Homogenization of the linearized ionic transport equations in random porous media

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Abstract

In this paper we obtain the homogenization results for a system of partial differential equations describing the transport of a N-component electrolyte in a dilute Newtonian solvent through a rigid random disperse porous medium. We present a study of the nonlinear Poisson-Boltzmann equation in a random medium, establish convergence of the stochastic homogenization procedure and prove well-posedness of the two-scale homogenized equations. In addition, after separating scales, we prove that the effective tensor satisfies the so-called Onsager properties, that is the tensor is symmetric and positive definite. This result shows that the Onsager theory applies to random porous media. The strong convergence of the fluxes is also established. In the periodic case homogenization results for the mentioned system have been obtained in [7].

Keywords: Boltzmann-Poisson equation, homogenization, electro-osmosis, random porous media.

1 Introduction

The quasi-static electroosmotic phenomena in porous media are present in many problems of applied interest. As examples we mention electromigration of solutes (see for instance Ottosen et al [43]), dewatering (see Mahmoud et al [34]) and permeability reduction in concretes (see Cardenas & Struble [16]).

We consider a porous material saturated by an N-component dilute electrolyte. The solid surfaces of the porous skeleton are electrically charged, attract ions of the opposite charge and repel the ones of the same charge. Simultaneously an external electrical field $\mathbf{E}$ and a body force $\mathbf{f}$ are applied, generating a hydrodynamical flow, a migration of ions and creation of an electrical double layer (EDL).

The modeling of phenomena at the pore level is well understood (Lyklema [33], Karniadakis et al [30]). The porous media are heterogeneous and have a pore structure consisting of a very large number of pores. In studying flows generated by the electro-osmotic phenomena, the pore size is of the same order as the size of the Debye layer (≈ 100 nanometers). Solving the partial differential equations of the model is a difficult task even for simple geometries and out of reach for realistic porous media with nanoscopic pores. The remedy for the complexity of the problem is to homogenize or upscale the equations posed at the pore scale and derive a new upscaled system valid at every point. The microscopic geometry would influence the effective coefficients.

The homogenization approach allows finding out how close the solutions to upscaled model are to those of the original physical equations, given at the pore scale. In the existing literature on the charged porous media, the homogenization technique was mainly applied under the hypothesis of the *periodic* porous media and for the so-called *ideal model, i.e. with ions at infinite dilution.* There are works which concern the study and homogenization of the Nernst-Planck-Poisson system (see for instance Ray et al [44]-[45] and Schmuck et al [46]-[49]). In the physical chemistry literature, its semi-linearized form, due to O'Brien & White [42], replaces the full model from Lyklema [33] and Karniadakis et al [30]. Looker & Carnie studied O'Brien's model using the two-scale expansion in [32]. They derived constitutive laws linking the effective fluxes with the pressure gradient and the gradient of the chemical potential. In addition, the Onsager tensor, containing the effective coefficients, was obtained. The rigorous convergence result for the homogenization process is due to Allaire et al [7], where the positive definiteness of the Onsager tensor and its symmetry were proved too. A numerical and qualitative analysis of the effective coefficients is in Allaire et al [9]. The results of Looker & Carnie [32] and Allaire et al [7], [9] support findings in the earlier work of Adler et al [18], [35], [2] and [24], on the particular flows and the computation of the effective coefficients forming Onsager's tensor.

The electro-osmotic phenomena are important also in *deformable* porous media, like clays. For the homogenization studies of the electro-osmosis with swelling, using two-scale expansions, we refer to the series of articles by Moyne & Murad [36], [37], [38], [39] and [40]. A further reference is the work of Dormieux et al [19]. A rigorous homogenization of the electro-osmotic flows in deformable porous media, with a derivation of the Onsager relation and determination of the swelling pressure, was obtained in Allaire et al [11].

We note that models that are more realistic involve electrolytes at *finite dilution*. Several complicated mathematical models were developed to take into account the finite size of the ions. One of them is the *Mean Spherical Approximation (MSA)* model (Dufrêche et al [20]). Its well-posedness and homogenization was studied by Allaire et al [10].

Finally, the effects of nonlinearities were studied for equilibrium solution (Ern et al [21], Allaire et al [8]).

The realistic materials have *random geometric structures*, and questions related to the upscaling of the electro-osmotic flows in such materials were studied only through numerical simulations. We mention the publications of Adler and al in [18], [35] and [2], where the electrokinetic flows through random packings of spheres and ellipsoids were considered.

The goal of the present paper is to study *upscaling of the ideal model describing the transport of a dilute N-component electrolyte in a rigid* **random** *porous medium.* We first briefly recall the dimensionless equations, which were the starting point for the periodic homogenization in [7], [9] and [10]. Since the modeling and derivation of the dimensionless form were undertaken in detail in these references (and especially in [10]) we do not dwell on the subject and simply start with the dimensionless ideal model equations from the above references.

The (dimensionless) equations are given on a typical realization of the porous medium G, the meaning of the quantities used in these equations can be found below in Table 1. If ω takes values in the probability space, we denote by $G_f^\varepsilon(\omega)$ a realization of the pore space, which is an open set filled with a fluid. Here ε is a small positive parameter defined as the ratio between the pore size and the size the material area.

The equations then read as follows:

$$\varepsilon^2 \Delta \mathbf{u}^\varepsilon - \nabla p^\varepsilon = \mathbf{f}^* - \sum_{j=1}^N z_j n_j^\varepsilon(x) \nabla \Psi^\varepsilon \quad \text{in } G_f^\varepsilon(\omega), \tag{1}$$

$$\mathbf{u}^\varepsilon = 0 \quad \text{on } \partial G_f^\varepsilon(\omega), \quad \text{div } \mathbf{u}^\varepsilon = 0 \quad \text{in } G_f^\varepsilon(\omega), \tag{2}$$

$$-\varepsilon^2 \Delta \Psi^\varepsilon = \beta \sum_{j=1}^N z_j n_j^\varepsilon(x) \quad \text{in } G_f^\varepsilon(\omega); \tag{3}$$

$$\varepsilon \nabla \Psi^\varepsilon \cdot \nu = -N_\sigma \sigma \quad \text{on } \partial G_f^\varepsilon(\omega) \setminus \partial G, \quad \Psi^\varepsilon = -\Psi^{\text{ext}} \quad \text{on } \partial G, \tag{4}$$

$$\text{div } \left(n_j^\varepsilon \nabla \ln(n_j^\varepsilon e^{\Psi^\varepsilon z_j}) - \text{Pe}_j n_j^\varepsilon \mathbf{u}^\varepsilon \right) = 0 \quad \text{in } G_f^\varepsilon(\omega), \tag{5}$$

$$\nabla \ln(n_j^\varepsilon e^{\Psi^\varepsilon z_j}) \cdot \nu = 0 \quad \text{on } \partial G_f^\varepsilon(\omega). \tag{6}$$

System (1)-(6) is the dimensionless model that we will homogenize in the sequel. More precisely, we will undertake study of its O'Brien's semi-linearized form. We assume that all constants appearing in (1)-(6) are independent of ε, namely N_σ and Pe_j are of order 1 with respect to ε.

For the comfort of the reader, we recall the meaning of the unknowns (which are dimensionless in our equations) on Table 1.

SYMBOL	*QUANTITY*
Pe_j	is the Péclet number for jth electrolyte component
n_i	ith concentration
$n_i^c \in (0,1)$	ith infinite dilution concentration
$\mathbf{u}$	fluid velocity
p	fluid pressure
ℓ	pore size
λ_D	Debye's length
$\beta = (\ell/\lambda_D)^2$	ratio of the pore scale to the Debye's length
N_σ	ratio of the pore scale to the Gouy length
$z_j \in \mathbb{Z}$	j-th electrolyte valence
σ	given surface charge density
$\mathbf{f}^*$	given applied force
Ψ	electrochemical potential
Ψ^{ext}	exterior potential

Table 1: *Description of the parameters and the unknowns*

In Section 2, we recall in subsection 2.1 the basic results on the stochastic two-scale convergence technique, which will be used to prove convergence of the homogenization process in random geometry. In subsection 2.2 their partial linearization in the spirit of the seminal work of O'Brien & White [42] is given. The section is completed with subsection 2.3, where the well-posedness of the non-linear Poisson-Boltzmann equation in a random geometry is studied (see Theorem 11). In Section 3, we undertake stochastic homogenization of the linearized electrokinetic equations around equilibrium (Theorem 16). The scale separation and derivation of Onsager's relations, linking the ionic current, filtration velocity and ionic fluxes with gradients of the electrical potential, pressure and ionic concentrations, are the subject of Section 4. In Proposition 24 we prove that the full homogenized Onsager tensor is symmetric positive definite for disperse random structures. The article is concluded with Section 5, where in Theorem 28 we prove strong convergence of the velocities and the ionic fluxes.

2 Description of the problem and of the results

In this section we give a precise formulation of our microscopic problem (the ε-problem) and of the results.

We start with defining the geometrical structure

2.1 The stochastic two-scale convergence in mean and a random porous medium

Let $(\Omega, \Xi, \mathbb{P})$ denote a probability space, with probability measure $\mathbb{P}$ and sigma-algebra Ξ, the symbol $\mathbb{E}$ stands for the corresponding expectation. In what follows $L^2(\Omega)$ is supposed to be separable. We assume that an ergodic dynamical system $\mathcal{T}$ with n-dimensional time is given on Ω, i.e. a family of invertible measurable maps $\mathcal{T}(x) : \Omega \to \Omega$, for each $x \in \mathbb{R}^n$, such that

1. $\mathcal{T}(0)$ is the identity map on Ω and $\mathcal{T}(x_1 + x_2) = \mathcal{T}(x_1)\mathcal{T}(x_2)$ for all $x_1, x_2 \in \mathbb{R}^n$;
2. $\forall\, x \in \mathbb{R}^n$ and $\forall\, E \in \Xi$,
$$\mathbb{P}(\mathcal{T}(x)^{-1}(E)) = \mathbb{P}(E), \quad \text{i.e. } \mathbb{P} \text{ is an invariant measure for } \mathcal{T} \text{ (endomorphism property).}$$
3. $\forall\, E \in \Xi$ the set $\{(x, \omega) \in \mathbb{R}^n \times \Omega \, : \, \mathcal{T}(x)\omega \in E\}$ is an element of the sigma-algebra $\mathcal{L} \times \Xi$ on $\mathbb{R}^n \times \Omega$, where $\mathcal{L}$ is the usual Lebesgue σ-algebra on $\mathbb{R}^n$.
4. $\mathcal{T}$ is *ergodic*, i.e. any set $E \in \Xi$ such that $\mathbb{P}\big((\mathcal{T}(x)E \cup E) \setminus (\mathcal{T}(x)E \cap E)\big) = 0$, $\forall x \in \mathbb{R}^n$ satisfies either $\mathbb{P}(E) = 0$ or $\mathbb{P}(E) = 1$.

With the measurable dynamics introduced above we associate a n-parameters group of unitary operators on $L^2(\Omega) \equiv L^2(\Omega, \Xi, \mathbb{P})$, as follows

$$(U(x)f)(\omega) = f(\mathcal{T}(x)\omega), \qquad f \in L^1(\Omega).$$

The map $x \to U(x)$ is continuous in the strong operator topology, see for instance Jikov et al [29].

Next, let $\mathcal{F} \in \Xi$ be such that $\mathbb{P}(\mathcal{F}) > 0$ and $\mathbb{P}(\Omega \setminus \mathcal{F}) > 0$. Starting from the set $\mathcal{F}$, we define a random pore structure $F(\omega) \subset \mathbb{R}^n$, $\omega \in \Omega$. It is obtained from $\mathcal{F}$ by setting

$$F(\omega) = \{x \in \mathbb{R}^n \, : \, \mathcal{T}(x)\omega \in \mathcal{F}\}. \tag{7}$$

In what follows we suppose that $F(\omega)$ is open and connected a.s. (for almost all $\omega \in \Omega$). In a complementary way, the random rigid skeleton structure $M(\omega)$ is introduced by setting

$$\mathcal{M} = \Omega \setminus \mathcal{F}, \qquad M(\omega) = \mathbb{R}^n \setminus F(\omega). \tag{8}$$

We assume that a.s. $M(\omega)$ is a disperse medium, i.e. a union of mutually disjoint components, called grains, satisfying the following conditions:

R1. A.s. $M(\omega)$ is a union of non-intersecting C^2-smooth bounded domains.

R2. The curvature of the boundary of these domains admits a deterministic upper bound.

R3. The distance between any two domains is greater than a positive deterministic constant.

R4. The diameter of any domain is not greater than a positive deterministic constant.

R5. There exists $r_0 > 0$ such that a.s. any ball of radius r_0 in $\mathbb{R}^n$ has a nontrivial intersection with $M(\omega)$.

Under condition **R1.** the connected components of $M(\omega)$ can be enumerated, so that

$$M(\omega)=\bigcup_{j=1}^{\infty} M_j(\omega).$$

In connection with the random set $M(\omega)$ we introduce a homothetic structure $M_\varepsilon(\omega)$, $\omega\in\Omega$, by

$$M_\varepsilon(\omega)=\{x\in\mathbb{R}^n\,:\,\varepsilon^{-1}x\in M(\omega)\}; \tag{9}$$

For more details on the homogenization of linear PDEs in perforated random domains we refer to the monograph Jikov et al [29].

Remark 1. *The system considered in* (1)–(6) *is a rather complicated system of nonlinear equations that requires essential technical work when obtaining a priori estimates and passing to the limit. Due to this reason, in order to make the presentation clear, we preferred to consider random disperse media that admit deterministic estimates of their geometric characteristics, such as the distance between inclusions and their diameters. We strongly believe that, using the approaches developed in recent articles [25], [26] and [27] (see also the earlier pioneer work [23]), one can consider a wider class of random geometries and generalize the results of this work to the case of more general random perforated domains.*

Here we provide several examples of random disperse media.

$\mathcal{E}$**1**. *Random perturbation of periodic structure.* Let ζ^j, $j\in\mathbb{Z}^n$, be a collection of independent and identically distributed (i.i.d.) random vectors in $\mathbb{R}^n$ taking on values in $[-\frac14,\frac14]^n$. Denote by r^j, $j\in\mathbb{Z}^n$, a collection of i.i.d. random variables such that $\frac13\le r^j\le\frac23$, and by η a random vector that is uniformly distributed on the cube $[-\frac12,\frac12]^n$ and independent with respect to ζ^j and r^j. Taking for $M(\omega)$ the union of balls centered at $j+\zeta^j+\eta$ of radius r^j we obtain an example of random statistically homogeneous disperse medium. In this case $\mathcal{M}=\{\omega\,:\,0\in M(\omega)\}$.

$\mathcal{E}$**2**. *Bernoulli spherical structure.* Consider a collection of i.i.d. random variables ξ^j, $j\in\mathbb{Z}^d$, with values 0 and 1, and an independent vector η uniformly distributed on $[-\frac12,\frac12]^n$. We say that a vertex $j\in\mathbb{Z}^n$ is open if $\xi^j=1$, and closed if $\xi^j=0$. We then define $M(\omega)$ by

$$M(\omega)=\bigcup_{j\text{ is open}}\big\{x\in\mathbb{R}^n\,:\,|x-j-\eta|\le\frac12\big\}\cup\bigcup_{j\text{ is closed}}\big\{x\in\mathbb{R}^n\,:\,|x-j-\eta|\le\frac14\big\}.$$

Then we set $\mathcal{M}=\{\omega\,:\,0\in M(\omega)\}$.

$\mathcal{E}$**3**. *Poisson spherical structure.* Let $\mathcal{P}$ be a Poisson process in $\mathbb{R}^n$ with intensity one, the averaged density of points is equal to one. By definition $\mathcal{P}$ is a random locally finite subset of $\mathbb{R}^n$ such that for any bounded Borel set $G\subset\mathbb{R}^n$ the number of points in $\mathcal{P}\cap G$ has a Poisson distribution with intensity $|G|$, and for any finite collection of bounded disjoint Borel sets $G_1,\dots,G_N$ the random variables defined as the number of points of $\mathcal{P}$ in each of these sets are independent.
Denote the points of $\mathcal{P}$ by z_k, $k=1,2\dots$, the cells of the Voronoi tessellation generated by $\{z_k\}$ are denoted Ξ_k. Given $r>0$ we choose those z_k for which the ball of radius $2r$ centered at z_k belongs to Ξ_k, and for chosen z_k denote the union of balls $\{x\in\mathbb{R}^d\,:\,|x-z_k|\leqslant r\}$ by $M(\omega)$. Again, $\mathcal{M}=\{\omega\,:\,0\in M(\omega)\}$. Notice that in this case all the inclusions are balls of radius r.

By construction the random domains $M(\omega)$ defined in the first two examples are stationary and satisfy conditions [**R1.**]–[**R5.**] that is in both examples $M(\cdot)$ is a random disperse medium. To analyse the ergodic properties of these media we consider for the sake of definiteness the second example and introduce a probability space by setting $\Omega=\{0,1\}^{\mathbb{Z}^n}$ and taking the cylindrical σ-algebra Ξ and the product measure $\mathbb{P}$. It suffices to check the ergodicity for the integer shifts only. We then define $\mathcal{T}$ by $\mathcal{T}_z\omega(\cdot) = \omega(\cdot + z)$, $z \in \mathbb{Z}^n$. It is straightforward to show that this dynamical system is ergodic. Indeed, if there exists a measurable invariant set A such that $\mathcal{T}_z(A) = A$, then for any $\delta > 0$ there exists a set A_δ supported by a finite number of $z \in \mathbb{Z}^n$ such that $\mathbb{P}(A \boldsymbol{\Delta} A_\delta) \leq \delta$; here $\boldsymbol{\Delta}$ stands for the symmetric difference. Since $\mathcal{T}$ preserves measure $\mathbb{P}$, $\mathbb{P}(A \boldsymbol{\Delta} \mathcal{T}_z(A_\delta)) \leq \delta$ for any $z \in \mathbb{Z}^n$. For sufficiently large z_0 we have $\mathbb{P}(A \cap \mathcal{T}_{z_0}(A)) = \mathbb{P}(A)$ and $\mathbb{P}(A_\delta \cap \mathcal{T}_{z_0}(A_\delta)) = (\mathbb{P}(A_\delta))^2$. Therefore, $\mathbb{P}(A)$ is equal to either 0 or 1. The ergodicity of the medium in the first example can be checked in the same way.

The random medium introduced in the third example is stationary, ergodic and satisfies conditions [**R1.**]–[**R4.**]; however, condition [**R5.**] fails to hold. Filling large empty areas with a regular grid of balls one can rearrange this random medium to make condition [**R5.**] also hold.

Notice that in all the above examples the balls can be replaced with smooth bounded domains whose geometry might be random and should satisfy a number of natural conditions. For more examples see Bourgeat et al [15].

Let G be a smooth bounded domain in $\mathbb{R}^n$. After having chosen the random structure in $\mathbb{R}^n$, we set

$$G_1^\varepsilon = \{x \in G \,:\, \mathrm{dist}(x, \partial G) \geq \varepsilon\}. \tag{10}$$

The random fluid filled pore system in G given by

$$G_f^\varepsilon(\omega) = G \setminus (\overline{\bigcup_{j \in \mathcal{J}(\varepsilon)} \varepsilon M_j(\omega)}), \tag{11}$$

where

$$\mathcal{J}(\varepsilon) = \{j \in \mathbb{Z}^+ \,:\, \varepsilon M_j(\omega) \subset G_1^\varepsilon\}.$$

Then, the random rigid solid skeleton part of G is defined as the complement of $G_f^\varepsilon(\omega)$ in G:

$$G_m^\varepsilon(\omega) = G \setminus \overline{G_f^\varepsilon(\omega)}. \tag{12}$$

Before giving the convergence results, we recall the definition and some properties of the stochastic two-scale convergence in the mean (see Bourgeat et al [13] for more details).

Let D_j denote the infinitesimal generator in $L^2(\Omega)$ of the one-parameter group of translations in x_j, with the other coordinates held equal to zero. $\mathcal{D}_j$ is its respective domain of definition in $L^2(\Omega)$, i.e. for $f \in \mathcal{D}_j$

$$(D_j f)(\omega) = \frac{\partial}{\partial x_j}(U(x)f)(\omega)|_{x=0} \tag{13}$$

Then $\{\sqrt{-1}\, D_j,\ j = 1, \ldots, n\}$ are closed, densely–defined and self–adjoint operators which commute pairwise on $\mathcal{D}(\Omega) = \bigcap_{j=1}^{n} \mathcal{D}_j$. $\mathcal{D}(\Omega)$ is a Hilbert space with respect to the scalar product

$$(f, g)_{\mathcal{D}(\Omega)} = (f, g)_{L^2(\Omega)} + \sum_{j=1}^{n} (D_j f, D_j g)_{L^2(\Omega)} \tag{14}$$

After (13), the stochastic gradient $\{\nabla_\omega f\}$, divergence $\{\mathrm{div}_\omega f\}$ and curl $\{\mathrm{curl}_\omega f\}$, read as follows

$$\left\{ \begin{array}{ll} \nabla_\omega f & = (D_1 f, \ldots, D_n f) \\ \mathrm{div}_\omega g & = \sum_j D_j g_j \\ \mathrm{curl}_\omega g & = (D_i g_j - D_j g_i),\ i \neq j,\ i, j = 1, \ldots, n \end{array} \right. \tag{15}$$

In addition, we use the following spaces:

$$\mathcal{V}^2_{\rm pot}(\Omega) = \{f \in L^2_{\rm pot}(\Omega),\ I\!E\{f\} = 0\} \tag{16}$$

$$\mathcal{V}^2_{\rm sol}(\Omega) = \{f \in L^2_{\rm sol}(\Omega),\ I\!E\{f\} = 0\} \tag{17}$$

where $L^2_{\rm pot}(\Omega)$ (respectively $L^2_{\rm sol}(\Omega)$) is the set of all $f \in (L^2(\Omega))^n$ such that almost all realizations $f(\mathcal{T}(x)\omega)$ are potential (resp. solenoidal) in $\mathbb{R}^n$; for more details see Jikov et al [29]. Notice that $L^2_{\rm sol}(\Omega)) = L^2_{\rm pot}(\Omega)^\perp$.

Remark 2. *In the case of disperse media the functions from $\mathcal{D}(\Omega)$ vanishing on $\mathcal{M}$ are dense in $L^2(\mathcal{F})$, see [15, Remark 2.4]. In the terminology of [52], $\mathcal{F}$ is called $\mathcal{T}$-open in Ω. Furthermore, due to connectivity of $F(\omega)$, $\nabla_\omega \psi = 0$ in $\mathcal{F}$ implies that ψ does not depend on ω in $\mathcal{F}$. Hypothesis* **R1.-R3.** *are sufficient to make the assumptions on $\mathcal{F}$ in Proposition 4 below satisfied.*

Next, following [13], we say that an element $\psi \in L^2(G \times \Omega)$ is admissible if the function

$$\psi_{\mathcal{T}} : (x,\omega) \longrightarrow \psi(x, \mathcal{T}(x)\omega), \qquad (x,\omega) \in G \times \Omega,$$

defines an element of $L^2(G \times \Omega)$.
Examples of admissible two-scale functions are elements from $C(\overline{G}\,;\ L^\infty(\Omega))$, and finite linear combination of functions of the form

$$(x,\omega) \longrightarrow f(x)g(\omega), \quad (x,\omega) \in G \times \Omega,\ f \in L^2(G),\ g \in L^2(\Omega),$$

(see Bourgeat et al [13]).

The notion of stochastic two–scale convergence in the mean was introduced in Bourgeat et al [13]. It generalizes the two-scale convergence in the periodic setting introduced by Nguetseng in [41] and Allaire in [5]. We recall it for comfort of the reader

Definition 3. *A bounded sequence $\{u^\varepsilon\}$ of functions from $L^2(G \times \Omega)$ is said to converge stochastically two-scale in the mean (s.2-s.m.) towards $u \in L^2(G \times \Omega)$ if for any admissible $\psi \in L^2(G \times \Omega)$ we have*

$$\lim_{\varepsilon \to 0} \int\limits_{G\times\Omega} u^\varepsilon(x,\omega)\psi(x, \mathcal{T}(\frac{x}{\varepsilon})\omega)dxd\mathbb{P} = \int\limits_{G\times\Omega} u(x,\omega)\psi(x,\omega)dxd\mathbb{P}. \tag{18}$$

Our functions are defined on $G^\varepsilon_f(\omega)$ and not on G. It is the well-known complication appearing in homogenization of Neumann problem in perforated domains. Let X be the closure of the space $\mathcal{V}^2_{\rm pot}(\Omega)$ in $L^2(\Omega \setminus \mathcal{M})^n$. Motivated by Jikov et al [29] and Bourgeat et al [14], [15, Proposition 2.2] we have the following $2-$scale compactness result.

Proposition 4. *Let $\{\Phi^\varepsilon\} \subset H^1(G)$ and $\{\Psi^\varepsilon\} \subset H^1(G)$ be such sequences that*

$$\begin{cases} \|\Phi^\varepsilon\|_{L^2(G)} + \|\nabla\Phi^\varepsilon\|_{L^2(G^\varepsilon_f(\omega))} \le C, \\ \|\Psi^\varepsilon\|_{L^2(G)} + \varepsilon\|\nabla\Psi^\varepsilon\|_{L^2(G^\varepsilon_f(\omega))} \le C, \end{cases} \tag{19}$$

and assume that assumptions **R1.**–**R4.** *are fulfilled.*

Then there exist functions $\Phi_0 \in H^1(G)$, $\Psi_1 \in L^2(G; \mathcal{D}(\Omega))$, $\Psi_1 = 0$ on $\mathcal{M}$, and $\Phi_1 \in L^2(G; X)$, $\Phi_1 = 0$ on $\mathcal{M}$, such that, up to a subsequence,

$$\chi_{G^\varepsilon_f(\omega)}\Phi^\varepsilon \quad \overset{s.2\text{-}s.m.}{\longrightarrow} \quad \chi_{\mathcal{F}}(\omega)\Phi_0(x), \tag{20}$$

$$\chi_{G^\varepsilon_f(\omega)}\Psi^\varepsilon \quad \overset{s.2\text{-}s.m.}{\longrightarrow} \quad \Psi_1(x,\omega), \tag{21}$$

$$\chi_{G^\varepsilon_f(\omega)}\nabla\Phi^\varepsilon \quad \overset{s.2\text{-}s.m.}{\longrightarrow} \quad \chi_{\mathcal{F}}[\nabla_x\Phi_0(x) + \Phi_1(x,\omega)] \tag{22}$$

$$\varepsilon\chi_{G^\varepsilon_f(\omega)}\nabla\Psi^\varepsilon \quad \overset{s.2\text{-}s.m.}{\longrightarrow} \quad \chi_{\mathcal{F}}(\omega)\nabla_\omega\Psi_1(x,\omega); \tag{23}$$

here $\chi_{\mathcal{F}}$ stands for the characteristic function of $\mathcal{F}$.

Proof. As it was shown in [29], [52] and [53], under assumptions **R1.–R4.** the set of all functions $\psi \in \mathcal{D}(\Omega)$ such that $\psi = 0$ on $\mathcal{M}$, is dense in $L^2(\Omega \setminus \mathcal{M})$, and whenever $\nabla_\omega \psi = 0$ in $\mathcal{F}$, then ψ does not depend on ω in $\mathcal{F}$.

Under that same assumptions **R1.–R4.** the tensor $\mathcal{A}_N^0$ associated to the homogenized Neumann problem and defined by

$$\xi \cdot \mathcal{A}_N^0 \xi = \inf_{v \in X} \int_{\Omega \setminus \mathcal{M}} |\xi + v|^2 d\mathbb{P}, \qquad \xi \in \mathbb{R}^n, \tag{24}$$

is positive definite. Using the above a priori estimates and the stochastic two-scale in the mean compactness theorem from Bourgeat et al [13], we conclude that, after taking a proper subsequence, the sequences $\{\Phi^\varepsilon\}$, $\{\chi_{G_f^\varepsilon(\omega)} \nabla \Phi^\varepsilon\}$, $\{\Psi^\varepsilon\}$ and $\{\varepsilon \chi_{G_f^\varepsilon(\omega)} \nabla \Psi^\varepsilon\}$ have stochastic two-scale limits. We have then:

- $\Phi^\varepsilon \overset{\text{s.2-s.m.}}{\longrightarrow} \Phi_0(x, \omega)$
- $\chi_{G_f^\varepsilon(\omega)} \nabla \Phi^\varepsilon \overset{\text{s.2-s.m.}}{\longrightarrow} \xi_0(x, \omega)$
- $\Psi^\varepsilon \overset{\text{s.2-s.m.}}{\longrightarrow} \Psi_1(x, \omega)$
- $\varepsilon \chi_{G_m^\varepsilon(\omega)} \nabla \Psi^\varepsilon \overset{\text{s.2-s.m.}}{\longrightarrow} z_0(x, \omega)$

We should find relations between Φ_0, ξ_0, v and z_0.

Concerning the relation between ξ_0 and Φ_0, it was considered in [15, Proposition 2.1], and it was proved that

$$\xi_0 = 0 \quad \text{on} \quad G \times \mathcal{M} \quad \text{and} \quad \xi_0(x, \omega) - \nabla_x \Phi_0(x) \in L^2(G; X)^n.$$

It remains to identify z_0 .

Taking into account the ergodicity of the dynamical system and connectivity of the solid skeleton, by the same arguments as in Bourgeat et al [13], Bourgeat et al [15] and Wright [52], we conclude

$$z_0(x, \omega) = 0 \quad \text{on} \quad G \times \mathcal{M} \quad \text{and} \quad z_0(x, \omega) = \chi_{\mathcal{F}} \Psi_1(x, \omega) \in \mathcal{D}(\Omega).$$

□

Remark 5. *It should be noted that $\mathcal{A}_N^0$ is always positive definite in the periodic case if the solid part is connected.*

2.2 Linearization

In this subsection we follow the lead of O'Brien & White [42] and proceed with semi-linearization of system (1)-(6). The static electric potential $\Psi^{\text{ext}}(x)$ and the applied fluid force $\mathbf{f}^*(x)$ are assumed to be sufficiently small. No smallness condition is imposed on $N_\sigma \sigma$ and the Poisson-Boltzmann equation (3) remains non-linear.

After O'Brien & White [42], we write the electrokinetic unknowns as

$$\begin{aligned} n_i^\varepsilon(x) &= n_i^{0,\varepsilon}(x) + \delta n_i^\varepsilon(x), \quad \Psi^\varepsilon(x) = \Psi^{0,\varepsilon}(x) + \delta \Psi^\varepsilon(x), \\ \mathbf{u}^\varepsilon(x) &= \mathbf{u}^{0,\varepsilon}(x) + \delta \mathbf{u}^\varepsilon(x), \quad p^\varepsilon(x) = p^{0,\varepsilon}(x) + \delta p^\varepsilon(x), \end{aligned}$$

where $n_i^{0,\varepsilon}, \Psi^{0,\varepsilon}, \mathbf{u}^{0,\varepsilon}, p^{0,\varepsilon}$ are the equilibrium quantities, corresponding to $\mathbf{f}^* = 0$ and $\Psi^{\text{ext}} = 0$. The δ prefix indicates the size of perturbation.

At zero order, corresponding to $\mathbf{f}^* = 0$ and $\Psi^{\rm ext} = 0$, we search for an equilibrium solution of the form

$$\mathbf{u}^{0,\varepsilon} = 0\,, \quad p^{0,\varepsilon} = \sum_{j=1}^{N} n_j^c \exp\{-z_j \Psi^{0,\varepsilon}\}\,,$$
$$n_j^{0,\varepsilon}(x) = n_j^c \exp\{-z_j \Psi^{0,\varepsilon}(x)\}\,, \tag{25}$$

where $\Psi^{0,\varepsilon}$ solves the Poisson-Boltzmann equation

$$\begin{cases} -\varepsilon^2 \Delta \Psi^{0,\varepsilon} = \beta \displaystyle\sum_{j=1}^{N} z_j n_j^c e^{-z_j \Psi^{0,\varepsilon}} \quad \text{in} \;\; G_f^\varepsilon(\omega), \\ \varepsilon \nabla \Psi^{0,\varepsilon} \cdot \nu = -N_\sigma \sigma \;\; \text{on} \;\; \partial G_f^\varepsilon(\omega) \setminus \partial G, \quad \Psi^{0,\varepsilon} = 0 \;\; \text{on} \;\; \partial G. \end{cases} \tag{26}$$

Furthermore, we assume that all valences z_j are different and

$$z_1 < z_2 < ... < z_N, \quad z_1 < 0 < z_N. \tag{27}$$

It should be noted that the second relation here is crucial while the strict inequalities in the first one can be assumed without loss of generality. Indeed, if for two species the valencies were the same we could relabel them as the same specie. We denote by j^+ and j^- the sets of positive and negative valencies.

Our goal is to show that problem (26) is well posed, and that its solution admits uniform a priori estimates and two-scale converges in G to a statistically homogeneous function being a solution to problem (37) below. To prove this we consider an auxiliary problem (38) and obtain L^∞ estimates for its solution. With the help of these estimates, using auxiliary variational problems in (45)–(46), we derive L^∞ estimates for the solution of (26). After that, we exploit the standard two-scale compactness arguments.

We note that problem (26) is equivalent to the following minimization problem:

$$\inf_{\varphi \in W_\varepsilon} J_\varepsilon(\varphi), \tag{28}$$

with $W_\varepsilon = \{z \in H^1(G_f^\varepsilon(\omega)) \mid z = 0 \;\text{ on }\; \partial G\}$ and

$$J_\varepsilon(\varphi) = \frac{\varepsilon^2}{2} \int_{G_f^\varepsilon(\omega)} |\nabla \varphi|^2 \, dx + \beta \sum_{j=1}^{N} \int_{G_f^\varepsilon(\omega)} n_j^c e^{-z_j \varphi} \, dx + \varepsilon N_\sigma \int_{\partial G_f^\varepsilon(\omega)} \sigma \varphi \, dS.$$

The functional J_ε is strictly convex, which gives the uniqueness of the minimizer. Nevertheless, for arbitrary non-negative β, n_j^c and N_σ, J_ε may be **not coercive** on W_ε if all z_j's have the same sign. Indeed, in the case $\sigma = 0$ it suffices to take as φ constants of the same sign as z_1 and z_N's and tending to infinity. Following the literature, this degeneracy is handled by imposing the **bulk electroneutrality condition**

$$\sum_{j=1}^{N} z_j n_j^c = 0, \tag{29}$$

which guarantees that for $\sigma = 0$, the unique solution is $\Psi^{0,\varepsilon} = 0$.

The second difficulty is that J_ε is not defined on W_ε, but rather on $W_\varepsilon \cap L^\infty(G_f^\varepsilon(\omega))$.

Remark 6. *The bulk electroneutrality condition (29) is not a restriction. Actually, all our results hold under the much weaker assumption that all valences z_j do not have the same sign. We refer to [9] for the argument how to reduce the general case to (29).*

Remark 7. *Assume that the electroneutrality condition (29) holds true and σ be a smooth bounded function. Then, in the deterministic setting, it was proved in Allaire et al [8] that problem (28) has a unique solution $\Psi^{0,\varepsilon} \in W_\varepsilon \cap L^\infty(G_f^\varepsilon(\omega))$.*

Motivated by the computation of $n_i^{0,\varepsilon}$, having the form of the Boltzmann equilibrium distribution, we follow again lead of [42] and introduce the so-called ionic potential Φ_i^ε which is defined in terms of n_i^ε by

$$n_i^\varepsilon = n_i^c \exp\{-z_i(\Psi^\varepsilon + \Phi_i^\varepsilon + \Psi^{\text{ext}})\}. \tag{30}$$

After linearization (30) leads to

$$\delta n_i^\varepsilon(x) = -z_i n_i^{0,\varepsilon}(x)(\delta\Psi^\varepsilon(x) + \Phi_i^\varepsilon(x) + \Psi^{\text{ext},*}(x)). \tag{31}$$

Introducing (31) into (1)-(5) and linearizing yields the following system

$$\varepsilon^2 \Delta \mathbf{u}^\varepsilon - \nabla P^\varepsilon = \mathbf{f}^* - \sum_{j=1}^N z_j n_j^{0,\varepsilon}(x)(\nabla \Phi_j^\varepsilon + \mathbf{E}^*) \quad \text{in } \ G_f^\varepsilon(\omega), \tag{32}$$

$$\text{div}\mathbf{u}^\varepsilon = 0 \ \text{ in } \ G_f^\varepsilon(\omega), \quad \mathbf{u}^\varepsilon = 0 \ \text{ on } \ \partial G_f^\varepsilon(\omega), \tag{33}$$

$$\text{div}\left(n_j^{0,\varepsilon}(x)\big(\nabla\Phi_j^\varepsilon + \mathbf{E}^* + \frac{\text{Pe}_j}{z_j}\mathbf{u}^\varepsilon\big)\right) = 0 \ \text{ in } \ G_f^\varepsilon(\omega), \ j = 1, \dots, N, \tag{34}$$

$$(\nabla\Phi_j^\varepsilon + \mathbf{E}^*) \cdot \nu = 0 \ \text{ on } \ \partial G_f^\varepsilon(\omega) \setminus \partial G, \quad \Phi_j^\varepsilon = 0 \ \text{ on } \ \partial G, \ j = 1, \dots, N, \tag{35}$$

where

$$\mathbf{E}^*(x) = \nabla \Psi^{\text{ext},*}(x),$$

the perturbed velocity is actually equal to the overall velocity and, for convenience, we introduced a global pressure P^ε

$$\delta\mathbf{u}^\varepsilon = \mathbf{u}^\varepsilon\,, \quad P^\varepsilon = \delta p^\varepsilon + \sum_{j=1}^N z_j n_j^{0,\varepsilon}\left(\delta\Psi^\varepsilon + \Phi_j^\varepsilon + \Psi^{\text{ext},*}\right). \tag{36}$$

For the choice of the boundary conditions on ∂G, we have followed O'Brien & White [42]. It is important to remark that, after the global pressure P^ε has been introduced, $\delta\Psi^\varepsilon$ does not enter equations (32)-(35) and thus is decoupled from the main unknowns $\mathbf{u}^\varepsilon$, P^ε and Φ_i^ε.

2.3 Poisson-Boltzmann equation in the random geometry

Rescaling of the Poisson-Boltzmann equation in (26) yields its form valid in $F(\omega)$:

$$\begin{cases} -\Delta_y \Psi^0 = \beta \sum_{j=1}^N z_j n_j^c e^{-z_j\Psi^0} = -\beta n_H(\Psi^0) \ \text{ in } \ F(\omega), \\ \nabla_y \Psi^0 \cdot \nu = -N_\sigma \boldsymbol{\sigma}(\mathcal{T}(y)\omega) \ \text{ on } \ \partial F(\omega), \\ \Psi^0 \ \text{ is statistically homogeneous.} \end{cases} \tag{37}$$

Here and in what follows we assume that $\sigma(y) = \sigma(y,\omega) = \boldsymbol{\sigma}(\mathcal{T}(y)\omega)$ with $\boldsymbol{\sigma} \in \mathcal{D}(\Omega) \cap L^\infty(\omega)$. Problem (37) does not have boundary conditions at infinity. They are hidden in the statistical homogeneity of a solution. This means that there exists a function $\boldsymbol{\Psi}^0 \in L^2(\mathcal{F})$ such that $\Psi^0(y,\omega) = \boldsymbol{\Psi}^0(\mathcal{T}(y)\omega)$. We recall that in the periodic case, one can search for globally bounded smooth solution in the space and it turns out that they are necessarily the periodic ones.

We will show below that problem (37) is well-posed and has a unique solution and that the solution $\Psi^{0,\varepsilon}$ of problem (26) stochastically two-scale converges as $\varepsilon \to 0$ to the function $\chi_G(x)\mathbf{\Psi}^0(\omega)$ with Ψ^0 being a solution of (37).

Clearly, the function $\Psi^0\big(\frac{x}{\varepsilon}\big) = \mathbf{\Psi}^0\big(\mathcal{T}(x/\varepsilon)\omega\big)$ satisfies a.s. the Poisson-Boltzmann equation and Neumann's boundary condition in (26). Since we are interested in the effective bulk behavior of the potential, $\Psi^0\big(\frac{x}{\varepsilon}\big)$ is in fact the desired approximation to be used in the concentration coefficients $n_j^{0,\varepsilon}$ of the equations (32) and (34).

In order to derive L^∞-bounds for problem (26), we first handle the non-homogeneous Neumann condition and study the following ε-problem

$$\begin{cases} -\Delta_y V^\varepsilon + V^\varepsilon = 0 \ \text{ in } \ G_f(\omega) = \dfrac{1}{\varepsilon}\, G_f^\varepsilon(\omega), \\ \nabla_y V^\varepsilon \cdot \nu = -N_\sigma \sigma(y) \ \text{ on } \ \dfrac{1}{\varepsilon}\,\partial G_m^\varepsilon(\omega) = \displaystyle\bigcup_{j\in\mathcal{J}(\varepsilon)} \partial M_j(\omega), \\ V^\varepsilon = 0 \ \text{ on } \ \frac{1}{\varepsilon}\,\partial G. \end{cases} \tag{38}$$

Proposition 8. *Let σ be a bounded function such that $\|\sigma\|_{L^\infty(\partial F(\omega))} \le C_0$ a.s. and assume that conditions* **R1.**–**R3.** *are fulfilled. Then a.s. there exist constants V_m and V_M, independent of ε, such that*

$$V_m \le V^\varepsilon(y,\omega)) \le V_M \quad \text{a.e. on} \quad F(\omega). \tag{39}$$

Proof. First, we recall that problem (38) has a unique solution $V^\varepsilon \in H^1(G_f(\omega)$, $V^\varepsilon = 0$ on $\partial G/\varepsilon$. We search a L^∞-bound independent of ε.

Under conditions **R1.**–**R3.**, results from Gilbarg & Trudinger [22, Appendix 14.6] yield that the distance d is an element of $C^2(\Gamma_k)$, with $\Gamma_k = \{\, x \in F(\omega) \mid d(x) = \text{dist}(x, \partial F(\omega)) \le 2k\}$ and $1/k$ bounds the positive curvature of $\partial F(\omega)$. We assume that $2k < \min\{1, r_3\}$, where r_3 is a deterministic lower bound in condition **R3.**

Let $h \in C^2[0,+\infty)$ be a nonnegative function such that $h(t) = 0$ for $t \ge k$ and $h'(0) = N_\sigma ||\sigma||_{L^\infty(\partial F(\omega))} = C_0$. Then there is a constant $\hat{C}$, independent of ε, such that the function

$$\hat{a}_\varepsilon = \begin{cases} h(d(x)), & \text{if } \text{dist}(x, \varepsilon^{-1}\partial G_m^\varepsilon(\omega)) \le 2k, \\ 0, & \text{otherwise} \end{cases}$$

satisfies

$$0 \le \hat{a}_\varepsilon \le \hat{C} \quad \text{in} \quad F(\omega), \qquad |\Delta \hat{a}_\varepsilon| < \hat{C} \quad \text{in} \quad F(\omega), \qquad \nabla_y \hat{a}_\varepsilon \cdot \nu = h'(0) = C_0 \ \text{ on } \bigcup_{j\in\mathcal{J}(\varepsilon)} \partial M_j(\omega).$$

Furthermore

$$-\Delta(\hat{a}_\varepsilon + \hat{C}) + (\hat{a}_\varepsilon + \hat{C}) > 0,$$

$\dfrac{\partial}{\partial \nu}\hat{a}_\varepsilon \ge N_\sigma \sigma$ and $\hat{a}_\varepsilon = 0$ on $\varepsilon^{-1}\partial G$. By the maximum principle

$$V^\varepsilon \le \hat{a}_\varepsilon + \hat{C} \le 2\hat{C}.$$

In the same way one can show that $-2\hat{C} \le V^\varepsilon \le 2\hat{C}$ and the solution V^ε of problem (38) satisfies (39). □

Remark 9. *Let us suppose, in addition, that σ is bounded in $C^1(\partial F(\omega))$. The function V^ε, solving problem* (38)*, is in fact the solution for the variational problem*

$$\text{Find } V^\varepsilon \in H^1(G_f(\omega)),\ V^\varepsilon = 0 \text{ on } \frac{1}{\varepsilon}\ \partial G \text{ such that}$$

$$\int_{G_f(\omega)} (\nabla_y V^\varepsilon \cdot \nabla_y \varphi + V^\varepsilon \varphi)\ dy = -N_\sigma \sum_{j\in\mathcal{J}(\varepsilon)} \int_{\partial M_j(\omega)} \sigma\varphi\ dS_y = -N_\sigma \sum_{j\in\mathcal{J}(\varepsilon)} \int_{\partial M_j(\omega)} \sigma \nabla d\cdot \nu\varphi\ dS_y =$$
$$-N_\sigma \int_{G_f(\omega)} \big(\frac{\sigma}{C_0} h(d)\ \nabla_y d\cdot\nabla_y\varphi + \frac{\sigma}{C_0} h(d)\ \Delta d\ \varphi + \frac{\sigma}{C_0} h'(d)\ |\nabla_y d|^2\varphi + \frac{\nabla_y \sigma}{C_0}\cdot \nabla_y d\ h(d)\varphi\big)\ dy. \qquad (40)$$

In addition to estimate (39)*, we have*

$$\int_{G_f(\omega)} (|\nabla_y V^\varepsilon|^2 + |V^\varepsilon|^2)\ dy \le C|G_f(\omega)|, \quad \text{(a.s.) in } \omega; \qquad (41)$$

here and later on the notation $|B|$ is used for the volume of a set $B \subset \mathbb{R}^n$. Estimates (39) and (41) allow passing to the limit $\varepsilon \to 0$ of the sequence $\{V^\varepsilon\}$.

Having L^∞-bounds for the solution of auxiliary problem (38) allows proving existence of a bounded solution to problem (37).

We start with the ε-problem:

$$\begin{cases} -\Delta_y \Psi^\varepsilon = \beta \sum_{j=1}^N z_j n_j^c e^{-z_j\Psi^\varepsilon} = -\beta n_H(\Psi^\varepsilon) \ \text{ in } \ G_f(\omega), \\ \nabla_y \Psi^\varepsilon \cdot \nu = -N_\sigma \sigma(y) \ \text{ on } \ \frac{1}{\varepsilon}\ \partial G_m^\varepsilon(\omega), \\ \Psi^\varepsilon = 0 \ \text{ on } \ \frac{1}{\varepsilon}\ \partial G. \end{cases} \qquad (42)$$

Theorem 10. *Under the electroneutrality condition (29) and hypotheses* **R1.**–**R4.***, a.s., there exists a unique solution $\Psi^\varepsilon \in H^1(G_f(\omega)) \cap L^\infty(G_f(\omega))$ of problem (42), such that*

$$\int_{G_f(\omega)} (|\nabla_y \Psi^\varepsilon|^2 + |\Psi^\varepsilon|^2)\ dy \le C|G_f(\omega)|, \qquad (43)$$

$$||\Psi^\varepsilon||_{L^\infty(G_f(\omega))} \le C, \qquad (44)$$

where C is a deterministic constant, independent of ε.

Proof.

Step 1.

Let $L > 0$ be a sufficiently large constant to be specified later on. We introduce the cut-off nonlinearity n_{HL} by

$$n_{HL}(z) = \begin{cases} n_H(z) \quad \text{for} \quad |z| \le L; \\ n_H(L) + z - L \quad \text{for} \quad z > L; \\ n_H(-N) + z + L \quad \text{for} \quad z < -L. \end{cases} \qquad (45)$$

The cut-off functional is defined by

$$J_L(\varphi) = \frac{1}{2}\int_{G_f(\omega)} |\nabla_y\varphi|^2\ dy + \beta\int_{G_f(\omega)} \Gamma_L(\varphi)\ dy - \int_{G_f(\omega)} (V^\varepsilon\varphi + \nabla_y V^\varepsilon\cdot\nabla_y\varphi)\ dy,$$

where

$$\Gamma_L(z) = \begin{cases} \sum_{j\in j^+\cup j^-} n_j(z) - \sum_{j\in j^-\cup j^+} n_j(0) \quad \text{for} \quad |z| \le L; \\ \sum_{j\in j^+\cup j^-} n_j(L) + n_H(L)(z-L) + \frac{1}{2}(z-L)^2 - \sum_{j\in j^-\cup j^+} n_j(0) \quad \text{for} \quad z > L; \\ \sum_{j\in j^+\cup j^-} n_j(-L) + n_H(-L)(z+L) + \frac{1}{2}(z+L)^2 - \sum_{j\in j^-\cup j^+} n_j(0) \quad \text{for} \quad z < -L. \end{cases}$$

Then the problem

$$\min_{\varphi\in W} J_L(\varphi) \tag{46}$$

where $W = \{ \varphi \in H^1(G_f(\omega)) \mid \varphi = 0 \text{ on } \partial G/\varepsilon \}$, has a unique solution φ_L. Furthermore

$$||\nabla_y \varphi_L||^2_{L^2(G_f(\omega))^n} + ||\varphi_L||^2_{L^2(G_f(\omega))} \le C|G_f(\omega)|, \tag{47}$$

where C does not depend on L and ε. The latter inequality follows from the evident fact that $\min_{\varphi\in W} J_L(\varphi) \le 0$ and the lower bound $\Gamma_L(z) \ge \tilde{c}|z|^2$ with a constant $\tilde{c} > 0$ that does not depend on L.

Step 2.

Our next goal is to establish L^∞ estimates for φ_L, independent of L and ε. We begin with the variational problem

$$\int_{G_f(\omega)} \nabla_y(\varphi_L - V^\varepsilon)\cdot\nabla_y\phi \, dy + \beta \int_{G_f(\omega)} n_{HL}(\varphi_L)\phi \, dy = \int_{G_f(\omega)} V^\varepsilon \phi \, dy, \tag{48}$$

for all $\phi \in H^1(G_f(\omega))$, $\phi = 0$ on $\partial G/\varepsilon$. We take $\phi = (\varphi_L - V^\varepsilon + C_m)_-$, where

$$C_m = V_M + \frac{1}{z_N}\log\Big(-(V_m + z_1 \sum_{j\in j^-} n_j^c)_-/(z_N n_N^c) + 1\Big);$$

we recall that z_1 and z_N are valences that satisfy relations (27). Inserting this particular test function into equation (48) and using that $|V^\varepsilon - C_m| \le L$, yield

$$\int_{G_f(\omega)} |\nabla_y(\varphi_L - V^\varepsilon + C_m)_-|^2 \, dy + \beta \int_{G_f(\omega)} \big(n_{HL}(\varphi_L) - n_H(V^\varepsilon - C_m)\big) \underbrace{(\varphi_L - V^\varepsilon + C_m)_-}_{=0 \text{ if } \varphi_L \ge L} \, dy =$$

$$\int_{G_f(\omega)} (V^\varepsilon - n_H(V^\varepsilon - C_m))(\varphi_L - V^\varepsilon + C_m)_- \, dy. \tag{49}$$

Next

$$V^\varepsilon - n_H(V^\varepsilon - C_m) = V^\varepsilon + \sum_j z_j n_j^c e^{-z_j(V^\varepsilon - C_m)} \ge V_m + z_N n_N^c e^{-z_N(V_M - C_m)} + z_1 \sum_{j\in j^-} n_j^c$$

$$\ge 0 \quad \text{for} \quad C_m = V_M + \frac{1}{z_N}\log\Big(-(V_m + z_1 \sum_{j\in j^-} n_j^c)_-/(z_N n_N^c) + 1\Big)$$

and we conclude that, under a proper choice of L,

$$\varphi_L(y) \ge V_m - V_M + \frac{1}{z_N}\log\Big(-(V_m + z_1 \sum_{j\in j^-} n_j^c)_-/(z_N n_N^c) + 1\Big) > -L.$$

The upper bound is analogous.

Step3.

With a priori bounds (47) and the uniform L^∞-bounds, there exists a subsequence of φ_L, again denoted by the same subscript, and $\varphi_\varepsilon \in W \cap L^\infty(G_f(\omega))$, such that

$$\begin{cases} \nabla\varphi_L \rightharpoonup \nabla\varphi_\varepsilon \quad \text{weakly in} \quad L^2(G_f(\omega))^n; \\ \varphi_L \rightharpoonup \varphi_\varepsilon \quad \text{weakly in} \quad L^2(G_f(\omega)); \\ \varphi_L \rightharpoonup \varphi_\varepsilon \quad \text{weak-* in} \quad L^\infty(G_f(\omega)); \\ \varphi_L \to \varphi_\varepsilon \quad \text{strongly in} \quad L^2(G_f(\omega)), \end{cases} \tag{50}$$

as $L \to +\infty$. Next

$$\lim\inf_{L\to+\infty} J_L(\varphi_L) = \lim\inf_{L\to+\infty} J(\varphi_L) \geq J(\varphi_\varepsilon)$$

and

$$J(g) \geq J_{L(g)}(g) \geq J_{L(g)}(\varphi_{L(g)}) \geq J(\varphi_\varepsilon), \quad \forall g \in W, \ \Gamma(g) \in L^1(G_f(\omega)).$$

Hence φ_ε solves variational problem (42) and provides the minimum in the corresponding minimization problem. The strict convexity implies uniqueness and $\varphi_\varepsilon = \Psi^\varepsilon$.

□

It remains passing to the limit in problem (42) as $\varepsilon \to 0$.

We are interested in homogenization of a problem posed in $G_f^\varepsilon(\omega)$. We set

$$\Psi^{0,\varepsilon}(x) = \Psi^\varepsilon(\frac{x}{\varepsilon}), \quad x \in G. \tag{51}$$

Estimates (43)-(44) then read

$$\int_{G_f^\varepsilon(\omega)} (|\varepsilon\nabla\Psi^{0,\varepsilon}|^2 + |\Psi^{0,\varepsilon}|^2)\ dx \leq C, \tag{52}$$

$$||\Psi^{0,\varepsilon}||_{L^\infty(G_f^\varepsilon(\omega))} \leq C, \tag{53}$$

where C is a deterministic constant, independent of ε.

Theorem 11. *Let $\Psi^{0,\varepsilon}$ be defined by (51). Then there exists $\Psi^0 \in L^2(G, \mathcal{D}(\Omega)) \cap L^\infty(\Omega \times G)$ such that*

$$\Psi^{0,\varepsilon} \xrightarrow{s.2-s.m.} \Psi^0(x,\omega), \tag{54}$$

$$\varepsilon\nabla\Psi^{0,\varepsilon} \xrightarrow{s.2-s.m.} \nabla_\omega\Psi^0(x,\omega), \tag{55}$$

The limit function Ψ^0 is the unique solution to the variational equation

$$\int_{\mathcal{F}} \nabla_\omega\Psi^0 \cdot \nabla_\omega g\ d\mathbb{P} + \beta\int_{\mathcal{F}} n_H(\Psi^0) g\ d\mathbb{P} = -N_\sigma \int_{\mathcal{F}} \Big(\frac{\boldsymbol{\sigma}}{C_0} h(d(\omega))\nabla_\omega d(\omega)\cdot\nabla_\omega g + \frac{\boldsymbol{\sigma}}{C_0} h(d(\omega))\Delta_\omega d(\omega) g + \frac{\boldsymbol{\sigma}}{C_0} h'(d(\omega))|\nabla_\omega d|^2 g + \frac{\nabla_\omega \boldsymbol{\sigma}}{C_0}\cdot\nabla_\omega d h(d) g\Big)\ d\mathbb{P}, \quad \forall g \in \mathcal{D}(\Omega)\cap L^\infty(\Omega). \tag{56}$$

Proof. Using a priori estimates (52)-(53) and the stochastic two-scale convergence in the mean compactness theorem 3.7 from [13] and Proposition 4, we conclude that, after taking a proper subsequence, the sequences $\{\Psi^{0,\varepsilon}\}$ and $\{\varepsilon\nabla\Psi^{0,\varepsilon}\}$ have stochastic two-scale limits in the mean Ψ^0 and $\nabla_\omega\Psi^0$. Furthermore, $\chi_{G_f^\varepsilon(\omega)}$ converges in stochastic two-scale in the mean toward $\chi_{\mathcal{F}}$.

Because of the lower-semicontinuity with respect to the stochastic two-scale convergence in the mean of the L^q-norms, $1 < q < +\infty$, and estimate (53), the $L^q(G\times\mathcal{F})$-norms of Ψ^0 are bounded uniformly with respect to q. Hence, $\Psi^0 \in L^\infty(\Omega\times G)$, with the same constant as in the bound (53).

Let now $\zeta \in C_0^\infty(G)$ and $g \in \mathcal{D}(\Omega) \cap L^\infty(\Omega)$. Let $g^\varepsilon(x,\omega) = g(\mathcal{T}(\frac{x}{\varepsilon})\omega)\zeta(x)$, $\sigma^\varepsilon(x,\omega) = \boldsymbol{\sigma}(\mathcal{T}(\frac{x}{\varepsilon})\omega)$ and $d^\varepsilon(x,\omega) = d(\mathcal{T}(\frac{x}{\varepsilon})\omega)$. Using Minty's lemma we write the scaled back problem (42) in the equivalent form

$$\int_\Omega \int_G \varepsilon \chi_{G_f^\varepsilon(\omega)} \nabla g^\varepsilon \cdot \varepsilon \nabla (g^\varepsilon - \Psi^{0,\varepsilon}) \ dxd\mathbb{P} + \beta \int_\Omega \int_G \chi_{G_f^\varepsilon(\omega)} n_H(g^\varepsilon)(g^\varepsilon - \Psi^{0,\varepsilon}) \ dxd\mathbb{P} +$$
$$N_\sigma \int_\Omega \int_G \big(\frac{\sigma^\varepsilon}{C_0} h(d^\varepsilon) \varepsilon \nabla d^\varepsilon \cdot \varepsilon \nabla (g^\varepsilon - \Psi^{0,\varepsilon}) + \frac{\sigma^\varepsilon}{C_0} (h(d^\varepsilon)\varepsilon^2 \Delta d^\varepsilon + h'(d^\varepsilon)|\varepsilon \nabla d^\varepsilon|^2)(g^\varepsilon - \Psi^{0,\varepsilon}) +$$
$$\frac{\varepsilon \nabla \sigma^\varepsilon}{C_0} \cdot \varepsilon \nabla d^\varepsilon h(d^\varepsilon)(g^\varepsilon - \Psi^{0,\varepsilon}) \big) \ dxd\mathbb{P} \geq 0, \quad \forall g \in \mathcal{D}(\Omega) \cap L^\infty(\Omega), \ \zeta \in C_0^\infty(G). \tag{57}$$

Passing to the limit $\varepsilon \to 0$ is now straightforward (see for instance [14] and [15]). It yields

$$\int_G \int_{\mathcal{F}} \nabla_\omega g(\omega)\zeta(x) \cdot \nabla_\omega (g(\omega)\zeta(x) - \Psi^0) \ dxd\mathbb{P} + \beta \int_G \int_{\mathcal{F}} n_H(g(\omega)\zeta(x))(g(\omega)\zeta(x) - \Psi^0) \ dxd\mathbb{P} +$$
$$N_\sigma \int_G \int_{\mathcal{F}} \big(\frac{\boldsymbol{\sigma}(\omega)}{C_0} h(d(\omega)) \nabla_\omega d(\omega) \cdot \nabla_\omega (g(\omega)\zeta(x) - \Psi^0) +$$
$$\frac{\boldsymbol{\sigma}(\omega)}{C_0} (h(d(\omega))\Delta_\omega d(\omega) + h'(d(\omega))|\nabla_\omega d(\omega)|^2)(g(\omega)\zeta(x) - \Psi^0) +$$
$$\frac{\nabla_\omega \boldsymbol{\sigma}(\omega)}{C_0} \cdot \nabla_\omega d(\omega) h(d(\omega))(g(\omega)\zeta(x) - \Psi^0) \big) \ dxd\mathbb{P} \geq 0, \quad \forall g \in \mathcal{D}(\Omega) \cap L^\infty(\Omega), \ \zeta \in C_0^\infty(G).' \tag{58}$$

Using again Minty's lemma we obtain that Ψ^0 satisfies problem (56). Due to the strict convexity, Ψ^0 is unique and the whole sequence converges. Moreover, Ψ^0 does not depend on x, and, by construction, the function $\Psi^0\big(\mathcal{T}(y)\omega\big)$ satisfies a.s. the Poisson-Boltzmann equation and the Neumann condition in (37). □

Remark 12. *For passing to the stochastic 2-scale limits for more complicated problems with convex structure, we refer to Hudson et al [28].*

3 Homogenization

In Subsection 2.3 we solved the nonlinear Poisson-Boltzmann equation, for the equilibrium potential $\Psi^{0,\varepsilon}(x)$. It allowed computation of the equilibrium concentrations $n_j^{0,\varepsilon}(x) = n_j^c \exp\{-z_j \Psi^{0,\varepsilon}(x)\}$. Furthermore, we established that, as $\varepsilon \to 0$, $\Psi^{0,\varepsilon}(x)$ converges stochastically two-scales to $\Psi^0(\omega)$, the unique solution of the variational problem (56). Since the goal of this section is to homogenize the system of linearized equations (32)-(35) and of Section 4 to establish Onsager's relationship between the fluxes and the gradients of potentials in the **bulk**, we make a further simplification of the original system and replace in the linearized system the function $n_j^{0,\varepsilon}$ with $n_\varepsilon^j(x) = n_j^c \exp\{-z_j \Psi^0\big(\mathcal{T}(x/\varepsilon)\omega\big)\}$.

The formal two-scale asymptotic expansion method follows the periodic case (see Looker & Carnie [32]). The fast variable is now $y = \mathcal{T}(x/\varepsilon)\omega$ and the expansion of the solutions of (32)-(35) now reads

$$\begin{cases} \mathbf{u}^\varepsilon(x) = \mathbf{u}^0(x,y) + \varepsilon \mathbf{u}^1(x,y) + \dots, \\ P^\varepsilon(x) = p^0(x) + \varepsilon p^1(x,y) + \dots, \\ \Phi_j^\varepsilon(x) = \Phi_j^0(x) + \varepsilon \Phi_j^1(x,y) + \dots. \end{cases}$$

We do not dwell on formal expansions and start by introducing the functional spaces related to the velocity field and the ionic potentials:

$$\mathcal{H}^\varepsilon = \{ \mathbf{z} \in H_0^1(G_f^\varepsilon(\omega))^n, \ \text{div}\, \mathbf{z} = 0 \ \text{ in } \ G_f^\varepsilon(\omega) \}, \quad W^\varepsilon = \{ z \in H^1(G_f^\varepsilon(\omega)), \ z = 0 \text{ on } \partial G \}.$$

Then, summing the variational formulation of (34)-(35) with that of (32)-(33) (weighted by z_j^2/Pe_j) yields a.s in ω the variational formulation for the coupled problem that reads

$$\begin{gathered}\text{Find } \mathbf{u}^\varepsilon \in \mathcal{H}^\varepsilon \text{ and } \{\Phi_j^\varepsilon\}_{j=1,\dots,N} \in (W^\varepsilon)^N \text{ such that}\\ \mathbf{a}\left((\mathbf{u}^\varepsilon, \{\Phi_j^\varepsilon\}), (\xi, \{\phi_j\})\right) = \langle \mathcal{L}, (\xi, \{\phi_j\})\rangle\end{gathered} \tag{59}$$

for any test functions $\xi \in \mathcal{H}^\varepsilon$ and $\{\phi_j\}_{j=1,\dots,N} \in (W^\varepsilon)^N$; here

$$\mathbf{a}\left((\mathbf{u}^\varepsilon, \{\Phi_j^\varepsilon\}), (\xi, \{\phi_j\})\right) := \varepsilon^2 \int_{G_f^\varepsilon(\omega)} \nabla \mathbf{u}^\varepsilon : \nabla \xi \, dx + \sum_{j=1}^N z_j \int_{G_f^\varepsilon(\omega)} \big(\mathbf{u}^\varepsilon \cdot \nabla \phi_j - \xi \cdot \nabla \Phi_j^\varepsilon\big) n_\varepsilon^j \, dx + \sum_{j=1}^N \frac{z_j^2}{\mathrm{Pe}_j} \int_{G_f^\varepsilon(\omega)} n_\varepsilon^j \nabla \Phi_j^\varepsilon \cdot \nabla \phi_j \, dx$$

and

$$\langle \mathcal{L}, (\xi, \{\phi_j\})\rangle := \sum_{j=1}^N z_j \int_{G_f^\varepsilon(\omega)} n_\varepsilon^j \mathbf{E}^* \cdot \left(\xi - \frac{z_j}{\mathrm{Pe}_j} \nabla \phi_j\right) dx - \int_{G_f^\varepsilon(\omega)} \mathbf{f}^* \cdot \xi \, dx.$$

We recall that the concentrations $n_j^{0,\varepsilon}$ are replaced with the statistically homogeneous concentrations n_ε^j.

Before studying problem (59), we briefly discuss Poincaré inequality in $G_f^\varepsilon(\omega)$. For a general class of random domains, it was studied in Beliaev & Kozlov [12].

With assumptions **R1.–R5.**, the proof of this inequality is analogous to that in the periodic case (see Allaire [4, Sec 3.1.3, Lemma 1.6]):

Lemma 13. *Under assumptions* **R1.–R5.** *a.s. in* ω,

$$||\xi||_{L^2(G_f^\varepsilon(\omega))^n} \le C\varepsilon ||\nabla \xi||_{L^2(G_f^\varepsilon(\omega))^{n^2}}, \quad \forall \xi \in H_0^1(G_f^\varepsilon(\omega))^n, \tag{60}$$

where C is a deterministic constant.

Proof. First, we rescale $\xi(x)$ to $\tilde{\xi}$ being defined on $\varepsilon^{-1}G$ Next, we extend $\xi(x)$ by zero to the complement of $G_f(\omega)$. Let $F_j(\omega)$ be a subset of $F(\omega)$ contained the points having $M_j(\omega)$ as the closest fluid block. This way we obtain a tessellation of the whole space. Now we have Poincaré inequality for every domain $F_j(\omega) \cup \overline{M}_j(\omega)$, with a deterministic constant independent of j. Hence, we have Poincaré's inequality for all $j \in \mathcal{J}(\varepsilon)$. Next we add the complement of the closure of the union of all domains $F_j(\omega) \cup \overline{M}_j(\omega)$, with $j \in \mathcal{J}(\varepsilon)$, in $\varepsilon^{-1}G$. It yields Poincaré's inequality in $\varepsilon^{-1}G$ for $\tilde{\xi}$, with deterministic constant independent of ε. Rescaling back with respect to ε, gives inequality (60). □

Proposition 14. *Let us assume* **R1.–R5.** *and let* **E**$= \nabla \Psi^{\mathrm{ext}}$ *and* $\mathbf{f}^*$ *be given elements of* $L^2(G)^n$*. Then variational problem (59) admits a unique solution* $(\mathbf{u}^\varepsilon, \{\Phi_j^\varepsilon\}_{1\le j\le N}) \in \mathcal{H}^\varepsilon \times (W^\varepsilon)^N$*. Furthermore, there exists a deterministic constant C, which does not depend on ε, nor on $\mathbf{f}^*$ and $\mathbf{E}^*$, such that the solution satisfies the following a priori estimates*

$$||\mathbf{u}^\varepsilon||_{L^2(G_f^\varepsilon(\omega))^n} + \varepsilon ||\nabla \mathbf{u}^\varepsilon||_{L^2(G_f^\varepsilon(\omega))^{n^2}} \le C\bigg(||\mathbf{E}^*||_{L^2(G)^n} + ||\mathbf{f}^*||_{L^2(G)^n}\bigg) \tag{61}$$

$$\max_{1\le j\le N} ||\Phi_j^\varepsilon||_{H^1(G_f^\varepsilon(\omega))} \le C\bigg(||\mathbf{E}^*||_{L^2(G)^n} + ||\mathbf{f}^*||_{L^2(G)^n}\bigg). \tag{62}$$

Proof. The Cauchy-Schwartz inequality yields continuity of the bilinear form a and the linear form $\mathcal{L}$ on $\mathcal{H}^\varepsilon \times (H^1(G_f^\varepsilon(\omega))/\mathbb{R})^N$. Furthermore for $\xi = \mathbf{u}^\varepsilon$ and $\phi_j = \Phi_j^\varepsilon$, we find out that the second integral (the cross-term) in the definition of a cancels. Next, because of the L^∞-bounds on $\Phi^{0,\varepsilon}$, $n_\varepsilon^j \geq C > 0$, for a deterministic constant C, and the bilinear form $a((\mathbf{u}^\varepsilon, \{\Phi_j^\varepsilon\}_{1\leq j\leq N}), (\mathbf{u}^\varepsilon, \{\Phi_j^\varepsilon\}_{1\leq j\leq N}))$ is $\mathcal{H}^\varepsilon \times (H^1(G_f^\varepsilon(\omega))/\mathbb{R})^N$-elliptic. Now, the Lax-Milgram lemma implies existence and uniqueness of solution of problem (59).

The a priori estimates (61)-(62) follow by testing the problem (59) by the solution, using the L^∞-estimate for Ψ^0 and using Poincaré's inequality (60). □

As in Subsection 2.3, to simplify the presentation we use an extension operator from the perforated domain $G_f^\varepsilon(\omega)$ into Ω (although it is not necessary). Using hypothesis **R1.-R4.**, in analogy with the periodic case, (studied for instance in Acerbi et al [1], Cionarescu & Saint-Jean-Paulin [17] and Jikov et al [29]), there exists an extension operator T^ε from $H^1(G_f^\varepsilon(\omega))$ in $H^1(G)$ satisfying $T^\varepsilon\phi|_{G_f^\varepsilon(\omega)} = \phi$ and the inequalities

$$\|T^\varepsilon\phi\|_{L^2(G)} \leq C\|\phi\|_{L^2(G_f^\varepsilon(\omega))}, \ \|\nabla(T^\varepsilon\phi)\|_{L^2(G)} \leq C\|\nabla\phi\|_{L^2(G_f^\varepsilon(\omega))}$$

with a deterministic constant C independent of ε, for any $\phi \in H^1(G_f^\varepsilon(\omega))$. We keep for the extended function $T^\varepsilon\Phi_j^\varepsilon$ the same notation Φ_j^ε.

We extend $\mathbf{u}^\varepsilon$ by zero in $G\backslash G_f^\varepsilon(\omega)$. It is well known that extension by zero preserves L^q and $W_0^{1,q}$ norms for $1 < q < \infty$. Therefore, we can replace $G_f^\varepsilon(\omega)$ by G in estimate (61).

The pressure field P^ε is reconstructed using de Rham's theorem, see Temam [51]. It is thus unique up to an additive constant. The *a priori* estimates for the pressure are not easy to obtain and in the case of periodic porous media require Tartar's construction from [50] (see also Allaire [3] or Allaire [4, Sec 3.1.3]). Here we deal with a random porous medium and the pressure extension was constructed only for checkerboard type random domains in Beliaev & Kozlov [12]. Nevertheless, assumptions **R1.-R4.** allow to construct a "security domain" $Y_j(\omega)$ of the fixed deterministic size surrounding every $M_j(\omega)$, $j \in \mathcal{J}(\varepsilon)$. It is such that its distance to neighboring solid inclusions M_ℓ is bigger than a strictly positive deterministic constant. For instance, one can choose $Y_j(\omega)$ to be the k-neighbourhood of $M_j(\omega)$: $Y_j(\omega) = \{x \in F(\omega) \, : \, \mathrm{dist}(x, M_j(\omega)) < k\}$, where k is a constant defined in the proof of Proposition 8.

Then we repeat Tartar's construction of the restriction operator, developed originally for periodic porous media (see Allaire [4]), for every $j \in \mathcal{J}(\varepsilon)$. Next, by gluing all the pieces, the restriction operator is defined as a continuous operator $R : H_0^1(\frac{1}{\varepsilon}G)^n \to H_0^1(G_f(\omega))^n$. Note that if div $\varphi = 0$ in G/ε, then div $(R_\varepsilon\varphi) = 0$ in $G_f(\omega)$. Rescaling in exactly the same way as in the periodic case yields the restriction operator $R_\varepsilon : H_0^1(G)^n \to H_0^1(G_f^\varepsilon(\omega))^n$, such that div $\varphi = 0$ in G implies div $(R_\varepsilon\varphi) = 0$ in $G_f^\varepsilon(\omega)$. ∇P^ε is then extended using duality, as in the periodic case, and an extended pressure is $\tilde{P}^\varepsilon$ is obtained and the following estimate holds

$$|\langle\nabla\tilde{P}^\varepsilon, \varphi\rangle_{H^{-1}(G),H_0^1(G)}| \leq (||\varphi||_{L^2(G)^n} + \varepsilon||\nabla\varphi||_{L^2(G)^{n^2}}), \quad \forall\varphi \in H_0^1(G). \tag{63}$$

Furthermore, a slight modification of the argument from Avellaneda & Lipton [31] gives that the pressure extension $\tilde{P}^\varepsilon$ is given by

$$\tilde{P}^\varepsilon = \begin{cases} P^\varepsilon & \text{in } G_f^\varepsilon(\omega), \\ \dfrac{1}{|\varepsilon Y_i(\omega)|}\displaystyle\int_{\varepsilon Y_i(\omega)} P^\varepsilon & \text{in } \varepsilon M_i(\omega), \end{cases} \tag{64}$$

for each $i \in \mathcal{J}(\varepsilon)$. The results are summarized in

Lemma 15. *Let $\tilde{P}^\varepsilon$ be defined by (64). Then (a.s.) in ω it satisfies the estimates*

$$\|\tilde{P}^\varepsilon - \frac{1}{|G|}\int_G \tilde{P}^\varepsilon dx\|_{L^2(G)} \leq C\Big(||\mathbf{E}^*||_{L^2(G)^n} + ||\mathbf{f}^*||_{L^2(G)^n}\Big),$$
$$\|\nabla \tilde{P}^\varepsilon\|_{H^{-1}(G)^n} \leq C\Big(||\mathbf{E}^*||_{L^2(G)^n} + ||\mathbf{f}^*||_{L^2(G)^n}\Big).$$

Using the a priori estimates and the notion of two-scale convergence, we are able to prove our main convergence result.

Theorem 16. *Let us assume* **R1.–R5.** *Let $n_j^0 = n_j^c \exp\{-z_j\Psi^0\}$ and $\{\mathbf{u}^\varepsilon, \{\Phi_j^\varepsilon\}_{j=1,\dots,N}\}$ be the variational solution of (59). We extend the velocity $\mathbf{u}^\varepsilon$ by zero in $G \setminus G_f^\varepsilon(\omega)$ and the pressure P^ε by $\tilde{P}^\varepsilon$, given by (64) and normalized by $\int_{G\setminus G_f^\varepsilon(\omega)} \tilde{P}^\varepsilon = 0$. Then there exist limits $(\mathbf{u}^0, P^0) \in V \times L_0^2(G)$ and $\{\Phi_j^0, \Phi_j^1\}_{j=1,\dots,N} \in \big(H_0^1(G) \times L^2(G;X)\big)^N$ such that the following convergences hold*

$$\mathbf{u}^\varepsilon \to \mathbf{u}^0(x,\omega) \qquad \text{in the stochastic two-scale sense} \tag{65}$$
$$\varepsilon\nabla\mathbf{u}^\varepsilon \to \nabla_\omega \mathbf{u}^0(x,\omega) \qquad \text{in the stochastic two-scale sense} \tag{66}$$
$$\tilde{P}^\varepsilon \to P^0(x) \text{ strongly in } L_0^2(G), \text{ (a.s.) in } \omega, \tag{67}$$
$$\Phi_j^\varepsilon \to \Phi_j^0(x) \text{ in the stochastic two-scale sense} \tag{68}$$
$$\chi_{G_f^\varepsilon(\omega)}\nabla\Phi_j^\varepsilon \to \chi_{\mathcal{F}}(\omega)\{\nabla_x\Phi_j^0(x) + \Phi_j^1(x,\omega)\} \text{ in the stochastic two-scale sense.} \tag{69}$$

In addition, for $j = 1, \dots, N$,

$$\chi_{\mathcal{M}}(\omega)\Phi_j^1(x,\omega) = 0, \quad \chi_{\mathcal{M}}(\omega)\mathbf{u}^0(x,\omega) = 0 \quad \text{and} \quad P^0(x,\omega) = P^0(x) \quad \text{a.e. on } G\times\Omega. \tag{70}$$

Furthermore, $(\mathbf{u}^0, P^0, \{\Phi_j^0, \Phi_j^1\}_{j=1,\dots,N})$ is the unique solution of the two-scale homogenized problem

$$-\Delta_\omega\mathbf{u}^0(x,\omega) + \nabla_\omega p^1(x,\omega) = -\nabla_x P^0(x) - \mathbf{f}^*(x)$$
$$+\sum_{j=1}^N z_j n_j^0(\omega)(\nabla_x\Phi_j^0(x) + \Phi_j^1(x,\omega) + \mathbf{E}^*(x)) \quad \text{in } G\times\mathcal{F}, \tag{71}$$
$$\mathrm{div}_\omega \mathbf{u}^0(x,\omega) = 0 \text{ in } G\times\mathcal{F}, \ \mathbf{u}^0(x,\omega) = 0 \text{ on } G\times\mathcal{M}, \tag{72}$$
$$\mathrm{div}_x\big(\mathbb{E}(\mathbf{u}^0)\big) = 0 \text{ in } G, \quad \mathbb{E}^*(\mathbf{u}^0)\cdot\nu = 0 \text{ and } \Phi_j^0 = 0 \text{ on } \partial G, \tag{73}$$
$$-\mathrm{div}_\omega\Big(n_j^0(\omega)\big(\Phi_j^1(x,\omega) + \nabla_x\Phi_j^0(x) + \mathbf{E}^*(x) + \frac{\mathrm{Pe}_j}{z_j}\mathbf{u}^0\big)\Big) = 0 \text{ in } G\times\mathcal{F}, \tag{74}$$
$$\mathrm{curl}_\omega\ \Phi_j^1 = 0 \text{ in } G\times\mathcal{F}, \tag{75}$$
$$n_j^0(\omega)(\Phi_j^1 + \nabla_x\Phi_j^0 + \mathbf{E}^*) = 0 \quad \text{in } G\times\mathcal{M}, \tag{76}$$
$$-\mathrm{div}_x\mathbb{E}(n_j^0\big(\Phi_j^1 + \nabla_x\Phi_j^0 + \mathbf{E}^*(x) + \frac{\mathrm{Pe}_j}{z_j}\mathbf{u}^0\big)) = 0 \text{ in } G, \tag{77}$$

for $j = 1, \dots, N$.

Remark 17. *Following the terminology of Allaire [4], the limit problem introduced in Theorem 16 is called the two-scale, two-pressure homogenized problem. It is well posed because the two incompressibility constraints (72) and (73) are exactly dual to the two pressures $P^0(x)$ and $p^1(x,\omega)$ which are their corresponding Lagrange multipliers.*

The separation of scales from the above two-scale limit problem and extracting the purely macroscopic homogenized problem will be done later in Proposition 24, Section 4.

The proof of Theorem 16 will follow from several auxiliary lemmas.

We start by rewriting the variational formulation (59) with a velocity test function which is not divergence-free, so we can still take into account the pressure

$$\varepsilon^2 \int_{G_f^\varepsilon(\omega)} \nabla \mathbf{u}^\varepsilon : \nabla \xi \, dx - \int_{G_f^\varepsilon(\omega)} P^\varepsilon \operatorname{div} \xi \, dx + \sum_{j=1}^N \int_{G_f^\varepsilon(\omega)} z_j \big(- \xi \cdot \nabla \Phi_j^\varepsilon + \mathbf{u}^\varepsilon \cdot \nabla \phi_j \big) n_\varepsilon^j \ dx +$$
$$\sum_{j=1}^N \frac{z_j^2}{\mathrm{Pe}_j} \int_{G_f^\varepsilon(\omega)} n_\varepsilon^j \nabla \Phi_j^\varepsilon \cdot \nabla \phi_j \ dx = - \sum_{j=1}^N \frac{z_j^2}{\mathrm{Pe}_j} \int_{G_f^\varepsilon(\omega)} n_\varepsilon^j \mathbf{E}^* \cdot \nabla \phi_j \ dx$$
$$+ \sum_{j=1}^N \int_{G_f^\varepsilon(\omega)} z_j n_j^\varepsilon \mathbf{E}^* \cdot \xi \, dx - \int_{G_f^\varepsilon(\omega)} \mathbf{f}^* \cdot \xi \, dx, \tag{78}$$

for any test functions $\xi \in H_0^1(G_f^\varepsilon(\omega))$ and $\phi_j \in W^\varepsilon$, $1 \le j \le N$. Of course, one keeps the divergence constraint $\operatorname{div}\mathbf{u}^\varepsilon = 0$ in $G_f^\varepsilon(\omega)$. Next we define the two-scale test functions:

$$\xi^\varepsilon(x) = \xi(x, \mathcal{T}(\frac{x}{\varepsilon}\omega)), \ \xi \in C_0^\infty(G; \mathcal{D}(\Omega)^n), \ \ \xi = 0 \ \text{ on } \ G \times \mathcal{M}, \operatorname{div}_\omega \xi(x,\omega) = 0 \ \text{ on } \ G \times \mathcal{F}, \tag{79}$$
$$\phi_j^\varepsilon = \varphi_j(x) + \varepsilon \gamma_j (x, \mathcal{T}(\frac{x}{\varepsilon})\omega), \ \varphi_j \in C_0^\infty(G), \ \gamma_j \in C_0^\infty(G; \mathcal{D}(\Omega)), \ j = 1, \dots N. \tag{80}$$

Lemma 18. *Let us suppose the assumptions of Theorem 16 and convergences (65)-(69). Then any cluster point* $\{\mathbf{u}^0, P^0\}$ *satisfies (70).*

Proof. If we take ξ which is with support in $\mathcal{M}$, then passing to the two-scale limit immediately gives $\chi_{\mathcal{M}}(\omega)\mathbf{u}^0(x,\omega) = 0$. Next we take as test function $\xi^\varepsilon = \varepsilon \xi(x, \mathcal{T}(\frac{x}{\varepsilon}\omega))$, where ξ is given by (79) and $\phi_j = 0$, for each j, then passing to the two-scale limit gives

$$0 = \int_G \int_\Omega P^0 \operatorname{div}_\omega \ \xi(x,\omega) \ d\mathbb{P} dx.$$

Remark 2 and the ergodicity assumption on $\mathcal{F}$ yields $P^0(x,\omega) = P^0(x)$ a.e. on $G \times \Omega$. For a detailed computation see Wright [52, Lemma 2.4]. □

Lemma 19. *Let us suppose the assumptions of Theorem 16 and convergences (65)-(69). Then any cluster point* $\{\mathbf{u}^0, P^0, \{\Phi_j^0, \Phi_j^1\}_{j=1,\dots N}\}$ *satisfies incompressibility constraints (72)-(73) and the variational equation*

$$\int_{G\times\mathcal{F}} \nabla_\omega \mathbf{u}^0(x,\omega) : \nabla_\omega \xi \ dx d\mathbb{P} - \int_{G\times\mathcal{F}} P^0(x) \operatorname{div}_x \xi \ dx d\mathbb{P} +$$
$$\sum_{j=1}^N \int_{G\times\mathcal{F}} z_j n_j^0(\omega) \Big(- \xi(x,\omega) \cdot (\nabla_x \Phi_j^0(x) + \Phi_j^1(x,\omega)) + \mathbf{u}^0(x,\omega) \cdot (\nabla_x \varphi_j(x) + g_j(x,\omega)) \Big) \ dx d\mathbb{P}$$
$$+ \sum_{j=1}^N \frac{z_j^2}{\mathrm{Pe}_j} \int_{G\times\mathcal{F}} n_j^0(\omega) (\nabla_x \Phi_j^0(x) + \Phi_j^1(x,\omega)) \cdot (\nabla_x \varphi_j + g_j) \ dx d\mathbb{P} =$$
$$- \sum_{j=1}^N \frac{z_j^2}{\mathrm{Pe}_j} \int_{G\times\mathcal{F}} n_j^0(\omega) \mathbf{E}(x) \cdot (\nabla_x \varphi_j(x) + g_j(x,\omega)) \ dx d\mathbb{P} + \sum_{j=1}^N \int_{G\times\mathcal{F}} z_j n_j^0(\omega) \mathbf{E}^*(x) \cdot \xi \ dx d\mathbb{P}$$
$$- \int_{G\times\mathcal{F}} \mathbf{f}^*(x) \cdot \xi(x,\omega) \, dx d\mathbb{P}, \tag{81}$$

for any test functions ξ given by (79) and ϕ_j given by (80). Notice that $\nabla_\omega \gamma_j$ was replaced with the element from the corresponding closed subspace: $g_j \in L^2(G;X)$.

Proof. If we multiply div $\mathbf{u}^\varepsilon = 0$ by ξ^ε, integrate over $G_f^\varepsilon(\omega) \times \mathcal{F}$ and pass to the two-scale limit, incompressibility constraint (72) follows immediately.

The incompressibility constraint (73) follows analogously, but with a choice of test function $\mathbb{E}(\xi^\varepsilon)$ and $\phi_j^\varepsilon = 0$ for each j.

Using convergences (65)-(69) and by recalling that $n_\varepsilon^j = n_j^0(\mathcal{T}(x/\varepsilon)\omega)$ we pass to the two-scale limit in equation (78) without difficulty. □

The next step is to prove the well-posedness of variational equation (81), which by uniqueness of the limit automatically implies that the entire sequence $(\mathbf{u}^\varepsilon, P^\varepsilon, \{\Phi_j^\varepsilon\}_{1\le j\le N})$ converges.

Let the functional space for the velocity $\mathbf{u}^0$ be given by

$$V = \{\mathbf{z}^0(x,\omega) \in L^2\left(G; \mathcal{D}(\Omega)^n\right) \ \text{ satisfying (72)} - (73)\},$$

Lemma 20. *Let $\eta \in L_0^2(G)$. Then there exists $\Theta \in V$ such that*

$$\mathrm{div}_x \mathbb{E}\{\Theta\} = \eta \ \ in \ \ G, \quad \mathbb{E}\{\Theta\}\cdot\nu = 0 \ \ on \ \ \partial G. \tag{82}$$

Proof. Let $\mathcal{W}$ be the Hilbert space given by

$$\mathcal{W} = \{ \ \mathbf{z} \in \mathcal{D}(\Omega)^n \mid \mathrm{div}_\omega \mathbf{z} = 0 \ \text{ in } \ \mathcal{F} \quad \text{and} \quad \mathbf{z} = 0 \ \text{ on } \ \mathcal{M}\}. \tag{83}$$

We define the random variables $\mathbf{q}^i \in \mathcal{W}$, $i = 1, \ldots, n$ by

$$\int_{\mathcal{F}} \nabla_\omega \mathbf{q}^i : \nabla_\omega \psi \ d\mathbb{P} + \int_{\mathcal{F}} \mathbf{q}^i \cdot \psi \ d\mathbb{P} = \int_{\mathcal{F}} \psi_i \ d\mathbb{P}, \quad \forall \psi \in \mathcal{W}. \tag{84}$$

Then we have

$$\mathbb{E}\{q_i^j\} = \int_{\mathcal{F}} \nabla_\omega \mathbf{q}^i : \nabla_\omega \mathbf{q}^j \ d\mathbb{P} + \int_{\mathcal{F}} \mathbf{q}^i \cdot \mathbf{q}^j \ d\mathbb{P} = \mathbb{E}\{q_j^i\}$$

and for all $\lambda \in \mathbb{R}^n$

$$\sum_{i,j=1}^n \lambda_i \lambda_j \mathbb{E}\{q_i^j\} = \int_{\mathcal{F}} |\nabla_\omega (\sum_{i=1}^n \lambda_i \mathbf{q}^i)|^2 \ d\mathbb{P} + \int_{\mathcal{F}} |\sum_{i=1}^n \lambda_i \mathbf{q}^i|^2 \ d\mathbb{P}.$$

Hence the matrix $K_q = \left[\mathbb{E}\{q_i^j\}\right]_{i,j=1,\ldots,n}$ is symmetric positive definite.

Now we set $\Theta = \sum_{i=1}^n \mathbf{q}^i(\omega) \dfrac{\partial q}{\partial x_i}(x)$, where $q \in H^1(G)\mathbb{R}$ solves the problem

$$\mathrm{div}_x\{K_q \nabla_x q\} = \eta \ \text{ in } \ G, \quad K_q \nabla_x q \cdot \nu = 0 \ \text{ on } \ \partial G. \tag{85}$$

As $\mathbb{E}\{\Theta\} = K_q \nabla_x q$, the Lemma is proved.

We notice the analogy with Allaire [4, Section 3.1.2]. □

Proposition 21. *Problem (81) with incompressibility constraints (72) and (73) has a unique solution*

$$(\mathbf{u}^0, P^0, \{\Phi_j^0, \Phi_j^1\}_{j=1,\ldots,N}) \in V \times L_0^2(G) \times (H_0^1(G) \times L^2(G;X))^N.$$

Proof. We study variational problem (81) with $\xi \in V$, $\varphi_j \in H^1(G)/\mathbb{R}$ and with $\nabla_\omega \gamma_j$, $j = 1, \dots, N$, replaced by arbitrary element of $L^2(G; X)$. We notice that for $\xi \in V$, $\int_{G\times\mathcal{F}} P^0(x)\,\mathrm{div}_x \xi \; dx d\mathbb{P} = 0$. Hence we apply the Lax- Milgram lemma to prove the existence and uniqueness of $(\mathbf{u}^0, \{\Phi_j^0, \Phi_j^1\})$ in $V \times (H_0^1(G) \times L^2(G;X))^N$. The only point which requires to be checked is the coercivity of the bilinear form. We take $\xi = \mathbf{u}^0$, $\varphi_j = \Phi_j^0$ and $g_j = \Phi_j^1$ as the test functions in (81). Using the incompressibility constraints (73) and the anti-symmetry of the third integral in (81), we obtain the quadratic form

$$\int_{G\times\mathcal{F}} |\nabla_\omega \mathbf{u}^0(x,\omega)|^2 \; dx d\mathbb{P} + \sum_{j=1}^{N} \frac{z_j^2}{\mathrm{Pe}_j} \int_{G\times\mathcal{F}} n_j^0(\omega) |\nabla_x \Phi_j^0(x) + \Phi_j^1(x,\omega)|^2 \; dx d\mathbb{P}. \tag{86}$$

Recalling from Lemma 7 that $n_j^0(\omega) \geq C > 0$ in $\mathcal{F}$, it is easy to check that each term in the sum on the second line of (86) is bounded from below by

$$C \left(\int_G |\nabla_x \Phi_j^0(x)|^2 \; dx + \int_{G\times\mathcal{F}} |\Phi_j^1(x,y)|^2 \; dx d\mathbb{P} \right),$$

which proves the coerciveness of the bilinear form in the required space.

It remains to prove uniqueness of the pressure P^0. It is sufficient to prove that for the homogeneous data, $P^0 = 0$ in $L_0^2(G)$.

By the above result and using equation (81), we have

$$0 = \int_{G\times\mathcal{F}} P^0(x)\,\mathrm{div}_x \xi \; dx d\mathbb{P} = \int_G P^0(x) \mathbb{E}\{\mathrm{div}_x \; \xi\} \; dx.$$

Hence, by Lemma 20, P^0 is orthogonal to all elements of $L_0^2(G)$ and, as such, equal to zero. □

Remark 22. *In analogy with Allaire [6] (see also Allaire [4, Section 3.1.2]), the space V is orthogonal in $L^2\ (G; \mathcal{D}(\Omega)^n)$ to the space of gradients of the form $\nabla_x q(x) + \nabla_\omega q_1(x,\omega)$ with $q(x) \in H^1(G)/\mathbb{R}$ and $q_1(x,\omega) \in L^2\,(G \times \mathcal{F})$.*

Proof of Theorem 16:

By virtue of the a priori estimates in Lemmas 14 and 15, and using the compactness of Proposition 4 and Lemma 18, there exist a subsequence, still denoted by ε, and limits $(\mathbf{u}^0, P^0, \{\Phi_j^0, \Phi_j^1\}_{1\leq j\leq N}) \in V \times L_0^2(G) \times (H_0^1(G) \times L^2(G;X))^N$ such that the convergences in Theorem 16 are satisfied. Using Lemma 19 we pass to the two-scale limit in (78) we get that the limit $(\mathbf{u}^0, P^0, \{\Phi_j^0, \Phi_j^1\}_{1\leq j\leq N})$ satisfy the two-scale variational formulation (81).

According to Proposition 21, the limit system has a unique solution and the whole sequence converges.

It remains to recover the two-scale homogenized system (71)-(77) from the variational formulation (81). In order to get the Stokes equations (71) we choose $\varphi_j = 0$ and $\gamma_j = 0$ in (81). Using Corollary 2.7 from [52] we deduce the existence of a pressure field $p^1(x,\omega)$ in $L^2(G\times\Omega)$ such that

$$-\Delta_\omega \mathbf{u}^0 + \nabla_\omega p^1 = -\nabla_x P^0 - \mathbf{f}^* + \sum_{j=1}^{N} z_j n_j^0 (\nabla_x \Phi_j^0 + \Phi_j^1 + \mathbf{E}^*) \;\; \text{in} \;\; G \times \mathcal{F}.$$

The incompressibility constraints (72) and (73) are simple consequences of passing to the two-scale limit in the equation $\mathrm{div}\mathbf{u}^\varepsilon = 0$ in $G_f^\varepsilon(\omega)$. To obtain the cell convection-diffusion equation (74) we now choose $\xi = 0$ and $\varphi_j = 0$ in (81) while the macroscopic convection-diffusion equation (77) is obtained by taking $\xi = 0$ and $\gamma_j = 0$. This finishes the proof of Theorem 16.

4 Scale separation and Onsager's relations

The limit problem obtained in Theorem 16 contains the two-scales and a large set if unknowns. Furthermore, it is a system of PDEs in a random geometry. For the practical purposes (which are overall the computational ones), it is important to extract from (71)-(77) the macroscopic homogenized problem, if possible. It requires to separate the slow (x-) and fast (ω-) scale. This was undertaken in Looker & Carnie [32] for periodic porous media. In Allaire et al [7] their analysis was simplified and Onsager properties for the effective fluxes were established. In addition, the scale separation results allowed establishing further qualitative properties of the effective coefficients and eliminating the fast scale. In this article, our goal is to generalize results from Allaire et al [7] to stochastic porous media.

The main idea is identifying in two-scale homogenized problem (71)-(77) the two different sets of macroscopic fluxes, namely $(\nabla_x P^0(x) + \mathbf{f}^*(x))$ and $\{\nabla_x \Phi_j^0(x) + \mathbf{E}^*(x)\}_{1\le j\le N}$. Therefore, we introduce two families of random geometry problems, indexed by $k \in \{1, ..., n\}$ for each component of these fluxes. We denote by $\{\mathbf{e}^k\}_{1\le k\le n}$ the canonical basis of $\mathbb{R}^n$.

Recalling the definition of the space $\mathcal{W}$ in (83) and the space X that is given just after Definition 3, the first family of random geometry problem, corresponding to the macroscopic pressure gradient, is

$$\text{Find } \{\mathbf{v}^{0,k}, \Theta_j^{0,k}\} \in \mathcal{W}\times X,\ j=1,\dots,N, \quad \text{such that}$$

$$\int_{\mathcal{F}} \nabla_\omega \mathbf{v}^{0,k}(\omega) : \nabla_\omega \xi(\omega)\, d\mathbb{P} - \sum_{j=1}^{N} \int_{\mathcal{F}} z_j n_j^0(\omega)\Theta_j^{0,k}(\omega)\cdot \xi(\omega)\, d\mathbb{P} = \int_{\mathcal{F}} \mathbf{e}^k \cdot \xi(\omega)\, d\mathbb{P},$$

$$\int_{\mathcal{F}} n_j^0(\omega)\Theta_j^{0,k}(\omega)\cdot \frac{z_j}{\mathrm{Pe}_j}\zeta_j(\omega)\, d\mathbb{P} + \int_{\mathcal{F}} n_j^0(\omega)\mathbf{v}^{0,k}(\omega)\cdot\zeta_j(\omega)\, d\mathbb{P} = 0 \tag{87}$$

$$\text{for all } \xi\in\mathcal{W} \text{ and } \zeta_j \in X,\ j=1,\dots,N.$$

The second family of random geometry problem, corresponding to the macroscopic diffusive flux, is for each species $i \in \{1, ..., N\}$

$$\text{Find } \{\mathbf{v}^{i,k}, \Theta_j^{i,k}\} \in \mathcal{W}\times X,\ j=1,\dots,N, \quad \text{such that}$$

$$\int_{\mathcal{F}} \nabla_\omega \mathbf{v}^{i,k}(\omega) : \nabla_\omega \xi(\omega)\, d\mathbb{P} - \sum_{j=1}^{N} \int_{\mathcal{F}} z_j n_j^0(\omega)\Theta_j^{i,k}(\omega)\cdot \xi(\omega)\, d\mathbb{P} = z_i \int_{\mathcal{F}} n_i^0(\omega)\mathbf{e}^k \cdot \xi(\omega)\, d\mathbb{P},$$

$$-\int_{\mathcal{F}} n_j^0(\omega)\Big(\Theta_j^{i,k}(\omega)\cdot\frac{z_j}{\mathrm{Pe}_j}\zeta_j(\omega) + \mathbf{v}^{i,k}(\omega)\cdot\zeta_j(\omega)\Big)\, d\mathbb{P} = -\delta_{ij}\int_{\mathcal{F}} n_j^0(\omega)\mathbf{e}^k\cdot\frac{z_j}{\mathrm{Pe}_i}\zeta_j(\omega)\, d\mathbb{P}, \tag{88}$$

$$\text{for all } \xi\in\mathcal{W} \text{ and } \zeta_j \in X, j=1,\dots,N;$$

here δ_{ij} is the Kronecker symbol. These two problems can be rewritten as follows:

$$\text{Find } \{\mathbf{v}^{0,k}, \Theta_j^{0,k}\} \in \mathcal{W}\times X,\ j=1,\dots,N, \quad \text{such that}$$

$$\int_{\mathcal{F}} \nabla_\omega \mathbf{v}^{0,k}(\omega) : \nabla_\omega \xi(\omega)\, d\mathbb{P} - \sum_{j=1}^{N} \int_{\mathcal{F}} z_j n_j^0(\omega)\Theta_j^{0,k}(\omega)\cdot (\xi(\omega) - \frac{z_j}{\mathrm{Pe}_j}\zeta_j(\omega))\, d\mathbb{P}$$

$$+\sum_{j=1}^{N} z_j \int_{\mathcal{F}} n_j^0(\omega)\mathbf{v}^{0,k}(\omega)\cdot\zeta_j(\omega)\, d\mathbb{P} = \int_{\mathcal{F}} \mathbf{e}^k\cdot\xi(\omega)\, d\mathbb{P}, \quad \forall \xi\in\mathcal{W},\, \zeta_j\in X, j=1,\dots,N. \tag{89}$$

and

$$\text{Find } \{\mathbf{v}^{i,k}, \Theta_j^{i,k}\} \in \mathcal{W} \times X,\ j = 1, \dots, N, \quad \text{such that}$$

$$\int_{\mathcal{F}} \nabla_\omega \mathbf{v}^{i,k}(\omega) : \nabla_\omega \xi(\omega)\ d\mathbb{P} - \sum_{j=1}^{N} \int_{\mathcal{F}} z_j n_j^0(\omega) \Big(\Theta_j^{i,k}(\omega) \cdot (\xi(\omega) - \frac{z_j}{\text{Pe}_j} \zeta_j(\omega)) - \mathbf{v}^{i,k}(\omega) \cdot \zeta_j(\omega) \Big)\ d\mathbb{P}$$
$$= z_i \int_{\mathcal{F}} n_i^0(\omega) \mathbf{e}^k \cdot (\xi(\omega) - \frac{z_i}{\text{Pe}_i} \zeta_i(\omega))\ d\mathbb{P}, \quad \forall \xi \in \mathcal{W},\ \zeta_j \in X, j = 1, \dots, N. \tag{90}$$

Lemma 23. *Problems (89) and (90) admit a unique solution.*

Then, denoting the components of $\mathbf{E}^*$ by E_k^* we can decompose the solution of (71)-(77) as

$$\mathbf{u}^0(x, \omega) = \sum_{k=1}^{n} \left(-\mathbf{v}^{0,k}(\omega) \left(\frac{\partial P^0}{\partial x_k} + f_k^* \right)(x) + \sum_{i=1}^{N} \mathbf{v}^{i,k}(\omega) \left(E_k^* + \frac{\partial \Phi_i^0}{\partial x_k} \right)(x) \right) \tag{91}$$

$$\Phi_j^1(x, \omega) = \sum_{k=1}^{n} \left(-\Theta_j^{0,k}(\omega) \left(\frac{\partial P^0}{\partial x_k} + f_k^* \right)(x) + \sum_{i=1}^{N} \Theta_j^{i,k}(\omega) \left(E_k^* + \frac{\partial \Phi_i^0}{\partial x_k} \right)(x) \right). \tag{92}$$

We average (91)-(92) in order to get a purely macroscopic homogenized problem. We introduce the the total electrochemical potential μ_j^ε of the linearized system as

$$\mu_j^\varepsilon = -z_j (\Phi_j^\varepsilon + \Psi^{\text{ext},*})$$

and the ionic flux of the jth species

$$\mathbf{j}_j^\varepsilon = \frac{z_j}{\text{Pe}_j} n_j^\varepsilon \left(\nabla \Phi_j^\varepsilon + \mathbf{E}^* + \frac{\text{Pe}_j}{z_j} \mathbf{u}^\varepsilon \right).$$

The corresponding homogenized quantities are defined as

$$\mu_j(x) = -z_j (\Phi_j^0(x) + \Psi^{\text{ext},*}(x)),$$
$$\mathbf{j}_j(x) = \frac{z_j}{\text{Pe}_j} \mathbb{E} \Big\{ n_j^0(\omega) (\nabla_x \Phi_j^0(x) + \mathbf{E}^* + \Phi_j^1(x, \omega) + \frac{\text{Pe}_j}{z_j} \mathbf{u}^0(x, \omega)) \Big\}, \quad \mathbf{u}(x) = \mathbb{E}\{\mathbf{u}^0\}.$$

From (91)-(92) we deduce the homogenized or upscaled equations for the above effective fields.

Proposition 24. *Introducing the flux $\mathcal{J}(x) = (\mathbf{u}, \{\mathbf{j}_j\}_{1 \le j \le N})$ and the gradient $\mathcal{F}(x) = (\nabla_x P^0, \{\nabla_x \mu_j\}_{1 \le j \le N})$, the macroscopic equations are*

$$\text{div}_x \mathcal{J} = 0 \quad \text{in } G, \tag{93}$$
$$\mathcal{J} = -\mathcal{B}\mathcal{F} - \mathcal{B}(\mathbf{f}^*, \{0\}) \tag{94}$$

with a symmetric positive definite $\mathcal{B}$, defined by

$$\mathcal{B} = \begin{pmatrix} \mathbb{K} & \frac{\mathbb{J}_1}{z_1} & \dots & \frac{\mathbb{J}_N}{z_N} \\ \mathbb{L}_1 & \frac{\mathbb{D}_{11}}{z_1} & \dots & \frac{\mathbb{D}_{1N}}{z_N} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{L}_N & \frac{\mathbb{D}_{N1}}{z_1} & \dots & \frac{\mathbb{D}_{NN}}{z_N} \end{pmatrix}, \tag{95}$$

and complemented with the boundary conditions for P^0 and $\{\Phi_j^0\}_{1\le j\le N}$. The matrices $\mathbb{J}_i$, $\mathbb{K}$, $\mathbb{D}_{ji}$ and $\mathbb{L}_j$ are defined by their entries

$$\{\mathbb{J}_i\}_{lk} = \mathbb{E}\{\mathbf{v}^{i,k}\cdot\mathbf{e}^l\}, \quad \{\mathbb{K}\}_{lk} = \mathbb{E}\{\mathbf{v}^{0,k}\cdot\mathbf{e}^l\},$$
$$\{\mathbb{D}_{ji}\}_{lk} = \mathbb{E}\{n_j^0(\mathbf{v}^{i,k} + \frac{z_j}{\mathrm{Pe}_j}\left(\delta_{ij}\mathbf{e}^k + \Theta_j^{i,k}\right))\cdot\mathbf{e}^l\}, \quad \{\mathbb{L}_j\}_{lk} = \mathbb{E}\{n_j^0(\mathbf{v}^{0,k} + \frac{z_j}{\mathrm{Pe}_j}\Theta_j^{0,k})\cdot\mathbf{e}^l\}.$$

Remark 25. *The tensor $\mathbb{K}$ is called permeability tensor, $\mathbb{D}_{ji}$ are the electrodiffusion tensors. The symmetry of the tensor $\mathcal{B}$ is equivalent to the famous Onsager's reciprocal relations. In the periodic case the symmetry of $\mathcal{B}$ was proved in [32] and its positive definiteness in [7]. It is essential in order to state that (93)-(94) is an elliptic system which admits a unique solution.*

Proof of Proposition 24. The relation in (93) is an immediate consequence of (73) and (77). Taking the expectation on the left- and right-hand sides of equalities (91) and (92) and considering the definitions of the homogenized functions $\{\mu_j\}$ and $\{\mathbf{j}_j\}$ and of the entries of the matrix $\mathcal{B}$, after elementary computations we arrive at (94).

In order to justify positive definiteness of $\mathcal{B}$ we fix an arbitrary vector $\eta = \left(\eta^0, \eta^1, \dots, \eta^N\right)$, $\eta^j \in \mathbb{R}^d$, and denote by $\mathbf{v}^\eta$ and Θ_j^η the following functions:

$$\mathbf{v}^\eta = \sum_{l=1}^d \Big\{\eta_l^0 \mathbf{v}^{0,l}(\omega) + \sum_{i=1}^N \eta_l^i \mathbf{v}^{i,l}(\omega)\Big\},$$

$$\Theta_j^\eta = \sum_{l=1}^d \Big\{\eta_l^0 \Theta_j^{0,l}(\omega) + \sum_{i=1}^N \eta_l^i \Theta_j^{i,l}(\omega)\Big\}.$$

From (89) we derive that

$$\{\mathbf{v}^\eta, \Theta_j^\eta\} \in \mathcal{W}\times X, \ j = 1,\dots,N,$$

and

$$\begin{aligned}
&\int_{\mathcal{F}} \nabla_\omega \mathbf{v}^\eta(\omega) : \nabla_\omega \xi(\omega)\ d\mathbb{P} - \sum_{j=1}^N \int_{\mathcal{F}} z_j n_j^0(\omega)\Theta_j^\eta(\omega)\cdot(\xi(\omega) - \frac{z_j}{\mathrm{Pe_j}}\zeta_j(\omega))\ d\mathbb{P} \\
&+ \sum_{j=1}^N z_j \int_{\mathcal{F}} n_j^0(\omega)\mathbf{v}^\eta(\omega)\cdot\zeta_j(\omega)\ d\mathbb{P} + \sum_{j=1}^N \int_{\mathcal{F}} z_j n_j^0(\omega)\big(\eta^j\cdot\xi(\omega) - \eta^j\cdot\frac{1}{\mathrm{Pe}_j}\zeta_(\omega)\big)\, d\mathbb{P} \\
&= \int_{\mathcal{F}} \eta^0\cdot\xi(\omega)\ d\mathbb{P}, \quad \forall\xi\in\mathcal{W},\ \zeta_j\in X, j = 1,\dots,N.
\end{aligned} \tag{96}$$

Substituting $\mathbf{v}^\eta$ for ξ and Θ_j^η for ζ_j in this integral relation yields

$$\begin{aligned}
&\int_{\mathcal{F}} |\nabla_\omega \mathbf{v}^\eta(\omega)|^2\ d\mathbb{P} + \sum_{j=1}^N \int_{\mathcal{F}} \frac{z_j^2}{\mathrm{Pe}_j} n_j^0(\omega)|\Theta_j^\eta(\omega)|^2\, d\mathbb{P} \\
&= \int_{\mathcal{F}} \eta^0\cdot\mathbf{v}^\eta(\omega)\, d\mathbb{P} + \sum_{j=1}^N \int_{\mathcal{F}} n_j^0(\omega)\eta^j\cdot\Big(z_j\mathbf{v}^\eta(\omega) - \frac{z_i^2}{\mathrm{Pe}_j}\Theta_j^\eta(\omega)\Big)\, d\mathbb{P};
\end{aligned}$$

here the quadratic form on the left-hand side have been obtained in the same way as the quadratic form in (86). This implies the following relation:

$$\int_{\mathcal{F}} |\nabla_\omega \mathbf{v}^\eta(\omega)|^2 \, d\mathbb{P} + \sum_{j=1}^N \int_{\mathcal{F}} \frac{z_j^2}{\mathrm{Pe}_j} n_j^0(\omega) |\Theta_j^\eta(\omega) + \eta^j|^2 \, d\mathbb{P}$$

$$= \int_{\mathcal{F}} \eta^0 \cdot \mathbf{v}^\eta(\omega) \, d\mathbb{P} + \sum_{j=1}^N \int_{\mathcal{F}} n_j^0(\omega) \eta^j \cdot \Big(z_j \mathbf{v}^\eta(\omega) + \frac{z_i^2}{\mathrm{Pe}_j} \big(\Theta_j^\eta(\omega) + \eta^j \big) \Big) \, d\mathbb{P},$$

$$= \mathbb{K}\eta^0 \cdot \eta^0 + \sum_{j=1}^N \mathbb{J}_j \eta^j \cdot \eta^0 + \sum_{i,\, j=1}^N \mathbb{J}_j \eta^j \cdot \eta^0 z_i \eta^i \cdot \mathbb{D}_{ij} \eta^j + \sum_{j=1}^N z_j \eta^j \cdot \mathbb{L}_j \eta^0$$

$$= \mathcal{B}\big(\eta^0,\, z_j \eta^j\big)^T \cdot \big(\eta^0,\, z_j \eta^j\big)^T,$$

which in turn yields the desired positive definiteness.

In order to show that $\mathcal{B}$ is symmetric we consider $\mathbf{v}^{\check\eta}$ and $\Theta_j^{\check\eta}$ with another set of vectors $\check\eta^0$, $\{\check\eta^j\}_{j=1}^N$ in $\mathbb{R}^d$. Then we substitute $\mathbf{v}^{\check\eta}$ and 0 for ξ and ζ_j, respectively, in (96). In a similar integral relation corresponding to $\check\eta^0$, $\{\check\eta^j\}_{j=1}^N$ we substitute 0 for ξ and Θ_j^λ for ζ_j. Summing up the resulting relations we get

$$\int_{\mathcal{F}} \Big(\nabla_\omega \mathbf{v}^\eta \cdot \nabla_\omega \mathbf{v}^{\check\eta} + \sum_{j=1}^N n_j^0(\omega) \Theta_j^\eta(\omega) \cdot \Theta_j^{\check\eta}(\omega) \Big) \, d\mathbb{P}$$

$$= \int_{\mathcal{F}} \eta^0 \cdot \nabla_\omega \mathbf{v}^{\check\eta} \, d\mathbb{P} + \sum_{j=1}^N \int_{\mathcal{F}} z_j n_j^0(\omega) \big(\eta^j \cdot \mathbf{v}^{\check\eta} - \frac{z_j}{\mathrm{Pe}_j} \check\eta \cdot \Theta_j^\eta(\omega) \big) \, d\mathbb{P}.$$

Exchanging η and $\check\eta$ and considering the symmetry of the integral on the left-hand side we obtain

$$\int_{\mathcal{F}} \eta^0 \cdot \nabla_\omega \mathbf{v}^{\check\eta} \, d\mathbb{P} + \sum_{j=1}^N \int_{\mathcal{F}} z_j n_j^0(\omega) \big(\eta^j \cdot \mathbf{v}^{\check\eta} + \frac{z_j}{\mathrm{Pe}_j} \eta \cdot \Theta_j^{\check\eta}(\omega) \big) \, d\mathbb{P}$$

$$= \int_{\mathcal{F}} \check\eta^0 \cdot \nabla_\omega \mathbf{v}^\eta \, d\mathbb{P} + \sum_{j=1}^N \int_{\mathcal{F}} z_j n_j^0(\omega) \big(\check\eta^j \cdot \mathbf{v}^\eta + \frac{z_j}{\mathrm{Pe}_j} \check\eta \cdot \Theta_j^\eta(\omega) \big) \, d\mathbb{P}.$$

This implies the following equality

$$\eta^0 \cdot \mathbb{K}\check\eta^0 + \sum_{j=1}^N \eta^0 \cdot \mathbb{J}_j \check\eta^j + \sum_{i=1}^N z_i \eta^i \cdot \Big(\mathbb{L}_i \check\eta^0 + \sum_{j=1}^N \mathbb{D}_{ij} \check\eta^j \Big)$$

$$= \check\eta^0 \cdot \mathbb{K}\eta^0 + \sum_{j=1}^N \check\eta^0 \cdot \mathbb{J}_j \eta^j + \sum_{i=1}^N z_i \check\eta^i \cdot \Big(\mathbb{L}_i \eta^0 + \sum_{j=1}^N \mathbb{D}_{ij} \eta^j \Big).$$

Therefore,

$$\mathcal{B}\big(\eta^0, z_1\eta^1, \ldots, z_N\eta^N\big)^t \cdot \big(\check\eta^0, z_1\check\eta^1, \ldots, z_N\check\eta^N\big)^t = \mathcal{B}\big(\check\eta^0, z_1\check\eta^1, \ldots, z_N\check\eta^N\big)^t \cdot \big(\eta^0, z_1\eta^1, \ldots, z_N\eta^N\big)^t$$

This yields the desired symmetry of the matrix $\mathcal{B}$. □

Corollary 26. *The homogenized equations in Proposition 24 form a symmetric elliptic system*

$$\mathrm{div}_x\{\mathbb{K}(\nabla_x P^0 + \mathbf{f}^*) + \sum_{i=1}^N \mathbb{J}_i(\nabla_x \Phi_i^0 + \mathbf{E}^*)\} = 0 \ \ in \ \ G,$$

$$\mathrm{div}_x\{\mathbb{L}_j(\nabla_x P^0 + \mathbf{f}^*) + \sum_{i=1}^N \mathbb{D}_{ji}(\nabla_x \Phi_i^0 + \mathbf{E}^*)\} = 0 \ \ in \ \ G,$$

with boundary conditions (73). In particular, $\mathbf{E}^*$ *and* $\mathbf{f}^* \in H^1(G)^n$*, with* $\sigma \in C^1(\partial F(\omega))$ *bounded, imply that the pressure field* $P^0 \in H^2(G)$.

5 Strong convergence and correctors

Besides the stochastic two-scale convergence of the microscopic fluxes and pressures to the effective ones, we also prove convergences of the energies.

First, we recall that the L^2-norm squared is lower semi continuous with respect to the stochastic two-scale convergence. In our situation, where after Corollary 26 the limit functions are smooth, it can be seen through a simple direct argument. First, we notice that the formulas of scale separations (91) and (92) imply that the functions $\mathbf{u}^0(x, \mathcal{T}(\frac{x}{\varepsilon})\omega)$ and $\Phi_j^1(x, \mathcal{T}(\frac{x}{\varepsilon})\omega)$ are admissible. Hence from

$$\varepsilon^2 \int_\Omega \int_G |\nabla \mathbf{u}^\varepsilon|^2 \, dx d\mathbb{P} \geq \varepsilon^2 \int_\Omega \int_G |\nabla \mathbf{u}^0(x, \mathcal{T}(\frac{x}{\varepsilon})\omega)|^2 \, dx d\mathbb{P} +$$
$$2 \int_\Omega \int_G \varepsilon \nabla \mathbf{u}^0(x, \mathcal{T}(\frac{x}{\varepsilon})\omega) \cdot \varepsilon \nabla \big(\mathbf{u}^\varepsilon - \mathbf{u}^0(x, \mathcal{T}(\frac{x}{\varepsilon})\omega) \big) \, dx d\mathbb{P}$$

and passing to the limit $\varepsilon \to 0$ gives

$$\lim_{\varepsilon \to 0} \varepsilon^2 \int_\Omega \int_G |\nabla \mathbf{u}^\varepsilon|^2 \, dx d\mathbb{P} \; \geq \int_{\Omega \times \mathcal{F}} |\nabla_\omega \mathbf{u}^0(x, \omega)|^2 \, d\mathbb{P} dx.$$

Similarly,

$$\lim_{\varepsilon \to 0} \int_\Omega \int_{G_f^\varepsilon(\omega)} n_\varepsilon^j |\nabla \Phi_j^\varepsilon|^2 \, dx d\mathbb{P} \geq \int_{\Omega \times \mathcal{F}} n_j^0(\omega) |\nabla_x \Phi_j^0(x) + \Phi_j^1(x, \omega)|^2 \, dx d\mathbb{P}.$$

A stronger result is

Proposition 27. *We have the for* $j = 1, \dots, N$,

$$\lim_{\varepsilon \to 0} \varepsilon^2 \int_\Omega \int_G |\nabla \mathbf{u}^\varepsilon|^2 \, dx d\mathbb{P} \; = \int_{\Omega \times \mathcal{F}} |\nabla_\omega \mathbf{u}^0(x, \omega)|^2 \, d\mathbb{P} dx, \tag{97}$$

$$\lim_{\varepsilon \to 0} \int_\Omega \int_{G_f^\varepsilon(\omega)} n_\varepsilon^j |\nabla \Phi_j^\varepsilon|^2 \, dx d\mathbb{P} = \int_{\Omega \times \mathcal{F}} n_j^0(\omega) |\nabla_x \Phi_j^0(x) + \Phi_j^1(x, \omega)|^2 \, dx d\mathbb{P}. \tag{98}$$

Proof. We follow the proof from the periodic case (Allaire [5, Theorem 2.6] and Allaire et al [7, Sec 5]). We start from the energy equality corresponding to the variational equation (59):

$$\varepsilon^2 \int_G |\nabla \mathbf{u}^\varepsilon|^2 \, dx + \sum_{j=1}^N \frac{z_j^2}{\mathrm{Pe}_j} \int_{G_f^\varepsilon(\omega)} n_\varepsilon^j |\nabla \Phi_j^\varepsilon|^2 \, dx = - \sum_{j=1}^N \frac{z_j^2}{\mathrm{Pe}_j} \int_{G_f^\varepsilon(\omega)} n_\varepsilon^j \mathbf{E}^* \cdot \nabla \Phi_j^\varepsilon \, dx +$$
$$\sum_{j=1}^N z_j \int_{G_f^\varepsilon(\omega)} n_j^\varepsilon \mathbf{E}^* \cdot \mathbf{u}^\varepsilon \, dx - \int_{G_f^\varepsilon(\omega)} \mathbf{f}^* \cdot \mathbf{u}^\varepsilon \, dx. \tag{99}$$

For the homogenized variational problem (81) the energy equality reads

$$\int_{\Omega \times \mathcal{F}} |\nabla_\omega \mathbf{u}^0|^2 \, dx d\mathbb{P} + \sum_{j=1}^N \frac{z_j^2}{\mathrm{Pe}_j} \int_{\Omega \times \mathcal{F}} n_j^0(\omega) |\nabla_x \Phi_j^0(x) + \Phi_j^1(x, \omega)|^2 \, dx d\mathbb{P} = - \sum_{j=1}^N \frac{z_j^2}{\mathrm{Pe}_j} \int_{\Omega \times \mathcal{F}} n_j^0(\omega) \mathbf{E}^* \cdot$$
$$\cdot (\nabla_x \Phi_j^0(x) + \Phi_j^1(x, \omega)) \, dx d\mathbb{P} + \sum_{j=1}^N z_j \int_{\Omega \times \mathcal{F}} n_j^0(\omega) \mathbf{E}^* \cdot \mathbf{u}^0(x, \omega) \, dx d\mathbb{P} - \int_{\Omega \times \mathcal{F}} \mathbf{f}^* \cdot \mathbf{u}^0(x, \omega) \, dx d\omega. \tag{100}$$

In (99) we observe the convergence of the right-hand side to the right-hand side of (100). Next we use the lower semicontinuity, with respect to the stochastic two-scale convergence, of the left-hand side and the equality (100) to conclude (97)-(98). □

Theorem 28. *Under the assumptions of Section 3, the following strong two-scale convergences hold*

$$\lim_{\varepsilon\to 0}\int_\Omega\int_G \left|\mathbf{u}^\varepsilon(x)-\mathbf{u}^0(x,\mathcal{T}(\frac{x}{\varepsilon})\omega))\right|^2 dxd\mathbb{P}=0 \tag{101}$$

and

$$\lim_{\varepsilon\to 0}\int_\Omega\int_{G_f^\varepsilon(\omega)} \left|\nabla\left(\Phi_j^\varepsilon(x)-\Phi_j^0(x)\right)-\Phi_j^1(x,\mathcal{T}(\frac{x}{\varepsilon})\omega)\right|^2 dxd\mathbb{P}=0. \tag{102}$$

Proof. We have

$$\int_\Omega\int_G \varepsilon^2|\nabla[\mathbf{u}^0(x,\mathcal{T}(\frac{x}{\varepsilon})\omega)]-\nabla\mathbf{u}^\varepsilon(x)|^2\,dxd\mathbb{P}=\int_\Omega\int_G |[\nabla_y\mathbf{u}^0](x,\mathcal{T}(\frac{x}{\varepsilon})\omega)|^2\,dxd\mathbb{P}$$
$$+\int_\Omega\int_G \varepsilon^2|\nabla\mathbf{u}^\varepsilon(x)|^2\,dxd\mathbb{P}-2\int_\Omega\int_G \varepsilon[\nabla_y\mathbf{u}^0](x,\mathcal{T}(\frac{x}{\varepsilon})\omega)\cdot\nabla\mathbf{u}^\varepsilon(x)\,dxd\mathbb{P}+O(\varepsilon). \tag{103}$$

Using Proposition 27 for the second term in the right-hand side of (103) and passing to the two-scale limit in the third term in the right-hand side of (103), we deduce

$$\lim_{\varepsilon\to 0}\int_\Omega\int_G \varepsilon^2\left|\nabla\left(\mathbf{u}^\varepsilon(x)-\mathbf{u}^0(x,\mathcal{T}(\frac{x}{\varepsilon})\omega)\right)\right|^2 dxd\mathbb{P}=0$$

Now application of Poincaré inequality (60) in $G_f^\varepsilon(\omega)$ yields (101).

On the other hand, by virtue of Theorem 10, n_j^ε is uniformly positive, i.e., there exists a constant $C>0$, which does not depend on ε, such that

$$\int_\Omega\int_{G_f^\varepsilon(\omega)}\int\left|\nabla\left(\Phi_j^\varepsilon(x)-\Phi_j^0(x)\right)-\Phi_j^1(x,\mathcal{T}(\frac{x}{\varepsilon})\omega)\right|^2 dxd\mathbb{P}\le C$$
$$\int_\Omega\int_{G_f^\varepsilon(\omega)} n_j^\varepsilon\left|\nabla\left(\Phi_j^\varepsilon(x)-\Phi_j^0(x)\right)-\Phi_j^1(x,\mathcal{T}(\frac{x}{\varepsilon})\omega)\right|^2 dxd\mathbb{P}. \tag{104}$$

Developing the right-hand side of (104) as we just did for the velocity and using the fact that $n_\varepsilon^j(x)=n_j^0(\mathcal{T}(\frac{x}{\varepsilon})\omega)$ is a two-scale test function, we easily deduce (102). □

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