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# Spectrum of One-Dimensional Potential Perturbed by a Small Convolution Operator: General Structure

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**Abstract:** We consider an operator of multiplication by a complex-valued potential in $L_2(\mathbb{R})$, to which we add a convolution operator multiplied by a small parameter. The convolution kernel is supposed to be an element of $L_1(\mathbb{R})$, while the potential is a Fourier image of some function from the same space. The considered operator is not supposed to be self-adjoint. We find the essential spectrum of such an operator in an explicit form. We show that the entire spectrum is located in a thin neighbourhood of the spectrum of the multiplication operator. Our main result states that in some fixed neighbourhood of a typical part of the spectrum of the non-perturbed operator, there are no eigenvalues and no points of the residual spectrum of the perturbed one. As a consequence, we conclude that the point and residual spectrum can emerge only in vicinities of certain thresholds in the spectrum of the non-perturbed operator. We also provide simple sufficient conditions ensuring that the considered operator has no residual spectrum at all.

**Keywords:** convolution operator; potential; perturbation; spectrum; emerging eigenvalues

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## 1. Introduction

Over the last 20 years, there has been growing interest in non-local operators since they arise in various applications. Among such operators, there are convolution operators with integrable kernels. They appear in population dynamics, ecological problems and porous media theory. One of the interesting models of a nonlocal operator is a convolution operator perturbed by a potential, i.e., an operator

$$(\mathcal{L}u)(x) = \int\limits_{\mathbb{R}^d} a(x-y)u(y)\,dy + V(x)u(x) \quad \text{in} \quad L_2(\mathbb{R}^d). \tag{1}$$

While the spectra of the convolution operator and of the operator of multiplication by the potential can be found and characterized very easily, the description of the spectrum of their sum is a very non-trivial problem. At the same time, the spectral properties of such sums are not only of pure mathematical interest, but are important also for many applications. For instance, such operators arise in the mathematical theory of population dynamics and it is important to know whether a given operator of the form (1) possesses positive eigenvalues; such questions were studied in [1–4].

A more general problem regards the spectral properties of Schrödinger type operators, which are perturbations of a given pseudo-differential operator by a potential; see [5–8] and the references therein. The assumptions made in the cited papers ensured that the essential spectrum of the perturbed operator coincides with that of the unperturbed pseudo-differential operator. The main results described the existence of the discrete spectrum and Cwikel–Lieb–Rozenblum-type inequalities. A similar result was obtained in [9] for perturbations of a rather general class of Schrödinger type operators defined on a $\sigma$-compact metric space. In [10], various bounds were obtained for the number of negative eigenvalues produced by a perturbation of an operator $\mathcal{H}_0$ under the assumption that the Markov process with generator $-\mathcal{H}_0$ is recurrent.

In our recent works [11,12], we studied spectral properties of operator (1) assuming that it was self-adjoint. The essential spectrum was found explicitly. We established several sufficient conditions ensuring the existence of the discrete spectrum and obtained upper and lower bounds for the number of points of the discrete spectrum. We also provided sufficient conditions guaranteeing that the considered operator had infinitely many discrete eigenvalues accumulating to the thresholds of the essential spectrum. The structure of such sufficient conditions was quite different from similar well-known sufficient conditions for differential operators perturbed by localized potentials. The reason is that in the latter case, the unperturbed differential operator is unbounded and is perturbed by a bounded multiplication operator. In the case of the operator in (1), both the convolution operator and multiplication are equipollent and this essentially changes the spectral properties in comparison with the classical model of perturbed elliptic differential operators.

It is well known that a small localized perturbation of a differential operator with a non-empty essential spectrum can create eigenvalues emerging from certain thresholds in this essential spectrum. There are hundreds of works, in which such bifurcation was investigated for various models. Not trying to mention all such works, we cite only a few very classical ones, where this phenomenon was first rigorously studied [13–16]. In view of such results for differential operators, a natural and reasonable continuation of our studies in [11,12] is to consider similar the issue for operators (1), i.e., to study the operator

$$(\mathcal{L}^\varepsilon u)(x) = \int\limits_{\mathbb{R}^d} a(x-y)u(y)\,dy + \varepsilon V(x)u(x)$$

on $L_2(\mathbb{R}^d)$, where $\varepsilon$ is a small parameter. Here, again, the unperturbed operator and the perturbed one are equipollent and we naturally expect that the mechanisms of the eigenvalue's emergence from the essential spectrum can be rather different from ones for differential operators. This is indeed the case; for instance, using the Fourier transform, we can replace the operator $\mathcal{L}^\varepsilon$ with a unitary equivalent one, in which the original convolution operator is replaced by the multiplication operator, while the potential generates a convolution operator with a small coupling constant:

$$(\hat{\mathcal{L}}^\varepsilon u)(x) = \hat{a}(x)u(x) + \varepsilon \int\limits_{\mathbb{R}^d} \hat{V}(x-y)u(y)\,dy.$$

Exactly this operator in the one-dimensional case $(d = 1)$ is the main object of the study in the present work. We succeed in dropping the condition of self-adjointness of the operator and treating a general operator with a complex-valued potential and a general convolution kernel. For such a general non-self-adjoint operator, we explicitly find its essential spectrum; it turns out to be the union of the ranges of the potential and of the Fourier image of the convolution kernel. Then, we show that the entire spectrum is located in a thin neighbourhood of the spectrum of the unperturbed multiplication operator. Our most nontrivial result states that in some fixed neighbourhood of a typical part of the spectrum of the unperturbed operator, there are no eigenvalues and no residual spectrum. As a consequence, we conclude that the eigenvalues and the residual spectrum can emerge only in vicinities of certain thresholds in the essential spectrum of the unperturbed operator. We also provide simple sufficient conditions ensuring that the considered operator has no residual spectrum at all, and not only in the aforementioned vicinities.

The issue of the existence and behaviour of possible eigenvalues and the residual spectrum emerging from the aforementioned threshold is an interesting problem that deserves an independent study. We shall present such a study in our next paper, which is being prepared now.

## 2. Problem and Main Results

Let $V = V(x)$ and $a = a(x)$ be measurable complex-valued functions defined on $\mathbb{R}$. On the space $L_1(\mathbb{R})$, we introduce a Fourier transform by the formula

$$\mathcal{F}[u](x) := \int\limits_{\mathbb{R}^d} u(\xi)e^{-\mathrm{i}x\cdot\xi}\,d\xi$$

and then extend it to $L_2(\mathbb{R})$. We assume that the function $a$ belongs to $L_1(\mathbb{R})$, while the function $V$ is an image of some function $\hat{V} \in L_1(\mathbb{R}^d)$, i.e., $V = \mathcal{F}[\hat{V}]$. We let $\hat{a}(\xi) := \mathcal{F}[a](\xi)$.

The paper is devoted to studying an operator in $L_2(\mathbb{R})$ defined by the formula

$$\mathcal{L}^\varepsilon := \mathcal{L}_V + \varepsilon\mathcal{L}_{a\star}, \qquad (\mathcal{L}_{a\star}u)(x) := \int\limits_{\mathbb{R}^d} a(x-y)u(y)\,dy, \qquad (\mathcal{L}_V u)(x) := V(x)u(x),$$

where $\varepsilon$ is a small positive parameter. This operator is bounded in $L_2(\mathbb{R})$; this fact can be easily proved by literally reproducing the proof of Lemma 4.1 in [11]. Our main aim is to describe the behaviour of the spectrum of this operator for sufficiently small $\varepsilon$.

Since the functions $a$ and $V$ are complex-valued, the operator $\mathcal{L}^\varepsilon$ is non-self-adjoint. In this paper, we follow a usual classification of the spectrum of a non-self-adjoint operator. Namely, the spectrum $\sigma(\,\cdot\,)$ of a given operator is introduced as a complement to its resolvent set. The point spectrum $\sigma_{\mathrm{pnt}}(\,\cdot\,)$ is the set of all eigenvalues. The essential spectrum $\sigma_{\mathrm{ess}}(\,\cdot\,)$ is defined in terms of the characteristic sequences, i.e., $\lambda \in \sigma_{\mathrm{ess}}(\mathcal{A})$ of a closed operator $\mathcal{A}$ in $L_2(\mathbb{R})$ if there exists a bounded non-compact sequence $u_n$ in $L_2(\mathbb{R})$ such that $(\mathcal{A}-\lambda)u_n \to 0$ in $L_2(\mathbb{R})$ as $n \to \infty$. The residual spectrum $\sigma_{\mathrm{res}}(\,\cdot\,)$ is defined as

$$\sigma_{\mathrm{res}}(\,\cdot\,) := \sigma(\,\cdot\,) \setminus \big(\sigma_{\mathrm{pnt}}(\,\cdot\,) \cup \sigma_{\mathrm{ess}}(\,\cdot\,)\big).$$

We shall show in Section 4.3, see Lemma 8, that the residual spectrum is given by the formula

$$\sigma_{\mathrm{res}}(\mathcal{A}) = (\sigma_{\mathrm{pnt}}(\mathcal{A}^*))^\dagger \setminus \big(\sigma_{\mathrm{pnt}}(\mathcal{A}) \cup \sigma_{\mathrm{ess}}(\mathcal{A})\big), \tag{2}$$

where for an arbitrary set $S \subset \mathbb{C}$, the set $S^\dagger$ is obtained by the symmetric reflection with respect to the real axis, i.e., $S^\dagger := \{\overline{\lambda} :\ \lambda \in S\}$.

We first describe the essential spectrum of the operator $\mathcal{L}^\varepsilon$. In order to do this, we introduce two curves in the complex plane as the ranges of the functions $V$ and $\hat{a}$:

$$\Upsilon := \{V(x) :\ x \in \mathbb{R}\}, \qquad \gamma := \{\hat{a}(x) :\ x \in \mathbb{R}\}.$$

**Theorem 1.** *The spectrum of the operator $\mathcal{L}^\varepsilon$ is located in a small neighbourhood of $\Upsilon$, namely,*

$$\sigma(\mathcal{L}^\varepsilon) \subseteq \big\{\lambda \in \mathbb{C} :\ \operatorname{dist}(\lambda,\Upsilon) \leqslant \varepsilon\|a\|_{L_1(\mathbb{R})}\big\}. \tag{3}$$

*For all $\varepsilon$ the essential spectrum of the operator $\mathcal{L}^\varepsilon$ is given by the identity*

$$\sigma_{\mathrm{ess}}(\mathcal{L}^\varepsilon) = \Upsilon \cup \varepsilon\gamma. \tag{4}$$

*The sets $\Upsilon$ and $\gamma$ are continuous closed curves in the complex plane that contain the origin.*

Apart of the essential spectrum described in Theorem 1, the operator $\mathcal{L}^\varepsilon$ can also have point and residual spectra. Our second main result states that the eigenvalues of the operator $\mathcal{L}^\varepsilon$ and its residual spectrum can exist only in the vicinities of certain thresholds on the curve $\Upsilon$ and they are absent in certain neighbourhoods of finite pieces of this curve. In order to state such a result, we classify all points $x_0 \in \mathbb{R}$ by a behaviour of the function $V$ in their vicinities. Namely, given two pairs $\alpha=(\alpha_-,\alpha_+)$ and $\beta=(\beta_-,\beta_+)$ with $\alpha_\pm\in\mathbb{C}\setminus\{0\}$ and $\beta_\pm\in(0,+\infty)$, a point $x_0\in\mathbb{R}$ is called a $(\beta,\alpha)$ threshold if there exists a $\rho$-neighbourhood of the point $x_0$ such that

$$V(x)-V_0=\alpha_\pm|x-x_0|^{\beta_\pm}v_\pm(x)\quad\text{as}\quad 0\leqslant\pm(x-x_0)\leqslant\rho, \tag{5}$$

where $v_-\in C^2[x_0-\rho,x_0]$, $v_+\in C^2[x_0,x_0+\rho]$ are some complex-valued functions such that

$$v_\pm(x_0)=1,\qquad |v_-'(x)|\leqslant C\quad\text{on}\quad [x_0-\rho,x_0],\qquad |v_+'(x)|\leqslant C\quad\text{on}\quad [x_0,x_0+\rho], \tag{6}$$

where $C$ is some constant independent of $x$.

A point $x_0\in\mathbb{R}$ is called regular if there exists a $\rho$-neighbourhood of the point $x_0$ such that

$$V\in C^2[x_0-\rho,x_0+\rho],\qquad V'(x_0)\neq 0. \tag{7}$$

Let $S$ be a connected close piece of the curve $\Upsilon$ not containing the origin. We assume that this piece is the image of finitely many disjoint segment $J_j:=[b_j^-,b_j^+]$ on the real axis, i.e.,

$$S=\big\{V(x):\,x\in J\big\},\qquad V(x)\notin S\quad\text{as}\quad x\notin J:=\bigcup_{j=1}^{n}J_j, \tag{8}$$

where $n\in\mathbb{N}$ and $b_j^\pm\in\mathbb{R}$ are fixed numbers and $b_j^-<b_j^+$. For $\delta>0$, we let

$$S^\delta:=\big\{\lambda\in\mathbb{C}:\,\mathrm{dist}(\lambda,S)\leqslant\delta\big\}.$$

By $B_r(y)$, we denote an open ball in the complex plane of a radius $r$ centred at a point $y$.

Now, we are in a position to formulate our second main result.

**Theorem 2.** *Let $S$ be a connected close piece of the curve $\Upsilon$ not containing the origin and obeying (8), each segment $J_j$ contains only regular points and finitely many $(\beta,\alpha)$ thresholds, and for each of such thresholds, we have $\beta_\pm<1$. Suppose that there exists a natural $m$ such that for each $\lambda\in S$, each of the segment $J_j$ contains at most $m$ points $x$ such that $V(x)=\lambda$. Suppose also that the generalize derivative $a'$ exists and*

$$a\in L_1(\mathbb{R})\cap W_2^1(\mathbb{R}),\qquad \operatorname*{esssup}_{\substack{(x,y)\in\mathbb{R}^2\\0<|x-y|<1}}\frac{|a'(x)-a'(y)|}{|x-y|^\theta}<\infty, \tag{9}$$

*where $\theta\in(0,1]$ is some fixed number. Then, there exists a sufficiently small $\delta>0$ such that for all sufficiently small $\varepsilon$, the closed $\delta$-neighbourhood $S^\delta$ of the set $S$ intersects neither with the point spectrum of the operator $\mathcal{L}^\varepsilon$, nor with its residual spectrum, i.e.,*

$$\sigma_{\mathrm{pnt}}(\mathcal{L}^\varepsilon)\cap S^\delta=\emptyset,\qquad \sigma_{\mathrm{res}}(\mathcal{L}^\varepsilon)\cap S^\delta=\emptyset.$$

Our third result concerns the residual spectrum. It is well known that such a spectrum is always absent for self-adjoint operators. In view of the absence of the residual spectrum in the set $S^\delta$ stated in Theorem 2, there arises a natural question on sufficient conditions ensuring the absence of the residual spectrum for the operator $\mathcal{L}^\varepsilon$. The answer to this question is our third main result formulated in the following theorem.

**Theorem 3.** *Assume that one of the following conditions holds:*

$$V(x)=\overline{V(x)},\qquad a(x)=\overline{a(-x)}, \tag{10}$$

*or*

$$V(\tau x+\varrho)=V(x),\qquad a(-\tau x)=a(x),\qquad x\in\mathbb{R}, \tag{11}$$

*for some $\varrho \in \mathbb{R}$ and $\tau \in \{-1, +1\}$. Then, the residual spectrum of the operator $\mathcal{L}^\varepsilon$ is empty for all $\varepsilon$.*

Let us briefly discuss the problem and the main results. The main feature of our operator $\mathcal{L}^\varepsilon$ is its non-self-adjointness, and in the general situation, both functions $V$ and $a$ are complex-valued. The convolution operator is multiplied by the small parameter and our operator $\mathcal{L}^\varepsilon$ is to be treated as a perturbation of the multiplication operator by a small convolution operator. As mentioned in the introduction, by applying the Fourier transform to the operator $\mathcal{L}^\varepsilon$, we can reduce it to a unitarily equivalent operator, in which the convolution and the potential parts interchange; then, we obtain a convolution operator perturbed by a small potential. The results of this work serve as a first step in studying how such a small perturbation deforms the spectrum of the unperturbed operator.

Our first result, Theorem 1, describes explicitly the location of the essential spectrum of the operator $\mathcal{L}^\varepsilon$. It turns out to be the union of the essential spectra of the unperturbed multiplication operator $\mathcal{L}_V$ and of the perturbed operator $\varepsilon\mathcal{L}_{a\star}$. These parts of the essential spectrum are the curves $\Upsilon$ and $\varepsilon\gamma$. The latter curve is small and is located in the vicinity of the origin. The spectrum of the operator $\mathcal{L}^\varepsilon$ also satisfies inclusion (3), which means that this spectrum is located in a thin tubular neighbourhood of the limiting spectrum $\Upsilon$.

Our most nontrivial result is Theorem 2. It states that in a typical situation, there are fixed neighbourhoods of finite pieces of the curve $\Upsilon$, which contain no point and residual spectra of the operator $\mathcal{L}^\varepsilon$. The choice of such finite pieces is characterized by the presence of $(\beta, \alpha)$ thresholds, and these pieces are to be generated by regular point and finitely many $(\beta, \alpha)$ thresholds with $\beta_\pm < 1$. The latter condition means that the function $V$ approaches such threshold with a not very high rate; see (5). The fact that there should be finitely many such thresholds is important and is employed essentially in the proof of Theorem 2. Another important point is that the considered piece of the curve $\Upsilon$ should not pass the origin; the presence of an additional curve $\varepsilon\gamma$ of the essential spectrum seems to play a nontrivial role in the existence of the discrete and residual spectrum in the vicinity of the origin. Assumption (9) is also essentially employed in the proof, and what can happen once they are violated is an interesting open question. We conjecture that violation of these conditions can dramatically change the spectral picture for the operator $\mathcal{L}^\varepsilon$.

We also observe that the second condition in (9) means that the first generalized derivative $a'$ is Hölder-continuous almost everywhere, and this can be guaranteed by assuming that the second generalized derivative $a''$ exists and belongs to $L_p(\mathbb{R})$ with some $p \in (1, +\infty)$ including the case $p = +\infty$. Indeed, if the second derivative is an element of $L_\infty(\mathbb{R})$, then the second condition in (9) is satisfied with $\theta = 1$, while for $1 < p < +\infty$, it is implied by the Hölder inequality:

$$|a'(x) - a'(y)| = \left| \int_x^y a''(t)\, dt \right| \leqslant |x - y|^{1 - \frac{1}{p}} \|a'\|_{L_p(\mathbb{R})}.$$

An important consequence of Theorem 2 is that the eigenvalues and the points of the residual spectrum can arise only in the vicinity of $(\beta, \alpha)$ thresholds, when at least one of the numbers $\beta_+$ and $\beta_-$ exceeds or equal to 1; in the case $\beta_+ = \beta_- = 1$, we should additionally assume that $\alpha_+ \neq -\alpha_-$ to avoid the case of a regular point. This means that typically, the spectrum of the operator $\mathcal{L}^\varepsilon$ is as follows: there is the essential spectrum described in Theorem 1, and along the curve $\Upsilon$, there are no eigenvalues and residual spectrum except vicinities of the origin and $(\beta, \alpha)$ thresholds with $\beta_+ \geqslant 1$ or/and $\beta_- \geqslant 1$. In such vicinities, the eigenvalues can indeed emerge; see an example in our recent work [12]. However, the study of possible emerging eigenvalues in the general situation is a non-trivial problem, which we postpone for our next paper.

Theorem 3 addresses one more question on the absence of the residual spectrum for the operator $\mathcal{L}^\varepsilon$. In contrast to Theorem 2, here we aim to find cases where the residual spectrum is completely absent rather than only in some neighbourhoods of some pieces of

$\Upsilon$. Condition (10) guarantees that the operator $\mathcal{L}^\varepsilon$ is self-adjoint. Condition (11) is more delicate and, in fact, it means that the operator $\mathcal{L}^\varepsilon$ is $\mathcal{PT}$-symmetric, namely,

$$\mathcal{PT}(\mathcal{L}^\varepsilon)^* = \mathcal{L}^\varepsilon\mathcal{PT}. \tag{12}$$

Here $\mathcal{T}$ is the operator of the complex conjugation, i.e., $\mathcal{T}u = \overline{u}$. The symbol $\mathcal{P}$ is an operator acting as

$$(\mathcal{P}u)(x) = u(\tau x + \varrho). \tag{13}$$

We also observe that once condition (12) holds for some other operator $\mathcal{P}$, it also ensures the absence of the residual spectrum for the operator $\mathcal{L}^\varepsilon$. Indeed, if $\lambda$ and $\phi$ are an eigenvalue and an associated eigenfunction of the adjoint operator $(\mathcal{L}^\varepsilon)^*$, then

$$\mathcal{L}^\varepsilon\mathcal{PT}\phi = \mathcal{PT}(\mathcal{L}^\varepsilon)^*\phi = \overline{\lambda}\mathcal{PT}\phi. \tag{14}$$

Hence, $\overline{\lambda}$ is an eigenvalue of the operator $\mathcal{L}^\varepsilon$, and by Formula (2), we see that the residual spectrum of the operator $\mathcal{L}^\varepsilon$ is empty.

## 3. Location of Spectrum and Essential Spectrum

In this section, we prove Theorem 1. We begin with checking identity (3). The spectrum of the operator $\mathcal{L}_V$ obviously coincides with $\Upsilon$. As $\lambda \notin \Upsilon$, the inverse operator $(\mathcal{L}_V - \lambda)^{-1}$ is the multiplication by $(V-\lambda)^{-1}$ and it is easy to see that the norm of the operator $(\mathcal{L}_V - \lambda)^{-1}$ satisfies the estimate

$$\|(\mathcal{L}_V - \lambda)^{-1}\| \leqslant \frac{1}{\operatorname{dist}(\lambda, \Upsilon)}. \tag{15}$$

For $\lambda \notin \Upsilon$, we consider the resolvent equation

$$(\mathcal{L}^\varepsilon - \lambda)u = f$$

with an arbitrary $f \in L_2(\Omega)$, and we rewrite it as

$$u + \varepsilon(\mathcal{L}_V - \lambda)^{-1}\mathcal{L}_{a\star}u = (\mathcal{L}_V - \lambda)^{-1}f. \tag{16}$$

By $\|\cdot\|_{X\to Y}$, we denote the norm of a bounded operator acting from a Banach space $X$ into a Banach space $Y$. As it was shown in the proof of Lemma 4.1 in [11], once $a \in L_1(\mathbb{R})$, the operator $\mathcal{L}_{a\star}$ is bounded in $L_2(\mathbb{R})$ and

$$\|\mathcal{L}_{a\star}\|_{L_2(\mathbb{R})\to L_2(\mathbb{R})} \leqslant \|a\|_{L_1(\mathbb{R})}. \tag{17}$$

This estimate and (15) yield that as

$$\varepsilon\|(\mathcal{L}_V - \lambda)^{-1}\mathcal{L}_{a\star}\| \leqslant \varepsilon\frac{\|a\|_{L_1(\mathbb{R})}}{\operatorname{dist}(\lambda,\Upsilon)} < 1,$$

the inverse operator $\big(\mathcal{I} + \varepsilon(\mathcal{L}_V - \lambda)^{-1}\mathcal{L}_{a\star}\big)^{-1}$ is well defined, where $\mathcal{I}$ is the identity operator. This allows us to solve Equation (16) and to find the resolvent of the operator $\mathcal{L}^\varepsilon$:

$$(\mathcal{L}^\varepsilon - \lambda)^{-1} = \big(\mathcal{I} + \varepsilon(\mathcal{L}_V - \lambda)^{-1}\mathcal{L}_{a\star}\big)^{-1}(\mathcal{L}_V - \lambda)^{-1} \qquad \text{as} \qquad \varepsilon\|a\|_{L_1(\mathbb{R})} < \operatorname{dist}(\lambda,\Upsilon).$$

Hence, each point in the spectrum of the operator $\mathcal{L}^\varepsilon$ satisfies the inequality $\operatorname{dist}(\lambda,\Upsilon) \leqslant \varepsilon\|a\|_{L_1(\mathbb{R})}$ and this proves inclusion (3).

In order to prove identity (4), we adapt the proof of Theorem 2.1 from [11] and below, we reproduce the main milestones from the cited work. It follows from our assumptions on $a$ and $\hat{V}$ that the functions $V$ and $\hat{a}$ are bounded and continuous on $\mathbb{R}$ and decay at infinity. We also observe the following unitary equivalence:

$$\left(\frac{1}{(2\pi)^{\frac{d}{2}}}\mathcal{F}\right)\mathcal{L}_{a\star}\left(\frac{1}{(2\pi)^{\frac{d}{2}}}\mathcal{F}\right)^{-1}=\mathcal{L}_{\hat{a}},\qquad \left(\frac{1}{(2\pi)^{\frac{d}{2}}}\mathcal{F}\right)\mathcal{L}_{V}\left(\frac{1}{(2\pi)^{\frac{d}{2}}}\mathcal{F}\right)^{-1}=\mathcal{L}_{\hat{V}\star}. \tag{18}$$

Hence,

$$\begin{aligned}&\sigma(\mathcal{L}_{\varepsilon\hat{a}})=\sigma_{\text{ess}}(\mathcal{L}_{\varepsilon\hat{a}})=\sigma(\mathcal{L}_{\varepsilon a\star})=\sigma_{\text{ess}}(\mathcal{L}_{\varepsilon a\star})=\varepsilon\gamma,\\ &\sigma(\mathcal{L}_{V})=\sigma_{\text{ess}}(\mathcal{L}_{V})=\sigma(\mathcal{L}_{\hat{V}\star})=\sigma_{\text{ess}}(\mathcal{L}_{\hat{V}\star})=\Upsilon.\end{aligned} \tag{19}$$

We are going to prove the inclusion

$$\Upsilon\cup\varepsilon\gamma\subseteq\sigma_{\text{ess}}(\mathcal{L}^{\varepsilon}). \tag{20}$$

We let

$$\varphi_n(x):=\begin{cases}(2n)^{\frac{1}{2}} & \text{as}\quad |x|<\dfrac{1}{n},\\ 0 & \text{as}\quad |x|>\dfrac{1}{n}\end{cases}$$

for all natural $n$. For an arbitrary $\lambda\in\Upsilon$, there exists $x_0\in\mathbb{R}$ such that $V(x_0)=\lambda$. The sequence $\varphi_n(x-x_0)$, normalized and non-compact in $L_2(\mathbb{R})$, is obviously a characteristic one of the operator $\mathcal{L}_V$ at the point $\lambda$. We also have:

$$\begin{aligned}\|\mathcal{L}_{a\star}\varphi_n(\cdot-x_0)\|^2_{L_2(\mathbb{R})}\leqslant&2n\int\limits_{\mathbb{R}}dx\left(\int\limits_{x_0-\frac{1}{n}}^{x_0+\frac{1}{n}}|a(x-y)|\,dy\right)^2=2n\int\limits_{\mathbb{R}}dx\left(\int\limits_{x-\frac{1}{n}}^{x+\frac{1}{n}}|a(y)|\,dy\right)^2\\ \leqslant&2n\left(\sup_{x\in\mathbb{R}}\int\limits_{x-\frac{1}{n}}^{x+\frac{1}{n}}|a(y)|\,dy\right)\int\limits_{\mathbb{R}^d}dx\int\limits_{x-\frac{1}{n}}^{x+\frac{1}{n}}|a(y)|\,dy\\ =&2n\left(\sup_{x\in\mathbb{R}}\int\limits_{x-\frac{1}{n}}^{x+\frac{1}{n}}|a(y)|\,dy\right)\int\limits_{\mathbb{R}^d}dy|a(y)|\int\limits_{y-\frac{1}{n}}^{y+\frac{1}{n}}dx\\ =&\|a\|_{L_1(\mathbb{R})}\left(\sup_{x\in\mathbb{R}}\int\limits_{x-\frac{1}{n}}^{x+\frac{1}{n}}|a(y)|\,dy\right)\to0,\quad n\to\infty,\end{aligned}$$

where the latter convergence is due to the absolute continuity of the Lebesgue integral. Hence, $\varphi_n(x-x_0)$ is a characteristic sequence of the operator $\mathcal{L}^{\varepsilon}$ at $\lambda$ and

$$\sigma_{\text{ess}}(\mathcal{L}_V)\subseteq\sigma_{\text{ess}}(\mathcal{L}^{\varepsilon}). \tag{21}$$

By unitary equivalence (18) and identity (19), we similarly obtain $\sigma_{\text{ess}}(\mathcal{L}_{\varepsilon a\star})\subseteq\sigma_{\text{ess}}(\mathcal{L})$, and in view of (21), this proves (20).

It remains to show that

$$\sigma_{\text{ess}}(\mathcal{L}^{\varepsilon})\setminus\left(\Upsilon\cup\varepsilon\gamma\right)=\emptyset.$$

If $\lambda\in\sigma_{\text{ess}}(\mathcal{L}^{\varepsilon})\setminus(\Upsilon\cup\varepsilon\gamma)$, there exists a bounded non-compact sequence $u_n\in L_2(\mathbb{R})$ such that

$$f_n:=(\mathcal{L}-\lambda)u_n\to0,\qquad n\to\infty. \tag{22}$$

Since $\lambda\notin\left(\sigma_{\text{ess}}(\mathcal{L}_V)\cup\sigma_{\text{ess}}(\mathcal{L}_{\varepsilon a\star})\right)$, in view of (19), the resolvents $(\mathcal{L}_V-\lambda)^{-1}$ and $(\mathcal{L}_{\varepsilon a\star}-\lambda)^{-1}$ are well defined and bounded. Then, we rewrite (22) as

$$\frac{1}{V-\lambda}\mathcal{L}_{\varepsilon a\star}u_n+u_n=\frac{f_n}{V-\lambda}\to0,\qquad n\to+\infty,\qquad V(x)\neq\lambda,\quad x\in\mathbb{R},$$

and we get

$$(\mathcal{L}_{\varepsilon a\star} - \lambda)u_n + V_1\mathcal{L}_{\varepsilon a\star}u_n = \frac{\lambda}{V-\lambda}f_n, \qquad V_1 := \frac{V}{V-\lambda}, \qquad \frac{1}{V-\lambda} = -\frac{1}{\lambda} + \frac{V_1}{\lambda}$$

where we have used zero as in $\sigma_{\text{ess}}(\mathcal{L}_V)$ and, therefore, $\lambda \neq 0$. Applying, then, the resolvent $(\mathcal{L}_{a\star} - \lambda)^{-1}$ to the obtained identity, we finally find:

$$u_n = (\mathcal{L}_{\varepsilon a\star} - \lambda)^{-1}\left(\frac{\lambda}{V-\lambda}f_n - V_1\mathcal{L}_{\varepsilon a\star}u_n\right).$$

Since the function $V$ decays as infinity, the same holds for $V_1$. This ensures the compactness of the operator $V_1\mathcal{L}_{\varepsilon a\star}$ in $L_2(\mathbb{R})$ and, hence, by the above identity, the sequence $u_n$ is compact, which is impossible. The proof is complete.

## 4. Absence of Point and Residual Spectrum

In this section, we prove Theorem 2. The proof consists of three main parts and we present them as separate subsections. After the proof of Theorem 2, we provide the proof of Theorem 3.

### *4.1. Absence of Embedded Eigenvalues*

By our assumptions, the segment $J_j$ contains only regular points and possibly finitely many $(\beta,\alpha)$ thresholds. We denote the latter thresholds by $x^{(j,i)}$, $i = 1,\ldots,m_j$, $j = 1,\ldots,n$, while the symbols $\beta_\pm^{(j,i)}$ and $\alpha_\pm^{(j,i)}$ stand for the corresponding values of $\beta_\pm$ and $\alpha_\pm$. The mentioned structure of the segment $J_j$ implies that the function $V$ is continuous on each of the segments $J_j$ and is continuously differentiable on the same segments except the $(\beta,\alpha)$ thresholds. It also follows from the definition of the $(\beta,\alpha)$ thresholds and the regular points that

$$|V'(x)| \geqslant c_0 > 0, \qquad x \in J_j \setminus \{x^{(j,i)},\, i = 1,\ldots,m_i\}, \quad j = 1,\ldots,n, \tag{23}$$

where $c_0$ is a fixed constant independent of $x$. As $x$ approaches one of the thresholds $x^{(j,i)}$, the derivative $V'$ blows up in the sense $|V'(x)| \to +\infty$ as $x \to x^{(j,i)}$.

It follows from (8) that there exists a small fixed $\delta_0$ such that

$$V(x) \notin S \quad \text{as} \quad x \in [b_j^- - \delta_0, b_j^-) \cup (b_j^+, b_j^+ + \delta_0], \quad j = 1,\ldots,n,$$

and, by (24),

$$\operatorname{dist}(V(x), S) \geqslant c_1|x - b_j^\pm| \quad \text{as} \quad 0 < \pm(x - b_j^\pm) < \delta_0, \quad j = 1,\ldots,n. \tag{24}$$

with some fixed positive constant $c_1$ independent of $x$ and $j$. We can additionally choose $\delta_0$ small enough so that for all $j = 1,\ldots,n$, the intervals $[b_j^- - \delta_0, b_j^-) \cup (b_j^+, b_j^+ + \delta_0]$ contain only regular points and, if necessary, reducing the constant $c_0$, we can extend estimate (23) to $\tilde{J}_j$, namely,

$$|V'(x)| \geqslant c_0 > 0, \qquad x \in \tilde{J}_j \setminus \{x^{(j,i)},\, i = 1,\ldots,m_i\}, \quad j = 1,\ldots,n. \tag{25}$$

Since $S$ is a closed connected piece of the curve $\Upsilon$, there exist two small fixed positive numbers $\delta_1$ and $c_2$ such that

$$\operatorname{dist}(V(x), S^{\delta_1}) \geqslant c_2 \qquad \text{as} \qquad x \notin \tilde{J} := \bigcup_{j=1}^{n} \tilde{J}_j, \qquad \tilde{J}_j := [b_j^- - \delta_0, b_j^+ + \delta_0]. \tag{26}$$

We consider the eigenvalue equation for the operator $\mathcal{L}^\varepsilon$ with the spectral parameter ranging in $S^{\delta_1}$:

$$(V-\lambda)\psi + \varepsilon\mathcal{L}_{a\star}\psi = 0. \tag{27}$$

Given an arbitrary measurable set $X \subseteq \mathbb{R}$, by $\mathcal{P}_X$, we denote the operator of restriction to $X$. This operator is considered as acting from $L_2(\mathbb{R})$ into $L_2(X)$ by the rule $(\mathcal{P}_X\psi)(x) := \psi(x)$, $x \in X$. Representing the real axis as $\mathbb{R} = \tilde{J} \cup (\mathbb{R} \setminus \tilde{J})$ and using an obvious decomposition $L_2(\mathbb{R}) = L_2(\tilde{J}) \oplus L_2(\mathbb{R} \setminus \tilde{J})$, we denote

$$\psi_{\tilde{J}} := \mathcal{P}_{\tilde{J}}\psi, \qquad \psi_{\mathbb{R}\setminus\tilde{J}} := \mathcal{P}_{\mathbb{R}\setminus\tilde{J}}\psi$$

and equivalently rewrite Equation (27) as a pair of two equations

$$\begin{aligned} &(V-\lambda)\psi_{\tilde{J}} + \varepsilon\mathcal{P}_{\tilde{J}}\mathcal{M}_{\tilde{J}}\psi_{\tilde{J}} + \varepsilon\mathcal{P}_{\tilde{J}}\mathcal{M}_{\mathbb{R}\setminus\tilde{J}}\psi_{\mathbb{R}\setminus\tilde{J}} = 0, \\ &(V-\lambda)\psi_{\mathbb{R}\setminus\tilde{J}} + \varepsilon\mathcal{P}_{\mathbb{R}\setminus\tilde{J}}\mathcal{M}_{\mathbb{R}\setminus\tilde{J}}\psi_{\mathbb{R}\setminus\tilde{J}} + \varepsilon\mathcal{P}_{\mathbb{R}\setminus\tilde{J}}\mathcal{M}_{\tilde{J}}\psi_{\tilde{J}} = 0, \end{aligned} \tag{28}$$

where for an arbitrary measurable set $X \subseteq \mathbb{R}$, the symbol $\mathcal{M}_X$ denotes a convolution operator acting from $L_2(X)$ into $L_2(\mathbb{R})$ by the rule

$$(\mathcal{M}_X\psi)(x) := \int\limits_X a(x-y)\psi(y)\,dy, \qquad x \in \mathbb{R}. \tag{29}$$

The first equation in (28) is to be treated as that in $L_2(\tilde{J})$, while the other equation is that in $L_2(\mathbb{R} \setminus \tilde{J})$.

Owing to (26), the norm of the operator of multiplication by $(V-\lambda)^{-1}$ in $L_2(\mathbb{R} \setminus \tilde{J})$ is bounded uniformly in $\lambda \in S^{\delta_1}$ by the constant $c_2^{-1}$. Applying this operator to the second equation in (28), we obtain an equivalent equation

$$\big(\mathcal{I}_{\mathbb{R}\setminus\tilde{J}} + \varepsilon(V-\lambda)^{-1}\mathcal{P}_{\mathbb{R}\setminus\tilde{J}}\mathcal{M}_{\mathbb{R}\setminus\tilde{J}}\big)\psi_{\mathbb{R}\setminus\tilde{J}} + \varepsilon(V-\lambda)^{-1}\mathcal{P}_{\mathbb{R}\setminus\tilde{J}}\mathcal{M}_{\tilde{J}}\psi_{\tilde{J}} = 0, \tag{30}$$

where $\mathcal{I}_{\mathbb{R}\setminus\tilde{J}}$ is the identity operator in $L_2(\mathbb{R} \setminus \tilde{J})$ and by estimate (17) we immediately see that $(V-\lambda)^{-1}\mathcal{P}_{\mathbb{R}\setminus\tilde{J}}\mathcal{M}_{\mathbb{R}\setminus\tilde{J}}$ is a bounded operator in $L_2(\mathbb{R} \setminus \tilde{J})$, and $(V-\lambda)^{-1}\mathcal{P}_{\mathbb{R}\setminus\tilde{J}}\mathcal{M}_{\tilde{J}}$ is a bounded operator from $L_2(\tilde{J})$ into $L_2(\mathbb{R} \setminus \tilde{J})$; both operators are bounded uniformly in $\lambda \in S^{\delta_1}$. Hence, for sufficiently small $\varepsilon$, the operator $\mathcal{I}_{\mathbb{R}\setminus\tilde{J}} + \varepsilon(V-\lambda)^{-1}\mathcal{P}_{\mathbb{R}\setminus\tilde{J}}\mathcal{M}_{\mathbb{R}\setminus\tilde{J}}$ is invertible for each $\lambda \in S^{\delta_1}$ and the inverse operator

$$\mathcal{Q}(\varepsilon,\lambda) := \big(\mathcal{I}_{\mathbb{R}\setminus\tilde{J}} + \varepsilon(V-\lambda)^{-1}\mathcal{P}_{\mathbb{R}\setminus\tilde{J}}\mathcal{M}_{\mathbb{R}\setminus\tilde{J}}\big)^{-1}$$

is bounded uniformly in $\varepsilon$ and $\lambda \in S^{\delta_1}$ as an operator in $L_2(\mathbb{R} \setminus \tilde{J})$. Applying this operator to Equation (30), we immediately find $\psi_{\mathbb{R}\setminus\tilde{J}}$:

$$\psi_{\mathbb{R}\setminus\tilde{J}} = -\varepsilon\mathcal{Q}(\varepsilon,\lambda)(V-\lambda)^{-1}\mathcal{P}_{\mathbb{R}\setminus\tilde{J}}\mathcal{M}_{\tilde{J}}\psi_{\tilde{J}}, \tag{31}$$

and the operator $\mathcal{Q}(\varepsilon,\lambda)(V-\lambda)^{-1}\mathcal{P}_{\mathbb{R}\setminus\tilde{J}}\mathcal{M}_{\tilde{J}}$ from $L_2(\tilde{J})$ into $L_2(\mathbb{R} \setminus \tilde{J})$ is bounded uniformly in $\varepsilon$ and $\lambda \in S^{\delta_1}$. Substituting this formula into the first equation in (28), we arrive at a single equation for $\psi_{\tilde{J}}$:

$$(V-\lambda)\psi_{\tilde{J}} + \varepsilon\mathcal{P}_{\tilde{J}}\mathcal{M}_{\tilde{J}}\psi_{\tilde{J}} - \varepsilon^2\mathcal{P}_{\tilde{J}}\mathcal{M}_{\mathbb{R}\setminus\tilde{J}}\mathcal{Q}(\varepsilon,\lambda)(V-\lambda)^{-1}\mathcal{P}_{\mathbb{R}\setminus\tilde{J}}\mathcal{M}_{\tilde{J}}\psi_{\tilde{J}} = 0. \tag{32}$$

We observe that the second and the third terms in the above equation can be rewritten as

$$\varepsilon\mathcal{P}_{\tilde{J}}\mathcal{M}_{\tilde{J}}\psi_{\tilde{J}} - \varepsilon^2\mathcal{P}_{\tilde{J}}\mathcal{M}_{\mathbb{R}\setminus\tilde{J}}\mathcal{Q}(\varepsilon,\lambda)(V-\lambda)^{-1}\mathcal{P}_{\mathbb{R}\setminus\tilde{J}}\mathcal{M}_{\tilde{J}}\psi_{\tilde{J}} = \varepsilon\mathcal{P}_{\tilde{J}}\mathcal{L}_{a\star}\mathcal{A}(\varepsilon,\lambda)\psi_{\tilde{J}},$$

where $\mathcal{A}$ is an operator from $L_2(\tilde{J})$ into $L_2(\mathbb{R})$ defined by the formula

$$\mathcal{A}(\varepsilon,\lambda)\psi_{\tilde{J}} := \begin{cases} \psi_{\tilde{J}} & \text{on} \quad \tilde{J}, \\ -\varepsilon\mathcal{Q}(\varepsilon,\lambda)(V-\lambda)^{-1}\mathcal{P}_{\mathbb{R}\setminus\tilde{J}}\mathcal{M}_{\tilde{J}}\psi_{\tilde{J}} & \text{on} \quad \mathbb{R}\setminus\tilde{J}. \end{cases} \tag{33}$$

This operator is bounded uniformly in $\varepsilon$ and $\lambda \in S^{\delta_1}$, namely,

$$\|\mathcal{A}(\varepsilon,\lambda)\|_{L_2(\tilde{J})\to L_2(\mathbb{R})} \leqslant c_3, \tag{34}$$

where $c_3$ is a constant independent of $\varepsilon$ and $\lambda$. Hence, Equation (32) becomes

$$(V-\lambda)\psi_{\tilde{J}} + \varepsilon \mathcal{P}_{\tilde{J}} \mathcal{L}_{a\star} \mathcal{A}(\varepsilon,\lambda)\psi_{\tilde{J}} = 0. \tag{35}$$

Our main aim is to prove that there exists a fixed positive $\delta \in (0, \delta_1]$ such that for $\lambda \in S^{\delta}$, Equation (35) can have only trivial solutions. First, we are going to show that such a statement holds for $\lambda$ located on the curve $\Upsilon \cap S^{\delta_1}$; such a curve obviously contains $S$.

We arbitrarily choose $\lambda \in \Upsilon \cap S^{\delta_1}$ and let $z^{(j,i)}$ be all points of the segment $\tilde{J}_j$ such that $V(z^{(j,i)}) = \lambda$. Here, the superscript $j$ ranges in some subset of $\{1, \ldots, n\}$ and $i$ ranges from 1 to some natural number depending on $j$. Let us show that the total number of points $z^{(j,i)}$ in each segment $\tilde{J}_j$ is bounded by some constant $\tilde{m} \geqslant m$ independent of $j$ and $\lambda$ provided $\delta_1$ and $\delta_0$ are chosen small enough. Indeed, according to our assumptions, the total number of the points $z^{(j,i)}$ located in the segment $J_j$ is bounded by $m$ and we only need to estimate the total number of such points located in $\tilde{J}_j \setminus J_j$. If $\lambda$ is such that one of the corresponding points $z^{(j,i)}$ is located in $[b_j^- - \delta_0, b_j^-)$ or in $(b_j^+, b_j^+ + \delta_0]$ for some $j$, then each of the mentioned intervals can contain at most one point $z^{(j,i)}$. This will be ensured by the inequality

$$V(x) \neq V(y) \quad \text{as} \quad x \neq y, \qquad x, y \in [b_j^- - \delta_0, b_j^-) \quad \text{or} \quad x, y \in (b_j^+, b_j^+ + \delta_0], \tag{36}$$

which we are going to prove. The point $b_j^+$ can be regular or a $(\beta, \alpha)$ threshold, and in both cases, owing to (5) and (7), for $x \in (b_j^+, b_j^+ + \delta_0]$ the function $V$ can be represented as

$$V(x) = \alpha_0 (x - b_j^+)^{\beta_0} v_0(x), \qquad v_0 \in C^2[b_j^+, b_j^+ + \delta_0],$$

provided $\delta_0$ is small enough. Here, $\alpha_0$ is some non-zero complex number, $\beta_0 \in (0,1]$ is some real number and $v_0$ is some complex-valued function such that $v_0(b_j^+) = 1$. Choosing $x, y \in (b_j^+, b_j^+ + \delta_0]$ arbitrarily, we have

$$\begin{aligned} V^{\frac{1}{\beta_0}}(x) - V^{\frac{1}{\beta_0}}(y) =& \alpha_0^{\frac{1}{\beta_0}} \Big( (x - b_j^+) v_0^{\frac{1}{\beta_0}}(x) - (y - b_j^+) v_0^{\frac{1}{\beta_0}}(y) \Big) \\ =& \alpha_0^{\frac{1}{\beta_0}} \Big( (x-y) v_0^{\frac{1}{\beta_0}}(x) + (y - b_j^+) \Big( v_0^{\frac{1}{\beta_0}}(x) - v_0^{\frac{1}{\beta_0}}(y) \Big) \Big). \end{aligned}$$

Applying the Lagrange rule, we obtain:

$$V^{\frac{1}{\beta_0}}(x) - V^{\frac{1}{\beta_0}}(y) = \alpha_0^{\frac{1}{\beta_0}} (x-y) \Big( v_0^{\frac{1}{\beta_0}}(x) + (y - b_j^+) \tilde{v}_0(x,y) \Big), \tag{37}$$

where $\tilde{v}_0(x,y)$ is some function obeying the uniform estimate

$$|\tilde{v}_0(x,y)| \leqslant \frac{1}{\beta_0} \|v_0\|_{C[b_j^+, b_j^+ + \delta_0]}^{\frac{1}{\beta_0} - 1} \|v_0'\|_{C[b_j^+, b_j^+ + \delta_0]}.$$

Since each segment $\tilde{J}_j$ can contain only finitely many $(\beta, \alpha)$ thresholds and all other points are regular, the right-hand side of this inequality can be estimated from the above by some constant independent of $j$. Hence, in view of the identity $v_0(b_j^+) = 1$, the expression in the brackets on the right-hand side of (37) is close to 1 and can not vanish once we choose a small enough $\delta_0$. This confirms inequality (36).

Let $\delta_2$ be a fixed positive number such that the intervals $U^{(j,i)} := \tilde{J}_j \cap \big( z^{(j,i)} - \delta_2, z^{(j,i)} + \delta_2 \big)$ are disjoint and each of these intervals contains no $(\beta, \alpha)$ thresholds except possibly that at $z^{(j,i)}$. Assume that $z^{(j,i)}$ is a regular point and let $x$ range outside $U^{(j,i)}$, but still in some bigger neighbourhood of $z^{(j,i)}$. By the Lagrange rule, we then have

$$V(x) - \lambda = V(x) - V(z^{(j,i)}) = (x - z^{(j,i)}) \big( \operatorname{Re} V'(x_{\mathrm{r}}^{(j,i)}) + \mathrm{i} \operatorname{Im} V'(x_{\mathrm{i}}^{(j,i)}) \big),$$

where $x_{\mathrm{r}}^{(j,i)}$ and $x_{\mathrm{i}}^{(j,i)}$ are some points between $x$ and $z^{(j,i)}$. By inequality (25), we see that for such $x$, the inequality holds:

$$|V(x) - \lambda| \geqslant c_2 |x - z^{(j,i)}|. \tag{38}$$

If $z^{(j,i)}$ is a $(\beta, \alpha)$ threshold, we choose $\delta_2$ small enough, so that in the interval $U^{(j,i)}$, representation (5) holds true. This representation implies immediately that

$$|V(x) - \lambda| \geqslant c_4 |x - z^{(j,i)}|$$

again for $x$ outside $U^{(j,i)}$, but still in some bigger neighbourhood of $z^{(j,i)}$; here, $c_4$ is a fixed positive constant independent of $x$, $j$ and $i$. This estimate and (38) imply the existence of a positive constant $c_5$ depending on $\delta_2$ but independent of the choice of $\lambda$ such that

$$|V(x) - \lambda| \geqslant c_5 > 0 \qquad \text{as} \qquad x \in \tilde{J} \setminus U, \qquad U := \bigcup_{j,i} U^{(j,i)}. \tag{39}$$

By $\chi^{(j,i)} = \chi^{(j,i)}(x)$, we denote the characteristic functions of the intervals $U^{(j,i)}$, while $M_0$ is the set of the superscripts $(j,i)$ such that either the point $z^{(j,i)}$ is regular or it is a $(\beta, \alpha)$ threshold with at least one of $\beta_\pm$ obeying $\beta_\pm \in [\frac{1}{2}, 1]$. We return back to Equation (35) with $\lambda \in \Upsilon \cap S^{\delta_1}$ and let $\psi_{\tilde{J}}$ be its solution in $L_2(\tilde{J})$. Since the function $V - \lambda$ vanishes only at the corresponding points $z^{(j,i)}$, which form a set of zero measures, we can rewrite this equation as

$$\psi_{\tilde{J}} + \frac{\varepsilon}{V - \lambda} \mathcal{P}_{\tilde{J}} \mathcal{L}_{a\star} \mathcal{A}(\varepsilon, \lambda) \psi_{\tilde{J}} = 0. \tag{40}$$

The second term in this equation can be represented as follows:

$$\frac{1}{V - \lambda} \mathcal{P}_{\tilde{J}} \mathcal{L}_{a\star} \mathcal{A}(\varepsilon, \lambda) \psi_{\tilde{J}} = \mathcal{B}_0(\varepsilon, \lambda) \psi_{\tilde{J}} + \mathcal{B}_1(\varepsilon, \lambda) \psi_{\tilde{J}}, \qquad \mathcal{B}_1(\varepsilon, \lambda) \psi_{\tilde{J}} := \frac{1}{V - \lambda} \mathcal{B}_2(\varepsilon, \lambda) \psi_{\tilde{J}},$$

where

$$\big(\mathcal{B}_0(\varepsilon, \lambda) \psi_{\tilde{J}}\big)(x) := \frac{1}{V(x) - \lambda} \sum_{(j,i) \in M_0} \chi^{(j,i)}(x) \int\limits_{\mathbb{R}} a(z^{(j,i)} - y) \big(\mathcal{A}(\varepsilon, \lambda) \psi_{\tilde{J}}\big)(y)\, dy, \qquad x \in J, \tag{41}$$

$$\big(\mathcal{B}_2(\varepsilon, \lambda) \psi_{\tilde{J}}\big)(x) := \sum_{(j,i) \in M_0} \int\limits_{\mathbb{R}} \big(a(x - y) - a(z^{(j,i)} - y) \chi^{(j,i)}(x)\big) \big(\mathcal{A}(\varepsilon, \lambda) \psi_{\tilde{J}}\big)(y)\, dy, \qquad x \in J.$$

Let us show that $\mathcal{B}_1(\varepsilon, \lambda)$ is a bounded operator in $L_2(\tilde{J})$ and, moreover, its norm is bounded uniformly in $\lambda \in \Upsilon \cap S^{\delta_1}$. Indeed, as $x \in \tilde{J} \setminus U$, the function $\big(\mathcal{B}_1(\varepsilon, \lambda) \psi_{\tilde{J}}\big)(x)$ reads as

$$\big(\mathcal{B}_1(\varepsilon, \lambda) \psi_{\tilde{J}}\big)(x) = \frac{1}{V(x) - \lambda} \int\limits_{\mathbb{R}} a(x - y) \big(\mathcal{A}(\varepsilon, \lambda) \psi_{\tilde{J}}\big)(y)\, dy.$$

Estimates (17), (34) and (39) then imply

$$\big\| \mathcal{B}_1(\varepsilon, \lambda) \psi_{\tilde{J}} \big\|_{L_2(\tilde{J} \setminus U)} \leqslant c_5^{-1} \|a\|_{L_1(\mathbb{R})} \|\mathcal{A} \psi_{\tilde{J}}\|_{L_2(\tilde{J})} \leqslant c_3 c_5^{-1} \|a\|_{L_1(\mathbb{R})} \|\psi_{\tilde{J}}\|_{L_2(\tilde{J})}. \tag{42}$$

As $x \in U^{(j,i)}$, $(j,i) \in M_0$, the function $\big(\mathcal{B}_1(\varepsilon, \lambda) \psi_{\tilde{J}}\big)(x)$ is given by the formula

$$\begin{aligned} \big(\mathcal{B}_2(\varepsilon, \lambda) \psi_{\tilde{J}}\big)(x) = & \frac{1}{V(x) - \lambda} \int\limits_{\mathbb{R}} \big(a(x - y) - a(z^{(j,i)} - y)\big) \big(\mathcal{A}(\varepsilon, \lambda) \psi_{\tilde{J}}\big)(y)\, dy \\ = & \frac{1}{V(x) - \lambda} \int\limits_{\mathbb{R}} dy\, \big(\mathcal{A}(\varepsilon, \lambda) \psi_{\tilde{J}}\big)(y) \int\limits_0^{x - z^{(j,i)}} a'\big(t + z^{(j,i)} - y\big)\, dt. \end{aligned} \tag{43}$$

Using, then, the definition of the regular points and $(\beta,\alpha)$ thresholds and estimate (25), by the Cauchy–Schwarz inequality and the uniform boundedness of the operator $\mathcal{A}$, we obtain:

$$
\begin{aligned}
\big|\big(\mathcal{B}_1(\varepsilon,\lambda)\psi_{\tilde{J}}\big)(x)\big|^2 \leqslant & \frac{C}{|x-z^{(j,i)}|^2}\left(\int\limits_{\mathbb{R}} dy\,\big|\big(\mathcal{A}(\varepsilon,\lambda)\psi_{\tilde{J}}\big)(y)\big| \int\limits_{-|x-z^{(j,i)}|}^{|x-z^{(j,i)}|} \big|a'\big(t+z^{(j,i)}-y\big)\big|\,dt\right)^2 \\
\leqslant & \frac{C}{|x-z^{(j,i)}|^2}\big\|\mathcal{A}(\varepsilon,\lambda)\psi_{\tilde{J}}\big\|^2_{L_2(\mathbb{R})} \int\limits_{\mathbb{R}} dy \left(\int\limits_{-|x-z^{(j,i)}|}^{|x-z^{(j,i)}|} \big|a'\big(t+z^{(j,i)}-y\big)\big|\,dt\right)^2 \\
\leqslant & \frac{C}{|x-z^{(j,i)}|}\big\|\mathcal{A}(\varepsilon,\lambda)\psi_{\tilde{J}}\big\|^2_{L_2(\mathbb{R})} \int\limits_{\mathbb{R}} dy \int\limits_{-|x-z^{(j,i)}|}^{|x-z^{(j,i)}|} \big|a'\big(t+z^{(j,i)}-y\big)\big|^2\,dt \\
= & \frac{C}{|x-z^{(j,i)}|}\big\|\mathcal{A}(\varepsilon,\lambda)\psi_{\tilde{J}}\big\|^2_{L_2(\mathbb{R})} \int\limits_{-|x-z^{(j,i)}|}^{|x-z^{(j,i)}|} dt \int\limits_{\mathbb{R}} \big|a'\big(t+z^{(j,i)}-y\big)\big|^2\,dy \\
\leqslant & C\|a'\|_{L_2(\mathbb{R})}\|\psi_{\tilde{J}}\|^2_{L_2(\tilde{J})},
\end{aligned}
\tag{44}
$$

where the symbol $C$ stands for various constants independent of $x$, $\lambda \in \Upsilon$ and $\psi_{\tilde{J}}$. Integrating the obtained estimate over $U^{(j,i)}$ and summing up the result over $(j,i) \in M_0$, we finally arrive at the inequality

$$\big\|\mathcal{B}_1(\varepsilon,\lambda)\psi_{\tilde{J}}\big\|_{L_2(U)} \leqslant c_6\|\psi_{\tilde{J}}\|_{L_2(\tilde{J})},$$

where $c_6$ is a constant independent of $\lambda \in \Upsilon \cap S^{\delta_1}$ and $\psi_{\tilde{J}}$. This inequality and (42) imply that the operator $\mathcal{B}_1$ is bounded in $L_2(\tilde{J})$ and its norm is bounded uniformly in $\lambda \in \Upsilon \cap S^{\delta_1}$.

Let us study the function $\mathcal{B}_0\psi_{\tilde{J}}$ defined in (41). If $\psi_{\tilde{J}}$ is a solution of Equation (40) in the space $L_2(\tilde{J})$, then the function $\mathcal{B}_1\psi_{\tilde{J}}$ is also an element of this space and, hence, $\mathcal{B}_0(\varepsilon,\lambda)\psi_{\tilde{J}}$ is necessarily in $L_2(\tilde{J})$. At the same time, as $x \in U^{(j,i)}$, this function reads

$$\big(\mathcal{B}_0(\varepsilon,\lambda)\psi_{\tilde{J}}\big)(x) = \frac{1}{V(x)-\lambda}\int\limits_{\mathbb{R}} a\big(z^{(j,i)}-y\big)\big(\mathcal{A}(\varepsilon,\lambda)\psi_{\tilde{J}}\big)(y)\,dy \tag{45}$$

and the integral is independent of $x$. The function $(V(x)-\lambda)^{-1}$ has a singularity at the point $z^{(j,i)}$ and since $z^{(j,i)}$ is either a regular point or a $(\beta,\alpha)$ threshold with at least one of $\beta_\pm$ not less than $\frac{1}{2}$, this function is not an element of $L_2(U^{(j,i)})$. Hence, the only possibility is that the integral in (45) necessarily vanishes. Then, $\mathcal{B}_0\psi_{\tilde{J}} = 0$ and Equation (40) becomes

$$\big(\mathcal{I}_{\tilde{J}} + \varepsilon\mathcal{B}_1(\varepsilon,\lambda)\big)\psi_{\tilde{J}} = 0,$$

where $\mathcal{I}_{\tilde{J}}$ is the identity mapping in $L_2(\tilde{J})$. Since the operator $\mathcal{B}_1$ is bounded uniformly in $\lambda$, for sufficiently small $\varepsilon$, the operator $\mathcal{I} + \varepsilon\mathcal{B}_1(\varepsilon,\lambda)$ is boundedly invertible and the above equation can have only the trivial solution. Therefore, Equations (35) and (40) also have only the trivial solution as $\lambda \in \Upsilon \cap S^{\delta_1}$.

### *4.2. Reduction to System of Linear Algebraic Equations*

We proceed to proving the existence of a small fixed positive $\delta \leqslant \delta_1$ such that the set $S^\delta \setminus \Upsilon$ contains no eigenvalues of the operator $\mathcal{L}^\varepsilon$. Namely, we are going to show that for $\lambda \in S^\delta \setminus \Upsilon$, Equation (35) possesses only the trivial solution. In this subsection, we make the first important step in studying this equation, i.e., we reduce it to a system of linear algebraic equations.

We choose a sufficiently small $\delta_3 \leqslant \min\left\{\frac{\delta_1}{2}, 1\right\}$ and introduce a finite covering of the curve $S$ by open balls $B_{\delta_3}(P_k)$ with centers at some points $P_k \in S$, $k = 1,\ldots,N$, where

$N \in \mathbb{N}$ is the number of the balls. By our assumptions, for each $k$, the point $P_k$ is the image of finitely many points in the segment $J_j$ and, hence, the piece of curve $B_{2\delta_3}(P_k) \cap \Upsilon$ is the image of finitely many segments in $\tilde{J}_j$, namely,

$$B_{2\delta_3}(P_k) \cap \Upsilon = \bigcup_{j=1}^{n} \bigcup_{i=1}^{N_k^{(j)}} \left\{ V(x) : x \in I_k^{(j,i)} \right\}, \qquad P_k = V(Y_k^{(j,i)}), \qquad Y_k^{(j,i)} \in I_k^{(j,i)},$$

where $I_k^{(j,i)} \subset \tilde{J}_j$ are some open intervals, $N_k^{(j)}$ are some given natural numbers, and $Y_k^{(j,i)}$ are some points. Owing to inequality (25) and the assumed smoothness of the function $V$, by choosing a small enough $\delta_3$, we can gain the following properties:

P1. The intervals $I_k^{(j,i)}$ are disjoint for different $i$, their lengths satisfy the estimate $|I_k^{(j,i)}| < 1$ and all possible thresholds in the interval $J_j$ are among the points $Y_k^{(j,i)}$;

P2. The end points of the intervals $I_k^{(j,i)}$ do not coincide with the $(\beta, \alpha)$ thresholds located in the segment $J_j$, each of the intervals $I_k^{(j,i)}$ contains at most one $(\beta, \alpha)$ threshold and the distance from this threshold to other intervals $I_k^{(j,i)}$ is at least $c_7\delta_3$, where $c_7 > 0$ is a constant independent of $\delta_3$, $k$, $j$, $i$; the image of each end point of each interval $I_k^{(j,i)}$ is located on $\partial B_{2\delta_3}(P_k)$;

P3. If some interval $I_k^{(j,i)}$ contains a $(\beta, \alpha)$ threshold, then the corresponding identity (5) holds true for the entire interval.

In what follows, given a curve and a point in the complex plane, we say that this point is projected onto this curve orthogonally to some non-zero complex number if this projection is made along the straight line orthogonal to the vector connecting the origin and this non-zero complex number. We suppose an extra two properties of $\delta_3$.

P4. If a given interval $I_k^{(j,i)}$ contains only regular points, then for all $\lambda \in B_{\delta_3}(Y_k^{(j,i)}) \setminus \Upsilon$, there exists a unique projection of $\frac{\lambda}{V'(Y_k^{(j,i)})}$ onto the curve $\Gamma_k^{(j,i)} := \{V(x) : x \in I_k^{(j,i)}\}$ orthogonally to the number $V'(Y_k^{(j,i)})$ and the inequality holds:

$$\operatorname{Re} \frac{V'(x)}{V'(Y_k^{(j,i)})} \geqslant \frac{1}{2} \quad \text{for all } x \in I_k^{(j,i)}; \tag{46}$$

P5. If a given interval $I_k^{(j,i)}$ contains a $(\beta, \alpha)$ threshold at $Y_k^{(j,i)} \in I_k^{(j,i)}$ with corresponding $\alpha_\pm = \alpha_{\pm,k}^{(j,i)}$, then for all $\lambda \in B_{\delta_3}(Y_k^{(j,i)}) \setminus \Upsilon$ such that $\operatorname{Re} \frac{\lambda - P_k}{\alpha_{\pm,k}^{(j,i)}} > 0$ there exists a unique projection of $\frac{\lambda}{\alpha_{k,\pm}^{(j,i)}}$ onto the curve

$$\Gamma_{k,\pm}^{(j,i)} := \{V(x) : x \in I_{k,\pm}^{(j,i)}\}, \quad \text{where} \quad I_{k,\pm}^{(j,i)} := I_k^{(j,i)} \cap \{x : \pm(x - Y_k^{(j,i)}) > 0\}$$

orthogonally to the number $\alpha_{k,\pm}^{(j,i)}$; the functions $v_\pm = v_{k,\pm}^{(j,i)}$ from (5) corresponding to the $(\beta, \alpha)$ threshold at $Y_k^{(j,i)}$ satisfy the estimates

$$\begin{aligned} & v_{k,\pm}^{(j,i)} \geqslant \frac{1}{2}, \quad |v_{k,\pm}^{(j,i)}(x)| \leqslant 2, \quad |\operatorname{Im} v_{k,\pm}^{(j,i)}(x)| \leqslant \tan \frac{\pi\beta_0}{2} \operatorname{Re} v_{k,\pm}^{(j,i)}(x) \quad \text{as} \quad x \in I_{k,+}^{(j,i)}, \\ & |I_{k,\pm}^{(j,i)}| \, \|(v_{k,\pm}^{(j,i)})'\|_{C(\overline{I_{k,+}^{(j,i)}})} \leqslant \frac{1}{4^{1+\frac{1}{\beta_0}}}, \end{aligned} \tag{47}$$

where

$$\beta_0 := \frac{1}{2} \min_{k,j,i} \{\beta_{k,+}^{(j,i)}; \beta_{k,-}^{(j,i)}\}. \tag{48}$$

We observe that the definition of intervals $I_k^{(j,i)}$ implies immediately that

$$|V(x)-\lambda|\geqslant\delta_3 \quad \text{as} \quad \lambda\in B_{\delta_3}(P_k),\quad x\in\tilde{J}\setminus I_k^{(j)},\quad j=1,\ldots,n,\qquad I_k^{(j)}:=\bigcup_{i=1}^{N_k^{(j)}} I_k^{(j,i)}. \tag{49}$$

Property P4 can be equivalently formulated as follows: there exists a unique solution to the equation

$$\operatorname{Re}\frac{V(Z)-P_k}{V'(Y_k^{(j,i)})}=\operatorname{Re}\frac{\lambda-P_k}{V'(Y_k^{(j,i)})} \tag{50}$$

for all $\lambda\in B_{\delta_3}(Y_k^{(j,i)})\setminus\Upsilon$. In view of the definition of a regular point, this equation is uniquely solvable, since for $Z$ close to $Y_k^{(j,i)}$ the quotient on the left hand side behaves as

$$\frac{V(Z)-P_k}{V'(Y_k^{(j,i)})}=Z-Y_k^{(j,i)}+O\big((Z-Y_k^{(j,i)})^2\big).$$

The latter identity also ensures the possibility of satisfying (46). We denote the unique solution of (50) by $Z_k^{(j,i)}=Z_k^{(j,i)}(\lambda)$.

Property P5 can be also equivalently formulated as follows: there exists a unique solution to the equation

$$\operatorname{Re}\frac{V(Z_\pm)-P_k}{\alpha_{\pm,k}^{(j,i)}}=\operatorname{Re}\frac{\lambda-P_k}{\alpha_{\pm,k}^{(j,i)}} \tag{51}$$

for all $\lambda\in B_{\delta_3}(Y_k^{(j,i)})\setminus\Upsilon$ obeying an additional condition $\operatorname{Re}\frac{\lambda-P_k}{\alpha_{\pm,k}^{(j,i)}}>0$. These equations are again locally uniquely solvable owing to the definition of $(\beta,\alpha)$ threshold, which also ensures (47). We denote the solutions of (51) by $Z_{\pm,k}^{(j,i)}=Z_{\pm,k}^{(j,i)}(\lambda)$. We also let

$$Z_{\pm,k}^{(j,i)}(\lambda):=Y_k^{j,i} \qquad \text{as} \qquad \operatorname{Re}\frac{\lambda-P_k}{\alpha_{\pm,k}^{(j,i)}}\leqslant 0. \tag{52}$$

In what follows, we consider Equation (35) for $\lambda\in E_{k,\delta_3}$, where

$$E_{k,\delta_3}:=B_{\delta_3}(P_k)\setminus\Upsilon. \tag{53}$$

We rewrite this equation in form (40) and then we represent the second term in the latter equation as

$$\psi_{\tilde{J}}+\varepsilon\mathcal{B}_{3,k}(\varepsilon,\lambda)\psi_{\tilde{J}}+\varepsilon\mathcal{B}_{4,k}(\varepsilon,\lambda)\psi_{\tilde{J}}=0, \tag{54}$$

$$\mathcal{B}_{3,k}(\varepsilon,\lambda):=\sum_{j=1}^{n}\sum_{i=1}^{N_k^{(j)}}\frac{\xi_k^{(j,i)}}{V-\lambda}\mathcal{P}_{\tilde{J}}\mathcal{L}_{a\star}\mathcal{A}(\varepsilon,\lambda),$$

$$\mathcal{B}_{4,k}(\varepsilon,\lambda):=\sum_{j=1}^{n}\sum_{i=1}^{N_k^{(j)}}\frac{1-\xi_k^{(j,i)}}{V-\lambda}\mathcal{P}_{\tilde{J}}\mathcal{L}_{a\star}\mathcal{A}(\varepsilon,\lambda),$$

where $\xi_k^{(j,i)}$ are the characteristic functions of the intervals $I_k^{(j,i)}$. It follows immediately from the definitions of the operators $\mathcal{B}_{4,k}$ and the function $\xi_k^{(j,i)}$ and estimates (17), (34) and (49) that

$$\|\mathcal{B}_{4,k}\|_{L_2(\tilde{J})\to L_2(\tilde{J})}\leqslant\frac{c_8}{\delta_3}, \tag{55}$$

where $c_8$ is a constant independent of $\lambda$, $k$, $\delta_3$.

We proceed to studying the operators $\mathcal{B}_{3,k}(\varepsilon,\lambda)$. Let $M_1$ be the set of all superscripts $(j,i)$ such that the intervals $I_k^{(j,i)}$, $(j,i)\in M_1$, contain only regular points, while $M_2$ is the set of all superscripts $(j,i)$ such that the intervals $I_k^{(j,i)}$, $(j,i) \in M_2$, possesses a $(\beta,\alpha)$ threshold at $Y_k^{(j,i)} \in I_k^{(j,i)}$. Bearing in mind Properties P4 and P5, we represent the operator $\mathcal{B}_{3,k}$ as a sum

$$\mathcal{B}_{3,k}(\varepsilon,\lambda) = \mathcal{B}_{5,k}(\varepsilon,\lambda) + \mathcal{B}_{6,k}(\varepsilon,\lambda)\mathcal{A}(\varepsilon,\lambda), \tag{56}$$

where $\mathcal{B}_{5,k}(\varepsilon,\lambda)$ and $\mathcal{B}_{6,k}(\varepsilon,\lambda)$ are operators in $L_2(\tilde{J})$ defined by the formulas

$$\begin{aligned}
\mathcal{B}_{5,k}(\varepsilon,\lambda) :=& \sum_{(j,i)\in M_1} \phi_k^{(j,i)} \ell(Z_k^{(j,i)}(\lambda),\varepsilon,\lambda) + \sum_{(j,i)\in M_2} \phi_{k,+}^{(j,i)} \ell(Z_{k,+}^{(j,i)}(\lambda),\varepsilon,\lambda) \\
&+ \sum_{(j,i)\in M_2} \phi_{k,-}^{(j,i)} \ell(Z_{k,-}^{(j,i)}(\lambda),\varepsilon,\lambda), \\
\mathcal{B}_{6,k}(\varepsilon,\lambda) :=& \sum_{(j,i)\in M_1} \mathcal{B}_{6,k}^{(j,i)}(\varepsilon,\lambda) + \sum_{(j,i)\in M_2} \mathcal{B}_{6,k,+}^{(j,i)}(\varepsilon,\lambda) + \sum_{(j,i)\in M_2} \mathcal{B}_{6,k,-}^{(j,i)}(\varepsilon,\lambda),
\end{aligned} \tag{57}$$

where

$$\begin{aligned}
&\phi_k^{(j,i)} := \frac{\xi_k^{(j,i)}}{V-\lambda}, \qquad \phi_{k,\pm}^{(j,i)} := \frac{\xi_{k,\pm}^{(j,i)}}{V-\lambda}, \\
&\big(\mathcal{B}_{6,k}^{(j,i)}(\varepsilon,\lambda)\psi\big)(x) := \xi_k^{(j,i)}(x) \int\limits_{\mathbb{R}} \frac{a(x-y) - a\big(Z_k^{(j,i)}(\lambda) - y\big)}{V(x)-\lambda} \psi(y)\,dy, \\
&\big(\mathcal{B}_{6,k,\pm}^{(j,i)}(\varepsilon,\lambda)\psi\big)(x) := \xi_k^{(j,i)}(x) \int\limits_{\mathbb{R}} \frac{a(x-y) - a\big(Z_{k,\pm}^{(j,i)}(\lambda) - y\big)}{V(x)-\lambda} \psi(y)\,dy,
\end{aligned} \tag{58}$$

$\xi_{k,\pm}^{(j,i)}$ are the characteristic functions of the intervals $I_{k,\pm}^{(j,i)}$, and $\ell(z,\varepsilon,\lambda)$, $z \in \mathbb{R}$, is a bounded linear functional on $L_2(\tilde{J})$ defined as

$$\ell(z,\varepsilon,\lambda)\psi_{\tilde{J}} := \int\limits_{\mathbb{R}} a(z-y)\big(\mathcal{A}(\varepsilon,\lambda)\psi_{\tilde{J}}\big)(y)\,dy.$$

In order to study the properties of the operators $\mathcal{B}_{5,k}(\varepsilon,\lambda)$ and $\mathcal{B}_{6,k}(\varepsilon,\lambda)$, we shall need the following lemma.

**Lemma 1.** *There exists $\delta_4 > 0$ independent of $k$ such that for all $\lambda$ßn$E_{k,\delta_3}$, all $k$ and all $\delta_3 \leqslant \delta_4$ the estimates hold:*

$$\begin{aligned}
|V(x)-\lambda| \geqslant c_9 |x - Z_k^{(j,i)}(\lambda)| \qquad &\text{as} \qquad x \in I_k^{(j,i)}, \quad (j,i) \in M_1, \\
|V(x)-\lambda| \geqslant c_9 |x - Z_{k,\pm}^{(j,i)}(\lambda)| \qquad &\text{as} \qquad x \in I_{k,\pm}^{(j,i)}, \quad (j,i) \in M_2,
\end{aligned} \tag{59}$$

*where $c_9$ is a positive constant independent of $\delta_3$, $x$, $\lambda$, $k$, $j$ and $i$.*

**Proof.** We first consider the case $(j,i) \in M_1$. By Equation (50), estimate (25) and the Lagrange rule, we have:

$$\begin{aligned}
|V(x)-\lambda| =& \big|V'(Y_k^{(j,i)})\big| \left| \frac{V(x)-P_k}{V'(Y_k^{(j,i)})} - \frac{\lambda - P_k}{V'(Y_k^{(j,i)})} \right| \\
\geqslant& \big|V'(Y_k^{(j,i)})\big| \left| \mathrm{Re}\, \frac{V(x)-P_k}{V'(Y_k^{(j,i)})} - \mathrm{Re}\, \frac{\lambda - P_k}{V'(Y_k^{(j,i)})} \right| \\
=& \big|V'(Y_k^{(j,i)})\big| \left| \mathrm{Re}\, \frac{V(x)-P_k}{V'(Y_k^{(j,i)})} - \mathrm{Re}\, \frac{V(Z_k^{(j,i)}) - P_k}{V'(Y_k^{(j,i)})} \right| \geqslant \frac{c_0}{2} |x - Z_k^{(j,i)}|.
\end{aligned} \tag{60}$$

We proceed to the case $(j,i)\in M_2$. We shall prove the second inequality in (59) only for $x\in I_{k,+}^{(j,i)}$; the case of the interval $I_{k,-}^{(j,i)}$ can be treated in the same way. In the considered case, the interval $I_k^{(j,i)}$ contains a $(\beta,\alpha)$ threshold at some internal point $Y_k^{(j,i)}$. We first suppose that $\operatorname{Re}\lambda\beta_+^{-1}\leqslant 0$. In view of (5) and (52), we have:

$$|V(x)-\lambda|\geqslant|\alpha_{k,+}^{(j,i)}|\left|\operatorname{Re}\frac{V(x)-\lambda}{\alpha_{k,+}^{(j,i)}}\right|\geqslant C|x-Y_k^{(j,i)}|^{\beta_{\pm,k}^{(j,i)}}\geqslant C|x-Y_k^{(j,i)}|, \tag{61}$$

where $C$ is a constant independent of $k$, $j$, $i$ and $\lambda$. This proves the second inequality in (59) as $\operatorname{Re}\lambda\beta_+^{-1}\leqslant 0$.

Suppose that $\operatorname{Re}\lambda\beta_+^{-1}>0$. Then, we argue similarly to (60):

$$\begin{aligned}
|V(x)-\lambda| =&|\alpha_{+,k}^{(j,i)}|\left|\frac{V(x)-P_k}{\alpha_{+,k}^{(j,i)}}-\frac{\lambda-P_k}{\alpha_{+,k}^{(j,i)}}\right|\\
\geqslant&|\alpha_{+,k}^{(j,i)}|\left|\operatorname{Re}\frac{V(x)-P_k}{\alpha_{+,k}^{(j,i)}}-\operatorname{Re}\frac{V(Z_{+,k}^{(j,i)})-P_k}{\alpha_{+,k}^{(j,i)}}\right|\\
\geqslant&|\alpha_{+,k}^{(j,i)}|\left|\left((x-Y_k^{(j,i)})^{\beta_{k,+}^{(j,i)}}\operatorname{Re}v_{k,+}^{(j,i)}(x)\right)'\Big|_{x=\zeta}\right||x-Z_{k,+}^{(j,i)}|\\
=&|\alpha_{+,k}^{(j,i)}|\left|\beta_{k,+}^{(j,i)}\operatorname{Re}v_{k,+}^{(j,i)}(\zeta)+(\zeta-Y_k^{(j,i)})\operatorname{Re}(v_{k,+}^{(j,i)})'(\zeta)\right|\frac{|x-Z_{k,+}^{(j,i)}|}{|\zeta-Y_k^{(j,i)}|^{1-\beta_{k,+}^{(j,i)}}},
\end{aligned}$$

where $\zeta$ is some point between $x$ and $Z_{k,+}^{(j,i)}$. It follows from the first and fourth inequalities in (47) and (48) that

$$\left|\beta_{k,+}^{(j,i)}\operatorname{Re}v_{k,+}^{(j,i)}(\zeta)+(\zeta-Y_k^{(j,i)})\operatorname{Re}(v_{k,+}^{(j,i)})'(\zeta)\right|\geqslant\frac{\beta_{k,+}^{(j,i)}}{2}-|I_{k,+}^{(j,i)}|\|(v_{k,+}^{(j,i)})'\|_{C(\overline{I_{k,+}^{(j,i)}})}\geqslant\beta_0-\frac{1}{4^{1+\frac{1}{\beta_0}}}>\frac{\beta_0}{2}.$$

This inequality and the inequality $|I_{k,+}^{(j,i)}|<|I_k^{(j,i)}|<1$, see Property P1, allows us to continue the above estimating:

$$|V(x)-\lambda|\geqslant\frac{\beta_0|\alpha_{+,k}^{(j,i)}|}{2}\frac{|x-Z_{k,+}^{(j,i)}|}{|\zeta-Y_k^{(j,i)}|^{1-\beta_{k,+}^{(j,i)}}}\geqslant\frac{\beta_0|\alpha_{+,k}^{(j,i)}|}{2|I_{k,+}^{(j,i)}|^{1-\beta_{k,+}^{(j,i)}}}|x-Z_{k,+}^{(j,i)}|\geqslant\frac{\beta_0|\alpha_{+,k}^{(j,i)}|}{2}|x-Z_{k,+}^{(j,i)}|.$$

The proof is complete. □

Using this lemma and arguing as in (43) and (44), we easily see that the operators $\mathcal{B}_{6,k}(\varepsilon,\lambda)$ are bounded uniformly in $\varepsilon$ and $\lambda\in E_{k,\delta_3}$ once $\delta_3\leqslant\delta_4$, namely,

$$\|\mathcal{B}_{6,k}(\varepsilon,\lambda)\|_{L_2(\tilde{J})\to L_2(\tilde{J})}\leqslant c_{10}, \tag{62}$$

where $c_{10}$ is a constant independent of $\varepsilon$ and $\lambda$. This inequality and (55), (34) yield that the operator

$$\mathcal{G}(\varepsilon,\lambda):=\left(\mathcal{I}_J+\varepsilon\mathcal{B}_{4,k}+\varepsilon\mathcal{B}_{6,k}\mathcal{A}(\varepsilon,\lambda)\right)^{-1}$$

is well defined and bounded in $L_2(\tilde{J})$ provided

$$\varepsilon\leqslant\frac{\delta_3}{2(c_8+c_{10}c_3\delta_3)},\qquad\lambda\in E_{k,\delta_3},\qquad 0<\delta_3\leqslant\delta_4$$

and for such values of $\varepsilon$, $\delta_3$ and $\lambda$, it satisfies the estimate

$$\|\mathcal{G}(\varepsilon,\lambda)\|_{L_2(\tilde{J})\to L_2(\tilde{J})}\leqslant 2.$$

We substitute identity (56) into Equation (54) and then apply the operator $\mathcal{G}(\varepsilon,\lambda)$ to the resulting relation and use the definition of the operator $\mathcal{B}_{5,k}$. This implies one more equation:

$$\begin{aligned}\psi_{\tilde{J}} &+ \varepsilon \sum_{(j,i)\in M_1} \Phi_k^{(j,i)}(\varepsilon,\lambda)\ell\big(Z_k^{(j,i)}(\lambda),\varepsilon,\lambda\big)\psi_{\tilde{J}} + \varepsilon \sum_{(j,i)\in M_2} \Phi_{k,+}^{(j,i)}(\varepsilon,\lambda)\ell\big(Z_{k,+}^{(j,i)}(\lambda),\varepsilon,\lambda\big)\psi_{\tilde{J}} \\ &+ \varepsilon \sum_{(j,i)\in M_2} \Phi_{k,-}^{(j,i)}(\varepsilon,\lambda)\ell\big(Z_{k,-}^{(j,i)}(\lambda),\varepsilon,\lambda\big)\psi_{\tilde{J}} = 0,\end{aligned} \tag{63}$$

where

$$\Phi_k^{(j,i)}(\varepsilon,\lambda) := \mathcal{G}(\varepsilon,\lambda)\phi_k^{(j,i)}, \qquad \Phi_{k,\pm}^{(j,i)}(\varepsilon,\lambda) := \mathcal{G}(\varepsilon,\lambda)\phi_{k,\pm}^{(j,i)}. \tag{64}$$

We arbitrarily choose $p \in \{1,\ldots,n\}$ and $i \in \{1,\ldots,N_k^{(p)}\}$ and if $(p,q) \in M_1$, we apply the functional $\ell\big(Z_k^{(p,q)}(\lambda),\varepsilon,\lambda\big)$ to Equation (63), while for $(p,q) \in M_2$ we apply the functionals $\ell\big(Z_{k,\pm}^{(p,q)}(\lambda),\varepsilon,\lambda\big)$ to the same equation. This gives the following identities:

$$\begin{aligned}
\ell\big(Z_k^{(p,q)}(\lambda),\varepsilon,\lambda\big)\psi_{\tilde{J}} &+ \varepsilon \sum_{(j,i)\in M_1} A_k^{(p,q,j,i)}(\varepsilon,\lambda)\ell\big(Z_k^{(j,i)}(\lambda),\varepsilon,\lambda\big)\psi_{\tilde{J}} \\
&+ \varepsilon \sum_{(j,i)\in M_2} A_{k,+}^{(p,q,j,i)}(\varepsilon,\lambda)\ell\big(Z_{k,+}^{(j,i)}(\lambda),\varepsilon,\lambda\big)\psi_{\tilde{J}} \\
&+ \varepsilon \sum_{(j,i)\in M_2} A_{k,-}^{(p,q,j,i)}(\varepsilon,\lambda)\ell\big(Z_{k,-}^{(j,i)}(\lambda),\varepsilon,\lambda\big)\psi_{\tilde{J}} = 0, \qquad (p,q)\in M_1, \\
\ell\big(Z_{k,+}^{(p,q)}(\lambda),\varepsilon,\lambda\big)\psi_{\tilde{J}} &+ \varepsilon \sum_{(j,i)\in M_1} A_{k,+}^{(p,q,j,i)}(\varepsilon,\lambda)\ell\big(Z_k^{(j,i)}(\lambda),\varepsilon,\lambda\big)\psi_{\tilde{J}} \\
&+ \varepsilon \sum_{(j,i)\in M_2} A_{k,+,+}^{(p,q,j,i)}(\varepsilon,\lambda)\ell\big(Z_{k,+}^{(j,i)}(\lambda),\varepsilon,\lambda\big)\psi_{\tilde{J}} \\
&+ \varepsilon \sum_{(j,i)\in M_2} A_{k,+,-}^{(p,q,j,i)}(\varepsilon,\lambda)\ell\big(Z_{k,-}^{(j,i)}(\lambda),\varepsilon,\lambda\big)\psi_{\tilde{J}} = 0, \qquad (p,q)\in M_2, \\
\ell\big(Z_{k,-}^{(p,q)}(\lambda),\varepsilon,\lambda\big)\psi_{\tilde{J}} &+ \varepsilon \sum_{(j,i)\in M_1} A_{k,-}^{(p,q,j,i)}(\varepsilon,\lambda)\ell\big(Z_k^{(j,i)}(\lambda),\varepsilon,\lambda\big)\psi_{\tilde{J}} \\
&+ \varepsilon \sum_{(j,i)\in M_2} A_{k,-,+}^{(p,q,j,i)}(\varepsilon,\lambda)\ell\big(Z_{k,+}^{(j,i)}(\lambda),\varepsilon,\lambda\big)\psi_{\tilde{J}} \\
&+ \varepsilon \sum_{(j,i)\in M_2} A_{k,-,-}^{(p,q,j,i)}(\varepsilon,\lambda)\ell\big(Z_{k,-}^{(j,i)}(\lambda),\varepsilon,\lambda\big)\psi_{\tilde{J}} = 0, \qquad (p,q)\in M_2,
\end{aligned} \tag{65}$$

where

$$A_k^{(p,q,j,i)}(\varepsilon,\lambda) := \ell\big(Z_k^{(p,q)}(\lambda),\varepsilon,\lambda\big)\Phi_k^{(j,i)}(\varepsilon,\lambda), \qquad A_{k,\pm}^{(p,q,j,i)}(\varepsilon,\lambda) := \ell\big(Z_k^{(p,q)}(\lambda),\varepsilon,\lambda\big)\Phi_{k,\pm}^{(j,i)}(\varepsilon,\lambda) \tag{66}$$

as $(p,q) \in M_1$ and

$$A_{k,\pm}^{(p,q,j,i)}(\varepsilon,\lambda) := \ell\big(Z_{k,\pm}^{(p,q)}(\lambda),\varepsilon,\lambda\big)\Phi_k^{(j,i)}(\varepsilon,\lambda), \qquad A_{k,\flat,\natural}^{(p,q,j,i)}(\varepsilon,\lambda) := \ell\big(Z_{k,\flat}^{(p,q)}(\lambda),\varepsilon,\lambda\big)\Phi_{k,\natural}^{(j,i)}(\varepsilon,\lambda) \tag{67}$$

as $(p,q) \in M_2$, where the symbols $\flat$ and $\natural$ are to be independently replaced by '$+$' or '$-$'. Identity (65) is a system of linear equations for the numbers $\ell\big(Z_k^{(j,i)}(\lambda),\varepsilon,\lambda\big)$ and $\ell\big(Z_{k,\pm}^{(j,i)}(\lambda),\varepsilon,\lambda\big)$. Once we find these numbers, we can recover the function $\psi_{\tilde{J}}$ for Equation (63). If system (65) has only the trivial solution, this immediately implies that $\psi_{\tilde{J}}$ vanishes identically, and by Formula (31), Equation (27) can have only the trivial solution; hence, the set $E_{k,\delta_3}$ contains no eigenvalues of the operator $\mathcal{L}^{\varepsilon}$. In order to prove that system (65) has only the trivial solution, it is sufficient to show that all functions $A_{k,\pm}^{(p,q,j,i)}(\varepsilon,\lambda)$ and $A_{k,\flat,\natural}^{(p,q,j,i)}(\varepsilon,\lambda)$ are bounded uniformly in $\varepsilon$ and $\lambda$. The proof of this fact is our next important step.

*4.3. Trivial Solution and Absence of the Spectrum*

In this subsection, we prove the uniform boundedness of the functions $A_{k,\pm}^{(p,q,j,i)}(\varepsilon,\lambda)$ and $A_{k,\flat,\natural}^{(p,q,j,i)}(\varepsilon,\lambda)$ and this will allow us to complete the proof of Theorem 2. We first rewrite Formula (64) for the functions $\Phi_k^{(j,i)}(\varepsilon,\lambda)$ and $\Phi_{k,\pm}^{(j,i)}(\varepsilon,\lambda)$ as

$$\begin{aligned}\Phi_k^{(j,i)}(\varepsilon,\lambda) &= \phi_k^{(j,i)} - \varepsilon\mathcal{G}(\varepsilon,\lambda)\big(\mathcal{B}_{4,k}(\varepsilon,\lambda)+\mathcal{B}_{6,k}(\varepsilon,\lambda)\big)\phi_k^{(j,i)},\\ \Phi_{k,\pm}^{(j,i)}(\varepsilon,\lambda) &= \phi_{k,\pm}^{(j,i)} - \varepsilon\mathcal{G}(\varepsilon,\lambda)\big(\mathcal{B}_{4,k}(\varepsilon,\lambda)+\mathcal{B}_{6,k}(\varepsilon,\lambda)\big)\phi_{k,\pm}^{(j,i)}.\end{aligned} \tag{68}$$

The prove of the uniform boundedness of $A_k^{(p,q,j,i)}(\varepsilon,\lambda)$, $A_{k,\pm}^{(p,q,j,i)}(\varepsilon,\lambda)$, $A_{k,\pm}^{(p,q,j,i)}(\varepsilon,\lambda)$, $A_{k,\flat,\natural}^{(p,q,j,i)}(\varepsilon,\lambda)$ is based on a series of the following lemmas.

**Lemma 2.** *There exists $\delta_5 > 0$ such that as $\delta_3 \leqslant \delta_5$, for all $\lambda \in E_{k,\delta_3}$ and $(j,i) \in M_1$ the estimates hold*

$$\left|\int\limits_{\mathbb{R}} \phi_k^{(j,i)}(x)\,dx\right| \leqslant \frac{c_{11}}{\delta_3},$$

*where $c_{11}$ is a constant independent of $k$, $j$, $i$, $\delta_3$ and $\lambda$.*

**Proof.** We begin with representing the considered integral as

$$\begin{aligned}\int\limits_{\mathbb{R}} \phi_k^{(j,i)}(x)\,dx &= \int\limits_{I_k^{(j,i)}} \frac{dx}{V(x)-\lambda}\\ &= \frac{1}{V'(Z_k^{(j,i)})}\int\limits_{I_k^{(j,i)}} \frac{V'(x)}{V(x)-\lambda}\,dx + \frac{1}{V'(Z_k^{(j,i)})}\int\limits_{I_k^{(j,i)}} \frac{V'(Z_k^{(j,i)})-V'(x)}{V(x)-\lambda}\,dx.\end{aligned} \tag{69}$$

The first integral in the right hand side of the above representation can be immediately rewritten as

$$\int\limits_{I_k^{(j,i)}} \frac{V'(x)}{V(x)-\lambda}\,dx = \int\limits_{\Gamma_k^{(j,i)}} \frac{dt}{t-\lambda}.$$

The above integral over the curve $\Gamma_k^{(j,i)}$ is holomorphic in $\lambda \in B_{\delta_3}(P_k)\setminus\Upsilon$. As $\lambda$ is such that $\operatorname{dist}(\lambda,\Upsilon) \geqslant \frac{\delta_3}{2}$, we have an obvious estimate

$$\left|\int\limits_{\Gamma_k^{(j,i)}} \frac{dt}{t-\lambda}\right| \leqslant \frac{C}{\delta_3}, \tag{70}$$

where $C$ is a constant independent of $\lambda$, $k$, $j$, $i$ and $\delta_3$. We also easily find that

$$\frac{d}{d\lambda}\int\limits_{\Gamma_k^{(j,i)}} \frac{dt}{t-\lambda} = \int\limits_{\Gamma_k^{(j,i)}} \frac{dt}{(t-\lambda)^2} = \frac{1}{\partial_-\Gamma_k^{(j,i)}-\lambda} - \frac{1}{\partial_+\Gamma_k^{(j,i)}-\lambda}, \tag{71}$$

where $\partial_\pm\Gamma_k^{(j,i)}$ are the end-points of the curve $\Gamma_k^{(j,i)}$. Definition (53) of the set $E_{k,\delta_3}$ ensures that

$$\frac{1}{|\partial_\pm\Gamma_k^{(j,i)}-\lambda|} \geqslant \frac{1}{\delta_3}.$$

Having this estimate and (70) in mind and integrating (71) with respect to $\lambda$, in view of (25), we immediately find

$$\left|\frac{1}{V'(Z_k^{(j,i)})}\int\limits_{I_k^{(j,i)}}\frac{V'(x)}{V(x)-\lambda}\,dx\right|\leqslant\frac{C}{\delta_3},\tag{72}$$

where $C$ is a constant independent of $\lambda$, $k$, $j$, $i$ and $\delta_3$.

In order to estimate the second integral in the right hand side of (69), we employ estimate (59) and the Lagrange rule:

$$\left|\frac{1}{V'(Z_k^{(j,i)})}\int\limits_{I_k^{(j,i)}}\frac{V'(Z_k^{(j,i)})-V'(x)}{V(x)-\lambda}\,dx\right|\leqslant\frac{C}{|V'(Z_k^{(j,i)})|}\sup_{t\in \overline{I_k^{(j,i)}}}|V''(t)|,\tag{73}$$

where $C$ is a constant independent of $\lambda$, $k$, $j$, $i$ and $\delta_3$. According to the definition of the regular points, the function $V$ is twice continuously differentiable on $J_j$ except for $(\beta,\alpha)$ thresholds, which are denoted, we recall, by $x^{(j,i)}$. In the vicinity of the latter points, the first and the second derivatives of the function $V$ have singularities of orders $O(|x-x^{(j,i)}|^{\beta_\pm^{(j,i)}-1})$ and $O(|x-x^{(j,i)}|^{\beta_\pm^{(j,i)}-2})$. According to Property P2, the minimal distance from the interval $I_k^{(j,i)}$ to the nearest $(\beta,\alpha)$ threshold is at least $c_7\delta_3$, and since the total number of the thresholds is finite, we conclude on the existence of $\delta_5>0$ such that for $\delta_3\leqslant\delta_5$ the estimate

$$\frac{\sup\limits_{t\in \overline{I_k^{(j,i)}}}|V''(t)|}{|V'(Z_k^{(j,i)})|}\leqslant\frac{C}{\delta_3}$$

holds true, where $C$ is a constant independent of $\delta_3$, $k$, $j$, $i$. Substituting this estimate into (73), we obtain:

$$\left|\frac{1}{V'(Z_k^{(j,i)})}\int\limits_{I_k^{(j,i)}}\frac{V'(Z_k^{(j,i)})-V'(x)}{V(x)-\lambda}\,dx\right|\leqslant\frac{C}{\delta_3},$$

where $C$ is a constant independent of $\delta_3$, $k$, $j$, $i$. This estimate and (72) yield the desired estimate from the statement of the lemma. The proof is complete. □

**Lemma 3.** *For all $\lambda\in E_{k,\delta_3}$ and $(j,i)\in M_2$ the estimates hold*

$$\left|\int\limits_{\mathbb{R}}\phi_{k,\pm}^{(j,i)}(x)\,dx\right|\leqslant c_{12},\tag{74}$$

*where $c_{12}$ is a constant independent of $k$, $j$, $i$, and $\lambda$ but depending on $\delta_3$.*

**Proof.** We provide the proof only for the integral with $\phi_{k,+}^{(j,i)}$; the other case can be treated in the same way. We first suppose that

$$\operatorname{Re}\frac{\lambda-P_k}{\alpha_{+,k}^{(j,i)}}\leqslant 0.$$

Then, by (61) and the assumed inequality $\beta_{k,+}^{(j,i)}<1$ we immediately obtain:

$$\left|\int\limits_{\mathbb{R}}\frac{\xi_{k,\pm}^{(j,i)}(x)}{V(x)-\lambda}\,dx\right|\leqslant C\int\limits_{\Gamma_{k,+}^{(j,i)}}\frac{dx}{(x-Y_k^{(j,i)})^{\beta_{k,+}^{j,i}}}\leqslant C,$$

where by $C$ we denote some constants independent of $\lambda$, $k$, $j$, $i$ and $\delta_3$.

Suppose now that

$$\operatorname{Re}\eta > 0, \qquad \text{where} \qquad \eta := \frac{\lambda - P_k}{\alpha_{+,k}^{(j,i)}}. \tag{75}$$

Owing to the third inequality in (47) and (48) the function

$$w(x) := (x - Y_k^{(j,i)})\big(v_{k,+}^{(j,i)}(x)\big)^{\frac{1}{\beta_{k,+}^{(j,i)}}}$$

is well defined and

$$w^{\beta_{k,+}^{(j,i)}}(x) = \big(x - Y_k^{(j,i)}\big)^{\beta_{k,+}^{(j,i)}} v_{k,+}^{(j,i)}(x) = \frac{V(x) - P_k}{\alpha_{k,+}^{(j,i)}}. \tag{76}$$

The assumed smoothness of $v_{k,+}^{(j,i)}$, see (5) and (6) yields that

$$w \in C^2(\overline{I_{k,+}^{(j,i)}}), \qquad \|w\|_{C^2(\overline{I_{k,+}^{(j,i)}})} \leqslant C, \tag{77}$$

where $C$ is a constant independent of $k$, $j$, $i$. The first, second and fourth inequalities in (47) and identity (5) imply that for $x \in \overline{I_{k,+}^{(j,i)}}$, the estimates hold:

$$\begin{aligned}
\left(\frac{|x - Y_{k,+}^{(j,i)}|}{\beta_{k,+}^{(j,i)}} \big|v_{k,+}^{(j,i)}(x)\big|^{\frac{1}{\beta_{k,+}^{(j,i)}} - 1} \big|(v_{k,+}^{(j,i)})'(x)\big|\right)^{\beta_{k,+}^{(j,i)}} &\leqslant \big|(v_{k,+}^{(j,i)})'(x)\big|^{\beta_{k,+}^{(j,i)}} \big|v_{k,+}^{(j,i)}(x)\big| \big|I_{k,+}^{(j,i)}\big|^{\beta_{k,+}^{(j,i)}} \\
&\leqslant 2\big|(v_{k,+}^{(j,i)})'(x)\big|^{\beta_{k,+}^{(j,i)}} \big|I_{k,+}^{(j,i)}\big|^{\beta_{k,+}^{(j,i)}} \\
&\leqslant \frac{2}{4^{\frac{\beta_{k,+}^{(j,i)}}{\beta_0} + \beta_{k,+}^{(j,i)}}} \leqslant \frac{1}{4^{\beta_{k,+}^{(j,i)}}} < \frac{1}{2^{\beta_{k,+}^{(j,i)}}} \leqslant \big|v_{k,+}^{(j,i)}(x)\big|^{\beta_{k,+}^{(j,i)}}.
\end{aligned}$$

Hence,

$$|w'(x)| \geqslant C, \qquad x \in \overline{I_{k,+}^{(j,i)}},$$

where $C$ is a positive constant independent of $x$, $k$, $j$ and $i$. We denote

$$\tilde{\Gamma} := \big\{w(x) : x \in I_{k,+}^{(j,i)}\big\}, \qquad \tilde{\alpha} := \alpha_{k,+}^{(j,i)}, \qquad \tilde{\beta} := \beta_{k,+}^{(j,i)}.$$

We rewrite the considered integral as follows:

$$\int\limits_{\mathbb{R}} \phi_{k,\pm}^{(j,i)}(x)\,dx = \int\limits_{I_{k,+}^{(j,i)}} \frac{dx}{V(x) - \lambda} = \frac{1}{w'(Z_{k,+}^{(j,i)})} \int\limits_{I_{k,+}^{(j,i)}} \frac{w'(x)\,dx}{V(x) - \lambda} + \int\limits_{I_{k,+}^{(j,i)}} \frac{w'(Z_{k,+}^{(j,i)}) - w'(x)}{V(x) - \lambda}\,dx.$$

Using, then, identity (76) and making the change in variable $t = w(x)$ in the first integral in the right hand side of the above identity, we obtain:

$$\int\limits_{\mathbb{R}} \phi_{k,\pm}^{(j,i)}(x)\,dx = \frac{1}{\tilde{\alpha} w'(Z_{k,+}^{(j,i)})} \int\limits_{\tilde{\Gamma}} \frac{dt}{t^{\tilde{\beta}} - \eta} + \int\limits_{I_{k,+}^{(j,i)}} \frac{w'(Z_{k,+}^{(j,i)}) - w'(x)}{V(x) - \lambda}\,dx. \tag{78}$$

Owing to the above established smoothness of the function $w$, see (77), and the second inequality in (59), by applying the Lagrangue rule, we immediately estimate the second integral in the right hand side of the above identity:

$$\left|\int\limits_{I_{k,+}^{(j,i)}} \frac{w'(Z_{k,+}^{(j,i)}) - w'(x)}{V(x)-\lambda}\,dx\right| \leqslant C \int\limits_{I_{k,+}^{(j,i)}} \frac{|x-Z_{k,+}^{(j,i)}|}{|V(x)-\lambda|}\,dx \leqslant C, \tag{79}$$

where the symbol $C$ denotes various constants independent of $\delta_3$, $\lambda$, $k$, $j$ and $i$.

Let us estimate the first integral in the right hand side of (78). Suppose that the point $\eta$ is located above the curve $\tilde{\Gamma}$. Then, we choose the branch of the analytic function $z^{\tilde{\beta}}$ with the cut along the positive imaginary semi-axis and the argument of $z$ ranging in $(-\frac{3\pi}{2}, \frac{\pi}{2}]$. Let $\tilde{z}$ be the end-point of the curve $\tilde{\Gamma}$ not coinciding with the origin. In the complex plane, we introduce extra two curves:

$$\tilde{\Gamma}_1 := \{z:\, z = e^{-\frac{\pi \mathrm{i}}{\tilde{\beta}}} s,\ s\in(0,|\tilde{z}|)\}, \qquad \tilde{\Gamma}_2 := \{z:\, |z|=|\tilde{z}|,\ \arg z \in (-\pi, \arg\tilde{z})\}.$$

Then, the closure of the union of these two curves and $\tilde{\Gamma}$ is a closed contour, and by the Cauchy integral theorem, we obtain:

$$\int\limits_{\tilde{\Gamma}} \frac{dt}{t^{\tilde{\beta}}-\eta} = -\int\limits_{\tilde{\Gamma}_1} \frac{dt}{t^{\tilde{\beta}}-\eta} - \int\limits_{\tilde{\Gamma}_2} \frac{dt}{t^{\tilde{\beta}}-\eta} = e^{-\frac{\pi\mathrm{i}}{\tilde{\beta}}} \int\limits_0^{|\tilde{z}|} \frac{ds}{s^{\tilde{\beta}}+\eta} - \int\limits_{\tilde{\Gamma}_2} \frac{dt}{t^{\tilde{\beta}}-\eta}. \tag{80}$$

Since $\lambda \in E_{k,\delta_3}$, it follows from the definition of $\eta$ in (75) and Property P2 that

$$|\eta| \leqslant \frac{\delta_3}{|\tilde{\alpha}|}, \qquad |\tilde{z}|^{\tilde{\beta}} = \frac{2\delta_3}{|\tilde{\alpha}|}.$$

Hence, $|t^{\tilde{\beta}} - \eta| \geqslant \frac{\delta_3}{|\tilde{\alpha}|}$ on the curve $\tilde{\Gamma}_2$ and

$$\left|\int\limits_{\tilde{\Gamma}_2} \frac{dt}{t^{\tilde{\beta}}-\eta}\right| \leqslant \frac{2\pi|\tilde{\alpha}|}{\delta_3}. \tag{81}$$

Since $\operatorname{Re}\eta > 0$ by (75), the first integral in the right hand side of the above identity can be immediately estimated as

$$\left| e^{-\frac{\pi\mathrm{i}}{\tilde{\beta}}} \int\limits_0^{|\tilde{z}|} \frac{ds}{s^{\tilde{\beta}}+\eta}\right| \leqslant \int\limits_0^{|\tilde{z}|} \frac{ds}{s^{\tilde{\beta}}} = \frac{1}{1-\tilde{\beta}} \frac{1}{|\tilde{z}|^{1-\tilde{\beta}}} = \frac{1}{1-\tilde{\beta}} \frac{|\tilde{\alpha}|^{\frac{1}{\tilde{\beta}}-1}}{(2\delta_3)^{\frac{1}{\tilde{\beta}}-1}} \leqslant \frac{C}{\delta_3^{\frac{1}{\beta_0}-1}},$$

where $C$ is a constant independent of $\delta_3$, $k$, $j$ and $i$. This estimate (80) and (81), (80) prove the uniform boundedness of the first integral in the right hand side of (78), and in view of (79), we arrive at estimate (74) for $\phi_{k,+}^{(j,i)}$. The proof is complete. □

**Lemma 4.** *The function $a$ is an element of $C(\overline{\tilde{J}})$.*

**Proof.** Since $a \in W_2^1(\mathbb{R})$, by the standard embedding theorems, we conclude that $a \in C(\mathbb{R})$ and this proves the lemma. □

**Lemma 5.** *As $\delta_3 \leqslant \min\{\delta_4, \delta_5\}$, for $\lambda \in E_{k,\delta_3}$ the estimates hold:*

$$\|\mathcal{L}_{a\star}\phi_k^{(j,i)}\|_{L_\infty(\tilde{J}_k)} \leqslant c_{13}, \qquad \|\mathcal{L}_{a\star}\phi_{k,\pm}^{(j,i)}\|_{L_\infty(\tilde{J}_k)} \leqslant c_{13}, \tag{82}$$

$$\|\mathcal{L}_{a\star}\phi_k^{(j,i)}\|_{L_2(\mathbb{R})} \leqslant c_{13}, \qquad \|\mathcal{L}_{a\star}\phi_{k,\pm}^{(j,i)}\|_{L_2(\mathbb{R})} \leqslant c_{13}, \tag{83}$$

*where $c_{13}$ is a constant independent of $\lambda$, $k$, $j$, $i$ but depending on $\delta_3$.*

**Proof.** We fix $k$ and some $(j,i)$ in the corresponding set $M_1$ and represent the function $\mathcal{L}_{a\star}\phi_k^{(j,i)}$ as

$$(\mathcal{L}_{a\star}\phi_{k,\pm}^{(j,i)})(x) = a(x - Z_k^{(j,i)}) \int\limits_{I_k^{(j,i)}} \phi_k^{(j,i)}(y)\,dy + \int\limits_{I_k^{(j,i)}} \frac{a(x-y) - a(x - Z_k^{(j,i)})}{V(y) - \lambda}\,dy. \tag{84}$$

By Lemmas 2 and 4, we immediately estimate the first integral in the right hand side of the above identity:

$$\begin{aligned} &\left| a(x - Z_k^{(j,i)}) \int\limits_{I_k^{(j,i)}} \phi_k^{(j,i)}(y)\,dy \right| \leqslant \frac{c_{11}}{\delta_3}\|a\|_{C(\bar{J})}, \\ &\left\| a(\,\cdot\, - Z_k^{(j,i)}) \int\limits_{I_k^{(j,i)}} \phi_k^{(j,i)}(y)\,dy \right\|_{L_2(\mathbb{R})} \leqslant \frac{c_{11}}{\delta_3}\|a\|_{L_2(\mathbb{R})}. \end{aligned} \tag{85}$$

To estimate the second integral in the right hand side of (84), we employ a representation similar to (43):

$$\int\limits_{I_k^{(j,i)}} \frac{a(x-y) - a(x - Z_k^{(j,i)})}{V(y) - \lambda}\,dy = \int\limits_{I_k^{(j,i)}} \frac{dy}{V(y) - \lambda} \int\limits_0^{y - Z_k^{(j,i)}} a'(x - Z_k^{(j,i)} - t)\,dt$$

and use then the Cauchy–Schwarz inequality and the first estimate from (59):

$$\begin{aligned} \left| \int\limits_{I_k^{(j,i)}} \frac{a(x-y) - a(x - Z_k^{(j,i)})}{V(y) - \lambda}\,dy \right| &\leqslant \frac{1}{c_9} \int\limits_{I_k^{(j,i)}} \frac{1}{|y - Z_k^{(j,i)}|^{\frac{1}{2}}} \left| \int\limits_{-|y - Z_k^{(j,i)}|}^{|y - Z_k^{(j,i)}|} \left| a'(x - Z_k^{(j,i)} - t) \right|^2 dt \right|^{\frac{1}{2}} dy \\ &\leqslant \frac{\|a'\|_{L_2(\mathbb{R})}}{c_9} \int\limits_{I_k^{(j,i)}} \frac{dy}{|y - Z_k^{(j,i)}|^{\frac{1}{2}}} \leqslant C, \end{aligned}$$

$$\begin{aligned} \left\| \int\limits_{I_k^{(j,i)}} \frac{a(\,\cdot\, - y) - a(\,\cdot\, - Z_k^{(j,i)})}{V(y) - \lambda}\,dy \right\|_{L_2(\mathbb{R})}^2 &\leqslant C \int\limits_{\mathbb{R}} dx \int\limits_{I_k^{(j,i)}} \frac{dy}{|y - Z_k^{(j,i)}|} \int\limits_{-|y - Z_k^{(j,i)}|}^{|y - Z_k^{(j,i)}|} \left| a'(x - Z_k^{(j,i)} - t) \right|^2 dt \\ &\leqslant C\|a'\|_{L_2(\mathbb{R})}^2 \int\limits_{I_k^{(j,i)}} \frac{dy}{|y - Z_k^{(j,i)}|} \int\limits_{-|y - Z_k^{(j,i)}|}^{|y - Z_k^{(j,i)}|} dt \leqslant C, \end{aligned}$$

where by $C$ we denote various constants independent of $\lambda$, $k$, $j$ and $i$. These estimates (84) and (85) prove the first inequalities in (82) and (83).

The proof of the second inequalities in (82) and (83) follows the same lines. Namely, in (84), we just replace $I_k^{(j,i)}$, $Z_k^{(j,i)}$, $\phi_k^{(j,i)}$ by $I_{k,+}^{(j,i)}$, $Z_{k,+}^{(j,i)}$, $\phi_{k,+}^{(j,i)}$. Then, a corresponding analogue of inequality (85) is implied by Lemmas 3 and 4, while estimating the second integral literally reproduces the above argument. The proof is complete. □

**Lemma 6.** *As $\delta_3 \leqslant \min\{\delta_4, \delta_5\}$, for $\lambda \in E_{k,\delta_3}$ the estimates hold:*

$$\|\mathcal{B}_{6,k}(\varepsilon,\lambda)\mathcal{A}(\varepsilon,\lambda)\phi_k^{(j,i)}\|_{L_2(\bar{J})} \leqslant c_{14}, \qquad \|\mathcal{B}_{6,k}(\varepsilon,\lambda)\mathcal{A}(\varepsilon,\lambda)\phi_{k,\pm}^{(j,i)}\|_{L_2(\bar{J})} \leqslant c_{14},$$

*where $c_{14}$ is a constant independent of $\lambda$, $k$, $j$, $i$ but depending on $\delta_3$.*

**Proof.** In view of the definition of the operator $\mathcal{B}_{6,k}$ in (57), it is sufficient to prove the uniform boundedness of the norms

$$\begin{aligned}
&\|\mathcal{B}_{6,k}^{(p,q)}(\varepsilon,\lambda)\mathcal{A}(\varepsilon,\lambda)\phi_k^{(j,i)}\|_{L_2(\tilde{J})}, && \|\mathcal{B}_{6,k}^{(p,q)}(\varepsilon,\lambda)\mathcal{A}(\varepsilon,\lambda)\phi_{k,\pm}^{(j,i)}\|_{L_2(\tilde{J})}, && (p,q)\in M_1,\\
&\|\mathcal{B}_{6,k,\flat}^{(p,q)}(\varepsilon,\lambda)\mathcal{A}(\varepsilon,\lambda)\phi_k^{(j,i)}\|_{L_2(\tilde{J})}, && \|\mathcal{B}_{6,k,\flat}^{(p,q)}(\varepsilon,\lambda)\mathcal{A}(\varepsilon,\lambda)\phi_{k,\pm}^{(j,i)}\|_{L_2(\tilde{J})}, && (p,q)\in M_2, \quad \flat\in\{+,-\}.
\end{aligned} \tag{86}$$

Bearing in mind the definition of the operators $\mathcal{B}_{6,k}^{(p,q)}$ and $\mathcal{B}_{6,k,\pm}^{(p,q)}$ in (58), the definition of the operator $\mathcal{A}$ in (33), and inequalities (26) and (62), we can estimate the first of the above norms as follows:

$$\begin{aligned}
\|\mathcal{B}_{6,k}^{(p,q)}(\varepsilon,\lambda)\mathcal{A}(\varepsilon,\lambda)\phi_k^{(j,i)}\|_{L_2(\tilde{J})} \leqslant& \|\mathcal{B}_{6,k}^{(p,q)}(\varepsilon,\lambda)\phi_k^{(j,i)}\|_{L_2(\tilde{J})} + \|\mathcal{B}_{6,k}^{(p,q)}(\varepsilon,\lambda)\mathcal{P}_{\mathbb{R}\setminus\tilde{J}}\mathcal{A}(\varepsilon,\lambda)\phi_k^{(j,i)}\|_{L_2(\tilde{J})}\\
\leqslant& \|\mathcal{B}_{6,k}^{(p,q)}(\varepsilon,\lambda)\phi_k^{(j,i)}\|_{L_2(\tilde{J})} + C\|\mathcal{M}_{\tilde{J}}\phi_k^{(j,i)}\|_{L_2(\mathbb{R}\setminus\tilde{J})},
\end{aligned}$$

where $C$ is a constant independent of $\lambda$, $\varepsilon$, $k$, $j$, $i$ and $\delta_3$. In the same way, we can estimate other norms in (86) and, hence, it is sufficient to prove the uniform boundedness only for the norms

$$\begin{aligned}
&\|\mathcal{B}_{6,k}^{(p,q)}(\varepsilon,\lambda)\phi_k^{(j,i)}\|_{L_2(\tilde{J})}, && \|\mathcal{B}_{6,k}^{(p,q)}(\varepsilon,\lambda)\phi_{k,\pm}^{(j,i)}\|_{L_2(\tilde{J})}, && (p,q)\in M_1,\\
&\|\mathcal{B}_{6,k,\flat}^{(p,q)}(\varepsilon,\lambda)\phi_k^{(j,i)}\|_{L_2(\tilde{J})}, && \|\mathcal{B}_{6,k,\flat}^{(p,q)}(\varepsilon,\lambda)\phi_{k,\pm}^{(j,i)}\|_{L_2(\tilde{J})}, && (p,q)\in M_2, \quad \flat\in\{+,-\},\\
&\|\mathcal{M}_{\tilde{J}}\phi_k^{(j,i)}\|_{L_2(\mathbb{R}\setminus\tilde{J})}, && \|\mathcal{M}_{\tilde{J}}\phi_{k,\pm}^{(j,i)}\|_{L_2(\mathbb{R}\setminus\tilde{J})}.
\end{aligned} \tag{87}$$

The uniform boundedness of the latter two norms follows immediately from (83) and definition (29) of the operator $\mathcal{M}_{\tilde{J}}$.

According to the definition of the operators $\mathcal{B}_{6,k}^{(j,i)}$ in (58), the identity holds:

$$\begin{aligned}
\big(\mathcal{B}_{6,k}^{(p,q)}(\varepsilon,\lambda)\phi_k^{(j,i)}\big)(x) =& \xi_k^{(j,i)}(x)\int\limits_{I_k^{(j,i)}} \frac{a(x-y)-a\big(Z_k^{(j,i)}(\lambda)-y\big)}{(V(x)-\lambda)(V(y)-\lambda)}\,dy\\
=& \xi_k^{(j,i)}(x)\frac{a(x-Z_k^{(j,i)}(\lambda))-a(0)}{(V(x)-\lambda)}\int\limits_{I_k^{(j,i)}}\frac{dy}{V(y)-\lambda}\\
&+\xi_k^{(j,i)}(x)\int\limits_{I_k^{(j,i)}}\frac{a(x-y)-a(x-Z_k^{(j,i)}(\lambda))-a\big(Z_k^{(j,i)}(\lambda)-y\big)+a(0)}{(V(x)-\lambda)(V(y)-\lambda)}\,dy\\
=& \xi_k^{(j,i)}(x)\frac{a(x-Z_k^{(j,i)}(\lambda))-a(0)}{(V(x)-\lambda)}\int\limits_{I_k^{(j,i)}}\frac{dy}{V(y)-\lambda}\\
&+\xi_k^{(j,i)}(x)\int\limits_{I_k^{(j,i)}}\frac{dy}{(V(x)-\lambda)(V(y)-\lambda)}\int\limits_0^{x-Z_k^{(j,i)}}\big(a'(t+Z_k^{(j,i)}-y)-a'(t)\big)\,dt.
\end{aligned}$$

Since $\xi_k^{(j,i)}$ is the characteristic function of a bounded interval $I_k^{(j,i)}$ and $a\in L_1(\tilde{J})$ by Lemma 4, in view of Lemma 2, we immediately conclude that the first term in the right hand side of the above identity is an element of $L_2(\mathbb{R})$ and it is bounded uniformly in $\lambda$, $k$, $j$, $i$ in the norm of this space. The norm of the second term is estimated by using (59) and the second condition in (9):

$$\int\limits_{\mathbb{R}}\left|\xi_k^{(j,i)}(x)\int\limits_{I_k^{(j,i)}}\frac{dy}{(V(x)-\lambda)(V(y)-\lambda)}\int\limits_0^{x-Z_k^{(j,i)}}\left(a'(t+Z_k^{(j,i)}-y)-a'(t)\right)dt\right|^2 dx$$

$$\leqslant C\int\limits_{I_k^{(j,i)}}dx\left|\int\limits_{I_k^{(j,i)}}\frac{dy}{|x-Z_k^{(j,i)}||y-Z_k^{(j,i)}|}\int\limits_{-|x-Z_k^{(j,i)}|}^{|x-Z_k^{(j,i)}|}\left|a'(t+Z_k^{(j,i)}-y)-a'(t)\right|dt\right|^2$$

$$\leqslant C\int\limits_{I_k^{(j,i)}}dx\left|\int\limits_{I_k^{(j,i)}}\frac{dy}{|x-Z_k^{(j,i)}||y-Z_k^{(j,i)}|}\int\limits_{-|x-Z_k^{(j,i)}|}^{|x-Z_k^{(j,i)}|}|Z_k^{(j,i)}-y|^\theta\,dt\right|^2$$

$$\leqslant C\int\limits_{I_k^{(j,i)}}dx\left|\int\limits_{I_k^{(j,i)}}\frac{dy}{|y-Z_k^{(j,i)}|^{1-\theta}}\right|^2\leqslant C,$$

where the symbol $C$ stands for various constants independent of $\lambda$, $k$, $j$, $i$. Hence, the functions $\mathcal{B}_{6,k}^{(p,q)}(\varepsilon,\lambda)\phi_k^{(j,i)}$ are bounded in $L_2(\mathbb{R})$ uniformly in $\lambda$, $k$, $j$, $i$. Similar boundedness for remaining functions in (87) is established in the same way, and one should just use the second estimate from (59) and Lemma 3. The proof is complete. □

**Lemma 7.** *As $\delta_3\leqslant\min\{\delta_4,\delta_5\}$, for $\lambda\in E_{k,\delta_3}$ the estimates hold:*

$$\|\mathcal{B}_{4,k}(\varepsilon,\lambda)\mathcal{A}(\varepsilon,\lambda)\phi_k^{(j,i)}\|_{L_2(\tilde{J})}\leqslant c_{15},\qquad \|\mathcal{B}_{4,k}(\varepsilon,\lambda)\mathcal{A}(\varepsilon,\lambda)\phi_{k,\pm}^{(j,i)}\|_{L_2(\tilde{J})}\leqslant c_{15},$$

*where $c_{15}$ is a constant independent of $\lambda$, $k$, $j$, $i$ but depending on $\delta_3$.*

**Proof.** Denoting

$$\mathcal{B}_{4,k}^{(p,q)}:=\frac{1-\xi_k^{(p,q)}}{V-\lambda}\mathcal{P}_{\tilde{J}}\mathcal{L}_{a\star},$$

we observe that

$$\mathcal{B}_{4,k}=\sum_{p=1}^{n}\sum_{q=1}^{N_k^{(j)}}\mathcal{B}_{4,k}^{(j,i)}\mathcal{A}(\varepsilon,\lambda). \tag{88}$$

Then, using inequality (49) and the definition of the operator $\mathcal{A}(\varepsilon,\lambda)$, we obtain:

$$\|\mathcal{B}_{4,k}^{(p,q)}\mathcal{A}(\varepsilon,\lambda)\phi_k^{(j,i)}\|_{L_2(\tilde{J})}\leqslant C\|\mathcal{L}_{a\star}\phi_k^{(j,i)}\|_{L_2(\mathbb{R})},\qquad \|\mathcal{B}_{4,k}^{(p,q)}\mathcal{A}(\varepsilon,\lambda)\phi_{k,\pm}^{(j,i)}\|_{L_2(\tilde{J})}\leqslant C\|\mathcal{L}_{a\star}\phi_k^{(j,i)}\|_{L_2(\mathbb{R})},$$

where the symbol $C$ denotes some constants independent of $\lambda$, $k$, $j$ and $i$. Applying, then, estimates (83), we see that the norms in the above inequality are uniformly bounded, and together with, (88) this completes the proof. □

We substitute Formula (68) into (66) and (67) and apply Lemmas 6 and 7 and estimate (82). This yields the desired uniform boundedness of the functions $A_k^{(p,q,j,i)}(\varepsilon,\lambda)$, $A_{k,\pm}^{(p,q,j,i)}(\varepsilon,\lambda)$, $A_{k,\pm}^{(p,q,j,i)}(\varepsilon,\lambda)$, $A_{k,\flat,\natural}^{(p,q,j,i)}(\varepsilon,\lambda)$ with some fixed sufficiently small $\delta_3$. All these functions are bounded by some constant $c_{16}$ independent of $\varepsilon$, $\lambda\in E_{k,\delta_3}$, $k$, $j$, $i$. Hence, there exists $\varepsilon_0>0$ independent of $k$, $\lambda$, $j$, $i$ such that as $\varepsilon<\varepsilon_0$, system (65) possesses only trivial solution simultaneously for all $k$. Therefore, there exists $\delta>0$ such that the set $S^\delta$ contains no eigenvalues of the operator $\mathcal{L}^\varepsilon$.

In order to prove the absence of the residual spectrum, we first need to establish Formula (2). By $\mathrm{Ker}(\,\cdot\,)$ and $\mathrm{Ran}(\,\cdot\,)$, we denote the kernel and the range of a given closed operator.

**Lemma 8.** *Identity (2) is true.*

**Proof.** Given a closed operator $\mathcal{A}$ in a Hilbert space $H$, let $\lambda \notin \sigma_{\text{ess}}(\mathcal{A}) \cup \sigma_{\text{pnt}}(\mathcal{A})$. Then, $\text{Ker}(\mathcal{A}-\lambda) = \{0\}$ and hence, the inverse operator $(\mathcal{A}-\lambda)^{-1}$ is well defined on the range $\text{Ran}(\mathcal{A}-\lambda)$. This inverse operator is bounded. Indeed, if this operator was unbounded, this would mean the existence of a sequence $u_n \in \mathfrak{D}(\mathcal{A})$ such that $\|u_n\|_H = 1$ and $\|(\mathcal{A}-\lambda)u_n\|_H \to 0$ as $n \to \infty$. Since $\lambda \notin \sigma_{\text{ess}}(A)$, the sequence $\{u_n\}$ is compact, and choosing a subsequence if needed, we can suppose that $u_n$ converges to some $u_*$ in $H$. Then, by the closedness of the operator $\mathcal{A}$ and the normalization of $u_n$, we immediately conclude that $\|u_*\|_H = 1$ and $(\mathcal{A}-\lambda)u_* = 0$, i.e., $u_*$ is an eigenfunction of $\mathcal{A}$ associated with its eigenvalue $\lambda$. This is impossible, since $\lambda \notin \sigma_{\text{pnt}}(\mathcal{A})$ and, therefore, the inverse operator $(\mathcal{A}-\lambda)^{-1}$ is bounded on the range $\text{Ran}(\mathcal{A}-\lambda)$. By [17] (Ch. 3, Sect. 2, Thm. 9), this yields that the range $\text{Ran}(\mathcal{A}-\lambda)$ is closed. Hence, as $\lambda \notin \sigma_{\text{ess}}(\mathcal{A}) \cup \sigma_{\text{pnt}}(\mathcal{A})$, it belongs to the spectrum $\sigma(\mathcal{A})$ if and only if $\overline{\text{Ran}(\mathcal{A}-\lambda)} = \text{Ran}(\mathcal{A}-\lambda) \neq H$, which is equivalent to $\text{Ker}(\mathcal{A}^*-\overline{\lambda}) \neq \{0\}$, i.e., if and only if $\overline{\lambda}$ is an eigenvalue of the adjoint operator $\mathcal{A}^*$. This completes the proof. □

In view of Formula (2), we observe that the adjoint operator for $\mathcal{L}^\varepsilon$ reads as

$$(\mathcal{L}^\varepsilon)^* = \mathcal{L}_{\overline{V}} + \varepsilon \mathcal{L}_{a^*\star}, \qquad a^*(z) := \overline{a(-z)}. \tag{89}$$

This adjoint operator is of the same structure as $\mathcal{L}^\varepsilon$ in particular, the essential spectrum of the operator $\mathcal{L}_{\overline{V}}$ is just the complex conjugation of the curve $\Upsilon$, namely,

$$\sigma(\mathcal{L}_{\overline{V}}) = \sigma_{\text{ess}}(\mathcal{L}_{\overline{V}}) = \Upsilon^\dagger.$$

Then, we choose the complex conjugation of the piece $S$ of this curve and we see that it also satisfies the assumptions of Theorem 2. The function $a^*$ obeys Assumption (9). Then, lessening if needed the number $\delta$, we conclude that the set $(S^\delta)^\dagger$ contains no eigenvalues of the operator $(\mathcal{L}^\varepsilon)^*$. Then, Formula (2) implies that the set $S^\delta$ also contains no points of the residual spectrum of the operator $\mathcal{L}^\varepsilon$ and this completes the proof of Theorem 2.

### *4.4. Absence of Residual Spectrum*

In this subsection, we prove Theorem 3. We recall Formula (89) for the adjoint operator $\mathcal{L}^\varepsilon$ and immediately see that Condition (10) guarantees the self-adjointness of the operator $\mathcal{L}^\varepsilon$. This implies the absence of the residual spectrum.

Suppose that Condition (11) is obeyed. As it was stated in Section 2, see identities (12)–(14), it is sufficient to check the validity of $\mathcal{PT}$-symmetricity condition (12) with the operator $\mathcal{P}$ given in (13). This can be carried out by straightforward calculations for an arbitrary $\psi \in L_2(\mathbb{R})$ using conditions (11):

$$\begin{aligned}
\big(\mathcal{PT}(\mathcal{L}^\varepsilon)^*\psi\big)(x) =& \mathcal{PT}\left(\overline{V(x)}\phi(x) + \varepsilon \int_{\mathbb{R}} \overline{a(y-x)}\phi(y)\,dy\right) \\
=& V(\tau x+\varrho)\overline{\phi(\tau x+\varrho)} + \varepsilon \int_{\mathbb{R}} a(y-\tau x-\varrho)\overline{\phi(y)}\,dy \\
=& V(\tau x+\varrho)\overline{\phi(\tau x+\varrho)} + \varepsilon \int_{\mathbb{R}} a(\tau(y-x))\overline{\phi(\tau y+\varrho)}\,dy \\
=& V(x)(\mathcal{PT}\phi)(x) + \varepsilon \int_{\mathbb{R}} a(x-y)(\mathcal{PT}\phi)(y)\,dy = (\mathcal{L}^\varepsilon\mathcal{PT}\phi)(x).
\end{aligned}$$

This completes the proof.

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