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Reflection groups and cones of sums of squares ☆

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We consider cones of real forms which are sums of squares and invariant under a (finite) reflection group. Using the representation theory of these groups we are able to use the symmetry inherent in these cones to give more efficient descriptions. We focus especially on the A_n, B_n, and D_n case where we use so-called higher Specht polynomials to give a uniform description of these cones. These descriptions allow us, to deduce that the description of the cones of sums of squares of fixed degree $2d$ stabilizes with $n > 2d$. Furthermore, in cases of small degree, we are able to analyze these cones more explicitly and compare them to the cones of non-negative forms.

1. Introduction

A real form (homogeneous polynomial) $f \in \mathbb{R}[X_1, \dots, X_n]$ is called a *sum of squares* if it admits a representation in the form $f = f_1^2 + \dots + f_m^2$ for some real forms $f_1, \dots, f_m \in \mathbb{R}[X_1, \dots, X_n]$ and it is called *positive semidefinite* or *non-negative* if it assumes only non-negative values on $\mathbb{R}^n$. We will denote by $\Sigma_{n,2d}$ the cone of sums of squares forms in n variables of degree $2d$ and by $\mathcal{P}_{n,2d}$ the corresponding cone of non-negative forms. Clearly, every sum of squares is also non-negative, and we therefore have the inclusion $\Sigma_{n,2d} \subset \mathcal{P}_{n,2d}$. Hilbert (1888) addressed and solved the question to characterize the cases, when the two cones coincide. As it turns out this only seldom happens, namely only in the case of bivariate forms ($n = 2$), quadratic forms ($2d = 2$), and ternary quartics

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($n=3, 2d=4$). Sums of squares play a fundamental role in real algebraic geometry and have in the last two decades become also a very important tool for polynomial optimisation (see for example Scheiderer, 2009). Several authors have considered situations in which one supposes that the forms are invariant under the action of a group: For a group $G \subset \mathrm{GL}_n(\mathbb{R})$ we denote by $\mathcal{P}_{n,2d}^G$ and $\Sigma_{n,2d}^G$ the invariant forms in the respective cones. Since this additional requirement can shrink the dimensions of the cones, their study may become more tractable. Furthermore, as presented in Gatermann and Parrilo (2004), representation theory of groups can be particularly used to simplify the sums of squares decomposition. Building on this, it was found in Riener (2011); Riener et al. (2013) that sums of squares invariant under the symmetric group are highly structured, and the complexity of a sum of squares decomposition, in this case, stabilizes with $n \geq 2d$. Furthermore, symmetric sums of squares appear quite naturally in various contexts (for example Raymond et al., 2018). This makes these cones an interesting object of study. Choi and Lam (1977) initiated a systematic study of Hilbert's classification restricted to the case of symmetric forms, and in a collaboration with Reznick they further provided a complete study of the cone of even symmetric sextics (Choi et al., 1987). Whereas they could show that in the sextic case there exists a form that is non-negative but not a sum of squares Harris (1999), who studied the case of even symmetric octics, was able to show that the cone of even symmetric ternary octics that are sums of squares coincides with the non-negative cone. Recently, Goel et al. (2017) constructed even symmetric polynomials of every degree $2d > 8$ and every number of variables $n \geq 3$ which are non-negative but not a sum of squares, so for even symmetric forms Harris' example and the quartics in any number of variables remain the only exceptional cases compared to Hilbert's classification. Despite the classical case analysis done by Hilbert, it can also be interesting to study the quantitative comparison of sums of squares on non-negative polynomials in an asymptotic situation, i.e., when the number of variables grows to infinity. Contrary to the general situation, where for large numbers of variables almost every non-negative form is not a sum of squares (see Blekherman, 2006), a detailed analysis of the symmetric sum of squares cone and symmetric non-negative cone in Blekherman and Riener (2021) showed that this is not the case in the symmetric case.

In this article, we advance the previously mentioned lines of research by focusing on the situation of sums of squares invariant under some families of finite real reflection groups $G \subset \mathrm{GL}_n(\mathbb{R})$. Such groups are generated by a set of orthogonal reflections across hyperplanes passing through the origin. The invariant theory of these groups is well understood and generalizes the theory of symmetric polynomials. Therefore, our setup provides a natural unification and extension to the previously mentioned works on symmetric and even symmetric forms.

Outline of the article and contributions: The beginning of the next section gives a short general introduction to the machinery of symmetry reduction for sums of squares based on linear representation theory. In the case of finite reflection groups these techniques combined with results from invariant theory, and in particular the coinvariant algebra and harmonic polynomials, allow for a concrete description of the quadratic module of invariant sums of squares in Theorem 2.24. The results we give in this second section are similar to previous works, notably (Blekherman and Riener, 2021; Dostert et al., 2017; Gatermann and Parrilo, 2004; Vallentin, 2009).

Section 3 then turns to the special situation of the three infinite families A_n, B_n and D_n of irreducible reflection groups for which we can integrate the notion of the higher Specht polynomials (Ariki et al., 1997; Morita et al., 1998) with the previously mentioned techniques. These polynomials allow for a convenient way to combinatorially describe an isotypic decomposition of the coinvariant algebra in the case of finite reflection groups whose irreducible components fall to the classes A_n, B_n, D_n (see Theorem 3.8). As we show in Theorem 3.11 this combinatorial description then in turn implies a concrete characterization of the cone of invariant sums of squares. In particular, we show in Theorem 3.22 that if the degree $2d$ is fixed and the number of variables n is growing, a stabilization of the isotypic decomposition and a resulting combinatorial stabilization of the structure of the cone of invariant sums of squares is happening in the case of all three families.

Building on these general results we study the cone of even symmetric (i.e., B_n-invariant) forms of degree 8 in more detail in Subsection 4.1. In Theorem 4.1 we obtain an explicit description of the dual cone of even symmetric ternary octics. As one application of this result, we are able to revisit the remarkable finding of Harris, which follow immediately from our description. Furthermore, we provide a complete description of the cone of even symmetric octic sums of squares for all number of variables in Theorem 4.14. Following our discussion of even symmetric forms we turn to forms that are D_n-invariant in Subsection 4.2. We first show in addition to the case of even symmetric ternary quartics also all ternary quartics invariant by the slightly smaller group D_3 are positive semidefinite if and only if they can be written as a sum of squares (see Theorem 4.18). We then examine the dual cone of D_4-invariant quartic sums of squares in Theorem 4.22, which turns out to be simplicial. Similarly to our approach in the even symmetric case this yields in particular that every D_4-invariant quaternary quartic non-negative form is a sum of squares. These results allow us to conclude a complete charaterization of the cases in which for D_n-invariant forms we have an equality between the cones of sums of squares and non-negative forms (see Theorem 4.26). To conclude our considerations, we highlight some connections to non-negativity testing of forms with the help of semidefinite programming in the last subsection. It follows from recent work of Scheiderer (2018) that the cone of non-negative forms in general is not a so-called spectrahedral shadow, i.e., it can in general not be represented as a feasibility set of semidefinite programming. In contrast to this result, we observe that additionally to the cases where the cone of invariant sums of squares coincides with the corresponding cone of non-negative forms, there are cases where we can represent the cone of non-negative forms by projections of sets defined by linear matrix inequalities.

Some of the results presented here are also included in the master's thesis (Debus, 2019) written by the first author at Universität Wien under the supervision of the second author.

2. Invariant sums of squares

2.1. General symmetry reduction

Let $X = (X_1, \ldots, X_n)$ always denote a tuple of variables and write $\mathbb{R}[\underline{X}] = \mathbb{R}[X_1, \ldots, X_n] = \bigoplus_{d \in \mathbb{N}_0} H_{n,d}$ for the polynomial ring in these variables, where $H_{n,d}$ denotes the subspace of forms of degree d. Let $G \subset \mathrm{GL}_n(\mathbb{R})$ be a finite group acting linearly on $\mathbb{R}^n$. This action then naturally gives rise to an action of G on the polynomial ring $\mathbb{R}[X_1, \ldots X_n]$ and thus we can view this $\mathbb{R}$-vector space as a G-module. It follows from Maschke's theorem that this G-module is completely reducible, and thus for any degree d, there exists an isotypic decomposition, i.e., the G-module $H_{n,d}$ decomposes into a direct sum of the form

$$H_{n,d} = V^{(1)} \oplus V^{(2)} \oplus \cdots \oplus V^{(h)} \tag{2.1}$$

with $V^{(j)} = \theta_1^{(j)} \oplus \cdots \oplus \theta_{\eta_j}^{(j)}$ and $\vartheta_j := \dim \theta_i^{(j)}$, where $\theta_{i_1}^{(u)}, \theta_{i_2}^{(v)}$ are G-isomorphic if and only if $u = v$ i.e., we denote by η_j the *multiplicity* of an irreducible G-module and by ϑ_j its *dimension*. Here, the $\theta_i^{(j)}$ are the *irreducible components* and the $V^{(j)}$ are the *isotypic components*, i.e., the direct sum of isomorphic irreducible components. The component with respect to the trivial irreducible representation in $\mathbb{R}[\underline{X}]$ is the invariant ring $\mathbb{R}[\underline{X}]^G$. Note that an irreducible representation $\theta_i^{(j)}$ will occur with infinite multiplicity in $\mathbb{R}[\underline{X}]$. For $f \in \mathbb{R}[\underline{X}]$ we write $\langle f \rangle_G$ for the G-module which is the linear span of $\{\sigma f : \sigma \in G\}$.

It is classically known that $\mathbb{R}[\underline{X}]^G$ is a finitely generated $\mathbb{R}$-algebra, and furthermore each isotypic component in $\mathbb{R}[\underline{X}]$ is a finitely generated $\mathbb{R}[\underline{X}]^G$-module (see Stanley, 1979, Theorem 1.3). These properties follow from the existence of a linear projection onto $\mathbb{R}[\underline{X}]^G$, called the Reynolds operator.

Definition 2.1. For a finite group G the linear map

$$\begin{array}{rccc} \mathcal{R}_G : & H_{n,d} & \longrightarrow & H_{n,d}^G \\ & f & \longmapsto & \frac{1}{|G|} \sum_{\sigma \in G} \sigma(f) \end{array}$$

is called the *Reynolds operator* of G.

Remark 2.2. Although we restrict to finite groups, most of the theory presented in this section can be directly translated to the more general setup of reductive groups.

An important tool for the study of invariant sums of squares is Schur's lemma, which we include for the convenience of the reader.

Lemma 2.3 *(Schur's lemma). Let $\mathbb{K}$ be a field that is algebraically closed and V be a G-module defined over $\mathbb{K}$. Further, let $\mathcal{V}, \mathcal{W}$ denote two irreducible G-submodules of V. Then the G-module $\mathrm{Hom}_G(\mathcal{V}, \mathcal{W})$ of G-homomorphism between $\mathcal{V}$ and $\mathcal{W}$ satisfies $\mathrm{Hom}_G(\mathcal{V}, \mathcal{W}) \simeq \mathbb{K}$ if and only if $\mathcal{V}$ and $\mathcal{W}$ are G-isomorphic. Otherwise, we have $\mathrm{Hom}_G(\mathcal{V}, \mathcal{W}) = 0$.*

Remark 2.4. In the sequel, we will mostly work with G-modules defined over the real numbers. In this setup, one devotes some care to the fact that irreducible representations defined over the reals may be reducible over the complex numbers. This additional difficulty is in fact not hard to overcome and, in particular, in the case of real reflection groups, which are the main focus of this work, all complexifications of real irreducible G-modules remain irreducible (Humphreys, 1990).

Let $\mathcal{V} = \langle f_1 \rangle_G$ be irreducible. As a consequence of Schur's lemma, we obtain that any G-homomorphism $\phi \in \mathrm{Hom}_G(\mathcal{V}, \mathcal{W})$ is uniquely defined by $f_2 := \phi(f_1)$. If further $\phi \neq 0$ then for any $\psi \in \mathrm{Hom}_G(\mathcal{V}, \mathcal{W})$, we have $\psi = \lambda\phi$ for a scalar $\lambda \in \mathbb{K}$. This motivates the following:

Definition 2.5. Let V be a finite dimensional G-module with isotypic decomposition

$$V = \bigoplus_{j=1}^{l} \bigoplus_{i=1}^{\eta_j} \theta_i^{(j)}$$

and $f_{ji} \in \theta_i^{(j)}$ be such that for every j each f_{ji} is the image of f_{j1} under a G-isomorphism. Then $\left(f_{11}, \dots, f_{1\eta_1}, f_{21}, \dots, f_{l\eta_l}\right)$ is called a *symmetry adapted basis* of V.

We point out that while a symmetry-adapted basis of a G-module is usually not a vector space basis, a system of linear generators is given by its G-orbit.

For a $\mathbb{R}$-vector space W we write $\sum W^2$ for the sums of squares of elements in W. Note, an invariant polynomial which can be expressed as a sum of squares in the ring $\mathbb{R}[\underline{X}]$ will not necessarily have a sum of squares decomposition in invariant polynomials, i.e.,

$$\mathbb{R}[\underline{X}]^G \bigcap \sum \mathbb{R}[\underline{X}]^2 \neq \sum (\mathbb{R}[\underline{X}]^G)^2.$$

For instance, the symmetric polynomial $X_1^2 + X_2^2$ cannot be a sum of squares of symmetric polynomials of degree 1.

By integrating the idea of a symmetry-adapted basis together with Schur's lemma, one arrives at the following observation more or less directly (see also Blekherman and Riener, 2021; Cimprič et al., 2009; Gatermann and Parrilo, 2004; Riener et al., 2013 for more details on the following statement).

Theorem 2.6. *Let $\{f_{11}, f_{12}, \dots, f_{l\eta_l}\}$ be a symmetry adapted basis of the G-module $H_{n,d}$ of forms of degree d. Then any G-invariant sum of squares form in $H_{n,2d}^G$ is contained in the set*

$$\sum_{j=1}^{l} \mathcal{R}_G\left(\langle f_{j1}, \dots, f_{j\eta_j} \rangle_{\mathbb{R}}^2\right)$$

In some situations, it is convenient to formulate Theorem 2.6 in terms of matrix polynomials, i.e., matrices with polynomial entries. For two $k \times k$ symmetric matrices A and B we define their inner product as $\langle A, B\rangle = \mathrm{Tr}(AB)$. We define a block-diagonal symmetric matrix B with j blocks $B^{(1)}, \ldots, B^{(j)}$ and

$$B^{(j)} = \big(\mathcal{R}_G(f_{ju} \cdot f_{jv})\big)_{u,v}. \tag{2.2}$$

Then Theorem 2.6 is equivalent to the following statement:

Corollary 2.7. *$g \in \Sigma^G_{n,2d}$ if and only if $g = \langle A_1, B^{(1)}\rangle + \ldots + \langle A_l, B^{(l)}\rangle$ for some $A_j \in \mathbb{R}^{\eta_j \times \eta_j}$ symmetric and positive semidefinite matrices.*

2.2. Representation theory of finite reflection groups

The aim of this subsection is to provide an introduction to the representation theory of finite real reflection groups and how their symmetry can be exploited to reduce complexity in calculations. The presented material is mainly based on work in Blekherman and Riener (2021); Dostert et al. (2017); Gatermann and Parrilo (2004); Riener et al. (2013). For a general overview on reflection groups and their invariant theory, the reader is advised to consult also Humphreys (1990); Lehrer and Taylor (2009).

Definition 2.8. A *real reflection group* is a pair (G, ρ), where G is a finite group and $\rho : G \to \mathrm{GL}_n$ a linear representation of G such that $\rho(G)$ is generated by a set of reflections. A reflection group is *essential* if $\mathbb{R}^n$ does not contain a non-trivial G-submodule.

Usually, we just say that a group G is a *reflection group* and the relevant linear map ρ should be understood from the context.

Example 2.9.

(i) The symmetric group $\mathfrak{S}_n$ on n letters is a reflection group acting via coordinate permutation on $\mathbb{R}^n$. The action of $\mathfrak{S}_n$ on $\mathbb{R}^n$ is not essential, as the linear subspace $\mathbb{R} \cdot (1, \ldots, 1)$ is fixed pointwise. The induced action of $\mathfrak{S}_n$ on $\mathbb{R}^n/\mathbb{R} \cdot (1, \ldots, 1)$ is known as the reflection group of type A_{n-1} and is essential.
(ii) The symmetry group of the regular m-gon is a reflection group denoted by $I_2(m)$ and called *dihedral group*.

Remark 2.10. Any real reflection group can be identified with a direct product of essential reflection groups. The essential real reflection groups have been classified and are precisely the infinite series $A_{n-1}, B_n, D_n, I_2(m)$ and the six exceptional reflection groups $E_6, E_7, E_8, F_4, H_3, H_4$ (see e.g. Humphreys, 1990).

The reflection group of type B_n can be identified with the hyperoctahedral group $\mathfrak{S}_2 \wr \mathfrak{S}_n$ acting on $\mathbb{R}^n$ via sign changing and permutation of coordinates. Then B_n is generated by the reflections at $\{x_i = \pm x_j\}$, for $1 \le i \le j \le n$. Furthermore, D_n can be identified with the subgroup of B_n of index 2, generated by the reflections at $\{x_i = \pm x_j\}$, for $1 \le i < j \le n$. D_n is usually called the group of "even sign changes".

Theorem 2.11 *(Chevalley-Shephard-Todd). Let G be a finite group and let G act linearly on $\mathbb{R}^n$. Then the invariant ring $\mathbb{R}[\underline{X}]^G$ is $\mathbb{R}$-algebra isomorphic to a polynomial ring if and only if G is a real reflection group. Moreover, in this case $\mathbb{R}[\underline{X}]^G$ is generated by n algebraically independent forms $\psi_1, \ldots, \psi_n$, i.e., $\mathbb{R}[\underline{X}]^G = \mathbb{R}[\psi_1, \ldots, \psi_n]$.*

While the generators are not unique but well explored (e.g., the elementary symmetric or the power sum polynomials are generators of $\mathbb{R}[\underline{X}]^{\mathcal{S}_n}$), the multisets of their degrees $\{d_1, \ldots, d_n\}$ are unique and $\prod_i d_i = |G|$ (see e.g. Humphreys, 1990 for further details).

Definition 2.12. Let G be a reflection group which acts linearly on $\mathbb{R}^n$ and $\mathbb{R}[\underline{X}]^G = \mathbb{R}[\psi_1, \dots, \psi_n]$. The forms $\psi_1, \dots, \psi_n$ are the *fundamental invariants* of G. Let $(d_1, \dots, d_n)$ be the ordered *sequence of degrees* of the fundamental invariants. We define

$$N_G(k) := |\{(\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n \,:\, \alpha_1 d_1 + \ldots + \alpha_n d_n = k\}|.$$

With this definition, the following is a direct consequence of Theorem 2.11.

Corollary 2.13. *Let G be a finite reflection group. The dimension of the vector space of G-invariant forms of degree d equals $N_G(d)$, i.e.,* $\dim H^G_{n,d} = N_G(d)$.

Example 2.14.

(i) $\mathbb{R}[\underline{X}]^{\mathfrak{S}_n} = \mathbb{R}[e_1, e_2, \dots, e_n] = \mathbb{R}[p_1, p_2, \dots, p_n]$, where $e_j(\underline{X}) := \sum_{I \subset [n]:|I|=j} \prod_{i \in I} X_i$ are the elementary symmetric and $p_j(\underline{X}) := \sum_{i=1}^n X_i^j$ are the power sum polynomials.
(ii) $\mathbb{R}[\underline{X}]^{B_n} = \mathbb{R}[e_1(\underline{X}^2), e_2(\underline{X}^2), \dots, e_n(\underline{X}^2)] = \mathbb{R}[p_2, p_4, \dots, p_{2n}]$, where $\underline{X}^2 := (X_1^2, \dots, X_n^2)$.
(iii) $\mathbb{R}[\underline{X}]^{D_n} = \mathbb{R}[p_2, p_4, \dots, p_{2n-2}, e_n]$.
(iv) $\mathbb{R}[\underline{X}]^{I_2(m)} = \mathbb{R}[X_1^2 + X_2^2, (X_1 + \sqrt{-1}X_2)^m + (X_1 - \sqrt{-1}X_2)^m]$.

Remark 2.15. For $\lambda := (\lambda_1, \dots, \lambda_l) \in \mathbb{N}^l$ we write $p_\lambda := p_{\lambda_1} \cdots p_{\lambda_l}$ for the l products of the power sums p_{λ_i} and analogously e_λ for the products of elementary symmetric polynomials.

From a computational perspective, invariant theory as outlined above can be used to reduce computations for polynomials in $\mathbb{R}[\underline{X}]$ to the smaller ring $\mathbb{R}[\underline{X}]^G$. Since $\mathbb{R}[\underline{X}]$ is in general a finite $\mathbb{R}[\underline{X}]^G$-module, the quadratic module $\mathbb{R}[\underline{X}]^G \bigcap \sum \mathbb{R}[\underline{X}]^2$ can be described conveniently. We outline this in the case of reflection groups below by using the coinvariant algebra and a theorem of Chevalley.

Definition 2.16. The quotient algebra of the polynomial ring modulo the ideal generated by the non-constant elements of the invariant ring is called the *coinvariant algebra* of G and is denoted by $\mathbb{R}[\underline{X}]_G$, i.e.,

$$\mathbb{R}[\underline{X}]_G = \mathbb{R}[\underline{X}] / (\psi_1, \dots, \psi_n)_{\mathbb{R}[\underline{X}]}.$$

Note, by definition the coinvariant algebra of G has the structure of a G-module.

Theorem 2.17 *(Chevalley, 1995). Let G be a real reflection group acting linearly on $\mathbb{R}^n$. Then the coinvariant algebra $\mathbb{R}[\underline{X}]_G$ is isomorphic as a G-module to the regular representation and*

$$\mathbb{R}[\underline{X}] \simeq \mathbb{R}[\underline{X}]^G \otimes_{\mathbb{R}} \mathbb{R}[\underline{X}]_G$$

as graded $\mathbb{R}$-modules.

Remark 2.18. Note that any irreducible representation θ occurs $\dim\theta$ many times in the regular representation. Therefore, Theorem 2.17 yields that the multiplicities of the different irreducible representations $\theta^{(j)}$ appearing in $\mathbb{R}[X]_G$ are equal to the corresponding dimensions ϑ_j.

Corollary 2.19. *Let $\mathbb{R}[\underline{X}]^G = \mathbb{R}[\psi_1, \dots, \psi_n]$ be a polynomial ring in the fundamental invariants $\psi_1, \dots, \psi_n$ and let $\mathbb{R}[\underline{X}]_G = \bigoplus_{j=1}^l \vartheta_j \theta^{(j)}$ be the isotypic decomposition of the coinvariant algebra. Then there exists a symmetry adapted basis $f_{11}, \dots, f_{l\vartheta_l} \in \mathbb{R}[\underline{X}]$ of $\mathbb{R}[\underline{X}]_G$ and any $f \in \mathbb{R}[\underline{X}]$ can be written as a sum of polynomials of the form*

$$\sum_{j=1}^{l}\sum_{i=1}^{\vartheta_j}\sum_{\sigma\in G} g_{ji,\sigma}\sigma f_{ji}$$

for some $g_{ji,\sigma} \in \mathbb{R}[\underline{X}]^G$.

Proof. The existence of the symmetry adapted basis $(f_{11},\dots,f_{l\vartheta_l})$ of $\mathbb{R}[\underline{X}]_G$ follows by Schur's Lemma 2.3. Further, by definition, the G-orbit of $(f_{11},\dots,f_{l\vartheta_l})$ spans the coinvariant algebra. The claim follows from the graded tensor decomposition in Theorem 2.17, since the basic tensors of $\mathbb{R}[\underline{X}]$ are elements described above. □

Note that second summation in the representation of a polynomial in Corollary 2.19 goes actually up to ϑ_j as the multiplicity η_j of an irreducible representation $\theta^{(j)}$ in the coinvariant algebra equals its dimension ϑ_j.

Remark 2.20. The calculation of a symmetry-adapted basis of the coinvariant algebra easily allows computation of the isotypic decomposition of the G-module $H_{n,d}$ for any degree. As a rough general procedure, one needs to compute the products of elements from the symmetry-adapted basis with fundamental invariants of G, such that the degree of the obtained forms equals d.

Definition 2.21. Let $S := \{s_1,\dots,s_{|G|}\}$ be a basis of $\mathbb{R}[\underline{X}]_G$. Then we define the matrix polynomial $H^S(\psi_1,\dots,\psi_n) \in \mathbb{R}[\psi_1,\dots,\psi_n]^{|G|\times|G|}$ entry wise

$$H^S_{u,v} := \mathcal{R}_G(s_u \cdot s_v),$$

and each entry $\mathcal{R}_G(s_u \cdot s_v)$ is expressed as a polynomial in the fundamental invariants $\psi_1,\dots,\psi_n$.

Lemma 2.22. *Let $f \in \mathbb{R}[\underline{X}]$ be G-invariant and let $\gamma \in \mathbb{R}[\psi_1,\dots,\psi_n]$ with $\gamma(\psi_1,\dots,\psi_n) = f$ then f is a sum of squares if and only if $\gamma(\psi_1,\dots,\psi_n)$ admits a representation of the form*

$$\gamma = \langle T, H^S\rangle$$

where T is a sum of squares matrix polynomial, i.e., $T = L^T L$ for a matrix $L \in \mathbb{R}[\psi_1,\dots,\psi_n]^{n\times m}$ and an integer $1 \le m \le n$.

Proof. This follows from the decomposition $\mathbb{R}[\underline{X}] \simeq \mathbb{R}[\underline{X}]^G \otimes \mathbb{R}[\underline{X}]_G$ in Theorem 2.17. □

Definition 2.23. For every irreducible representation $\theta^{(j)}$ of G we construct a matrix polynomial $H^{\vartheta_j} \in \mathbb{R}[\psi_1,\dots,\psi_n]^{\eta_j\times\eta_j}$ in the following way: Let $\mathbb{R}[\underline{X}]_G = \bigoplus_{j=1}^{l} \mathbb{R}[\underline{X}]_G^{\vartheta_j}$ be the isotypic decomposition of the coinvariant algebra and $\{s_{1,1},\dots,s_{1,\eta_1},s_{2,1},\dots,s_{l,\eta_l}\}$ be a symmetry adapted basis of $\mathbb{R}[\underline{X}]_G$. Then we define

$$H^{\vartheta_j}_{u,v} = \mathcal{R}_G(s_{j,u} \cdot s_{j,v}).$$

Combining the above definition and lemma, and the results from Schur's lemma we immediately obtain

Theorem 2.24. *Let G be a finite reflection group with $\mathbb{R}[\underline{X}]^G = \mathbb{R}[\psi_1,\dots,\psi_n]$. Then,*

$$\Sigma\mathbb{R}[\underline{X}]^2 \cap \mathbb{R}[\underline{X}]^G = \left\{ g \in \mathbb{R}[\psi_1,\dots,\psi_n] \,:\, g = \sum_{j=1}^{l} \langle H^{\vartheta_j}, A_j\rangle \right\},$$

where $A_j \in \mathbb{R}[\psi_1,\dots,\psi_n]^{\eta_j\times\eta_j}$ is a sum of squares matrix polynomial.

Example 2.25. Let $f \in \mathbb{R}[X_1, X_2]$ be a form of degree $2d$ which is invariant under the dihedral group $I_2(k)$. The dihedral group $I_2(k)$ has only irreducible representations of dimension 1 or 2. In fact, if k is odd (resp. even), then 2 (resp. 4) representations of dimension one and $\frac{k-1}{2}$ (resp. $\frac{k-2}{2}$) representations of dimension two. By block-diagonalisation we end up with $H^S(z)$ having 2 (resp. 4) 1×1 blocks $H^{\theta_1}, H^{\theta_2}$ (resp. $H^{\theta_1}, \dots, H^{\theta_4}$) and $\frac{k-1}{2}$ (resp. $\frac{k-2}{2}$) 2×2 blocks $H^{\theta_3}, \dots, H^{\theta_{\frac{k+3}{2}}}$ (resp. $H^{\theta_5}, \dots, H^{\theta_{\frac{k+6}{2}}}$). Then for n odd (resp. even) f non-negative if and only if there exist sums of squares matrix polynomials $A_j \in \mathbb{R}[X_1^2 + X_2^2, (X_1 + \sqrt{-1}X_2)^k + (X_1 - \sqrt{-1}X_2)^k]^{\dim\theta_j \times \dim\theta_j}$ such that

$$f = \sum_{j=1}^{m} \langle H^{\theta_j}, A_j \rangle$$

and $m = \frac{k+3}{2}$ (resp. $m = \frac{k+6}{2}$).

For $k = 3$ the coinvariant algebra $\mathbb{R}[x, y]_{I_2(3)}$ decomposes into the direct sum of

$$\theta^{(1)} = \langle 1 \rangle, \theta^{(2)} = \langle -x^3 + 3xy^2 \rangle, \theta_1^{(3)} = \langle x \rangle_{I_2(3)}, \theta_2^{(3)} = \langle xy \rangle_{I_2(3)},$$

where $\theta_1^{(3)}$ and $\theta_2^{(3)}$ are $I_2(3)$-isomorphic via $x \mapsto xy$. Then we find:

$$H^{\theta^{(1)}} = 1, \;\; H^{\theta^{(2)}} = \mathcal{R}_{I_2(3)}(3xy^2 - x^3)^2,$$
$$H^{\theta^{(3)}} = \begin{pmatrix} \mathcal{R}_{I_2(3)}(x^2) & \mathcal{R}_{I_2(3)}(x^2 y) \\ \mathcal{R}_{I_2(3)}(x^2 y) & \mathcal{R}_{I_2(3)}(x^2 y^2) \end{pmatrix}.$$

Definition 2.26. Let G be a finite real reflection group and θ be an irreducible representation. We denote by h_k^ϑ the multiplicity of θ in the isotypic decomposition of the degree k part of the covariant algebra $(\mathbb{R}[\underline{X}]_G^\theta)_k$.

In order to study the sums of squares of a given degree the following direct consequence of Theorem 2.17 which relates the dimension of the space of G-invariant forms of degree d, $N_G(d)$ (see 2.13) to the multiplicity of an irreducible representation in $H_{n,d}$ will be helpful.

Corollary 2.27. *Let G be a finite reflection group and θ be an irreducible representation. Then the multiplicity of the corresponding irreducible representation in the G-module $H_{n,d}$ equals*

$$\sum_{k=0}^{d} N_G(d-k) \cdot h_k^\vartheta .$$

2.3. G-harmonic polynomials

In this subsection we present a specific basis of the coinvariant algebra for reflection groups which can be simply computed.

Definition 2.28. For a polynomial $f(\underline{X}) = \sum_\alpha c_\alpha \underline{X}^\alpha \in \mathbb{R}[\underline{X}]$ we define $f(\partial)$ as the linear operator

$$\begin{array}{rccc} f(\partial): & \mathbb{R}[\underline{X}] & \longrightarrow & \mathbb{R}[\underline{X}] \\ & g & \longmapsto & \sum_\alpha c_\alpha \frac{\partial^\alpha}{(\partial \underline{X})^\alpha} g \end{array}.$$

I.e., $f(\partial)$ is a linear map which is a formal sum of scaled partial derivatives.

Example 2.29. Let $f(\underline{X}) = X_1^2 + X_1X_2 \in \mathbb{R}[X_1, X_2, X_3]$, then $f(\partial) = \frac{\partial^2}{\partial X_1 \partial X_1} + \frac{\partial^2}{\partial X_1 \partial X_2}$ and $f(\partial)\big(X_1^2 + X_2^2 + X_3^2 + X_1X_2X_3\big) = 2 + X_3$.

Definition 2.30. Let G be a real reflection group and $\mathbb{R}[\underline{X}]^G = \mathbb{R}[\psi_1, \psi_2, \ldots, \psi_n]$. Let $\mathcal{H}_G$ be the $\mathbb{R}$-vector space defined as $\mathcal{H}_G := \left(\mathbb{R}[\underline{X}]^G_{>0}\right)^{\perp}$ with respect to the inner product

$$\begin{array}{rccc} \langle \cdot, \cdot \rangle : & \mathbb{R}[\underline{X}] \times \mathbb{R}[\underline{X}] & \longrightarrow & \mathbb{R} \\ & (f, g) & \longmapsto & \mathrm{ev}_{(0,\ldots,0)}\left(f(\partial) g(\underline{X})\right). \end{array}$$

Any $f \in \mathcal{H}_G$ is called a *harmonic polynomial* and $\mathcal{H}_G$ is called the *vector space of G-harmonic polynomials*.

Theorem 2.31 *(Bergeron, 2009). Let G be a real reflection group and $\Delta := \prod L_i$, be the product of a minimal system of linear polynomials defining the reflection hyperplanes. Then, the vector space of G-harmonic polynomials $\mathcal{H}_G$ is generated by all partial derivatives of Δ, i.e., $\mathcal{H}_G = \langle \frac{\partial^\alpha}{(\partial \underline{X})^\alpha} \Delta : \alpha \in \mathbb{N}_0^n \rangle_{\mathbb{R}}$. Furthermore, $\mathcal{H}_G$ is G-module isomorphic to the regular representation of G and $\mathbb{R}[\underline{X}] = \mathbb{R}[\underline{X}]^G \otimes_{\mathbb{R}} \mathcal{H}_G$.*

Remark 2.32. Note that $\mathcal{H}_G$ consists precisely of those polynomials which vanish under all G-invariant differential operators annihilating the constants. This notion of G-harmonic polynomials can be in fact defined for any group of transformations and under some general assumptions similar representations of polynomials in terms of invariant polynomials and harmonic polynomials can be obtained (Helgason, 2022, Chapter 3).

Remark 2.33. Let G be a real reflection group, $\psi_1, \ldots, \psi_n$ be the fundamental invariants and consider the map

$$\begin{array}{rccc} \Psi : & \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & X & \longmapsto & (\psi_1(X), \ldots, \psi_n(X)). \end{array}$$

Then by a statement of Steinberg (1960) we have

$$\Delta = c \cdot \det(\mathrm{Jac}\,\Psi)$$

where $\mathrm{Jac}\,\Psi$ denotes the Jacobian of Ψ and c is a non-zero scalar. Thus up to a scalar multiple, the determinant is independent of the choice of fundamental invariants $\psi_1, \ldots, \psi_n$.

Example 2.34. For $\mathfrak{S}_n$ the symmetric group acting on $\mathbb{R}^n$ via coordinate permutation and $\psi_i = \sum_{j=1}^n X_j^i$ the power sums, we obtain $\Delta = \prod_{i<j}(x_i - x_j)$ equals the determinant of the Vandermonde matrix, which is precisely the product over all reflections of $\mathfrak{S}_n$, and $\det(\mathrm{Jac}\,\Psi) = n! \cdot \Delta$.

Remark 2.35. Computing a basis of the coinvariant algebra $\mathbb{R}[\underline{X}]_G = \mathbb{R}[\underline{X}]/\mathbb{R}[\underline{X}]^G_{>0}$, which is defined as a quotient may be challenging and involves the calculation of a Gröbner basis. However, the approach using harmonic polynomials can be more efficient. As the fundamental invariants of real reflection groups are well-known, one can simply calculate the polynomial Δ and all its partial derivatives and one is only faced with a problem in linear algebra.

2.4. Convex geometric properties of Σ^G and $\mathcal{P}^G$

Convex cones and the dual cones have been studied extensively in the research on non-negativity versus sums of squares (see e.g. Blekherman, 2012 on Hilbert's inequality cases or Blekherman et al., 2012). In this subsection, we present known and adapted knowledge on the convex geometrical properties of $\Sigma^G_{n,2d}$ and $\mathcal{P}^G_{n,2d}$. We refer to Blekherman and Riener (2021, Subsection 4.5) for more details.

The sets $\Sigma^G_{n,2d}$ and $\mathcal{P}^G_{n,2d}$ are *convex cones*, i.e., they are convex sets which are closed under scalar multiplication by non-negative scalars. Moreover, these sets are closed and *pointed*, i.e., they do not contain a non-trivial linear subspace. We refer to Blekherman (2006) for details.

Remark 2.36. To study the set $\Sigma_{n,2d}^{G,*}$ we associate the elements in $\Sigma_{n,2d}^{G,*}$ with positive semidefinite quadratic forms. We associate a linear functional $\ell \in H_{n,2d}^{G,*}$ with a G-invariant quadratic form Q_ℓ defined as

$$\begin{array}{rccc} Q_\ell : & H_{n,d} & \longrightarrow & \mathbb{R} \\ & f & \longmapsto & \ell\left(\mathcal{R}_G(f^2)\right). \end{array}$$

Note, although ℓ is defined on the space of invariant forms, the quadratic form Q_ℓ is defined on the space of all forms.

Since our considered polynomials are homogeneous we have the following description of the dual cone of invariant non-negative forms. For $a \in \mathbb{R}^n$ we write ev_a for the *point-evaluation* of a, i.e.,

$$\begin{array}{rccc} \mathrm{ev}_a : & \mathbb{R}[\underline{X}] & \longrightarrow & \mathbb{R} \\ & f(\underline{X}) & \longmapsto & f(a). \end{array}$$

Recall that the dual cone of a convex cone $K \subset \mathbb{R}^N$ is denoted by K^* and is defined as

$$K^* = \left\{\ell \in \mathbb{R}^{N,*} : \ell(K) \subseteq \mathbb{R}_{\geq 0}\right\}.$$

Proposition 2.37 *(Blekherman, 2006). The dual cone of the non-negative invariant forms is the convex cone that is generated by all point-evaluations, i.e.,*

$$\mathcal{P}_{n,2d}^{G,*} = \mathrm{cone}\{\mathrm{ev}_a : a \in \mathbb{S}^{n-1}\}.$$

By duality any $f \in \mathcal{P}_{n,2d}^G$ contained in the boundary of $\mathcal{P}_{n,2d}^G$ has a real zero. We formulate the dual version of Theorem 2.6.

Lemma 2.38. *Let $\ell \in H_{n,2d}^{G,*}$ and $\{f_{11}, \dots, f_{1\eta_1}, f_{21}, \dots, f_{l\eta_l}\}$ be a symmetry adapted basis of $H_{n,d}$ and $B^{(j)} = \left(\mathcal{R}_G(f_{ju} \cdot f_{jv})\right)_{u,v}$. Then $\ell \in \Sigma_{n,2d}^{G,*}$ if and only if $\ell(B_j)$ is positive semidefinite for all $j = 1, \dots, l$.*

The following lemma enables the characterization of extremal elements via their kernels.

Lemma 2.39 *(Blekherman, 2012, Lemma 2.2). Let V be a $\mathbb{R}$-vector space, $\mathcal{A}$ the vector space of quadratic forms on V and $\mathcal{A}^+ \subset \mathcal{A}$ the cone of positive semidefinite quadratic forms. Let L be a linear subspace of $\mathcal{A}$ and K be the section of $\mathcal{A}^+$ with L, i.e., $K = \mathcal{A}^+ \cap L$. Then a quadratic form $Q \in K$ spans an extreme ray of K if and only if its kernel is maximal among all kernels of quadratic forms in L, i.e., if $\ker Q \subseteq \ker P$ for a $P \in L$, it is $P = \lambda Q$ for some $\lambda \in \mathbb{R}$.*

In order to examine the kernels of invariant quadratic forms, we use the following construction. For a linear subspace $W \subset H_{n,d}$, we define its *quadratic symmetrization* with respect to G as

$$W^{<2>} := \left\{h \in H_{n,2d}^G : h = \mathcal{R}_G\left(\sum f_i g_i\right) \text{ for } f_i \in W \text{ and } g_i \in H_{n,d}\right\}.$$

In order to characterize the extreme rays of $\Sigma_{n,2d}^{G,*}$ we use Lemma 2.38 to identify $\Sigma_{n,2d}^{G,*}$ with a linear section of the cone of positive semidefinite quadratic forms on $H_{n,d}$ with the subspace of G-invariant quadratic forms on $H_{n,d}$.

Proposition 2.40 *(Blekherman and Riener, 2021). An element $\ell \in \Sigma_{n,2d}^{G,*}$ is extremal if and only if $\ker Q_\ell$ is maximal among all kernels of G-invariant quadratic forms on $H_{n,d}$. Let $W := \ker Q_\ell$, then $W^{<2>}$ is equal to the kernel of ℓ. Moreover, if $\left(f_{11}, \dots, f_{l\eta_l}\right)$ is a symmetry adapted basis of $H_{n,d}$ and $\left(g_{11}, \dots, g_{l\eta_l'}\right)$ is a* *symmetry adapted basis of W such that $\langle g_{ji_1}\rangle_G \simeq_G \langle f_{ji_2}\rangle_G$ and $g_{ji_1} \mapsto f_{ji_2}$ define the unique G-isomorphism, then*

$$W^{\langle 2\rangle} = \langle \mathcal{R}_G(g_{ji_1} \cdot f_{ji_2}) : 1 \leq j \leq l, 1 \leq i_1 \leq \eta'_j, 1 \leq i_2 \leq \eta_j\rangle_{\mathbb{R}}.$$

Proof. The first claim follows from Lemma 2.39. The second claim follows from the positive semidefiniteness of the quadratic form Q_ℓ. The complexity reduction gives the above description of $W^{\langle 2\rangle}$ according to the use of a symmetry-adapted basis and by applying Schur's lemma. □

In order to prove equality or inequality between the sets $\Sigma^G_{n,2d}$ and $\mathcal{P}^G_{n,2d}$ we will make use the following dual approach.

Corollary 2.41. *Suppose the convex cones $\Sigma^G_{n,2d}, \mathcal{P}^G_{n,2d}$ are full dimensional. Then $\Sigma^G_{n,2d} = \mathcal{P}^G_{n,2d}$ if and only if any extremal ray in $\Sigma^{G,*}_{n,2d}$ is generated by a point-evaluation.*

Proof. The primal cones $\mathcal{P}^G_{n,2d}$ and $\Sigma^G_{n,2d}$ are equal if and only if the dual cones are equal. By Minkowski's theorem, any $\ell \in \Sigma^{G,*}_{n,2d}$ can be written as a sum of extremal elements. If any extremal ray in $\Sigma^{G,*}_{n,2d}$ is generated by a point-evaluation, then there exists a set $M \subset \mathbb{R}^n$ such that

$$\mathcal{P}^{G,*}_{n,2d} \subseteq \Sigma^{G,*}_{n,2d} = \text{cone}\{\text{ev}_a : a \in M\} \subset \text{cone}\{\text{ev}_a : a \in \mathbb{S}^{n-1}\} = \mathcal{P}^{G,*}_{n,2d}$$

where the last equality follows by Proposition 2.37.

Conversely, if $\Sigma^G_{n,2d} = \mathcal{P}^G_{n,2d}$ then also the dual cones are equal. However, $\mathcal{P}^{G,*}_{n,2d}$ is the convex cone that is generated by all point-evaluations. Hence, any extremal ray in $\Sigma^{G,*}_{n,2d}$ is generated by a point-evaluation. □

3. Sums of squares invariant under A_n, B_n, and D_n

In this section we present an algorithmic approach for calculating a symmetry-adapted basis of the coinvariant algebra for reflection groups of type A_{n-1}, B_n or D_n which was introduced by Morita et al. (1998). We prove stabilization of the isotypic decomposition for a fixed degree and a large enough number of variables, for these series of essential reflection groups.

3.1. Higher Specht polynomials

A well-known classical construction of the irreducible $\mathfrak{S}_n$-modules in the real polynomial ring is due to Specht (1935). The $\mathfrak{S}_n$-generators of these representations are called *Specht polynomials*. However, we are interested in the decomposition of the coinvariant algebra. An elegant combinatorial algorithm to decompose the coinvariant algebra into all irreducible submodules for all pseudoreflection groups of type $G(r,p,n)$ was introduced in Morita et al. (1998). In the following, we briefly present their work. We begin by recalling some basic definitions from combinatorics.

Definition 3.1. A non-increasing sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_l)$ is called a *partition* and l is the *length* of λ. We denote by $|\lambda| = \sum_{i=1}^{l} \lambda_i = n$ the *value* of λ and say that λ is a partition of n, which we denote by $\lambda \vdash n$, if $|\lambda| = n$. For partitions λ^1 and λ^2 we call the pair $\Lambda = (\lambda^1, \lambda^2)$ a *bipartition* and allow $\lambda^1 = \emptyset$ or $\lambda^2 = \emptyset$. We say that $|\Lambda| = |\lambda^1| + |\lambda^2| = n$ is the value of Λ and write $\Lambda \vdash n$ when Λ is a bipartition of n.

We always denote bipartitions with capital letters and partitions with small letters. Note that every partition λ naturally defines also a bipartition $(\lambda, \emptyset)$.

Definition 3.2.

(1) For a given partition $\lambda = (\lambda_1, \dots, \lambda_l) \vdash n$ the *Young diagram* associated to λ is an arrangement of n boxes that are left-aligned and top-aligned, such that the i-th row contains exactly λ_i many boxes. Filling the n boxes with the distinct integers in $[n]$, one obtains a *Young tableau* or *tableau* of shape λ. If the numbers in all columns and rows are increasing we call the tableau *standard*.
(2) Given a bipartition Λ we can associate in the same way a pair of Young diagrams. A *Young bitableau* or *bitableau* is a filling of both diagrams with all the numbers in $[n]$ and we call it standard if both diagrams are standard. We denote by $\mathrm{YT}(\Lambda)$ the set of (bi-)tableaux of shape Λ and by $\mathrm{SYT}(\Lambda)$ the subset of standard (bi-)tableaux.

Example 3.3.

(1) The Young diagram associated to $(3, 3, 1)$ is $\begin{array}{|c|c|c|}\hline &&\\\hline &&\\\hline \\\cline{1-1}\end{array}$.
(2) For the partition $(3, 1)$ the tableau $\begin{array}{|c|c|c|}\hline 1&4&3\\\hline 2\\\cline{1-1}\end{array}$ is not standard and the set of all standard tableaux of shape $(3, 1)$ is $\left\{\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2\\\cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4\\\cline{1-1}\end{array}\right\}$.
(3) The bitableau $\left(\begin{array}{|c|c|c|}\hline 2&3&4\\\hline 5\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&6\\\hline\end{array}\right)$ is standard, while $\left(\begin{array}{|c|c|c|}\hline 1&3&2\\\hline 4\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 5&6\\\hline\end{array}\right)$ is not standard.

In the following, we will denote an irreducible representation indexed by a (bi-)partition Λ by $\mathbb{S}^\Lambda$, i.e., $\mathbb{S}^\Lambda$ is a *Specht module*. The underlying group should be clear from the context.

The famous Robinson-Schensted correspondence gives a bijection between the standard tableaux of shape λ and the elements in the conjugacy class of $\mathfrak{S}_n$ which are labelled by λ. Hence, this number equals the multiplicity of the Specht module $\mathbb{S}^\lambda$ in the coinvariant algebra. The correspondence has been adapted to pseudoreflection groups of type $G(r, p, n)$ and in particular, for the contained series of reflection groups of types $B_n = G(2, 1, n)$ and $D_n = G(2, 2, n)$, e.g., see Caselli (2011, Section 10).

Following Ariki et al. (1997); Morita et al. (1998) we construct a symmetry-adapted basis of the coinvariant algebra. The group $\mathfrak{S}_n$ acts naturally on a tableau by replacing the entry i with $\sigma(i)$ for $\sigma \in \mathfrak{S}_n$.

Definition 3.4. Let T be a Young tableau of shape $\lambda \vdash n$. The $\mathfrak{S}_n$-subgroups

$$\mathcal{C}_T := \{\sigma \in \mathfrak{S}_n : \sigma T \textit{ is obtained by permutation of the columns of } T\}$$
$$\mathcal{R}_T := \{\sigma \in \mathfrak{S}_n : \sigma T \textit{ is obtained by permutation of the rows of } T\}$$

are the *column* and *row stabilizer* of T. We define the formal linear combination

$$\epsilon_T := \frac{f^\lambda}{n!} \sum_{\sigma \in \mathcal{C}_T, \tau \in \mathcal{R}_T} \mathrm{sgn}(\sigma)\sigma\tau \in \mathbb{R}[\mathfrak{S}_n],$$

where f^λ is the number of standard tableaux of shape λ. For a bitableau $T = (T^1, T^2)$ we define $\epsilon_{T^1}, \epsilon_{T^2} \in \mathbb{R}[\mathfrak{S}_n]$ analogously and set $\epsilon_T := \epsilon_{T^1} \cdot \epsilon_{T^2}$.

We associate (bi-)tableau with sequences, monomials, and polynomials:

Definition 3.5. Let $T = (T^1, T^2) \in \mathrm{YT}(\Lambda)$ be a (bi-)tableau.

(1) The *word* of T is the sequence $w(T) \in \mathbb{N}^{|\lambda|}$ where we read each column of the tableau T^1 from the bottom to the top, starting from the left. We continue with this procedure for the tableau T^2.
(2) We define the *index* $i(T) \in \mathbb{N}^{|\Lambda|}$ of T as follows. The number 1 in the word $w(T)$ has index 0. If k in the word has index p, then $k+1$ has index p or $p+1$ according as it lies to the right or the left of k. We call the sum of the entries of $i(T)$ the *charge* of T and write $\mathrm{ch}(T)$.

(3) We associate to a pair of (bi-)tableaux $(T,S) \in \mathrm{YT}(\Lambda) \times \mathrm{YT}(\Lambda)$ a monomial in n variables $\underline{X}_T^S := X_{w(T)_1}^{i(w(S))_1} \cdots X_{w(T)_{|\Lambda|}}^{i(w(S))_{|\Lambda|}}$. Moreover, we define polynomials associated to (T,S)

$$F_T^S := \epsilon_T \cdot \underline{X}_T^S \in \mathbb{R}[\underline{X}] \text{ and } \widehat{F}_T^S := F_T^S(\underline{X}^2) \cdot \prod_{j \in T^2} X_j$$

where $\underline{X}^2 := (X_1^2, \dots, X_n^2)$.

Note that associating tableaux with words is a standard technique in the combinatorics of tableaux (see e.g. Fulton, 1997).

Example 3.6. Let $\Lambda = ((2,1),(1)) \vdash 4$ be a bipartition and $S = \left(\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}, \boxed{3}\right)$, $T = \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1}\end{array}, \boxed{3}\right) \in \mathrm{SYT}(\Lambda)$. The word of S is $w(S) = (2,1,4,3)$ and the word of T is $w(T) = (4,1,2,3)$. We calculate the indices $i(S) = (1,0,2,1)$ and $i(T) = (1,0,0,0)$ and compute $\underline{X}_T^S = X_4^1 X_1^0 X_2^2 X_3^1 = X_2^2 X_3 X_4$, $F_T^S = X_1^2 X_3 X_4 + X_2^2 X_3 X_4 - X_1 X_2^2 X_3 - X_1 X_3 X_4^2$.

In Morita et al. (1998) introduced the following polynomials in analogy to Specht's polynomial representation of the irreducible $\mathfrak{S}_n$-modules.

Definition 3.7. Let $n \in \mathbb{N}$ and let $\mathcal{L} := \{(\lambda,\mu) \vdash n : \lambda \neq \mu, |\lambda| \geq |\mu|\}$.

(1) For A_{n-1} a complete list of *higher Specht polynomials* is given by the polynomials

$$\left\{ F_T^S : (T,S) \in \bigcup_{\lambda \vdash n} \mathrm{SYT}(\lambda) \times \mathrm{SYT}(\lambda) \right\}.$$

(2) For B_n a complete list of *higher Specht polynomials* is given by the polynomials

$$\left\{ \widehat{F}_T^S : (T,S) \in \bigcup_{\Lambda \vdash n} \mathrm{SYT}(\Lambda) \times \mathrm{SYT}(\Lambda) \right\}.$$

(3) For D_n a complete list of *higher Specht polynomials* is given by the polynomials

$$\left\{ \widehat{F}_T^S : (T,S) \in \bigcup_{\Lambda \in \mathcal{L}} \mathrm{SYT}(\Lambda) \times \mathrm{SYT}(\Lambda) \right\}, \text{ and if } n \text{ is even additionally also}$$

$$\left\{ \widehat{F}_{(T^1,T^2)}^S \pm \widehat{F}_{(T^2,T^1)}^S : ((T^1,T^2),S) \in \bigcup_{\lambda \vdash \frac{n}{2}} \mathrm{SYT}((\lambda,\lambda)) \times \mathrm{SYT}((\lambda,\lambda)) \right\}.$$

Note that for the D_n case the higher Specht polynomials differ in their structure depending on the parity of n.

Theorem 3.8 *(Morita et al., 1998, Theorem 3). For the reflection groups of type A_{n-1}, B_n or D_n the higher Specht polynomials form a vector space basis of the coinvariant algebra. For $(P,Q),(P',Q') \in \mathrm{SYT}(\lambda) \times \mathrm{SYT}(\lambda)$ and $(T,S),(T',S') \in \mathrm{SYT}(\Lambda) \times \mathrm{SYT}(\Lambda)$ we have*

$$\begin{aligned}
\mathbb{S}^\lambda &\simeq_{A_{n-1}} \langle F_{(P,Q)}^{(P',Q')} \rangle_{A_{n-1}} = \langle F_{(P'',Q'')}^{(P',Q')} : (P'',Q'') \in \mathrm{SYT}(\lambda) \rangle_{\mathbb{R}} \\
\mathbb{S}^\Lambda &\simeq_{B_n} \langle \widehat{F}_{(T,S)}^{(T',S')} \rangle_{B_n} = \langle \widehat{F}_{(T'',S'')}^{(T',S')} : (T'',S'') \in \mathrm{SYT}(\Lambda) \rangle_{\mathbb{R}}
\end{aligned}$$

Furthermore, for $\lambda \neq \mu$ the associated irreducible B_n-representations (λ, μ) and (μ, λ) remain D_n-irreducible, but are D_n-isomorphic. For a pair $((T^1, T^2), S)$ of standard bitableaux of shape $(\lambda, \lambda) \vdash n$ we have

$$\langle \widehat{F}_T^S \rangle_{D_n} = \langle \widehat{F}_T^S + \widehat{F}_{(T^2,T^1)}^S \rangle_{D_n} \oplus \langle \widehat{F}_T^S - \widehat{F}_{(T^2,T^1)}^S \rangle_{D_n} \simeq_{D_n} : \mathbb{S}_+^{(\lambda,\lambda)} \oplus \mathbb{S}_-^{(\lambda,\lambda)}$$

and the D_n-modules $\mathbb{S}_+^{(\lambda,\lambda)}, \mathbb{S}_-^{(\lambda,\lambda)}$ are D_n-irreducible and non-isomorphic.

Moreover, we find the following as a consequence of Schur's Lemma and the statements in Morita et al. (1998): For the groups A_{n-1}, B_n and D_n and standard (bi-)tableaux $T = (T^1, T^2), S_1, S_2$ of shape Λ (resp. λ) the maps

$$F_T^{S_1} \mapsto F_T^{S_2} \text{ for } A_{n-1} \text{ and } \widehat{F}_T^{S_1} \mapsto \widehat{F}_T^{S_2} \text{ for } B_n, D_n$$

define the (up to scalar multiplication) unique G-module isomorphisms. If $\Lambda = (\lambda, \lambda)$, then the unique D_n-isomorphisms are

$$\widehat{F}_{(T^1,T^2)}^{S_1} \pm \widehat{F}_{(T^2,T^1)}^{S_1} \mapsto \widehat{F}_{(T^1,T^2)}^{S_2} \pm \widehat{F}_{(T^2,T^1)}^{S_2}.$$

Definition 3.9. Let $G \in \{A_{n-1}, B_n, D_n\}$ and $\Lambda \vdash n$ be a (bi-)partition. We write q_d^Λ for the multiplicity of the G-module $\mathbb{S}^\Lambda$ in $H_{n,d}$.

Remark 3.10. From Theorem 3.8 we obtain a combinatorial description of h_k^θ, i.e., of the multiplicity of an irreducible representation θ in the subspace of the coinvariant algebra of forms of degree k. Namely, in the case of A_{n-1} θ is labelled by a partition $\lambda \vdash n$ and

$$h_k^\lambda = |\{T \in \mathrm{SYT}(\lambda) : \mathrm{ch}(T) = k\}|.$$

While for B_n and D_n θ is labelled by a bipartition $\Lambda = (\lambda, \mu) \vdash n$ and

$$h_k^\Lambda = |\{(T, S) \in \mathrm{SYT}(\Lambda) : 2\,\mathrm{ch}(T, S) + |\mu| = k\}|.$$

In particular, the multiplicity of $\mathbb{S}^\Lambda$ in $H_{n,d}$ can be described combinatorially through the number of standard (bi-)tableaux and the degrees of G

$$q_d^\Lambda = \sum_{k=0}^{d} N_G(d-k) \cdot h_k^\Lambda.$$

By integrating the above-presented construction with the general setup, the degrees of the considered reflection groups and the number of standard (bi-)tableaux combinatorially encode the following information about the invariant sums of squares.

Theorem 3.11. *Let $G \in \{A_{n-1}, B_n\}$.*

(1) *The isotypic decomposition of $H_{n,d}$ is*

$$\bigoplus_{\Lambda \vdash n} q_d^\Lambda \cdot \mathbb{S}^\Lambda,$$

where Λ ranges over partitions for A_{n-1} and bipartitions for B_n.

(2) *There exists a symmetry adapted basis of the coinvariant algebra $\mathbb{R}[\underline{X}]_G$ consisting of higher Specht polynomials $(s_1^\Lambda, \ldots, s_{\vartheta_\Lambda}^\Lambda)_{\Lambda \vdash n}$, where ϑ_Λ denotes the dimension of $\mathbb{S}^\Lambda$. By defining symmetric matrix polynomials $(H_{v,u}^\Lambda) = \left(\mathcal{R}_G(s_v^\Lambda \cdot s_u^\Lambda)\right) \in \mathbb{R}[\underline{X}]^{\vartheta_\Lambda \times \vartheta_\Lambda}$ we have*

$$\Sigma\mathbb{R}[\underline{X}]^2 \cap \mathbb{R}[\underline{X}]^G = \left\{ g \in \mathbb{R}[\psi_1, \dots, \psi_n] : g = \sum_{\Lambda \vdash n} \langle H^{\vartheta_j}, A_\Lambda \rangle \right\},$$

where $A_\Lambda \in \mathbb{R}[\psi_1, \dots, \psi_n]^{\vartheta_\Lambda \times \vartheta_\Lambda}$ are sums of squares matrix polynomial.

(3) *There exists a symmetry adapted basis of $H_{n,d} = \bigoplus_{\Lambda \vdash n} q_d^\Lambda \cdot \mathbb{S}^\Lambda$ such that its elements $\left(s_1^\Lambda, \dots, s_{q_d^\Lambda}^\Lambda\right)$ which belong to the isotypic component $q_d^\Lambda \cdot \mathbb{S}^\Lambda$ are products each of one higher Specht polynomial and a monomial in $\psi_1, \dots, \psi_n$. By defining matrix polynomials $B^\Lambda = \left(\mathcal{R}_G(s_v^\Lambda \cdot s_u^\Lambda)\right)_{v,u} \in \left(\mathbb{R}[\underline{X}]^G\right)^{q_d^\Lambda \times q_d^\Lambda}$ a form $f \in H_{n,2d}^G$ is a sum of squares if and only if*

$$f = \sum_{\Lambda \vdash n} \langle B^\Lambda, A_\Lambda \rangle$$

for some positive semidefinite matrices $A_\Lambda \in \mathbb{R}^{q_d^\Lambda \times q_d^\Lambda}$.

Proof. The isotypic decomposition of $H_{n,d}$ can be realized by multiplying the higher Specht polynomials of G of degree $\leq d$ with products of fundamental invariants by Theorems 3.8 and 2.17. For every k the multiplicity of G-modules G-isomorphic to $\mathbb{S}^\Lambda$ in the subspace of the coinvariant algebra of degree k is precisely h_k^Λ, while $N_G(d-k)$ gives the dimension of $H_{n,d-k}^G$. Now, (2) and (3) follow from Theorem 2.24 and Corollary 2.7. □

Remark 3.12. For D_n one can provide analogous assertions. The isotypic decomposition in (1) and the sizes of the matrices in (2) and (3) differ slightly, since then the D_n-module $\mathbb{S}^{(\lambda,\lambda)}$ decomposes into two irreducible D_n-modules, and since $\mathbb{S}^{(\lambda,\mu)}$ is D_n-isomorphic to $\mathbb{S}^{(\mu,\lambda)}$.

Example 3.13. The D_4 fundamental invariants are the following:

$$p_2 = X_1^2 + X_2^2 + X_3^2 + X_4^2,\ p_4 = X_1^4 + X_2^4 + X_3^4 + X_4^4,$$
$$p_6 = X_1^6 + X_2^6 + X_3^6 + X_4^6,\ e_4 = X_1X_2X_3X_4,$$

i.e., we have $\mathbb{R}[\underline{X}]^{D_4} = \mathbb{R}[p_2, p_4, p_6, e_4]$. By Corollary 2.19 and Theorem 3.11 the symmetry adapted basis of $H_{4,2}$ can be obtained by multiplying fundamental invariants with higher Specht polynomials such that the degree equals 2.

We apply Theorem 3.8 to calculate the D_4 higher Specht polynomials. For a bipartition $\Lambda \vdash 4$ the minimal degree of a higher Specht polynomial associated with Λ is given by the smallest integer in $\{2\,\mathrm{ch}(T) + |\lambda^2| : T \in \mathrm{SYT}(\Lambda)\}$.

Since the degrees of the fundamental invariants are at least 2, we need to compute all higher Specht polynomials of degrees 0 and 2. Therefore, we only need to consider bipartitions $(\lambda^1, \lambda^2) \vdash 4$ with $\lambda^2 \vdash m \in \{0, 2\}$ as otherwise the degree is odd. In the case $\lambda^2 \vdash 2$ we must have $\mathrm{ch}(T) = 0$. This can only occur for $w(T) = (1, 2, 3, 4)$ which forces $\Lambda = ((2), (2))$. The possible remaining cases are

$$\Lambda^1 = ((4), \emptyset),\ \Lambda^2 = ((3,1), \emptyset),\ \Lambda^3 = ((2,2), \emptyset),\ \Lambda^4 = ((2,1,1), \emptyset),\ \Lambda^5 = ((1,1,1,1), \emptyset).$$

Amongst the standard bitableaux T of shape Λ^j, $j \in \{1,2,3,4,5\}$ we only consider those with $\mathrm{ch}(T) \in \{0, 1\}$. Only for Λ^1 charge 0 is possible. In the remaining cases, $\mathrm{ch}(T) = 1$ if and only if $T = \left(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array},\ \emptyset\right)$. Then, the D_4-module $\mathbb{S}^{((2),(2))}$ decomposes, by Theorem 3.8, into two irreducible, non-isomorphic modules $\mathbb{S}_+^{((2),(2))}$ and $\mathbb{S}_-^{((2),(2))}$. Hence, the D_4-module $H_{4,2}$ has the isotypic decomposition

$$H_{4,2} = \mathbb{S}^{((4),\emptyset)} \oplus \mathbb{S}^{((3,1),\emptyset)} \oplus \mathbb{S}_+^{((2),(2))} \oplus \mathbb{S}_-^{((2),(2))}.$$

The relevant higher Specht polynomials are 1 for $\mathbb{S}^{((4),\emptyset)}$, $X_4^2 - X_1^2$ for $\mathbb{S}^{((3,1),\emptyset)}$ and $X_1X_2 \pm X_3X_4$ for $\mathbb{S}_+^{((2),(2))}$ and $\mathbb{S}_-^{((2),(2))}$.

3.2. Stabilization of the isotypic decompositions

In the following, we prove a stabilization of the isotypic decompositions of the Z_n-modules $H_{n,d}$ for large n and $(Z_n)_n \in \{(A_n)_n, (B_n)_n, (D_n)_n\}$.

Definition 3.14. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \vdash n$ we write $\lambda + 1$ for the partition of $n+1$ obtained from λ by replacing λ_1 with $\lambda_1 + 1$. For a bipartition $\Lambda \vdash n$ we define $\Lambda + 1$ as the bipartition $(\lambda + 1, \mu) \vdash n + 1$.

Note, $\lambda + 1 = (\lambda_1 + 1, \lambda_2, \ldots, \lambda_l)$. We use the combinatorial description of the degrees of a symmetry-adapted basis of $H_{n,d}$ from Remark 3.10. For A_{n-1} and a standard tableau T we have $\deg F_T^T = \mathrm{ch}(T)$, while for B_n and (T,S) we have $\deg \widehat{F}_{(T,S)}^{(T,S)} = 2\,\mathrm{ch}(T,S) + |\mu|$. Our aim is to identify the relevant standard (bi-)tableaux whose associated higher Specht polynomials occur in $H_{n,d}$.

Lemma 3.15. *Let $k \geq 1$ be an integer and $\lambda \vdash n = d + k$ be a partition. In the case that the first row of a tableau $T \in \mathrm{SYT}(\lambda)$ does not begin with $1, 2, \ldots, k$ we have $\deg F_T^T > d$.*

Proof. We assume that a standard tableau T of shape λ does not contain $1, 2, \ldots, k$ in the first row. Let $\tilde{k}$ be the first entry of T in the second row. We must have $\tilde{k} \leq k$ and $i(T)$ does contain at least $n - \tilde{k} + 1$ entries which are larger than or equal to 1. Therefore,

$$\deg F_T^T = \mathrm{ch}(T) \geq n - \tilde{k} + 1 \geq n - k + 1 = d + 1. \quad \square$$

We formulate Lemma 3.15 for bipartitions.

Lemma 3.16. *Let $(\lambda, \mu) \vdash n$ be a bipartition, with $|\mu| \leq d$ and $|\lambda| \geq \frac{d-1}{2} + j$ for an integer $j \geq 1$. Let (T,S) be a standard bitableau of shape (λ, μ) where $\alpha_1 < \ldots < \alpha_{|\lambda|}$ are all the entries in T. Suppose the first row of T does not begin with $\alpha_1, \ldots, \alpha_j$ then $\deg \widehat{F}_{(T,S)}^{(T,S)} > d$.*

Proof. We suppose that for some $i \leq j$ the i-th entry in the first row of T is not α_i and let i be minimal with this property. Then α_i must be the first entry in the second row and $|\lambda| - i + 1$ entries in $i(T,S)$ are at least 1. Hence

$$\deg \widehat{F}_{(T,S)}^{(T,S)} = 2\,\mathrm{ch}(T,S) + |\mu| \geq 2(|\lambda| - i + 1) \geq 2\left(\frac{d-1}{2} + j - j + 1\right) \geq d + 1. \quad \square$$

We write $T = (\alpha_{ij})$ for a standard tableau T of shape λ and α_{ij} denotes the entry in the i-th row and j-th column of T, counted from the left to the right and the top to the bottom. Analogously, we write $(T,S) = \big((\alpha_{ij}), (\beta_{ij})\big)$ for a standard bitableau.

Definition 3.17. For a partition $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash n = d + k$ we define

$$\Pi_k^\lambda := \{(\alpha_{ij}) \in \mathrm{SYT}(\lambda) : \alpha_{1j} = j, 1 \leq j \leq k\}.$$

For a bipartition $\Lambda \vdash n = d + k$ we define Π_k^Λ as the set

$$\{((\alpha_{ij}), (\beta_{ij})) \in \mathrm{SYT}(\Lambda) : (\alpha_{1j}) \text{ starts with the } k \text{ smallest integers in } \{\alpha_{ij}\}\}$$

and (α_{1j}) denotes the first row of T.

Example 3.18.

$$\Pi_3^{((3,1),(1))} = \left\{ \left(\begin{array}{|c|c|c|}\hline 2 & 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}, \begin{array}{|c|}\hline 1 \\ \hline\end{array}\right), \left(\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}, \begin{array}{|c|}\hline 2 \\ \hline\end{array}\right), \left(\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}, \begin{array}{|c|}\hline 3 \\ \hline\end{array}\right), \left(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 5 \\ \cline{1-1}\end{array}, \begin{array}{|c|}\hline 4 \\ \hline\end{array}\right), \left(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}, \begin{array}{|c|}\hline 5 \\ \hline\end{array}\right) \right\}$$

Lemma 3.19. *Let $n = d + k$, $\lambda = (\lambda_1, \dots, \lambda_l) \vdash n$ be a partition and*

$$\begin{array}{cccc} \rho^{\lambda}_{n,n+1}: & \Pi^{\lambda}_k & \longrightarrow & \Pi^{\lambda+1}_{k+1} \\ & S = (\alpha_{ij}) & \longmapsto & \widetilde{S} = (\widetilde{\alpha}_{ij}) \end{array}$$

where $\widetilde{\alpha}_{1j} = j$ for $1 \le j \le \alpha_{21}$. Further, $\widetilde{\alpha}_{1j} = \alpha_{1j-1} + 1$ for $j \ge \alpha_{21} + 1$ and $\widetilde{\alpha}_{ij} = \alpha_{ij} + 1$ for all $i \ge 2$ and $j \ge 1$.

The map $\rho^{\lambda}_{n,n+1}$ is injective, and $i(S), i(\widetilde{S})$ differ only by a 0, i.e., any non-zero entry in $i(S)$ occurs with the same multiplicity in $i(\widetilde{S})$, while 0 occurs once more. Furthermore, if $k > d - 1$ then for any $\widetilde{S} \in \Pi^{\lambda+1}_{k+1} \setminus \rho^{\lambda}_{n,n+1}(\Pi^{\lambda}_k)$, we have $\mathrm{ch}(\widetilde{S}) > d$.

Proof. Since $S \in \Pi^{\lambda}_k$ is standard, we observe that α_{21} is the smallest integer t for which $\alpha_{1t} \ne t$, if such a t exists, and otherwise $\alpha_{21} = \max_j\{\alpha_{1j}\} + 1$. For $S \in \Pi^{\lambda}_k$ the tableau $\widetilde{S}$ of shape $\lambda + 1$ is indeed standard: $\widetilde{S}$ is filled with $1, \dots, n+1$. Increasing rows and columns are inherited from S, as $\alpha_{1\alpha_{21}} > \alpha_{21}$, if $\alpha_{21} < \max_j\{\alpha_{1j}\}$. $\widetilde{S}$ is clearly increasing in any column from the second row onward. But also from the first row to the second. For $1 \le j \le \alpha_{21}$ this is clear from S. For $j \ge \alpha_{21} + 1$ this follows because $\widetilde{\alpha}_{1j} = \alpha_{1,j-1} + 1 < \alpha_{2,j-1} + 1 < \alpha_{2,j} + 1 = \widetilde{\alpha}_{2j}$.

The smallest p which is written left of $p - 1$ in $w(S)$ (resp. $w(\widetilde{S})$) is α_{21} if $\alpha_{21} < \max_j\{\alpha_{1j}\}$ and otherwise $\min\{\alpha_{22}, \alpha_{31}\}$. From there any $p > \alpha_{21}$ is left of $p - 1$ in $w(S)$ if and only if $p + 1$ is left of p in $w(\widetilde{S})$. Hence, $i(S)$ and $i(\widetilde{S})$ differ only by a 0.

Consider $\psi^{\lambda+1}_{n+1,n} : \Pi^{\lambda+1}_{k+1} \to \mathrm{YT}(\lambda)$ which maps a standard tableau $\widetilde{S}$ to a tableau S by removing the box of the first entry $\widetilde{\alpha}_{1j}$ in the first row of $\widetilde{S}$, that is strictly smaller than $\widetilde{\alpha}_{1j+1} - 1$, and if such an entry does not exist then the last entry. The boxes to the right are shifted to the left such that one obtains a diagram. Any entry that was to the right of $\widetilde{\alpha}_{1j}$ or in a row below is decreased by one. If $\psi^{\lambda+1}_{n+1,n}(\widetilde{S}) =: S$ is again standard, then $\psi^{\lambda+1}_{n+1,n} \circ \rho^{\lambda}_{n,n+1}(S) = S$. This shows the injectivity of $\rho^{\lambda}_{n,n+1}$.

If S is not standard, then one entry in the first column must be smaller than the entry below. Assume that this happens at S's entry α_{1j}. By assumption $j > k$, but this means $\lambda_2 \ge j > k$ and we observe

$$\mathrm{ch}(\widetilde{S}) \ge \lambda_2 + 1 \ge k + 2 \ge d + 1. \quad \square$$

We present the analogous assertion for bipartitions.

Lemma 3.20. *Let $n = d + k$, $\Lambda = (\lambda, \mu) = ((\lambda_1, \dots, \lambda_l), \mu) \vdash n$ be a bipartition and*

$$\begin{array}{cccc} \rho^{\Lambda}_{n,n+1}: & \Pi^{\Lambda}_k & \longrightarrow & \Pi^{\Lambda+1}_{k+1} \\ & (T, S) = ((\alpha_{ij}), (\beta_{ij})) & \longmapsto & (\widetilde{T}, \widetilde{S}) = ((\widetilde{\alpha}_{ij}), (\widetilde{\beta}_{ij})) \end{array},$$

where $(\widetilde{T}, \widetilde{S})$ is defined by: Let t be minimal with $\alpha_{1t} \ne t$, then $\widetilde{\alpha}_{1j} = j$, $1 \le j \le t$ and $\widetilde{\alpha}_{1j} = \alpha_{1j-1} + 1$, for $t + 1 \le j \le \lambda_1 + 1$, $\widetilde{\alpha}_{ij} = \alpha_{ij} + 1$, when $i \ge 2$, and $\widetilde{\beta}_{ij} = \beta_{ij} + 1$ for all i, j. If such a t does not exist, then $\widetilde{\alpha}_{1j} = j$ for all $1 \le j \le \lambda_1 + 1$ and $\widetilde{\alpha}_{ij} = \alpha_{ij} + 1$, if $i \ge 2$, $j \ge 1$ and $\widetilde{\beta}_{ij} = \beta_{ij} + 1$ for all i, j. Then the map $\rho^{\Lambda}_{n,n+1}$ is injective and $i(S, T), i(\widetilde{S}, \widetilde{T})$ differ only by a 0, i.e., any non-zero entry in $i(S, T)$ occurs with the same multiplicity in $i(\widetilde{S}, \widetilde{T})$ and 0 occurs once more. Furthermore, if $k > \frac{d}{2} - 2$ then for any $(\widetilde{T}, \widetilde{S}) \in \Pi^{\Lambda+1}_{k+1} \setminus \rho^{\Lambda}_{n,n+1}(\Pi^{\Lambda}_k)$, it holds that $2\,\mathrm{ch}(\widetilde{T}, \widetilde{S}) > d$.

Proof. For $(T, S) \in \Pi^{\Lambda}_k$ we note $(\widetilde{T}, \widetilde{S})$ is indeed a standard bitableau of shape $\Lambda + 1$, since increasing entries in all rows and columns are inherited from (T, S). An integer p occurs left of $p - 1$ in $w(T, S)$ if and only if $p + 1$ occurs left of p in $w(\widetilde{T}, \widetilde{S})$. In particular, $i(T, S)$ and $i(\widetilde{T}, \widetilde{S})$ differ only by an additional 0 entry and hence their charges are equal.

Consider $f : \Pi^{\Lambda+1}_{k+1} \to \mathrm{YT}(\Lambda)$ which maps an element $(\widetilde{T}, \widetilde{S}) \in \Pi^{\Lambda+1}_{k+1}$ to a bitableau of shape Λ by removing $\widetilde{\alpha}_{11}$, if $\widetilde{\alpha}_{11} \ne 1$. Otherwise, we remove the box containing the largest entry in the first row of $\widetilde{T}$ that is no longer the predecessor of the following number and subtract 1 from any larger entry $\widetilde{\alpha}_{ij}, \widetilde{\beta}_{ij}$. Then f is an inverse of $\rho^{\Lambda}_{n,n+1}$ and therefore $\rho^{\Lambda}_{n,n+1}$ is injective.

If $f(\widetilde{T}, \widetilde{S})$ is not standard, we have $\lambda_2 \geq k+1$. For $k > \frac{d}{2} - 2$ we have

$$2\,\mathrm{ch}(\widetilde{T}, \widetilde{S}) \geq 2(k+2) > d. \quad \square$$

Definition 3.21. For $m > n \geq d$ and (bi-)partitions $\Lambda, \lambda \vdash n$ we define $\rho^{\lambda}_{n,m} := \rho^{\lambda+m-n-1}_{m-1,m} \circ \cdots \circ \rho^{\lambda}_{n,n+1}$ and $\rho^{\Lambda}_{n,m} := \rho^{\Lambda+m-n-1}_{m-1,m} \circ \cdots \circ \rho^{\Lambda}_{n,n+1}$.

Now, we can prove the stabilization of the isotypic decomposition, which was already proven in Riener (2011); Riener et al. (2013) for the symmetric group with different methods.

Theorem 3.22. *For large enough integers $n \in \mathbb{N}$, $\Lambda \vdash n$ and $Z_n \in \{A_{n-1}, B_n, D_n\}$ the Z_n- and Z_{n+1}-isotypic decompositions remain stable, in the sense that $\mathbb{S}^{(\lambda,\mu)}$ occurs with the same multiplicity in $H_{n,d}$ as $\mathbb{S}^{(\lambda+1,\mu)}$ in $H_{n+1,d}$. The stabilization of the isotypic decomposition of $H_{n,d}$ occurs from $n = 2d$ for A_{n-1}, $n = d$ for B_n and $n = 2d+1$ for D_n.*

Proof. We restrict us to the cases A_{n-1} with $n \geq 2d$, and B_n with $n \geq d$. For $n > d$ the relevant fundamental invariants of degree $\leq d$ are equal for B_n and D_n. Thus, for D_n and $n > 2d$ the same argument as for B_n applies, since no bipartition of n can be of the form (λ, λ). By iteration, it is sufficient to compare the isotypic decompositions of $H_{n,d}$ and $H_{n+1,d}$.

Let $n \geq d$ and $\Lambda = (\lambda, \mu) \vdash n$ be a bipartition with $|\mu| \leq d$ and $\kappa \vdash n \geq 2d$ be a partition. Further, be $f_1, \ldots, f_m$ a symmetry adapted basis of the isotypic component $\bigoplus_{i=1}^{m} \mathbb{S}^{\Lambda}$ (resp. $\bigoplus_{i=1}^{m} \mathbb{S}^{\kappa}$) from Theorem 3.8. We suppose there exist m standard (bi-)tableaux $T := T_1$ and $T_2 \ldots, T_m$ of shape Λ (resp. κ) with $f_j = \pi \widehat{F}^{T_j}_{T}$ (resp. $f_j = \pi F^{T_j}_{T}$), for some $\pi \in \mathbb{R}[\underline{X}]^{Z_n}$. π can be chosen as a product of fundamental invariants of Z_n by a change of basis, since πf_j must be homogeneous. The degree of a polynomial f_j is determined by the degrees of fundamental invariants $d_1, \ldots, d_n$, the charge of the standard (bi-)tableau T_j and $|\mu|$.

The degrees $\leq d$ of fundamental invariants are equal for n and $n+1$. By Lemma 3.15, we have $T_1, \ldots, T_m \in \Pi^{\kappa}_{n-d}$ and by Lemma 3.19, for all i, the tableau $\rho^{\kappa}_{n,n+1}(T_i)$ is standard with the same charge as T_i. Furthermore, the map $\rho^{\kappa}_{n,n+1}$ is injective and any standard tableau that is not contained in the image has too large a charge. The claim follows since only the standard tableaux in $\rho^{\kappa}_{k,k+1}(\Pi^{\kappa}_{k})$ are possible options for higher Specht polynomials in $H_{n+1,d}$.

By the Lemmas 3.16 and 3.20 the standard bitableaux (T, S) of shape Λ with $2\,\mathrm{ch}(T, S) \leq d$ are in bijection with the standard bitableaux $(\widetilde{T}, \widetilde{S})$ of shape $\Lambda + 1$ with $2\,\mathrm{ch}(\widetilde{T}, \widetilde{S}) \leq d$ and the bijection preserves the charge. Furthermore, our bijection adds a 0 to the index of the image bitableau and preserves the other entries.

Finally, note that $\mathbb{S}^{(d,d)} \subset H_{2d,d}$, since the tableau $\begin{array}{|c|c|c|c|}\hline 1&2&\ldots&d\\\hline v&w&\ldots&2d\\\hline\end{array}$ with $v = d+1$ and $w = d+2$ has charge d, but $(d-1, d)$ is not a partition of $2d-1$. Similarly, $\mathbb{S}^{(\emptyset,(d))} \subset H_{d,d}$ for the bitableau $(\emptyset, \begin{array}{|c|c|c|c|}\hline 1&2&\ldots&d\\\hline\end{array})$ with charge 0. For D_n we observe $\mathbb{S}^{((d),(d))} \subset H_{2d,d}$ but the D_{2d}-module $\mathbb{S}^{((d),(d))}$ is special since it decomposes which does not happen for $\mathbb{S}^{((d-1),(d))}$. $\square$

We note that in the case of D_n and $n = d$, an additional fundamental invariant of degree d occurs, which does not occur for $n > d$ anymore. Thus, at least the trivial representation occurs with larger multiplicity in $H_{d,d}$ than in $H_{d+1,d}$. However, Example 3.23 shows that already for the symmetric group the stabilization does not occur in the step from d to $d+1$ in general.

Example 3.23. Consider the bitableau $T = \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array}$ of shape $\lambda + 1 = (3,2) \vdash 5$. We have $\mathrm{ch}(T) = 3$, i.e., $p_1 F^{T}_{T} \in H_{5,4}$. However, $\mathrm{SYT}(\lambda) = \left\{ \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}, \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array} \right\}$ with charges 2 and 4. For any $S \in \mathrm{SYT}(\lambda)$, we can construct a tableau $\widetilde{S} \in \mathrm{SYT}(\lambda+1)$ with the same charge, but T cannot be obtained in this way. In particular, the A_3-module S^{λ} has smaller multiplicity in $H_{4,4}$ than the A_4-module $S^{\lambda+1}$ in $H_{5,4}$.

Corollary 3.24. *For a fixed degree d and a sequence $(Z_n)_n$ of reflection groups $(A_{n-1})_n$, $(B_n)_n$ or $(D_n)_n$ the symmetry adapted description of the set $\Sigma_{n,2d}^{Z_n}$ are equal up to the map $\rho_{n,m}^{\Lambda}$, for $n \geq 2d$, $n \geq d$ or $n \geq 2d+1$ respectively.*

The corollary says that up to $\rho_{n,m}^{\Lambda}$ and $\rho_{n,m}^{\lambda}$ the same matrix polynomials can be used in a sum of squares representation.

Proof. This follows from Theorem 3.22 and Lemmas 3.15, 3.16, 3.19, 3.20. $\square$

The case $n = 2d$ is the last where $\Lambda \vdash n$ can be of the form $\Lambda = (\lambda, \lambda)$, i.e., the B_{2d}-module $\mathbb{S}^{\Lambda}$ is not D_{2d}-irreducible in $H_{n,d}$ but the B_{2d+1}-module $\mathbb{S}^{\Lambda+1}$ is D_{2d+1}-irreducible in $H_{n+1,d}$ (see Theorem 3.8). Nevertheless, the multiplicities of $\mathbb{S}^{\Lambda}$ in $H_{2d,d}$ and $H_{n,d}$ are equal for $n \geq 2d$. Moreover, whenever $n \geq d$ for B_n, or $n > d$ in case of D_n one can use that if $\mathbb{S}^{\Lambda} \subset H_{n,d}$, for $\Lambda = (\lambda, \mu) \vdash n$, and d even (odd), then $|\mu|$ must also be even (odd).

4. Concrete examples and applications

We apply the results from the preceding Section 3 to solve non-negativity versus sums of squares questions. In contrast to the non-equivariant case, the B_n-invariant forms have a non-trivial equality between the sets of even symmetric sums of squares and non-negative forms in 3 variables and degree 8. This was proven by Harris (1999). In fact, it turns out that this case and quartics are the only non-trivial equality cases (Goel et al., 2017). We will present a characterization of the dual and primal cones of B_3-invariant sums of squares ternary octics and obtain a new elementary proof of Harris' theorem. Moreover, we study D_n-invariant forms, prove that $\mathcal{P}_{4,4}^{D_4}$ is a simplicial cone, and answer the non-negativity versus sums of squares question there.

In general, testing the non-negativity of a polynomial in more than two variables is already for quartics an NP-hard problem (see e.g., Blum et al., 1998 or Murty and Kabadi, 1985). In equivariant situations, it is therefore of interest to exploit the symmetry of invariant polynomials to reduce this complexity. The works in Acevedo and Velasco (2016); Friedl et al. (2018); Harris (1999); Moustrou et al. (2021); Riener (2012, 2016); Timofte (2003) focus on providing test sets for verification of non-negativity of invariant polynomials. We also examine test sets for B_n and D_n invariant forms and small degrees.

We remark that each group in the infinite series $I_2(m)$ of dihedral groups acts on $\mathbb{R}^2$. In particular, any $I_2(m)$ invariant non-negative form is a sum of squares.

4.1. Even symmetric octics

One of the well-known and rare cases of equality between sums of squares and non-negative forms in equivariant situations was proven by Harris (1999). Harris' proof is quite analytical. In this subsection, we give s new proof of equality build on a detailed study of the dual cone. Furthermore, we present a uniform description of the cones of n-ary even symmetric sums of squares octics.

Theorem 4.1. *The dual cone of even symmetric ternary octic sums of squares has the following description*

$$\Sigma_{3,8}^{B_3,*} = \left\{ \mathrm{ev}_{\left(a,\sqrt{1-a^2},0\right)}, \mathrm{ev}_{(b,c,c)} : \frac{1}{2} \leq a \leq 1, 0 \leq b \leq 1, c = \sqrt{\frac{(1-b^2)}{2}} \right\}.$$

As a consequence of Theorem 4.1 we can give a new proof for Harris' result.

Corollary 4.2 *(Harris, 1999, Theorem 4.1). The sets of non-negative and sums of squares even symmetric ternary octics are equal, i.e., $\Sigma_{3,8}^{B_3} = \mathcal{P}_{3,8}^{B_3}$.*

Proof. By Theorem 4.1, the cone $\Sigma_{3,8}^{B_3,*}$ is generated by point-evaluations. Thus, the claim follows from Corollary 2.41. $\square$

Remark 4.3.

(1) Harris showed that $\Omega := \{(a,a,b),(0,a,b) : a,b \in \mathbb{R}_{\geq 0}\}$ is a test set for even symmetric ternary octics and used this as the main ingredient in his proof of equality (Harris, 1999). Theorem 4.1 establishes also this result of Harris directly. Note, that more generally test sets for D_n invariant polynomials are described in Friedl et al. (2018).

(2) The equality in Corollary 4.1 does not follow directly from Hilbert's equality case $\Sigma_{3,4}^{\mathfrak{S}_3} = \mathcal{P}_{3,4}^{\mathfrak{S}_3}$ under canonical identification through the $\mathfrak{S}_3$-isomorphism

$$\begin{array}{cccc} \Phi : & H_{3,8}^{B_3} & \longrightarrow & H_{3,4}^{\mathfrak{S}_3} \\ & \sum_{\alpha \in 2\mathbb{N}_0^3} c_\alpha x^\alpha & \longmapsto & \sum_{\alpha \in 2\mathbb{N}_0^3} c_\alpha \underline{X}^{\frac{1}{2}\alpha} \end{array}.$$

For $g \in H_{3,4}^{\mathfrak{S}_3}$ we have $\Phi^{-1}(g) = g(X_1^2, X_2^2, X_3^2)$. Then g is non-negative on $\mathbb{R}_{\geq 0}^3$ if and only if $\Phi^{-1}(g)$ is non-negative. However, the example

$$f(\underline{X}) := e_1(\underline{X}^2)e_3(\underline{X}^2) = (X_1^2 + X_2^2 + X_3^2)(X_1^2X_2^2X_3^2) \in \mathcal{P}_{3,8}^{B_3}$$

with $\Phi(f)(-1,-1,1) = -1 < 0$ shows $\mathcal{P}_{3,4}^{\mathfrak{S}_3} \subsetneq \Phi(\mathcal{P}_{3,8}^{B_3})$.

In order to show Theorem 4.1 we give a detailed study of the even symmetric ternary octics which are sums of squares. Note that the vector space dimension of $H_{3,8}^{B_3}$ is 4, while the dimension of $H_{n,8}^{B_n}$ is 5 for all $n \geq 4$.

Lemma 4.4. *The B_3-module $H_{3,4}$ has the isotypic decomposition*

$$H_{3,4} = 2 \cdot \mathbb{S}^{((3),\emptyset)} \oplus 2 \cdot \mathbb{S}^{((2,1),\emptyset)} \oplus 2 \cdot \mathbb{S}^{((1),(2))} \oplus \mathbb{S}^{((1),(1,1))}.$$

A symmetry adapted basis of $H_{3,4}$ realising the B_3-isotypic decomposition is given by the following polynomials:

$$\mathbb{S}^{((3),\emptyset)} : \left\{e_1(\underline{X}^2)^2, e_2(\underline{X}^2)\right\}, \qquad \mathbb{S}^{((2,1),\emptyset)} : \left\{e_1(\underline{X}^2)(X_3^2 - X_1^2), X_2^2X_3^2 - X_1^2X_2^2\right\},$$

$$\mathbb{S}^{((1),(2))} : \left\{e_1(\underline{X}^2)X_2X_3, X_1^2X_2X_3\right\}, \qquad \mathbb{S}^{((1),(1,1))} : \left\{(X_3^2 - X_2^2)X_2X_3\right\}.$$

Proof. We need to determine the multiplicity of the irreducible B_3-modules $\mathbb{S}^{(\lambda,\mu)}$ in $H_{3,4}$ for all bipartitions $(\lambda,\mu) \vdash 3$. We can immediately exclude some of them. Since we only need higher Specht polynomials of degree $0, 2$ or 4 by Theorem 3.11, the degree - which equals 2 times the charge of a standard bitableau of shape (λ,μ) plus $|\mu|$ - must be $0, 2$ or 4. However, this implies that only bipartitions with $\mu \in \{\emptyset, (2), (1,1)\}$ are feasible to obtain an even degree. By going through all the remaining cases one obtains precisely the following higher Specht polynomials of degree $0, 2$ and 4:

$$\left\{1, X_3^2 - X_1^2, X_2^2X_3^2 - X_1^2X_2^2, X_2X_3, X_1^2X_2X_3, (X_3^2 - X_2^2)X_2X_3\right\}.$$

Multiplying by the invariants $1, e_1(\underline{X}^2)^2$ and $e_2(\underline{X}^2)$ results accordingly in the above mentioned symmetry adapted basis. $\square$

Corollary 4.5. *A form $f \in H_{3,8}^{B_3}$ is a sum of squares if and only if there exist positive semidefinite matrices $A^{(1)}, A^{(2)}, A^{(3)} \in \mathbb{R}^{2\times 2}$ and $A^{(4)} \in \mathbb{R}^{1\times 1}$ such that*

$$f = \langle A^{(1)}, B^{(1)}\rangle + \langle A^{(2)}, B^{(2)}\rangle + \langle A^{(3)}, B^{(3)}\rangle + \langle A^{(4)}, B^{(4)}\rangle$$

where the $B^{(j)}$'s are the following matrix polynomials corresponding to the B_3-modules in $H_{3,4}$

$$B^{(1)} := \begin{pmatrix} e_1(\underline{X}^2)^4 & e_1(\underline{X}^2)^2 e_2(\underline{X}^2) \\ e_1(\underline{X}^2)^2 e_2(\underline{X}^2) & e_2(\underline{X}^2)^2 \end{pmatrix},$$

$$B^{(2)} := \begin{pmatrix} \frac{2}{3}e_1(\underline{X}^2)^4 - 2e_1(\underline{X}^2)^2 e_2(\underline{X}^2) & -3e_1(\underline{X}^2)e_3(\underline{X}^2) + \frac{1}{3}e_1(\underline{X}^2)^2 e_2(\underline{X}^2) \\ -3e_1(\underline{X}^2)e_3(\underline{X}^2) + \frac{1}{3}e_1(\underline{X}^2)^2 e_2(\underline{X}^2) & \frac{2}{3}e_2(\underline{X}^2)^2 - 2e_1(\underline{X}^2)e_3(\underline{X}^2) \end{pmatrix},$$

$$B^{(3)} := \begin{pmatrix} \frac{1}{3}e_1(\underline{X}^2)^2 e_2(\underline{X}^2) & e_1(\underline{X}^2)e_3(\underline{X}^2) \\ e_1(\underline{X}^2)e_3(\underline{X}^2) & \frac{1}{3}e_1(\underline{X}^2)e_3(\underline{X}^2) \end{pmatrix},$$

$$B^{(4)} := \left(e_1(\underline{X}^2)e_3(\underline{X}^2) - \frac{4}{3}e_2(\underline{X}^2)^2 + \frac{1}{3}e_1(\underline{X}^2)^2 e_2(\underline{X}^2) \right).$$

Proof. The matrices $B^{(1)}, \dots, B^{(4)}$ are the symmetrizations of the products of the symmetry adapted basis from Lemma 4.4. By Theorem 2.6, any invariant sum of squares form has such a representation. $\square$

Corollary 4.6. *A linear form $\ell \in H_{3,8}^{B_3,*}$ is contained in $\Sigma_{3,8}^{B_3,*}$ if and only if the following four matrices are positive semidefinite*

$$\begin{pmatrix} m_{(1^4)} & m_{(2,1^2)} \\ m_{(2,1^2)} & m_{(2^2)} \end{pmatrix}, \quad \begin{pmatrix} \frac{2}{3}m_{(1^4)} - 2m_{(2,1^2)} & \frac{1}{3}m_{(2,1^2)} - 3m_{(3,1)} \\ \frac{1}{3}m_{(2,1^2)} - 3m_{(3,1)} & \frac{2}{3}m_{(2^2)} - 2m_{(3,1)} \end{pmatrix},$$

$$\begin{pmatrix} \frac{1}{3}m_{(2,1^2)} & m_{(3,1)} \\ m_{(3,1)} & \frac{1}{3}m_{(3,1)} \end{pmatrix}, \quad \left(\frac{1}{3}m_{(2,1^2)} - \frac{4}{3}m_{(2^2)} + m_{(3,1)} \right),$$

where we write $m_{(1^4)} := \ell(e_1(\underline{X}^2)^4)$, $m_{(3,1)} := \ell(e_1(\underline{X}^2)e_3(\underline{X}^2))$, $m_{(2,1^2)} := \ell(e_1(\underline{X}^2)^2 e_2(\underline{X}^2))$ and $m_{(2^2)} := \ell(e_2(\underline{X}^2)^2)$.

Proof. By Lemma 2.38, this is precisely the dual statement to Corollary 4.5. $\square$

As remarked before $H_{3,8}^{B_3}$ is a 4-dimensional $\mathbb{R}$-vector space. We choose as fundamental invariants the elementary symmetric polynomials evaluated in $\underline{X}^2 = (X_1^2, X_2^2, X_3^2)$ and work with the $\mathbb{R}$-basis

$$\left(e_1(\underline{X}^2)^4, e_1(\underline{X}^2)e_3(\underline{X}^2), e_1(\underline{X}^2)^2 e_2(\underline{X}^2), e_2(\underline{X}^2)^2 \right)$$

of $H_{3,8}^{B_3}$. In order to establish the proof of Theorem 4.1 we study the extremal rays of $\Sigma_{3,8}^{B_3,*}$ and show that all of them are spanned by point-evaluations. Recall that for to an element $\ell \in \Sigma_{3,8}^{B_3,*}$ we associate a B_3-invariant quadratic form on $H_{3,4}$, denoted by $\mathcal{Q}_\ell$, and we will study $W_\ell := \ker \mathcal{Q}_\ell$ its kernel and

$$W_\ell^{\langle 2 \rangle} := \ker \ell = \left\{ \mathcal{R}_G \left(\sum f_i g_i \right) \in H_{n,2d}^G \ : \ f_i \in W, g_i \in H_{n,d} \right\}$$

(see Proposition 2.40). As the dimension of $H_{3,8}^{B_3}$ is 4 a hyperplane in $H_{3,8}^{B_3}$ is of dimension 3. Thus, it follows that we must have $\dim W_\ell^{\langle 2 \rangle} = 3$. By Lemma 4.4, the isotypic decomposition of the B_3-submodule W_ℓ of $H_{3,4}$ has the form

$$W_\ell = \ker \mathcal{Q}_\ell = \alpha \cdot \mathbb{S}^{((3),\emptyset)} \oplus \beta \cdot \mathbb{S}^{((2,1),\emptyset)} \oplus \gamma \cdot \mathbb{S}^{((1),(2))} \oplus \delta \cdot \mathbb{S}^{((1),(1,1))},$$

where $\alpha, \beta, \gamma \in \{0, 1, 2\}$ and $\delta \in \{0, 1\}$. In order to show that indeed, the extremal elements correspond to point evaluations we use that if $\ker \ell$ is maximal among all kernels of elements in $\Sigma_{3,8}^{G,*}$, i.e., if $\ker \ell$ contains a non-trivial zero then ℓ must be a scalar multiple of the point-evaluation at this point (see Lemma 2.39). In the following lemmas we analyze possible combinations of the integers α, β, γ, and δ through a case distinction to obtain a classification of all extremal elements in $\Sigma_{3,8}^{B_3,*}$.

Lemma 4.7. *Let $\ell \in \Sigma_{3,8}^{B_3,*}$ be an extremal element. Then $\alpha < 2$, i.e., the multiplicity of the trivial representation in W_ℓ is smaller than 2.*

Proof. If $\alpha = 2$ then $e_1(\underline{X}^2)^2 \in W_\ell$ and hence $e_1(\underline{X}^2)^4 \in W_\ell^{\langle 2\rangle} = \ker \ell$. However, any monomial of degree 8 that is a square occurs with positive coefficients in $e_1(\underline{X}^2)^4$, which implies $\ell = 0$ must be the 0 map. □

Lemma 4.8. *Let $\ell \in \Sigma_{3,8}^{B_3,*}$ be an extremal element and $\alpha = 0$. Then ℓ is a scalar multiple of the point-evaluation ev_z, where $z \in \{(1,1,1),(1,0,0),(1,1,0)\}$.*

Proof. In the case $\beta = 2$ we know by dimension reasons on $W_\ell^{\langle 2\rangle}$ that any other B_3-module occurring in W_ℓ must already be contained in $2 \cdot \mathbb{S}^{((2,1),\emptyset)}$. However, the forms in the module $2 \cdot \mathbb{S}^{((2,1),\emptyset)}$ have the common zero $(1,1,1)$.

If $\beta = 1$, then we must have $\gamma \geq 1$ or $\delta = 1$ such that $W_\ell^{\langle 2\rangle}$ is a hyperplane. For $\delta = 1$ the elements in W_ℓ have the common root $(1,1,1)$. Now, we consider the case $\beta = 1, \gamma \geq 1$. Thus for some pairs $(a,b),(c,d) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$$ae_1(\underline{X}^2)(X_3^2 - X_1^2) + b(X_2^2X_3^2 - X_1^2X_2^2), ce_1(\underline{X}^2)X_2X_3 + dX_1^2X_2X_3 \in W_\ell,$$

and the symmetrized products with elements in $H_{3,4}$ are contained in $W_\ell^{\langle 2\rangle}$, i.e.,

$$\begin{aligned}
0 =& a\left(\frac{2}{3}m_{(1^4)} - 2m_{(2,1^2)}\right) + b\left(\frac{1}{3}m_{(2,1^2)} - 3m_{(3,1)}\right),\\
0 =& a\left(\frac{1}{3}m_{(2,1^2)} - 3m_{(3,1)}\right) + b\left(\frac{2}{3}m_{(2^2)} - 2m_{(3,1)}\right),\\
0 =& \frac{c}{3}m_{(2,1^2)} + dm_{(3,1)},\\
0 =& cm_{(3,1)} + \frac{d}{3}m_{(3,1)}.
\end{aligned}$$

Now, we distinguish two cases, depending on $m_{(3,1)}$ vanishing or not-vanishing.

i) If $m_{(3,1)} \neq 0$ we have that $c + \frac{d}{3} = 0$. Since W_ℓ is a linear space we can set $c = 1$ and $d = -3$. However, then the B_3-module W_ℓ has the common zero $(1,1,1)$. Thus ℓ is a scalar multiple of the point-evaluation $\mathrm{ev}_{(1,1,1)}$.

ii) Assume that $m_{(3,1)} = 0$. We first consider the case when $c \neq 0$. Then $m_{(2,1^2)} = 0$ and since $m_{(1^4)} > 0$ we have $a = 0$. Hence, $b \neq 0$ and $m_{(2^2)} = 0$ which implies that the elements in W_ℓ all vanish at $(1,0,0)$ and ℓ is a scalar multiple of $\mathrm{ev}_{(1,0,0)}$. So we only the case with vanishing c remains. If $c = 0$ we have

$$\begin{aligned}
0 =& a\left(\frac{2}{3}m_{(1^4)} - 2m_{(2,1^2)}\right) + b\left(\frac{1}{3}m_{(2,1^2)}\right),\\
0 =& a\left(\frac{1}{3}m_{(2,1^2)}\right) + b\left(\frac{2}{3}m_{(2^2)}\right).
\end{aligned}$$

In this situation we find that if $a = 0$ then ℓ is a scalar multiple of $\mathrm{ev}_{(1,0,0)}$, since any form in $W_\ell^{\langle 2\rangle}$ has the zero $(1,0,0)$. If otherwise $a \neq 0$ we may assume that $a = 1$ since $W_\ell^{\langle 2\rangle}$ is a linear space. In this situation we have

$$\begin{aligned}
0 =& \frac{2}{3}m_{(1^4)} + (-2 + \frac{b}{3})m_{(2,1^2)},\\
0 =& \frac{1}{3}m_{(2,1^2)} + \frac{2b}{3}m_{(2^2)}.
\end{aligned}$$

By scaling ℓ and since $m_{(1^4)}>0$, we can assume that $m_{(1^4)}=1$. We first note that $b=0$ is impossible. Indeed, $b=0$ directly yields the contradiction $0=m_{(1^4)}=1$. So $b\neq 0$ and $m_{(2,1^2)}=\frac{2}{6-b}, m_{(2^2)}=\frac{1}{-6b+b^2}$, for a non zero $b\neq 6$. From the conditions in Corollary 4.6 we obtain for the first matrix

$$\det\begin{pmatrix} 1 & m_{(2,1^2)} \\ m_{(2,1^2)} & m_{(2^2)} \end{pmatrix}\geq 0,$$

which implies that $-2\leq b<0$. And the positive semidefiniteness of the last matrix in Corollary 4.6 yields

$$\frac{1}{3}m_{(2,1^2)}-\frac{4}{3}m_{(2^2)}+m_{(3,1)}\geq 0$$

implies that $b\leq -2$ or $0<b<6$. Thus $b=-2$ and ℓ is the point-evaluation $\mathrm{ev}_{(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0)}$.

Finally, if $\gamma\geq 1$, then $\beta=1$ or $\delta=1$. However, we have already examined the case $\beta=1$. For $\delta=1$ the elements in W_ℓ have the common zero $(1,0,0)$. Thus ℓ is a scalar multiple of $\mathrm{ev}_{(1,0,0)}$. $\square$

We now can proceed with the cases where $\alpha=1$, which implies that $ae_1(\underline{X}^2)^2+e_2(\underline{X}^2)\in W_\ell$ for an $a\in\mathbb{R}$, since $e_1(\underline{X}^2)^4\notin W_\ell$. Notice that this implies the following equations for $\ker\ell$:

$$am_{(1^4)}+m_{(2,1^2)}=0,$$
$$am_{(2,1^2)}+m_{(2^2)}=0.$$

Moreover, since $m_{(1^4)}>0$ and since ℓ is a linear form, without loss of generality we can suppose $m_{(1^4)}=1$, as ℓ is then just a positive scalar. The positive semidefiniteness conditions with the reductions $m_{(2,1^2)}=-am_{(1^4)}, m_{(2^2)}=a^2m_{(1^4)}$ and $m_{(1^4)}=1$ can then be expressed with the following four matrices all of which must be positive semidefinite.

$$\begin{pmatrix} 1 & -a \\ -a & a^2\end{pmatrix},\begin{pmatrix} \frac{2}{3}+2a & -\frac{1}{3}a-3m_{(3,1)} \\ -\frac{1}{3}a-3m_{(3,1)} & \frac{2}{3}a^2-2m_{(3,1)}\end{pmatrix},\begin{pmatrix} -\frac{1}{3}a & m_{(3,1)} \\ m_{(3,1)} & \frac{1}{3}m_{(3,1)}\end{pmatrix},\left(\frac{-a}{3}-\frac{4a^2}{3}+m_{(3,1)}\right) \tag{4.1}$$

Using the positive semidefiniteness of the second matrix and $-a=m_{(2,1^2)}\geq 0$ we obtain $a\in[-\frac{1}{3},0]$.

With these considerations, we now proceed with a case distinction on the parameters β,γ,δ. We show here that in certain cases the admissible linear forms in the dual cone correspond to point-evaluations. Recall that by Corollary 2.37 the evaluations on points on the sphere generate the dual cone of non-negative forms and the following lemmas show that the only possible linear forms in the dual cone are indeed point-evaluations. Notice that we may scale these points, as the polynomials are homogeneous and we thus can identify point-evaluations along a ray.

Lemma 4.9. *Let $\ell\in\Sigma_{3,8}^{B_3,*}$ be an extremal element. If $\alpha=\delta=1$, then ℓ is a scalar multiple of the point-evaluation in* $(1,1,0)$.

Proof. The condition $\delta=1$ yields $\mathbb{S}^{((1),(1,1))}\subset W_\ell$ which implies $(X_3^2-X_2^2)X_2X_3\in W_\ell$ and

$$-\frac{a}{3}-\frac{4a^2}{3}+m_{(3,1)}=0.$$

Positiveness yields $0\leq m_{(3,1)}=\frac{1}{3}(a+4a^2)$ and therefore that $a\leq-\frac{1}{4}$. We use that the determinant of the second matrix in (4.1) is non-negative, i.e.,

$$0\leq\left(\frac{2}{3}+2a\right)\left(\frac{2}{3}a^2-2m_{(3,1)}\right)-\left(-\frac{1}{3}a-3m_{(3,1)}\right)^2=-\frac{4}{9}a(1+3a)^2(1+4a).$$

This is not satisfied for $a<-\frac{1}{4}$. Hence $a=-\frac{1}{4}, m_{(1^4)}=1, m_{(3,1)}=0, m_{(2,1^2)}=\frac{1}{4}, m_{(2^2)}=\frac{1}{16}$ and ℓ is a scalar multiple of $\mathrm{ev}_{(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0)}$. $\square$

Lemma 4.10. *Let $\ell \in \Sigma_{3,8}^{B_3,*}$ be an extremal element. If $\alpha = 1, \gamma \geq 1$, then ℓ is a scalar multiple of a point-evaluation in $(1,0,0)$, $(1,1,1)$ or $\left(\sqrt{\frac{1}{2}+\sqrt{a+\frac{1}{4}}}, \sqrt{\frac{1}{2}-\sqrt{a+\frac{1}{4}}}, 0\right)$, for $-\frac{1}{4} \leq a \leq 0$.*

Proof. We have $\mathbb{S}^{((1),(2))} \subset W_\ell$, i.e., for a pair $(b,c) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$$be_1(\underline{X}^2)X_2X_3 + cX_1^2X_2X_3 \in W_\ell$$

and the symmetrized products with elements in $H_{3,4}$ are contained in $W_\ell^{\langle 2\rangle}$, i.e.,

$$\begin{aligned}0 &= \frac{-ab}{3} + cm_{(3,1)},\\ 0 &= bm_{(3,1)} + \frac{c}{3}m_{(3,1)}.\end{aligned}$$

Substituting $\frac{ab}{3} = cm_{(3,1)}$ in the second equation gives $b\left(\frac{a}{9} + m_{(3,1)}\right) = 0$.

a) We first assume that $b \neq 0$. Then $m_{(3,1)} = -\frac{a}{9}$. In this case, we obtain from the positive semidefiniteness of the second matrix in (4.1) that

$$0 \leq \frac{2}{3}a^2 - 2m_{(3,1)} = \frac{2}{3}a(a+\frac{1}{3}).$$

Thus $a \in \{0, -\frac{1}{3}\}$. If $a = 0$ then $m_{(3,1)} = m_{(2,1^2)} = m_{(2^2)} = 0$ and $\ell = \mathrm{ev}_{(1,0,0)}$. For $a = -\frac{1}{3}$ we have $m_{(3,1)} = \frac{1}{27}, m_{(2,1^2)} = \frac{1}{3}, m_{(2^2)} = \frac{1}{9}$ and $\ell = \mathrm{ev}_{\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)}$.

b) In the remaining case $b = 0$ we can assume by linearity of W_ℓ that $c = 1$, which implies $m_{(3,1)} = 0$. By the non-negativity of the last 1×1 matrix in (4.1), i.e.,

$$0 \leq -\frac{a}{3} - \frac{4a^2}{3} + m_{(3,1)}$$

we obtain $-\frac{1}{4} \leq a \leq 0$. However, for any such $-\frac{1}{4} \leq a \leq 0$, we have $m_{(1^4)} = 1, m_{(3,1)} = -a, m_{(2,1^2)} = a^2, m_{(2^2)} = 0$ and $\ell = \mathrm{ev}_{\left(\sqrt{\frac{1}{2}+\sqrt{a+\frac{1}{4}}},\sqrt{\frac{1}{2}-\sqrt{a+\frac{1}{4}}},0\right)}$. $\square$

Lemma 4.11. *Let $\ell \in \Sigma_{3,8}^{B_3,*}$ be an extremal element. If $\alpha = \beta = 1$, then ℓ is a scalar multiple of a point-evaluation at a point of the form*

$$\left(\sqrt{\frac{1+2\sqrt{1+3a}}{3}}, \sqrt{\frac{1-\sqrt{1+3a}}{3}}, \sqrt{\frac{1-\sqrt{1+3a}}{3}}\right), \textit{for } -\frac{1}{3} \leq a \leq 0, \textit{ or}$$
$$\left(\sqrt{\frac{1-2\sqrt{1+3b}}{3}}, \frac{\sqrt{1+\sqrt{1+3b}}}{3}, \frac{\sqrt{1+\sqrt{1+3b}}}{3}\right), \textit{for } -\frac{1}{3} \leq b \leq -\frac{1}{4}.$$

Proof. If $\beta = 1$ then $\mathbb{S}^{((2,1),\emptyset)} \subset W_\ell$, i.e., for a pair $(b,c) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$$be_1(\underline{X}^2)(X_3^2 - X_1^2) + c(X_2^2X_3^2 - X_1^2X_2^2) \in W_\ell$$

and the symmetrized products with elements in $H_{3,4}$ are contained in $W_\ell^{\langle 2\rangle}$, i.e.,

$$
\begin{aligned}
0 =& b\left(\frac{2}{3}+2a\right)+c\left(-\frac{1}{3}a-3m_{(3,1)}\right),\\
0 =& b\left(-\frac{1}{3}a-3m_{(3,1)}\right)+c\left(\frac{2}{3}a^2-2m_{(3,1)}\right).
\end{aligned}
$$

We distinguish two cases:

i) If $b=0, c=1$ or if $b=1, c=0$ then $-\frac{1}{3}=a, m_{(3,1)}=\frac{1}{27}$ and $\ell=\mathrm{ev}_{\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)}$.

ii) We continue with the remaining case $b\neq 0$ and $c\neq 0$. Since W_ℓ is a vector space we assume without loss of generality that $b=1$ and obtain

$$
m_{(3,1)}=\frac{2}{9c}+\frac{2a}{3c}-\frac{a}{9} \text{ and } \frac{2(1+3a)(-3-2c+ac^2)}{9c}=0.
$$

Hence $a=-\frac{1}{3}$ (in which case $\ell=\mathrm{ev}_{(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}})}$) or $-3-2c+ac^2=0$. If $a=0$ then $c=-\frac{3}{2}$ and $m_{(3,1)}=-\frac{4}{27}$ which does not satisfy the positive semidefiniteness conditions. If $-\frac{1}{3}<a<0$ then either $c=\frac{1}{a}-\sqrt{\frac{1+3a}{a^2}}$ or $c=\frac{1}{a}+\sqrt{\frac{1+3a}{a^2}}$.

In the first case we find

$$
m_{(1^4)}=1, m_{(3,1)}=\frac{a\left(1+a\left(6+\sqrt{\frac{1+3a}{a^2}}\right)\right)}{9-9a\sqrt{\frac{1+3a}{a^2}}}, m_{(2,1^2)}=-a, m_{(2^2)}=a^2.
$$

For any $-\frac{1}{3}<a<0$, ℓ is the point-evaluation at $\left(\sqrt{\frac{1+2\sqrt{1+3a}}{3}},\sqrt{\frac{1-\sqrt{1+3a}}{3}},\sqrt{\frac{1-\sqrt{1+3a}}{3}}\right)$.

In the second case we find

$$
m_{(1^4)}=1, m_{(3,1)}=\frac{a\left(1-a\left(-6+\sqrt{\frac{1+3a}{a^2}}\right)\right)}{9+9a\sqrt{\frac{1+3a}{a^2}}}, m_{(2,1^2)}=-a, m_{(2^2)}=a^2.
$$

However, $m_{(3,1)}\geq 0$ is equivalent to $-\frac{1}{3}<a\leq-\frac{1}{4}$. For any $-\frac{1}{3}<a\leq-\frac{1}{4}$, ℓ is the point-evaluation at $\left(\sqrt{\frac{1-2\sqrt{1+3a}}{3}},\frac{\sqrt{1+\sqrt{1+3a}}}{3},\frac{\sqrt{1+\sqrt{1+3a}}}{3}\right)$. $\square$

These results now allow us to conclude the proof of Theorem 4.1.

Proof of Theorem 4.1. By Lemmas 4.7, 4.8, 4.9, 4.10 and 4.11 we have established that the extremal rays in $\Sigma_{3,8}^{B_3,*}$ are all generated by point-evaluations. These generators are the point-evaluations at elements in the set

$$
\left\{\left(a,\sqrt{1-a^2},0\right),(b,c,c):\frac{1}{2}\leq a\leq 1, 0\leq b\leq 1, c=\frac{1}{\sqrt{2}}\sqrt{(1-b^2)}\right\}. \quad \square
$$

Corollary 4.12. *$\mathcal{P}_{3,8}^{B_3}$ is the convex cone generated by the following six forms*

$$
\begin{aligned}
&e_1(\underline{X}^2)^4-3e_1(\underline{X}^2)^2e_2(\underline{X}^2), -9e_1(\underline{X}^2)e_3(\underline{X}^2)+e_1(\underline{X}^2)^2e_2(\underline{X}^2), e_2(\underline{X}^2)^2-3e_1(\underline{X}^2)e_3(\underline{X}^2),\\
&e_1(\underline{X}^2)^2e_2(\underline{X}^2), e_1(\underline{X}^2)e_3(\underline{X}^2), 3e_1(\underline{X}^2)e_3(\underline{X}^2)-4e_2(\underline{X}^2)^2+e_1(\underline{X}^2)^2e_2(\underline{X}^2)
\end{aligned}
$$

and the following two families of forms

$$
\left(ae_1(\underline{X}^2)^4+e_1(\underline{X}^2)e_2(\underline{X}^2), ae_1(\underline{X}^2)e_2(\underline{X}^2)+e_2(\underline{X}^2)^2:-\tfrac{1}{3}\leq a\leq 0\right)
$$

Proof. These forms are precisely the sums of squares contained in the kernels of extremal rays of $\Sigma_{3,8}^{B_3,*}$. Since $\Sigma_{3,8}^{B_3} = \mathcal{P}_{3,8}^{B_3}$ by Corollary 4.2, these forms are also the extremal elements in the pointed convex cone $\mathcal{P}_{3,8}^{B_3}$. The claim follows from Minkowski's Theorem. $\square$

Theorem 3.22 established the stabilization of B_n-Specht modules in $H_{n,d}$ for a large enough number of variables for $d=4$. This allows a uniform description of the sets $\Sigma_{n,8}^{B_n}$ for all $n \geq 4$, as observed in Corollary 3.24.

Lemma 4.13. *The B_n-isotypic decomposition of $H_{n,4}$ for $n \geq 4$ is*

$$\begin{aligned} &2 \cdot \mathbb{S}^{((n),\emptyset)} \oplus 2 \cdot \mathbb{S}^{((n-1,1),\emptyset)} \oplus \mathbb{S}^{((n-2,2),\emptyset)} \oplus 2 \cdot \mathbb{S}^{((n-2),(2))} \\ &\oplus \mathbb{S}^{((n-2),(1,1))} \oplus \mathbb{S}^{((n-3,1),(2))} \oplus \mathbb{S}^{((n-4),(4))}. \end{aligned}$$

A symmetry adapted basis of $H_{n,4}$ realising the B_n-isotypic decomposition is given by the following seven sets of polynomials

$$\begin{array}{llllll} \mathbb{S}^{((n),\emptyset)} & : & \left\{p_{(4)}^{(n)}, p_{(2^2)}^{(n)}\right\} & \mathbb{S}^{((n-1,1),\emptyset)} & : & \{(X_n^2 - X_1^2)p_{(2)}^{(n)}, X_n^4 - X_1^4\} \\ \mathbb{S}^{((n-2,2),\emptyset)} & : & \{(X_1^2 - X_3^2)(X_2^2 - X_4^2)\} & \mathbb{S}^{((n-2),(1,1))} & : & \left\{(X_n^2 - X_{n-1}^2)X_{n-1}X_n\right\} \\ \mathbb{S}^{((n-4),(4))} & : & \{X_1X_2X_3X_4\} & \mathbb{S}^{((n-3,1),(2))} & : & \left\{(X_n^2 - X_1^2)X_{n-2}X_{n-1}\right\} \\ \mathbb{S}^{((n-2),(2))} & : & \{X_{n-1}X_n p_{(2)}^{(n)}, (X_{n-1}^2 + X_n^2)X_{n-1}X_n\} & & & \end{array}$$

Proof. We determine the multiplicity of an irreducible B_n-module $\mathbb{S}^{(\lambda,\mu)}$ in $H_{n,4}$ for bipartitions $(\lambda,\mu) \vdash n$ using Theorem 3.8. We can immediately exclude some bipartitions. The fundamental invariants of degree ≤ 4 are of degree 2 and 4. Only (λ,μ) such that $\mu \vdash n_2$, with $n_2 \leq 4$ can occur, since a corresponding higher Specht polynomial has as a factor the monomial consisting of all products of the X_i's, where i ranges over the entries of the second bitableau. Furthermore, we only need to consider bipartitions (λ,μ) such that $|\mu|$ is even because a factor of the higher Specht polynomial is of degree $|\mu|$, while the additional factor has even degree. We can restrict us to bipartitions (λ,μ) such that there exist $(T,S) \in \mathrm{SYT}(\lambda,\mu)$ with $2\,\mathrm{ch}(T,S) + |\mu| \leq 4$. Therefore a charge ≤ 2 is necessary. We calculated all relevant higher Specht polynomials for $n \geq 4$:

$$\begin{array}{llllll} \mathbb{S}^{((n),\emptyset)} & : & \{1\} & \mathbb{S}^{((n-1,1),\emptyset)} & : & \left\{X_n^2 - X_1^2, \frac{1}{n}\sum_{i=2}^{n-1} X_i^2(X_n^2 - X_1^2)\right\} \\ \mathbb{S}^{((n-2,2),\emptyset)} & : & \left\{(X_1^2 - X_3^2)(X_2^2 - X_4^2)\right\} & \mathbb{S}^{((n-2),(1,1))} & : & \left\{(X_n^2 - X_{n-1}^2)X_{n-1}X_n\right\} \\ \mathbb{S}^{((n-4),(4))} & : & \{X_1X_2X_3X_4\} & \mathbb{S}^{((n-3,1),(2))} & : & \left\{(X_n^2 - X_1^2)X_{n-2}X_{n-1}\right\} \\ \mathbb{S}^{((n-2),(2))} & : & \left\{X_{n-1}X_n, \frac{1}{n-2}(X_1^2 + \ldots + X_{n-2}^2)X_{n-1}X_n\right\}. & & & \end{array}$$

Multiplying these polynomials with power means gives a B_n-symmetry adapted basis of $H_{n,4}$. However, since

$$X_n^4 - X_1^4 \in \langle p_2^{(n)}(X_n^2 - X_1^2), \frac{1}{n}\sum_{i=2}^{n-1} X_i^2(X_n^2 - X_1^2)\rangle_{\mathbb{R}},$$

$$(X_{n-1}^2 + X_n^2)X_{n-1}X_n \in \langle p_2^{(n)} X_{n-1}X_n, \frac{1}{n-2}(X_1^2 + \ldots + X_{n-2}^2)X_{n-1}X_n\rangle_{\mathbb{R}},$$

we can work with the above-mentioned symmetry adapted basis. $\square$

The isotypic decomposition now allows us directly to give a uniform description of the B_n-invariant sum of squares of degree 8.

Theorem 4.14. *For $n \geq 4$, $f \in H_{n,8}^{B_n}$ is a sum of squares if and only if there exist positive semidefinite matrices $A^{((n),\emptyset)}, A^{((n-1,1),\emptyset)}, A^{((n-2,2),\emptyset)}, A^{((n-2),(2))} \in \mathbb{R}^{2\times 2}$ and $A^{((n-2),(1,1))}, A^{((n-4),(4))}, A^{((n-3,1),(2))} \in \mathbb{R}^{1\times 1}_{\geq 0}$ such that*

$$
\begin{aligned}
\mathfrak{f} = {} & \langle A^{((n),\emptyset)}, B^{((n),\emptyset)}\rangle + \langle A^{((n-1,1),\emptyset)}, B^{((n-1,1),\emptyset)}\rangle + \langle A^{((n-2,2),\emptyset)}, B^{((n-2,2),\emptyset)}\rangle \\
& + \langle A^{((n-2),(2))}, B^{((n-2),(2))}\rangle + A^{((n-2),(1,1))}B^{((n-2),(1,1))} \\
& + A^{((n-4),(4))}B^{((n-4),(4))} + A^{((n-3,1),(2))}B^{((n-3,1),(2))}
\end{aligned}
$$

where

$$
\begin{aligned}
B^{((n),\emptyset)} &:= \begin{pmatrix} p^{(n)}_{(4^2)} & p^{(n)}_{(4,2^2)} \\ p^{(n)}_{(4,2^2)} & p^{(n)}_{(2^4)} \end{pmatrix}, \\
B^{((n-1,1),\emptyset)} &:= \begin{pmatrix} p^{(n)}_{(4,2^2)} - p^{(n)}_{(2^4)} & p^{(n)}_{(6,2)} - p^{(n)}_{(4,2^2)} \\ p^{(n)}_{(6,2)} - p^{(n)}_{(4,2^2)} & p^{(n)}_{(8)} - p^{(n)}_{(4^2)} \end{pmatrix}, \\
B^{((n-2,2),\emptyset)} &:= \left(\frac{-n+1}{n^2} p^{(n)}_{(8)} + \frac{4n-4}{n^2} p^{(n)}_{(6,2)} + \frac{n^2-3n+3}{n^2} p^{(n)}_{(4^2)} - 2p^{(n)}_{(4,2^2)} + p^{(n)}_{(2^4)} \right), \\
B^{((n-2),(2))} &:= \begin{pmatrix} p^{(n)}_{(2^4)} - \frac{1}{n} p^{(n)}_{(4,2^2)} & 2p^{(n)}_{(4,2^2)} - \frac{2}{n} p^{(n)}_{(6,2)} \\ 2p^{(n)}_{(4,2^2)} - \frac{2}{n} p^{(n)}_{(6,2)} & 2p^{(n)}_{(6,2)} + 2p^{(n)}_{(4^2)} - \frac{4}{n} p^{(n)}_{8} \end{pmatrix}, \\
B^{((n-2),(1,1))} &:= \left(p^{(n)}_{(6,2)} - p^{(n)}_{(4^2)} \right), \\
B^{((n-4),(4))} &:= \left(p^{(n)}_{(2^4)} - \frac{6}{n} p^{(n)}_{(4,2^2)} + \frac{3}{n^2} p^{(n)}_{(4^2)} + \frac{8}{n^2} p^{(n)}_{(6,2)} - \frac{6}{n^3} p^{(n)}_{(8)}) \right), \\
B^{((n-3,1),(2))} &:= \left(\frac{2}{n^2} p^{(n)}_{(8)} - \frac{2n+2}{n^2} p^{(n)}_{(6,2)} - \frac{1}{n} p^{(n)}_{(4^2)} + \frac{n+3}{n} p^{(n)}_{(4,2^2)} - p^{(n)}_{(2^4)} \right).
\end{aligned}
$$

Proof. The matrices $B^{(i)}$ are the matrices that contain the symmetrized products of the symmetry-adapted basis of the B_n-module $H_{n,4}$ from Lemma 4.13. By Theorem 2.6, any invariant sum of squares form has such a representation. □

We observe that for $n \geq 4$ the $\mathbb{R}$-vector spaces

$$H_{n,8}^{B_n} = \langle p^{(n)}_{(2^4)}, p^{(n)}_{(4,2^2)}, p^{(n)}_{(4^2)}, p^{(n)}_{(4,2)}, p^{(n)}_{(6,2)}, p^{(n)}_{8} \rangle_{\mathbb{R}}$$

have the same dimension. We identify the vector spaces with respect to the isomorphism

$$p^{(n)}_{\lambda} \mapsto p^{(m)}_{\lambda}$$

for $n, m \in \mathbb{N}_{\geq 4}$. Blekherman and the second author studied symmetric quartic forms (Blekherman and Riener, 2021) and defined a limit set as the linear span of all $\mathfrak{p}_\lambda := \lim_{n\to\infty} p^{(n)}_{\lambda}$. They showed that for symmetric quartics the limits of the cones of sums of squares and non-negative forms are equal. As a first step towards a similar result in the B_n case, we provide a classification of the limit of the cones of even symmetric octics which are sums of squares.

Remark 4.15. The matrices in Theorem 4.14 have the following limits for $n \to \infty$

$$
\begin{aligned}
\mathcal{B}^{((n),\emptyset)} &:= \begin{pmatrix} \mathfrak{p}_{(4^2)} & \mathfrak{p}_{(4,2^2)} \\ \mathfrak{p}_{(4,2^2)} & \mathfrak{p}_{(2^4)} \end{pmatrix} \\
\mathcal{B}^{((n-1,1),\emptyset)} &:= \begin{pmatrix} \mathfrak{p}_{(4,2^2)} - \mathfrak{p}_{(2^4)} & \mathfrak{p}_{(6,2)} - \mathfrak{p}_{(4,2^2)} \\ \mathfrak{p}_{(6,2)} - \mathfrak{p}_{(4,2^2)} & \mathfrak{p}_{(8)} - \mathfrak{p}_{(4^2)} \end{pmatrix},
\end{aligned}
$$

$$\begin{aligned}
\mathcal{B}^{((n-2,2),\emptyset)} &:= \left(\mathfrak{p}_{(4^2)} - 2\mathfrak{p}_{(4,2^2)} + \mathfrak{p}_{(2^4)} \right), \\
\mathcal{B}^{((n-2),(2))} &:= \begin{pmatrix} \mathfrak{p}_{(2^4)} & 2\mathfrak{p}_{(4,2^2)} \\ 2\mathfrak{p}_{(4,2^2)} & 2\mathfrak{p}_{(6,2)} + 2\mathfrak{p}_{(4^2)} \end{pmatrix}, \\
\mathcal{B}^{((n-2),(1,1))} &:= \left(\mathfrak{p}_{(6,2)} - \mathfrak{p}_{(4^2)} \right), \\
\mathcal{B}^{((n-4),(4))} &:= \left(\mathfrak{p}_{(2^4)} \right), \\
\mathcal{B}^{((n-3,1),(2))} &:= \left(\mathfrak{p}_{(4,2^2)} - \mathfrak{p}_{(2^4)} \right).
\end{aligned}$$

Corollary 4.16. *An even symmetric homogeneous octic limit sum of squares inequality $\mathfrak{f}$ has the form*

$$\begin{aligned}
\mathfrak{f} = {}& \alpha_1 \mathfrak{p}_{(4^2)} + 2\alpha_2 \mathfrak{p}_{(4,2^2)} + \alpha_3 \mathfrak{p}_{(2^4)} \\
&+ \beta_1(\mathfrak{p}_{(4,2^2)} - \mathfrak{p}_{(2^4)}) + 2\beta_2(\mathfrak{p}_{(6,2)} - \mathfrak{p}_{(4,2^2)}) + \beta_3(\mathfrak{p}_{(8)} - \mathfrak{p}_{(4^2)}) \\
&+ \delta(\mathfrak{p}_{(6,2)} - \mathfrak{p}_{(4^2)})
\end{aligned}$$

where $\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_2 & \beta_3 \end{pmatrix}, (\delta)$ *are positive semidefinite real matrices.*

Proof. We observe that an invariant limit sum of squares coming from the irreducible representation $\mathbb{S}^{((n-2,2),\emptyset)}$, i.e., $\mathfrak{p}_{(4^2)} - 2\mathfrak{p}_{(4,2^2)} + \mathfrak{p}_{(2^4)}$, is contained in the first line. The limit sum of squares $\mathfrak{p}_{(2^4)}$ from $\mathbb{S}^{((n-4),(4))}$ is also contained in the first line, while the limit form from $\mathbb{S}^{((n-3,1),(2))}$, i.e., $\mathfrak{p}_{(4,2^2)} - \mathfrak{p}_{(2^4)}$, is contained in the second line for $\beta_1 = 1$. Furthermore,

$$(\alpha, \beta) \begin{pmatrix} \mathfrak{p}_{(2^4)} & 2\mathfrak{p}_{(4,2^2)} \\ 2\mathfrak{p}_{(4,2^2)} & 2\mathfrak{p}_{(6,2)} + 2\mathfrak{p}_{(4^2)} \end{pmatrix} (\alpha, \beta)^T = 2\beta^2(\mathfrak{p}_{(6,2)} - \mathfrak{p}_{(4^2)}) + \langle \begin{pmatrix} 4\beta^2 & 2\alpha\beta \\ 2\alpha\beta & \alpha^2 \end{pmatrix}, \begin{pmatrix} \mathfrak{p}_{(4^2)} & \mathfrak{p}_{(4,2^2)} \\ \mathfrak{p}_{(4,2^2)} & \mathfrak{p}_{(2^4)} \end{pmatrix} \rangle. \qquad \square$$

Remark 4.17. Let $\Sigma_{\infty,8}^{B_\infty}$ denote the cone consisting of all limit forms from Corollary 4.16, $\Sigma_{\infty,4}^{\mathfrak{S}_\infty}$ the limit cone of symmetric sums of squares of degree 4 in Blekherman and Riener (2021) and $\Phi : H_{\infty,8}^{B_\infty} \to H_{\infty,4}^{\mathfrak{S}_\infty}$ be the canonical $\mathfrak{S}_\infty$-homomorphism. The cones $\Phi\left(\Sigma_{\infty,8}^{B_\infty}\right)$ and $\Sigma_{\infty,4}^{\mathfrak{S}_\infty} = \mathcal{P}_{\infty,4}^{\mathfrak{S}_\infty}$ are different. This is not surprising, since the cone $\Phi\left(\mathcal{P}_{\infty,8}^{B_\infty}\right)$ can be identified with the limit of all symmetric forms that are non-negative on the positive orthant (compare with Polya's Nichtnegativstellensatz; Pólya, 1928).

It is a question for further studies to determine the relationship between the limit cones of even symmetric sums of squares and non-negative octics.

4.2. Forms invariant under D_n

It is natural to wonder, to what extent Harris' result on ternary forms invariant under B_3 carries over to the slightly smaller group D_3. As is shown in the following theorem we obtain equality between the sets $\Sigma_{3,8}^{D_3}$ and $\mathcal{P}_{3,8}^{D_3}$. Furthermore, we prove that $\mathcal{P}_{4,4}^{D_4}$ is a simplicial cone that gives a test set for non-negativity consisting of three points. We prove that for quaternary quartics invariant under D_4 we also have that non-negativity implies having a sum of squares representation. We conclude with a full characterization of the non-negativity versus sums of squares question for forms invariant under D_n.

Theorem 4.18. *The sets of non-negative and sums of squares ternary octics invariant under D_3 are equal, i.e., $\Sigma_{3,8}^{D_3} = \mathcal{P}_{3,8}^{D_3}$.*

Proof. The invariant ring $\mathbb{R}[X_1, X_2, X_3]^{D_3} = \mathbb{R}[p_2, e_3, p_4]$ is a polynomial ring in the symmetric polynomials p_2, e_3 and p_4. A vector space basis of $H_{3,8}^{D_3}$ is given by $\left(p_{(2^4)}, p_{(4,2^2)}, p_{(4^2)}, p_2 e_3^2\right)$. Recall that

$H_{3,8}^{B_3} = \langle p_{(2^4)}, p_{(4,2^2)}, p_{(4^2)}, p_{(6,2)} \rangle_{\mathbb{R}}$. The functions p_6 and e_3^2 occur linearly in the following identity for symmetric functions in three variables

$$p_{(2^3)} - 3p_{(4,2)} + 2p_6 - 6e_3^2 = 0.$$

Hence we deduce that $H_{3,8}^{D_3} = H_{3,8}^{B_3}$. The claim follows by Corollary 4.2. □

Remark 4.19. We have the same conical generators and test set for non-negative ternary octics invariant under D_3 as for B_3, i.e., a form $f \in H_{3,8}^{D_3}$ is non-negative if and only if $f(y) \geq 0$ for all $y \in \{(a, a, b), (0, a, b) : a, b \in \mathbb{R}_{\geq 0}\}$.

In the following, we study quaternary quartics invariant under D_4.

Lemma 4.20. *The D_4-module $H_{4,2}$ has the isotypic decomposition*

$$H_{4,2} = \mathbb{S}^{((4),\emptyset)} \oplus \mathbb{S}^{((3,1),\emptyset)} \oplus \mathbb{S}_+^{((2),(2))} \oplus \mathbb{S}_-^{((2),(2))}.$$

A symmetry adapted basis which realizes the D_4-module decomposition of $H_{4,2}$ is the following:

$$\mathbb{S}^{((4),\emptyset)} : \left\{p_{(2)}\right\}, \qquad \mathbb{S}^{((3,1),\emptyset)} : \left\{X_4^2 - X_1^2\right\},$$
$$\mathbb{S}_+^{((2),(2))} : \{X_1X_2 + X_3X_4\}, \qquad \mathbb{S}_-^{((2),(2))} : \{X_1X_2 - X_3X_4\}.$$

Proof. By Theorem 3.8, we have to determine the multiplicity of the irreducible D_4-modules labelled by bipartitions $(\lambda, \mu) \vdash 4$ of the form $|\lambda| \geq |\mu|$. We are just interested in higher Specht polynomials of degree 0 or 2, since the only D_4 fundamental invariant of degree ≤ 2 is p_2. Thus, necessarily $|\mu| \in \{0, 2\}$. If $\mu = \emptyset$, then both bipartitions $((4), \emptyset), ((3, 1), \emptyset)$ have exactly one standard bitableau whose charge is at most 1, i.e., they occur precisely once in $H_{4,2}$. Any occurring module labelled by (λ, μ) with $|\mu| = 2$ must have a standard bitableau with index $(0, 0, 0, 0)$. This can only occur if the word equals $(1, 2, 3, 4)$. Thus, only the bipartition $((2), (2))$ has a standard bitableau with charge 0. By Theorem 3.8, the module $\mathbb{S}^{((2),(2))}$ decomposes into two irreducible D_4-modules $\mathbb{S}_+^{((2),(2))}$ and $\mathbb{S}_-^{((2),(2))}$. We calculated the relevant higher Specht polynomials according to Theorem 3.8

$$\left\{1,\ X_4^2 - X_1^2,\ X_1X_2 + X_3X_4,\ X_1X_2 - X_3X_4\right\},$$

and find accordingly the polynomials above. □

Corollary 4.21. *A D_4-invariant quaternary quartic $f \in H_{4,4}^{D_4}$ is a sum of squares if and only if there exist positive numbers $A^{(1)}, A^{(2)}, A^{(3)}, A^{(4)} \in \mathbb{R}_{\geq 0}$ such that $f = A^{(1)}B^{(1)} + A^{(2)}B^{(2)} + A^{(3)}B^{(3)} + A^{(4)}B^{(4)}$, where*

$$B^{((4),\emptyset)} := \left(p_{(2^2)}\right), \qquad B^{((3,1),\emptyset)} := \left(\frac{2}{3}p_{(4)} - \frac{1}{6}p_{(2^2)}\right),$$
$$B_+^{((2),(2))} := \left(\frac{1}{6}p_{(2^2)} - \frac{1}{6}p_{(4)} + 2e_4\right), \qquad B_-^{((2),(2))} := \left(\frac{1}{6}p_{(2^2)} - \frac{1}{6}p_{(4)} - 2e_4\right).$$

Proof. The matrices $B^{(i)}$ are obtained by calculating the Reynolds operator evaluated at squares of the symmetry adapted basis of the irreducible D_4-modules from Lemma 4.20. By Theorem 2.6, any invariant sum of squares form has such a representation. □

Theorem 4.22. *The cone of $\Sigma_{4,4}^{D_4,*}$ is a simplicial cone with the following description*

$$\Sigma_{4,4}^{D_4,*} = \operatorname{cone}\left\{\operatorname{ev}_{(1,0,0,0)},\ \operatorname{ev}_{(1,1,1,-1)}, \operatorname{ev}_{(1,1,1,1)}\right\}.$$

Proof. Let $\ell \in \Sigma_{4,4}^{D_4,*}$ denote an extremal element. Let

$$W_\ell := \alpha \cdot \mathbb{S}^{((4),\emptyset)} \oplus \beta \cdot \mathbb{S}^{((3,1),\emptyset)} \oplus \gamma \cdot \mathbb{S}_+^{((2),(2))} \oplus \delta \cdot \mathbb{S}_-^{((2),(2))}$$

be the D_4-submodule of $H_{4,2}$ which is the kernel of the associated quadratic form, for $\alpha, \beta, \gamma, \delta \in \{0,1\}$. Now, we show that ℓ must be a scalar multiple of one of the three point-evaluations above, respectively that $W_\ell^{\langle 2\rangle}$ must have one of the points as a zero.

Since $p_{(2^2)}$ is not contained in the boundary of $\Sigma_{4,4}^{D_4}$ we see that $\alpha = 0$. Furthermore, $\dim_{\mathbb{R}} W_\ell^{\langle 2\rangle} = 2$ and therefore we have that precisely two of the parameters are non-zero, because the symmetrized squares of the symmetry adapted basis elements belonging to the D_4-modules $\mathbb{S}^{((3,1),\emptyset)}$, $\mathbb{S}_+^{((2),(2))}$ and $\mathbb{S}_-^{((2),(2))}$ are linearly independent.

i) We start by examining the case $\gamma = \delta = 1$. Then $\ell(e_4) = 0$, $\ell(p_{(2^2)}) = \ell(p_{(4)})$ and

$$W_\ell^{\langle 2\rangle} = \langle e_4, p_{(2^2)} - p_{(4)} \rangle_{\mathbb{R}}.$$

$W_\ell^{\langle 2\rangle}$ has the root $(1,0,0,0)$.

We proceed with the cases $\gamma = \beta = 1$ or $\beta = \delta = 1$.

ii) We notice that if $\gamma = \beta = 1$ then

$$W_\ell = \langle X_4^2 - X_1^2, X_1X_2 + X_3X_4 \rangle_{D_4},$$

but all elements in W_ℓ have the common root $(1,1,1,-1)$.

iii) If $\beta = \delta = 1$ then

$$W_\ell = \langle X_4^2 - X_1^2, X_1X_2 - X_3X_4 \rangle_{D_4}$$

with the common root $(1,1,1,1)$. □

Corollary 4.23. *The set of non-negative and sums of squares quaternary quartics invariant under D_4 are equal, i.e., $\Sigma_{4,4}^{D_4} = \mathcal{P}_{4,4}^{D_4}$.*

The corollary does not already follow from the observation made in Harris (1999) that $\Sigma_{4,4}^{B_4} = \mathcal{P}_{4,4}^{B_4}$ because $H_{4,4}^{D_4} \setminus H_{4,4}^{B_4} \neq \emptyset$.

Proof. By Theorem 4.22, the cone $\Sigma_{4,4}^{D_4,*}$ is generated by point-evaluations. Hence any extremal ray in $\Sigma_{4,4}^{D_4,*}$ is spanned by a point-evaluation and the claim follows from Corollary 2.41. □

By reformulating Theorem 4.22 we obtain the following very simple test set for D_4-quartics:

Corollary 4.24. *A form $f = a(X_1^2+X_2^2+X_3^2+X_4^2)^2 + b(X_1^4+X_2^4+X_3^4+X_4^4) + cX_1X_2X_3X_4$, with $a,b,c \in \mathbb{R}$, is non-negative if and only if $f(z) \geq 0$ for all $z \in \{(1,0,0,0),(1,1,1,-1),(1,1,1,1)\}$.*

Proof. An invariant form $f \in H_{4,4}^{D_4}$ is non-negative if and only if $\ell(f) \geq 0$ for all $\ell \in \mathcal{P}_{4,4}^{D_4,*}$. By Corollary 4.23 we have $\mathcal{P}_{4,4}^{D_4,*} = \Sigma_{4,4}^{D_4,*}$. But we saw in Theorem 4.22 that $\Sigma_{4,4}^{D_4,*}$ has these 3 extreme rays. □

Corollary 4.25. *The convex cone $\mathcal{P}_{4,4}^{D_4}$ of non-negative D_4-quartics is a simplicial cone generated by*

$$4p_{(4)} - p_{(2^2)}, \quad p_{(2^2)} - p_{(4)} + 12e_4, \quad p_{(2^2)} - p_{(4)} - 12e_4.$$

Proof. The sets $\mathcal{P}_{4,4}^{D_4}$ and $\Sigma_{4,4}^{D_4}$ are equal by Corollary 4.23. The boundary of $\Sigma_{4,4}^{D_4}$ is equal to the union of all kernels of extremal elements in $\Sigma_{4,4}^{D_4,*}$ intersected with $\Sigma_{4,4}^{D_4}$. The claimed forms are precisely the invariant sums of squares contained in the kernels of the three extremal rays in Theorem 4.22. $\square$

The results from the previous two subsections allow to conclude the following classification of the equivariant non-negativity versus sums of squares question for the reflection group D_n.

Theorem 4.26. *The sets $\Sigma_{n,2d}^{D_n}$ and $\mathcal{P}_{n,2d}^{D_n}$ are equal if and only if $(n, 2d) \in \{(2, 2d), (n, 2), (n, 4), (3, 8)\}$.*

Proof. Suppose that there exists $f \in \mathcal{P}_{n,2d}^{B_n} \setminus \Sigma_{n,2d}^{B_n}$. This implies $f \in \mathcal{P}_{n,2d}^{D_n} \setminus \Sigma_{n,2d}^{D_n}$. Therefore, we can directly rely on the classification carried out in Goel et al. (2017) and we only need to consider these cases specifically, where all even symmetric positive semidefinite forms are sums of squares. These are only the following non-trivial cases: $(n, 2d) \in \{(3, 8), (n, 4)\}$. But we have shown in Theorem 4.18 that in the case $(3, 8)$ the equality does survive, and while following Corollary 4.23 it does also for $(4, 4)$. Furthermore, if $n > 4$ then the invariant quartics with respect to B_n are precisely the invariant quartics with respect to D_n as $H_{n,4}^{B_n} = \langle p_{(2^2)}, p_{(4)}\rangle_{\mathbb{R}} = H_{n,4}^{D_n}$ for $n \geq 5$, which completes the proof. $\square$

4.3. LMIs and non-negativity testing

In general testing non-negativity of a polynomial in more than two variables is already for quartics an NP-hard problem (see e.g. Blum et al., 1998 or Murty and Kabadi, 1985). On the other hand, certifying that a given polynomial is a sum of squares can be done with so-called semidefinite programming. Although the complexity status of this procedure in the Turing or in the real numbers model is not yet known (see Ramana, 1997) SDPs can be solved numerically in polynomial time to a given accuracy through the ellipsoid algorithm and this approach generally provides a tractable way to certify that a polynomial is non-negative, if it is a sum of squares. For real symmetric matrices $A, B \in \mathbb{R}^{n\times n}$ we write $A \succeq B$ if $A - B$ is positive semidefinite. The feasible region of a semidefinite program is given by the projection of a set defined by a *linear matrix inequality* (LMI), i.e., an inequality of the form $A_0 + x_1 A_1 + x_2 A_2 + \ldots + x_n A_n \succeq 0$, where $A_0, \ldots, A_n$ are real symmetric matrices all of the same size and $x_1, \ldots, x_n$ are supposed to be real scalars. The set of all $x \in \mathbb{R}^n$ satisfying a given LMI is called a *spectrahedron*. For every $f \in H_{n,2d}$ one can construct an LMI (Powers and Wörmann, 1998) which possesses a solution if and only if f is a sum of squares. The corresponding spectrahedron is called the *Gram spectrahedron* of f (Chua et al., 2016), and it represents in fact all possible ways to decompose f into sums of squares. Accordingly, it is non-empty if and only if f is a sum of squares. The results presented in the article can be directly transferred into the setup of symmetry-adapted Gram-spectrahedra, which were, for example, recently studied by Heaton et al. (2020).

Theorem 4.27. *Let G be a finite reflection group and consider $f \in H_{n,2d}^{G}$ and $\theta_1, \ldots, \theta_l$ be all non G-isomorphic irreducible representations. Then the Gram spectrahedron of f can be defined by a block diagonal matrix, consisting of l blocks $B_1, \ldots, B_l$ and the size of the block B_i equals*

$$\sum_{k=0}^{d} N(d-k) \cdot h_k^{\vartheta_i}.$$

In particular, in the case $G \in \{A_{n-1}, B_n, D_n\}$ the size of the matrix is independent of n, for large n.

Proof. This follows from choosing a symmetry-adapted basis of $H_{n,d}$ and Corollary 2.27. When $G \in \{A_{n-1}, B_n, D_n\}$ the stabilization follows from Corollary 3.24. $\square$

A convex set that can be obtained as the projection of a higher dimensional spectrahedron is called *spectrahedral shadow*. Following a question by Nemirovski, which convex sets can be represented as projections of spectrahedra, Scheiderer (2018) showed that the cones of non-negative forms in general are not spectrahedral shadows. In the next theorem, we give some examples of invariant non-negative forms, which form spectrahedral shadows.

Theorem 4.28. *For all n, the families of cones $\mathcal{P}_{n,4}^{\mathfrak{S}_n}$, $\mathcal{P}_{n,6}^{B_n}$, $\mathcal{P}_{n,8}^{B_n}$ and $\mathcal{P}_{n,10}^{B_n}$ are spectrahedral shadows. Moreover, for forms in any of these families, there exists an LMI of size $O(n^3)$ certifying the non-negativity.*

Proof. For $n \leq 2$ this is trivial, and in the case $n = 3$ this follows either from Hilbert's Theorem in the $\mathfrak{S}_3$ case or from Harris' result 4.2 in the B_3 case. So we assume $n \geq 4$. By the half-degree principle, an element $f \in H_{n,4}^{\mathfrak{S}_n}$ is non-negative on $\mathbb{R}^n$ if and only if for any partition $\lambda \vdash n$ of length 2, the form $f^\lambda \in H_{2,4}$ is non-negative on $\mathbb{R}^2$, where $f^\lambda(x, y) := f(x, \ldots, x, y, \ldots, y)$ and x occurs precisely λ_1 times and y λ_2 times. Notice that each f^λ is non-negative if and only if it is a sum of squares, i.e., if we have $f^\lambda \in \Sigma_{2,4}$. If we denote by Φ^λ the linear map $f \mapsto \tilde{f}^\lambda(x, y)$ and if $\lambda^1, \ldots, \lambda^m$ are all partitions of n with length 2 then

$$\mathcal{P}_{n,4}^{\mathfrak{S}_n} = \bigcap_{i=1}^{m} \left(\Phi^{\lambda^i}\right)^{-1} (\Sigma_{2,4})$$

which proves the claim in the $\mathfrak{S}_n$ case. Using the half-degree principle (Riener, 2016, Theorem 3.1) for B_n and considering instead of $f(\underline{X}) \in \mathbb{R}[\underline{X}]^{B_n}$ the form $f(\sqrt{X_1}, \ldots, \sqrt{X_n}) \in \mathbb{R}[\underline{X}]^{\mathfrak{S}_n}$, one can argue analogously with slight modifications. □

Remark 4.29. In the case of symmetric polynomials, the above statement was implicitly already stated in Riener et al. (2013, Theorem 5.5) for symmetric quartic forms, albeit without mentioning of the term spectrahedral shadow.

The core of the proof above is the reduction to bivariate forms through test sets.

Theorem 4.30. *For the families of cones $\mathcal{P}_{n,6}^{\mathfrak{S}_n}$, $\mathcal{P}_{n,12}^{B_n}$ and $\mathcal{P}_{n,14}^{B_n}$ membership can be decided with $O(n^3)$ many LMIs, each of which has size bounded independent of n.*

Proof. Using the half-degree principle (Riener, 2016, Theorem 3.1) one finds that membership in each of the above-mentioned cones can be decided by reducing to $O(n^3)$ many ternary forms, similar to the proof above. For each of these ternary forms, one can decide non-negativity individually. de Klerk and Pasechnik (2004) provided a construction to decide the non-negativity of a ternary form of degree $2d$ by means of $d/4$ LMIs each of which is polynomial in d. Combining their construction with the arguments above thus yields an LMI of the announced size. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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