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ABSTRACT

We present decomposable (5, 6)-solutions $\widetilde{M}^{1,4} \times M^6$ in eleven-dimensional supergravity by solving the bosonic supergravity equations for a variety of non-trivial flux forms. Many of the bosonic backgrounds presented here are induced by various types of null flux forms on products of certain totally Ricci-isotropic Lorentzian Walker manifolds and Ricci-flat Riemannian manifolds. These constructions provide an analogy of the work performed by Chrysikos and Galaev [Classical Quantum Gravity **37**, 125004 (2020)], who made similar computations for decomposable (6, 5)-solutions. We also present bosonic backgrounds that are products of Lorentzian Einstein manifolds with a negative Einstein constant (in the "mostly plus" convention) and Riemannian Kähler–Einstein manifolds with a positive Einstein constant. This conclusion generalizes a result of Pope and van Nieuwenhuizen [Commun. Math. Phys. **122**, 281–292 (1989)] concerning the appearance of six-dimensional Kähler–Einstein manifolds in eleven-dimensional supergravity. In this setting, we construct infinitely many non-symmetric decomposable (5, 6)-supergravity backgrounds by using the infinitely many Lorentzian Einstein–Sasakian structures with a negative Einstein constant on the 5-sphere, known from the work of Boyer *et al.* [Commun. Math. Phys. **262**, 177–208 (2006)].

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I. INTRODUCTION

A. Motivation

The five established ten-dimensional superstring theories [type I, type IIA, type IIB, heterotic $\mathsf{SO}(32)$, and heterotic $\mathsf{E}_8 \times \mathsf{E}_8$] provide frameworks for uniting quantum theory and general relativity. By using string dualities such as T-duality, a unique eleven-dimensional superstring theory, called M-*theory*, unites these five theories. As a result, the eleven-dimensional supergravity theory, viewed as a low-energy limit of M-theory, has attracted much attention during the last half century (see, for example, Refs. 1–5).

The Lagrangian of eleven-dimensional supergravity was proposed in Ref. 6. The fields in the theory are a Lorentzian metric h, a closed 4-form F (called the flux form), and a Majorana spinor Ψ. They are defined on an eleven-dimensional manifold X and are subject to the equations of motion determined by the Lagrangian. A special class of supergravity solutions are those with a vanishing fermionic part, $\Psi = 0$. In this case, the equations of motion reduce to a simpler set of equations involving only h and F, which we call the bosonic supergravity equations. This set of equations closely resembles the Einstein–Maxwell equations in four dimensions. Solutions to the bosonic supergravity equations are called bosonic supergravity backgrounds. The bosonic backgrounds include the special class of eleven-dimensional Ricci-flat Lorentzian manifolds for which $\mathsf{F} = 0$.

Finding bosonic supergravity backgrounds is an important task, and the literature on bosonic supergravity backgrounds is vast. Several geometrical tools and constructions have been used for finding them, including manifolds with special holonomy or special G-structures, irreducible symmetric spaces, compactifications, Killing superalgebras, certain ansatzes on h and F, and others. We refer to some representative works,[7–21] and the reader can find more references therein.

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Supersymmetries also play an important role in these investigations. The maximally supersymmetric bosonic backgrounds are, in addition to flat Minkowski space, the Freund–Rubin backgrounds $(\mathsf{AdS})_7 \times \mathsf{S}^4$ and $(\mathsf{AdS})_4 \times \mathsf{S}^7$ (Ref. 22) and a particular pp-wave (see, for example, Ref. 23). These are locally homogeneous, something that is true for all backgrounds admitting more than half of the maximal amount of supersymmetries (Ref. 24). Other well-known examples are the M2-brane and the M5-brane (Refs. 25 and 26), whose near-horizon geometries are the Freund–Rubin backgrounds (see also Refs. 27 and 28). These have exactly half the maximal amount of supersymmetries. On the other side of the spectrum, with respect to the number of supersymmetries admitted, we have bosonic backgrounds such as $(\mathsf{AdS})_5 \times \mathbb{CP}^3$, which admit no supersymmetry (see Ref. 29).

B. Outline

In this article, we search for bosonic supergravity backgrounds that are products of an oriented Lorentzian manifold $(\widetilde{M}^{1,4}, \tilde{g})$ and an oriented Riemannian manifold (M^6, g), with a flux form $\mathsf{F} \in \Omega^4(\mathsf{X})$ of the type

$$\mathsf{F} = \varphi\tilde{\alpha} + \tilde{\beta} \wedge \nu + \tilde{\gamma} \wedge \delta + \tilde{\omega} \wedge \epsilon + \tilde{\psi}\theta,$$

where the ith term is the product of a $(5-i)$-form on $\widetilde{M}^{1,4}$ and an $(i-1)$-form on M^6, for $i = 1, \ldots, 5$. For various flux forms of the above-mentioned type, we write down the corresponding simplified form of the bosonic supergravity equations and find particular solutions to these equations. Our work can be considered a natural continuation of Ref. 30, where products of six-dimensional Lorentzian manifolds and five-dimensional Riemannian manifolds are treated in a similar way.

We begin by describing the general constraints that appear due to the bosonic supergravity equations (which we split up into the closedness condition, the Maxwell equation, and the supergravity Einstein equation). For a general 4-form F of the above-mentioned form, the resulting system is still quite complicated. See, for example, Proposition 3.3 for the Maxwell equation and Eqs. (4.3)–(4.5) for the supergravity Einstein equation. In order to obtain a more tractable system of equations, we specify F even further by letting three or four of its terms vanish. Then, as in Ref. 30, the constraints that occur due to the Maxwell equation in combination with the closedness condition are simplified (see Proposition 3.4), and the same applies to the supergravity Einstein equation (Proposition 4.3). It is worth mentioning that the form of F can impose non-trivial restrictions on the geometry of $\widetilde{M}^{1,4}$ or M^6 (see Corollary 4.5). For example, for $\mathsf{F} = \tilde{\alpha}$, the supergravity Einstein equation implies that (M^6, g) is an Einstein manifold, while for $\mathsf{F} = \theta$, the Einstein equation implies that $(\widetilde{M}^{1,4}, \tilde{g})$ is an Einstein manifold. In both cases, the scalar curvature of (X, h) is constant (Corollary 4.8).

In order to find explicit solutions to eleven-dimensional bosonic supergravity, we take two different approaches. First, we examine the case when F is composed of null forms. In this case, the bosonic supergravity equations simplify significantly, as shown in Proposition 5.1 and Theorems 5.2–5.5, 4.8, and 4.9. Moreover, the supergravity Einstein equation requires (M^6, g) to be Ricci-flat (Proposition 5.1). In addition, for a bosonic supergravity background $(\mathsf{X}^{1,10} = \widetilde{M}^{1,4} \times M^6, h = \tilde{g} + g, \mathsf{F} = \tilde{\omega} \wedge \epsilon)$, where $\tilde{\omega} \in \Omega^1(\widetilde{M}^{1,4})$ is null, we see in Corollary 5.7 that (X, h) is totally Ricci-isotropic, as an analog of Ref. 30, Corollary 5.10. To find concrete solutions to the bosonic supergravity equations, we follow Ref. 30 and assume that the Lorentzian part $(\widetilde{M}^{1,4}, \tilde{g})$ is a special type of Walker metric, since these come equipped with a distribution of null lines from which non-trivial null flux forms can be built. Propositions 6.4, 6.6, 6.8, 6.9, 6.11, 6.13, and 6.16 concern non-symmetric bosonic backgrounds that are direct products of a Ricci-isotropic Lorentzian Walker manifold and a Ricci-flat Riemannian manifold. These products have special holonomy properties and can potentially support supersymmetries (see also Ref. 9). We explicitly illustrate these results with examples involving five-dimensional pp-waves, while an investigation of supersymmetries will be left for a forthcoming work.

In the second approach, we study some cases where the Riemannian part (M^6, g, ω) is a Kähler manifold. In this case, we do not assume that F is null but rather that it is related to the Kähler form ω. In particular, we consider the cases $\mathsf{F} = \tilde{\gamma} \wedge \delta$ with $\delta = \omega$ and $\mathsf{F} = \theta = c \star_6 \omega$, where c is a constant. We see that if the flux form is given by $\mathsf{F} = \tilde{\gamma} \wedge \delta$ and $\|\tilde{\gamma}\|^2_{\tilde{g}}$ is not constant, then the supergravity Einstein equation forces M^6 to be a Ricci-flat almost Hermitian manifold (Corollary 4.5), so in Proposition 7.2, we write down the bosonic supergravity equations for the case when M^6 is a Kähler manifold. For the case where the flux form is given by $\mathsf{F} = c\star_6\omega$, Proposition 7.4 says that the bosonic supergravity equations are satisfied if and only if both $(\widetilde{M}^{1,4}, \tilde{g})$ and (M^6, g) are Einstein with Einstein constants $\frac{1}{6}c^2$ and $-\frac{1}{6}c^2$, respectively. Therefore, $(\widetilde{M}^{1,4}, \tilde{g})$ has positive scalar curvature, while (M^6, g) has negative scalar curvature. For instance, the symmetric spaces $\mathbb{CP}^3$ and $\mathsf{Gr}_+(2,5)$ endowed with their respective (unique) homogeneous Kähler–Einstein metrics can be used to obtain some of the decomposable symmetric supergravity backgrounds presented in Ref. 17. Note that in this paper, we use the "mostly minus" convention for the Lorentzian metric h, and thus a Riemannian metric g is viewed as a negative definite metric. Proposition 7.4 generalizes a result presented in Ref. 29, which involved bosonic backgrounds of the form $(\mathsf{AdS})_5 \times M^6$, where M^6 is a compact Kähler manifold. We discuss some possible candidates for the Einstein manifold $(\widetilde{M}^{1,4}, \tilde{g})$, other than $(\mathsf{AdS})_5$. In particular, we are based on *negative Sasakian geometries* and use the infinitely many different Lorentzian Einstein–Sasakian structures with negative Einstein constants (in the "mostly plus" convention) described on the 5-sphere S^5 by Boyer *et al.*[31] In this way, we get infinitely many new bosonic non-symmetric decomposable $(5,6)$-solutions in eleven-dimensional supergravity given by $\mathsf{S}^5 \times M^6$, where M^6 is any six-dimensional (de Rham irreducible) Kähler–Einstein manifold with positive scalar curvature (also in the "mostly plus" convention), and S^5 is endowed with one of the Lorentzian Einstein–Sasakian structures mentioned earlier. Note that all such solutions that are based on the same Kähler–Einstein manifold M^6 have equal flux forms. Other such examples can be obtained by using the connected sum $\sharp k(\mathsf{S}^2 \times \mathsf{S}^3)$, since this manifold also admits Lorentzian Einstein–Sasakian metrics for any integer $k \geq 1$.

We should finally mention that the above-mentioned conclusion fails if M^6 is a six-dimensional (strictly) nearly Kähler manifold since, in this case, the Kähler form ω is not closed, so the 4-form F indicated earlier cannot serve as a flux form (see the final section). As a consequence,

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and in line with the conclusion pointed out in Ref. 29 for $(\mathsf{AdS})_5 \times M^6$, bosonic solutions of the form $\widetilde{M}^{1,4} \times M^6$, where $\widetilde{M}^{1,4}$ is a Lorentzian Einstein manifold and M^6 is a compact Kähler–Einstein manifold, are not expected to admit supersymmetries. Essentially, this is because, in dimension 6, smooth spin manifolds admitting real Killing spinors are exhausted by nearly Kähler manifolds [see, for example, Ref. 32 and also Ref. 33 for the classification of eleven-dimensional supersymmetric supergravity solutions containing $(\mathsf{AdS})_5$].

The paper is structured as follows. In Sec. II, we lay out the framework that will be used throughout the paper and establish some notation. We introduce the eleven-dimensional bosonic supergravity equations and write down the ansatz of the general flux form, which we use in this paper. The bosonic supergravity equations corresponding to this ansatz are computed in Secs. III and IV, respectively. There, we also investigate the form of the equations after further simplification of the flux form and state some general consequences of the equations. As mentioned earlier, the bosonic supergravity equations are simpler when the flux form is composed of null forms, and in Sec. V, we present some general results for such flux forms. Next, in Sec. VI, we apply these results to Ricci-isotropic Lorentzian Walker manifolds and produce several explicit examples of decomposable (5, 6)-supergravity backgrounds. In Sec. VII, we drop the requirement that F is null and analyze the appearance of Kähler–Einstein manifolds and of negative Einstein–Sasakian geometries in our decomposable (5, 6)-solutions.

II. PRELIMINARIES

In this work, we study connected eleven-dimensional Lorentzian manifolds of the form

$$\mathsf{X}^{1,10} = \widetilde{M}^{1,4} \times M^6,$$

where $(\widetilde{M}^{1,4}, \tilde{g})$ is a five-dimensional connected oriented Lorentzian manifold and (M^6, g) is a six-dimensional connected oriented Riemannian manifold. Our aim is to present on such products a systematic examination of the bosonic supergravity equations, i.e., of the following system of field equations (see, for example, Refs. 10 and 34):

$$\begin{cases} d\,\mathsf{F} & = & 0, \\ d \star \mathsf{F} & = & \dfrac{1}{2}\mathsf{F} \wedge \mathsf{F}, \\ \mathsf{Ric}_h(X, Y) & = & -\dfrac{1}{2}\langle X \lrcorner \mathsf{F}, Y \lrcorner \mathsf{F}\rangle_h + \dfrac{1}{6}h(X, Y)\|\mathsf{F}\|_h^2. \end{cases} \tag{2.1}$$

Here, the Lorentzian metric on $\mathsf{X}^{1,10}$ is the product metric $h = \tilde{g} + g$, and $\star : \Omega^k(\mathsf{X}^{1,10}) \to \Omega^{11-k}(\mathsf{X}^{1,10})$ is the Hodge-star operator on $(\mathsf{X}^{1,10}, h)$, defined by $\alpha \wedge \star\beta = \langle \alpha, \beta\rangle_h \,\mathsf{vol}_{\mathsf{X}}$, where $\mathsf{vol}_{\mathsf{X}} = \mathsf{vol}_{\widetilde{M}} \wedge \mathsf{vol}_M$ denotes the volume form on $(\mathsf{X}^{1,10}, h)$. We also have $\|\mathsf{F}\|_h^2 = \langle \mathsf{F}, \mathsf{F}\rangle_h$. The bosonic field F is a global 4-form on $\mathsf{X}^{1,10}$, called the *flux form*, which, together with the Lorentzian metric h, forms the bosonic sector of eleven-dimensional supergravity. We will refer to the three conditions appearing in (2.1) as the *closedness condition*, the *Maxwell equation*, and the *supergravity Einstein equation*, respectively. Triples $(\mathsf{X}^{1,10}, h, \mathsf{F})$ solving this system of equations are called *bosonic supergravity backgrounds*.

Remark 2.1. In this paper, we apply the "mostly minus" convention. That is, the signature for h is $(+, -, \dots, -)$, and hence g is a negative definite Riemannian metric on M. Recall that for any two k-forms ω and ϕ, we have

$$\langle \omega, \phi\rangle_h = \frac{1}{k!}\sum_{1\le i_\alpha, j_\beta \le 11} \omega_{i_1\dots i_k}\phi_{j_1\dots j_k} h^{i_1 j_1}\dots h^{i_k j_k}.$$

Then, for a k-form $\omega \in \Omega^k(M^6)$, the sign of $\|\omega\|_h^2 = \|\omega\|_g^2$ is equal to that of $(-1)^k$. Note that this choice of convention makes the right-hand side of the last equation of (2.1) different from how it usually appears in the literature, by a minus sign.

Before we discuss closed 4-forms $\mathsf{F} \in \Omega^4_{\mathrm{cl}}(\mathsf{X}^{1,10})$ on $\mathsf{X}^{1,10}$, let us decompose the tangent space $\mathbb{V} := T_x\mathsf{X} \simeq \mathbb{R}^{1,10}$ of $\mathsf{X}^{1,10}$ at a point $x \in \mathsf{X}$ as

$$\mathbb{V} = \mathbb{L}^{1,4} \oplus \mathbb{E}^6,$$

where we identify $\mathbb{L}$ with the five-dimensional Minkowski tangent space of $\widetilde{M}^{1,4}$ and $\mathbb{E}$ with the six-dimensional Euclidean tangent space of M^6. Then, one has an orthogonal decomposition

$$\Lambda^4\mathbb{V} = \Lambda^4\mathbb{R}^{1,10} = \Lambda^4\mathbb{L} \bigoplus (\Lambda^3\mathbb{L} \wedge \Lambda^1\mathbb{E}) \bigoplus (\Lambda^2\mathbb{L} \wedge \Lambda^2\mathbb{E}) \bigoplus (\Lambda^1\mathbb{L} \wedge \Lambda^3\mathbb{E}) \bigoplus \Lambda^4\mathbb{E}.$$

In this paper, we consider global differential 4-forms $\mathsf{F} \in \Omega^4(\mathsf{X})$, given by

$$\mathsf{F} = \varphi\tilde{\alpha} + \tilde{\beta} \wedge \nu + \tilde{\gamma} \wedge \delta + \tilde{\omega} \wedge \epsilon + \tilde{\psi}\theta, \tag{2.2}$$

for some

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$$\tilde{\alpha} \in \Omega^4(\widetilde{M}^{1,4}), \quad \tilde{\beta} \in \Omega^3(\widetilde{M}^{1,4}), \quad \tilde{\gamma} \in \Omega^2(\widetilde{M}^{1,4}), \quad \tilde{\omega} \in \Omega^1(\widetilde{M}^{1,4}), \quad \tilde{\psi} \in C^\infty(\widetilde{M}^{1,4}),$$
$$\varphi \in C^\infty(M^6), \quad \nu \in \Omega^1(M^6), \quad \delta \in \Omega^2(M^6), \quad \epsilon \in \Omega^3(M^6), \quad \theta \in \Omega^4(M^6).$$

We assume that all these differential forms are smooth and defined globally on $\widetilde{M}^{1,5}$ and M^6, respectively. We will show that the chosen class of 4-forms is large enough to allow for a variety of non-trivial bosonic supergravity backgrounds. Note that the difference between (2.2) and a general 4-form on $\mathsf{X}^{1,10}$ is, first, that a general 4-form may have more terms taking values in each of the subspaces $\Lambda^i\mathbb{L} \wedge \Lambda^{4-i}\mathbb{E}$ and, second, that each term can be multiplied by a function on $\mathsf{X}^{1,10}$.

III. THE CLOSEDNESS CONDITION AND THE MAXWELL EQUATION

We begin by writing down the closedness condition and the Maxwell equation on $(\mathsf{X}^{1,10} = \widetilde{M}^{1,4} \times M^6, h = \tilde{g} + g)$ for the 4-form F given by (2.2). The closedness condition can be found by computing $\mathrm{d}\,\mathsf{F}$ and comparing terms of similar type, a procedure that gives the following.

Lemma 3.1. The 4-form F *defined by (2.2) is closed if and only if the following system is satisfied:*

$$\begin{cases} \varphi \mathrm{d}\tilde{\alpha} = 0, & \tilde{\gamma} \wedge \mathrm{d}\delta + \mathrm{d}\tilde{\omega} \wedge \epsilon = 0, \\ \tilde{\alpha} \wedge \mathrm{d}\varphi + \mathrm{d}\tilde{\beta} \wedge \nu = 0, & \tilde{\omega} \wedge \mathrm{d}\epsilon - \mathrm{d}\tilde{\psi} \wedge \theta = 0, \\ \mathrm{d}\tilde{\gamma} \wedge \delta - \tilde{\beta} \wedge \mathrm{d}\nu = 0, & \tilde{\psi}\mathrm{d}\theta = 0. \end{cases}$$

In particular, we notice that F given by (2.2) is closed in the case that $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\omega}, \tilde{\psi}$ and $\varphi, \nu, \delta, \epsilon, \theta$ are closed on their respective manifolds. Before studying the Maxwell equation, we recall some useful basic useful formulas (see also Refs. 30 and 34).

Lemma 3.2. Let $\mathsf{X} = \widetilde{M}^p \times M^q$ *be a product of two pseudo-Riemannian manifolds* $(\tilde{M}, \tilde{g})$ *and* (M, g) *of dimensions* p, q*, and let* $\tilde{s}$*,* s *be the number of negative eigenvalues of* $\tilde{g}$*,* g*, respectively. Let us denote by* $\star$*,* $\star_p$*,* $\star_q$ *the Hodge operator on* $(\mathsf{X}, h = \tilde{g} + g)$*,* $(\widetilde{M}, \tilde{g})$*, and* (M, g)*, respectively. Then, for any* $\tilde{\alpha} \in \Omega^{\tilde{k}}(\widetilde{M})$*, and* $\beta \in \Omega^k(M)$ *the following hold:*

$$\begin{aligned} &\star\tilde{\alpha} = \star_p\tilde{\alpha} \wedge \mathrm{vol}_M, && \star\,\mathrm{vol}_M = (-1)^s(-1)^{pq}\,\mathrm{vol}_{\widetilde{M}}, \\ &\star\,\mathrm{vol}_{\widetilde{M}} = (-1)^{\tilde{s}}\,\mathrm{vol}_M, && \langle \tilde{\alpha} \wedge \beta, \tilde{\alpha} \wedge \beta \rangle_h = \langle \tilde{\alpha}, \tilde{\alpha} \rangle_{\tilde{g}} \langle \beta, \beta \rangle_g, \\ &\star\beta = (-1)^{pq}\star_q\beta \wedge \mathrm{vol}_{\widetilde{M}}, && \star(\tilde{\alpha} \wedge \beta) = (-1)^{k(p-\tilde{k})}\star_p\tilde{\alpha} \wedge \star_q\beta. \end{aligned}$$

With the help of Lemma 3.2, we compute $\star\mathsf{F}$ and $\mathrm{d}\star\mathsf{F}$ for F being of the form (2.2). We obtain

$$\star\mathsf{F} = \star_5\tilde{\alpha} \wedge \star_6\varphi + \star_5\tilde{\beta} \wedge \star_6\nu + \star_5\tilde{\gamma} \wedge \star_6\delta + \star_5\tilde{\omega} \wedge \star_6\epsilon + \star_5\tilde{\psi} \wedge \star_6\theta,$$

and consequently,

$$\begin{aligned} \mathrm{d}\star\mathsf{F} = {} & \mathrm{d}\star_5\tilde{\alpha} \wedge \star_6\varphi + \mathrm{d}\star_5\tilde{\beta} \wedge \star_6\nu + \star_5\tilde{\beta} \wedge \mathrm{d}\star_6\nu + \mathrm{d}\star_5\tilde{\gamma} \wedge \star_6\delta \\ & - \star_5\tilde{\gamma} \wedge \mathrm{d}\star_6\delta + \mathrm{d}\star_5\tilde{\omega} \wedge \star_6\epsilon + \star_5\tilde{\omega} \wedge \mathrm{d}\star_6\epsilon - \star_5\tilde{\psi} \wedge \mathrm{d}\star_6\theta. \end{aligned}$$

Notice that $\mathrm{d}\star_6\varphi = 0$ since $\star_6\varphi$ is a 6-form on a six-dimensional manifold. Similarly, we have $\mathrm{d}\star_5\tilde{\psi} = 0$. We also compute

$$\begin{aligned} \frac{1}{2}\mathsf{F} \wedge \mathsf{F} = {} & \varphi\,\tilde{\alpha} \wedge \tilde{\omega} \wedge \epsilon + \varphi\,\tilde{\psi}\,\tilde{\alpha} \wedge \theta + \tilde{\beta} \wedge \tilde{\gamma} \wedge \delta \wedge \nu + \tilde{\beta} \wedge \tilde{\omega} \wedge \epsilon \wedge \nu \\ & + \tilde{\psi}\,\tilde{\beta} \wedge \theta \wedge \nu + \tilde{\psi}\,\tilde{\gamma} \wedge \theta \wedge \delta + \tilde{\gamma} \wedge \tilde{\omega} \wedge \epsilon \wedge \delta + \frac{1}{2}\tilde{\gamma} \wedge \tilde{\gamma} \wedge \delta \wedge \delta. \end{aligned}$$

After collecting the terms according to the subspace $\Lambda^i\mathbb{L} \wedge \Lambda^{4-i}\mathbb{E}$ in which they take values, we get the following proposition.

Proposition 3.3. The Maxwell equation on the Lorentzian manifold $(\mathsf{X}^{1,10} = \widetilde{M}^{1,4} \times M^6, h = \tilde{g} + g)$ *with 4-form* F *given by (2.2) is equivalent to the following system of equations:*

$$\begin{array}{r|l} \textit{Type}\ (\tilde{2}, 6) & \mathrm{d}\star_5\tilde{\alpha} \wedge \star_6\varphi + \star_5\tilde{\beta} \wedge \mathrm{d}\star_6\nu \;=\; \tilde{\psi}\,\tilde{\gamma} \wedge \theta \wedge \delta, \\ \textit{Type}\ (\tilde{3}, 5) & \mathrm{d}\star_5\tilde{\beta} \wedge \star_6\nu - \star_5\tilde{\gamma} \wedge \mathrm{d}\star_6\delta \;=\; \tilde{\psi}\,\tilde{\beta} \wedge \theta \wedge \nu + \tilde{\gamma} \wedge \tilde{\omega} \wedge \epsilon \wedge \delta, \\ \textit{Type}\ (\tilde{4}, 4) & \mathrm{d}\star_5\tilde{\gamma} \wedge \star_6\delta + \star_5\tilde{\omega} \wedge \mathrm{d}\star_6\epsilon \;=\; \tilde{\psi}\,\varphi\,\tilde{\alpha} \wedge \theta + \tilde{\beta} \wedge \tilde{\omega} \wedge \epsilon \wedge \nu + \frac{1}{2}\tilde{\gamma} \wedge \tilde{\gamma} \wedge \delta \wedge \delta, \\ \textit{Type}\ (\tilde{5}, 3) & \mathrm{d}\star_5\tilde{\omega} \wedge \star_6\epsilon - \star_5\tilde{\psi} \wedge \mathrm{d}\star_6\theta \;=\; \varphi\,\tilde{\alpha} \wedge \tilde{\omega} \wedge \epsilon + \tilde{\beta} \wedge \tilde{\gamma} \wedge \delta \wedge \nu. \end{array}$$

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Of course, the system of equations given in Lemma 3.1 and Proposition 3.3 is significantly simplified when some terms of F vanish. Let us list some of the important cases and write down the corresponding equations.

Proposition 3.4. Consider the Lorentzian manifold $(\mathsf{X}^{1,10} = \widetilde{M}^{1,4} \times M^6, h = \tilde{g} + g)$ *as earlier. Then the following hold:*

(1) *The 4-form* $\mathsf{F} \in \Omega^4(\mathsf{X})$ *defined by*

$$\mathsf{F} = \varphi\tilde{\alpha}, \tag{3.1}$$

satisfies the Maxwell equation and the closedness condition if and only if φ *is constant and* $\tilde{\alpha}$ *is closed and coclosed,*

$$\mathrm{d}\,\tilde{\alpha} = \mathrm{d} \star_5 \tilde{\alpha} = 0.$$

(2) *The 4-form* $\mathsf{F} \in \Omega^4(\mathsf{X})$ *defined by*

$$\mathsf{F} = \tilde{\beta} \wedge \nu, \tag{3.2}$$

satisfies the Maxwell equation and the closedness condition if and only if $\tilde{\beta}$ *and* ν *are closed and coclosed,*

$$\mathrm{d}\,\tilde{\beta} = \mathrm{d} \star_5 \tilde{\beta} = 0, \qquad \mathrm{d}\,\nu = \mathrm{d} \star_6 \nu = 0.$$

(3) *The 4-form* $\mathsf{F} \in \Omega^4(\mathsf{X})$ *defined by*

$$\mathsf{F} = \tilde{\gamma} \wedge \delta, \tag{3.3}$$

satisfies the Maxwell equation and the closedness condition if and only if

$$\mathrm{d}\,\tilde{\gamma} = \mathrm{d}\,\delta = \mathrm{d} \star_6 \delta = 0, \qquad \mathrm{d} \star_5 \tilde{\gamma} \wedge \star_6 \delta = \frac{\tilde{\gamma} \wedge \tilde{\gamma} \wedge \delta \wedge \delta}{2}.$$

If $\tilde{\gamma} \wedge \tilde{\gamma} = 0$, *then the last equation implies* $\mathrm{d} \star_5 \tilde{\gamma} = 0$, *and it puts no additional constraints on* δ. *If* $\tilde{\gamma} \wedge \tilde{\gamma}$ *is nonzero, the last condition is equivalent to*

$$\mathrm{d} \star_5 \tilde{\gamma} = \kappa \tilde{\gamma} \wedge \tilde{\gamma}, \qquad \kappa \star_6 \delta = \frac{\delta \wedge \delta}{2},$$

for some constant $\kappa \in \mathbb{R}$.

(4) *The 4-form* $\mathsf{F} \in \Omega^4(\mathsf{X})$ *defined by*

$$\mathsf{F} = \tilde{\omega} \wedge \epsilon, \tag{3.4}$$

satisfies the Maxwell equation and the closedness condition if and only if $\tilde{\omega}$ *and* ϵ *are closed and coclosed,*

$$\mathrm{d}\,\tilde{\omega} = \mathrm{d} \star_5 \tilde{\omega} = 0, \qquad \mathrm{d}\,\epsilon = \mathrm{d} \star_6 \epsilon = 0.$$

(5) *The 4-form* $\mathsf{F} \in \Omega^4(\mathsf{X})$ *defined by*

$$\mathsf{F} = \tilde{\psi}\theta, \tag{3.5}$$

satisfies the Maxwell equation and the closedness condition if and only if $\tilde{\psi}$ *is constant and* θ *is closed and coclosed,*

$$\mathrm{d}\,\theta = \mathrm{d} \star_6 \theta = 0.$$

(6) *The 4-form* $\mathsf{F} \in \Omega^4(\mathsf{X})$ *defined by*

$$\mathsf{F} = \varphi\tilde{\alpha} + \tilde{\beta} \wedge \nu, \tag{3.6}$$

satisfies the Maxwell equation and the closedness condition if and only if

$$\mathrm{d}\,\tilde{\alpha} = \mathrm{d} \star_5 \tilde{\beta} = \mathrm{d}\,\nu = 0, \quad \mathrm{d}\,\varphi = \kappa\nu, \quad \mathrm{d}\,\tilde{\beta} = -\kappa\tilde{\alpha}, \quad \mathrm{d} \star_5 \tilde{\alpha} = -\lambda \star_5 \tilde{\beta}, \quad \mathrm{d} \star_6 \nu = \lambda \star_6 \varphi, \tag{3.7}$$

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for some constants $\kappa, \lambda \in \mathbb{R}$. The last four conditions imply

$$\star_6 \mathrm{d} \star_6 \mathrm{d}\, \varphi = \kappa\lambda\varphi, \qquad \star_5 \mathrm{d} \star_5 \mathrm{d}\, \tilde{\beta} = \kappa\lambda\tilde{\beta}.$$

(7) *The 4-form $\mathsf{F} \in \Omega^4(\mathsf{X})$ defined by*

$$\mathsf{F} = \tilde{\omega} \wedge \epsilon + \tilde{\psi}\theta, \tag{3.8}$$

satisfies the Maxwell equation and the closedness condition if and only if

$$\mathrm{d}\,\theta = \mathrm{d}\,\tilde{\omega} = \mathrm{d} \star_6 \epsilon = 0, \quad \mathrm{d}\,\tilde{\psi} = \kappa\tilde{\omega}, \quad \mathrm{d}\,\epsilon = \kappa\theta, \quad \mathrm{d} \star_5 \tilde{\omega} = \lambda \star_5 \tilde{\psi}, \quad \mathrm{d} \star_6 \theta = \lambda \star_6 \epsilon, \tag{3.9}$$

for some constants $\kappa, \lambda \in \mathbb{R}$. The last four conditions imply

$$\star_5 \mathrm{d} \star_5 \mathrm{d}\, \tilde{\psi} = \kappa\lambda\tilde{\psi}, \qquad \star_6 \mathrm{d} \star_6 \mathrm{d}\, \epsilon = -\kappa\lambda\epsilon.$$

(8) *The 4-form $\mathsf{F} \in \Omega^4(\mathsf{X})$ defined by*

$$\mathsf{F} = \varphi\tilde{\alpha} + \tilde{\psi}\theta. \tag{3.10}$$

satisfies the Maxwell equation and the closedness condition if and only if either $\varphi\tilde{\alpha} = 0$ or $\tilde{\psi}\theta = 0$. Therefore, this case reduces to case (5) or case (1), respectively.

(9) *The 4-form $\mathsf{F} \in \Omega^4(\mathsf{X})$ defined by*

$$\mathsf{F} = \tilde{\beta} \wedge \nu + \tilde{\omega} \wedge \epsilon, \tag{3.11}$$

satisfies the Maxwell equation and the closedness condition if and only if

$$\mathrm{d}\,\tilde{\beta} = \mathrm{d}\,\nu = \mathrm{d}\,\tilde{\omega} = \mathrm{d}\,\epsilon = 0, \quad \mathrm{d} \star_6 \nu = \mathrm{d} \star_5 \tilde{\beta} = \mathrm{d} \star_5 \tilde{\omega} = 0, \quad \star_5 \tilde{\omega} \wedge \mathrm{d} \star_6 \epsilon = \tilde{\beta} \wedge \tilde{\omega} \wedge \epsilon \wedge \nu.$$

If $\epsilon \wedge \nu = 0$, then the last equation implies $\mathrm{d} \star_6 \epsilon = 0$. If $\epsilon \wedge \nu$ is nonzero, then the last equation is equivalent to

$$\mathrm{d} \star_6 \epsilon = \kappa\epsilon \wedge \nu, \qquad \kappa \star_5 \tilde{\omega} = \tilde{\beta} \wedge \tilde{\omega},$$

for some constant $\kappa \in \mathbb{R}$.

Proof. All the cases are direct consequences of Lemma 3.1 and Proposition 3.3, i.e., the closedness condition and Maxwell equation for F of the form (2.2). We show the calculations for (6) in detail. For $\mathsf{F} = \varphi\tilde{\alpha} + \tilde{\beta} \wedge \nu$, the first three equations of Lemma 3.1 reduce to

$$\varphi\, \mathrm{d}\,\tilde{\alpha} = 0, \quad \tilde{\alpha} \wedge \mathrm{d}\,\varphi + \mathrm{d}\,\tilde{\beta} \wedge \nu = 0, \quad \tilde{\beta} \wedge \mathrm{d}\,\nu = 0,$$

while the last three equations hold automatically. In particular, we see that $\mathrm{d}\,\tilde{\alpha} = 0$ and $\mathrm{d}\nu = 0$ (assuming $\varphi \neq 0$ and $\tilde{\beta} \neq 0$). We also have $\mathrm{d}\varphi = \kappa\nu$ and $\mathrm{d}\,\tilde{\beta} = -\kappa\tilde{\alpha}$ for some constant $\kappa \in \mathbb{R}$. The first two equations of Proposition 3.3 reduce to $\mathrm{d} \star_5 \tilde{\alpha} \wedge \star_6 \varphi + \star_5 \tilde{\beta} \wedge \mathrm{d} \star_6 \nu = 0$ and $\mathrm{d} \star_5 \tilde{\beta} \wedge \star_6 \nu = 0$, respectively, while the last two hold automatically. This implies $\mathrm{d} \star_5 \tilde{\beta} = 0$, $\mathrm{d} \star_6 \nu = \lambda \star_6 \varphi$, and $\mathrm{d} \star_5 \tilde{\alpha} = -\lambda \star_5 \tilde{\beta}$. All together, we obtain (3.7). The other cases are treated similarly. □

Remark 3.5. In comparison with the examination of $(6, 5)$-decomposable supergravity backgrounds presented in Ref. 30, i.e., Lorentzian manifolds of the form $\mathsf{Y} = \widetilde{M}^{1,5} \times M^5$, we see that the system of the closedness condition and the Maxwell equation are very similar, although non-identical. In particular, a comparison of our Proposition 4.3 with Ref. 30, Proposition 3.5, shows that when the 4-form F is determined via one of the cases (1), (2), or (4)–(8), then we obtain very similar constraints. On the other hand, cases (3) and (9) are quite different. For example, in our case, $\mathsf{X} = \widetilde{M}^{1,4} \times M^6$, and the Maxwell equation contains the new term $\tilde{\psi}\tilde{\gamma} \wedge \theta \wedge \delta$. On the other hand, it does not contain the term corresponding to $\tilde{\alpha} \wedge \tilde{\gamma} \wedge \delta$, which appears in the Maxwell equation for $\mathsf{Y} = \widetilde{M}^{1,5} \times M^5$.

For certain flux forms, imposing topological restrictions on M^6 may result in additional conditions on F. For example, let $\mathsf{F} = \varphi\tilde{\alpha} + \tilde{\beta} \wedge \nu$ be a 4-form satisfying the Maxwell equation and the closedness condition. We assume that $\tilde{\alpha}, \tilde{\beta}, \nu$ are nonzero differential forms, and also that the function φ is nonzero at every point in M^6. Equation (3.7) of Proposition 3.4 implies that $\mathrm{d} \star_6 \nu = \lambda \star_6 \varphi$ or, equivalently, $\mathrm{d} \star_6 \nu = \lambda\varphi\, \mathrm{vol}_M$, for some constant $\lambda \in \mathbb{R}$. Now assume that M^6 is a closed manifold. We will see that this implies $\lambda = 0$. If $\lambda \neq 0$, then there exists a constant $K > 0$ such that $|\lambda\varphi| \geq K$. Therefore,

$$K\, \mathrm{vol}(M) = K \int_M \mathrm{vol}_M \leq \int_M |\lambda\varphi|\, \mathrm{vol}_M.$$

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The function $\lambda\varphi$ is either positive at each point in M^6 or negative at each point. Therefore, by Stokes' theorem, we deduce that for a connected closed manifold M^6, the right-hand side is (up to an overall sign) equal to

$$\int_M \lambda\varphi \,\mathsf{vol}_M = \int_M \mathrm{d}\star_6\nu = \int_{\partial M}\star_6\nu = 0.$$

This implies $\mathsf{vol}(M) = 0$, a contradiction. In particular, if φ is a nonzero constant, we get the following statement (after absorbing the constant φ into $\tilde{\alpha}$).

Proposition 3.6. Let $(\mathsf{X}^{1,10} = \widetilde{M}^{1,4}\times M^6, h = \tilde{g}+g, \mathsf{F} = \tilde{\alpha}+\tilde{\beta}\wedge\nu)$ be an eleven-dimensional bosonic supergravity background. If M^6 is closed, then $\lambda = 0$ in Eq. (3.7), i.e., $\mathrm{d}\star_5\tilde{\alpha} = 0$ and $\mathrm{d}\star_6\nu = 0$.

A similar phenomenon occurs for the flux form $\mathsf{F} = \tilde{\omega}\wedge\epsilon+\theta$.

Proposition 3.7. Let $(\mathsf{X}^{1,10} = \widetilde{M}^{1,4}\times M^6, h = \tilde{g}+g, \mathsf{F} = \tilde{\omega}\wedge\epsilon+\theta)$ be an eleven-dimensional bosonic supergravity background. If $\widetilde{M}^{1,4}$ is closed, then $\lambda = 0$ in equation (3.9), i.e., $\mathrm{d}\star_5\tilde{\omega} = 0$ and $\mathrm{d}\star_6\theta = 0$.

Proof. By assumption, we have $\tilde{\psi} = 1$ in (3.9), which implies $\mathrm{d}\star_5\tilde{\omega} = \lambda\star_5 1 = \lambda\,\mathsf{vol}_{\widetilde{M}}$. We have

$$\lambda\,\mathsf{vol}(\widetilde{M}) = \int_{\widetilde{M}}\lambda\,\mathsf{vol}_{\widetilde{M}} = \int_{\widetilde{M}}\mathrm{d}\star_5\tilde{\omega} = \int_{\partial\widetilde{M}}\star_5\tilde{\omega} = 0,$$

where the last equality follows since $\widetilde{M}^{1,4}$ has been assumed to be closed. This implies $\lambda = 0$. □

Before we treat the supergravity Einstein equation, let us consider one more special case for which the Maxwell equation significantly simplifies. Namely, assume that $\tilde{\psi}\theta = 0$ and that $\tilde{\alpha},\tilde{\beta},\tilde{\gamma},\tilde{\omega}$ share a common factor $\tilde{\omega}\in\Omega^1(\widetilde{M}^{1,4})$, meaning that $\tilde{\alpha} = \tilde{\omega}\wedge\hat{\alpha}, \tilde{\beta} = \tilde{\omega}\wedge\hat{\beta}, \tilde{\gamma} = \tilde{\omega}\wedge\hat{\gamma}$, $\tilde{\omega} = \hat{\omega}\tilde{\omega}$. Since $\tilde{\omega}$ is a 1-form, we have $\tilde{\omega}\wedge\tilde{\omega} = 0$. Therefore, the right-hand sides of Proposition 3.3 vanish. We summarize this in a proposition that we will take advantage of in Sec. VI F.

Proposition 3.8. Consider the Lorentzian manifold $(\mathsf{X}^{1,10} = \widetilde{M}^{1,4}\times M^6, h = \tilde{g}+g)$ with 4-form $\mathsf{F} = \varphi\tilde{\alpha}+\tilde{\beta}\wedge\nu+\tilde{\gamma}\wedge\delta+\tilde{\omega}\wedge\epsilon$ and assume that $\tilde{\alpha} = \tilde{\omega}\wedge\hat{\alpha}, \tilde{\beta} = \tilde{\omega}\wedge\hat{\beta}, \tilde{\gamma} = \tilde{\omega}\wedge\hat{\gamma}, \tilde{\omega} = \hat{\omega}\tilde{\omega}$ for a non-trivial 1-form $\tilde{\omega}\in\Omega^1(\widetilde{M}^{1,4})$ and $\hat{\alpha}\in\Omega^3(\widetilde{M}^{1,4})$, $\hat{\beta}\in\Omega^2(\widetilde{M}^{1,4})$, $\hat{\gamma}\in\Omega^1(\widetilde{M}^{1,4})$, $\hat{\omega}\in C^\infty(\widetilde{M}^{1,4})$. Then, the Maxwell equation is equivalent to the following system of equations:

$$\begin{aligned} &\mathrm{d}\star_5\tilde{\alpha}\wedge\star_6\varphi+\star_5\tilde{\beta}\wedge\mathrm{d}\star_6\nu = 0, \qquad &&\mathrm{d}\star_5\tilde{\beta}\wedge\star_6\nu-\star_5\tilde{\gamma}\wedge\mathrm{d}\star_6\delta = 0,\\ &\mathrm{d}\star_5\tilde{\gamma}\wedge\star_6\delta+\star_5\tilde{\omega}\wedge\mathrm{d}\star_6\epsilon = 0, \qquad &&\mathrm{d}\star_5\tilde{\omega}\wedge\star_6\epsilon = 0.\end{aligned}$$

In particular, Proposition 3.8 shows that if $\tilde{\alpha},\tilde{\beta},\tilde{\gamma},\tilde{\omega},\nu,\delta,\epsilon$ are coclosed on their respective manifolds and if $\tilde{\alpha},\tilde{\beta},\tilde{\gamma},\tilde{\omega}$ share a common factor $\tilde{\omega}$ as earlier, then $\mathsf{F} = \varphi\tilde{\alpha}+\tilde{\beta}\wedge\nu+\tilde{\gamma}\wedge\delta+\tilde{\omega}\wedge\epsilon$ satisfies the Maxwell equation.

IV. THE SUPERGRAVITY EINSTEIN EQUATION

In this section, we present the supergravity Einstein equation for an oriented Lorentzian manifold of the form $\mathsf{X} = \widetilde{M}^{1,4}\times M^6$, endowed with the product metric $h = \tilde{g}+g$ and the 4-form F defined by (2.2). We recall that the supergravity Einstein equation has the form

$$\mathsf{Ric}_h(X,Y) = -\frac{1}{2}\langle X\lrcorner\mathsf{F}, Y\lrcorner\mathsf{F}\rangle_h+\frac{1}{6}h(X,Y)\|\mathsf{F}\|_h^2, \tag{4.1}$$

where X, Y are vector fields on $\mathsf{X}^{1,10}$. Note that Lemma 3.2 implies the following:

Lemma 4.1. Let F be the 4-form on $(\mathsf{X}^{1,10} = \widetilde{M}^{1,4}\times M^6, h = \tilde{g}+g)$ defined by (2.2). Then,

$$\|\mathsf{F}\|_h^2 = \varphi^2\|\tilde{\alpha}\|_{\tilde{g}}^2+\|\tilde{\beta}\|_{\tilde{g}}^2\|\nu\|_g^2+\|\tilde{\gamma}\|_{\tilde{g}}^2\|\delta\|_g^2+\|\tilde{\omega}\|_{\tilde{g}}^2\|\epsilon\|_g^2+\tilde{\psi}^2\|\theta\|_g^2. \tag{4.2}$$

Since $\mathsf{X}^{1,10} = \widetilde{M}^{1,4}\times M^6$ is a direct product of pseudo-Riemannian manifolds, we have

$$\begin{aligned} &\mathsf{Ric}_h(X,Y) = \mathsf{Ric}_g(X,Y), &&\forall\ X,Y\in\Gamma(TM^6),\\ &\mathsf{Ric}_h(\tilde{X},\tilde{Y}) = \mathsf{Ric}_{\tilde{g}}(\tilde{X},\tilde{Y}), &&\forall\ \tilde{X},\tilde{Y}\in\Gamma(T\widetilde{M}^{1,4}),\\ &\mathsf{Ric}_h(X,\tilde{Y}) = 0, &&\forall\ X\in\Gamma(TM^6),\ \tilde{Y}\in\Gamma(T\widetilde{M}^{1,4}).\end{aligned}$$

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This lets us split Eq. (4.1) into three parts. By using (4.2), we explicitly obtain each of the parts of the supergravity Einstein equation. In particular:

For any $X, Y \in \Gamma(TM^6)$, we compute

$$\begin{aligned}\mathrm{Ric}_h(X,Y) = &\frac{\varphi^2\|\tilde{\alpha}\|_{\tilde{g}}^2}{6}g(X,Y)\\ &+\left(\frac{\|\nu\|_g^2}{6}g(X,Y)-\frac{1}{2}\nu(X)\nu(Y)\right)\|\tilde{\beta}\|_{\tilde{g}}^2\\ &+\left(\frac{\|\delta\|_g^2}{6}g(X,Y)-\frac{1}{2}\langle X\lrcorner\delta, Y\lrcorner\delta\rangle_g\right)\|\tilde{\gamma}\|_{\tilde{g}}^2.\\ &+\left(\frac{\|\epsilon\|_g^2}{6}g(X,Y)-\frac{1}{2}\langle X\lrcorner\epsilon, Y\lrcorner\epsilon\rangle_g\right)\|\tilde{\omega}\|_{\tilde{g}}^2\\ &+\left(\frac{\|\theta\|_g^2}{6}g(X,Y)-\frac{1}{2}\langle X\lrcorner\theta, Y\lrcorner\theta\rangle_g\right)\tilde{\psi}^2.\end{aligned} \tag{4.3}$$

For any $\tilde{X}, \tilde{Y} \in \Gamma(T\widetilde{M}^{1,4})$, we obtain

$$\begin{aligned}\mathrm{Ric}_h(\tilde{X},\tilde{Y}) = &\left(\frac{\|\tilde{\alpha}\|_{\tilde{g}}^2}{6}\tilde{g}(\tilde{X},\tilde{Y})-\frac{1}{2}\langle X\lrcorner\tilde{\alpha}, Y\lrcorner\tilde{\alpha}\rangle_{\tilde{g}}\right)\varphi^2\\ &+\left(\frac{\|\tilde{\beta}\|_{\tilde{g}}^2}{6}\tilde{g}(\tilde{X},\tilde{Y})-\frac{1}{2}\langle \tilde{X}\lrcorner\tilde{\beta}, \tilde{Y}\lrcorner\tilde{\beta}\rangle_{\tilde{g}}\right)\|\nu\|_g^2\\ &+\left(\frac{\|\tilde{\gamma}\|_{\tilde{g}}^2}{6}\tilde{g}(\tilde{X},\tilde{Y})-\frac{1}{2}\langle \tilde{X}\lrcorner\tilde{\gamma}, \tilde{Y}\lrcorner\tilde{\gamma}\rangle_{\tilde{g}}\right)\|\delta\|_g^2\\ &+\left(\frac{\|\tilde{\omega}\|_{\tilde{g}}^2}{6}\tilde{g}(\tilde{X},\tilde{Y})-\frac{1}{2}\tilde{\omega}(\tilde{X})\tilde{\omega}(\tilde{Y})\right)\|\epsilon\|_g^2\\ &+\frac{\tilde{\psi}^2\|\theta\|_g^2}{6}\tilde{g}(\tilde{X},\tilde{Y}).\end{aligned} \tag{4.4}$$

Finally, for any $X \in \Gamma(TM^6)$ and $\tilde{Y} \in \Gamma(T\widetilde{M}^{1,4})$, we get the following condition:

$$\begin{aligned}0 = \mathrm{Ric}_h(X,\tilde{Y}) = \frac{1}{2}\Big(&\varphi\nu(X)\langle\tilde{\beta}, \tilde{Y}\lrcorner\tilde{\alpha}\rangle_{\tilde{g}} - \langle\tilde{\gamma}\wedge(X\lrcorner\delta), (\tilde{Y}\lrcorner\tilde{\beta})\wedge\nu\rangle_h\\ &+\langle\tilde{\omega}\wedge(X\lrcorner\epsilon), (\tilde{Y}\lrcorner\tilde{\gamma})\wedge\delta\rangle_h - \tilde{\psi}\tilde{\omega}(\tilde{Y})\langle X\lrcorner\theta, \epsilon\rangle_g\Big).\end{aligned} \tag{4.5}$$

As a summary, we state the following:

Theorem 4.2. *Consider the manifold $\mathsf{X}^{1,10} = \widetilde{M}^{1,4} \times M^6$ with the product metric $h = \tilde{g} + g$, where $\tilde{g}$ is a Lorentzian metric on $\widetilde{M}^{1,4}$ and g is a Riemannian metric on M^6, and let F be the 4-form defined by (2.2). Then the eleven-dimensional supergravity Einstein equation (4.1) decomposes into Eqs. (4.3)–(4.5).*

Regarding the supergravity Einstein equation for the various special cases of F discussed in Proposition 3.4, we present the following result (all equations below hold for general vector fields X, Y on M^6, and $\tilde{X}, \tilde{Y}$ on $\widetilde{M}^{1,4}$, which, for brevity, we will not repeat).

Proposition 4.3. Consider the Lorentzian manifold $(\mathsf{X}^{1,10} = \widetilde{M}^{1,4} \times M^6, h = \tilde{g} + g)$.

(1) *The 4-form $\mathsf{F} \in \Omega^4(\mathsf{X})$ defined by*

$$\mathsf{F} = \tilde{\alpha}, \tag{4.6}$$

satisfies the Einstein condition if and only if the following equations hold:

$$\begin{aligned}\mathrm{Ric}_h(X,Y) &= \frac{\|\tilde{\alpha}\|_{\tilde{g}}^2}{6}g(X,Y),\\ \mathrm{Ric}_h(\tilde{X},\tilde{Y}) &= \frac{\|\tilde{\alpha}\|_{\tilde{g}}^2}{6}\tilde{g}(\tilde{X},\tilde{Y})-\frac{1}{2}\langle X\lrcorner\tilde{\alpha}, Y\lrcorner\tilde{\alpha}\rangle_{\tilde{g}}.\end{aligned} \tag{4.7}$$

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(2) *The 4-form* $\mathsf{F} \in \Omega^4(\mathsf{X})$ *defined by*

$$\mathsf{F} = \tilde{\beta} \wedge \nu, \tag{4.8}$$

satisfies the Einstein condition if and only if the following equations hold:

$$\begin{aligned} \mathrm{Ric}_h(X, Y) &= \left(\frac{\|\nu\|_g^2}{6} g(X, Y) - \frac{1}{2} \nu(X)\nu(Y) \right) \|\tilde{\beta}\|_{\tilde{g}}^2, \\ \mathrm{Ric}_h(\tilde{X}, \tilde{Y}) &= \left(\frac{\|\tilde{\beta}\|_{\tilde{g}}^2}{6} \tilde{g}(\tilde{X}, \tilde{Y}) - \frac{1}{2} \langle \tilde{X} \lrcorner \tilde{\beta}, \tilde{Y} \lrcorner \tilde{\beta} \rangle_{\tilde{g}} \right) \|\nu\|_g^2. \end{aligned} \tag{4.9}$$

(3) *The 4-form* $\mathsf{F} \in \Omega^4(\mathsf{X})$ *defined by*

$$\mathsf{F} = \tilde{\gamma} \wedge \delta, \tag{4.10}$$

satisfies the Einstein condition if and only if the following equations hold:

$$\begin{aligned} \mathrm{Ric}_h(X, Y) &= \left(\frac{\|\delta\|_g^2}{6} g(X, Y) - \frac{1}{2} \langle X \lrcorner \delta, Y \lrcorner \delta \rangle_g \right) \|\tilde{\gamma}\|_{\tilde{g}}^2, \\ \mathrm{Ric}_h(\tilde{X}, \tilde{Y}) &= \left(\frac{\|\tilde{\gamma}\|_{\tilde{g}}^2}{6} \tilde{g}(\tilde{X}, \tilde{Y}) - \frac{1}{2} \langle \tilde{X} \lrcorner \tilde{\gamma}, \tilde{Y} \lrcorner \tilde{\gamma} \rangle_{\tilde{g}} \right) \|\delta\|_g^2. \end{aligned} \tag{4.11}$$

(4) *The 4-form* $\mathsf{F} \in \Omega^4(\mathsf{X})$ *defined by*

$$\mathsf{F} = \tilde{\omega} \wedge \epsilon, \tag{4.12}$$

satisfies the Einstein condition if and only if the following equations hold:

$$\begin{aligned} \mathrm{Ric}_h(X, Y) &= \left(\frac{\|\epsilon\|_g^2}{6} g(X, Y) - \frac{1}{2} \langle X \lrcorner \epsilon, Y \lrcorner \epsilon \rangle_g \right) \|\tilde{\omega}\|_{\tilde{g}}^2, \\ \mathrm{Ric}_h(\tilde{X}, \tilde{Y}) &= \left(\frac{\|\tilde{\omega}\|_{\tilde{g}}^2}{6} \tilde{g}(\tilde{X}, \tilde{Y}) - \frac{1}{2} \tilde{\omega}(\tilde{X})\tilde{\omega}(\tilde{Y}) \right) \|\epsilon\|_g^2. \end{aligned} \tag{4.13}$$

(5) *The 4-form* $\mathsf{F} \in \Omega^4(\mathsf{X})$ *defined by*

$$\mathsf{F} = \theta, \tag{4.14}$$

satisfies the Einstein condition if and only if the following equations hold:

$$\begin{aligned} \mathrm{Ric}_h(X, Y) &= \frac{\|\theta\|_g^2}{6} g(X, Y) - \frac{1}{2} \langle X \lrcorner \theta, Y \lrcorner \theta \rangle_g, \\ \mathrm{Ric}_h(\tilde{X}, \tilde{Y}) &= \frac{\|\theta\|_g^2}{6} \tilde{g}(\tilde{X}, \tilde{Y}). \end{aligned} \tag{4.15}$$

(6) *The 4-form* $\mathsf{F} \in \Omega^4(\mathsf{X})$ *defined by*

$$\mathsf{F} = \varphi\, \tilde{\alpha} + \tilde{\beta} \wedge \nu, \tag{4.16}$$

satisfies the Einstein condition if and only if the following equations hold:

$$\begin{aligned} \mathrm{Ric}_h(X, Y) &= \frac{\varphi^2 \|\tilde{\alpha}\|_{\tilde{g}}^2}{6} g(X, Y) + \left(\frac{\|\nu\|_g^2}{6} g(X, Y) - \frac{1}{2} \nu(X)\nu(Y) \right) \|\tilde{\beta}\|_{\tilde{g}}^2, \\ \mathrm{Ric}_h(\tilde{X}, \tilde{Y}) &= \left(\frac{\|\tilde{\alpha}\|_{\tilde{g}}^2}{6} \tilde{g}(\tilde{X}, \tilde{Y}) - \frac{1}{2} \langle \tilde{X} \lrcorner \tilde{\alpha}, \tilde{Y} \lrcorner \tilde{\alpha} \rangle_{\tilde{g}} \right) \varphi^2 + \left(\frac{\|\tilde{\beta}\|_{\tilde{g}}^2}{6} \tilde{g}(\tilde{X}, \tilde{Y}) - \frac{1}{2} \langle \tilde{X} \lrcorner \tilde{\beta}, \tilde{Y} \lrcorner \tilde{\beta} \rangle_{\tilde{g}} \right) \|\nu\|_g^2, \\ 0 &= \varphi\, \nu(X) \langle \tilde{\beta}, \tilde{Y} \lrcorner \tilde{\alpha} \rangle_{\tilde{g}}. \end{aligned} \tag{4.17}$$

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(7) *The 4-form* $\mathsf{F} \in \Omega^4(\mathsf{X})$ *defined by*

$$\mathsf{F} = \tilde{\omega} \wedge \epsilon + \tilde{\psi}\,\theta, \tag{4.18}$$

satisfies the Einstein condition if and only if the following equations hold:

$$\begin{aligned} \mathrm{Ric}_h(X,Y) &= \left(\frac{\|\epsilon\|_g^2}{6} g(X,Y) - \frac{1}{2}\langle X \lrcorner \epsilon, Y \lrcorner \epsilon\rangle_g\right)\|\tilde{\omega}\|_{\tilde{g}}^2 + \left(\frac{\|\theta\|_g^2}{6} g(X,Y) - \frac{1}{2}\langle X \lrcorner \theta, Y \lrcorner \theta\rangle_g\right)\tilde{\psi}^2, \\ \mathrm{Ric}_h(\tilde{X},\tilde{Y}) &= \left(\frac{\|\tilde{\omega}\|_{\tilde{g}}^2}{6}\tilde{g}(\tilde{X},\tilde{Y}) - \frac{1}{2}\tilde{\omega}(\tilde{X})\tilde{\omega}(\tilde{Y})\right)\|\epsilon\|_g^2 + \frac{\tilde{\psi}^2\|\theta\|_g^2}{6}\tilde{g}(\tilde{X},\tilde{Y}), \\ 0 &= \tilde{\psi}\,\tilde{\omega}(\tilde{Y})\langle X \lrcorner \theta, \epsilon\rangle_g. \end{aligned} \tag{4.19}$$

(8) *The 4-form* $\mathsf{F} \in \Omega^4(\mathsf{X})$ *defined by*

$$\mathsf{F} = \tilde{\alpha} + \theta, \tag{4.20}$$

satisfies the Einstein condition if and only if the following equations hold:

$$\begin{aligned} \mathrm{Ric}_h(X,Y) &= \frac{\|\tilde{\alpha}\|_{\tilde{g}}^2}{6} g(X,Y) + \frac{\|\theta\|_g^2}{6} g(X,Y) - \frac{1}{2}\langle X \lrcorner \theta, Y \lrcorner \theta\rangle_g, \\ \mathrm{Ric}_h(\tilde{X},\tilde{Y}) &= \frac{\|\tilde{\alpha}\|_{\tilde{g}}^2}{6}\tilde{g}(\tilde{X},\tilde{Y}) - \frac{1}{2}\langle X \lrcorner \tilde{\alpha}, Y \lrcorner \tilde{\alpha}\rangle_{\tilde{g}} + \frac{\|\theta\|_g^2}{6}\tilde{g}(\tilde{X},\tilde{Y}). \end{aligned} \tag{4.21}$$

(9) *The 4-form* $\mathsf{F} \in \Omega^4(\mathsf{X})$ *defined by*

$$\mathsf{F} = \tilde{\beta} \wedge \nu + \tilde{\omega} \wedge \epsilon, \tag{4.22}$$

satisfies the Einstein condition if and only if the following equations hold:

$$\begin{aligned} \mathrm{Ric}_h(X,Y) &= \left(\frac{\|\nu\|_g^2}{6} g(X,Y) - \frac{1}{2}\nu(X)\nu(Y)\right)\|\tilde{\beta}\|_{\tilde{g}}^2 + \left(\frac{\|\epsilon\|_g^2}{6} g(X,Y) - \frac{1}{2}\langle X \lrcorner \epsilon, Y \lrcorner \epsilon\rangle_g\right)\|\tilde{\omega}\|_{\tilde{g}}^2, \\ \mathrm{Ric}_h(\tilde{X},\tilde{Y}) &= \left(\frac{\|\tilde{\beta}\|_{\tilde{g}}^2}{6}\tilde{g}(\tilde{X},\tilde{Y}) - \frac{1}{2}\langle \tilde{X} \lrcorner \tilde{\beta}, \tilde{Y} \lrcorner \tilde{\beta}\rangle_{\tilde{g}}\right)\|\nu\|_g^2 + \left(\frac{\|\tilde{\omega}\|_{\tilde{g}}^2}{6}\tilde{g}(\tilde{X},\tilde{Y}) - \frac{1}{2}\tilde{\omega}(\tilde{X})\tilde{\omega}(\tilde{Y})\right)\|\epsilon\|_g^2. \end{aligned} \tag{4.23}$$

Proof. For each case, the equations involving the Ricci tensor follow directly from Eqs. (4.3)–(4.5). Note that we have simplified the cases (1) and (5) by setting φ and $\tilde{\psi}$ equal to 1 since we are only interested in solutions to the supergravity Einstein equation that also satisfy the Maxwell equation and the closedness condition (recall from Lemma 3.1 that the closedness condition implies that φ and $\tilde{\psi}$ are constant in these particular cases). □

We see that the supergravity Einstein equation simplifies significantly when the form of F is further specified, as earlier. In fact, the special form of many of the equations in Proposition 4.3 leads to some particular consequences that we will now investigate.

Let $(\tilde{x}, x)$ denote a general point on $\widetilde{M}^{1,4} \times M^6$. For each equation in Proposition 4.3, we observe that the left-hand side depends either on $\tilde{x}$ or x (but not both), while the right-hand side is either of the form $f_1(\tilde{x})g_1(x)$ or $f_1(\tilde{x})g_1(x) + f_2(\tilde{x})g_2(x)$ for every pair of vector fields. From this, we draw some conclusions about f_i and g_i, which are functions on $\widetilde{M}^{1,4}$ and M^6, respectively. They are based on the following simple observation.

Lemma 4.4. Assume that $r(\tilde{x}) = f_1(\tilde{x})g_1(x)$ *(for every* $\tilde{x} \in \widetilde{M}^{1,4}$ *and every* $x \in M^6$*). Then either* f_1 *is identically equal to zero or* g_1 *is constant.*

Assume that $r(\tilde{x}) = f_1(\tilde{x})g_1(x) + f_2(\tilde{x})g_2(x)$ *and that none of the functions* f_1 *and* f_2 *are identically equal to zero (if one of them is, then the situation is the same as earlier). Then either* g_1 *and* g_2 *are both constant, or* $f_1(\tilde{x}) = Cf_2(\tilde{x})$ *for some* $C \in \mathbb{R}\backslash\{0\}$*, and* $g_2(x) = -Cg_1(x) + D$ *for some constant* $D \in \mathbb{R}$*. In the latter case, we have* $r(\tilde{x}) = Df_2(\tilde{x})$*.*

Notice that a similar statement holds after switching x with $\tilde{x}$. The application of these statements to the equations of Proposition 4.3 results in the following corollary.

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Corollary 4.5. Assume that $(\mathsf{X}^{1,10} = \widetilde{M}^{1,4} \times M^6, h = \tilde{g} + g, \mathsf{F})$ *is a solution of the supergravity Einstein equations (4.1).*

(1) If $\mathsf{F} = \tilde{\alpha}$, *then* (M^6, g) *is Einstein with Einstein constant* $\|\tilde{\alpha}\|_{\tilde{g}}^2/6$.

(2) If $\mathsf{F} = \tilde{\beta} \wedge \nu$, *then* $\|\tilde{\beta}\|_{\tilde{g}}^2$ *is constant.*

(3) If $\mathsf{F} = \tilde{\gamma} \wedge \delta$ *and* $\|\tilde{\gamma}\|_{\tilde{g}}^2$ *is not constant, then we have* $g(X, Y) = \frac{3}{\|\delta\|_g^2} \langle X \lrcorner \delta, Y \lrcorner \delta \rangle_g$ *for all* $X, Y \in \Gamma(TM^6)$, *which implies that* (M^6, g, ω, J) *is a Ricci-flat almost Hermitian manifold with an almost complex structure* J *defined by* $\omega(X, Y) = g(JX, Y)$ *and Kähler form* $\omega = \sqrt{\frac{3}{\|\delta\|_g^2}}\,\delta$.

(4) If $\mathsf{F} = \tilde{\omega} \wedge \epsilon$, *then* $\|\epsilon\|_g^2$ *is constant and negative.*

(5) If $\mathsf{F} = \theta$, *then* $(\widetilde{M}^{1,4}, \tilde{g})$ *is Einstein with Einstein constant* $\|\theta\|_g^2/6 > 0$.

(6) If $\mathsf{F} = \varphi\tilde{\alpha} + \tilde{\beta} \wedge \nu$, *then* $\|\tilde{\beta}\|_{\tilde{g}}^2$ *and* $\|\tilde{\alpha}\|_{\tilde{g}}^2$ *are constant.*

(7) If $\mathsf{F} = \tilde{\omega} \wedge \epsilon + \tilde{\psi}\theta$, *then* $\|\epsilon\|_g^2$ *and* $\|\theta\|_g^2$ *are constant.*

Proof. Statements (1) and (5) are obvious. We prove the rest of them.

(2) When $Y = X$, the first equation of (4.9) reduces to

$$\mathrm{Ric}_h(X, X) = \left(\frac{\|\nu\|_g^2}{6} \|X\|_g^2 - \frac{1}{2} \nu(X)^2 \right) \|\tilde{\beta}\|_{\tilde{g}}^2,$$

which must hold for every $X \in \Gamma(TM^6)$. By Lemma 4.4, either $\|\tilde{\beta}\|_{\tilde{g}}^2$ is constant, or

$$\left(\frac{\|\nu\|_g^2}{6} \|X\|_g^2 - \frac{1}{2} \nu(X)^2 \right) = 0.$$

Let X be a nonzero vector field in the kernel of ν, that is, $\nu(X) = 0$. Since g is negative definite, the functions $\|\nu\|_g^2$ and $\|X\|_g^2$ are not identically zero. Therefore, $\|\tilde{\beta}\|_{\tilde{g}}^2$ must be constant.

(3) If $\mathsf{F} = \tilde{\gamma} \wedge \delta$, we have from the first equation of (4.11) that

$$\mathrm{Ric}_h(X, Y) = \left(\frac{\|\delta\|_g^2}{6} g(X, Y) - \frac{1}{2} \langle X \lrcorner \delta, Y \lrcorner \delta \rangle_g \right) \|\tilde{\gamma}\|_{\tilde{g}}^2.$$

If $\|\tilde{\gamma}\|_{\tilde{g}}^2$ is not constant, it follows from Lemma 4.4 that $g(X, Y) = \frac{3}{\|\delta\|_g^2} \langle X \lrcorner \delta, Y \lrcorner \delta \rangle_g$. Consequently, (M^6, g) is Ricci-flat. Since δ is a 2-form, $\|\delta\|_g^2$ is positive. Using the definition of ω we can write the condition as $g(X, Y) = \langle X \lrcorner \omega, Y \lrcorner \omega \rangle_g$. If $\omega(X, Z) = 0$ for every Z, then $g(X, Y) = 0$ for every Y, so the non-degeneracy of g implies the non-degeneracy of ω. This allows us to define an almost complex structure J via $\omega(X, Y) = g(JX, Y)$, since we then get

$$J_i^j J_j^k = \omega_{ia} g^{aj} \omega_{jb} g^{bk} = -g^{aj} \omega_{ia} \omega_{bj} g^{bk} = -\mathrm{Id}_i^k,$$

where the last equality follows from $g(X, Y) = \langle X \lrcorner \omega, Y \lrcorner \omega \rangle_g$. Moreover,

$$g(JX, JY) = \omega(JX, J^2 Y) = \omega(JX, -Y) = \omega(Y, JX) = g(Y, X) = g(X, Y),$$

and consequently (M^6, g, J) is an almost Hermitian manifold.

(4) For $\tilde{Y} = \tilde{X}$, the second equation of (4.13) reduces to

$$\mathrm{Ric}_h(\tilde{X}, \tilde{X}) = \left(\frac{\|\tilde{\omega}\|_{\tilde{g}}^2}{6} \tilde{g}(\tilde{X}, \tilde{X}) - \frac{1}{2} \tilde{\omega}(\tilde{X})^2 \right) \|\epsilon\|_g^2.$$

If $\|\tilde{\omega}\|_{\tilde{g}}^2 = 0$, let $\tilde{X}$ be a vector with the property $\tilde{\omega}(\tilde{X}) \neq 0$. If $\|\tilde{\omega}\|_{\tilde{g}}^2$ is different from 0, let $\tilde{X}$ be the $\tilde{g}$-dual of the 1-form $\tilde{\omega}$. Then we obtain the equation

$$\mathrm{Ric}_h(\tilde{X}, \tilde{X}) = -\|\tilde{\omega}\|_{\tilde{g}}^4 \|\epsilon\|_g^2/3.$$

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In both cases, we see that $\|\epsilon\|_g^2$ is constant due to Lemma 4.4, and it is negative since g is negative definite on 3-forms.

(6) To see that $\|\tilde{\alpha}\|_{\tilde{g}}^2$ is constant, let us assume that X is given by $X^j = \frac{1}{3}g^{ij}v_i$. Then, the first equation of (4.17) reduces to

$$\begin{aligned}\mathsf{Ric}_h(X,Y) &= \frac{\varphi^2\|\tilde{\alpha}\|_{\tilde{g}}^2}{6}g(X,Y) + \left(\frac{\|v\|_g^2}{6}g(X,Y) - \frac{1}{2}v(X)v(Y)\right)\|\tilde{\beta}\|_{\tilde{g}}^2\\ &= \left(\frac{\varphi^2\|\tilde{\alpha}\|_{\tilde{g}}^2}{6} + \left(\frac{\|v\|_g^2}{6} - \frac{1}{2}v(X)\right)\|\tilde{\beta}\|_{\tilde{g}}^2\right)v(Y)\\ &= \frac{\varphi^2\|\tilde{\alpha}\|_{\tilde{g}}^2}{6}v(Y),\end{aligned}$$

since $v(X) = \|v\|_g^2/3$. By choosing Y such that $v(Y) = 3g(X,Y) \neq 0$, we see that $\|\tilde{\alpha}\|_{\tilde{g}}^2$ is constant by Lemma 4.4. It is then clear that also $\|\tilde{\beta}\|_{\tilde{g}}^2$ must be constant.

(7) The proof is similar to that of (6). □

Remark 4.6. The sign of the Einstein constant depends on the signature convention. For example, if the manifold $(\widetilde{M}^{1,4},\tilde{g})$ in point (5) has positive Einstein constant, it will have negative Einstein constant in the "mostly plus" convention.

Remark 4.7. For point (3) in Corollary 4.5, we observe the following. If $(\tilde{X}^{1,10},h,\mathsf{F} = \tilde{\gamma}\wedge\delta)$ satisfies the closedness condition, then $d\delta = 0$. If $\|\delta\|_g^2$ is constant, this implies that ω is closed, and so (M^6,g,J,ω) must be an almost Kähler manifold.

From the trace of the Einstein equation, consequently, the scalar curvature Scal_h of a bosonic supergravity background $(\mathsf{X}^{1,10},h,\mathsf{F})$ satisfies the relation $\mathsf{Scal}_h = \frac{1}{6}\|\mathsf{F}\|_h^2$ (see, for example, Refs. 23 and 35). For the cases (1) and (5) in Corollary 4.5, we obtain that $\|\mathsf{F}\|_h^2$ is constant.

Corollary 4.8. Assume that $(\mathsf{X}^{1,10} = \widetilde{M}^{1,4}\times M^6, h = \tilde{g}+g,\mathsf{F})$ is a solution of the supergravity Einstein equations (4.1).
If $\mathsf{F} = \tilde{\alpha}$, then $\mathsf{Scal}_h = \frac{1}{6}\|\tilde{\alpha}\|_{\tilde{g}}^2$ is constant.
If $\mathsf{F} = \theta$, then $\mathsf{Scal}_h = \frac{1}{6}\|\theta\|_g^2$ is constant and positive.

V. GENERAL THEOREMS REGARDING FLUX FORMS COMPOSED OF NULL FORMS

We continue our investigation of manifolds of the form $\mathsf{X}^{1,10} = \widetilde{M}^{1,4}\times M^6$, endowed with the product metric $h = \tilde{g}+g$ and the 4-form F given in (2.2). Since $\tilde{g}$ is a Lorentzian metric, there exist differential forms on $\widetilde{M}^{1,4}$ that are null. Recall that a k-form $\omega\in\Omega^k(\widetilde{M}^{1,4})$ is called *null* if $\langle\omega,\omega\rangle_{\tilde{g}} = 0$. In this section, we will show that if F is composed of such forms, then the supergravity Einstein equation simplifies. These results are of particular relevance whenever $\widetilde{M}^{1,4}$ comes equipped with a distribution of null lines. Indeed, this is the case, for example, if $\widetilde{M}^{1,4}$ is a *Walker manifold* or a *Kundt spacetime*. The case when $\widetilde{M}^{1,4}$ is a Walker manifold will be further explored in Sec. VI in a way analogous to what was performed in Ref. 30.

The following proposition concerns a particular type of null flux form and is a direct consequence of the relations (4.3) and (4.4).

Proposition 5.1. Consider the Lorentzian manifold $\mathsf{X}^{1,10} = \widetilde{M}^{1,4}\times M^6$ with metric $h = \tilde{g}+g$ and the 4-form F, given by (2.2). Assume that $\tilde{\alpha},\tilde{\beta},\tilde{\gamma},\tilde{\omega}$ are null and, moreover, that $\tilde{\psi} = 0$. Then, $(\mathsf{X}^{1,10},h,\mathsf{F})$ satisfies the supergravity Einstein equations if and only if (M^6,g) is Ricci-flat and

$$\begin{aligned}\mathsf{Ric}_h(\tilde{X},\tilde{Y}) &= -\frac{1}{2}\langle\tilde{X}\lrcorner\mathsf{F},\tilde{Y}\lrcorner\mathsf{F}\rangle_h\\ &= -\frac{1}{2}\Big(\langle\tilde{X}\lrcorner\tilde{\alpha},\tilde{Y}\lrcorner\tilde{\alpha}\rangle_{\tilde{g}}\varphi^2 + \langle\tilde{X}\lrcorner\tilde{\beta},\tilde{Y}\lrcorner\tilde{\beta}\rangle_{\tilde{g}}\|v\|_g^2 + \langle\tilde{X}\lrcorner\tilde{\gamma},\tilde{Y}\lrcorner\tilde{\gamma}\rangle_{\tilde{g}}\|\delta\|_g^2 + \tilde{\omega}(\tilde{X})\tilde{\omega}(\tilde{Y})\|\epsilon\|_g^2\Big),\\ 0 &= \varphi v(Z)\langle\tilde{\beta},\tilde{Y}\lrcorner\tilde{\alpha}\rangle_{\tilde{g}} - \langle\tilde{\gamma}\wedge(Z\lrcorner\delta),(\tilde{Y}\lrcorner\tilde{\beta})\wedge v\rangle_h + \langle\tilde{\omega}\wedge(Z\lrcorner\epsilon),(\tilde{Y}\lrcorner\tilde{\gamma})\wedge\delta\rangle_h,\end{aligned}$$

for any $\tilde{X},\tilde{Y}\in\Gamma(T\widetilde{M}^{1,4})$ and $Z\in\Gamma(TM^6)$.

One notable consequence is that the component (M^6,g) of bosonic supergravity backgrounds of the type described in Proposition 5.1 is required to be Ricci-flat. Moreover, by combining Proposition 5.1 with Proposition 3.4, we arrive at the following general statements.

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Theorem 5.2. *Consider the Lorentzian manifold* $\mathsf{X}^{1,10} = \widetilde{M}^{1,4} \times M^6$ *with metric* $h = \tilde{g} + g$ *and the 4-form* $\mathsf{F} = \tilde{\alpha}$*, where* $\tilde{\alpha}$ *is null. Then* $(\mathsf{X}^{1,10}, h, \mathsf{F})$ *is a bosonic supergravity background if and only if* M^6 *is Ricci-flat,* $\tilde{\alpha}$ *is closed and co-closed on* M^6*, and*

$$\mathsf{Ric}_h(\tilde{X}, \tilde{Y}) = -\frac{1}{2}\langle \tilde{X}\lrcorner\tilde{\alpha}, \tilde{Y}\lrcorner\tilde{\alpha}\rangle_{\tilde{g}}, \quad \forall\ \tilde{X}, \tilde{Y} \in \Gamma(T\widetilde{M}). \tag{5.1}$$

Note that if $\mathsf{F} = \varphi\tilde{\alpha}$, then Proposition 3.4 implies that φ is constant, and it can thus be absorbed into $\tilde{\alpha}$. Therefore, the above-mentioned condition $\mathsf{F} = \tilde{\alpha}$ is considered without loss of generality.

Theorem 5.3. *Consider the Lorentzian manifold* $\mathsf{X}^{1,10} = \widetilde{M}^{1,4} \times M^6$ *with metric* $h = \tilde{g} + g$ *and the 4-form* $\mathsf{F} = \tilde{\beta} \wedge \nu$*, where* $\tilde{\beta}$ *is null. Then* $(\mathsf{X}^{1,10}, h, \mathsf{F})$ *is a bosonic supergravity background if and only if* M^6 *is Ricci-flat,* $\tilde{\beta}$ *and* ν *are closed and co-closed (on* $\widetilde{M}^{1,4}$ *and* M^6*, respectively), and*

$$\mathsf{Ric}_h(\tilde{X}, \tilde{Y}) = -\frac{1}{2}\langle \tilde{X}\lrcorner\tilde{\beta}, \tilde{Y}\lrcorner\tilde{\beta}\rangle_{\tilde{g}}\|\nu\|_g^2, \quad \forall\ \tilde{X}, \tilde{Y} \in \Gamma(T\widetilde{M}). \tag{5.2}$$

Theorem 5.4. *Consider the Lorentzian manifold* $\mathsf{X}^{1,10} = \widetilde{M}^{1,4} \times M^6$ *with metric* $h = \tilde{g} + g$ *and the 4-form* $\mathsf{F} = \tilde{\gamma} \wedge \delta$*, where* $\tilde{\gamma}$ *is null. Then* $(\mathsf{X}^{1,10}, h, \mathsf{F})$ *is a bosonic supergravity background if and only if* M^6 *is Ricci-flat,* $\tilde{\gamma}$ *and* δ *are closed (on* $\widetilde{M}^{1,4}$ *and* M^6*, respectively),* δ *is co-closed on* M *and the following two equations hold:*

$$\mathrm{d} \star_5\tilde{\gamma} \wedge \star_6\delta = \frac{1}{2}\tilde{\gamma} \wedge \tilde{\gamma} \wedge \delta \wedge \delta, \quad \mathsf{Ric}_h(\tilde{X}, \tilde{Y}) = -\frac{1}{2}\langle \tilde{X}\lrcorner\tilde{\gamma}, \tilde{Y}\lrcorner\tilde{\gamma}\rangle_{\tilde{g}}\|\delta\|_g^2, \quad \forall\ \tilde{X}, \tilde{Y} \in \Gamma(T\widetilde{M}). \tag{5.3}$$

Note that the first condition of (5.3) is satisfied if $\tilde{\gamma}$ is co-closed and either $\tilde{\gamma} \wedge \tilde{\gamma} = 0$ or $\delta \wedge \delta = 0$.

Theorem 5.5. *Consider the Lorentzian manifold* $\mathsf{X}^{1,10} = \widetilde{M}^{1,4} \times M^6$ *with metric* $h = \tilde{g} + g$ *and the 4-form* $\mathsf{F} = \tilde{\omega} \wedge \epsilon$*, where* $\tilde{\omega}$ *is null. Then* $(\mathsf{X}^{1,10}, h, \mathsf{F})$ *is a bosonic supergravity background if and only if* M^6 *is Ricci-flat,* $\tilde{\omega}$ *and* ϵ *are closed and co-closed (on* $\widetilde{M}^{1,4}$ *and* M^6*, respectively), and*

$$\mathsf{Ric}_h(\tilde{X}, \tilde{Y}) = -\frac{1}{2}\tilde{\omega}(\tilde{X})\tilde{\omega}(\tilde{Y})\|\epsilon\|_g^2, \quad \forall\ \tilde{X}, \tilde{Y} \in \Gamma(T\widetilde{M}). \tag{5.4}$$

Observe that Eq. (5.4) implies that $\|\epsilon\|_g^2$ is constant, and without loss of generality, one may assume that it is equal to -1 (by absorbing the constant into $\tilde{\omega}$). Let us also recall the following definition (see, for example, Ref. 36).

Definition 5.6. A Lorentzian manifold (X, h) is called *totally Ricci-isotropic* if the Ricci endomorphism $\mathrm{ric}^h : T\mathsf{X} \to T\mathsf{X}$ corresponding to Ric^h satisfies the relation

$$h(\mathrm{ric}^h(X), \mathrm{ric}^h(Y)) = 0, \quad \forall\ X, Y \in \Gamma(T\mathsf{X}).$$

In Theorem 5.5, we observe that a bosonic supergravity background $(\mathsf{X}^{1,10} = \widetilde{M}^{1,4} \times M^6, h = \tilde{g} + g)$, with flux form $\mathsf{F} = \tilde{\omega} \wedge \epsilon$ and $\tilde{\omega}$ null, is totally Ricci-isotropic. This is essentially the same claim as Ref. 30, Corollary 5.10, and the proof is similar.

Corollary 5.7. Let $\tilde{\omega} \in \Omega^2(\widetilde{M}^{1,4})$ *be null. Then, a bosonic supergravity background* $(\mathsf{X}^{1,10} = \widetilde{M}^{1,4} \times M^6, h = \tilde{g} + g, \mathsf{F} = \tilde{\omega} \wedge \epsilon)$ *is totally Ricci-isotropic.*

Next, we state two results concerning flux forms of the form $\varphi\tilde{\alpha} + \tilde{\beta} \wedge \nu$ and $\tilde{\beta} \wedge \nu + \tilde{\omega} \wedge \epsilon$, respectively, where $\tilde{\alpha}, \tilde{\beta}$ and $\tilde{\omega}$ are null.

Theorem 5.8. *Consider the Lorentzian manifold* $\mathsf{X}^{1,10} = \widetilde{M}^{1,4} \times M^6$ *with metric* $h = \tilde{g} + g$ *and the 4-form* $\mathsf{F} = \varphi\tilde{\alpha} + \tilde{\beta} \wedge \nu$*, where* $\tilde{\alpha}$ *and* $\tilde{\beta}$ *are null. Then* $(\mathsf{X}^{1,10}, h, \mathsf{F})$ *is a bosonic supergravity background if and only if* M^6 *is Ricci-flat,*

$$\mathrm{d}\,\tilde{\alpha} = \mathrm{d}\,\nu = 0, \quad \mathrm{d}\,\varphi = \kappa\nu, \quad \mathrm{d}\,\tilde{\beta} = -\kappa\tilde{\alpha}, \quad \mathrm{d} \star_5\tilde{\beta} = 0, \quad \mathrm{d} \star_5\tilde{\alpha} = -\lambda\star_5\tilde{\beta}, \quad \mathrm{d} \star_6\nu = \lambda\star_6\varphi,$$

for some constants $\kappa, \lambda \in \mathbb{R}$ *and*

$$\begin{aligned}\mathsf{Ric}_h(\tilde{X}, \tilde{Y}) &= -\frac{1}{2}\langle \tilde{X}\lrcorner\tilde{\alpha}, \tilde{Y}\lrcorner\tilde{\alpha}\rangle_{\tilde{g}}\varphi^2 - \frac{1}{2}\langle \tilde{X}\lrcorner\tilde{\beta}, \tilde{Y}\lrcorner\tilde{\beta}\rangle_{\tilde{g}}\|\nu\|_g^2, \quad \forall \tilde{X}, \tilde{Y} \in \Gamma(T\widetilde{M}),\\ 0 &= \langle \tilde{\beta}, \tilde{X}\lrcorner\tilde{\alpha}\rangle_{\tilde{g}}, \quad \forall \tilde{X} \in \Gamma(T\widetilde{M}).\end{aligned}$$

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Theorem 5.9. *Consider the Lorentzian manifold $\mathsf{X}^{1,10} = \widetilde{M}^{1,4} \times M^6$ with metric $h = \tilde{g} + g$ and the 4-form $\mathsf{F} = \tilde{\beta} \wedge \nu + \tilde{\omega} \wedge \epsilon$, where $\tilde{\beta}$ and $\tilde{\omega}$ are null. Then $(\mathsf{X}^{1,10}, h, \mathsf{F})$ is a bosonic supergravity background if and only if M^6 is Ricci-flat,*

$$\mathrm{d}\,\tilde{\beta} = \mathrm{d}\,\nu = \mathrm{d}\,\tilde{\omega} = \mathrm{d}\,\epsilon = 0, \quad \mathrm{d} \star_6 \nu = \mathrm{d} \star_5 \tilde{\beta} = \mathrm{d} \star_5 \tilde{\omega} = 0, \quad \star_5 \tilde{\omega} \wedge \mathrm{d} \star_6 \epsilon = \tilde{\beta} \wedge \tilde{\omega} \wedge \epsilon \wedge \nu,$$

and

$$\mathsf{Ric}_h(\tilde{X}, \tilde{Y}) = -\frac{1}{2}\langle \tilde{X} \lrcorner \tilde{\beta}, \tilde{Y} \lrcorner \tilde{\beta} \rangle_{\tilde{g}} \|\nu\|_g^2 - \frac{1}{2}\tilde{\omega}(\tilde{X})\tilde{\omega}(\tilde{Y})\|\epsilon\|_g^2, \quad \forall \tilde{X}, \tilde{Y} \in \Gamma(T\widetilde{M}).$$

Notice that the last equation in Proposition 5.1, the one coming from Eq. (4.5), is satisfied automatically in Theorems 5.2–5.9. The only exception is Theorem 5.8, where the consequence $\langle \tilde{\beta}, \tilde{X} \lrcorner \tilde{\alpha} \rangle_{\tilde{g}} = 0$ gives additional constraints.

Let us also pose the following:

Corollary 5.10. All bosonic supergravity backgrounds appearing in this section have vanishing scalar curvature.

Proof. This is a simple consequence of the relation $\mathsf{Scal}_h = \frac{1}{6}\|\mathsf{F}\|_h^2$ and the fact that the flux form F is null for all the backgrounds presented earlier. □

VI. BOSONIC SUPERGRAVITY BACKGROUNDS FOR WHICH $\tilde{M}^{1,4}$ IS A RICCI-ISOTROPIC WALKER MANIFOLD

In order to construct explicit examples of bosonic supergravity backgrounds for which the flux form is composed of null forms, as treated in Sec. V, we assume in this section that $\widetilde{M}^{1,4}$ is a *Lorentzian Walker manifold*. Lorentzian Walker manifolds admit a parallel distribution of isotropic lines, which we will use to build the Lorentzian part of the flux form F. With the aim of further simplifying the supergravity Einstein equations, we follow Ref. 30 and will work with a special type of totally Ricci-isotropic Walker manifolds, defined below. In Secs. VI B–VI E, we consider the simplest type of flux forms, namely, those of the form $\tilde{\alpha}$, $\tilde{\beta} \wedge \nu$, $\tilde{\gamma} \wedge \delta$, and $\tilde{\omega} \wedge \epsilon$. In Sec. VI F, we unify these results by considering the more general flux form $\varphi\tilde{\alpha} + \tilde{\beta} \wedge \nu + \tilde{\gamma} \wedge \delta + \tilde{\omega} \wedge \epsilon$ under the additional condition that the eight involved differential forms $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\omega}, \varphi, \nu, \delta, \epsilon$ are closed and coclosed on their respective manifolds. Finally, in Sec. VI G, we consider the flux form $\varphi\tilde{\alpha} + \tilde{\beta} \wedge \nu$ without the strict assumption of closedness and coclosedness for each of its components.

A. Ricci-isotropic Walker manifolds and null forms

Let us recall the definition of a Lorentzian Walker manifold.

Definition 6.1. A Lorentzian Walker manifold is a Lorentzian manifold that admits a parallel distribution of isotropic (or null) lines.

Next, we focus on the five-dimensional case. If $(\widetilde{M}^{1,4}, \tilde{g})$ is a Lorentzian Walker manifold, then it is locally diffeomorphic to a product $\mathbb{R} \times N^3 \times \mathbb{R}$ of manifolds with coordinates u, $x = (x^1, x^2, x^3)$, and v, respectively, on which the metric takes the form

$$\tilde{g} = 2\,\mathrm{d}u\,\mathrm{d}v + \rho + 2A\,\mathrm{d}\,u + H\,\mathrm{d}u^2. \tag{6.1}$$

Here, $\rho = \rho_{ij}(u, x)\mathrm{d}x^i\mathrm{d}x^j$ is a family of Riemannian metrics on N^3 [parametrized by u and of signature $(-,-,-)$], $A = A_i(u, x)\mathrm{d}x^i$ is a family of 1-forms on N^3, and $H = H(u, x, v)$ is a smooth function on $\widetilde{M}^{1,4}$ (see Refs. 30 and 37–39). In these coordinates, the distribution spanned by ∂_v consists of isotropic lines and is parallel since $\nabla^{\tilde{g}}\partial_v = \frac{1}{2}H_v\partial_v \otimes \mathrm{d}\,u$, where $\nabla^{\tilde{g}}$ denotes the Levi–Cività connection with respect to $\tilde{g}$. Observe also that $\mathrm{d}\,u = \partial_v \lrcorner \tilde{g}$ is null, that is, $\langle \mathrm{d}\,u, \mathrm{d}\,u \rangle_{\tilde{g}} = 0$.

Following Ref. 30, we will further assume the following equation:

$$\partial_v H = 0, \quad A_i = 0 \ \text{ for any } i = 1, 2, 3, \quad \rho \ \text{ is a family of Ricci}-\text{flat metrics.} \tag{6.2}$$

The first condition implies $\nabla^{\tilde{g}}\partial_v = 0$, and consequently $\nabla^{\tilde{g}}\mathrm{d}\,u = 0$. Under these assumptions, the Ricci tensor is significantly simplified to

$$\mathsf{Ric}_{\tilde{g}} = -\frac{1}{2}\Delta_\rho(H)\,\mathrm{d}u^2, \tag{6.3}$$

where $\Delta_\rho(H) = \sum_{i,j,k=1}^{3}\rho^{ij}\left(\partial_{x^i}\partial_{x^j}H - \Gamma_{ij}^k\partial_{x^k}H\right)$ is the Laplace–Beltrami operator of the metric ρ applied to H (see Ref. 38). Consequently, the Ricci endomorphism $\mathrm{ric}_{\tilde{g}}$ is null, i.e., $\langle \mathrm{ric}_{\tilde{g}}, \mathrm{ric}_{\tilde{g}} \rangle_{\tilde{g}} = 0$. As in Ref. 30, we shall slightly abuse the terminology and call Walker metrics satisfying conditions (6.2) *Ricci-isotropic Walker metrics*, referring to the property that the image of the Ricci endomorphism related to the Walker metric is totally null (see Definition 5.6). Note that since the 1-form $\mathrm{d}u$ is null, we can use it to build other null differential forms on $\widetilde{M}^{1,4}$. In particular, if we use $\mathrm{d}u$ to construct a null 4-form F, we may take advantage of the results found in Sec. V.

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By Proposition 5.1, we know that if $(X^{1,10} = \widetilde{M}^{1,4} \times M^6, h = \tilde{g} + g, \mathsf{F})$ is a bosonic supergravity background, then the Ricci tensor of $\widetilde{M}^{1,4}$ satisfies the equation

$$\mathrm{Ric}_h(\tilde{X}, \tilde{Y}) = -\frac{1}{2}\Big(\langle \tilde{X}\lrcorner\tilde{\alpha}, \tilde{Y}\lrcorner\tilde{\alpha}\rangle_{\tilde{g}}\varphi^2 + \langle \tilde{X}\lrcorner\tilde{\beta}, \tilde{Y}\lrcorner\tilde{\beta}\rangle_{\tilde{g}}\|\nu\|_g^2 + \langle \tilde{X}\lrcorner\tilde{\gamma}, \tilde{Y}\lrcorner\tilde{\gamma}\rangle_{\tilde{g}}\|\delta\|_g^2 + \tilde{\omega}(\tilde{X})\tilde{\omega}(\tilde{Y})\|\epsilon\|_g^2\Big).$$

When $(\widetilde{M}^{1,4}, \tilde{g})$ is a Ricci-isotropic Walker manifold of the type described earlier, the right-hand side of this equation must be the same type of tensor as in (6.3). The following lemma shows that this happens when $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\omega}$ are of the form $\mathrm{d}u \wedge \omega(u)$, for some $\omega(u) \in \Omega^k(N^3)$. Here, the notation indicates that the differential forms on N^3 are parametrized by u. We remind that the metric ρ, in general, also depends on u, even though our notation does not emphasize this.

Lemma 6.2. Let $\omega(u) \in \Omega^k(N^3)$ be a k-form and let $\tilde{g} = 2\,\mathrm{d}u\,\mathrm{d}v + \rho + H(u,x)\mathrm{d}u^2$ be a metric on $\widetilde{M}^{1,4} = \mathbb{R} \times N^3 \times \mathbb{R}$. Then

$$\langle \tilde{X}\lrcorner(\mathrm{d}\,u \wedge \omega(u)), \tilde{Y}\lrcorner(\mathrm{d}\,u \wedge \omega(u))\rangle_{\tilde{g}} = a_1a_2\langle \omega(u), \omega(u)\rangle_{\tilde{g}},$$

where $a_1 = \tilde{X}\lrcorner\mathrm{d}\,u$ and $a_2 = \tilde{Y}\lrcorner\mathrm{d}\,u$. In particular, the expression vanishes, unless both $\tilde{X}\lrcorner\mathrm{d}\,u$ and $\tilde{Y}\lrcorner\mathrm{d}\,u$ are nonzero.

Proof. We have $\langle \mathrm{d}\,u, \mathrm{d}\,u\rangle_{\tilde{g}} = 0$ and $\langle \mathrm{d}\,u, \mathrm{d}\,x^i\rangle_{\tilde{g}} = 0$, which implies $\langle \mathrm{d}\,u \wedge \omega_1, \omega_2\rangle_{\tilde{g}} = 0$ for every k-form ω_2 and every $(k-1)$-form ω_1 on $\mathbb{R} \times N^3$. Let $\tilde{X} = a_1\partial_u + \sum_{i=1}^3 b_1^i\partial_{x^i} + c_1\partial_v$ and $\tilde{Y} = a_2\partial_u + \sum_{i=1}^3 b_2^i\partial_{x^i} + c_2\partial_v$. Then, for $\tilde{\omega} = \mathrm{d}\,u \wedge \omega(u)$, we compute

$$\begin{aligned}\langle \tilde{X}\lrcorner\tilde{\omega}, \tilde{Y}\lrcorner\tilde{\omega}\rangle_{\tilde{g}} &= \langle a_1\omega(u) - \mathrm{d}\,u \wedge \left(\sum_{i=1}^3 b_1^i\partial_{x^i}\lrcorner\omega(u)\right), a_2\omega(u) - \mathrm{d}\,u \wedge \left(\sum_{i=1}^3 b_2^i\partial_{x^i}\lrcorner\omega(u)\right)\rangle_{\tilde{g}} \\ &= a_1a_2\langle \omega(u), \omega(u)\rangle_{\tilde{g}}.\end{aligned}$$

□

An important subclass of Ricci-isotropic Lorentzian Walker metrics, which we may use in our study to construct explicit examples of bosonic supergravity backgrounds, consists of the so-called *pp-waves* (see Refs. 9 and 40 for details). Locally, in five dimensions, such manifolds are of the form (6.1), with $A = 0$, $\rho = -(\mathrm{d}\,x^1)^2 - (\mathrm{d}\,x^2)^2 - (\mathrm{d}\,x^3)^2$, and $\partial_v H = 0$, and so topologically, $\widetilde{M}^{1,4} = \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R} \cong \mathbb{R}^5$. In particular, we have $\Delta_\rho H = -\sum_{i=1}^3 H_{x^ix^i}$.

Remark 6.3. Walker manifolds provide examples of indefinite metrics that exhibit various geometric aspects (see, for example, Refs. 39, 41, and 42 for the Lorentzian version of such manifolds). For instance, the pp-waves form one of the simplest and well-known classes of Lorentzian Walker manifolds. On the other hand, (*totally*) *Ricci-isotropic* Lorentzian manifolds are known to be important in the holonomy theory of indefinite metrics (see, for example, Ref. 36), and their Ricci tensor attains a simplified expression (Ref. 38). Due to this special holonomy feature, Ricci-isotropic Lorentzian Walker manifolds have many natural applications in supergravity theories (see, for instance, Refs. 9, 30, 41, and 43–45).

In the remainder of this section, we apply the results from the previous sections to the case where $(\widetilde{M}^{1,4}, \tilde{g})$ is a Lorentzian Walker manifold satisfying (6.2).

B. Results concerning the flux form $\mathsf{F} = \tilde{\alpha}$

Let us consider the non-trivial 4-form $\mathsf{F} = \tilde{\alpha} = \mathrm{d}\,u \wedge f(u,x)\,\mathrm{vol}_\rho$, where vol_ρ denotes the (in general, u-dependent) volume form of the metric ρ on N^3.

Proposition 6.4. Let (M^6, g) be a Ricci-flat Riemannian manifold and let $(\widetilde{M}^{1,4}, \tilde{g} = 2\,\mathrm{d}u\,\mathrm{d}v + \rho + H\,\mathrm{d}u^2)$ be a Walker manifold with ρ Ricci-flat and $\partial_v(H) = 0$. Then $(\mathrm{X}^{1,10} = \widetilde{M}^{1,4} \times M^6, h = \tilde{g} + g, \mathsf{F} = f(x,u)\mathrm{d}\,u \wedge \mathrm{vol}_\rho)$ is a bosonic supergravity background if and only if $\partial_{x^i}(f) = 0$ for $i = 1, 2, 3$ and

$$\Delta_\rho H = -f^2.$$

Proof. Since $\tilde{\alpha}$ does not depend on v, we have $\mathrm{d}\,\tilde{\alpha} = 0$. The condition $\mathrm{d} \star_5 \tilde{\alpha} = 0$ is equivalent to $\partial_{x^i}(f) = 0$ for $i = 1, 2, 3$. From Lemma 6.2, we see that $\langle \tilde{X}\lrcorner\tilde{\alpha}, \tilde{Y}\lrcorner\tilde{\alpha}\rangle_{\tilde{g}} = 0$ for every pair $\tilde{X}$, $\tilde{Y}$ on which $\mathrm{d}u^2$ vanishes. The statement then follows from Theorem 5.2 since we get

$$\mathrm{Ric}_{\tilde{g}} = \frac{1}{2}f^2\mathrm{d}u^2,$$

which by (6.3) is equivalent to $\Delta_\rho H = -f^2$. □

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Example 6.5. For an explicit example, let (M^6, g) be a Ricci-flat Riemannian manifold and let $(\widetilde{M}^{1,4}, \tilde{g} = 2\,\mathrm{d}u\,\mathrm{d}v - \sum_{i=1}^{3} (\mathrm{d}x^i)^2 + H\,\mathrm{d}u^2)$ be a five-dimensional pp-wave. Since $\Delta_\rho H = -\sum_{i=1}^{3} H_{x^i x^i}$, the equation $\Delta_\rho H = -f^2$ is satisfied when $H = \frac{1}{6} f(u)^2 \sum_{i=1}^{3} (x^i)^2$. Therefore, with this choice of H,

$$(\mathsf{X}^{1,10} = \widetilde{M}^{1,4} \times M^6, h = \tilde{g} + g, \mathsf{F} = f(u)\mathrm{d}u \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3),$$

is an eleven-dimensional bosonic supergravity background.

C. Results concerning the flux form $\mathsf{F} = \tilde{\beta} \wedge \nu$

Let us now consider a Ricci-flat Riemannian manifold $M^6 = \mathbb{R} \times \Sigma$ with metric $g = -\mathrm{d}t^2 - \mu$, where μ is a positive definite metric on the five-dimensional manifold Σ.

Proposition 6.6. Let $(M^6 = \mathbb{R} \times \Sigma, g)$ be a Ricci-flat Riemannian manifold with metric $g = -\mathrm{d}t^2 - \mu$ and let $(\widetilde{M}^{1,4} = \mathbb{R} \times N^3 \times \mathbb{R}$, $\tilde{g} = 2\,\mathrm{d}v\,\mathrm{d}u + \rho + H\,\mathrm{d}u^2)$ be a Walker manifold with $\partial_v(H) = 0$ and ρ u-independent and Ricci-flat. Set $\nu = \mathrm{d}t$ and $\tilde{\beta} = \mathrm{d}u \wedge \omega$ for a closed and coclosed 2-form ω on N^3. Then $(\mathsf{X}^{1,10} = \widetilde{M}^{1,4} \times M^6, h = \tilde{g} + g, \mathsf{F} = \tilde{\beta} \wedge \nu)$ is a bosonic supergravity background if and only if

$$\Delta_\rho H = -\|\omega\|_\rho^2.$$

Proof. It is clear that $\tilde{\beta}$ is closed and that ν is closed and coclosed. It follows from $\star_5 \tilde{\beta} = \star_2 \mathrm{d}u \wedge \star_\rho \omega = -\mathrm{d}u \wedge \star_\rho \omega$ that $\tilde{\beta}$ is coclosed. We also see that $\langle \tilde{X} \lrcorner \tilde{\beta}, \tilde{Y} \lrcorner \tilde{\beta} \rangle_{\tilde{g}} = 0$ for every pair $\tilde{X}, \tilde{Y}$ on which $\mathrm{d}u^2$ vanishes. Therefore, it follows from Theorem 5.3 that the Ricci tensor is given by

$$\mathsf{Ric}_h = -\frac{1}{2} \|\omega\|_{\tilde{g}}^2 \|\nu\|_g^2 \mathrm{d}u^2.$$

This equivalent to $\Delta_\rho H = -\|\omega\|_{\tilde{g}}^2 = -\|\omega\|_\rho^2$, since $\|\nu\|_g^2 = -1$. □

Example 6.7. Let $(\widetilde{M}^{1,4} = \mathbb{R} \times N^3 \times \mathbb{R}, \tilde{g} = 2\,\mathrm{d}v\,\mathrm{d}u - \sum_{i=1}^{3} (\mathrm{d}x^i)^2 + H\,\mathrm{d}u^2)$ be a pp-wave with $H = \frac{1}{6} \sum_{i=1}^{3} (x^i)^2$. Let (M^6, g) and ν be as described in the proposition earlier and set $\tilde{\beta} = \mathrm{d}u \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2$. Then

$$(\mathsf{X}^{1,10} = \widetilde{M}^{1,4} \times M^6, h = \tilde{g} + g, \mathsf{F} = \tilde{\beta} \wedge \nu),$$

is an eleven-dimensional bosonic supergravity background.

D. Results concerning the flux form $\mathsf{F} = \tilde{\gamma} \wedge \delta$

Let $\tilde{\gamma} = \mathrm{d}u \wedge \zeta$ for a 1-form ζ on N^3 and assume that M^6 is a Calabi–Yau manifold and that δ is its Kähler form.

Proposition 6.8. Consider a six-dimensional Calabi–Yau manifold (M^6, g, δ), where δ denotes the Kähler form, and a five-dimensional Walker manifold $(\widetilde{M}^{1,4} = \mathbb{R} \times N^3 \times \mathbb{R}, \tilde{g} = 2\,\mathrm{d}v\,\mathrm{d}u + \rho + H\,\mathrm{d}u^2)$ with ρ u-independent and Ricci-flat, and $\partial_v(H) = 0$. Set $\tilde{\gamma} = \mathrm{d}u \wedge \zeta$ for some closed and coclosed 1-form ζ on N^3. Then $(\mathsf{X}^{1,10} = \widetilde{M}^{1,4} \times M, h = \tilde{g} + g, \mathsf{F} = \tilde{\gamma} \wedge \delta)$ is a bosonic supergravity background if and only if

$$\Delta_\rho H = \|\zeta\|_\rho^2 \|\delta\|_g^2. \tag{6.4}$$

Proof. We use Theorem 5.4. Since $\tilde{\gamma} \wedge \tilde{\gamma} = 0$, the Maxwell equation and closedness condition are satisfied when $\tilde{\gamma}$ and δ are closed and coclosed. Since δ is the Kähler form, it is closed and coclosed, and the same holds for $\tilde{\gamma}$. Since (M^6, g, δ) is a Calabi–Yau manifold, it is Ricci-flat. We also see that $\langle \tilde{X} \lrcorner \tilde{\gamma}, \tilde{Y} \lrcorner \tilde{\gamma} \rangle_{\tilde{g}} = 0$ for every pair $\tilde{X}, \tilde{Y}$ on which $\mathrm{d}u^2$ vanishes. It follows from (5.3) that

$$\mathsf{Ric}_h = -\frac{1}{2} \|\zeta\|_{\tilde{g}}^2 \|\delta\|_g^2 \mathrm{d}u^2,$$

which is equivalent to (6.4) (note that the function $\|\delta\|_g^2$ is constant since it is the Kähler form). This proves our claim. □

E. Results concerning the flux form $\mathsf{F} = \tilde{\omega} \wedge \epsilon$

Proposition 6.9. Let (M^6, g) be a Riemannian Ricci-flat manifold and ϵ a closed and coclosed 3-form on M^6. Let $(\widetilde{M}^{1,4} = \mathbb{R} \times N^3 \times \mathbb{R}$, $\tilde{g} = 2\,\mathrm{d}v\,\mathrm{d}u + \rho + H\,\mathrm{d}u^2)$ be a Walker manifold with ρ u-independent and Ricci-flat and $\partial_v(H) = 0$. Set $\tilde{\omega} = \mathrm{d}u$. Then $(\mathsf{X}^{1,10} = \widetilde{M}^{1,4} \times M^6$, $h = \tilde{g} + g, \mathsf{F} = \tilde{\omega} \wedge \epsilon)$ is a bosonic supergravity background if and only if $\|\epsilon\|_g^2$ is constant, and

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$$\Delta H = \|\epsilon\|_g^2.$$

Proof. We use Theorem 5.5. It is clear that $\tilde{\omega} = \mathrm{d}\,u$ is both closed and coclosed. We have $\tilde{\omega}(\tilde{X})\tilde{\omega}(\tilde{Y}) = 0$ for every pair $\tilde{X}, \tilde{Y}$ on which $\mathrm{d}u^2$ vanishes. Therefore, by (5.4), the Ricci tensor takes the form

$$\mathrm{Ric}_h = -\frac{1}{2}\|\epsilon\|_g^2 \mathrm{d}u^2,$$

or, equivalently, $\Delta_\rho H = \|\epsilon\|_g^2$. Since the left-hand side of this equation is a function on $\widetilde{M}^{1,4}$, $\|\epsilon\|_g^2$ must be constant. This completes the proof. □

We illustrate Proposition 6.9 with the following example:

Example 6.10. Let $\tilde{M}^{1,4}$ be a pp-wave with metric

$$\tilde{g} = 2\,\mathrm{d}v\,\mathrm{d}u - \sum_{i=1}^{3}(\mathrm{d}\,x^i)^2 - \frac{E^2}{6}\left(\sum_{i=1}^{3}(x^i)^2\right)\mathrm{d}u^2,$$

let (M^6, g) be a Riemannian Ricci-flat manifold and let ϵ be a closed and coclosed 3-form on M^6 with $\|\epsilon\|_g^2 = -E^2$ constant. Then $(\mathsf{X}^{1,10} = \widetilde{M}^{1,4} \times M^6, h = \tilde{g} + g, \mathsf{F} = \mathrm{d}\,u \wedge \epsilon)$ is an eleven-dimensional bosonic supergravity background.

F. Results concerning the flux form $\mathsf{F} = \varphi\tilde{\alpha} + \tilde{\beta}\wedge\nu + \tilde{\gamma}\wedge\delta + \tilde{\omega}\wedge\epsilon$

In this section, we unify the previous four cases by considering a flux form $\mathsf{F} = \varphi\tilde{\alpha} + \tilde{\beta}\wedge\nu + \tilde{\gamma}\wedge\delta + \tilde{\omega}\wedge\epsilon$, where

$$\tilde{\alpha} = \mathrm{d}\,u \wedge \hat{\alpha}(u), \quad \tilde{\beta} = \mathrm{d}\,u \wedge \hat{\beta}(u), \quad \tilde{\gamma} = \mathrm{d}\,u \wedge \hat{\gamma}(u), \quad \tilde{\omega} = \mathrm{d}\,u \wedge \hat{\omega}(u),$$

with $\hat{\alpha}(u) \in \Omega^3(N^3)$, $\hat{\beta}(u) \in \Omega^2(N^3)$, $\hat{\gamma}(u) \in \Omega^1(N^3)$, and $\hat{\omega}(u) \in C^\infty(N^3)$, respectively. Recall that the notation indicates that the differential forms on N^3 are parametrized by u. Since $\dim N^3 = 3$, we have $\hat{\alpha}(u) = f(u,x)\,\mathrm{vol}_\rho$ for some function $f \in C^\infty(\mathbb{R}\times N^3)$, where vol_ρ is the (in general, u-dependent) volume form with respect to the metric ρ. In this case, the Maxwell equations (Proposition 3.3) simplify significantly: all right-hand sides vanish due to $\mathrm{d}u \wedge \mathrm{d}u = 0$, and we obtain the equations in Proposition 3.8. It is easily seen that both the closedness condition and the Maxwell equation are satisfied in the particular case that $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\omega}$ and $\varphi, \nu, \delta, \epsilon$ are closed and coclosed on their respective manifolds. Let us remark that the closedness of φ implies its constancy, and we can without loss of generality assume that it is equal to 1. Finally, notice that if $\omega = \mathrm{d}\,u \wedge \hat{\omega}(u)$ for some $\hat{\omega}(u) \in \Omega^k(N^3)$, then closedness of ω with respect to the exterior derivative on $\widetilde{M}^{1,4}$ is equivalent to closedness of $\hat{\omega}(u)$ with respect to the exterior derivative d_N on N^3,

$$\mathrm{d}\,\omega = -\mathrm{d}\,u \wedge \mathrm{d}\,\hat{\omega}(u) = -\mathrm{d}\,u \wedge \mathrm{d}_N\hat{\omega}(u).$$

A similar statement can be made for coclosedness of ω: $\mathrm{d}\star_5\omega = 0$ if and only if $\mathrm{d}_N\star_\rho\hat{\omega}(u) = 0$.

Proposition 6.11. Consider the 4-form $\mathsf{F} = \mathrm{d}\,u \wedge (\hat{\alpha}(u) + \hat{\beta}(u)\wedge\nu + \hat{\gamma}(u)\wedge\delta + \hat{\omega}(u)\epsilon)$, *where*

$$\hat{\alpha}(u) \in \Omega^3(N^3), \quad \hat{\beta}(u) \in \Omega^2(N^3), \quad \hat{\gamma}(u) \in \Omega^1(N^3), \quad \hat{\omega}(u) \in C^\infty(N^3),$$
$$\nu \in \Omega^1(M^6), \quad \delta \in \Omega^2(M^6), \quad \epsilon \in \Omega^3(M^6),$$

are closed and coclosed on N^3 *and* M^6, *respectively. Let also* $(\widetilde{M}^{1,4}, \tilde{g})$ *be a Walker metric of the form (6.1) with* ρ *Ricci-flat,* $A = 0$, *and* $\partial_v(H) = 0$. *Then,*

$$(\mathsf{X}^{1,10} = \widetilde{M}^{1,4} \times M^6, h = g + \tilde{g}, \mathsf{F}),$$

is a bosonic supergravity background if and only if (M^6, g) *is Ricci-flat and*

$$\Delta_\rho H = \|\hat{\alpha}(u)\|_\rho^2 + \|\hat{\beta}(u)\|_\rho^2\|\nu\|_g^2 + \|\hat{\gamma}(u)\|_\rho^2\|\delta\|_g^2 + \hat{\omega}(u)^2\|\epsilon\|_g^2.$$

Proof. It is clear that the closedness condition in Lemma 3.1 is satisfied, as is the Maxwell equation (Proposition 3.8). Therefore, the condition for being a bosonic supergravity background boils down to the supergravity Einstein equation, which, by Proposition 5.1, consists of the following system of equations:

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$$\begin{cases} \mathrm{Ric}_h(\tilde{X}, \tilde{Y}) = -\frac{1}{2}\Big(\langle \tilde{X}\lrcorner\tilde{\alpha}, \tilde{Y}\lrcorner\tilde{\alpha}\rangle_{\tilde{g}} + \langle \tilde{X}\lrcorner\tilde{\beta}, \tilde{Y}\lrcorner\tilde{\beta}\rangle_{\tilde{g}}\|\nu\|_g^2 + \langle \tilde{X}\lrcorner\tilde{\gamma}, \tilde{Y}\lrcorner\tilde{\gamma}\rangle_{\tilde{g}}\|\delta\|_g^2 + \tilde{\omega}(\tilde{X})\tilde{\omega}(\tilde{Y})\|\epsilon\|_g^2\Big), \\ \qquad 0 = \varphi\nu(Z)\langle\tilde{\beta}, \tilde{Y}\lrcorner\tilde{\alpha}\rangle_{\tilde{g}} - \langle\tilde{\gamma}\wedge(Z\lrcorner\delta), (\tilde{Y}\lrcorner\tilde{\beta})\wedge\nu\rangle_h + \langle\tilde{\omega}\wedge(Z\lrcorner\epsilon), (\tilde{Y}\lrcorner\tilde{\gamma})\wedge\delta\rangle_h. \end{cases}$$

The first equation reduces to

$$\Delta_\rho H = -2\mathrm{Ric}_h(\partial_u, \partial_u) = \|\hat{\alpha}(u)\|_\rho^2 + \|\hat{\beta}(u)\|_\rho^2\|\nu\|_g^2 + \|\hat{\gamma}(u)\|_\rho^2\|\delta\|_g^2 + \hat{\omega}(u)^2\|\epsilon\|_g^2,$$

due to (6.3) and Lemma 6.2, while the second one holds automatically. Therefore, we get our claim. □

Notice that since $\hat{\alpha}(u) = f(u,x)\,\mathrm{vol}_\rho$, we have $\mathrm{d}_N\hat{\alpha}(u) = 0$. We also see that $\|\hat{\alpha}(u)\|_\rho^2 = -f^2$, and the condition $\mathrm{d}_N\star_\rho\hat{\alpha}(u) = 0$ is equivalent to condition $\partial_{x^i}(f) = 0$ for $i = 1,2,3$, which we recognize from Proposition 6.4. Notice that closedness of $\hat{\omega}(u)$ on N^3 means that $\hat{\omega}(u)$ is a function of u only. Propositions 6.4, 6.6, 6.8, and 6.9 can now be viewed as corollaries of Proposition 6.11. Moreover, their corresponding examples are special cases of the following more general example.

Example 6.12. Let (M^6, g) be a Ricci-flat Riemannian manifold and assume that there exist differential forms $\nu\in\Omega^1(M^6)$, $\delta\in\Omega^2(M^6)$, $\epsilon\in\Omega^3(M^6)$, which are closed and coclosed, satisfying $\|\nu\|_g^2 = -1$, $\|\delta\|_g^2 = 1$, and $\|\epsilon\|_g^2 = -1$, respectively. Let $\widetilde{M}^{1,4}$ be a five-dimensional pp-wave with metric

$$\tilde{g} = 2\,\mathrm{d}u\,\mathrm{d}v - \sum_{i=1}^{3}(\mathrm{d}x^i)^2 + \frac{f_1(u)^2 - f_2(u)^2 + f_3(u)^2 - f_4(u)^2}{6}\left(\sum_{i=1}^{3}(x^i)^2\right)\mathrm{d}u^2,$$

and set $\hat{\alpha}(u) = f_1(u)\,\mathrm{vol}_\rho, \hat{\beta}(u) = f_2(u)\mathrm{d}x^1\wedge\mathrm{d}x^2, \hat{\gamma}(u) = f_3(u)\mathrm{d}x^1, \hat{\omega}(u) = f_4(u)$. If

$$\begin{aligned} \mathsf{F} &= \mathrm{d}u\wedge(\hat{\alpha}(u) + \hat{\beta}(u)\wedge\nu + \hat{\gamma}(u)\wedge\delta + \hat{\omega}(u)\epsilon) \\ &= \mathrm{d}u\wedge(f_1(u)\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3 + f_2(u)\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\nu + f_3(u)\mathrm{d}x^1\wedge\delta + f_4(u)\epsilon), \end{aligned}$$

then $(\mathsf{X}^{1,10} = \widetilde{M}^{1,4}\times M^6, h = \tilde{g} + g, \mathsf{F})$ is an eleven-dimensional bosonic supergravity background. Notice that when all but one of the functions f_i vanish, the example reduces to the ones previously considered. We can specify the example even further by considering $M^6 = \mathbb{R}^3\times\Sigma$ with metric $g = -(\mathrm{d}y^1)^2 - (\mathrm{d}y^2)^2 - (\mathrm{d}y^3)^2 - \nu$, where ν is a Ricci-flat positive definite metric on a three-dimensional manifold Σ, and $\nu = \mathrm{d}y^3$, $\delta = \mathrm{d}y^2\wedge\mathrm{d}y^3$, $\epsilon = \mathrm{d}y^1\wedge\mathrm{d}y^2\wedge\mathrm{d}y^3$.

G. Results concerning the flux form $\mathsf{F} = \varphi\tilde{\alpha} + \tilde{\beta}\wedge\nu$

Now, set $\tilde{\alpha} = \mathrm{d}u\wedge f(u,x)\,\mathrm{vol}_\rho$ and $\tilde{\beta} = \mathrm{d}u\wedge\omega(u)$, where $f\in C^\infty(\mathbb{R}\times N^3)$ is a function, $\omega(u)\in\Omega^2(N^3)$ is a 2-form depending smoothly on the parameter u, and $f, \omega(u), \nu, \varphi$ are nonzero. Then we get the following statement regarding bosonic supergravity backgrounds with flux form $\mathsf{F} = \varphi\tilde{\alpha} + \tilde{\beta}\wedge\nu$.

Proposition 6.13. Let (M^6, g) be a Riemannian Ricci-flat manifold, φ a function on M^6 satisfying $\star_6\mathrm{d}\star_6\mathrm{d}\varphi = \kappa\lambda\varphi$, and set $\nu = \frac{1}{\kappa}\mathrm{d}\varphi$ for a nonzero constant κ. Let also $(\widetilde{M}^{1,4} = \mathbb{R}\times N^3\times\mathbb{R}, \tilde{g} = 2\,\mathrm{d}v\,\mathrm{d}u + \rho + H\,\mathrm{d}u^2)$ be a Walker manifold with ρ Ricci-flat and $\partial_v(H) = 0$. Set as above $\tilde{\alpha} = \mathrm{d}u\wedge f\,\mathrm{vol}_\rho$ and $\tilde{\beta} = \mathrm{d}u\wedge\omega(u)$ with $\omega(u) = -\frac{1}{\lambda}\star_\rho\mathrm{d}_N f$ for a nonzero constant λ, and assume that f satisfies $\star_\rho\mathrm{d}_N\star_\rho\mathrm{d}_N f = -\kappa\lambda f$. Then $(\mathsf{X}^{1,10} = \widetilde{M}^{1,4}\times M^6, h = \tilde{g} + g, \mathsf{F} = \varphi\tilde{\alpha} + \tilde{\beta}\wedge\nu)$ is a bosonic supergravity background if and only if

$$\Delta_\rho H = \|\omega(u)\|_\rho^2\|\nu\|_g^2 - f^2\varphi^2.$$

Proof. The proof is mainly based on Theorem 5.8. The Maxwell equation and the closedness condition reduce to

$$\mathrm{d}_N\star_\rho\omega(u) = 0,\quad \mathrm{d}_N\omega(u) = \kappa f\,\mathrm{vol}_\rho,\quad \mathrm{d}_N f = \lambda\star_\rho\omega(u),\quad \mathrm{d}\nu = 0,\quad \mathrm{d}\varphi = \kappa\nu,\quad \mathrm{d}\star_6\nu = \lambda\star_6\varphi.$$

These equations are satisfied due to the definitions of ν and $\omega(u)$ and the two differential equations

$$\star_6\mathrm{d}\star_6\mathrm{d}\varphi = \kappa\lambda\varphi,\qquad \star_\rho\mathrm{d}_N\star_\rho\mathrm{d}_N f = -\kappa\lambda f,$$

constraining them. Next, the supergravity Einstein equation consists of the following system of equations:

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$$\begin{cases} \mathrm{Ric}_h(\tilde{X},\tilde{Y}) = -\frac{1}{2}\langle \tilde{X}\lrcorner\tilde{\alpha}, \tilde{Y}\lrcorner\tilde{\alpha}\rangle_{\tilde{g}}\varphi^2 - \frac{1}{2}\langle \tilde{X}\lrcorner\tilde{\beta}, \tilde{Y}\lrcorner\tilde{\beta}\rangle_{\tilde{g}}\|v\|_g^2, \quad \forall\ \tilde{X},\tilde{Y}\in\Gamma(T\widetilde{M}), \\ \qquad 0 = \langle\tilde{\beta}, \tilde{X}\lrcorner\tilde{\alpha}\rangle_{\tilde{g}}, \quad \forall\ \tilde{X}\in\Gamma(T\widetilde{M}). \end{cases}$$

By Lemma 6.2 and by assuming that $(\widetilde{M}^{1,4},\tilde{g})$ is of the form (6.1) with ρ Ricci-flat and $\partial_v(H)=0$, we see that the only nonzero component of the Ricci tensor has the form

$$\mathrm{Ric}_h(\partial_u,\partial_u) = -\frac{1}{2}\big(f^2\varphi^2\|\operatorname{vol}_\rho\|_{\tilde{g}}^2 + \|\omega(u)\|_{\tilde{g}}^2\|v\|_g^2\big) = \frac{1}{2}\big(f^2\varphi^2 - \|\omega(u)\|_{\tilde{g}}^2\|v\|_g^2\big),$$

while for $\tilde{\alpha}$ and $\tilde{\beta}$ of the chosen form, the equation $\langle\tilde{\beta},\tilde{X}\lrcorner\tilde{\alpha}\rangle_{\tilde{g}}=0$ holds automatically. However, then, by (6.3), it turns out that the above-mentioned relation is equivalent to the condition $\Delta_\rho H = \|\omega(u)\|_{\tilde{g}}^2\|v\|_g^2 - f^2\varphi^2$, which proves our claim. □

Remark 6.14. By assumption, the function f^2 is nonzero, and $\|\omega(u)\|_{\tilde{g}}^2 = \|\omega(u)\|_\rho^2 > 0$, since ρ is positive definite on 2-forms. By Lemma 4.4, we see that either φ and $\|v\|_g^2$ are both constants, or $f^2 = C\|\omega(u)\|_{\tilde{g}}^2$ for some nonzero constant C, which can only be positive. In the latter case, we obtain

$$\|v\|_g^2 = C\varphi^2 + D,$$

for some constant D. In this equation, the left-hand side is negative. For this reason, we look for an example involving trigonometric functions.

Example 6.15. Let $(M^6 = \mathsf{S}^1\times\mathbb{R}^5, g)$ be a flat Riemannian manifold with coordinates $y^1,\ldots,y^6$ and metric $g = -(\mathrm{d}y^1)^2 - \sum_{i=2}^6(\mathrm{d}y^i)^2$. Set

$$\varphi = \sin(y^1),\quad v = \frac{1}{\kappa}\mathrm{d}\varphi = \frac{1}{\kappa}\cos(y^1)\mathrm{d}y^1.$$

Consider also a pp-wave $(\widetilde{M}^{1,4},\tilde{g})$ with metric $\tilde{g} = 2\,\mathrm{d}u\,\mathrm{d}v - \sum_{i=1}^3(\mathrm{d}x^i)^2 + H(u,x)\mathrm{d}u^2$. Set $f=\exp(x^1)$ and $\omega = \kappa\star_\rho\ \mathrm{d}f = -\kappa\exp(x^1)\mathrm{d}x^2\wedge\mathrm{d}x^3$. In terms of the notation in Proposition 6.13, we have $\lambda = -1/\kappa$. Now, the supergravity Einstein condition gives

$$\Delta_\rho H = \|\omega\|_{\tilde{g}}^2\|v\|_g^2 - f^2\varphi^2 = -\exp(2x^1).$$

This equation is satisfied if, for example, $H = \frac{1}{4}\exp(2x^1)$. Therefore, $(\mathsf{X}^{1,10} = \widetilde{M}^{1,4}\times M^6, h = g+\tilde{g}, \mathsf{F} = \varphi\tilde{\alpha}+\tilde{\beta}\wedge v)$ is a bosonic supergravity background with these choices of φ, v, f, ω, H, and then the flux form F reads as $\mathsf{F} = \exp(x^1)\mathrm{d}u\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3\wedge\big(\sin(y^1)\mathrm{d}x^1 + \cos(y^1)\mathrm{d}y^1\big)$.

Let us now consider the case where $\mathrm{d}\varphi = 0$ (and $\kappa=0$). By absorbing the constant into $\tilde{\alpha}$, we can, without loss of generality, assume that $\varphi=1$. As long as $\lambda\neq 0$, the 2-form $\omega(u)$ is determined by f via the relation $\omega(u) = -\frac{1}{\lambda}\star_\rho\mathrm{d}_N f$. This implies at once that $\mathrm{d}_N\star_\rho\omega(u)=0$. The equation $\mathrm{d}_N\omega(u)=0$ is then equivalent to $\mathrm{d}_N\star_\rho\ \mathrm{d}_N f = 0$, or $\star_\rho\mathrm{d}_N\star_\rho\ \mathrm{d}_N f = 0$. With this simplification, we obtain the following statement.

Proposition 6.16. Let (M^6,g) be a Riemannian Ricci-flat manifold endowed with a closed 1-form $v\in\Omega^1(M)$ satisfying $\star_6\mathrm{d}\star_6 v = \lambda$ for some constant $\lambda\neq 0$. Let also $(\widetilde{M}^{1,4} = \mathbb{R}\times N^3\times\mathbb{R}, \tilde{g} = 2\,\mathrm{d}v\,\mathrm{d}u + \rho + H\,\mathrm{d}u^2)$ be a Walker manifold with ρ Ricci-flat and $\partial_v(H)=0$, and assume that f is a smooth function on $\mathbb{R}\times N^3$, such that

$$\star_\rho\mathrm{d}_N\star_\rho\mathrm{d}_N f = 0.$$

Set $\tilde{\alpha} = \mathrm{d}u\wedge f\operatorname{vol}_\rho, \tilde{\beta} = \mathrm{d}u\wedge\omega(u)$, with $\omega(u) = -\frac{1}{\lambda}\star_\rho\mathrm{d}_N f$. Then $(\mathsf{X}^{1,10} = \widetilde{M}^{1,4}\times M^6, h = \tilde{g}+g, \mathsf{F} = \tilde{\alpha}+\tilde{\beta}\wedge v)$ is a bosonic supergravity background if and only if

$$\Delta_\rho H = \|\omega(u)\|_\rho^2\|v\|_g^2 - f^2.$$

Consequently, $\|v\|_g^2$ must be constant for such a bosonic supergravity background.

Recall that by Proposition 3.6, the six-dimensional Riemannian manifold M^6 appearing in Proposition 6.16 must be non-closed. Notice that Examples 6.12 and 6.15 can easily be modified to make M^6 a compact manifold, for example, a flat six-dimensional torus. This is *not* the case for the following example.

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Example 6.17. Consider the Riemannian manifold $M^6 = (-L, L) \times \mathbb{R}^5$ with metric $g = -(\mathrm{d}y^1)^2 - \sum_{i=2}^{6} (\mathrm{d}y^i)^2$. We set $\lambda = 1$ and $\nu = -y^1 \mathrm{d}y^1 + \sqrt{L^2 - (y^1)^2}\mathrm{d}y^2$ so that $\|\nu\|_g^2 = -L^2$. Let $(\widetilde{M}^{1,4}, \tilde{g})$ be a pp-wave, that is, $\tilde{g} = 2\,\mathrm{d}u\,\mathrm{d}v - \sum_{i=1}^{3} (\mathrm{d}x^i)^2 + H(u,x)\mathrm{d}u^2$. If we set $f = x^1$ and $\omega = -\star_\rho \mathrm{d}_N f = \mathrm{d}x^2 \wedge \mathrm{d}x^3$, then the Einstein equation is given by

$$\Delta_\rho H = \|\omega\|_\rho^2 \|\nu\|_g^2 - f = -L^2 - (x^1)^2.$$

This equation is satisfied when, for example, $H = \frac{1}{12}(x^1)^4 + \frac{L^2}{2}(x^1)^2$. Therefore, $(\mathsf{X}^{1,10} = \widetilde{M}^{1,4} \times M^6, h = \tilde{g} + g, \mathsf{F} = \tilde{\alpha} + \tilde{\beta} \wedge \nu)$ is a bosonic supergravity background with these choices of ν, f, ω, H. In this case, the flux form is given by $\mathsf{F} = \mathrm{d}u \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge (x^1 \mathrm{d}x^1 - y^1 \mathrm{d}y^1 + \sqrt{L^2 - (y^1)^2}\mathrm{d}y^2)$.

VII. BOSONIC BACKGROUNDS INVOLVING KÅHLER MANIFOLDS AND NON-NULL FLUX FORMS

In this section, we treat flux forms of types $\mathsf{F} = \tilde{\gamma} \wedge \delta$ and $\mathsf{F} = \theta$ for the case when (M^6, g) is a Kähler manifold with Kähler form ω. We will assume that the part of F taking values in TM^6 is given by ω. More precisely, we consider the cases for which $\delta = \omega$ and $\theta = c \star_6 \omega$ for some nonzero constant $c \in \mathbb{R}$.

A. Results concerning $\mathsf{F} = \tilde{\gamma} \wedge \delta$

Inspired by Corollary 4.5, we consider bosonic supergravity backgrounds with $\mathsf{F} = \tilde{\gamma} \wedge \delta$, where $\|\tilde{\gamma}\|_{\tilde{g}}^2$ is not (necessarily) constant, and hence not null. The following theorem follows directly from the bosonic supergravity equations (Propositions 3.4 and 4.3).

Theorem 7.1. *Consider the Lorentzian manifold $\mathsf{X}^{1,10} = \widetilde{M}^{1,4} \times M^6$ with metric $h = \tilde{g} + g$ and a 4-form $\mathsf{F} = \tilde{\gamma} \wedge \delta$. Then $(\mathsf{X}^{1,10}, h, \mathsf{F})$ is a bosonic supergravity background if and only if δ is closed and coclosed, $\tilde{\gamma}$ is closed and satisfies*

$$\mathrm{d} \star_5 \tilde{\gamma} \wedge \star_6 \delta = \frac{\tilde{\gamma} \wedge \tilde{\gamma} \wedge \delta \wedge \delta}{2},$$

and the following equations hold:

$$\begin{aligned} \mathsf{Ric}_h(X, Y) &= \left(\frac{\|\delta\|_g^2}{6} g(X, Y) - \frac{1}{2}\langle X \lrcorner \delta, Y \lrcorner \delta\rangle_g\right)\|\tilde{\gamma}\|_{\tilde{g}}^2, \\ \mathsf{Ric}_h(\tilde{X}, \tilde{Y}) &= \left(\frac{\|\tilde{\gamma}\|_{\tilde{g}}^2}{6} \tilde{g}(\tilde{X}, \tilde{Y}) - \frac{1}{2}\langle \tilde{X} \lrcorner \tilde{\gamma}, \tilde{Y} \lrcorner \tilde{\gamma}\rangle_{\tilde{g}}\right)\|\delta\|_g^2. \end{aligned} \tag{7.1}$$

Now, Corollary 4.5 tells us that if $\|\tilde{\gamma}\|_{\tilde{g}}^2$ is not constant, then M^6 is a Ricci-flat, almost Hermitian manifold. If we let M^6 be a Kähler manifold with Kähler form δ, we get the following statement.

Proposition 7.2. Let (M^6, g, δ) be a Kähler manifold and let $(\widetilde{M}^{1,4}, \tilde{g})$ be a Lorentzian manifold. Assume also that $\tilde{\gamma}$ is a closed 2-form on $\widetilde{M}^{1,4}$. Then $(\mathsf{X}^{1,10} = \widetilde{M}^{1,4} \times M^6, h = g + \tilde{g}, \mathsf{F} = \tilde{\gamma} \wedge \delta)$ is a bosonic supergravity background if and only if (M^6, g) is Ricci-flat,

$$\mathrm{d} \star_5 \tilde{\gamma} = \tilde{\gamma} \wedge \tilde{\gamma},$$

and

$$\mathsf{Ric}_h(\tilde{X}, \tilde{Y}) = \left(\frac{\|\tilde{\gamma}\|_{\tilde{g}}^2}{2} \tilde{g}(\tilde{X}, \tilde{Y}) - \frac{3}{2}\langle \tilde{X} \lrcorner \tilde{\gamma}, \tilde{Y} \lrcorner \tilde{\gamma}\rangle_{\tilde{g}}\right). \tag{7.2}$$

Proof. Let us first mention that the Kähler form δ is closed and coclosed and that $\|\delta\|_g^2 = 3$. The equation $\mathrm{d} \star_5 \tilde{\gamma} \wedge \star_6 \delta = \frac{\tilde{\gamma} \wedge \tilde{\gamma} \wedge \delta \wedge \delta}{2}$ is in this case equivalent to $\mathrm{d} \star_5 \tilde{\gamma} = \tilde{\gamma} \wedge \tilde{\gamma}$ because of the identity

$$\star_6 \delta = \frac{1}{2}\delta \wedge \delta, \tag{7.3}$$

on the Kähler form. The first of Eq. (7.1) is satisfied since M^6 is Kähler, while the second reduces to (7.2). This completes the proof. □

This construction illustrates the well-known fact that Ricci-flat Kähler manifolds and, more specifically, Calabi–Yau manifolds play an important role as components of bosonic supergravity backgrounds.

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B. Results concerning the flux form $\mathsf{F} = \theta$ and Kähler–Einstein metrics

By Corollary 4.5, we know that $(\mathsf{X}^{1,10} = \widetilde{M}^{1,4} \times M^6, h = \tilde{g} + g, \mathsf{F} = \theta)$ is a solution of the supergravity Einstein equations only if $\|\theta\|_g^2$ is constant and $(\widetilde{M}^{1,4}, \tilde{g})$ is an Einstein manifold with positive Einstein constant $\|\theta\|_g^2/6$ (in our "mostly minus" convention). By also taking into account Propositions 3.4 and 4.3, we obtain the following theorem.

Theorem 7.3. *Consider the Lorentzian manifold $\mathsf{X}^{1,10} = \widetilde{M}^{1,4} \times M^6$ with metric $h = \tilde{g} + g$ and a 4-form $\mathsf{F} = \theta \in \Omega^4(M^6)$. Then $(\mathsf{X}^{1,10}, h, \mathsf{F})$ is a bosonic supergravity background if and only if θ is closed and coclosed, $\|\theta\|_g^2$ is constant, $(\widetilde{M}^{1,4}, \tilde{g})$ is an Einstein manifold with Einstein constant $\frac{1}{6}\|\theta\|_g^2$ and*

$$\mathrm{Ric}_h(X, Y) = \frac{\|\theta\|_g^2}{6} g(X, Y) - \frac{1}{2}\langle X \lrcorner \theta, Y \lrcorner \theta\rangle_g, \qquad \forall\ X, Y \in \Gamma(TM^6). \tag{7.4}$$

As a consequence, we obtain the following proposition.

Proposition 7.4. Consider the Lorentzian manifold $\mathsf{X}^{1,10} = \widetilde{M}^{1,4} \times M^6$, where $(\widetilde{M}^{1,4}, \tilde{g})$ is a Lorentzian manifold and (M^6, g, ω) is a Kähler manifold. Then the triple $(\mathsf{X}^{1,10}, h = \tilde{g} + g, \mathsf{F} = \theta = c\star_6\omega)$ is a bosonic supergravity background if and only if $(\widetilde{M}^{1,4}, \tilde{g})$ and (M^6, g) are Einstein with Einstein constants $\frac{1}{2}c^2$ and $-\frac{1}{2}c^2$, respectively.

Proof. Let M^6 be a Kähler–Einstein manifold with a complex structure J and Kähler 2-form ω, defined by $\omega_{ik}J_j^k = g_{ij}$. Obviously, the 4-form $\theta = c \star_6 \omega$ is coclosed. By identity (7.3), we learn that the flux form is also closed. Hence, the Maxwell equation and closedness condition are satisfied. Recall now that (M^6, g, ω) admits an adapted orthonormal frame,

$$e_1, \quad e_2 := Je_1, \quad e_3, \quad e_4 := Je_3, \quad e_5, \quad e_6 := Je_5,$$

such that each $\omega_{ij} = 0$ if $(i, j) \notin \{(1, 2), (2, 1), (3, 4), (4, 3), (5, 6), (6, 5)\}$. Then, by (7.3), we see that

$$\theta_{ijkl} = \omega_{ij}\omega_{kl} - \omega_{ik}\omega_{jl} - \omega_{il}\omega_{kj}.$$

This relation can be used to compute $\frac{1}{3!}\theta_{iabc}\theta_j{}^{abc}$, where with g we may raise or lower the indices. In particular, having in mind the relation $\omega_{ik}J_j^k = g_{ij}$, it is straightforward to verify that

$$\langle\theta_i, \theta_j\rangle_h = \frac{1}{3!}\theta_{iabc}\theta_j{}^{abc} = 2g_{ij} = \frac{2}{3}\|\theta\|_g^2 g_{ij}.$$

Note that $\star_6$ is an isometry and we have $c^2\|\omega\|_g^2 = \|\theta\|_g^2$. Hence (7.4), together with the second equation of (4.15), reduces to the following system of equations:

$$\begin{cases} \mathrm{Ric}_h(X, Y) &= -\frac{c^2}{2} g(X, Y), \\ \mathrm{Ric}_h(\tilde{X}, \tilde{Y}) &= \frac{c^2}{2}\tilde{g}(\tilde{X}, \tilde{Y}). \end{cases} \tag{7.5}$$

This completes our proof. □

By Corollary 4.8, the bosonic background $(\mathsf{X}^{1,10} = \widetilde{M}^{1,4} \times M^6, h = \tilde{g} + g, \mathsf{F} = \theta = c\star_6\omega)$ discussed earlier must have constant positive scalar curvature given by $\mathrm{Scal}_h = \frac{1}{6}\|\theta\|_g^2 = \frac{c^2}{2}$. Moreover, we see that $(\widetilde{M}^{1,4}, \tilde{g})$ has positive scalar curvature while (M^6, g) has negative scalar curvature.

Corollary 7.5. Let $(\widetilde{M}^{1,4}, \tilde{g})$ be a Lorentzian Einstein manifold with Einstein constant $\frac{1}{2}c^2$ and let (M^6, g, ω) be an Einstein Kähler manifold with with Einstein constant $-\frac{1}{2}c^2$. Then the triple

$$(\widetilde{M}^{1,4} \times M^6, \tilde{g} + g, \mathsf{F} = c\star_6\omega),$$

is a bosonic supergravity background.

Remark 7.6. Several bosonic backgrounds of the type appearing in Corollary 7.5 are already known. In Ref. 29, it was shown that $(\mathrm{AdS})_5 \times M^6$, where (M^6, g, ω) is a Kähler manifold, defines a bosonic background with flux form $\mathsf{F} = \frac{1}{2}c\omega \wedge \omega$ (which is proportional to $\star_6\omega$) if and only if (M^6, g) is Einstein with negative scalar curvature (in the "mostly minus" setting). In particular, Eqs. (4) and (5) in Ref. 29 are the same as (7.5), up to a scalar and signature convention. Therefore, Corollary 7.5 can be viewed as a slight generalization of the backgrounds found in Ref. 29, in the sense that $\widetilde{M}^{1,4}$ is here allowed to be any Lorentzian Einstein manifold with a positive Einstein constant (in the "mostly minus" setup), not only $(\mathrm{AdS})_5$.

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In order to get other bosonic supergravity backgrounds from Corollary 7.5, we need Lorentzian Einstein manifolds $(\widetilde{M}^{1,4}, \tilde{g})$ with positive scalar curvature that are different from $(\mathsf{AdS})_5$. The class of Lorentzian Einstein–Sasakian manifolds provides us with many candidates.

Lorentzian Einstein–Sasakian structures were studied in Refs. 46–48 under the geometric perspective of Killing and twistor spinors on Lorentzian manifolds. Such manifolds admit a cone characterization and, in the simply connected case, a spin structure (Ref. 47, Lemma 12), as in the Riemannian case. Moreover, there is an analog of the well-known construction of Riemannian Einstein–Sasakian manifolds (see, for example, Ref. 32), which provides Lorentzian Einstein–Sasakian manifolds in terms of circle bundles over Kähler–Einstein manifolds of negative scalar curvature (Ref. 47, Lemma 14).

Lorentzian Einstein–Sasakian manifolds with positive (in the "mostly minus" convention) Einstein constants also occur within the framework of *η-Einstein Sasakian geometry*, and we refer to Ref. 31 for many details and notions that we omit. A particularly important result, from our perspective, is that every *negative Sasakian manifold* M^{2n+1} admits a Lorentzian Einstein–Sasakian structure with positive Einstein constant $2n$ in the "mostly minus" convention (Ref. 31, Corollary 24). In particular, the 5-sphere S^5 admits infinitely many different Lorentzian Einstein–Sasakian structures, with Einstein constant 4 (all of them are inhomogeneous). Each of these can be used as the five-dimensional Lorentzian manifold $\widetilde{M}^{1,4}$ in Corollary 7.5, providing us with infinitely many decomposable backgrounds (carrying the same flux form).

Example 7.7. Let $(\widetilde{M}^{1,4}, \tilde{g})$ be the 5-sphere S^5 with an Einstein metric coming from any of the infinitely many Lorentzian Einstein–Sasakian structures mentioned earlier with Einstein constant 4, and let (M^6, g, ω) be any Einstein Kähler manifold with Einstein constant -4. Then the triple

$$(\widetilde{M}^{1,4} \times M^6, \tilde{g} + g, \mathsf{F} = 2\sqrt{2}\star_6\omega),$$

is a bosonic supergravity background.

The connected sum $\sharp k(\mathsf{S}^2 \times \mathsf{S}^3)$ also admits Lorentzian Einstein–Sasakian metrics for any integer $k \geq 1$ (see Refs. 31 and 49), giving another class of Lorentzian Einstein–Sasakian manifolds that can potentially be used as ingredients in bosonic supergravity backgrounds. Example 7.7 highlights the appearance of Lorentzian Einstein–Sasakian geometries in supergravity, and more applications of such structures in eleven-dimensional supergravity are described in Ref. 50. For further details on the applications of five-dimensional Lorentzian manifolds in certain supergravity theories, the reader may consult the recent work[21] and the references therein.

Remark 7.8. Some of the bosonic backgrounds of the type appearing in Corollary 7.5 are symmetric. For example, if $\widetilde{M}^{1,4} = (\mathsf{AdS})_5$ and M^6 is one of the symmetric spaces $\mathbb{CP}^3 = \mathsf{SU}(4)/\mathsf{U}(3)$ or $\mathsf{Gr}_+(2,5)$, endowed with their respective homogeneous Kähler–Einstein metrics, then we obtain decomposable symmetric backgrounds (see also Ref. 17, Sec. 4.4). To construct decomposable, homogeneous but non-symmetric, (5,6)-supergravity backgrounds, we may use the full flag manifold $\mathbb{F} = \mathsf{SU}(3)/T_{\mathsf{max}}$ (see, for example, Ref. 51 for the corresponding Kähler–Einstein metrics). In addition, in Ref. 52, the reader can find families of non-supersymmetric bosonic backgrounds with non-relativistic symmetry based on products involving $(\mathsf{AdS})_5$. Finally, it is worth mentioning that (7.3) also holds for a strictly nearly Kähler manifold, and $\|\theta\|_g^2 = c^2\|\omega\|_g^2$ is a constant as well (see Ref. 53, Corollary 2.7). However, for a strictly nearly Kähler manifold, the Kähler form is not closed. Therefore, in this case, the 4-form F chosen earlier is not coclosed and cannot serve as a flux form.

VIII. CONCLUSION

The aim of this paper was to find new bosonic supergravity backgrounds by searching for decomposable (5, 6)-solutions to the bosonic supergravity equations (2.1). Following the ideas proposed in Ref. 30, where they were applied to find decomposable (6, 5)-solutions, we analyzed the bosonic supergravity equations for a variety of different types of flux 4-form F. We singled out some special cases that we analyzed more carefully in order to get concrete new supergravity backgrounds.

For several different types of null 4-forms, we found supergravity backgrounds whose five-dimensional Lorentzian part was a special type of Ricci-isotropic Walker manifold. These results, which can be found in Sec. VI, are very close in nature to those of Ref. 30, and one may conjecture that similar results and examples can be found for decomposable $(m, 11-m)$-solutions with other choices of m. A couple of places where we differed from Ref. 30 were Corollary 4.5 and Proposition 6.11 (and the corresponding Example 6.12), which contain insights that were not used in Ref. 30. These solutions may support supersymmetries, but we leave this investigation for a later paper.

In Sec. VII, we investigated two particular types of flux forms that were not null. Here we relied on some known results that are very specific to the particular pair of dimensions (5, 6). For example, by assuming that the six-dimensional Riemannian component is a Kähler manifold and by choosing a 4-form F that depended explicitly on the Kähler form, we showed that any Einstein Kähler manifold with Einstein constant $-c^2/2$ (in the "mostly minus" convention) can be paired with any Lorentzian Einstein manifold with Einstein constant $c^2/2$ to give an eleven-dimensional bosonic supergravity background. One background of this type that was already known is $(\mathsf{AdS})_5 \times M^6$, which was treated in Ref. 29 (the symmetric ones among these are also listed in Sec. 4.4 of Ref. 17), and we describe infinitely many more: From Ref. 31, we know that the 5-sphere admits infinitely many different Lorentzian Einstein–Sasakian structures with Einstein constant 4, and each of them give rises to a bosonic supergravity background when paired with an Einstein Kähler manifold with Einstein constant -4.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Hanci Chi: Writing – original draft (equal); Writing – review & editing (equal). **Ioannis Chrysikos**: Writing – original draft (equal); Writing – review & editing (equal). **Eivind Schneider**: Writing – original draft (equal); Writing – review & editing (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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