On a parametric estimation for a convolution of exponential densities

Alexander Andronov, Nadezda Spiridovska and Diana Santalova

Abstract Broad application of the continuous time Markov chain is caused by memoryless property of exponential distribution. An employment of non-exponential distributions leads to remarkable analytical difficulties. The usage of arbitrary nonnegative density approximation by a convolution of exponential densities is a way of considerable interest. Two aspects of the problem solution are considered. Firstly, the parametrical estimation of the convolution on the basis of given statistical data. Secondly, an approximation of fixed non-negative density. An approximation and estimation are performed by the method of the moments, maximum likelihood method, and fitting of a density. An empirical analysis of different approaches has been performed with the use of simulation. The efficiency of the considered approach is illustrated by the task of the queuing theory.

1 Introduction

Broad application of the continuous time Markov chain is caused by exponential distribution properties, see for example, [3], [4] and [6]. The employment of non-exponential distributions leads to considerable analytical difficulties. The usage of arbitrary nonnegative density approximation by a convolution of exponential densities is a way of considerable interest. Such approach is considered in this manuscript. More general approach consists in application so called phase type (PH) distribution,

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see [5]. Unfortunately, parameter's estimation of PH type distribution is a difficult statistical problem. We confine ourselves to a sum of exponential distributed random variables. Two aspects of the problem are considered. Firstly, the parametrical estimation of the convolution on the basis of given statistical data. Secondly, an approximation of fixed non-negative density. Different approaches to such approximation and estimation are under consideration: maximum likelihood method, the method of the moments, fitting of densities. An empirical analysis of different approach has been performed using the simulation. The efficiency of the considered approach is illustrated by the task of the queuing theory.

2 Convolution of the exponential densities

Let $Z_1, \ldots . Z_m$ be independet random variables having exponential distribution with parameters $\lambda = (\lambda_1, \ldots, \lambda_m)$, where all components $\{\lambda_i\}$ are different. A distribution of their sum $S = Z_1 + \cdots + Z_m$ has the following density and cumulative distribution function for $z \geq 0$, see [2]:

$$f(z; \lambda) = \left(\prod_{i=1}^{m} \lambda_i\right) \frac{\sum_{i=1}^{m} exp(-\lambda_i z)}{\prod_{j=1, j\neq i}^{m} (\lambda_j - \lambda_i)}, \quad (1)$$

$$f(z; \lambda) = \left(\prod_{i=1}^{m} \lambda_i\right) \frac{\sum_{i=1}^{m} (1 - exp(-\lambda_i z))}{\lambda_i \prod_{j=1, j\neq i}^{m} (\lambda_j - \lambda_i)}, \quad (2)$$

Moment of order r is calculated by formula

$$\mu_r(\lambda) = \left(\prod_{i=1}^{m} \lambda_i\right) \frac{\sum_{i=1}^{m} r!}{\lambda_i^{r+1} \prod_{j=1, j\neq i}^{m} (\lambda_j - \lambda_i)}. \quad (3)$$

3 ML-estimation of the parameters

Firstly, we consider maximum likelihood method of parameter's estimation for density (1) when a sample corresponds this distribution, for example, see [8]. For this aim it is necessary to calculate partial derivative with respect to estimated parameters $\{\lambda_i\}$:

$$\frac{d}{d\lambda_k} f(z;\lambda) = \frac{1}{\lambda_k} f(z;\lambda) +$$
$$\left(\prod_{i=1}^{m} \lambda_i\right) exp(-\lambda_k z) \left(\frac{-z}{\prod_{j=1, j\neq k}^{m} (\lambda_j - \lambda_k)} + \frac{d}{d\lambda_k} \frac{1}{\prod_{j=1, j\neq k}^{m} (\lambda_j - \lambda_k)} \right) +$$
$$\left(\prod_{i=1}^{m} \lambda_i\right) \sum_{i=1, i\neq k}^{m} exp(-\lambda_i z) \frac{d}{d\lambda_k} \frac{1}{\prod_{j=1, j\neq i}^{m} (\lambda_j - \lambda_i)}.$$

Further:

$$\frac{d}{d\lambda_k} \frac{1}{\prod_{j=1, j\neq i}^{m} (\lambda_j - \lambda_i)} = \frac{-1}{(\lambda_k - \lambda_i)} \frac{1}{\prod_{j=1, j\neq i}^{m} (\lambda_j - \lambda_i)},$$

$$\frac{d}{d\lambda_k} \frac{1}{\prod_{j=1, j\neq k}^{m} (\lambda_j - \lambda_k)} = \frac{d}{d\lambda_k} \prod_{j=1, j\neq k}^{m} \frac{1}{(\lambda_j - \lambda_k)} =$$
$$\left(\prod_{j=1, j\neq k}^{m} \frac{1}{(\lambda_j - \lambda_k)} \right) \sum_{j=1, j\neq k}^{m} \frac{1}{(\lambda_j - \lambda_k)}. \tag{4}$$

Finally

$$\frac{d}{d\lambda_k} f(z;\lambda) = \frac{1}{\lambda_k} f(z;\lambda) - \left(\prod_{i=1}^{m} \lambda_i\right) exp(-\lambda_k z) \left(\frac{z}{\prod_{j=1, j\neq k}^{m} (\lambda_j - \lambda_k)} \right) +$$
$$\left(\prod_{i=1}^{m} \lambda_i\right) exp(-\lambda_k z) \left(\prod_{j=1, j\neq k}^{m} \frac{1}{(\lambda_j - \lambda_k)} \right) \sum_{j=1, j\neq k}^{m} \frac{1}{(\lambda_j - \lambda_k)} -$$
$$\left(\prod_{i=1}^{m} \lambda_i\right) \sum_{j=1, j\neq k}^{m} exp(-\lambda_i z) \frac{1}{(\lambda_k - \lambda_i)} \frac{1}{\prod_{j=1, j\neq i}^{m} (\lambda_j - \lambda_i)}.$$

Now we can use a gradient method for searching maximum likelihood estimates of $\lambda = (\lambda_1, \ldots, \lambda_m)$.

4 Parameter's estimation by the moments' method

The moments' method does not request explanations. We will use some modification of this method, using more moments as a number of unknown parameters.

Let μ_r^* be an empirical moment of the r-th order, $k \geq m$ and $\mu^* = (\mu_1^*, \ldots, \mu_k^*)$. As a criterion of the estimation we consider the following:

$$R_M(\lambda) = \sum_{r=1}^{k} \left|\mu_r(\lambda) - \mu_r^*\right|^{1/r}. \tag{5}$$

A value of λ, which minimizes this criterion, gives necessary estimate.

If distribution type of the given sample is known, we can act as follows: firstly, it is necessary to estimate the unknown distribution parameters; the second phase involves the approximation of corresponding density by a convolution of exponents. Such approximation is considered below.

5 Approximation of the density

The following situation is considered here. We have some differentiable density $g(z)$ of a non-negative continuous random variable. It is necessary to approximate the above mentioned density $g(z)$ by the density (1).

Firstly, we can approximate it using moments of the density $g(z)$. With this aim in view these moments are used instead of the empirical moments $\{\mu_r^*\}$ in formula (5). Secondly, as criterion of the estimation we can use a square between curves $g(z)$ and $f(z;\lambda)$:

$$R_S(\lambda) = \int_0^\infty \upsilon(z)|f(z;\lambda) - g(z)|dz, \tag{6}$$

where $\upsilon(z) \geq 0$ is a known "weight" function.

Here the multiplier $\upsilon(z)$ allows the deviations to get various weights $|f(z;\lambda) - g(z)|$ for various z. If $\upsilon(z) = z$ then the stress is laid on big value of the argument. If $\upsilon(z) = 1/z$ – on of the neighbourhood of zero. A case $\upsilon(z) = g(z)$ means that the emphasis is placed on big values of $g(z)$.

A minimization of integral (6) can be done through gradient method. If sign $(f(z;\lambda) - g(z)) = \pm 1$ gives a sign of the difference $f(z;\lambda) - g(z)$, then a partial derivative of (6) is as follows:

$$\frac{d}{d\lambda_k} R_S(\lambda) = \int_0^\infty \upsilon(z) sign\big(f(z;\lambda) - g(z)\big) \frac{d}{d\lambda_k} f(z;\lambda)dz. \tag{7}$$

Concerning the above considered approaches it is critical to underscore that it is possible to use a mixture of various estimates of vector parameter λ. It is easier to apply equal weights of mixture's components. But at the same time, it is possible to select weights based on some assumptions or speculations. We can control this procedure by calculating the criterion (5) and (6).

Further we present results of experiment's investigation for above considered approaches.

6 Experimental study

We use a simulation for a verification of received estimates. Further n means a size of given sample.

Firstly, we present results of ML-estimations using density (1). Let vector $\lambda 1 = (1, 1.6, 2.1)$ be true values of unknown parameters of the distribution (1). Estimates $\lambda 2 = (1.02, 1.114, 3.12)$, $\lambda 3 = (0.9, 1.26, 3.66)$, and $\lambda 4 = (1.08, 1.32, 2.16)$ were received on samples of sizes 100, 200, 250, correspondingly. Graphs of the density function (1) for these parameters are presented in Figure 1.

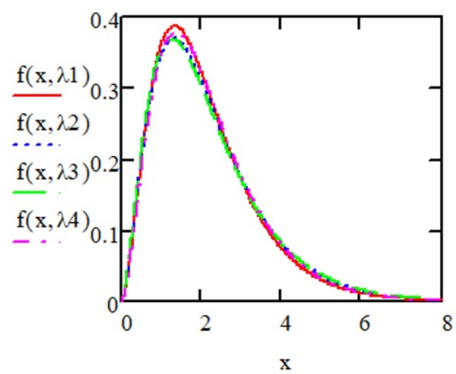


Fig. 1 True $f(x, \lambda 1)$ and estimated densities

Table 1 contains corresponding values of expectation μ and variance σ^2. We see that the difference in these values has a small influence on the form of the densities.

Table 1 Expectation μ and variance σ^2 for different estimates

	$\lambda 1$	$\lambda 2$	$\lambda 3$	$\lambda 4$
μ	2.101	2.178	2.178	2.146
σ^2	1.617	1.833	1.939	1.646

Further we consider the approximation results of some distribution by density (1). The Weibull-Gnedenko distribution (see [7]) will be considered as an approximated distribution. This distribution has the following density $fW(z; a, c)$ and moments $\mu W_r(a, c)$:

$$fW(z; a, c) = \frac{c}{a}\left(\frac{z}{a}\right)^{c-1} exp\left(-\left(\frac{z}{a}\right)^{c}\right), z \geq 0. \tag{8}$$

$$\mu W_r(a, c) = a^r \Gamma\left(\frac{r}{c} + 1\right), r = 1, 2, \ldots, \tag{9}$$

where parameters $a, c > 0$.

Let us set $a = 5.7$, $c = 2.2$. In this case $\mu W_1(a, c) = 5.048$, $\mu W_2(a, c) = 31.35$, $\mu W_3(a, c) = 224.721$, $\mu W_4(a, c) = 1.797 \times 10^3$.

Firstly, we consider results of approximations by maximum likelihood method. With this purpose in mind used sample was simulated with respect to this distribution. The size of the sample equals 250, the number m of convolution‘s components equals 3. The necessary estimates of the parameter are gotten through the procedure of log-likelihood function minimization. As result we have estimate

$$\lambda^* = (0.57\ 0.58\ 0.59)^T.$$

Fig. 2 contains graphs of density functions $fW(x; 5.7, 2.2)$ and $f(x; \lambda^*)$. Additionally the density of Erlang distribution

$$f\,Er(x, \alpha, k) = \frac{\alpha}{(k-1)!}(\alpha x)^{k-1}(\alpha e)exp(-\alpha x), x \geq 0$$

with parameters $\alpha = 0.85$ and $k = 3$ is presented.

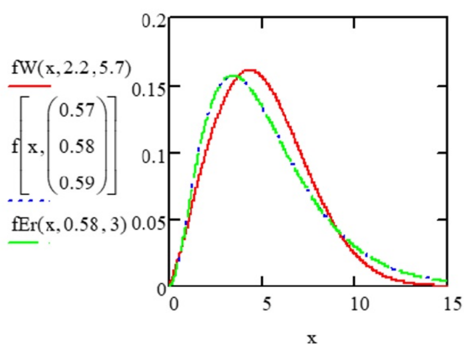


Fig. 2 True $f\,W\,(x; 2.2, 5.7)$ and estimated densities

Results of approximations of the density (8) by the density (1) for different used moments are represented in the Tab.2. Values of the parameters $\lambda = (\lambda_1, \ldots, \lambda_m)$ estimation for 3 moments μW_1, μW_2, μW_3 and $m = 3$ are denoted by $\lambda 1_1$, $\lambda 1_2$, $\lambda 1_3$, for 4 moments μW_1, μW_2, μW_3, μW_4 and $m = 4$ are denoted by $\lambda 2_1$, $\lambda 2_2$, $\lambda 2_3$,

$\lambda 2_4$. Corresponding moments $\mu_r(\lambda 1)$ and $\mu_r(\lambda 2)$ are represented too. Graphs of the corresponding densities are presented in Fig.3.

Table 2 Approximations of the density (8) using criterion (5)

r	$\lambda 1_r$	$\mu_r(\lambda 1)$	$\lambda 2_r$	$\mu_r(\lambda 2)$
1	0.56	4.63	0.70	4.789
2	0.61	28.764	0.725	28.864
3	0.83	224.723	0.90	210.187
4			1.15	1.797×10^3

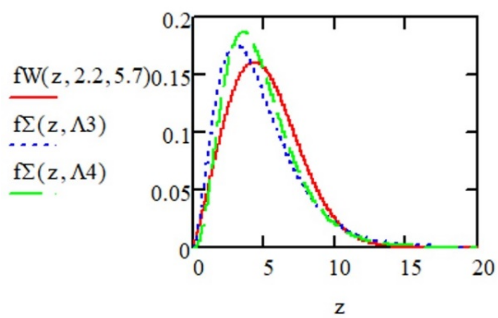


Fig. 3 Density of Weibull-Gnedenko distribution and its approximations

These graphs can be improved if the procedure of the approximation by means criteria (6) is employed. It gives graphs, presented in Fig.4 for $\upsilon(z) = 1$. Table 3 contains values of estimated parameters and moments.

Table 3 Approximations of the density (8) using criterion (6)

r	$\lambda 3_r$	$\mu_r(\lambda 3)$	$\lambda 4_r$	$\mu_r(\lambda 4)$
1	0.59	5.001	0.70	5.188
2	0.60	33.347	0.75	33.684
3	0.61	277.971	0.80	262.703
4			0.85	2.393×10^3

Having examined presented graphs, we conclude that parameters $\lambda = (0.57, 0.58, 0.59)$ or $\lambda = (0.59, 0.60, 0.61)$ give the best approximation. This example can be considered as a "hard" case, because the gotten approximation isn't very good.

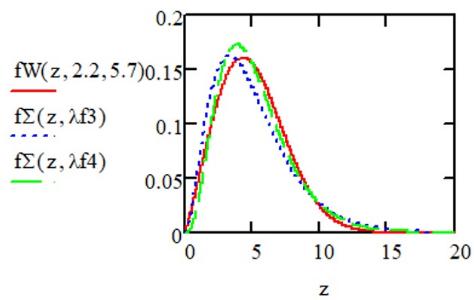


Fig. 4 Density of Weibull-Gnedenko distribution and its approximations

We present another example, where gotten approximation is good. Namely let us consider a log-normal distribution's density (see [7])

$$f\,LN(x,\mu,\sigma) = \frac{1}{x\sigma\sqrt{2\pi}} exp\left(-\frac{\left(ln\frac{x}{m}\right)^2}{2\sigma^2}\right), x \geq 0,$$

with parameters $m = 4.552$, $\sigma = 0.650$.

Using criterion (6) and weight function $\upsilon(z) = z$, we get the following parameter vector of the exponentially distributed addends:

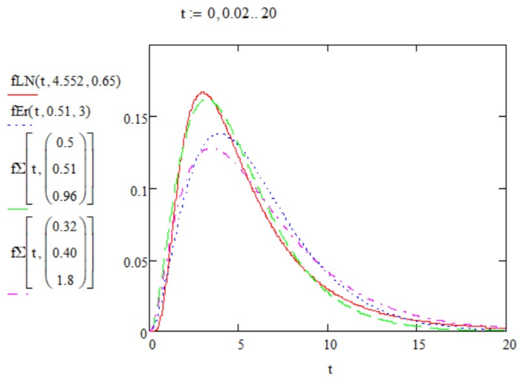
$$\lambda = (\lambda_1, \lambda_2, \lambda_3) = (0.5, 0.51, 0.96).$$

If we use the criterion of the moments (5), we get the following estimates:

$$\lambda = (\lambda_1, \lambda_2, \lambda_3) = (0.32, 0.40, 1.80).$$

Corresponding densities are presented in Fig.5. Below moments of the distributions are presented too.

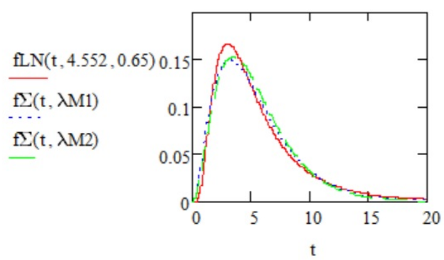
We see that approximation of the density's curve is good, but the moments are different. So, it is necessary to make a sacrifice. Further it is possible to use mixture of the exponential densities (Fig.6). It allows us finding a compromise between requests to densities and moments. For example, mixture (0.5 0.51 0.96) and (0.32 0.40 1.80) with equal weight 0.5 gives estimate $\lambda M1$ =(0.41 0.455 1.38). Mixture (0.5 0.51 0.96) and (0.5 0.51 0.52) with equal weight 0.5 gives estimate $\lambda M2$ =(0.50 0.51 0.74). Corresponding approximating densities and their moments are presented below. This example shows that several types of good approximation are possible. Both examples tell that the described approach isn't universal one. To apply it successfully, it is necessary to verify that used approximation is well.



$$\mu LN(3, 4.552, 0.65) = \begin{pmatrix} 5.623 \\ 48.237 \\ 631.407 \end{pmatrix} \qquad \mu\Sigma All\left[\begin{pmatrix} 0.5 \\ 0.51 \\ 0.96 \end{pmatrix}, 3\right] = \begin{pmatrix} 5.002 \\ 33.954 \\ 292.533 \end{pmatrix}$$

$$\mu\Sigma All\left[\begin{pmatrix} 0.5 \\ 0.51 \\ 0.52 \end{pmatrix}, 3\right] = \begin{pmatrix} 5.884 \\ 46.163 \\ 452.75 \end{pmatrix} \qquad \mu\Sigma All\left[\begin{pmatrix} 0.32 \\ 0.40 \\ 1.8 \end{pmatrix}, 3\right] = \begin{pmatrix} 6.181 \\ 54.524 \\ 631.4 \end{pmatrix}$$

Fig. 5 Density of Log-normal distribution and its approximations



$$\mu\Sigma All(\lambda M1, 3) = \begin{pmatrix} 5.361 \\ 40.05 \\ 386.951 \end{pmatrix} \qquad \mu\Sigma All(\lambda M2, 3) = \begin{pmatrix} 5.312 \\ 37.89 \\ 340.033 \end{pmatrix}$$

Fig. 6 Density and moments of the mixtures

7 Application to a single queueing system M/G/1/k

A single-server queueing system with a Poisson flow of the customers with arrival intensity α is under consideration. A distribution of the service time is arbitrary. Different service times are independent, and do not dependent on arrival flow too. Maximal number of the customers in the system equals k.

It is necessary to consider nonstationary regime, namely, to calculate a distribution of the number of customers in the system $X(t)$ at time moment t.

This system was studied by many authors, see, for example, [3]. Usually the stationary regime is considered. The nonstationary regime excepts any simple analysis. We use the above described approach for that.

Let assume that the convolution of the exponential distributions with parameters $\lambda = (\lambda_1, \dots, \lambda_m)$ adequately describes service time S. Then this time is presented as $S = Z_1 + \dots + Z_m$.

We will be concerned with m sequential phases of the service with lengths $Z_1, \dots, Z_m$.

Let $I(t)$ and $J(t)$ represent the number of customers in the queue and the phase of service at time moment t, correspondingly. $J(t) = 0$ means that the queueing system is empty, so $I(t) = 0$ too. Obviously, $Y(t) = (I(t), J(t))$ is a two-dimensional Markovian process.

To simplify the next presentation, we input a one-dimensional process $N(t) = mI(t) + J(t)$, that gives a total number of the phases of the forthcoming services. A set of values of this random variable is $\Omega = 0, 1, \dots, n^*$, where $n^* = mk$. The following relations take place:

$$\begin{aligned} n = 0 &\rightarrow I = 0 \text{ and } J = 0, \\ n > 0 \text{ and } \frac{n}{m} \text{ is an integer} &\rightarrow J = m \text{ and } I = \frac{n}{m}, \\ n > 0 \text{ and } \frac{n}{m} \text{ is not an integer} &\rightarrow \; J = mod(n, m) \text{ and} \\ & \qquad I = \frac{1}{m}(n - mod(n, m)). \end{aligned} \tag{10}$$

These relations give two functions φ and ψ on n which define I and J:

$$\varphi(n) = I \text{ and } \psi(n) = J.$$

Process $N(t)$ is continuous-time ergodic finite Markov chain. Its $(n^*+1) \times (n^*+1)$-matrix of the transition intensities $\Lambda = (\Lambda_{n,n'})$ is as follows:

$$\begin{aligned} \Lambda_{n,n+m} &= \alpha, \quad n = 0, \dots, n^* - m, \\ \Lambda_{n,n-1} &= \lambda_{\psi(n)}, \quad n = 1, \dots, n^*, \end{aligned}$$

the remaining components of the matrix equal zero.

Now we are able to calculate the conditional probability $P_{n,n'}(t) = P\{(N(t) = n' | N(0) = n)\}$ that a total number of the phases $N(t)$ at time moment t equals n' if initially one equals n. Let $P(t) = (P_{n,n'}(t))$ be the corresponding $(n^*+1) \times (n^*+1)$-matrix and ΛD be a diagonal $(n^* + 1) \times (n^* + 1)$-matrix with the diagonal $\Lambda 1$, where $\Lambda 1$ is column-vector of dimension $n + 1$ from units. If all eigenvalues of matrix (generator) $G = \Lambda - \Lambda D$ are different then probabilities $P(t) = (P_{n,n'}(t))$ can be represented simply. Let ν_η and Z_η, $n = 0, \dots, n^*$, be the eigenvalue and the corresponding eigenvector of G, $Z = (Z_0, \dots, Z_{n^*})$ be the matrix of the eigenvectors and $\widetilde{Z} = Z^{-1} = (\widetilde{Z}_0^T, \ldots, \widetilde{Z}_{n^*}^T)$ be the corresponding inverse matrix (here $\widetilde{Z}_n$ is the n–th row of $\widetilde{Z}$).

Then, see [1],

$$P(t) = \left(P_{n,n'}(t)\right) = \sum_{n=0}^{n^*} exp(\nu_\eta t) Z_n \widetilde{Z}_n. \tag{11}$$

Now a distribution of the number of the customers in the system $X(t)$ at time moment t can be calculated. Let us suppose that initially at time moment $t = 0$ i customers are in the system. If $i > 0$ we suppose additionally that the service of the customer begins only now.

If $\Omega(i) = \{n \in \Omega : \varphi(n) = i\}$, then

$$P(X(t) = i'|X(0) = i) = \sum_{n' \in \Omega(i')} P_{ik,n'}(t), \quad 0 \le i, \quad i' \le k. \tag{12}$$

Below the numerical example is considered. Our aim is to study how the expectation $E(X(t))$ of the number of customers in the system depends on time t. The famous Pollaczek-Khinchin formula gives an answer for the stationary case. If α is intensity of Poisson flow, μ and σ are average and standard deviation of service time, and load coefficient $\rho = \alpha\mu$ is less than one, then

$$E(X(\infty)) = \rho + \frac{\rho^2 + (\lambda\sigma)^2}{2(1-\rho)}. \tag{13}$$

We consider the following initial data: $\alpha = 0.2$, $m = 3$, $\lambda = (\lambda_1, \lambda_2, \lambda_3)^T = (1, 1.6, 2.1)^T$, $k = 4$, $n^* = 12$. The first and the second moments of the service time, calculated by formula (3), are as follows: $\mu = \mu_1(\lambda) = 2.101$, $\mu_2(\lambda) = 6.032$, so $\sigma = 1.272$.

Fig.7 contains the generator G. Expression (14) and Fig.9 (see Appendix) contain the vector $v = (v_0, \ldots, v_{12})$ of the eigenvalues and the matrix Z of the eigenvectors. The matrix of the transition probabilities for $t = 2$ is presented in Fig.10 (see Appendix).

$$\begin{aligned}\chi = (&-2.99,\ -2.69,\ 0,\ -1.65 + 1.13i,\ -1.65 - 1.13i,\ -0.34,\ -1.61 + 0.81i,\\ &-1.61 - 0.81i,\ -2.21,\ -0.66,\ -1.59 + 0.26i,\ -1.59 - 0.26i,\ -1.20)\end{aligned} \tag{14}$$

The Fig.8 contains final graphs of $E(X(\infty))$ and $E(X(t))$, named as $MeanN(t, 01, \lambda 1)_0$ and $AvrNumber(t, 0.1, \lambda 1)$ correspondingly. We see how non-stationary expectation $E(X(t))$ tends to stationary limit.

$$G := \begin{pmatrix} -0.2 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2.1 & -2.3 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.6 & -1.8 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1.2 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.1 & -2.3 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.6 & -1.8 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1.2 & 0 & 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2.1 & -2.3 & 0 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.6 & -1.8 & 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1.2 & 0 & 0 & 0.2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.1 & -2.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.6 & -1.6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

Fig. 7 The table with generator G

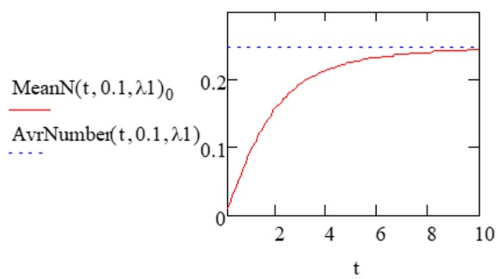


Fig. 8 The graph of the dependence $MeanN(t, 01, \lambda 1)_0 = E(X(t))$ on t and stationary value $AvrNumber(t, 0.1, \lambda 1) = E(X(\infty))$

8 Conclusions

Two aspects of the problem were considered. Firstly, the parametrical estimation of the convolution on the basis of given statistical data. Secondly, an approximation of fixed non-negative density. Different approaches to such approximation and estimation was considered: maximum likelihood method, the method of the moments, fitting of densities. An empirical analysis of different approaches has been performed using the simulation. The efficiency of the considered approach was illustrated by the task of the queuing theory.

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Appendix

$$Z := \begin{pmatrix}
0 & 0 & 0.277 & 0 & 0 & -3.594\times 10^{-3} & 0 & 0 & 0 & -1.912\times 10^{-3} & 0 & 0 & 0 \\
-0.012 & -7.244\times 10^{-3} & 0.277 & 2.99i\times 10^{-3} & -2.99i\times 10^{-3} & -3.215\times 10^{-3} & 1.813i\times 10^{-3} & 1.813i\times 10^{-3} & 2.497\times 10^{-3} & -1.475\times 10^{-3} & 0 & 0 & 0 \\
0.024 & 0.014 & 0.277 & 0.011 & 0.011 & -1.539\times 10^{-3} & 0 & 0 & -4.544\times 10^{-3} & 0 & 0 & 1.59\times 10^{-3} & 0 \\
-0.017 & -9.582\times 10^{-3} & 0.277 & -0.01i & 0.01i & 8.585\times 10^{-3} & 0 & 0 & 2.856\times 10^{-3} & 0.011 & 0 & 1.05i\times 10^{-3} & 4.094\times 10^{-3} \\
0.079 & 0.028 & 0.277 & -0.035 & -0.035 & 0.016 & -0.016 & -0.016 & 2.568\times 10^{-3} & 0.017 & 0 & 0 & 6.808\times 10^{-3} \\
-0.128 & -0.027 & 0.277 & 0.074i & -0.074i & 0.03 & 0.011+0.02i & 0.011-0.02i & -0.017 & 0.03 & 0 & 0 & 2.694\times 10^{-3} \\
0.079 & 8.26\times 10^{-3} & 0.277 & 0.059-0.048i & 0.059+0.048i & 0.081 & 0 & 0 & 0.014 & 0.041 & 0 & -0 & -6.172\times 10^{-3} \\
-0.274 & 0.065 & 0.277 & -0.025-0.16i & -0.025+0.16i & 0.111 & -0.015+0.011i & -0.015-0.011i & -0.06 & 0.043 & 0.034i & 0.034i & -0.018 \\
0.384 & -0.177 & 0.277 & -0.28+0.043i & -0.28-0.043i & 0.165 & 0.098+0.023i & 0.098-0.023i & 0.044 & 0.037 & 0.059-0.051i & 0.059+0.051i & -0.095 \\
-0.219 & 0.138 & 0.277 & 0.166+0.193i & 0.166-0.193i & 0.31 & -0.102-0.11i & -0.102+0.11i & 0.015 & -0.126 & 0.012+0.021i & 0.012-0.021i & -0.021 \\
0.514 & -0.492 & 0.277 & 0.417-0.144i & 0.417+0.144i & 0.369 & -0.326+0.066i & -0.326-0.066i & -0.282 & -0.183 & 0.074+0.047i & 0.074-0.047i & -0.049 \\
-0.59 & 0.721 & 0.277 & -0.231-0.581i & -0.231+0.581i & 0.468 & 0.138+0.643i & 0.138-0.643i & 0.737 & -0.312 & 0.303-0.445i & 0.303+0.445i & -0.194 \\
0.296 & -0.426 & 0.277 & -0.298+0.377i & -0.298-0.377i & 0.708 & 0.425-0.492i & 0.425+0.492i & -0.609 & -0.92 & -0.706+0.442i & -0.706-0.442i & 0.975
\end{pmatrix}$$

Fig. 9 The matrix of the eigenvectors

$$P := \begin{pmatrix}
0.854 & 0.022 & 0.037 & 0.074 & 2.67\times 10^{-3} & 3.777\times 10^{-3} & 5.117\times 10^{-3} & 1.86\times 10^{-4} & 2.397\times 10^{-4} & 2.782\times 10^{-4} & 9.751\times 10^{-6} & 1.185\times 10^{-5} & 1.25\times 10^{-5} \\
0.836 & 0.033 & 0.037 & 0.077 & 5.029\times 10^{-3} & 3.894\times 10^{-3} & 5.701\times 10^{-3} & 4.259\times 10^{-4} & 2.532\times 10^{-4} & 3.298\times 10^{-4} & 2.655\times 10^{-5} & 1.237\times 10^{-5} & 1.449\times 10^{-5} \\
0.737 & 0.084 & 0.067 & 0.077 & 0.016 & 0.01 & 6.333\times 10^{-3} & 1.493\times 10^{-3} & 9.025\times 10^{-4} & 3.947\times 10^{-4} & 1.017\times 10^{-4} & 5.761\times 10^{-5} & 1.525\times 10^{-5} \\
0.46 & 0.135 & 0.149 & 0.166 & 0.026 & 0.028 & 0.027 & 2.598\times 10^{-3} & 2.724\times 10^{-3} & 2.54\times 10^{-3} & 1.805\times 10^{-4} & 1.87\times 10^{-4} & 1.657\times 10^{-4} \\
0.323 & 0.139 & 0.179 & 0.23 & 0.04 & 0.035 & 0.041 & 5.173\times 10^{-3} & 3.425\times 10^{-3} & 4.044\times 10^{-3} & 4.456\times 10^{-4} & 2.235\times 10^{-4} & 2.412\times 10^{-4} \\
0.179 & 0.11 & 0.174 & 0.29 & 0.089 & 0.068 & 0.055 & 0.016 & 0.01 & 5.561\times 10^{-3} & 1.582\times 10^{-3} & 9.147\times 10^{-4} & 2.696\times 10^{-4} \\
0.064 & 0.055 & 0.107 & 0.239 & 0.137 & 0.15 & 0.158 & 0.026 & 0.028 & 0.027 & 2.767\times 10^{-3} & 2.871\times 10^{-3} & 2.553\times 10^{-3} \\
0.031 & 0.033 & 0.072 & 0.189 & 0.14 & 0.18 & 0.226 & 0.04 & 0.035 & 0.042 & 5.404\times 10^{-3} & 3.252\times 10^{-3} & 3.458\times 10^{-3} \\
0.012 & 0.015 & 0.037 & 0.116 & 0.111 & 0.175 & 0.289 & 0.09 & 0.068 & 0.057 & 0.017 & 0.01 & 3.588\times 10^{-3} \\
2.831\times 10^{-3} & 4.281\times 10^{-3} & 0.012 & 0.045 & 0.055 & 0.107 & 0.239 & 0.138 & 0.151 & 0.16 & 0.029 & 0.03 & 0.027 \\
1.059\times 10^{-3} & 1.822\times 10^{-3} & 5.732\times 10^{-3} & 0.023 & 0.034 & 0.074 & 0.195 & 0.145 & 0.188 & 0.235 & 0.039 & 0.029 & 0.03 \\
3.028\times 10^{-4} & 5.907\times 10^{-4} & 2.044\times 10^{-3} & 9.376\times 10^{-3} & 0.016 & 0.039 & 0.123 & 0.119 & 0.189 & 0.315 & 0.098 & 0.062 & 0.027 \\
5.471\times 10^{-5}+10i\times 10^{-15} & 1.207\times 10^{-4} & 4.597\times 10^{-4} & 2.366\times 10^{-3} & 4.575\times 10^{-3} & 0.013 & 0.049 & 0.061 & 0.119 & 0.271 & 0.161 & 0.168 & 0.151
\end{pmatrix}$$

Fig. 10 Matrix of the transition probabilities for $t = 2$

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