^{1,a} + E^{0,1} \wedge E^{2,a} \right) \otimes e_a{}^1$
≥ 3	$\mathbb{A}_2^{\mathrm{tr}}$	$\alpha \sum_{i=0}^{n} \left[\Phi^{2,i} + (\frac{n}{2} - i)\Phi^{1,i} - \frac{1}{2} i(n+1-i)\Phi^{0,i} \right]$ $+\beta \sum_{i=0}^{n} \left[(n+1-i)(\Phi^{i,0} - \Phi^{0,i}) + \Phi^{i,1} - \Phi^{1,i} \right],$ where $\Phi^{i,j} := \sum_{a=1}^{m} E^{i,1} \wedge E^{j,a} \otimes E_{i+j-1,a}$ and $\alpha = \frac{-6(n-1)(m+1)}{mn(n+1)+6} \beta$
≥ 2	$\mathbb{A}_2^{\mathrm{tf}}$	$\sum_{j=0}^{n} [(n+1-j)\Phi^{0,j} + \Phi^{1,j}],$ where $\Phi^{i,j} := E^{i,1} \wedge E^{j,1} \otimes E_{i+j-1,m}$

TABLE 4. Classification of lowest weight vectors $\Phi_{\mathbb{U}}$ for irreducible C-class modules $\mathbb{U} \subset \mathbb{E}$

Let us give a brief summary of the computations to follow. For $\Phi_{\mathbb{U}} \in \mathbb{U}$ lying in the appropriate module given in (3.26), we use the bi-grade and $\mathfrak{sl}(W)$-lowest weight data for $\mathbb{U}$ from Table 2 to first write a general form for $\Phi_{\mathbb{U}}$. (The reader should recall the bi-grades given in §3.3.2, e.g. $E_{i,a}$ has bi-grade $(-i, -1)$, and so $E_{i,a} \in \mathfrak{g}_{-i-1} \subset \mathfrak{g}^{-i-1}$.) We then further constrain this form by imposing additional linear conditions coming from harmonicity, strong regularity, and the DM condition (3.30). (For example, since $\Phi_{\mathbb{U}} \in \bigwedge^2 V^* \otimes V$ in the $\mathbb{A}_2^{\mathrm{tr}}, \mathbb{A}_2^{\mathrm{tf}}$ cases, then $\partial \Phi_{\mathbb{U}} = 0$ if and only if $\mathsf{X} \cdot \Phi_{\mathbb{U}} = 0$ by Lemma 3.3.2. Imposing X-annihilation will be a detailed calculation involving the relations $\mathsf{X} \cdot E_{i,a} = E_{i+1,a}$ and $\mathsf{X} \cdot E^{i,a} = -E^{i-1,a}$.) This calculation will be involved, but we remark that in fact not all such conditions will need to be explicitly imposed:

REMARK 7. If $\Phi_{\mathbb{U}}$ can be constrained to a 1-dimensional subspace by imposing some of the conditions above, then $\Phi_{\mathbb{U}}$ necessarily satisfies all the remaining linear conditions (harmonicity, strong regularity, and (3.30)). This follows from *existence* of the module $\mathbb{U} \subset \mathbb{E}$ for ODE systems, which was established in [17, 41].

Let us now carry out the indicated computations and establish Theorem 3.4.1 above.

3.4.1 $\mathbb{B}_4$ case

Since $\mathbb{U} := \mathbb{B}_4 \subset \bigwedge^2 V^* \otimes \mathfrak{q}$ has bi-grade $(2,2)$, then $\Phi_{\mathbb{U}}$ must be a linear combination of:

$$E^{2,a} \wedge E^{1,b} \otimes \mathsf{X}, \quad E^{1,a} \wedge E^{0,b} \otimes \mathsf{Y}, \quad E^{2,a} \wedge E^{0,b} \otimes \mathsf{H}, \quad E^{1,a} \wedge E^{1,b} \otimes \mathsf{H},$$
$$E^{2,a} \wedge E^{0,b} \otimes e_c{}^d, \quad E^{1,a} \wedge E^{1,b} \otimes e_c{}^d \quad (1 \le a, b, c, d \le m). \tag{3.31}$$

Since $\mathbb{U}$ has $\mathfrak{sl}(W)$-lowest weight $\lambda = -2\epsilon_1$ (Table 2), then $\Phi_{\mathbb{U}}$ lies in the subspace spanned by

$$E^{2,1} \wedge E^{1,1} \otimes \mathsf{X}, \quad E^{2,1} \wedge E^{0,1} \otimes \mathsf{H}, \quad E^{1,1} \wedge E^{0,1} \otimes \mathsf{Y}, \quad E^{2,1} \wedge E^{0,1} \otimes e_1{}^1,$$
$$E^{2,1} \wedge E^{0,1} \otimes e_a{}^a, \quad E^{2,1} \wedge E^{0,a} \otimes e_a{}^1, \quad E^{2,a} \wedge E^{0,1} \otimes e_a{}^1,$$
$$E^{1,a} \wedge E^{1,1} \otimes e_a{}^1. \tag{3.32}$$

For $\mathfrak{ann}(\Phi_{\mathbb{U}})$ from Table 3, requiring $\mathfrak{ann}(\Phi_{\mathbb{U}}) \cdot \Phi_{\mathbb{U}} = 0$ further constrains $\Phi_{\mathbb{U}}$ to lie in span of:

$$E^{2,1} \wedge E^{1,1} \otimes \mathsf{X}, \quad E^{2,1} \wedge E^{0,1} \otimes \mathsf{H}, \quad E^{1,1} \wedge E^{0,1} \otimes \mathsf{Y},$$
$$\sum_{a=1}^m E^{2,1} \wedge E^{0,1} \otimes e_a{}^a, \quad \sum_{a=1}^m E^{2,1} \wedge E^{0,a} \otimes e_a{}^1, \quad \sum_{a=1}^m E^{2,a} \wedge E^{0,1} \otimes e_a{}^1,$$
$$\sum_{a=1}^m E^{1,a} \wedge E^{1,1} \otimes e_a{}^1. \tag{3.33}$$

Let us briefly explain this. From Table 3, $\mathfrak{sl}_{m-1}$ embeds into $\mathfrak{ann}(\Phi_{\mathbb{U}})$ via $A \mapsto \mathrm{diag}(0, A)$, which acts trivially on the first 4 elements of (3.32). The remaining tensors in (3.32) lie in a direct sum of 4 $\mathfrak{sl}_{m-1}$-reps equivalent to the sum of 4 copies of $\mathfrak{gl}_{m-1}$. (Namely, consider the span of $E^{2,1} \wedge E^{0,1} \otimes e_a{}^b$, $E^{2,1} \wedge E^{0,b} \otimes e_a{}^1$, etc.) Since $\mathfrak{gl}_{m-1} \cong \mathbb{R} \oplus \mathfrak{sl}_{m-1}$, then the aforementioned subspace contains a 4-dimensional subspace annihilated by $\mathfrak{sl}_{m-1}$. This is clearly spanned by the elements in the second line of (3.33) except taking the sum over $2 \le a \le m$. Finally, forcing annihilation with respect to $e_f{}^1$ for $f \ge 2$ yields (3.33).

Let $\Phi_{\mathbb{U}}$ be a general linear combination of all elements of (3.33), with μ_i denoting the coefficient of the i-th term, i.e. $\Phi_{\mathbb{U}} = \mu_1 E^{2,1} \wedge E^{1,1} \otimes \mathsf{X} + \mu_2 E^{2,1} \wedge E^{0,1} \otimes \mathsf{H} + \ldots + \mu_7 \sum_{a=1}^m E^{1,a} \wedge E^{1,1} \otimes e_a{}^1$. We conclude our computation by imposing ∂-closedness for $\Phi_{\mathbb{U}}$ using Lemma 3.3.2:

- X-annihilation: This yields $\mu_2 = \mu_3 = -\frac{\mu_1}{2}$, $\mu_4 = 0$ and $\mu_7 = \mu_5 = -\mu_6$.
- ∂_V-closedness: $0 = \partial_V \Phi_{\mathbb{U}}(E_{1,2}, E_{2,1}, E_{1,1}) = (\mu_5 - \mu_1) E_{2,2}$, and hence $\mu_1 = \mu_5$.

This uniquely pins down $\Phi_{\mathbb{U}}$ (as stated in Table 4), up to a nonzero scaling. From Remark 7, we in particular have that $\Phi_{\mathbb{U}}$ is normal and strongly regular. (The condition (3.30) does not apply for 3rd order ODE systems.)

3.4.2 $\mathbb{A}_2^{\rm tr}$ case

This case proceeds similarly, but is more involved than the $\mathbb{B}_4$ case. In particular, more conditions are required to pin down the lowest weight vector (up to scale).

Let $n \geq 3$. Since $\mathbb{U} := \mathbb{A}_2^{\rm tr} \subset \bigwedge^2 V^* \otimes V$ has bi-grade $(1,1)$ and $\mathfrak{sl}(W)$-lowest weight $\lambda = -\epsilon_1$, then $\Phi_{\mathbb{U}}$ must be a linear combination of

$$E^{i,1} \wedge E^{j,a} \otimes E_{i+j-1,a} \quad (0 \leq i,j \leq n,\ 1 \leq i+j \leq n+1,\ 1 \leq a \leq m). \tag{3.34}$$

Moreover, $\mathfrak{ann}(\Phi_{\mathbb{U}})$ from Table 3 annihilates $\Phi_{\mathbb{U}}$, so $\Phi_{\mathbb{U}}$ is in fact constrained to be a linear combination of

$$\Phi^{i,j} := \sum_{a=1}^{m} E^{i,1} \wedge E^{j,a} \otimes E_{i+j-1,a}. \tag{3.35}$$

Recalling our convention in §3.3.2 that $E_k = 0$ for $k < 0$ or $k > n$, we have:

PROPOSITION 3.4.2. *Fix $n \geq 3$ and $m \geq 2$. Let $\mathbb{U} = \mathbb{A}_2^{\rm tr}$ and define $\Phi_{\mathbb{U}} = \sum_{i,j=0}^{n} c_{i,j}\Phi^{i,j}$ for $\Phi^{i,j}$ as in (3.35), where we may assume that $c_{0,0} = 0 = c_{i,j}$ for $i+j > n+1$. Since $\Phi_{\mathbb{U}}$ is ∂-closed and satisfies the strong regularity and DM conditions, then we have*

$$\begin{cases} c_{i+1,j} + c_{i,j+1} = c_{i,j}; & \text{(XA): annihilation by } \mathsf{X}; \\ c_{i,j} = 0, \quad \text{for} \quad \min(i,j) \geq 3; & \text{(SR) : strong regularity;} \\ c_{n-1,2} = 0, \quad \text{for} \quad n \geq 4. & \text{(DM): DM conditions beyond (SR).} \end{cases} \tag{3.36}$$

PROOF. By Lemma 3.3.2, ∂-closedness of $\Phi_{\mathbb{U}}$ is equivalent to its X-annihilation, so using $\mathsf{X} \cdot E_{i,a} = E_{i+1,a}$ and $\mathsf{X} \cdot E^{i,a} = -E^{i-1,a}$, Leibniz rule, and re-indexing the summation, we straightforwardly obtain:

$$0 = \mathsf{X} \cdot \Phi_{\mathbb{U}} = \sum_{i=0}^{n}\sum_{j=0}^{n}(c_{i,j} - c_{i+1,j} - c_{i,j+1})\Phi^{i,j}. \tag{3.37}$$

This proves the first relations.

Next, recall that $E_{k,a} \in \mathfrak{g}^{-k-1}$. Strong regularity (3.17) of $\Phi^{i,j}$ forces $E_{i+j-1,a} \in \mathfrak{g}^{\min(-i-1,-j-1)-1}$, i.e.

$$\begin{aligned} &-i-j \geq \min(-i-1,-j-1) - 1 \geq -\max(i,j) - 2 \iff \\ &\qquad i+j \leq \max(i,j) + 2, \end{aligned} \tag{3.38}$$

or equivalently $\min(i,j) \leq 2$. All other terms are not present in the summation.

Finally, for the last relations we force (3.30) for $A = \Phi_{\mathbb{U}}$, i.e. $\vartheta(\Phi_{\mathbb{U}}) \in \mathrm{im}(\delta)$. Recall the maps δ and ϑ given in (3.28) and (3.29), and $F = \mathrm{span}\{E_0, \dots, E_{n-1}\} \otimes W \subset V$. Modulo F,

- $\vartheta(\Phi_{\mathbb{U}}) = \sum_{i,j=0}^{n} c_{i,j}\vartheta(\Phi^{i,j}) \equiv \sum_{i,j=0}^{n} c_{i,j}\Phi^{i,j}|_{\bigwedge^2 F} \equiv \sum_{i=2}^{n-1} c_{i,n+1-i}\Phi^{i,n+1-i} \overset{\text{(SR)}}{\equiv} c_{2,n-1}\Phi^{2,n-1} + c_{n-1,2}\Phi^{n-1,2}$.
- $\delta(E^{i,a} \otimes \mathsf{X}) \equiv \sum_{b=1}^{m} E^{i,a} \wedge E^{n-1,b} \otimes E_{n,b}$, i.e. bi-grade $(i-1,1)$ tensors for $0 \leq i \leq n-1$.

Since $\vartheta(\Phi_{\mathbb{U}})$ only consists of bi-grade $(1,1)$ tensors, it suffices to examine the $(1,1)$ subspace of $\mathrm{im}(\delta)$. From above, this always contains $\Phi^{2,n-1}$ (modulo F), but does not contain $\Phi^{n-1,2}$ when $n \geq 4$. Hence, beyond (SR), DM condition implies $\vartheta(\Phi_{\mathbb{U}}) \in \mathrm{im}(\delta)$, which forces $c_{n-1,2} = 0$ for $n \geq 4$. ■

We now solve (3.36):

PROPOSITION 3.4.3. *Fix $n \geq 3$. Then $(c_{i,j})_{0\leq i,j\leq n}$ from Proposition 3.4.2 is of the following form:*

$$
\begin{aligned}
c_{i,0} &= \begin{cases} (n-i+1)\beta, & 3 \leq i \leq n; \\ \alpha + (n-1)\beta, & i = 2; \\ \frac{n\alpha}{2} + (n-1)\beta, & i = 1; \end{cases} \\
c_{0,i} &= \begin{cases} -c_{1,0}, & i = 1; \\ (i-n-1)(\beta + \frac{i\alpha}{2}), & 2 \leq i \leq n; \end{cases} \\
c_{1,i} &= \left(\frac{n}{2} - i\right)\alpha + ({\delta_i}^1 - 1)\beta, \quad 1 \leq i \leq n; \\
c_{i,1} &= \beta + {\delta_i}^2\alpha, \quad 2 \leq i \leq n; \\
c_{2,i} &= (1 - {\delta_i}^n)\alpha, \quad 2 \leq i \leq n; \\
c_{i,2} &= 0, \quad 3 \leq i \leq n.
\end{aligned}
\tag{3.39}
$$

where $\alpha := c_{2,n-1}$ and $\beta := c_{n,1}$, and all other coefficients are trivial.

PROOF. Since (DM) is only present for $n \geq 4$, we split our proof into two cases:

- $\boxed{n=3}$: The system (3.36) becomes:

$$
\begin{array}{llll}
c_{1,0} + c_{0,1} = 0, & c_{1,1} + c_{0,2} = c_{0,1}, & c_{2,0} + c_{1,1} = c_{1,0}, & \\
c_{1,2} + c_{0,3} = c_{0,2}, & c_{2,1} + c_{1,2} = c_{1,1}, & c_{3,0} + c_{2,1} = c_{2,0}, & \\
c_{1,3} = c_{0,3}, & c_{2,2} + c_{1,3} = c_{1,2}, & c_{3,1} + c_{2,2} = c_{2,1}, & c_{3,1} = c_{3,0}.
\end{array}
\tag{3.40}
$$

Solving this in terms of $\alpha = c_{2,2}$ and $\beta = c_{3,1}$ gives (3.39).

- $\boxed{n \geq 4}$:

Step 1: Start with the assumed conditions $c_{0,0} = 0 = c_{i,j}$ for $i + j > n + 1$, the (SR) relations, as well as the (DM) relation $c_{n-1,2} = 0$. Using (XA), determine the entries above $c_{n-1,2} = 0$ and left of $c_{2,n-1} =: \alpha$ (until the $(2,2)$-position), as shown below.

$$\left(\begin{array}{ccc|ccccccc} 0 & * & * & * & * & \cdots & * & * & * & * \\ * & * & * & * & * & \cdots & * & * & * & * \\ * & * & * & * & * & \cdots & * & * & \alpha & 0 \\ \hline * & * & * & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & 0 & 0 & \iddots & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccccccc} 0 & * & * & * & * & \cdots & * & * & * & * \\ * & * & * & * & * & \cdots & * & * & * & * \\ * & * & \alpha & \alpha & \alpha & \cdots & \alpha & \alpha & \alpha & 0 \\ \hline * & * & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & 0 & 0 & \iddots & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \tag{3.41}$$

Step 2: Using (XA), we have $\beta := c_{n,1} = c_{n,0}$. Use (XA) to determine the entries above $c_{n,1}$ and left of $\gamma := c_{1,n}$ (until the $(1,1)$-position), as shown below.

$$\left(\begin{array}{ccc|ccccccc} 0 & * & * & * & * & \cdots & * & * & * & * \\ * & \begin{matrix}(n-1)\alpha \\ +\beta+\gamma\end{matrix} & \begin{matrix}(n-2)\alpha \\ +\gamma\end{matrix} & \begin{matrix}(n-3)\alpha \\ +\gamma\end{matrix} & \begin{matrix}(n-4)\alpha \\ +\gamma\end{matrix} & \cdots & \begin{matrix}3\alpha \\ +\gamma\end{matrix} & \begin{matrix}2\alpha \\ +\gamma\end{matrix} & \begin{matrix}\alpha \\ +\gamma\end{matrix} & \gamma \\ * & \alpha+\beta & \alpha & \alpha & \alpha & \cdots & \alpha & \alpha & \alpha & 0 \\ \hline * & \beta & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & 0 & 0 & \iddots & 0 & 0 & 0 & 0 \\ * & \beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & \beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & \beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta & \beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \tag{3.42}$$

More precisely, we have

$$c_{1,i} = (n-i)\alpha + \delta_i^1 \beta + \gamma, \quad 1 \leq i \leq n. \tag{3.43}$$

Step 3: Using (XA), determine all entries above $c_{n,0} = \beta$. This yields:

$$c_{i,0} = \begin{cases} (n-i+1)\beta, & 3 \leq i \leq n; \\ \alpha + (n-1)\beta, & i = 2; \\ n(\alpha+\beta) + \gamma, & i = 1. \end{cases} \tag{3.44}$$

Step 4: Impose $c_{0,1} \overset{\text{(XA)}}{=} -c_{1,0} = -n(\alpha+\beta) - \gamma$. For $2 \leq i \leq n$, we have the telescoping sum:

$$c_{0,i} - c_{0,1} = \sum_{k=2}^{i} (c_{0,k} - c_{0,k-1}) \overset{\text{(XA)}}{=} -\sum_{k=2}^{i} c_{1,k-1} \tag{3.45}$$

$$\stackrel{(3.43)}{=} -\sum_{k=2}^{i}[(n-k+1)\alpha + \delta^1_{k-1}\beta + \gamma] \tag{3.46}$$

$$\begin{aligned} c_{0,i} &= c_{0,1} - \beta - (i-1)\gamma - \alpha[(n-1) + ... + (n-i+1)] \\ &= -(n+1)\beta - i\gamma - \alpha\left[\binom{n+1}{2} - \binom{n-i+1}{2}\right] \end{aligned} \tag{3.47}$$

Step 5: Impose $c_{0,n} \stackrel{(XA)}{=} c_{1,n} = \gamma$. Solving this yields $\boxed{\gamma = -\beta - \frac{n\alpha}{2}}$. Substituting this into (3.43), (3.44) and (3.47) then gives the stated result.

■

We conclude our computation by imposing the coclosedness condition, i.e. $\partial^*\Phi_{\mathbb{U}} = 0$.

PROPOSITION 3.4.4. *Let $n \geq 3$ and $m \geq 2$. Take $\Phi_{\mathbb{U}}$ from Proposition 3.4.2 with coefficients (3.39). Then:*

$$\alpha = \frac{-6(n-1)(m+1)}{mn(n+1)+6}\beta. \tag{3.48}$$

PROOF. From Lemma 3.9.2, we have:

$$\begin{aligned} 0 = \sum_{k=0}^{n-1} \frac{(n-k)(k+1)}{n(n-1)}(c_{k,2} - mc_{2,k}) + \sum_{k=0}^{n} \frac{2k-n}{n}(mc_{1,k} - c_{k,1}) + \\ \sum_{k=1}^{n}(mc_{0,k} - c_{k,0}). \end{aligned} \tag{3.49}$$

We now substitute (3.39) into (3.49) and simplify. The computations are straightforward but tedious, and for the respective summations above, this leads to:

$$0 = -\frac{(n+2)\Omega}{6n(n-1)} - \frac{(n+2)\Omega}{6n} - \frac{(n+2)\Omega}{12} = -\frac{(n+2)(n+1)\Omega}{12(n-1)}, \tag{3.50}$$

where $\Omega := (mn(n+1)+6)\alpha + 6(n-1)(m+1)\beta$. This implies $\Omega = 0$, and hence the result. ■

Combining (3.48), (3.39), and Proposition 3.4.2 uniquely determines $\Phi_{\mathbb{U}}$ (given in Table 4), up to a nonzero scaling. Note that (3.49) (derived in Appendix 3.9) was only a small part of the coclosedness condition, but using Remark 7, we deduce that indeed $\partial^*\Phi_{\mathbb{U}} = 0$.

3.4.3 $\mathbb{A}_2^{\mathrm{tf}}$ case

The trace-free case proceeds analogously to the trace case. Since $\mathbb{U} := \mathbb{A}_2^{\mathrm{tf}} \subset \bigwedge^2 V^* \otimes V$ has bi-grade $(1,1)$ and $\mathfrak{sl}(W)$-lowest weight $\lambda = \epsilon_m - 2\epsilon_1$, then $\Phi_{\mathbb{U}}$

is a linear combination of

$$\Phi^{i,j} := E^{i,1} \wedge E^{j,1} \otimes E_{i+j-1,m} \quad (0 \le i,j \le n, \quad 1 \le i+j \le n+1). \quad (3.51)$$

Note that $\Phi^{i,j} = -\Phi^{j,i}$ is annihilated by all elements of $\mathfrak{ann}(\Phi_{\mathbb{U}})$ given in Table 3.

PROPOSITION 3.4.5. *Fix $n, m \ge 2$. Let $\mathbb{U} = \mathbb{A}_2^{\text{tf}}$ and define $\Phi_{\mathbb{U}} = \sum_{i,j=0}^n c_{i,j}\Phi^{i,j}$ for $\Phi^{i,j}$ as in (3.51), where we may assume that $c_{i,j} = -c_{j,i}$, and $c_{i,j} = 0$ for $i+j > n+1$. Since $\Phi_{\mathbb{U}}$ is ∂-closed and satisfies the strong regularity and DM conditions, then we have*

$$\begin{cases} c_{i+1,j} + c_{i,j+1} = c_{i,j}; & \textit{(XA): annihilation by } \mathsf{X};\\ c_{i,j} = 0, \quad \textit{for} \quad \min(i,j) \ge 3; & \textit{(SR): strong regularity;}\\ c_{n-1,2} = c_{2,n-1} = 0, \quad \textit{for} \quad \boxed{n \ge 3}. & \textit{(DM'): DM conditions beyond (SR).} \end{cases} \quad (3.52)$$

PROOF. The proof is very similar to Proposition 3.4.2, as we now explain. Recall that $\mathsf{X} \cdot E_{i,a} = E_{i+1,a}$ and $\mathsf{X} \cdot E^{i,a} = -E^{i-1,a}$, i.e. the X-action on these basis elements is independent of the second index. Consequently, comparing (3.51) and (3.35), it is immediate that $\mathsf{X} \cdot \Phi_{\mathbb{U}} = 0$ yields the same conditions (XA). Strong regularity similarly does not involve the second index, and so we obtain the same conditions (SR).

Finally, let us focus on (3.30). As in the proof of Proposition 3.4.2,

- $\vartheta(\Phi_{\mathbb{U}}) \equiv \sum_{i=2}^{n-1} c_{i,n+1-i}\Phi^{i,n+1-i} \mod F$, which consists of bi-grade $(1,1)$ tensors;
- the bi-grade $(1,1)$ tensors in $\operatorname{im}(\delta)$ are spanned by $\sum_{b=1}^m E^{2,a} \wedge E^{n-1,b} \otimes E_{n,b} \mod F$.

Since $m \ge 2$, then $\vartheta(\Phi_{\mathbb{U}}) \in \operatorname{im}(\delta)$ forces $\vartheta(\Phi_{\mathbb{U}}) \equiv 0$. This is automatic for $n = 2$, while for $n \ge 3$, we have $c_{i,n+1-i} = 0$ for $2 \le i \le n-1$. Beyond (SR), we have merely $c_{2,n-1} = c_{n-1,2} = 0$. ■

PROPOSITION 3.4.6. *Fix $n \ge 2$. Then $(c_{i,j})_{0 \le i,j \le n}$ from Proposition 3.4.5 is of the following form:*

$$\begin{aligned} c_{i,0} = -c_{0,i} &= \begin{cases} (n-i+1)\beta, & 2 \le i \le n \\ (n-1)\beta, & i = 1 \end{cases} \quad \textit{and} \\ c_{i,1} = -c_{1,i} &= \beta, \quad 2 \le i \le n \end{aligned} \quad (3.53)$$

and all other coefficients are trivial.

PROOF. We split our proof into two cases:

- $\boxed{n=2}$: the system (3.52) reduces to

$$c_{1,0}+c_{0,1}=0, \quad c_{2,0}=c_{1,0}, \quad c_{0,2}=c_{0,1}, \quad c_{1,2}=c_{0,2}, \quad c_{2,1}=c_{2,0}, \tag{3.54}$$

and solving the system in terms of $c_{2,1}$ proves the claim.

- $\boxed{n\geq 3}$: The conditions on $c_{i,j}$ in Proposition 3.4.5 can be viewed as (3.36) with additionally $c_{i,j} = -c_{j,i}$ (and $c_{2,n-1} = 0$ when $n = 3$). Consequently, the solution to (3.52) can be obtained from the solution (3.39) to (3.36) by merely imposing $\alpha := c_{2,n-1} = 0$.

■

Combining (3.53) and Proposition 3.4.5 uniquely determines $\Phi_{\mathbb{U}}$ (given in Table 4), up to a nonzero scaling. As before, using Remark 7, we deduce that $\partial^*\Phi_{\mathbb{U}} = 0$. This completes our proof of Theorem 3.4.1.

3.5 Homogeneous structures and Cartan-theoretic descriptions

Our method for proving Theorems 3.2.1 and 3.2.2 will rely on the fact that Cartan geometries (see §3.3.3) associated to submaximally symmetric vector ODEs are locally homogeneous. In this section, we summarize all relevant symmetry-based facts about such geometries and their corresponding algebraic models of ODE type. We will use G, P, G_0 and $\mathfrak{g}$ from §3.3.2, and the filtration and grading on $\mathfrak{g}$ defined there.

3.5.1 Symmetry gaps for ODE

An *infinitesimal symmetry* of a given Cartan geometry $(\mathcal{G}\to\mathcal{E},\omega)$ of type (G,P) is a P-invariant vector field $\xi\in\mathfrak{X}(\mathcal{G})^P$ on $\mathcal{G}$ that preserves ω under Lie differentiation, i.e. $\mathcal{L}_\xi\omega=0$. The collection of all such vector fields forms a Lie algebra, which we denote by

$$\mathfrak{inf}(\mathcal{G},\omega) := \{\xi\in\mathfrak{X}(\mathcal{G})^P \,:\, \mathcal{L}_\xi\omega=0\}\subset\mathfrak{X}(\mathcal{G}). \tag{3.55}$$

The submaximal symmetry dimension is

$$\begin{aligned}\mathfrak{S} := \max\{\dim\mathfrak{inf}(\mathcal{G},\omega) \,:\, (\mathcal{G}\to\mathcal{E},\omega) \quad &\text{strongly regular, normal of type}\\ (G,P) \quad \text{associated to a vector ODE } \mathcal{E}\text{ (3.2)}, \quad &\text{with} \quad \kappa_H\not\equiv 0\}.\end{aligned} \tag{3.56}$$

Recall that $\mathbb{E}$ decomposes into G_0-irreducible submodules $\mathbb{U}\subset\mathbb{E}$. Analogous to $\mathfrak{S}$ above, we define:

$$\begin{aligned}\mathfrak{S}_{\mathbb{U}} := \max\{\dim\mathfrak{inf}(\mathcal{G},\omega) \,:\, (\mathcal{G}\to\mathcal{E},\omega) \quad &\text{strongly regular, normal of type}\\ (G,P) \quad \text{associated to a vector ODE } \mathcal{E}\text{ (3.2)}, \quad &\text{with} \quad 0\not\equiv\mathrm{im}(\kappa_H)\subset\mathbb{U}\}.\end{aligned} \tag{3.57}$$

In order to define suitable algebraic upper bounds, we will need the following notion from [36]:

DEFINITION 3.5.1. *Given a subspace* $\mathfrak{a}_0 \subset \mathfrak{g}_0$*, the graded subalgebra* $\mathfrak{a} = \mathrm{pr}(\mathfrak{g}_-, \mathfrak{a}_0) := \mathfrak{a}_- \oplus \mathfrak{a}_0 \oplus \mathfrak{a}_1 \subset \mathfrak{g}$*, where* $\mathfrak{a}_- := \mathfrak{g}_- = \mathfrak{g}_{-n-1} \oplus \ldots \oplus \mathfrak{g}_{-1}$ *and* $\mathfrak{a}_1 := \{x \in \mathfrak{g}_1 \,:\, [x, \mathfrak{g}_{-1}] \subset \mathfrak{a}_0\}$ *is called the Tanaka prolongation algebra. For* ϕ *in some* $\mathfrak{g}_0$*-module, we define* $\mathfrak{a}^\phi := \mathrm{pr}(\mathfrak{g}_-, \mathfrak{ann}(\phi))$*, where* $\mathfrak{ann}(\phi) \subset \mathfrak{g}_0$ *is the annihilator of* ϕ*.*

Now, we define

$$\mathfrak{U} := \max\left\{\dim \mathfrak{a}^\phi \,:\, 0 \neq \phi \in \mathbb{E}\right\} \quad \text{and} \quad \mathfrak{U}_{\mathbb{U}} := \max\left\{\dim \mathfrak{a}^\phi \,:\, 0 \neq \phi \in \mathbb{U}\right\}. \tag{3.58}$$

By [26, Thm 2.11], we conclude that

$$\mathfrak{S} \leq \mathfrak{U} < \dim \mathfrak{g} \quad \text{and} \quad \mathfrak{S}_{\mathbb{U}} \leq \mathfrak{U}_{\mathbb{U}} \quad \text{for all } G_0\text{- irreducible modules} \quad \mathbb{U} \subset \mathbb{E}. \tag{3.59}$$

Note that $\mathfrak{U} = \max_{\mathbb{U} \subset \mathbb{E}} \mathfrak{U}_{\mathbb{U}}$. In fact, by [26, Thm 1.2], in all of the *vector* cases we have equality:

$$\mathfrak{S} = \mathfrak{U} \qquad \text{and} \qquad \mathfrak{S}_{\mathbb{U}} = \mathfrak{U}_{\mathbb{U}}. \tag{3.60}$$

Examples of some vector ODEs realizing these can be found in [26, Table 8 & 10].

3.5.2 Local homogeneity and algebraic models of ODE type

Recall that a Cartan geometry $(\mathcal{G} \to \mathcal{E}, \omega)$ of type (G, P) is said to be locally *homogeneous* if there exists a (left) action by a local Lie group F on $\mathcal{G}$ by principal bundle morphisms preserving ω that projects onto a transitive action down on $\mathcal{E}$. We then have [26, Lemma A.1]:

LEMMA 3.5.2. *Fix a* G_0*-irrep* $\mathbb{U} \subset \mathbb{E}$*. Then any regular, normal Cartan geometry* $(\mathcal{G} \to \mathcal{E}, \omega)$ *of type* (G, P) *with* $0 \not\equiv \mathrm{im}(\kappa_H) \subset \mathbb{U}$ *and* $\dim(\mathfrak{inf}(\mathcal{G}, \omega)) = \mathfrak{U}_{\mathbb{U}}$ *is locally homogeneous about any point* $u \in \mathcal{G}$ *with* $\kappa_H(u) \neq 0$*.*

By [26, §A.1], such a homogeneous Cartan geometry can be encoded *Cartan-theoretically* by:

DEFINITION 3.5.3. *An algebraic model* $(\mathfrak{f}; \mathfrak{g}, \mathfrak{p})$ *of ODE type is a Lie algebra* $(\mathfrak{f}, [\cdot, \cdot]_{\mathfrak{f}})$ *such that:*

(i) $\mathfrak{f} \subset \mathfrak{g}$ *is a filtered subspace whose associated graded* $\mathfrak{s} := \mathrm{gr}(\mathfrak{f}) \subset \mathfrak{g}$ *has* $\mathfrak{s}_- = \mathfrak{g}_-$*;*

(ii) $\mathfrak{f}^0$ *inserts trivially into* $\kappa(x, y) := [x, y] - [x, y]_{\mathfrak{f}}$*, i.e.* $\kappa(z, \cdot) = 0$ *for all* $z \in \mathfrak{f}^0$*;*

(iii) κ *is normal and strongly regular :* $\partial^* \kappa = 0$ *and* $\kappa(\mathfrak{g}^i, \mathfrak{g}^j) \subset \mathfrak{g}^{i+j+1} \cap \mathfrak{g}^{\min(i,j)-1}$, $\forall i, j$*.*

Let $\mathcal{N}$ denote the set of all algebraic models $(\mathfrak{f}; \mathfrak{g}, \mathfrak{p})$ of ODE type for fixed (G, P). Then $\mathcal{N}$ admits a P-action and is partially ordered:

(1) P-action: for $p \in P$ and $\mathfrak{f} \in \mathcal{N}$, we have $p \cdot \mathfrak{f} := \mathrm{Ad}_p(\mathfrak{f})$. We will regard all algebraic models $(\mathfrak{f}; \mathfrak{g}, \mathfrak{p})$ of ODE type in the same P-orbit to be equivalent.
(2) partial order relation $\leq$: for $\mathfrak{f}, \widetilde{\mathfrak{f}} \in \mathcal{N}$ regard $\mathfrak{f} \leq \widetilde{\mathfrak{f}}$ if there exists a map $\mathfrak{f} \hookrightarrow \widetilde{\mathfrak{f}}$ of Lie algebras. We will focus on maximal elements in $(\mathcal{N}, \leq)$.

Combining (3.59), Lemma 3.5.2, and Definition 3.5.3, we obtain the following key existence result:

THEOREM 3.5.4. *Fix an irreducible G_0-module $\mathbb{U}$ in the effective part $\mathbb{E}$ for vector ODEs (3.2) of order ≥ 3. Then there* exists *an algebraic model $(\mathfrak{f}; \mathfrak{g}, \mathfrak{p})$ of ODE type with $0 \not\equiv \mathrm{im}(\kappa_H) \subset \mathbb{U}$ and $\dim \mathfrak{f} = \mathfrak{U}_{\mathbb{U}} = \mathfrak{S}_{\mathbb{U}}$.*

REMARK 8. Conversely, by [36, Lemma 4.1.4], for a given algebraic model $(\mathfrak{f}; \mathfrak{g}, \mathfrak{p})$ of ODE type, there exists a locally homogeneous strongly regular, normal Cartan geometry $(\mathcal{G} \to \mathcal{E}, \omega)$ of type (G, P) with $\mathfrak{inf}(\mathcal{G}, \omega)$ containing a subalgebra isomorphic to $\mathfrak{f}$.

We caution that such a geometry may not arise from an ODE (3.2). (For instance, the Doubrov–Medvedev condition must additionally hold.) Consequently, our strategy involves:

- classifying (up to the P-action) the corresponding algebraic models $(\mathfrak{f}; \mathfrak{g}, \mathfrak{p})$ of ODE type, and then
- providing vector ODEs of C-class realizing these algebraic models.

We will use the following results from [26, §A.1] in carrying out the classifications.

DEFINITION 3.5.5. *A filtered linear space $\mathfrak{f} \subset \mathfrak{g}$ can be described as the graph of some linear map on $\mathfrak{s}$ into $\mathfrak{g}$ as follows. Let $\mathfrak{s}^\perp \subset \mathfrak{g}$ be a graded subalgebra such that $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{s}^\perp$. Then $\mathfrak{f} := \bigoplus_i \mathrm{span}\,\{x + \mathfrak{D}(x) : x \in \mathfrak{s}_i\}$, for some* unique *linear (deformation) map $\mathfrak{D} : \mathfrak{s} \to \mathfrak{s}^\perp$ such that $\mathfrak{D}(x) \in \mathfrak{s}^\perp \cap \mathfrak{g}^{i+1}$ for $x \in \mathfrak{s}_i$.*

LEMMA 3.5.6. *Let $T \in \mathfrak{f}^0$ and suppose that the complementary graded subspaces $\mathfrak{s}, \mathfrak{s}^\perp \subset \mathfrak{g}$ are ad_T-invariant, then the map $\mathfrak{D} : \mathfrak{s} \to \mathfrak{s}^\perp$ is ad_T-invariant, i.e. $T \cdot \mathfrak{D} = 0 \iff \mathrm{ad}_T \circ \mathfrak{D} = \mathfrak{D} \circ \mathrm{ad}_T$.*

Recall from §3.3.3 that $\kappa_H := \kappa \mod \mathrm{im}\,\partial^*$, where ∂^* is the codifferential. We then have:

PROPOSITION 3.5.7. *Let $(\mathfrak{f}; \mathfrak{g}, \mathfrak{p})$ be an algebraic model of ODE type. Then*

(i) $(\mathfrak{f}, [\cdot, \cdot]_{\mathfrak{f}})$ *is a filtered Lie algebra.*
(ii) $\mathfrak{f}^0$ *annihilates κ, i.e.* $\mathfrak{f}^0 \cdot \kappa = 0 \iff [z, \kappa(x, y)]_{\mathfrak{f}} = \kappa([z, x]_{\mathfrak{f}}, y) + \kappa(x, [z, y]_{\mathfrak{f}})$, $\forall x, y \in \mathfrak{f}$, $\forall z \in \mathfrak{f}^0$.
(iii) $\mathfrak{s} \subset \mathfrak{a}^{\kappa_H}$.

Adapting the terminology from [46, §2.2], we shall refer to $\mathfrak{f}$ as a (constrained) *filtered sub-deformation* of $\mathfrak{s}$.

3.5.3 Characterizing maximality of the Tanaka prolongation

Fix a G_0-irrep $\mathbb{U} \subset \mathbb{E}$, and recall $\mathfrak{U}_{\mathbb{U}}$ defined in (3.58). For vector ODEs (3.2), $\mathfrak{U}_{\mathbb{U}}$ were computed in [26, §3.4] using the fact that $\mathfrak{U}_{\mathbb{U}} = \dim \mathfrak{a}^{\Phi_{\mathbb{U}}}$, where $\Phi_{\mathbb{U}} \in \mathbb{U}$ is an extremal (lowest or highest) weight vector. For the purpose of our goal in §3.6, we next prove that $\mathfrak{U}_{\mathbb{U}}$ is achieved precisely in this way:

LEMMA 3.5.8. *Let $\mathbb{U} \subset \mathbb{E}$ be a G_0-irrep and $\Phi_{\mathbb{U}} \in \mathbb{U}$ be a lowest weight vector. Then, $\mathfrak{U}_{\mathbb{U}} = \dim \mathfrak{a}^{\Phi_{\mathbb{U}}}$. Moreover, if $0 \neq \phi \in \mathbb{U}$, then $\dim \mathfrak{a}^{\phi} = \mathfrak{U}_{\mathbb{U}}$ iff $[\phi]$ is contained in the G_0-orbit of $[\Phi_{\mathbb{U}}] \in \mathbb{P}(\mathbb{U})$.*

PROOF. The proof used in [36, Prop 3.1.1] can be applied for our purposes here. (We note that the initial hypothesis of G complex semisimple Lie group and $P \leq G$ a parabolic subgroup is not necessary. We use our G_0 here for the G_0 appearing there.) Over $\mathbb{C}$, the same proof yields the result. Over $\mathbb{R}$, the essential fact used in the proof is that $\mathrm{SL}_m \subset G_0$ acts with a *unique closed orbit* $\mathcal{O}$ (of minimal dimension) in $\mathbb{P}(\mathbb{U})$. We can directly verify this for the modules $\mathbb{U} \subset \mathbb{E}$ in Table 2:

$\mathbb{U}$	$\mathfrak{sl}(W)$-module structure	$\mathcal{O} \subset \mathbb{P}(\mathbb{U})$
$\mathbb{B}_4$	S^2W^*	$\{[\eta^2] : [\eta] \in \mathbb{P}(W^*)\}$
$\mathbb{A}_2^{\mathrm{tr}}$	W^*	$\mathbb{P}(W^*)$
$\mathbb{A}_2^{\mathrm{tf}}$	$(S^2W^* \otimes W)_0$	$\{[\eta^2 \otimes w] : [\eta] \in \mathbb{P}(W^*),$ $[w] \in \mathbb{P}(W),\ \eta(w) = 0\}$

■

3.5.4 Prolongation-rigidity

In terms of the Tanaka prolongation algebra $\mathfrak{a}^{\phi}$ (Definition 3.5.1), we define:

DEFINITION 3.5.9. *A G_0-module $\mathbb{U} \subset \mathbb{E}$ is said to be prolongation-rigid (PR) if $\mathfrak{a}_1^{\phi} = 0$ for* all *non-zero $\phi \in \mathbb{U}$.*

Let $\mathbb{U} \subset \mathbb{E}$ be an irreducible C-class module (see §3.3.5). To study prolongation-rigidity, it suffices by Lemma 3.5.8 to consider the lowest weight vector $\phi = \Phi_{\mathbb{U}} \in \mathbb{U}$. By [26, Lemma 3.3], we have $\mathfrak{a}_1^{\Phi_{\mathbb{U}}} = \mathbb{R}\mathsf{Y}$ if and only if $\mathbb{U}$ has bi-grade that is a multiple of $(n, 2)$. From Table 2, the bi-grade of $\mathbb{U}$ is a multiple of $(1, 1)$, so $\mathbb{U}$ is not PR if and only if $n = 2$. A summary is given in Table 5, with $\mathfrak{a}^{\Phi_{\mathbb{U}}}$ in each case, and $\mathfrak{ann}(\Phi_{\mathbb{U}})$ stated in Table 3.

3.6 Embeddings of filtered sub-deformations

By §3.5.2 above, all submaximally symmetric vector ODEs (3.2) can be encoded using algebraic models of ODE type. Consequently, proving our main results (Theorems 3.2.1 and 3.2.2) boils down to classifying these corresponding algebraic

n	$\mathbb{U}$	$\mathbb{U}$ PR?	$\mathfrak{a}^{\Phi_{\mathbb{U}}}$
2	$\mathbb{B}_4$	$\times$	$\mathfrak{g}_- \oplus \mathfrak{ann}(\Phi_{\mathbb{U}}) \oplus \mathbb{R}\mathsf{Y}$
≥ 3	$\mathbb{A}_2^{\rm tr}$	$\checkmark$	$\mathfrak{g}_- \oplus \mathfrak{ann}(\Phi_{\mathbb{U}})$
2	$\mathbb{A}_2^{\rm tf}$	$\times$	$\mathfrak{g}_- \oplus \mathfrak{ann}(\Phi_{\mathbb{U}}) \oplus \mathbb{R}\mathsf{Y}$
≥ 3	$\mathbb{A}_2^{\rm tf}$	$\checkmark$	$\mathfrak{g}_- \oplus \mathfrak{ann}(\Phi_{\mathbb{U}})$

TABLE 5. Prolongation-rigidity for irreducible C-class modules $\mathbb{U} \subset \mathbb{E}$

models (Theorem 3.5.4). More precisely, in view of Lemma 3.5.8, for each irreducible C-class module $\mathbb{U} \subset \mathbb{E}_C \subsetneq \mathbb{E}$ (Definition 3.3.6), our goal is to classify (up to the P-action) all algebraic models $(\mathfrak{f}; \mathfrak{g}, \mathfrak{p})$ of ODE type with $\kappa_H = \Phi_{\mathbb{U}} \in \mathbb{U}$, where $\Phi_{\mathbb{U}}$ is the lowest weight vector from Table 4, and $\dim \mathfrak{f} = \mathfrak{S}_{\mathbb{U}}$.

In this section, we classify all possible (filtered) linear embeddings $\mathfrak{f} \subset \mathfrak{g}$ for such $(\mathfrak{f}; \mathfrak{g}, \mathfrak{p})$. The possibilities for curvature κ of $(\mathfrak{f}; \mathfrak{g}, \mathfrak{p})$ are then classified in §3.7. Recall the canonical filtration and the grading structure on $\mathfrak{g}$ from §3.3.2. Having computed graded subalgebras $\mathfrak{a}^{\Phi_{\mathbb{U}}} \subset \mathfrak{g}$ in Table 5, we next classify, up to the P-action, possible filtered linear subspaces $\mathfrak{f} \subset \mathfrak{g}$ for algebraic models $(\mathfrak{f}; \mathfrak{g}, \mathfrak{p})$ satisfying $\mathrm{gr}(\mathfrak{f}) = \mathfrak{a}^{\Phi_{\mathbb{U}}}$:

PROPOSITION 3.6.1. *Fix an irreducible C-class module $\mathbb{U} = \mathbb{B}_4$, $\mathbb{A}_2^{\rm tf}$ or $\mathbb{A}_2^{\rm tr}$ in $\mathbb{E}_C \subsetneq \mathbb{E}$, viewed as a G_0-submodule of $\ker \Box \subset C^2(\mathfrak{g}_-, \mathfrak{g})$ via (3.21), and consider an algebraic model $(\mathfrak{f}; \mathfrak{g}, \mathfrak{p})$ of ODE type with $\kappa_H = \Phi_{\mathbb{U}} \in \mathbb{U}$, for $\Phi_{\mathbb{U}}$ from Table 4, and $\dim \mathfrak{f} = \mathfrak{S}_{\mathbb{U}}$. Then using the P-action, $\mathfrak{f} \mapsto \mathrm{Ad}_p \mathfrak{f}$, we may normalize to*

- *(a)* $\mathfrak{f} = \mathfrak{a}^{\Phi_{\mathbb{U}}}$ *when* $\mathbb{U} = \mathbb{B}_4$ *or* $\mathbb{A}_2^{\rm tf}$;
- *(b)* $\mathfrak{f} = \mathrm{span}\{E_{n,a},\, \dots,\, E_{2,a},\, E_{1,1}+(n-2)\zeta \mathsf{Z}_1,\, E_{1,b},\, E_{0,1}+\zeta\mathsf{Y},\, E_{0,b},\, \mathsf{X} : \zeta \in \mathbb{R},\, 1 \leq a \leq m,\, 2 \leq b \leq m\} \oplus \mathfrak{ann}(\Phi_{\mathbb{A}_2^{\rm tr}})$ *when* $\mathbb{U} = \mathbb{A}_2^{\rm tr}$.

PROOF. Since $\mathbb{U}$ has a bi-grade that is a multiple of $(1,1)$ (see Table 2), then $T := \mathsf{Z}_1 - \mathsf{Z}_2 \in \mathfrak{ann}(\Phi_{\mathbb{U}}) = \mathfrak{a}_0^{\Phi_{\mathbb{U}}}$. We note that $\mathfrak{a}^{\Phi_{\mathbb{U}}} \subset \mathfrak{g}$ is a graded subalgebra, and denote by $\widehat{T}$ the element in $\mathfrak{f}^0$ with the *leading part* T, i.e. $\mathrm{gr}_0(\widehat{T}) = T$. Since $\mathfrak{g}_1 = \mathbb{R}\mathsf{Y}$ and $\mathfrak{g}_i = 0$ for all $i \geq 2$ (see §3.3.2), then necessarily $\widehat{T} = T + s\mathsf{Y} \in \mathfrak{f}^0$. We claim that without loss of generality, i.e. using the P-action (for P defined in §3.3.2), we may assume that $T \in \mathfrak{f}^0$. This is immediate when $\mathbb{U}$ is *not* PR, since $\mathsf{Y} \in \mathfrak{f}^0$ and therefore $T = \widehat{T} - s\mathsf{Y}$ is a linear combination of $\widehat{T}$ and Y. Otherwise, when $\mathbb{U}$ is PR ($\mathsf{Y} \notin \mathfrak{f}^0$), using the P_+-action and $[\mathsf{Y}, T] = -\mathsf{Y}$, we have:

$$\begin{aligned} \mathrm{Ad}_{\exp(t\mathsf{Y})}(\widehat{T}) &= \exp(\mathrm{ad}_{t\mathsf{Y}})(\widehat{T}) = \widehat{T} + [t\mathsf{Y}, \widehat{T}] + \frac{1}{2!}[t\mathsf{Y}, [t\mathsf{Y}, \widehat{T}]] + \cdots \\ &= \mathsf{Z}_1 - \mathsf{Z}_2 + (s-t)\mathsf{Y}, \end{aligned} \tag{3.61}$$

then choosing $t = s$ normalizes the right-hand side to T. So, relabeling the left hand side by $\widehat{T}$, gives $T = \widehat{T} \in \mathfrak{f}^0$.

By Definition 3.5.3 (ii), we have $\kappa(T, z) = 0$, i.e. $[T, z]_{\mathfrak{f}} = [T, z]$, $\forall z \in \mathfrak{f}$. Then, by exploiting the semi-simplicity of ad_T, we next determine the remaining basis elements $\widehat{x} \in \mathfrak{f}^i$ with the leading parts $x \in \mathfrak{a}_i^{\Phi_{\mathbb{U}}}$, i.e. $\mathrm{gr}_i(\widehat{x}) = x$.

We first consider $\widehat{x} \in \mathfrak{f}^0$. We claim that without loss of generality, as it was for $\widehat{T} \in \mathfrak{f}^0$ above, we may assume that $x \in \mathfrak{f}^0$ for all $x \in \mathfrak{a}_0^{\Phi_{\mathbb{U}}}$. We let $\widehat{x} = x + c_x \mathsf{Y}$. Then, for $\mathbb{U}$ that is *not* PR the claim holds, since $\mathsf{Y} \in \mathfrak{f}^0$, and so $x = \widehat{x} - c_x \mathsf{Y} \in \mathfrak{f}^0$. In order to give the argument for the case when $\mathbb{U}$ is PR, we recall that $[T, \mathsf{Y}] = \mathsf{Y}$ and $[T, x] = 0$ for all $x \in \mathfrak{a}_0^{\Phi_{\mathbb{U}}}$. So, $[T, \widehat{x}]_{\mathfrak{f}} = [T, \widehat{x}] = c_x \mathsf{Y} \in \mathfrak{f}^0$. Now, since $(\mathfrak{f}, [\cdot, \cdot]_{\mathfrak{f}})$ is a Lie algebra and $\mathsf{Y} \notin \mathfrak{f}^0$, then the closure condition $[T, \widehat{x}] \in \mathfrak{f}^0$ implies that $c_x = 0$. So, $x = \widehat{x} \in \mathfrak{f}^0$.

Next, we similarly consider $\widehat{x} \in \mathfrak{f}^i$ for $i < 0$. Recall that by Definition 3.5.1 for these cases we have $\mathfrak{a}_i^{\Phi_{\mathbb{U}}} = \mathfrak{g}_i$, for $\mathfrak{g}_i$ as was defined in §3.3.2. In view of Lemma 3.5.6, we fix ad_T-invariant subspaces $\mathfrak{s}^\perp \subset \mathfrak{g}$ in Table 6 such that $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{s}^\perp$, where $\mathfrak{s} := \mathfrak{a}^{\Phi_{\mathbb{U}}}$, and define the deformation map $\mathfrak{D} : \mathfrak{s} \to \mathfrak{s}^\perp$ (Definition 3.5.5). Let $E^{i,a}$ and X^* denote the dual basis elements to $E_{i,a}$ and X respectively, and recall bi-grades for the basis elements from Figure 1. Since, for $0 \le i \le n$, $1 \le a, b, c \le m$, the eigenvalues of ad_T on

$$E^{i,a} \otimes \mathsf{Z}_1, \quad E^{i,a} \otimes e_c{}^b, \quad E^{i,a} \otimes \mathsf{Y}, \quad \mathsf{X}^* \otimes \mathsf{Z}_1, \quad \mathsf{X}^* \otimes e_c{}^b, \quad \mathsf{X}^* \otimes \mathsf{Y}, \tag{3.62}$$

are $i-1,\ i-1,\ i,\ 1,\ 1,\ 2$ respectively, then we have zero eigenvalues only when $i = 0$ or 1. Then $T \cdot \mathfrak{D} = 0$ (Lemma 3.5.6) implies $\mathsf{X} = \widehat{\mathsf{X}} \in \mathfrak{f}$ and $E_{i,a} = \widehat{E}_{i,a} \in \mathfrak{f}$ for all i except possibly when $i = 0$ or 1.

n	Irreducible C-class module $\mathbb{U}$	Generators for $\mathfrak{s}^\perp \subset \mathfrak{g}$	Ranges
2	$\mathbb{B}_4$	$\mathsf{Z}_1,\ e_1{}^b$	$2 \le b \le m$
≥ 3	$\mathbb{A}_2^{\mathrm{tr}}$	$\mathsf{Z}_1,\ e_1{}^b,\ \mathsf{Y}$	$2 \le b \le m$
2	$\mathbb{A}_2^{\mathrm{tf}}$	$\mathsf{Z}_1,\ e_1{}^b,\ e_d{}^m$	$2 \le b \le m,\ 2 \le d \le m-1$
≥ 3	$\mathbb{A}_2^{\mathrm{tf}}$	$\mathsf{Z}_1,\ e_1{}^b,\ e_d{}^m,\ \mathsf{Y}$	$2 \le b \le m,\ 2 \le d \le m-1$

TABLE 6. ad_T-invariant subspace $\mathfrak{s}^\perp \subset \mathfrak{g}$ complementary to $\mathfrak{s} = \mathfrak{a}^{\Phi_{\mathbb{U}}}$

Now, consider the above exceptional cases. Based on the eigenvalues for ad_T given in (3.62), we must have

$$\widehat{E}_{0,a} = E_{0,a} + \lambda_a \mathsf{Y}. \tag{3.63}$$

Since $\mathbb{U}$ is a C-class module, then $\kappa(\mathsf{X}, \cdot) = 0$ (Definition 3.3.4), which implies $[\mathsf{X}, \cdot]_{\mathfrak{f}} = [\mathsf{X}, \cdot]$. Recall that $(\mathfrak{f}, [\cdot,\cdot]_{\mathfrak{f}})$ is a Lie algebra and $[\mathsf{X}, E_{0,a}] = E_{1,a}$. Then for $\mathbb{U}$ that is

(a) *not* PR ($\mathbb{U} = \mathbb{B}_4$ or $\mathbb{A}_2^{\rm tf}$ when $n = 2$): we have that $E_{0,a} = \widehat{E}_{0,a} - \lambda_a \mathsf{Y} \in \mathfrak{f}$, since $\mathsf{Y} \in \mathfrak{f}$ and $\widehat{E}_{0,a} \in \mathfrak{f}$. Since $\mathsf{X} \in \mathfrak{f}$, then $[\mathsf{X}, E_{0,a}]_{\mathfrak{f}} = [\mathsf{X}, E_{0,a}] = E_{1,a} \in \mathfrak{f}$. Hence, $E_{1,a} = \widehat{E}_{1,a}$ and so $\mathfrak{f} = \mathfrak{a}^{\Phi_{\mathbb{U}}}$.

(b) PR ($\mathbb{U} = \mathbb{A}_2^{\rm tf}$ or $\mathbb{A}_2^{\rm tr}$ when $n \geq 3$): Recall from our discussion above that for *any* $\widehat{x} \in \mathfrak{f}^0$ with the leading part $x \in \mathfrak{a}_0^{\Phi_{\mathbb{U}}}$, we may assume without loss of generality that $x \in \mathfrak{f}^0$. Hence, Table 3 yields:

$$\begin{aligned} &q := {e_1}^1 - {e_2}^2 + \mathsf{Z}_2 \in \mathfrak{f}^0 \quad \text{for} \quad \mathbb{A}_2^{\rm tr} \\ &p := {e_1}^1 - {e_2}^2 + (2 + {\delta_{m-1}}^1)\mathsf{Z}_2 \in \mathfrak{f}^0 \quad \text{for} \quad \mathbb{A}_2^{\rm tf} \end{aligned} \tag{3.64}$$

Recall that by Definition 3.5.3 (ii) we have $\kappa(\mathfrak{f}^0, \cdot) = 0$, which implies $[z, \cdot]_{\mathfrak{f}} = [z, \cdot]$ for all $z \in \mathfrak{f}^0$. Now, since both p and q commute with Y, $[\mathsf{Z}_2, E_{i,a}] = -E_{i,a}$ and $[{e_1}^1 - {e_2}^2, E_{i,a}] = ({\delta_a}^1 - {\delta_a}^2)E_{i,a}$ (see §3.3.2), then for

(i) $\mathbb{U} = \mathbb{A}_2^{\rm tf}$: we have $[p, \widehat{E}_{0,a}]_{\mathfrak{f}} = [p, \widehat{E}_{0,a}] = ({\delta_a}^1 - {\delta_a}^2 - {\delta_{m-1}}^1 - 2)E_{0,a}$. So, the closure condition $[p, \widehat{E}_{0,a}] \in \mathfrak{f}$ forces $\lambda_a = 0$, i.e. $E_{0,a} = \widehat{E}_{0,a} \in \mathfrak{f}$. Then $[\mathsf{X}, E_{0,a}] = E_{1,a} \in \mathfrak{f}$, which implies $E_{1,a} = \widehat{E}_{1,a} \in \mathfrak{f}$. Hence, $\mathfrak{f} = \mathfrak{a}^{\Phi_{\mathbb{A}_2^{\rm tf}}}$. This completes the proof for (a).

(ii) $\mathbb{U} = \mathbb{A}_2^{\rm tr}$: we have $[q, \widehat{E}_{0,a}]_{\mathfrak{f}} = [q, \widehat{E}_{0,a}] = (\delta_a^1 - \delta_a^2 - 1)E_{0,a}$. So, $[q, \widehat{E}_{0,a}] \in \mathfrak{f}$ implies that $\lambda_a = 0$ *except* for $a = 1$, so for these cases we have $E_{0,a} = \widehat{E}_{0,a} \in \mathfrak{f}$. Consequently, $[\mathsf{X}, E_{0,a}] = E_{1,a} \in \mathfrak{f}$ implies that $E_{1,a} = \widehat{E}_{1,a} \in \mathfrak{f}$ except for $a = 1$.
Finally, we consider the case when $a = 1$. Based on the eigenvalues for ad_T given in (3.62) and setting $\lambda_1 = \zeta$, we necessarily have

$$\widehat{E}_{0,1} = E_{0,1} + \zeta\mathsf{Y} \quad \text{and} \quad \widehat{E}_{1,1} = E_{1,1} + \beta\mathsf{Z}_1 + \sum_{b=2}^{m} \alpha_b {e_1}^b. \tag{3.65}$$

Since from (3.11) we have $\mathsf{H} = n\mathsf{Z}_2 - 2\mathsf{Z}_1$ and $T = \mathsf{Z}_1 - \mathsf{Z}_2$, then we have

$$\begin{aligned} [\mathsf{X}, \widehat{E}_{0,1}]_{\mathfrak{f}} = [\mathsf{X}, E_{0,1} + \zeta\mathsf{Y}] = E_{1,1} + \zeta\mathsf{H} = E_{1,1} + \zeta(n\mathsf{Z}_2 - 2\mathsf{Z}_1), \\ = E_{1,1} + \zeta(n-2)\mathsf{Z}_1 - \zeta n T. \end{aligned} \tag{3.66}$$

So, the closure condition $[\mathsf{X}, \widehat{E}_{0,1}]_{\mathfrak{f}} \in \mathfrak{f}$ holds only if $\widehat{E}_{1,1} = E_{1,1} + \zeta(n-2)\mathsf{Z}_1 - \zeta n T$. This implies $\beta = (n-2)\zeta$ and $\alpha_b = 0$ for *all* b, which proves (b) and concludes our proof.

■

We have the following result for the curvature κ of such an algebraic model. This result is essential for our study of curvatures in §3.7.

COROLLARY 3.6.2. *Fix an irreducible C-class module $\mathbb{U} \subset \mathbb{E}_C \subsetneq \mathbb{E}$, viewed as a G_0-submodule of* $\ker \Box \subset C^2(\mathfrak{g}_-, \mathfrak{g})$ *via (3.21), and consider an algebraic model* $(\mathfrak{f}; \mathfrak{g}, \mathfrak{p})$ *of ODE type with $\kappa_H = \Phi_{\mathbb{U}} \in \mathbb{U}$, for $\Phi_{\mathbb{U}}$ from Table 4, and* $\dim \mathfrak{f} = \mathfrak{S}_{\mathbb{U}}$, *normalized according to Proposition 3.6.1. Then $\mathsf{X} \cdot \kappa = 0$.*

PROOF. Since $\mathbb{U}$ is a C-class module, then, $\kappa \in \bigwedge^2 V^* \otimes \mathfrak{g}$ (Remark 6) for V from §3.3.2. So, for $\mathsf{X} \in \mathfrak{f}$ we have $[\mathsf{X}, z]_{\mathfrak{f}} = [\mathsf{X}, z]$ for all $z \in \mathfrak{f}$. Then, as a consequence of the Jacobi identity we get the claim as follows:

$$(\mathsf{X} \cdot \kappa)(x, y) = [\mathsf{X}, \kappa(x, y)] - \kappa([\mathsf{X}, x], y) - \kappa(x, [\mathsf{X}, y]) = [\mathsf{X}, [x, y]] - \underbrace{[\mathsf{X}, [x, y]_{\mathfrak{f}}]}_{[\mathsf{X}, [x,y]_{\mathfrak{f}}]_{\mathfrak{f}}} + \underbrace{[[\mathsf{X}, x]}_{[\mathsf{X}, x]_{\mathfrak{f}}}, y]_{\mathfrak{f}} - [[\mathsf{X}, x], y] + [x, \underbrace{[\mathsf{X}, y]}_{[\mathsf{X}, y]_{\mathfrak{f}}}]_{\mathfrak{f}} - [x, [\mathsf{X}, y]] = 0. \quad (3.67)$$

■

3.7 Classification of submaximally symmetric vector ODEs of C-class

In this section, we classify (up to the P-action) all algebraic models of ODE type for submaximally symmetric vector ODEs (3.2) of C-class (see the introduction to §3.6) and establish Theorems 3.2.1, and 3.2.2.

3.7.1 Algebraic curvature constraints

For the algebraic models $(\mathfrak{f}; \mathfrak{g}, \mathfrak{p})$ whose possible filtered linear subspaces $\mathfrak{f} \subset \mathfrak{g}$ have been classified in Proposition 3.6.1, we classify their possible curvatures κ below.

PROPOSITION 3.7.1. *Fix an irreducible C-class module $\mathbb{U} = \mathbb{B}_4$, $\mathbb{A}_2^{\mathrm{tf}}$ or $\mathbb{A}_2^{\mathrm{tr}}$ in $\mathbb{E}_C \subsetneq \mathbb{E}$, viewed as a G_0-submodule of* $\ker \Box \subset C^2(\mathfrak{g}_-, \mathfrak{g})$ *via (3.21), and consider an algebraic model* $(\mathfrak{f}; \mathfrak{g}, \mathfrak{p})$ *of ODE type with $\kappa_H = \Phi_{\mathbb{U}} \in \mathbb{U}$, for $\Phi_{\mathbb{U}}$ from Table 4, and* $\dim \mathfrak{f} = \mathfrak{S}_{\mathbb{U}}$, *normalized according to Proposition 3.6.1. Then κ is*

(a) $\mathbb{U} = \mathbb{B}_4 : \ \kappa = \pm \Phi_{\mathbb{U}}$ *(over $\mathbb{C}$, we can take $\kappa = \Phi_{\mathbb{U}}$);*
(b) $\mathbb{U} = \mathbb{A}_2^{\mathrm{tf}} : \ \kappa = \Phi_{\mathbb{U}}$*;*
(c) $\mathbb{U} = \mathbb{A}_2^{\mathrm{tr}} : \ \kappa = \Phi_{\mathbb{U}} + \kappa_4$*, where*

$$\begin{aligned} \kappa_4 = {} & \mu_1 E^{3,1} \wedge E^{0,1} \otimes \mathsf{X} + \mu_2 E^{2,1} \wedge E^{1,1} \otimes \mathsf{X} \\ & - \frac{\mu_1 + \mu_2}{2} \left(E^{2,1} \wedge E^{0,1} \otimes \mathsf{H} + E^{1,1} \wedge E^{0,1} \otimes \mathsf{Y} \right) \\ & + \mu_3 \sum_{a=1}^{m} \left(E^{2,1} \wedge E^{0,a} - E^{2,a} \wedge E^{0,1} + E^{1,a} \wedge E^{1,1} \right) \otimes e_a{}^1, \end{aligned} \quad (3.68)$$

for some $\mu_1, \mu_2, \mu_3 \in \mathbb{R}$.

PROOF. The majority of the proof will consist of evaluating the annihilation conditions $\mathfrak{f}^0 \cdot \kappa = 0$.

Recall from Table 2 that $\mathbb{U}$ has bi-grade either $(1,1)$ or $(2,2)$, so $\mathsf{Z}_1 - \mathsf{Z}_2 \in \mathfrak{ann}(\Phi_{\mathbb{U}})$, which is contained in $\mathfrak{f}^0$ by Proposition 3.6.1. Hence, $(\mathsf{Z}_1 - \mathsf{Z}_2) \cdot \kappa = 0$ implies that κ is the sum of terms with bi-grades that are multiples of $(1,1)$. But κ is regular, lies in $\bigwedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$, and Z_2 acts on the latter with eigenvalues $0, 1$ or 2. Thus, the terms in κ can only have bi-grades $(1,1)$ or $(2,2)$. By Theorem 3.8.3, κ_H can be identified with the lowest Z-degree component of κ. Moreover, κ_H is a nonzero multiple of $\Phi_{\mathbb{U}}$. Using the G_0-action by $\exp(\mathsf{Z}t)$, where $\mathsf{Z} \in \mathfrak{z}(\mathfrak{g}_0)$ is the grading element (§3.3.2), this multiple can be re-scaled to ± 1. For $\mathbb{U} = \mathbb{A}_2^{\rm tf}$ or $\mathbb{A}_2^{\rm tr}$, we can further normalize this multiple to $+1$. (Use the diagonal elements in $g = \operatorname{diag}(a_1, ..., a_m) \in \mathrm{GL}_m \subset G_0$, i.e. $g \cdot \Phi^{i,j} = \frac{1}{a_1}\Phi^{i,j}$ for $\Phi^{i,j}$ from (3.35), while $g \cdot \Phi^{i,j} = \frac{a_m}{(a_1)^2}\Phi^{i,j}$ for $\Phi^{i,j}$ from (3.51).) Summarizing, we have

$$\kappa = \begin{cases} \pm\Phi_{\mathbb{U}}, & \text{when} \quad \mathbb{U} = \mathbb{B}_4, \\ \Phi_{\mathbb{U}} + \kappa_4, & \text{when} \quad \mathbb{U} = \mathbb{A}_2^{\rm tf} \quad \text{or} \quad \mathbb{A}_2^{\rm tr}. \end{cases} \tag{3.69}$$

where κ_4 is the bi-grade $(2,2)$ component of κ. The $\mathbb{B}_4$ case is complete, and we turn to the remaining cases.

Since $\mathbb{U}$ is a C-class module, then by Remark 6 we have $\kappa \in \bigwedge^2 V^* \otimes \mathfrak{g}$ in the notation of §3.3.2. Recall $\mathfrak{g} = \mathfrak{q} \ltimes V$, and $\mathfrak{q}$ and V have Z_2-degrees 0 and -1 respectively (Figure 1). In particular, $\Phi_{\mathbb{U}} \in \bigwedge^2 V^* \otimes V$ and $\kappa_4 \in \bigwedge^2 V^* \otimes \mathfrak{q}$. More precisely, since κ_4 has bi-grade $(2,2)$, then in terms of the dual basis elements $E^{i,a}$ to $E_{i,a}$, having bi-grades $(i,1)$ and $(-i,-1)$ respectively, κ_4 must lie in the subspace $K_4 \subset \bigwedge^2 V^* \otimes \mathfrak{q}$ spanned by

$$\begin{aligned} &E^{1,a} \wedge E^{0,b} \otimes \mathsf{Y}, \quad E^{3,a} \wedge E^{0,b} \otimes \mathsf{X}, \quad E^{2,a} \wedge E^{1,b} \otimes \mathsf{X}, \\ &E^{2,a} \wedge E^{0,b} \otimes \mathsf{H}, \quad E^{1,a} \wedge E^{1,b} \otimes \mathsf{H}, \quad E^{2,a} \wedge E^{0,b} \otimes e_c{}^d, \\ &E^{1,a} \wedge E^{1,b} \otimes e_c{}^d, \end{aligned} \tag{3.70}$$

where $1 \leq a, b, c, d \leq m$. We will further constrain κ_4 as follows. Using Proposition 3.6.1, we have $\mathfrak{ann}(\Phi_{\mathbb{U}}) \subset \mathfrak{f}^0$. Such elements annihilate both $\Phi_{\mathbb{U}}$ and κ, and so

$$z \cdot \kappa_4 = 0, \quad \forall z \in \mathfrak{ann}(\Phi_{\mathbb{U}}). \tag{3.71}$$

Let us use these to find more explicit conditions on κ_4.

(1) $\boxed{\mathbb{U} = \mathbb{A}_2^{\rm tf}}$: from Table 3, we have $p := e_1{}^1 - e_2{}^2 + (2 + \delta_{m-1}{}^1)\mathsf{Z}_2 \in \mathfrak{ann}(\Phi_{\mathbb{U}})$. From $0 = p \cdot \kappa_4$ and $\mathsf{Z}_2 \cdot \kappa_4 = 2\kappa_4$, we have that κ_4 has eigenvalue $\lambda = -2(2 + \delta_{m-1}{}^1)$ for $e_1{}^1 - e_2{}^2$. We conclude that $\kappa_4 = 0$ (hence $\kappa = \Phi_{\mathbb{U}}$) from the following considerations:

(i) $\boxed{m=2}$: We have $\lambda = -6$. Noting that ${e_1}^1 - {e_2}^2$ commutes with $\{\mathsf{X}, \mathsf{H}, \mathsf{Y}\}$, and

$$({e_1}^1 - {e_2}^2) \cdot E^{i,a} = ({\delta_a}^2 - {\delta_a}^1) E^{i,a}, \tag{3.72}$$

$$({e_1}^1 - {e_2}^2) \cdot {e_a}^b = {\delta_a}^1 {e_1}^b - {\delta_1}^b {e_a}^1 - {\delta_a}^2 {e_2}^b + {\delta_2}^b {e_a}^2. \tag{3.73}$$

From (3.70), we conclude that the eigenvalues of ${e_1}^1 - {e_2}^2$ in K_4 lie between -4 and 4. Since -6 is not an eigenvalue, then $\kappa_4 = 0$.

(ii) $\boxed{m \geq 3}$: We have $\lambda = -4$. Proceeding as in (a), we observe that K_4 has -4-eigenspace for ${e_1}^1 - {e_2}^2$ spanned by $E^{2,1} \wedge E^{0,1} \otimes {e_2}^1$. But from Table 3, we also have ${e_{m-1}}^{m-1} - {e_m}^m + \mathsf{Z}_2 \in \mathfrak{ann}(\Phi_{\mathbb{U}})$, which must similarly annihilate κ and κ_4. But its eigenvalue on $E^{2,1} \wedge E^{0,1} \otimes {e_2}^1$ is ${\delta_{m-1}}^2 + 2$, which is nonzero, so $\kappa_4 = 0$ follows.

(2) $\boxed{\mathbb{U} = \mathbb{A}_2^{\mathrm{tr}}}$: from Table 3, we have $q_d := {e_d}^d - {e_{d+1}}^{d+1} + {\delta_1}^d \mathsf{Z}_2 \in \mathfrak{ann}(\Phi_{\mathbb{U}})$, so $0 = q_d \cdot \kappa_4$ for $1 \leq d \leq m-1$. Letting $\mathfrak{h} \subset \mathfrak{sl}_m$ denote the standard Cartan subalgebra consisting of diagonal trace-free matrices, and $\epsilon_a \in \mathfrak{h}^*$ the standard weights for $\mathfrak{h}$, we observe:

(i) $0 = q_d \cdot \kappa_4$ for $1 \leq d \leq m-1$ is equivalent to κ_4 having weight $-2\epsilon_1$;

(ii) the first five elements of (3.70) have weight $-\epsilon_a - \epsilon_b$.

(iii) the last two elements of (3.70) have weight $-\epsilon_a - \epsilon_b + \epsilon_c - \epsilon_d$.

Matching these weights with $-2\epsilon_1$, we deduce that κ_4 lies in the span of the following:

$$\begin{gathered} E^{3,1} \wedge E^{0,1} \otimes \mathsf{X}, \quad E^{2,1} \wedge E^{1,1} \otimes \mathsf{X}, \quad E^{2,1} \wedge E^{0,1} \otimes \mathsf{H}, \quad E^{1,1} \wedge E^{0,1} \otimes \mathsf{Y}, \\ E^{2,1} \wedge E^{0,1} \otimes {e_1}^1, \quad E^{2,1} \wedge E^{0,1} \otimes {e_a}^a, \quad E^{2,1} \wedge E^{0,a} \otimes {e_a}^1, \\ E^{2,a} \wedge E^{0,1} \otimes {e_a}^1, \quad E^{1,a} \wedge E^{1,1} \otimes {e_a}^1, \end{gathered} \tag{3.74}$$

where $2 \leq a \leq m$. Similarly as in §3.4.1, we conclude that imposing annihilation by all of $\mathfrak{ann}(\Phi_{\mathbb{A}_2^{\mathrm{tr}}}) \subset \mathfrak{f}^0$ forces κ_4 to lie in the subspace spanned by

$$\begin{gathered} E^{3,1} \wedge E^{0,1} \otimes \mathsf{X}, \quad E^{2,1} \wedge E^{1,1} \otimes \mathsf{X}, \quad E^{2,1} \wedge E^{0,1} \otimes \mathsf{H}, \quad E^{1,1} \wedge E^{0,1} \otimes \mathsf{Y}, \\ \sum_{a=1}^m E^{2,1} \wedge E^{0,1} \otimes {e_a}^a, \quad \sum_{a=1}^m E^{2,1} \wedge E^{0,a} \otimes {e_a}^1, \quad \sum_{a=1}^m E^{2,a} \wedge E^{0,1} \otimes {e_a}^1, \\ \sum_{a=1}^m E^{1,a} \wedge E^{1,1} \otimes {e_a}^1. \end{gathered} \tag{3.75}$$

Finally, we complete the proof by imposing $\mathsf{X} \cdot \kappa = 0$ (Corollary 3.6.2). Since $\mathsf{X} \cdot \Phi_{\mathbb{A}_2^{\mathrm{tr}}} = 0$ (Lemma 3.3.2), then $\mathsf{X} \cdot \kappa = 0$ implies that

$\mathsf{X} \cdot \kappa_4 = 0$. Now let κ_4 be a general linear combination of all elements of (3.75), i.e. $\kappa_4 = \nu_1 E^{3,1} \wedge E^{0,1} \otimes \mathsf{X} + \nu_2 E^{2,1} \wedge E^{1,1} \otimes \mathsf{X} + \dots + \nu_8 \sum_{a=1}^m E^{1,a} \wedge E^{1,1} \otimes {e_a}^1$, and impose $0 = \mathsf{X} \cdot \kappa_4$ using the actions given in §3.3.2. Namely, $\mathsf{X} \cdot \mathsf{Y} = \mathsf{H}$, $\mathsf{X} \cdot \mathsf{H} = -2\mathsf{X}$, and $\mathsf{X} \cdot {e_a}^b = 0$. Also, $\mathsf{X} \cdot E_{i,a} = E_{i+1,a}$, and so $\mathsf{X} \cdot E^{i,a} = -E^{i-1,a}$. We find that $0 = \mathsf{X} \cdot \kappa_4$ is equivalent to:

$$\nu_3 = \nu_4 = -\frac{\nu_1 + \nu_2}{2}, \quad \nu_5 = 0, \quad \nu_6 = \nu_8 = -\nu_7. \tag{3.76}$$

Setting $(\nu_1, \nu_2, \nu_8) = (\mu_1, \mu_2, \mu_3)$ then yields the result.

■

COROLLARY 3.7.2. *All parameters involved in an algebraic model $(\mathfrak{f}; \mathfrak{g}, \mathfrak{p})$ of ODE type from Proposition 3.7.1 for $\mathbb{U} = \mathbb{A}_2^{\mathrm{tr}}$ are uniquely determined.*

PROOF. Recall from Table 2 that $\mathbb{U} = \mathbb{A}_2^{\mathrm{tr}}$ arises for $n \geq 3$. By Proposition 3.6.1 (b) and Proposition 3.7.1 (c), any algebraic model $(\mathfrak{f}; \mathfrak{g}, \mathfrak{p})$ of ODE type with $\kappa_H = \Phi_{\mathbb{U}} \in \mathbb{U}$, for $\Phi_{\mathbb{U}}$ from Table 4, and $\dim \mathfrak{f} = \mathfrak{S}_{\mathbb{U}}$ has

$$\begin{aligned} \mathfrak{f} = \operatorname{span}\{&E_{n,a}, \dots, E_{2,a}, \widehat{E}_{1,1}, E_{1,b}, \widehat{E}_{0,1}, E_{0,b}, \mathsf{X} \,:\, 1 \leq a \leq m,\ 2 \leq b \leq m\} \\ &\oplus \mathfrak{ann}(\Phi_{\mathbb{U}}), \end{aligned} \tag{3.77}$$

where $\mathfrak{ann}(\Phi_{\mathbb{U}})$ was given in Table 3, and

$$\widehat{E}_{1,1} := E_{1,1} + (n-2)\zeta \mathsf{Z}_1 \in \mathfrak{f}, \qquad \widehat{E}_{0,1} := E_{0,1} + \zeta \mathsf{Y} \in \mathfrak{f} \tag{3.78}$$

for some $\zeta \in \mathbb{R}$. Curvature is $\kappa = \Phi_{\mathbb{A}_2^{\mathrm{tr}}} + \kappa_4$, for κ_4 given in (3.68), and $[\cdot,\cdot]_{\mathfrak{f}} = [\cdot,\cdot] - \kappa(\cdot,\cdot)$.

Let us now impose the Jacobi identity. We define

$$\operatorname{Jac}^{\mathfrak{f}}(x,y,z) := [x,[y,z]_{\mathfrak{f}}]_{\mathfrak{f}} - [[x,y]_{\mathfrak{f}},z]_{\mathfrak{f}} - [y,[x,z]_{\mathfrak{f}}]_{\mathfrak{f}}, \qquad \forall x,y,z \in \mathfrak{f}. \tag{3.79}$$

We calculate

$$\begin{aligned} [\widehat{E}_{1,1}, [E_{0,2}, E_{3,1}]_{\mathfrak{f}}]_{\mathfrak{f}} &= -(n-2)^2 \left(2\zeta + \frac{3(2m+3) - n(4m+3)}{mn(n+1)+6} \right) E_{2,2}, \\ [E_{0,2}, [\widehat{E}_{1,1}, E_{3,1}]_{\mathfrak{f}}]_{\mathfrak{f}} &= -(n-2)^2 \left(3\zeta + \frac{3(3m+5) - n(5m+3)}{mn(n+1)+6} \right) E_{2,2}, \\ [[\widehat{E}_{1,1}, E_{0,2}]_{\mathfrak{f}}, E_{3,1}]_{\mathfrak{f}} &= -\frac{(n-1)(n-2)^2(mn-3)}{mn(n+1)+6} E_{2,2}. \end{aligned} \tag{3.80}$$

so that

$$\operatorname{Jac}^{\mathfrak{f}}(\widehat{E}_{1,1}, E_{0,2}, E_{3,1}) = 0 \quad \text{implies} \quad \zeta = \frac{(2n - n^2 - 3)m + 3n - 9}{mn(n+1)+6}. \tag{3.81}$$

Continuing in a similar manner, we find that:

$$\begin{aligned}
&\operatorname{Jac}^{\mathfrak{f}}(\widehat{E}_{0,1}, E_{1,2}, E_{3,1}) = 0 \quad \text{implies} \quad \mu_1 = \frac{6(n-1)(n-2)(m+1)}{mn(n+1)+6}, \\
&\operatorname{Jac}^{\mathfrak{f}}(\widehat{E}_{1,1}, E_{2,2}, E_{2,1}) = 0 \quad \text{implies} \\
&\qquad \mu_2 = -\frac{6(n-1)(m+1)(m(n^3+n^2-6n+6)+6)}{(mn(n+1)+6)^2}.
\end{aligned} \tag{3.82}$$

Using ζ and μ_1 above, we then have

$$\operatorname{Jac}^{\mathfrak{f}}(\widehat{E}_{0,1}, E_{2,2}, E_{3,1}) = 0 \quad \text{implies} \quad \mu_3 = 1-n. \tag{3.83}$$

As claimed, the parameters $\zeta, \mu_1, \mu_2, \mu_3$ are uniquely determined functions of (n, m).

We remark that the remaining Jacobi identities for $\mathfrak{f}$ are necessarily satisfied because the existence of a submaximally symmetric ODE model in the $\mathbb{A}_2^{\text{tr}}$-branch (see Table 1) guarantees the existence of a corresponding algebraic model of ODE type. (Necessarily, this is equivalent to the one found above.) ■

3.7.2 Conclusion

Fix an irreducible C-class module $\mathbb{U} = \mathbb{B}_4, \mathbb{A}_2^{\text{tr}}$ or $\mathbb{A}_2^{\text{tf}}$ in the effective part $\mathbb{E}$, and recall the respective lowest weight vectors $\Phi_{\mathbb{U}} \in \mathbb{U}$ from Table 4. By Propositions 3.6.1 and 3.7.1, the classification of algebraic models $(\mathfrak{f}; \mathfrak{g}, \mathfrak{p})$ of ODE type with $0 \not\equiv \operatorname{im}(\kappa_H) \subset \mathbb{U}$ and $\dim \mathfrak{f} = \mathfrak{S}_{\mathbb{U}}$ is given in Table 7.

n	Irreducible C-class module $\mathbb{U} \subset \mathbb{E}$	$\mathfrak{f}$	κ
2	$\mathbb{B}_4$	$\mathfrak{a}^{\Phi_{\mathbb{U}}}$	$\begin{cases} \Phi_{\mathbb{U}}, & \text{over } \mathbb{C} \\ \pm\Phi_{\mathbb{U}}, & \text{over } \mathbb{R} \end{cases}$
≥ 3	$\mathbb{A}_2^{\text{tr}}$	$\mathfrak{f}$ in (3.77) with ζ in (3.81)	$\Phi_{\mathbb{U}} + \kappa_4$, with $\beta = 1, \kappa_4$ in (3.68), and μ_1, μ_2, μ_3 in (3.82) and (3.83).
≥ 2	$\mathbb{A}_2^{\text{tf}}$	$\mathfrak{a}^{\Phi_{\mathbb{U}}}$	$\Phi_{\mathbb{U}}$

TABLE 7. Classification of algebraic models of ODE type with $0 \not\equiv \operatorname{im}(\kappa_H) \subset \mathbb{U}$ and $\dim \mathfrak{f} = \mathfrak{S}_{\mathbb{U}}$

We now discuss how the ODE model classification in Table 1 is deduced from the abstract classification in Table 7. Using fundamental invariants described in §3.3.4, we confirm that these ODE lie in the claimed branches. In [26, Table 10], the point symmetries were given for all of these models with the exception of the second $\mathbb{B}_4$ model. (See below for this case.) We confirm submaximal symmetry and deduce the associated algebraic models. (This is immediate by uniqueness in the $\mathbb{A}_2^{\rm tr}$, $\mathbb{A}_2^{\rm tf}$ cases, as well as the $\mathbb{B}_4$ case over $\mathbb{C}$.)

To complete the proof of Theorem 3.2.2, we establish point-inequivalence over $\mathbb{R}$ of the following $\mathbb{B}_4$ models:

$$u_3^a = \frac{3u_2^1 u_2^a}{2u_1^1} \qquad \text{or} \qquad u_3^a = \frac{3u_1^1 u_2^1 u_2^a}{1+(u_1^1)^2} \qquad \text{for} \quad 1 \le a \le m. \tag{3.84}$$

The point symmetries of the former are given in [26, Table 10], from which we observe that the distribution $\ker(du^1)$ on $J^0(\mathbb{R},\mathbb{R}^m)$ is invariant under the action of the symmetry algebra $\mathcal{S}$. This implies that the foliation in $J^0(\mathbb{R},\mathbb{R}^m)$ by level sets of u^1 is also $\mathcal{S}$-invariant, and total differentiation implies that $\{u_1^1 = 0\} \subset J^1(\mathbb{R},\mathbb{R}^m)$ is $\mathcal{S}$-invariant, i.e. the prolonged action of $\mathcal{S}$ on $J^1(\mathbb{R},\mathbb{R}^m)$ is not locally transitive.

In contrast, we now establish that the latter ODE in (3.84) has symmetry algebra that acts locally transitively on $J^1(\mathbb{R},\mathbb{R}^m)$. Its point symmetries are (for $1 \le a \le m$ and $2 \le b \le m$):

$$\begin{gathered}
\partial_t, \quad \partial_{u^a}, \quad t\partial_{u^b}, \quad u^a\partial_{u^b}, \quad (t^2+(u^1)^2)\partial_{u^b}, \quad u^1\partial_t - t\partial_{u^1}, \\
t\partial_t + u^1\partial_{u^1} + 2\sum_{b=2}^m u^b\partial_{u^b}, \quad (t^2-(u^1)^2)\partial_t + 2t\sum_{a=1}^m u^a\partial_{u^a}, \\
tu^1\partial_t + \frac{1}{2}((u^1)^2 - t^2)\partial_{u^1} + u^1\sum_{b=2}^m u^b\partial_{u^b}.
\end{gathered} \tag{3.85}$$

In particular over $\mathbb{R}$, transitivity immediately follows from prolonging some of them to $J^1(\mathbb{R},\mathbb{R}^m)$:

$$\partial_t, \quad \partial_{u^a}, \quad t\partial_{u^b} + \partial_{u_1^b}, \quad u^1\partial_t - t\partial_{u^1} - (1+(u_1^1)^2)\partial_{u_1^1} - u_1^1\sum_{b=2}^m u_1^b\partial_{u_1^b}. \tag{3.86}$$

We conclude that the symmetry algebras of (3.84) are point-inequivalent, and hence the ODEs are point-inequivalent. We also remark that when $m=1$, the two 3rd order ODEs (3.84) are point-inequivalent scalar ODEs with symmetry algebra $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ and $\mathfrak{so}(3,1)$ respectively [2, p. 18].

This completes the proof of Theorem 3.2.2. We have also proven the remaining Theorem 3.2.1(b) since for vector ODEs (3.2) of C-class of order $n+1 \ge 3$, we have $\mathfrak{S} = \mathfrak{M} - 2 = \mathfrak{S}_{\mathbb{B}_4} = \mathfrak{S}_{\mathbb{A}_2^{\rm tf}}$ only when $(n,m) = (2,2)$. This completes our proofs for Theorems 3.2.1 and 3.2.2.

3.8 Appendix A: Harmonic curvature as the lowest degree component of curvature

Fix G and P as in §3.3.2 and recall from §3.3.3 some basic notions of Cartan geometries $(\mathcal{G} \to \mathcal{E}, \omega)$ of type (G, P) associated to ODEs (3.2). We formulate Theorem 3.8.3 below stating that the harmonic curvature κ_H can be identified with the *lowest degree* component (with respect to the grading element) of the curvature κ. (We note that this is used in the proof of Proposition 3.7.1, which is essential in proving Theorems 3.2.1 and 3.2.2.)

DEFINITION 3.8.1. *Let $(\mathcal{G} \to \mathcal{E}, \omega)$ be a Cartan geometry of type (G, P), let $\rho : G \to \mathrm{GL}(V)$ be a G-representation, and $\rho \circ \iota : P \to \mathrm{GL}(V)$ its restriction, where $\iota : P \hookrightarrow G$ is the canonical inclusion. A* tractor bundle *is an associated vector bundle $\mathcal{G} \times_P V$ with respect to the P-representation $\rho \circ \iota$. Given the adjoint representation $\rho = \mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$, the tractor bundle $\mathcal{A}\mathcal{E} := \mathcal{G} \times_P \mathfrak{g}$ is called the* adjoint tractor bundle *(see [8, §1.5.7] for further details).*

Using the Cartan connection ω, the tangent bundle $T\mathcal{E}$ can be identified with the bundle $\mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p})$. Then, the P-invariant quotient map from $\mathfrak{g}$ onto $\mathfrak{g}/\mathfrak{p}$ gives rise to the natural projection $\Pi : \mathcal{A}\mathcal{E} \to T\mathcal{E}$. And using this identification, the curvature $\kappa \in \Omega^2(\mathcal{E}, \mathcal{A}\mathcal{E})$ is a two-form on $\mathcal{E}$ with values in $\mathcal{A}\mathcal{E}$ [8, Prop 1.5.7].

DEFINITION 3.8.2. *Given a Cartan geometry $(\mathcal{G} \to \mathcal{E}, \omega)$ of type (G, P) with curvature $\kappa \in \Omega^2(\mathcal{E}, \mathcal{A}\mathcal{E})$. Then:*

(a) *ω is called* regular *if $\kappa \in (\Omega^2(\mathcal{E}, \mathcal{A}\mathcal{E}))^1$, i.e. $\kappa(T^i\mathcal{E}, T^j\mathcal{E}) \subset \mathcal{A}^{i+j+1}\mathcal{E}$, $\forall i, j < 0$.*

(b) *ω is called* normal *if $\partial^* \kappa = 0$.*

(c) *If ω is both regular and normal, then the* harmonic curvature *is $\kappa_H := \kappa \mod \mathrm{im}(\partial^*)$, which is a section of $\mathcal{G} \times_P \frac{\ker \partial^*}{\mathrm{im}\, \partial^*}$.*

Then, we have the following result.

THEOREM 3.8.3. *Fix G and P as in §3.3.2. Let $(\mathcal{G} \to \mathcal{E}, \omega)$ be a regular, normal Cartan geometry of type (G, P) whose curvature $\kappa \in \left(\Omega^2(\mathcal{E}, \mathcal{A}\mathcal{E})\right)^\ell$ for some $\ell \geq 1$, i.e. $\kappa\left(T^i\mathcal{E}, T^j\mathcal{E}\right) \subset \mathcal{A}^{i+j+\ell}\mathcal{E}$ for all $i, j < 0$. Then the induced section $\mathrm{gr}_\ell(\kappa) \in \mathrm{gr}_\ell\left(\Omega^2(\mathcal{E}, \mathcal{A}\mathcal{E})\right)$ coincides with the degree ℓ component of the harmonic curvature κ_H. Consequently, $\kappa_H \equiv 0$ implies $\kappa \equiv 0$.*

PROOF. The statement was proved in [8, Theorem 3.1.12] for parabolic geometries. The same proof works for our *non-parabolic* Cartan geometries associated to vector ODEs (3.2) of order ≥ 3. ■

3.9 Appendix B: A necessary condition for coclosedness

From §3.4.2, our strategy for computing a $\mathfrak{sl}(W)$-lowest weight vector $\Phi_{\mathbb{A}_2^{\mathrm{tr}}} \in \mathbb{A}_2^{\mathrm{tr}}$ involves imposing coclosedness, i.e. $\partial^*\Phi_{\mathbb{A}_2^{\mathrm{tr}}} = 0$, where ∂^* was defined in §3.3.3. By adjointness of ∂ and ∂^* with respect to the inner product $\langle \cdot, \cdot \rangle$ on cochains induced from Definition 3.3.1, we have:

$$\partial^*\Phi_{\mathbb{A}_2^{\mathrm{tr}}} = 0 \quad \Longleftrightarrow \quad \left\langle \Phi_{\mathbb{A}_2^{\mathrm{tr}}}, \partial\psi \right\rangle = 0, \quad \forall \psi \in \mathfrak{g}_-^* \otimes \mathfrak{g}. \tag{3.87}$$

In order to pin down $\Phi_{\mathbb{A}_2^{\mathrm{tr}}}$ in Proposition 3.4.4, only a small part of the conditions in (3.87) will be in fact required. In this section, we identify a key condition (Lemma 3.9.2) that is essential to the proof of Proposition 3.4.4.

Recalling $\mathfrak{ann}(\Phi_{\mathbb{A}_2^{\mathrm{tr}}})$ given in Table 3, let us restrict attention to ψ lying in the subspace below.

LEMMA 3.9.1. *Suppose that $\psi \in \mathfrak{g}_-^* \otimes \mathfrak{g}$ has bi-grade $(1,1)$, with $\mathsf{X} \cdot \psi = 0$ and $\mathfrak{ann}(\Phi_{\mathbb{A}_2^{\mathrm{tr}}}) \cdot \psi = 0$. Then ψ is a multiple of*

$$\Psi := -2E^{2,1} \otimes \mathsf{X} + E^{1,1} \otimes \mathsf{H} + E^{0,1} \otimes \mathsf{Y}. \tag{3.88}$$

PROOF. Any $\psi \in \mathfrak{g}_-^* \otimes \mathfrak{g}$ with bi-grade $(1,1)$ lies in the span of:

$$E^{2,a} \otimes \mathsf{X}, \quad E^{1,a} \otimes \mathsf{H}, \quad E^{0,a} \otimes \mathsf{Y}, \quad E^{1,a} \otimes e_b{}^c \quad (1 \leq a, b, c \leq m). \tag{3.89}$$

Since $\mathsf{X} \cdot E_{i,a} = E_{i+1,a}$, then $\mathsf{X} \cdot E^{i,a} = -E^{i-1,a}$. Imposing $\mathsf{X} \cdot \psi = 0$ forces ψ to lie in the span of:

$$-2E^{2,a} \otimes \mathsf{X} + E^{1,a} \otimes \mathsf{H} + E^{0,a} \otimes \mathsf{Y} \quad (1 \leq a \leq m). \tag{3.90}$$

Let us now impose $\mathfrak{ann}(\Phi_{\mathbb{A}_2^{\mathrm{tr}}}) \cdot \psi = 0$. Recall from Table 3 that $q_d = e_d{}^d - e_{d+1}{}^{d+1} + \delta_1{}^d \mathsf{Z}_2 \in \mathfrak{ann}(\Phi_{\mathbb{A}_2^{\mathrm{tr}}})$ for $1 \leq d \leq m-1$. Let $\mathfrak{h} \subset \mathfrak{sl}_m$ denote the standard Cartan subalgebra consisting of diagonal trace-free matrices, and $\epsilon_a \in \mathfrak{h}^*$ the standard weights for $\mathfrak{h}$. Since $\mathsf{Z}_2 \cdot \psi = \psi$, then

$$q_d \cdot \psi = 0 \quad \text{for} \quad 1 \leq d \leq m-1 \iff \psi \quad \text{has weight} \quad -\epsilon_1. \tag{3.91}$$

Since each element of (3.90) has weight $-\epsilon_a$, then being of weight $-\epsilon_1$ implies that ψ is a multiple of (3.88). We note that (3.88) is annihilated by all off-diagonal elements $e_f{}^d \in \mathfrak{ann}(\Phi_{\mathbb{A}_2^{\mathrm{tr}}})$, since we have $f \geq 2$ and $e_f{}^d$ commutes with $\{\mathsf{X}, \mathsf{H}, \mathsf{Y}\}$. This completes the proof. ■

In terms of $\Phi^{i,j} = \sum_{a=1}^m E^{i,1} \wedge E^{j,a} \otimes E_{i+j-1,a}$ defined in (3.35), and using ∂ (3.19), we get:

$$\partial\Psi = -2\sum_{k=0}^{n-1} \Phi^{2,k} + \sum_{k=0}^{n} (2k-n)\Phi^{1,k} + \sum_{k=1}^{n} k(n+1-k)\Phi^{0,k}. \tag{3.92}$$

We then have the following necessary condition, which will be used in the proof of Proposition 3.4.4.

LEMMA 3.9.2. *Take $\Phi_{\mathbb{A}_2^{\mathrm{tr}}}$ defined in Proposition 3.4.2, i.e. $\Phi_{\mathbb{A}_2^{\mathrm{tr}}} = \sum_{i,j=0}^{n} c_{i,j}\Phi^{i,j}$ with $c_{i,j}$ satisfying (3.36). Then*

$$
\begin{aligned}
0 = &\sum_{k=0}^{n-1} \frac{(n-k)(k+1)}{n(n-1)}(c_{k,2} - mc_{2,k}) + \sum_{k=0}^{n} \frac{2k-n}{n}(mc_{1,k} - c_{k,1}) \\
&+ \sum_{k=1}^{n}(mc_{0,k} - c_{k,0}).
\end{aligned} \tag{3.93}
$$

PROOF. We evaluate (3.87) for $\psi = \Psi$ given in (3.88). In preparation for this, note that from Definition 3.3.1, we have $\langle E_{k,a}, E_{k,a}\rangle = \frac{k!}{(n-k)!}$ and $\langle E^{k,a}, E^{k,a}\rangle = \frac{(n-k)!}{k!}$, and so

$$
\begin{aligned}
||E^{2,1} \wedge E^{k,a} \otimes E_{k+1,a}||^2 &= ||E^{2,1}||^2||E^{k,a}||^2||E_{k+1,a}||^2 \\
&= \frac{(n-k)(k+1)(n-2)!}{2}.
\end{aligned} \tag{3.94}
$$

Hence, by bilinearity of $\langle\cdot,\cdot\rangle$ and orthogonality of the basis elements for $\mathfrak{g}$ (Definition 3.3.1), we have

$$
\begin{aligned}
\langle\Phi^{2,k}, \Phi_{\mathbb{A}_2^{\mathrm{tr}}}\rangle &= \sum_{i,j=0}^{n}\sum_{a,b=1}^{m} c_{i,j}\left\langle E^{2,1} \wedge E^{k,b} \otimes E_{k+1,b}, E^{i,1} \wedge E^{j,a} \otimes E_{i+j-1,a}\right\rangle \\
&= \sum_{i,j=0}^{n}\sum_{a=1}^{m}({\delta_i}^2{\delta_j}^k - {\delta_i}^k{\delta_j}^2{\delta_a}^1)c_{i,j}||E^{2,1} \wedge E^{k,a} \otimes E_{k+1,a}||^2 \\
&= \sum_{a=1}^{m}(c_{2,k} - c_{k,2}{\delta_a}^1)\frac{(n-k)(k+1)(n-2)!}{2} \\
&= (mc_{2,k} - c_{k,2})\frac{(n-k)(k+1)(n-2)!}{2}
\end{aligned} \tag{3.95}
$$

Similarly, we have

$$
\begin{aligned}
\langle\Phi^{1,k}, \Phi_{\mathbb{A}_2^{\mathrm{tr}}}\rangle &= (n-1)!(mc_{1,k} - c_{k,1}), \\
\langle\Phi^{0,k}, \Phi_{\mathbb{A}_2^{\mathrm{tr}}}\rangle &= \frac{n!}{k(n+1-k)}(mc_{0,k} - c_{k,0}).
\end{aligned} \tag{3.96}
$$

We use these relations and (3.92) to evaluate $0 = \langle\partial\Psi, \Phi_{\mathbb{A}_2^{\mathrm{tr}}}\rangle$ and obtain the claimed result. ■

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