SOME NEW weak-(H_p–L_p) TYPE INEQUALITIES FOR WEIGHTED MAXIMAL OPERATORS OF FEJÉR MEANS OF WALSH–FOURIER SERIES

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Abstract. We introduce some new weighted maximal operators of the Fejér means of the Walsh–Fourier series. We prove that for some "optimal" weights these new operators are bounded from the martingale Hardy space $H_p(G)$ to the space weak-$L_p(G)$, for $0<p<1/2$. Moreover, we also prove sharpness of this result. As a consequence we obtain some new and well-known results.

1. Introduction

All symbols used in this introduction can be found in Section 2.

In the one-dimensional case, the weak (1,1)-type inequality for the maximal operator σ^* of Fejér means σ_n with respect to the Walsh system

$$\sigma^* f := \sup_{n\in\mathbb{N}} |\sigma_n f|$$

can be found in Schipp [19] and Pál, Simon [14] (see also [4], [13] and [16]). Fujii [7] and Simon [21] proved that σ^* is bounded from H_1 to L_1. Weisz [29] generalized this result and proved boundedness of σ^* from the martingale space H_p to the Lebesgue space L_p for $p>1/2$. Simon [20] gave a counterexample which shows that boundedness does not hold for $0<p<1/2$. A counterexample for $p=1/2$ was given by Goginava [9]. Moreover, in [10]

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(see also [23]) he proved that there exists a martingale $F \in H_p$ $(0 < p \leq 1/2)$ such that

$$\sup_{n\in\mathbb{N}} \|\sigma_n F\|_p = \infty.$$

Weisz [29,32] proved that the maximal operator σ^* of the Fejér means is bounded from the Hardy space $H_{1/2}$ to the space weak-$L_{1/2}$.

For $0 < p < 1/2$ in [25] it was investigated the weighted maximal operator

$$\widetilde{\sigma}^{*,p} F := \sup_{n\in\mathbb{N}} \frac{|\sigma_n F|}{(n+1)^{1/p-2}} \tag{1}$$

was investigated and it was proved that the following estimate holds:

$$\left\| \widetilde{\sigma}^* F \right\|_p \leq c_p \|F\|_{H_p}$$

and

$$\left\| \widetilde{\sigma}^* F \right\|_{\text{weak-}L_p} \leq c_p \|F\|_{H_p}. \tag{2}$$

Moreover, it was proved that the rate of sequence $\{(n+1)^{1/p-2}\}$, given in denominator of (1) can not be improved. In the case $p = 1/2$ analogical results for the maximal operator

$$\widetilde{\sigma}^* F := \sup_{n\in\mathbb{N}} \frac{|\sigma_n F|}{\log^2(n+1)}$$

was proved in [11] for Walsh system and [24] for Vilenkin systems.

In the study of convergence of subsequences of Fejér means and their restricted maximal operators on the martingale Hardy spaces $H_p(G)$ for $0 < p \leq 1/2$, the central role is played by the fact that any natural number $n \in \mathbb{N}$ can be uniquely expression as $n = \sum_{k=0}^{\infty} n_j 2^j, \quad n_j \in Z_2$ $(j \in \mathbb{N})$, where only a finite numbers of n_j differ from zero and their important characters $[n]$, $|n|$, $\rho(n)$ and $V(n)$ are defined by

$$[n] := \min\{j \in \mathbb{N}, n_j \neq 0\}, \quad |n| := \max\{j \in \mathbb{N}, n_j \neq 0\}, \quad \rho(n) = |n| - [n],$$

$$V(n) := n_0 + \sum_{k=1}^{\infty} |n_k - n_{k-1}|, \quad \text{for all } n \in \mathbb{N}.$$

Weisz [31] (see also [30]) also proved that for any $F \in H_p(G)$ $(p > 0)$, the maximal operator $\sup_{n\in\mathbb{N}} |\sigma_{2^n} F|$ is bounded from the Hardy space H_p to the Lebesgue space L_p. Persson and Tephnadze [15] (see also [4]) generalized this result and proved that if $0 < p \le 1/2$ and $\{n_k : k \ge 0\}$ is a sequence of positive numbers such that

$$\sup_{k\in\mathbb{N}} \rho(n_k) \le c < \infty, \tag{3}$$

then the restricted maximal operator $\widetilde{\sigma}^{*,\nabla}$, defined by

$$\widetilde{\sigma}^{*,\nabla} F := \sup_{k\in\mathbb{N}} |\sigma_{n_k} F| \tag{4}$$

is bounded from the Hardy space $H_p(G)$ to the space $L_p(G)$. Moreover, if $0 < p < 1/2$ and $\{n_k : k \ge 0\}$ is a sequence of positive numbers such that

$$\sup_{k\in\mathbb{N}} \rho(n_k) = \infty,$$

then there exists a martingale $F \in H_p$ such that

$$\sup_{k\in\mathbb{N}} \|\sigma_{n_k} F\|_p = \infty.$$

From these facts it follows that if $0 < p < 1/2$, $F \in H_p$ and $\{n_k : k \ge 0\}$ is any sequence of positive numbers, then the maximal operator defined by (4) is bounded from the Hardy space H_p to the Lebesgue space L_p if and only if the condition (3) is fulfilled.

For $0 < p < 1/2$ in [28] it was proved that if $F \in H_p$, then there exists an absolute constant c_p, depending only on p, such that

$$\|\sigma_n F\|_{H_p} \le c_p 2^{\rho(n)(1/p-2)} \|F\|_{H_p}.$$

Using this it follows that

$$\left\| \frac{\sigma_n F}{2^{\rho(n)(1/p-2)}} \right\|_p \le c_p \|F\|_{H_p}$$

and

$$\left\| \frac{\sigma_n F}{2^{\rho(n)(1/p-2)}} \right\|_{\text{weak-}L_p} \le c_p \|F\|_{H_p}. \tag{5}$$

Moreover, if $\{\Phi_n\}$ is any nondecreasing sequence such that

$$\sup_{k\in\mathbb{N}} \rho(n_k) = \infty, \quad \overline{\lim_{k\to\infty}} \frac{2^{\rho(n_k)(1/p-2)}}{\Phi_{n_k}} = \infty,$$

then there exists a martingale $F \in H_p$ $(0 < p < 1/2)$ such that

$$\sup_{k\in\mathbb{N}} \left\| \frac{\sigma_{n_k} F}{\Phi_{n_k}} \right\|_{\text{weak-}L_p} = \infty.$$

In [28] it was proved that if $F \in H_{1/2}$, then there exists an absolute constant c such that

$$\|\sigma_n F\|_{H_{1/2}} \leq cV^2(n) \|F\|_{H_{1/2}}.$$

Moreover, the rate of sequence $V^2(n)$ can not be improved.

The $(H_{1/2}$–$L_{1/2})$-type inequalities for the the restricted and weighted maximal operators of Walsh–Fejér means were studied in [2] and [3]. Analogical problems for partial sums of Walsh-Fourier series for $0 < p < 1$ were proved in [5] and [6] (see also [26,27]).

In this paper we generalize estimates (2) and (5). In particular, we prove that the weighted maximal operator $\widetilde{\sigma}^{*,\nabla}$, defined by

$$\widetilde{\sigma}^{*,\nabla} F := \sup_{n\in\mathbb{N}} \frac{|\sigma_n F|}{2^{\rho(n)(1/p-2)}} \tag{6}$$

of Fejér means of Walsh–Fourier series is bounded from the Hardy space $H_p(G)$ to the space weak-$L_p(G)$. Moreover, we prove that the rate of the sequence $\{2^{\rho(n)(1/p-2)}\}$ in (6) is sharp. We also prove that the maximal operator defined by (6) is not bounded from the Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$. As a consequence we obtain some new and well-known results.

This paper is organized as follows: In order not to disturb our discussions later on some preliminaries are presented in Section 2. The main result and some of its consequences can be found in Section 3. The detailed proof of the main result is given in Section 4. Some open questions and final remarks are given in Section 5.

2. Preliminaries

Let $\mathbb{N}_+$ denote the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Denote by Z_2 the discrete cyclic group of order 2, that is $Z_2 := \{0, 1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on Z_2 is given so that the measure of a singleton is 1/2.

Define the group G as the complete direct product of infinite copies of the group Z_2, with the product of the discrete topologies of Z_2 and product of the measures on Z_2 (it will be denoted by μ). The elements of G are represented by sequences $x := (x_0, x_1, \ldots , x_j, \ldots)$, where $x_k = 0 \vee 1$.

It is easy to give a base for the neighborhood of $x \in G$

$$I_0(x) := G, \quad I_n(x) := \{y \in G : y_0 = x_0, \dots, y_{n-1} = x_{n-1}\} \quad (n \in \mathbb{N}).$$

Denote $I_n := I_n(0)$, $\overline{I_n} := G \setminus I_n$ and $e_n := (0, \dots, 0, x_n = 1, 0, \dots) \in G$, for $n \in \mathbb{N}$. Then it is easy to show that

$$(7) \quad \overline{I_M} = \bigcup_{i=0}^{M-1} I_i \backslash I_{i+1} = \left(\bigcup_{k=0}^{M-2} \bigcup_{l=k+1}^{M-1} I_{l+1}(e_k + e_l) \right) \bigcup \left(\bigcup_{k=0}^{M-1} I_M(e_k) \right),$$

where

$$I_N^{k,l} =: \begin{cases} I_N(0, \dots, 0, x_k \neq 0, 0, \dots, 0, x_l \neq 0, x_{l+1}, \dots, x_{N-1}, \dots), \\ \qquad \text{for } k < l < N, \\ I_N(0, \dots, 0, x_k \neq 0, x_{k+1} = 0, \dots, x_{N-1} = 0, x_N, \dots), \\ \qquad \text{for } l = N. \end{cases}$$

If $n \in \mathbb{N}$, then every n can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j 2^j$, where $n_j \in Z_2$ $(j \in \mathbb{N})$ and only a finite numbers of n_j differ from zero.

Every $n \in \mathbb{N}$ can be also represented as $n = \sum_{i=1}^{r} 2^{n^i}$, $n^1 > n^2 > \cdots > n^r \geq 0$. For such representation of $n \in \mathbb{N}$, let denote numbers

$$n^{(i)} = 2^{n^{i+1}} + \cdots + 2^{n^r}, \quad i = 1, \dots, r.$$

The norms (or quasi-norms) of the spaces $L_p(G)$ and weak-$L_p(G)$ $(0 < p < \infty)$ are, respectively, defined by

$$\|f\|_p^p := \int_G |f|^p \, d\mu, \quad \|f\|_{\text{weak-}L_p(G)}^p := \sup_{\lambda > 0} \lambda^p \mu(f > \lambda) < +\infty,$$

The k-th Rademacher function is defined by

$$r_k(x) := (-1)^{x_k} \quad (x \in G, \ k \in \mathbb{N}).$$

Now, define the Walsh system $w := (w_n : n \in \mathbb{N})$ on G as:

$$w_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \quad (n \in \mathbb{N}).$$

The Walsh system is orthonormal and complete in $L_2(G)$ (see [18]).

If $f \in L_1(G)$, we can define the Fourier coefficients, partial sums of Fourier series, Fejér means, Dirichlet and Fejér kernels in the usual manner:

$$\widehat{f}(n) := \int_G f w_n \, d\mu, \ (n \in \mathbb{N}),$$

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}(k) w_k \quad (n \in \mathbb{N}_+,\ S_0 f := 0), \qquad \sigma_n f := \frac{1}{n} \sum_{k=1}^{n} S_k f,$$

$$D_n := \sum_{k=0}^{n-1} w_k\,, \quad K_n := \frac{1}{n} \sum_{k=1}^{n} D_k \qquad (n \in \mathbb{N}_+).$$

Recall that (see [8], [12] and [18]) for any $t, n \in \mathbb{N}$,

$$D_{2^n}(x) = \begin{cases} 2^n & \text{if } x \in I_n, \\ 0 & \text{if } x \notin I_n. \end{cases} \tag{8}$$

and

$$K_{2^n}(x) = \begin{cases} 2^{t-1}, & \text{if } x \in I_n(e_t),\ n > t,\ x \in I_t \backslash I_{t+1}, \\ (2^n + 1)/2, & \text{if } x \in I_n, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Let $n = \sum_{i=1}^{r} 2^{n^i}$, $n^1 > n^2 > \cdots > n^r \geq 0$. Then (see [12] and [18])

$$nK_n = \sum_{A=1}^{r} \left(\prod_{j=1}^{A-1} w_{2^{n^j}} \right) \left(2^{n^A} K_{2^{n^A}} + n^{(A)} D_{2^{n^A}} \right). \tag{10}$$

The next two lemmas can be found in [17] (see also [15]):

Lemma 1. *Let $n \geq 2^M$ and $x \in I_M^{k,l}$, $k = 0, \ldots, M-1$, $l = k+1, \ldots, M$. Then*

$$\int_{I_M} |K_n(x+t)| \, d\mu(t) \leq c 2^{k+l-2M}.$$

Lemma 2. *Let $n \in \mathbb{N}_+$, $[n] \neq |n|$ and $x \in I_{[n]+1}(e_{[n]-1} + e_{[n]})$. Then*

$$|nK_n(x)| = \left| \left(n - 2^{|n|} \right) K_{n-2^{|n|}}(x) \right| \geq \frac{2^{2[n]}}{4}.$$

The σ-algebra, generated by the intervals $\{I_n(x) : x \in G\}$ will be denoted by ζ_n $(n \in \mathbb{N})$. Denote by $F = (F_n, n \in \mathbb{N})$ a martingale with respect to ζ_n $(n \in \mathbb{N})$ (for details see e.g. [30]).

The maximal function F^* of a martingale F is defined by

$$F^* := \sup_{n \in \mathbb{N}} |F_n|.$$

In the case $f \in L_1(G)$ the maximal function f^* is given by

$$f^*(x) := \sup_{n\in\mathbb{N}} \frac{1}{\mu(I_n(x))} \left| \int_{I_n(x)} f(u)\, d\mu(u) \right|.$$

For $0 < p < \infty$ the Hardy martingale spaces $H_p(G)$ consists of all martingales for which (for details see e.g. [17], [22] and [30])

$$\|F\|_{H_p} := \|F^*\|_p < \infty.$$

It is easy to check that for every martingale $F = (F_n, n \in \mathbb{N})$ and every $k \in \mathbb{N}$ the limit

$$\widehat{F}(k) := \lim_{n\to\infty} \int_G F_n(x) w_k(x)\, d\mu(x)$$

exists and is called the k-th Walsh–Fourier coefficients of F.

If $F := (S_{2^n} f : n \in \mathbb{N})$ is a regular martingale, generated by $f \in L_1(G)$, then $\widehat{F}(k) = \widehat{f}(k), \quad k \in \mathbb{N}$.

A bounded measurable function a is called p-atom if there exists a dyadic interval I such that

$$\int_I a\, d\mu = 0, \quad \|a\|_\infty \le \mu(I)^{-1/p}, \quad \operatorname{supp}(a) \subset I.$$

The dyadic Hardy martingale spaces H_p for $0 < p \le 1$ have an atomic characterization. Namely, the following theorem holds (see [17], [30], [31]):

Lemma 3. *A martingale $F = (F_n, n \in \mathbb{N})$ belongs to H_p $(0 < p \le 1)$ if and only if there exists a sequence $(a_k, k \in \mathbb{N})$ of p-atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of real numbers such that for every $n \in \mathbb{N}$*

$$\sum_{k=0}^{\infty} \mu_k S_{2^n} a_k = F_n, \quad \sum_{k=0}^{\infty} |\mu_k|^p < \infty. \tag{11}$$

Moreover, $\|F\|_{H_p} \backsim \inf \left(\sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p}$, where the infimum is taken over all decomposition of F of the form (11).

From this result it follows the following important lemma.

Lemma 4 (Weisz [30]). *Suppose that an operator T is σ-sublinear and*

$$\sup_{\rho>0} \rho^p \mu\left\{ x \in \overline{I} : |Ta(x)| > \rho \right\} \le C_p < \infty,$$

for every p-atom a, where I denotes the support of the atom. If T is bounded from L_∞ to L_∞, then

$$\|TF\|_{\text{weak-}L_p} \le c_p \|F\|_{H_p}.$$

3. The main result and its consequences

Theorem 1. a) *Let* $0 < p < 1/2$ *and* $f \in H_p(G)$. *Then the weighted maximal operator* $\widetilde{\sigma}^{*,\nabla}$, *defined by* (6), *is bounded from the Hardy space* H_p *to the space* weak-L_p.

b) *Let* $\varphi\colon \mathbb{N} \to [1,\infty)$ *be a nondecreasing function, satisfying the condition*

$$\overline{\lim_{n\to\infty}} \frac{2^{\rho(n)(1/p-2)}}{\varphi(n)} = \infty.$$

Then there exist a sequence $\{f_{n_k}, k \in \mathbb{N}_+\}$ *of* p*-atoms and a sequence* $\{q_{n_k}, k \in \mathbb{N}_+\}$ *of real numbers satisfying the condition* $|q_{n_k}| = n_k$ *such that*

$$\sup_{k\in\mathbb{N}} \frac{\left\| \frac{\sigma_{q_{n_k}} f_{n_k}}{\varphi(q_{n_k})} \right\|_{\text{weak-}L_p}}{\|f_{n_k}\|_{H_p}} = \infty.$$

We also prove the following theorem.

Theorem 2. *Let* $0 < p < 1/2$. *There exists a sequence* $\{f_k, k \in \mathbb{N}_+\}$ *of* p*-atoms such that*

$$\sup_{k\in\mathbb{N}} \frac{\|\widetilde{\sigma}^{*,\nabla} f_k\|_p}{\|f_k\|_{H_p}} = \infty.$$

From Theorem 1 immediately follows the mentioned result of Weisz [31] (see also [30]):

Corollary 1. *Let* $0 < p < 1/2$ *and* $f \in H_p(G)$. *Then the maximal operator*

$$\sup_{n\in\mathbb{N}} |\sigma_{2^n} F|$$

is bounded from the Hardy space $H_p(G)$ *to the Lebesgue space* weak-$L_p(G)$.

We also obtain results of Persson and Tephnadze [15] (see also [4]):

Corollary 2. *Let* $0 < p < 1/2$ *and* $f \in H_p(G)$. *Then the maximal operator, defined by* (4) *is bounded from the Hardy space* $H_p(G)$ *to the space* weak-$L_p(G)$ *if and only if condition* (3) *is fulfilled.*

Corollary 3. a) *Let* $0 < p < 1/2$ *and* $f \in H_p(G)$. *Then the weighted maximal operator*

$$\sup_{n\in\mathbb{N}} \frac{|\sigma_{2^n+2^{n/2}} F|}{2^{\frac{n}{2}(1/p-2)}}$$

is bounded from the martingale Hardy space $H_p(G)$ *to the space* weak-$L_p(G)$.

b) *Let* $\varphi\colon \mathbb{N} \to [1,\infty)$ *be a nondecreasing function, satisfying the condition*

$$\overline{\lim_{n\to\infty}} \frac{2^{\frac{n}{2}(1/p-2)}}{\varphi(n)} = \infty.$$

Then, there exists a p*-atom* a *such that*

$$\sup_{n\in\mathbb{N}} \frac{\left\| \frac{\sigma_{2^n+2^{n/2}}a}{\varphi(2^n+2^{n/2})} \right\|_{\text{weak-}L_p}}{\|a\|_{H_p}} = \infty.$$

Corollary 4. a) *Let* $0 < p < 1/2$ *and* $f \in H_p(G)$. *Then the weighted maximal operator*

$$\sup_{n\in\mathbb{N}} \frac{|\sigma_{2^n+1}F|}{2^{n(1/p-2)}}$$

is bounded from the Hardy space H_p *to the space* weak-L_p.

b) *Let* $\varphi\colon \mathbb{N} \to [1,\infty)$ *be a nondecreasing function, satisfying the condition*

$$\overline{\lim_{n\to\infty}} \frac{2^{n(1/p-2)}}{\varphi(n)} = \infty.$$

Then, there exists a p*-atom* a *such that*

$$\sup_{n\in\mathbb{N}} \frac{\left\| \frac{\sigma_{2^n+1}a}{\varphi(2^n+1)} \right\|_{\text{weak-}L_p}}{\|a\|_{H_p}} = \infty.$$

Theorem 1 immediately follows result given in [25]:

Corollary 5. a) *Let* $0 < p < 1/2$ *and* $f \in H_p(G)$. *Then the weighted maximal operator* $\widetilde{\sigma}^*$, *defined by*

$$\widetilde{\sigma}^* F := \sup_{n\in\mathbb{N}} \frac{|\sigma_n F|}{(n+1)^{1/p-2}}$$

is bounded from the martingale Hardy space $H_p(G)$ *to the space* weak-$L_p(G)$.

b) *Let* $\{\varphi_n\}$ *be any nondecreasing sequence satisfying the condition*

$$\overline{\lim_{n\to\infty}} \frac{(n+1)^{1/p-2}}{\varphi_n} = +\infty.$$

Then there exists a martingale $f \in H_p$ *such that*

$$\sup_{n\in\mathbb{N}} \left\| \frac{\sigma_n f}{\varphi_n} \right\|_p = \infty.$$

4. Proof of the Theorems

Proof of Theorem 1. Since σ_n is bounded from L_∞ to L_∞, by Lemma 4, the proof of Theorem 1 will be complete, if we show that

$$t\mu\left\{x \in \overline{I_M} : \widetilde{\sigma}^{*,\nabla} a(x) \geq t^{1/p}\right\} \leq c < \infty, \quad t \geq 0 \tag{12}$$

for every p-atom a. We may assume that a is an arbitrary p-atom, with support I, $\mu(I) = 2^{-M}$ and $I = I_M$. It is easy to see that $\sigma_n a(x) = 0$ when $n < 2^M$. Therefore, we can suppose that $n \geq 2^M$. Since $\|a\|_\infty \leq 2^{M/p}$, we obtain that

$$\begin{aligned}\frac{|\sigma_n a(x)|}{2^{\rho(n)(1/p-2)}} &\leq \frac{1}{2^{\rho(n)(1/p-2)}}\|a\|_\infty \int_{I_M} |K_n(x+t)|\, d\mu(t)\\ &\leq \frac{1}{2^{\rho(n)(1/p-2)}} 2^{M/p} \int_{I_M} |K_n(x+t)|\, d\mu(t).\end{aligned}$$

Let $x \in I_{l+1}(e_k + e_l)$, $0 \leq k, l \leq [n] \leq M$ or $0 \leq k, l \leq M < [n]$. Then, it is easy to see that $x + t \in I_{l+1}(e_k + e_l)$ for $t \in I_M$ and if we combine (8) and (9) with (10) we get that

$$K_n(x+t) = 0, \quad \text{for } t \in I_M$$

and

$$\frac{|\sigma_n a(x)|}{2^{\rho(n)(1/p-2)}} = 0. \tag{13}$$

Let $x \in I_{l+1}(e_k + e_l)$, $[n] \leq k, l \leq M$ or $k \leq [n] \leq l \leq M$. By using Lemma 1 we can conclude that

$$\begin{aligned}\frac{|\sigma_n a(x)|}{2^{\rho(n)(1/p-2)}} &\leq c_p 2^{M/p} \frac{2^{k+l-2M}}{2^{\rho(n)(1/p-2)}} \leq c_p \frac{2^{[n](1/p-2)+k+l+M(1/p-2)}}{2^{|n|(1/p-2)}}\\ &\leq c_p 2^{[n](1/p-2)+k+l} \leq c_p 2^{k+l(1/p-1)}.\end{aligned} \tag{14}$$

By applying (13) and (14) for any $x \in I_{l+1}(e_k + e_l)$, $1 \leq k < l \leq M$ we find that

$$\widetilde{\sigma}^{*,\nabla} a(x) = \sup_{n \in \mathbb{N}} \left(\frac{|\sigma_n a(x)|}{2^{\rho(n)(1/p-2)}}\right) \leq c_p 2^{k+l\ (1/p-1)}.$$

It immediately follows that for such $k < l \leq M$ we have the estimate

$$\widetilde{\sigma}^{*,\nabla} a(x) \leq C_p 2^{M/p} \quad \text{for } x \in I_M^{k,l}$$

and also that

$$\mu\left\{x \in I_N^{k,l} : \widetilde{\sigma}^{*,\nabla} a(x) > C_p 2^{s/p}\right\} = 0, \quad s = M+1, M+2, \ldots. \tag{15}$$

Suppose that

$$2^{k+l\ (1/p-1)} > 2^{s/p} \quad \text{for some } s \leq M \tag{16}$$

It is evident that inequality (16) does not hold when $k < l \leq s$. On the other hand, inequality (16) holds for all $l > k \geq s$, that is,

$$2^{k+l\ (1/p-1)} > 2^{s/p}, \quad \text{where } l > k \geq s. \tag{17}$$

If $l > s > k$, from (16) we can conclude that

$$k + l\ (1/p - 1) > s/p, \quad l > (s/p - k)/(1/p - 1)$$

and

$$2^{k+l(1/p-1)} > 2^{s/p}, \quad \text{where } s > k,\ l > (s/p - k)/(1/p - 1). \tag{18}$$

By combining (7), (17) and (18) we get that

$$\begin{aligned}
&\left\{x \in \overline{I_M} : \widetilde{\sigma}^{*,\nabla} a(x) \geq C_p 2^{s/p}\right\} \\
&\subset \left(\bigcup_{k=s}^{M-1} \bigcup_{l=k+1}^{M} \left\{x \in I_M^{k,l} : \widetilde{\sigma}^{*,\nabla} a(x) \geq C_p 2^{s/p}\right\}\right) \\
&\cup \left(\bigcup_{k=0}^{s} \bigcup_{l>(s/p-k)(1/p-1)}^{M} \left\{x \in I_M^{k,l} : \widetilde{\sigma}^{*,\nabla} a(x) \geq C_p 2^{s/p}\right\}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\mu\left\{x \in \overline{I_M} : \widetilde{\sigma}^{*,\nabla} a(x) \geq C_p 2^{s/p}\right\} \\
&\leq \sum_{k=s}^{M-1} \sum_{l=k+1}^{M} \mu(I_M^{k,l}) + \sum_{k=0}^{s} \sum_{l>(s/p-k)/\ (1/p-1)}^{M} \mu(I_M^{k,l}) \\
&\leq \sum_{k=s}^{M-1} \sum_{l=k+1}^{M} \frac{1}{2^l} + \sum_{k=0}^{s} \sum_{l>(s/p-k)/\ (1/p-1)}^{M} \frac{1}{2^l} \\
&\leq \sum_{k=s}^{M-1} \frac{1}{2^k} + \sum_{k=0}^{s} \frac{1}{2^{(s/p-k)/\ (1/p-1)-1}} \leq \frac{c_p}{2^s}.
\end{aligned} \tag{19}$$

In view of (15) and (19) we can conclude that

$$2^{s}\mu\left\{x\in\overline{I_{M}}:\widetilde{\sigma}^{*,\nabla}a(x)\geq C_{p}2^{s/p}\right\}<c_{p}<\infty,$$

which shows (12) as well as part a).

Let $q_{n_k}\in\mathbb{N}$ be sequence such that $|q_{n_k}|=n_k$, $[q_{n_k}]=s_k$ and

$$\lim_{k\to\infty}\frac{2^{\rho(q_{n_k})(1/p-2)}}{\varphi\left(q_{n_k}\right)}=\infty. \tag{20}$$

Set

$$f_{n_k}(x)=D_{2^{n_k+1}}(x)-D_{2^{n_k}}(x),\quad n_k\geq 3.$$

It is evident

$$\widehat{f}_{n_k}(i)=\begin{cases}1, & \text{if } i=2^{n_k},\dots,2^{n_k+1}-1,\\ 0 & \text{otherwise.}\end{cases}$$

Then we can write that

$$S_i f_{n_k}(x)=\begin{cases}D_i(x)-D_{2^{n_k}}(x), & \text{if } i=2^{n_k},\dots,2^{n_k+1}-1,\\ f_{n_k}(x), & \text{if } i\geq 2^{n_k+1},\\ 0 & \text{otherwise.}\end{cases} \tag{21}$$

Since

$$D_{j+2^{n_k}}(x)-D_{2^{n_k}}(x)=w_{2^{n_k}}D_j(x),\quad j=1,2,..,2^{n_k}, \tag{22}$$

from (8) we get

$$\begin{aligned}\|f_{n_k}\|_{H_p}&=\left\|\sup_{n\in\mathbb{N}}S_{2^n}f_{n_k}\right\|_p=\|D_{2^{n_k+1}}-D_{2^{n_k}}\|_p\\ &=\|D_{2^{n_k}}\|_p\leq 2^{n_k(1-1/p)}.\end{aligned} \tag{23}$$

By applying (21) we can conclude that

$$\left|\sigma_{q_{n_k}}f_{n_k}(x)\right|=\frac{1}{q_{n_k}}\left|\sum_{j=0}^{q_{n_k}-1}S_jf_{n_k}(x)\right|=\frac{1}{q_{n_k}}\left|\sum_{j=2^{n_k}}^{q_{n_k}-1}S_jf_{n_k}(x)\right|$$

$$=\frac{1}{q_{n_k}}\left|\sum_{j=2^{n_k}}^{q_{n_k}-1}\left(D_j(x)-D_{2^{n_k}}(x)\right)\right|=\frac{1}{q_{n_k}}\left|\sum_{j=0}^{q_{n_k}-2^{n_k}-1}\left(D_{j+2^{n_k}}(x)-D_{2^{n_k}}(x)\right)\right|.$$

By using (22) we find that

$$(24)\quad \left|\sigma_{q_{n_k}} f_{n_k}(x)\right| = \frac{1}{q_{n_k}}\left|\sum_{j=0}^{q_{n_k}-2^{n_k}-1} D_j(x)\right| = \frac{q_{n_k}-2^{n_k}-1}{q_{n_k}}\left|K_{q_{n_k}-2^{n_k}-1}(x)\right|.$$

Let $x \in I_{[q_{n_k}]+1}\left(e_{[q_{n_k}]-1}+e_{[q_{n_k}]}\right)$. By using Lemma 2 we obtain that

$$\left|\sigma_{q_{n_k}} f_{n_k}(x)\right| \geq \frac{c\,2^{2s_k}}{2^{n_k}} \quad \text{and} \quad \frac{|\sigma_{q_{n_k}} f_{n_k}(x)|}{\varphi(q_{n_k})} \geq \frac{c\,2^{2s_k}}{2^{n_k}\varphi(q_{n_k})}.$$

Hence, we can conclude that

$$(25)\quad \begin{aligned} &\mu\left\{x \in G : \frac{|\sigma_{q_{n_k}} f_{n_k}(x)|}{\varphi(q_{n_k})} \geq \frac{c\,2^{2[q_{n_k}]}}{2^{n_k}\varphi(q_{n_k})}\right\} \\ &\geq \mu\left(I_{[q_{n_k}]+1}(e_{[q_{n_k}]-1}+e_{[q_{n_k}]})\right) > c/2^{[q_{n_k}]}. \end{aligned}$$

By combining (20), (23) and (25) we get that

$$\begin{aligned} &\frac{\frac{c\,2^{2[q_{n_k}]}}{2^{n_k}\varphi(q_{n_k})}\left(\mu\left\{x \in G : \frac{|\sigma_{q_{n_k}} f_{n_k}(x)|}{\varphi(q_{n_k})} \geq \frac{c\,2^{2[q_{n_k}]}}{2^{n_k}\varphi(q_{n_k})}\right\}\right)^{1/p}}{\|f_{n_k}(x)\|_{H_p}} \\ &\geq \frac{c_p\,2^{2[q_{n_k}]}}{2^{n_k}\varphi(q_{n_k})2^{n_k(1-1/p)}}\,\frac{1}{2^{[q_{n_k}]/p}} = \frac{c_p\,2^{n_k(1/p-2)}}{2^{[q_{n_k}](1/p-2)}\varphi(q_{n_k})} \\ &= \frac{c_p\,2^{\rho(q_{n_k})(1/p-2)}}{\varphi(q_{n_k})} \to \infty \quad \text{as } k \to \infty. \quad \square \end{aligned}$$

Proof of Theorem 2. Let f_{n_k} be the p-atom from part b) of Theorem 1. If we replace q_{n_k} by $q_{n_k}^s = 2^{n_k}+2^s$ (we note that $|q_{n_k}^s| = n_k$, $[q_{n_k}^s] = s$) from (24) we find that

$$\left|\sigma_{q_{n_k}^s} f_{n_k}(x)\right| \geq \frac{c\,2^{2s}}{2^{n_k}} \quad \text{for } x \in I_{s+1}(e_{s-1}+e_s)$$

and

$$\frac{|\sigma_{q_{n_k}^s} f_{n_k}(x)|}{2^{(1/p-2)\rho(q_{n_k}^s)}} \geq \frac{c_p\,2^{s/p}}{2^{n_k(1/p-1)}} \quad \text{for } x \in I_{s+1}(e_{s-1}+e_s).$$

Hence,

$$(26)\quad \int_G \left(\sup_{k\in\mathbb{N}} \frac{|\sigma_{q_{n_k}^s} f_{n_k}(x)|}{2^{(1/p-2)\rho(q_{n_k}^s)}}\right)^p d\mu(x)$$

$$\geq \sum_{s=1}^{n_k-1} \int_{I_{s+1}(e_{s-1}+e_s)} \left(\frac{|\sigma_{q_{n_k}^s} f_{n_k}(x)|}{2^{(1/p-2)\rho(q_{n_k}^s)}} \right)^p d\mu(x)$$

$$\geq c_p \sum_{s=1}^{n_k-1} \frac{1}{2^s} \frac{2^s}{2^{n_k(1-p)}} \geq \frac{C_p n_k}{2^{n_k(1-p)}}.$$

Finally, by combining (23) and (26) we find that

$$\frac{\left(\int_G \left(\sup_{k\in\mathbb{N}} \sup_{0<s<n_k} \frac{|\sigma_{q_{n_k}^s} f_{n_k}(x)|}{2^{(1/p-2)\rho(q_{n_k}^s)}} \right)^p d\mu(x) \right)^{1/p}}{\|f_{n_k}\|_{H_p}}$$

$$\geq \frac{\left(\frac{C_p n_k}{2^{n_k(1-p)}}\right)^{1/p}}{2^{n_k(1/p-1)}} \geq c_p {n_k}^{1/p} \to \infty, \quad \text{as } k \to \infty. \quad \square$$

5. Open questions and final remarks

Remark 1. This article can be regarded as a complement of the new book [17]. In this book also a number of open problems are raised. Also this new investigation implies some corresponding open questions.

From Theorem 2 we can conclude the following result:

Theorem 3. *Let $0 < p < 1/2$ and $f \in H_p(G)$. Then the weighted maximal operator $\widetilde{\sigma}^{*,\nabla}$ defined by (6) is not bounded from the Hardy space H_p to the Lebesgue space L_p.*

An open problem. Let us introduce some new weighted maximal operator of the Fejér means of the Walsh–Fourier series with some "optimal" weights such that this new operator is bounded from the martingale Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$, for $0 < p < 1/2$.

To study boundedness of restricted maximal operators from the martingale Hardy spaces $H_p(G)$ to the Lebesgue space $L_p(G)$, where $0 < p \leq 1/2$, for any natural number satisfying the condition

$$2^s \leq n_{s_1} \leq n_{s_2} \leq \cdots \leq n_{s_r} < 2^{s+1}, \quad s \in \mathbb{N},$$

we define numbers

$$s_- := \min\{[n_{s_j}]\}, \quad s_+ := \max\{[n_{s_j}]\} = s, \quad \rho_s(n_{s_j}) := s_+ - s_-. \tag{27}$$

Conjecture 1. *Let $0 < p < 1/2$, $f \in H_p(G)$ and $\{n_k : k \geq 0\}$ be a sequence of positive numbers and let $\{n_{s_i} : 1 \leq i \leq r\} \subset \{n_k : k \geq 0\}$ be numbers such that*

$$2^s \leq n_{s_1} \leq n_{s_2} \leq \cdots \leq n_{s_r} \leq 2^{s+1}, \quad s \in \mathbb{N}.$$

a) *The weighted maximal operator*

$$\widetilde{\sigma}^{*,\nabla}F := \sup_{s\in\mathbb{N}} \sup_{2^s\le n_{s_i}<2^{s+1}} \frac{|\sigma_n F|}{2^{\rho_s(n_{s_i})(1/p-2)}},$$

where $\rho_s(n_{s_i})$ *are defined by* (27), *is bounded from the Hardy space* $H_p(G)$ *to the Lebesgue space* $L_p(G)$.

b) *For any nonnegative and nondecreasing function* $\varphi\colon \mathbb{R}_+\to\mathbb{R}$ *satisfying the condition*

$$\sup_{s\in\mathbb{N}} \sup_{2^s\le n_{s_i}<2^{s+1}} \frac{2^{\rho_s\left(n_{s_i}\right)(1/p-2)}}{\varphi(n_{s_i})} = \infty, \tag{28}$$

there exists p-atoms f_s *such that*

$$\frac{\left\| \sup_{s\in\mathbb{N}} \sup_{2^s\le n_{s_i}<2^{s+1}} \frac{|\sigma_{n_{s_i}} f_s|}{\varphi(n_{s_i})} \right\|_p}{\|f_s\|_{H_p}} \to \infty, \quad \text{as } s\to\infty.$$

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