Slicing Orbit spaces

Geometry and combinatorics of hyperbolic and even-hyperbolic slices

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Abstract

Motivated by a connection to Timofte's degree and half-degree principle we study canonical hyperbolic slices, that is, sets of univariate hyperbolic polynomials that share the same first few coefficients. We study the geometric and combinatorial properties of a natural stratification of these slices and use these properties to improve upon the degree principle.

Amongst the geometric properties we establish is a description of the dimension and relative interior of the strata along with a characterisation of some natural points of "escapes" from these strata. And on the combinatorial side we show that the lattice of strata is determined by the zero-dimensional strata and that the boundary complex of the dual lattice is generically a combinatorial sphere.

We finish by showing that a similar story can be told about a natural stratification of even-hyperbolic slices. These are the subsets of hyperbolic slices consisting of the polynomials with only nonnegative roots and such sets arise in the context of the degree principle for the hyperoctahedral group.

Sammendrag

Grunnet en kobling til Timoftes grad- og halvgradprinsipp studerer vi såkalte hyperbolske stykker. Dette er mengder bestående av hyperbolske polynomer i en variabel som har de samme første koeffisientene. Vi studerer geometriske og kombinatoriske egenskaper ved en naturlig stratifikasjon av hyperbolske stykker og bruker disse egenskapene til å forbedre Timoftes gradprinsipp.

Innenfor geometri så viser vi hvilke dimensjoner stratene kan ha og vi beskriver det relative indre til strataene i tillegg til å karakterisere noen naturlige "rømningspunkter" fra strataene. Innen kombinatorikk så viser vi at stratifikasjonen er bestemt av de nulldimensionale strataene og at randkomplekset til den duale delordnede mengden av strata er en kombinatorisk sfære.

Vi avslutter med å vise at en naturlig stratifikasjon av parhyperbolske stykker har lignende geometriske og kombinatoriske egenskaper. Parhyperbolske stykker er delmengder av hyperbolske stykker bestående av de polynomene med kun ikke-negative røtter og slike mengder har en kobling til gradprinsippet for den hyperoktaedriske gruppen.

Contents

Preface

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The thesis contains work from the preprints [20] and [21], the last of which is joint work with fellow doctoral student Robin Schabert at UiT. The content from these works is interspersed among other work in chapter 2, 3 and 4 but roughly subchapter 2.1 and 3.1 correspond to the first preprint and subchapter 2.2 and 3.2 along with chapter 4 correspond to the second.

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[1]Untranslatable Norwegian word referring to a couples' last child

Introduction

In this thesis we will mainly study the surprisingly nice geometric and combinatorial structure of certain *hyperbolic slices*. That is, sets of univariate *hyperbolic* (real-rooted) polynomials that share the same first few coefficients. Since the coefficients of hyperbolic polynomials can be expressed with the elementary symmetric polynomials, the context in which hyperbolic slices appear is coming from the study of the natural action of the symmetric group on real multivariate polynomials. In particular, since the elementary symmetric polynomials generate the algebra of invariants of the symmetric group, hyperbolic slices can be seen as cross-sections of the orbit space of the symmetric group. In a similar manner is the action of the hyperoctahedral (signed symmetric) group on real multivariate polynomials connected to *even-hyperbolic slices*. That is, sets of univariate hyperbolic polynomials that share the same first few coefficients and has only nonnegative roots. We will study a natural stratification of both hyperbolic and even-hyperbolic slices, in particular the geometric structure of the strata and the combinatorial structure of the poset of strata.

The study of polynomials is of interest in general as polynomials and their zero sets occur in many areas of mathematics like algebra, geometry, analysis and number theory. Polynomials and their zero sets also occur outside of mathematics in areas such as optimisation, statistics, chemistry and physics and thus has uses in all the sciences as well as in medical fields, data security and finance. In the cases where the polynomials has an invariant structure, it is often possible to reduce questions about the polynomials or their zero sets to easier questions by modding out the invariant structure. In particular, the action of the symmetric group is of interest as it plays a fundamental role in Galois theory, combinatorics and even in the study of group theory itself, thus it makes sense to study the action of the symmetric group on polynomials. We also have that the symmetric group along with the hyperoctahedral group appear as two of the main examples of Weyl groups. The Weyl groups have been studied extensively in the past and are generated by the reflections through a certain set of hyperplanes in Euclidean space, thus their action is quite straightforward to visualise. These reflections offer a rich combinatorial structure which we will exploit in our study of hyperbolic and even-hyperbolic slices.

From invariant algebraic sets to slices of orbit spaces

The origin of the study of hyperbolic slices is their connection to symmetric real algebraic sets. This connection comes from the fact that the orbit space of the natural action of the symmetric group on $\mathbb{R}^n$ can be identified with the set of monic hyperbolic polynomials of degree n by using the *Vieta* map. So the image of a symmetric real algebraic set under the Vieta map is a particular set of hyperbolic polynomials and hyperbolic slices are certain subsets of such images. This connection was used in [27] to provide an elementary proof of *Timofte's degree* and *half-degree principle* for the symmetric group and it was used in [28] to generalise these results. After we have studied hyperbolic slices we will also exploit this connection to make further improvements on the degree principle.

To show how hyperbolic slices and the degree principle connect, let $f \in \mathbb{R}[t]$ be a monic hyperbolic polynomial of degree n, then by the Fundamental Theorem of Algebra

$$f := t^n + f_1 t^{n-1} + ... + f_n = \prod_{i=1}^{n} (t - a_i)$$

for some $a_1, ..., a_n \in \mathbb{R}$. We can see by expanding the product above that $f_i = (-1)^i E_i(a_1, ..., a_n)$, where $E_1, ..., E_n$ are the elementary symmetric polynomials in n variables. Thus we can think of the monic hyperbolic polynomials of degree n as the orbit space of the symmetric group $\mathcal{S}(n)$ acting on $\mathbb{R}^n$.

Suppose $F_1, ..., F_k \in \mathbb{R}[x_1, ..., x_n]$ are symmetric polynomials of degree at most $s \leq n$, then it follows from a closer study of a classical proof of the Fundamental Theorem for Symmetric Polynomials (see Proposition 2.3 in [27]) that for each $i \in [k] := \{1, 2, ..., k\}$ we have

$$F_i = G_i(E_1(x), ..., E_s(x))$$

for some $G_i \in \mathbb{R}[y_1, ..., y_s]$. Thus $F_1, ..., F_k$ has a common zero $b := (b_1, ..., b_n) \in \mathbb{R}^n$ if and only if $G_1, ..., G_k$ has the point $(E_1(b), ..., E_s(b)) \in \mathbb{R}^s$ as a common zero. In other words $F_1, ..., F_k$ has a common zero if and only if

$$E_1(x) = w_1, ..., E_s(x) = w_s$$

has a real solution for some common zero $(w_1, ..., w_s) \in \mathbb{R}^s$ of $G_1, ..., G_s$. As we have seen, these equations have a solution if and only if there exists a monic hyperbolic polynomial, f, of degree n with

$$f_1 = -w_1, ..., f_s = (-1)^s w_s.$$

This connection was used in [27] to prove Timofte's degree principle, namely that $F_1, ..., F_k$ has a common zero if and only if they have a common zero with at most s distinct coordinates. It also leads us to the main topic of this thesis which is to study the following sets:

Definition. *Let $\mathcal{H}$ denote the set of all monic hyperbolic polynomials of degree n and let $f \in \mathcal{H}$. Then for any $s \in [n]$, we call the subset of hyperbolic polynomials*

$$\mathcal{H}_s(f) := \{t^n + h_1 t^{n-1} + ... + h_n \in \mathcal{H} \mid h_i = f_i \ \forall \ i \in [s]\}$$

a (canonical) ***hyperbolic slice****.*

We introduce a natural stratification of hyperbolic slices defined by considering the order and multiplicities of the roots of the corresponding hyperbolic polynomial. Then we show that the strata of hyperbolic slices have much in common with faces of polytopes and that the poset of strata have a lot in common with the face lattice of polytopes despite the fact that hyperbolic slices and their strata are rarely convex sets.

Specifically, if f is a monic hyperbolic polynomial of degree n with distinct roots $a_1 < ... < a_l$ and respective multiplicities $m_1, ..., m_l$, we let

$$c(f) := (m_1, ..., m_l)$$

be the **composition** of f. For two compositions of n, μ and ν, we let $\mu \leq \nu$ if μ can be obtained from ν by replacing some of the commas in ν with plus signs and so we define the strata of $\mathcal{H}_s(f)$ as follows:

Definition. *Let μ be a composition of n, then*

$$\mathcal{H}_s^{\mu}(f) := \{h \in \mathcal{H}_s(f) \mid c(h) \leq \mu\}$$

is a stratum of $\mathcal{H}_s(f)$.

Hyperbolic slices have implicitly been studied in several articles before. In [3], [13] and [18] they studied the related *Vandermonde varieties* which are real algebraic sets given by weighted power sums. In particular they showed that the strata of hyperbolic slices are contractible sets. Also, in [3] and [23] they studied a particular kind of extremal points of hyperbolic slices that in a natural way represents an "escape" from the domain of hyperbolic polynomials. We build on these works and delve into other geometric properties of the strata, but more importantly we use these properties to study the somewhat unexplored combinatorial structure of the poset of strata of hyperbolic slices. This in turn leads us to make improvements on Timofte's degree principle.

The degree principle for the symmetric group can also be extended to other Weyl groups (see [12]), in particular to the hyperoctahedral group. And similar to the connection between the symmetric group and hyperbolic slices we can connect the hyperoctahedral group to even-hyperbolic slices. Specifically, if $F_1, ..., F_k \in \mathbb{R}[x_1, ..., x_n]$ are polynomials of degree at most $2s \leq 2n$ and they are invariant under the natural action of the hyperoctahedral group $\mathcal{B}(n)$, then for all $i \in [k]$ we have

$$F_i = G_i(E_1(x^2), ..., E_s(x^2))$$

for some $G_i \in \mathbb{R}[y_1, ..., y_s]$ and where $x^2 := (x_1^2, ..., x_n^2)$.

Thus, by applying the variable change $z_i = x_i^2$, we see that $F_1, ..., F_k$ has a common zero if and only if

$$E_1(z) = w_1, ..., E_s(z) = w_s$$

has a real solution with only nonnegative entries for some common zero $(w_1, ..., w_s) \in \mathbb{R}^s$ of $G_1, ..., G_s$. These equations have a real solution with only nonnegative entries if and only if there exists a monic hyperbolic polynomial, f, degree n and with only nonnegative roots such that

$$f_1 = -w_1, ..., f_s = (-1)^s w_s.$$

We call hyperbolic polynomials with only nonnegative roots, **even-hyperbolic** and analogously to the symmetric group we define the following sets:

Definition. *Let $\mathcal{N}$ denote the set of all monic even-hyperbolic polynomials of degree n and let $f \in \mathcal{N}$. Then for any $s \in [n]$, we call the subset*

$$\mathcal{N}_s(f) := \mathcal{H}_s(f) \cap \mathcal{N}$$

a (canonical) ***even-hyperbolic slice****.*

Such slices have implicitly been studied in [4], [13] and [29], however we will see that by considering even-hyperbolic polynomials as a subset of hyperbolic polynomials, many properties of even-hyperbolic slices follow easily from hyperbolic slices.

To see how we stratify even-hyperbolic slices, note that zero coordinates of a point in a hyperoctahedral-invariant real algebraic set plays a particular role since the orbits of points with zero coordinates are smaller. This leads us to stratify $\mathcal{N}_s(f)$ a bit differently from $\mathcal{H}_s(f)$. Namely, we will let $\mathcal{N}_s^{\mu}(f) := \mathcal{H}_s^{\mu}(f) \cap \mathcal{N}$ be a stratum of $\mathcal{N}_s(f)$, but we will also include strata of the form

$$\mathcal{N}_s^{-\mu}(f) := \{h \in \mathcal{N}_s^{\mu}(f) | h(0) = 0\}$$

and we call $\pm\mu$ a **signed composition**.

We show that many of the geometric properties of the hyperbolic strata transfer to the even-hyperbolic strata. Furthermore we will see that many of the combinatorial properties of the poset of strata of hyperbolic slices can be established for the poset of even-hyperbolic strata and we also obtain an improvement of the degree principle for the hyperoctahedral group.

Contribution

The main geometrical results we contribute with in this thesis is firstly a description of the relative interior of the strata (Theorem 2.1.9). Next, we show that the strata of hyperbolic slices are either empty, a point or of maximal possible dimension (Theorem 2.1.10) and we show that the strata equals the closure of their relative interior (Corollary 2.1.11). These results tell us a lot about how the polynomials in hyperbolic slices are distributed. We continue by generalising the main theorem from [23] and show that any stratum has a unique polynomial with a minimal $(s+1)^{th}$ coefficient and a unique polynomial with a maximal $(s+1)^{th}$ coefficient and we characterise the composition of these polynomials (Theorem 2.2.3). Finally, we show that similarly to the strata, the relative interior of the strata are also contractible when nonempty (Theorem 2.3.4).

The main combinatorial results we contribute with are firstly to show that the poset of strata of hyperbolic slices are graded, atomic and coatomic lattices (Theorem 3.1.7). This gives us a combinatorial algorithm to compute the lattice of strata from its zero-dimensional strata (Algorithm 3.1.8) which allows us to study more examples than we otherwise would have been able to. Next, we show that the boundary complex of the dual lattice is generically a shellable simplicial complex (Theorem 3.2.6) and thus a combinatorial sphere (Corollary 3.2.10). This lets us provide general bounds on the number of i-dimensional strata in a hyperbolic slice. That is, we get a “g-conjecture” and an “upper bound theorem” for generic (resp. general) hyperbolic slices (Corollary 3.2.14 and Corollary 3.2.16).

These results are what we use to derive improvements on the degree principle for the symmetric group. We use them to cut down on the number of orbit types needed to consider when checking if a symmetric real algebraic set is empty or not. We present a set of orbit types which is sufficient to check (Theorem 4.1.1) which gives an upper bound for the number of orbits needed to check and we provide lower bounds for the number of orbit types needed to check (Theorem 4.1.4). As neither of these bounds are sharp in general, we show how one may use our results to find smaller test sets of orbit types for given values of s and n (subchapter 4.2).

Before we look at the even-hyperbolic strata, we show that if we order any hyperbolic polynomials' roots from smallest to largest, then the polynomial with the minimal first root also in a hyperbolic slice have either the minimal or the maximal $(s+1)^{th}$ coefficient and conversely for the polynomial with the maximal first root (Theorem 5.1.3). This gives us a condition to check which even-hyperbolic strata are nonempty subsets of hyperbolic strata and allows us to extend many of the properties of hyperbolic strata to even-hyperbolic strata. Namely, we show that the even-hyperbolic strata are either empty, a point or of maximal possible dimension (Theorem 5.1.7) and we provide a new way to show that the strata are connected when nonempty (Theorem 5.1.9). We also show that the strata of nonnegative slices have a unique polynomial with a minimal $(s+1)^{th}$ coefficient and a unique polynomial with a maximal $(s+1)^{th}$ coefficient and we characterise their signed compositions (Theorem 5.1.11).

Finally, we use these properties to establish that the poset of even-hyperbolic strata is a lattice that can be computed combinatorially from its zero-dimensional strata (Algorithm 5.2.3). We also identify the posets of even-hyperbolic strata for s and n as a subset of potential posets of hyperbolic strata for $s+1$ and $n+1$ which means that the boundary complex of the dual lattice is a shellable simplicial complex and thus a combinatorial sphere (Theorem 5.2.5). And as with the hyperbolic slices, we show that we can make some improvements on the degree principle for the hyperoctahedral group (Theorem 5.2.10). In particular we find that improvements on the degree principle for $s+1$ and $n+1$ with the symmetric group can be translated to improvements on the degree principle for s and n with the hyperoctahedral group.

Chapter 1

Background

1.1 Symmetric and hyperoctahedral group

We start by introducing some important definitions and results from representation theory of the symmetric and hyperoctahedral group. Throughout the article, we denote by $\mathcal{S}(n)$ the symmetric group on the set $[n]$. We will consider the natural action of $\mathcal{S}(n)$ on $\mathbb{R}[x] := \mathbb{R}[x_1, ..., x_n]$ and so we let $\mathbb{R}[x]^{\mathcal{S}(n)}$ denote the ring of symmetric polynomials.

When $\mathcal{S}(n)$ acts on $\mathbb{R}[x]$ by permuting the variables, this induces an action of $\mathcal{S}(n)$ on $\mathbb{R}^n$ given by permutation of the coordinates. The orbit of a point $a \in \mathbb{R}^n$ is described by the partition $\lambda = (\lambda_1, ..., \lambda_k)$, where $a = \sigma(b_1, ..., b_1, ..., b_k, ..., b_k)$ for some $\sigma \in \mathcal{S}(n)$ and b_i occurs λ_i times and $b_i \neq b_j$ for any $i \neq j$. Therefore we call λ, the **orbit type** of a. We see that the only points in $\mathbb{R}^n$ that are invariant are the points whose orbit type is (n), namely the points with only one distinct coordinate. For invariant polynomials in $\mathbb{R}[x]$, the story is a bit more involved.

Definition 1.1.1. *For $i \in [n]$, we denote by*

$$E_i := \sum_{1 \le j_1 < \cdots < j_i \le n} x_{j_1} \cdots x_{j_i}$$

the ***i^{th} elementary symmetric polynomial*** *and by*

$$P_i := \sum_{j=1}^{n} x_j^i$$

the ***i^{th} power sum****.*

We see that both the power sums and the elementary symmetric polynomials are symmetric. Moreover, we can pass between them by using Newton's identities:

Proposition 1.1.2 (Newton's identities)**.** *For $i \in [n]$ we have*

$$iE_i = \sum_{j=1}^{i} (-1)^{j-1} E_{i-j} P_j,$$

where we set $E_0 := 1$.

Proof. See the proof of Theorem 8 in [8]. □

The Fundamental Theorem of Symmetric Polynomials states that every symmetric polynomial can be uniquely written in terms of the elementary symmetric polynomials, but it can also be strengthened to the following:

Theorem 1.1.3 (Fundamental Theorem of Symmetric Polynomials)**.** *Any symmetric polynomial $F \in \mathbb{R}[x]^{\mathcal{S}(n)}$ of degree $s \leq n$, can be uniquely written as*

$$F = G(E_1, \ldots, E_s),$$

where G is a polynomial in $\mathbb{R}[y_1, \ldots, y_s]$.

Proof. Proposition 2.3 in [27]. □

This formulation of the Fundamental Theorem of Symmetric Polynomials is a key tool in the proof of the degree principle in [27].

Theorem 1.1.4 (Degree principle)**.** *Let $F_1, \ldots, F_k \in \mathbb{R}[x]^{\mathcal{S}(n)}$ be symmetric polynomials of degree at most $s \leq n$. Then the real algebraic set*

$$V_{\mathbb{R}}(F_1, \ldots, F_k) := \{a \in \mathbb{R}^n | F_1(a) = \ldots = F_k(a) = 0\}$$

is nonempty if and only if it contains a point with at most s distinct coordinates.

To define the hyperoctahedral group, let us for a moment think of the symmetric group $\mathcal{S}(2n)$ as the permutations of the set $-[n] \cup [n] = \{-n, -n+1 \ldots, -1, 1, 2, \ldots, n\}$. Then the **hyperoctahedral** (or signed symmetric) group is the subgroup $\mathcal{B}(n) \subset \mathcal{S}(2n)$ consisting of the permutations σ such that $-\sigma(i) = \sigma(-i)$ for all $i \in -[n] \cup [n]$.

We let $\mathcal{B}(n)$ act on $\mathbb{R}[x]$ by permutation and sign change, that is, for $F \in \mathbb{R}[x]$ we let

$$\sigma(F) := F(\delta_\sigma(1)x_{|\sigma(1)|}, ..., \delta_\sigma(n)x_{|\sigma(n)|}),$$

where $\delta_\sigma(i) = \frac{\sigma(i)}{|\sigma(i)|}$ and we see that the reflections $x_i = \pm x_j$, for $1 \le i \le j \le n$, generate $\mathcal{B}(n)$. Thus the action on $\mathbb{R}[x]$ induces an action on $\mathbb{R}^n$ by sign change and permutation of coordinates. But due to the sign changes the orbit types under the hyperoctahedral group is slightly different. Just note that if $a \in \mathbb{R}^n$ has a coordinate equal to zero, then we can change the sign of this coordinate without changing a.

To find the generators of the ring of invariant $\mathbb{R}[x]^{\mathcal{B}(n)}$, note that if $F \in \mathbb{R}[x]^{\mathcal{B}(n)}$, then $F(x) = F(x_1, ..., x_{i_1}, -x_i, x_{i+1}, ..., x_n)$ for any $i \in [n]$. Thus every term of F that contains the variable x_i must contain x_i to an even degree. So we see that F is of even degree and we can replace each instance of x_i^2 with y_i and get a polynomial $G \in \mathbb{R}[y_1, ..., y_n]$ that is invariant with respect to the symmetric group. Thus we have a bijection $\phi : \mathbb{R}[x]^{\mathcal{B}(n)} \to \mathbb{R}[y]^{\mathcal{S}(n)}$ given by

$$x_i^2 \mapsto y_i,$$

which by Theorem 1.1.3 gives the following:

Theorem 1.1.5. *Any polynomial $F \in \mathbb{R}[x]^{\mathcal{B}(n)}$ of degree $2s \le 2n$, can be uniquely written as*

$$F = G(E_1(x^2), \dots, E_s(x^2)),$$

where G is a polynomial in $\mathbb{R}[y_1, \dots, y_s]$ and $x^2 := (x_1^2, ..., x_n^2)$.

Similarly to the method in [27], this can be used to prove the following:

Theorem 1.1.6. *Let $F_1, \dots, F_k \in \mathbb{R}[x]^{\mathcal{B}(n)}$ be polynomials of degree at most $2s \le 2n$. Then the real algebraic set*

$$V_{\mathbb{R}}(F_1, \dots, F_k) := \{a \in \mathbb{R}^n | F_1(a) = ... = F_k(a) = 0\}$$

is nonempty if and only if it contains a point with at most s distinct coordinates.

Proof. See Theorem 2 in [12]. $\square$

1.2 Polytopality and sphericity

Here we introduce some key concepts and results from combinatorics. As names and notations often seem to be an individual choice in combinatorics note that we will mostly be following conventions from [32] and [37].

Definition 1.2.1. *A* ***poset*** $(L, \leq)$*, or* ***partially ordered set****, is a set* L *equipped with a partial order* $\leq$*.*

We usually just write L if the partial order is clear from context. Also, we say that an element a, of a poset, L, **covers** $b \in L$ if $b \leq a$ and for any $c \in L$ with $b \leq c \leq a$, we have either $c = a$ or $c = b$.

Definition 1.2.2. *A totally ordered subset of a poset is a* ***chain*** *and if a chain is maximal with respect to inclusion, it is a* ***maximal chain****. A poset in which every maximal chain has the same length is called* ***graded*** *(or* ***pure****).*

To see why we call such a poset graded let $y_0 < \dots < y_l$ and $z_0 < \dots < z_l$ be two maximal chains of a finite graded poset L, where $y_i = z_j$ for some i and j. Then we have $i = j$, otherwise $y_0 < y_1 < \dots < y_i = z_j < z_{j+1} < \dots < z_l$ is a maximal chain which is not of length $l + 1$ contradicting the gradedness of L. Thus the **rank** of y_i, $\mathrm{rank}(y_i) := i$, is well defined and the poset is the union $L = \cup_{k \geq 0} L(k)$, where $L(k)$ contains the rank k elements. Also, we say that the **rank** of a graded poset is the length of a maximal chain.

We will largely concern ourselves with the following type of poset:

Definition 1.2.3. *A* ***lattice*** *is a poset* L *such that there exists a least upper bound for any subset* $Q \subseteq L$ *and a greatest lower bound. We call it the* ***join*** *(resp.* ***meet****) of* Q*.*

We will also need to look at the following subposet of lattices: If L has the maximal element 1, then we call $L \backslash \{1\}$, the **boundary complex** of L. Note that if a poset L is finite and any subset $Q \subseteq L$ has a join, then L is a lattice since the meet of Q will have to be the join of of all the elements $a \in L$ such that $a \leq b$ for all $b \in Q$. We will look at a particular type of lattice but before we introduce these, note that the poset $(L, \geq)$, where the partial order is reversed, is called the **dual** poset of $(L, \leq)$. Thus in particular, if L is a lattice, the dual poset is also a lattice as the notion of join and meet are dual.

Definition 1.2.4. *In a lattice with a smallest element* 0*, the elements covering* 0 *are called* ***atoms***. *The lattice is called* ***atomic*** *if any element can be expressed as the join of atoms. Conversely, a lattice with a greatest element* 1*, is called* ***coatomic*** *if if the dual lattice is atomic.*

Typical examples of atomic and coatomic lattices come from polytopes, namely the **face lattice** of a polytope $P \subset \mathbb{R}^n$ which is the poset of faces of P (including P and $\emptyset$) partially ordered by inclusion. In fact it is known that the poset of faces of a polytope is a graded, atomic and coatomic lattice (see [37], Theorem 2.7).

We will be concerned with the question of whether or not a lattice is "polytopal" in this thesis, so for the next definition note that we call two posets $(L, \leq)$ and $(Q, \leq^*)$ **isomorphic** if there exists an order-preserving bijection between L and Q whose inverse is also order-preserving.

Definition 1.2.5. *A lattice is* ***polytopal*** *if it is isomorphic to the face lattice of a polytope.*

The problem of determining whether or not a lattice is polytopal is often referred to as Steinitz problem and is generally considered to be a difficult problem. In fact for graded lattices of rank 5 or more there are no known combinatorial way of classifying these lattices, therefore we will introduce a slight weakening of this property.

Definition 1.2.6. *A* ***polytope complex*** *is a family,* C*, of polytopes in* $\mathbb{R}^m$*, such that each face of a polytope is in* C *and such that the intersection of two polytopes is a face of each.*

We see that in particular a polytope is a polytope complex. And just as for polytopes we call the elements of C **faces**. The **dimension** of C is the dimension of its highest-dimensional faces. The proper faces that are maximal with respect to inclusion are called **facets**, the second largest are called **ridges**. Similarly, the smallest nonempty faces are called **vertices** and the second smallest are called **edges**.

If a poset L is isomorphic to the poset of faces of a polytope complex C, we call C a **geometric realisation** of L. And as we will not be working with the geometric realisations directly we will abuse the terminology and call a poset a polytope complex if it has a geometric realisation. If we find it too difficult to determine if a lattice L is polytopal, we may instead ask if the boundary complex of L is a polytope complex. And as the following will show this can oddly enough be a much easier question to answer if our guess that L is polytopal was correct. Firstly we need another definition:

Definition 1.2.7. *A **simplicial complex** is a family of finite sets that is closed under taking subsets.*

Thus any simplicial complex may be identified with a family, C, of subsets of $[m]$ for some nonnegative integer m, such that if $A \subset B \in C$, then $A \in C$. We may identify the smallest nonempty sets of C with the points $e_1, \dots, e_m \in \mathbb{R}^m$, where e_i is the i-th standard basis vector, and then take the convex hull of $e_{i_1}, \dots, e_{i_k}$ whenever $\{i_1, \dots, i_k\}$ is an element of C. Then it is not too difficult to see that this is a geometric realisation of C so C is a polytope complex.

If a lattice L is polytopal, then it might very well be isomorphic to the face lattice of a simplicial polytope (a polytope whose proper faces are simplices) since simplicial polytopes are the generic polytopes with the appropriate notion of genericity (see Chapter 0 in [37]). And if L is isomorphic to a simplicial polytope, then its boundary complex must be a simplicial complex which is realisable. However, there are many simplicial complexes that are not isomorphic to the face poset of the boundary of a simplicial polytope so let us look a different weakening of the question of polytopality.

Definition 1.2.8. *A polytope complex is a **sphere** if it has a geometric realisation which is homeomorphic to a sphere.*

If a spherical polytope complex is also a simplicial complex, we call it a **simplicial sphere**. Since a the boundary of a polytope is homeomorphic to a sphere, we see that sphericity is also a weakening of polytopality. However, similarly to polytopality, sphericity is a property that is generally quite difficult to establish. For that reason we need to introduce a subclass of spheres that can be easier to recognize.

Definition 1.2.9. *A **subdivision** of a polytope complex C is a polytope complex S such that*

$$\bigcup_{I \in S} I = \bigcup_{J \in C} J \subset \mathbb{R}^m$$

*and such that each face of S is contained in a face of C. Moreover, we say a subdivision S is **simplicial** if S is a simplicial complex.*

Definition 1.2.10. *A **combinatorial** (or **PL**) **(m-1)-sphere** is a polytope complex for which there exists a simplicial subdivision which is isomorphic to a simplicial subdivision of the boundary of an m-dimensional simplex.*

A great tool for recognizing combinatorial spheres is the notion of shellability.

Definition 1.2.11. *A **shelling** of a pure simplicial complex, C, is an ordering of the facets, $F_1, \ldots, F_k$, such that for any $i \in \{2, \ldots, k\}$, the simplicial complex*

$$\bigcup_{j=1}^{i-1} F_j \cap F_i$$

is graded of dimension $\dim(C) - 1$. *If there exists a shelling of C, then C is called **shellable**.*

From Proposition 1.2 in [9] we have the following useful result:

Proposition 1.2.12. *A shellable simplicial complex of dimension m, where each ridge is contained in exactly two facets, is a combinatorial m-sphere.*

List of notations

- $n \in \mathbb{N} = \{1, 2, ...\}$
- $[n] = \{1, 2, ..., n\}$
- $\mathcal{S}(n)$ is the symmetric group
- $\mathcal{B}(n)$ is the hyperoctahedral group
- $\text{dist}(a, b) = \sqrt{\sum_{i=1}^{n}(a_i - b_i)^2}$ for $a, b \in \mathbb{R}^n$.
- $B_\epsilon^k(a)$ is the open ball around $a \in \mathbb{R}^k$ of radius ϵ.
- $\mathcal{H}$ the set of monic hyperbolic polynomials of degree n
- $f = t^n + f_1 t^{n-1} + ... + f_n \in \mathcal{H}$
- $c(f)$ is the composition of f
- $\mathcal{H}_s(f) = \{t^n + h_1 t^{n-1} + ... + h_n | h \in \mathcal{H}$ and $h_i = f_i \ \forall \ i \in [s]\}$
- $\mathcal{H}_s^\mu(f) = \{h \in \mathcal{H}_s(f) | c(h) \leq \mu\}$
- $\mu = (\mu_1, ..., \mu_l)$ and $\ell(\mu) = l$
- $\lambda/\mu = \nu$, where ν is such that $\lambda = (\mu_1 + ... + \mu_{\nu_1}, ..., \mu_{l-\nu_{\ell(\lambda)}+1} + ... + \mu_l)$
- $E_i(x)$ is the i^{th} elementary symmetric polynomial in n variables.
- For $x \in \mathbb{R}^l$, $E_i^\mu(x) = E_i(x_1, ..., x_1, ..., x_l, ..., x_l)$, where x_i is repeated μ_i times.
- $\mathcal{E}^\mu(x) = t^n - E_1^\mu(x) t^{n-1} + \cdots + (-1)^n E_n^\mu(x)$
- $P_i(x)$ is the i^{th} power sum in n variables.
- For $x \in \mathbb{R}^l$, $P_i^\mu(x) = P_i(x_1, ..., x_1, ..., x_l, ..., x_l)$, where x_i is repeated μ_i times.
- $\mathcal{V}_s(f) = \{x \in \mathbb{R}^n | -E_1(x) - f_1 = 0, ..., (-1)^s E_s(x) - f_s = 0\}$
- $\mathcal{V}_s^\mu(f) = \{x \in \mathbb{R}^l | -E_1^\mu(x) - f_1 = 0, ..., (-1)^s E_s^\mu(x) - f_s = 0\}$
- $\mathcal{W}_l = \{x \in \mathbb{R}^l | x_1 \leq ... \leq x_l\}$
- $\pi^r : \mathbb{R}^k \to \mathbb{R}^{k-r}$ denotes projection $(a_1, ..., a_k) \mapsto (a_1, .., a_{k-r})$
- $\iota_\mu : \mathcal{V}_s^\mu(f) \cap \mathcal{W}_l \to \mathbb{C}^n$ denotes the inclusion $(x_1, ..., x_l) \mapsto (x_1, ..., x_1, ..., x_l, ..., x_l)$, where x_i is repeated μ_i times
- $\mathcal{L}_s(f)$ is the lattice of strata of $\mathcal{H}_s(f)$
- $\mathcal{L}_s^\Delta(f)$ is the dual of $\mathcal{L}_s(f)$
- $\partial(\mathcal{L}_s^\Delta(f))$ is the boundary complex of $\mathcal{L}_s^\Delta(f)$

- For two elements, u and v, of a lattice, $u \vee v$ denotes the join and $u \wedge v$ denotes the meet.
- $\mathcal{C}(n)$ is the compositions of n
- $\mathcal{C}(n,s)$ is the compositions of n of length s
- $\mathcal{P}(n)$ is the partitions of n
- $\mathcal{P}(n,s)$ is the partitions of n of length s
- $\mathcal{C}_{\min}(n,s) = \{\mu \in \mathcal{C}(n,s) | \mu \text{ is alternate odd}\}$
- $\mathcal{C}_{\max}(n,s) = \{\mu \in \mathcal{C}(n,s) | \mu \text{ is alternate even}\}$
- $\mathcal{P}_{\min}(n,s) := \{\lambda \in \mathcal{P}(n,s) | \lambda_{\lfloor \frac{s}{2} \rfloor +1} = ... = \lambda_s = 1\}$
- $\mathcal{P}_{\max}(n,s) := \{\lambda \in \mathcal{P}(n,s) | \lambda_{\lceil \frac{s}{2} \rceil +1} = ... = \lambda_s = 1\}$
- $\mathcal{N}$ the set of monic even-hyperbolic polynomials of degree n
- $h = t^n + h_1 t^{n-1} + ... + h_n \in \mathcal{N}$ (in the last chapter)
- $\mathcal{N}_s(h) = \mathcal{H}_s(h) \cap \mathcal{N}$
- $sc(h)$ is the signed composition of h
- $\mathcal{N}_s^{\nu}(h) = \{g \in \mathcal{N}_s(h) | sc(g) \leq \nu\}$
- $\nu = c|\nu| = c(\nu_1, ..., \nu_l)$, $\ell(\nu) = l$ and $\operatorname{sgn}(\nu) = c \in \{-1, 1\}$ (in the last chapter)
- $\mathcal{K}_s(h)$ is the lattice of strata of $\mathcal{N}_s(h)$
- $\mathcal{K}_s^{\Delta}(h)$ is the dual of $\mathcal{K}_s(h)$
- $\partial(\mathcal{K}_s^{\Delta}(h))$ is the boundary complex of $\mathcal{K}_s^{\Delta}(h)$
- $\mathcal{SC}(n)$ is the signed compositions of n
- $\mathcal{SP}(n) = \cup_{i=0}^{n} (n-i) \times \mathcal{P}(i)$
- $\mathcal{SP}(n,s) = \{m \times \lambda \in \mathcal{SP}(n) | \ell(\lambda) = s\}$
- $\psi : \mathcal{SC}(n) \to \mathcal{C}(n)$ is the map given by

$$\nu \mapsto \begin{cases} (1, \nu_1, ..., \nu_l), & \text{if } \operatorname{sgn}(\nu) = 1, \\ (\nu_1 + 1, \nu_2, ..., \nu_l), & \text{if } \operatorname{sgn}(\nu) = -1. \end{cases}$$

Chapter 2

Geometry...

In this chapter we establish several geometric/topological properties of the strata of hyperbolic slices. In the first part we see that the previous study of *Vandermonde varieties* in [18] implies that the strata are contractible. This gives us a tool to determine the possible dimension of any stratum and describe the relative interior and its closure. In the second part we generalise the main theorem from [23] on the "escape from hyperbolic space" and show that any stratum has a unique polynomial with a maximal $(s+1)^{th}$ coefficient and a unique polynomial with a minimal $(s+1)^{th}$ coefficient. Lastly, in part three we use this result to show that the relative interior of the strata are also contractible.

Before that we will define the stratification and have a look at an example of a stratified hyperbolic slice. Throughout this thesis we will let $f := t^n + f_1 t^{n-1} + \cdots + f_n \in \mathbb{R}[t]$ be a monic **hyperbolic** polynomial of degree n. That is, f is a monic polynomial of degree n with only real roots. Then we will study the following sets of hyperbolic polynomials:

Definition 2.0.1. *Let $\mathcal{H}$ denote the set of all monic hyperbolic polynomials of degree n. Then for $s \in [n]$, we call the subset*

$$\mathcal{H}_s(f) := \{t^n + h_1 t^{n-1} + ... + h_n \in \mathcal{H} \mid h_i = f_i \ \forall \ i \in [s]\}$$

a (canonical) ***hyperbolic slice****.*

We will also let $s \in \{0, ..., n\}$ throughout this thesis. For an $h = t^n + h_1 t^{n-1} + ... + h_n \in \mathcal{H}_s(f)$, we will refer to h_i as the $\mathbf{i^{th}}$ **coefficient** of h and if $s < n$, we refer to h_{s+1} as the **first free**

coefficient of h. To introduce our stratification of hyperbolic slices we recall the notion of compositions and their partial order.

Definition 2.0.2. *A **composition** of n is a tuple of positive integers, $\mu = (\mu_1, ..., \mu_l)$, that sum up to n. We call $\ell(\mu) := l$ the **length** of μ and the μ_i's the **parts** of μ.*

We will let μ be a composition of length l throughout this thesis and we partially order the compositions the following way:

Definition 2.0.3. *For two compositions of n, μ and λ we let $\lambda \leq \mu$ if there is a composition, ν, of $l = \ell(\mu)$ of length $\ell(\lambda)$ such that*

$$\lambda = (\mu_1 + \cdots + \mu_{\nu_1}, \ldots, \mu_{l-\nu_{\ell(\lambda)}+1} + \cdots + \mu_l).$$

Thus $\mu \leq \lambda$ if μ can be obtained from λ by replacing some of the commas in λ with plus signs. For a hyperbolic polynomial h with distinct roots $b_1 < \cdots < b_l$ and respective multiplicities $m_1, \ldots, m_l$ we will let $c(h) := (m_1, \ldots, m_l)$ denote the **composition of** h. We stratify $\mathcal{H}_s(f)$ as follows:

Definition 2.0.4. *Let μ be a composition of n, then*

$$\mathcal{H}_s^\mu(f) := \{h \in \mathcal{H}_s(f) \mid c(h) \leq \mu\}$$

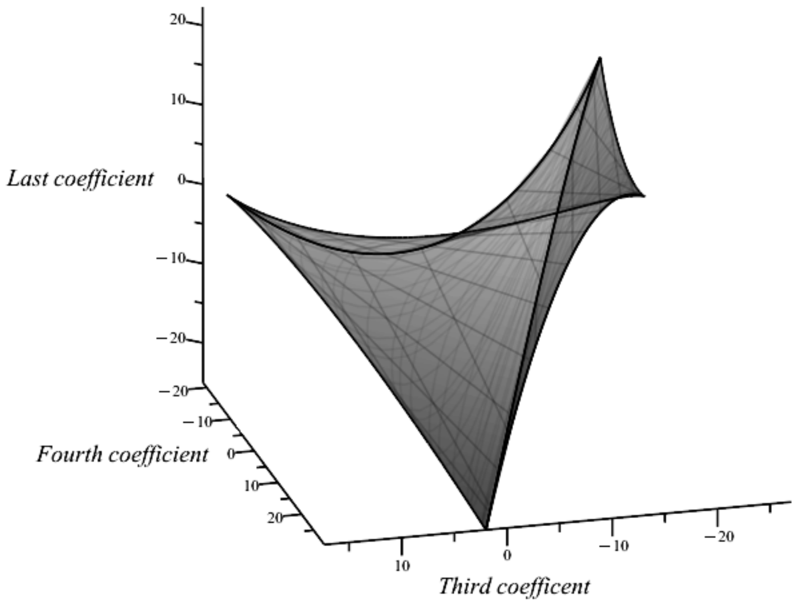
is a stratum of $\mathcal{H}_s(f)$.

Note that since the composition $(1^n) := (1, 1, ..., 1)$ is greater than any other composition then $\mathcal{H}_s^{(1^n)}(f) = \mathcal{H}_s(f)$. Also, by definition of the strata, if $\mu \leq \nu$, then $\mathcal{H}_s^\mu(f) \subseteq \mathcal{H}_s^\nu(f)$. However, the reverse statement does not need to hold as not all compositions need to occur in any hyperbolic slice. For instance, already in the following example of a stratified hyperbolic slice, not all the compositions occur:

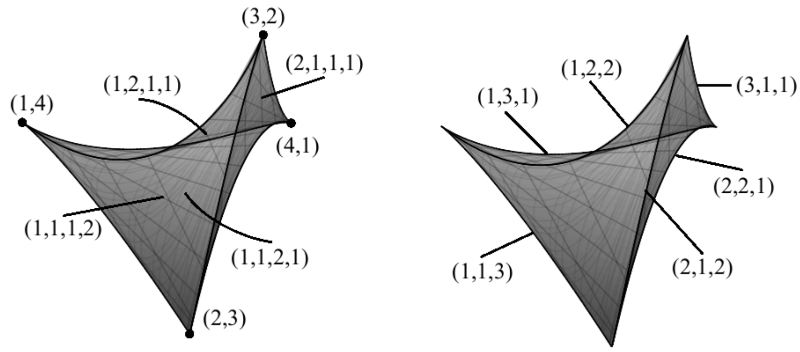
Example 2.0.5. *Let $d = 5$ and $s = 2$ and let*

$$h = (t + \pi)(t + \sqrt{2})t(t - 1{,}23456789123456789)(t - e),$$

then if we plot the last three coefficients of the polynomials in $\mathcal{H}_2(h)$, we get this picture:



There is no polynomial with only one distinct root in $\mathcal{H}_2(h)$, that is, the composition (5) *does not occur and so $\mathcal{H}_2^{(5)}(h) = \emptyset$. The other compositions does occur however and the polynomials with no repeated roots make up the interior, the other compositions occur in the parts of the boundary as indicated by the following picture:*



In the example above we see that the strata of $\mathcal{H}_2(h)$ have some nice geometrical properties. For instance, they are all compact and connected. Also, we can see that the poset of strata is isomorphic to the face lattice of a tetrahedron. Thus, other than not being convex, $\mathcal{H}_2(h)$ and its strata have a lot in common with a tetrahedron and its faces. Since the main point of this project is to explore hyperbolic slices, we can think of polytopes as a "guiding star" as we will establish some of the similarities between hyperbolic slices and polytopes.

2.1 Hyperbolic strata

To study the geometric properties of the strata of hyperbolic slices we need to introduce Vandermonde varieties. For this recall that we let E_i, for $i \in [n]$, denote the i^{th} elementary symmetric polynomial.

Definition 2.1.1. *For $s \in [n]$ we let*

$$\mathcal{V}_s(f) := V_{\mathbb{R}}(-E_1(x) - f_1, \ldots, (-1)^s E_s(x) - f_s) =$$
$$\{x \in \mathbb{R}^n | - E_1(x) - f_1 = \cdots = (-1)^s E_s(x) - f_s = 0\}.$$

Real algebraic sets as above is usually referred to as the **Vandermonde varieties** (see [3] for the general definition). If $a = (a_1, ..., a_n) \in \mathbb{R}^n$ is a tuple of the roots of $h \in \mathcal{H}$, then h can be written as $t^n - E_1(a)t^{n-1} + ... + (-1)^n E_n(a)$. Thus we see that the hyperbolic slice $\mathcal{H}_s(f)$ arise as the image of the Vandermonde variety $\mathcal{V}_s(f)$ under the **Vieta map**,

$$\mathcal{E} : \mathbb{R}^n \to \mathcal{H},$$

given by

$$x \mapsto t^n - E_1(x)t^{n-1} + \cdots + (-1)^n E_n(x).$$

Furthermore, we see that the Vieta map is a bijection between the hyperbolic slice and the intersection of the Vandermonde variety with the **Weyl chamber**

$$\mathcal{W}_n := \{x \in \mathbb{R}^n \mid x_1 \leq ... \leq x_n\}.$$

More generally we have the following definition:

Definition 2.1.2. *Let the map $\mathcal{E}^\mu : \mathbb{R}^l \to \mathcal{H}$ be given by*

$$x \mapsto t^n - E_1^\mu(x)t^{n-1} + \cdots + (-1)^n E_n^\mu(x),$$

where

$$E_i^\mu(x_1, ..., x_l) := E_i(x_1, ..., x_1, ..., x_l, ..., x_l)$$

and x_j is repeated μ_j times.

Then the image of $\mathcal{W}_l$ under $\mathcal{E}^\mu$ are the polynomials in $\mathcal{H}$ whose composition is smaller than or equal to μ. Furthermore, $\mathcal{E}^\mu$ is a bijection between the stratum $\mathcal{H}_s^\mu(f)$ and the intersection of $\mathcal{W}_l$ and:

$$\mathcal{V}_s^\mu(f) := \{x \in \mathbb{R}^l \mid (-1)^i E_i^\mu(x) = f_i \ \forall \ i \in [s]\}.$$

The reason for the terminology comes from the following: by Newtons identities (Proposition 1.1.2) we can find some $c_1, ..., c_s \in \mathbb{R}$ and rewrite $\mathcal{V}_s^\mu(f)$ as

$$\{x \in \mathbb{R}^l \mid P_i^\mu(x) = c_i \;\forall\; i \in [s]\},$$

where

$$P_i^\mu(x_1, ..., x_l) := P_i(x_1, ..., x_1, ..., x_l, ..., x_l)$$

and x_j is repeated μ_j times. As we will see later, the Jacobian of the first s power sums is a constant multiple of a Vandermonde determinant.

We can now get started at studying the hyperbolic strata and due the preceding discussion we immediately have the following:

Lemma 2.1.3. *The stratum $\mathcal{H}_s^\mu(f)$ is a semialgebraic set.*

Proof. We just saw that the stratum $\mathcal{H}_s^\mu(f)$ is image of the semialgebraic set $\mathcal{V}_s^\mu(f) \cap \mathcal{W}_l$ under a polynomial function. And a well known consequence of the *Projection Theorem*, see Theorem 2.2.1 in [6], is that a polynomial function maps semialgebraic sets to semialgebraic sets. □

Thus a hyperbolic stratum is the polynomial image of a subset of a real algebraic set given by s polynomials and we will call it **generic** if it contains no polynomial with at most $s-1$ distinct roots. Thus if a hyperbolic slice is generic, then all its strata are generic. It can be shown (using for instance Proposition 3.2.8) that there is an open dense subset $U \subset \mathcal{H}$ such that $\mathcal{H}_s(f)$ is generic for any $f \in U$.

We will often identify a polynomial $f = t^n + f_1 t^{n-1} + \cdots + f_n$ with $(f_1, ..., f_n) \in \mathbb{R}^n$ without mentioning this change of basis, thus we can consider $\mathcal{H}_s(f)$ as a subset of $\mathbb{R}^{n-s}$. Therefore we equip hyperbolic slices and their strata with the subspace topology of the Euclidean topology. Similarly, we equip the Weyl chamber and Vandermonde varieties with the subspace topology of the Euclidean topology.

We will soon see that work by [13] and [18] implies that the strata of hyperbolic slices are contractible. But first we need to show that the restriction of the Vieta map is a homeomorphism. So let $B_\epsilon^k(a)$ denote the open ball about $a \in \mathbb{R}^k$ of radius $\epsilon > 0$ and let its closure be denoted by $\overline{B_\epsilon^k(a)}$. Unless we work with balls of different dimensions, we omit the superscript k.

Lemma 2.1.4. *The stratum $\mathcal{H}_s^\mu(f)$ is closed in $\mathbb{R}^{n-s}$ and*

$$\mathcal{E}^\mu : \mathcal{V}_s^\mu(f) \cap \mathcal{W}_l \to \mathcal{H}_s^\mu(f),$$

where $l = \ell(\mu)$, is a homeomorphism.

Proof. Recall that $\mathcal{E}^\mu$ is a bijection and a polynomial mapping, thus it is a continuous bijection. To see that the inverse map is continuous and that $\mathcal{H}_s^\mu(f)$ is closed in $\mathbb{R}^{n-s}$, we show that the image of closed sets in $\mathcal{V}_s^\mu(f) \cap \mathcal{W}_l$ are closed in $\mathbb{R}^{n-s}$.

Let S be a closed subset of $\mathcal{V}_s^\mu(f) \cap \mathcal{W}_l$, then since $\mathcal{V}_s^\mu(f)$ and $\mathcal{W}_l$ are closed in $\mathbb{R}^l$, so is S. Let

$$\iota_\mu : \mathcal{V}_s^\mu(f) \cap \mathcal{W}_l \to \mathbb{C}^n$$

be the inclusion $(x_1, ..., x_l) \mapsto (x_1, ..., x_1, ..., x_l, ..., x_l)$, where x_i is repeated μ_i times. Then $\mathcal{E}^\mu(x) = (\mathcal{E} \circ \iota_\mu)(x)$ and clearly $\iota_\mu(S)$ is a closed subset of $\mathbb{C}^n$.

Let $h = t^n + h_1 t^{n-1} + ... + h_n \notin \mathcal{E}^\mu(S)$ have the roots $a = (a_1, ..., a_n)$ and let $h_i = f_i \ \forall\ i \in [s]$. Let $\epsilon > 0$ be such that $D_\epsilon^k(\sigma(a)) \cap \iota_\mu(S)$ is empty for any $\sigma \in \mathcal{S}(n)$, where $D_\epsilon^k(z)$ denotes the complex open ball about $z \in \mathbb{C}^k$ of radius ϵ. If $b_1, ..., b_k$ are the distinct roots of h with respective multiplicities $\nu_1, .., \nu_k$, then by [36] there is a $\delta > 0$ such that any polynomial, g, of degree n, with $|h_i - g_i| \leq \delta$ for all $i \in [n]$ has exactly ν_i zeroes in $D_\epsilon^1(b_i)$. Since $D_\epsilon^k(\sigma(a)) \cap \iota_\mu(S)$ is empty for any $\sigma \in \mathcal{S}(n)$, then so is $B_\delta(h) \cap \mathcal{E}^\mu(S)$ and therefore is $\mathcal{E}^\mu(S)$ closed in $\mathbb{R}^{n-s}$. □

Proposition 2.1.5. *The sets $\mathcal{V}_s^\mu(f) \cap \mathcal{W}_l$ and $\mathcal{H}_s^\mu(f)$ are contractible or empty.*

Proof. We have seen that we can use Newton's identities to define $\mathcal{V}_s^\mu(f)$ in terms of the first s power sums in n variables. Then the proof that $\mathcal{V}_s^\mu(f) \cap \mathcal{W}_l$ is contractible or empty can be found in [18] (Theorem 1.1). By Lemma 2.1.4 the map $\mathcal{E}^\mu : \mathcal{V}_s^\mu(f) \cap \mathcal{W}_l \to \mathcal{H}_s^\mu(f)$ is a homeomorphism, thus $\mathcal{H}_s^\mu(f)$ is contractible if it is nonempty. □

To see how this proposition can be used to further describe our strata we need some more definitions. First note that as $\mathcal{H}_s^\mu(f)$ is semialgebraic the dimension of the stratum is the maximum integer, d, such that $\mathcal{H}_s^\mu(f)$ contains a nonempty open set which is homeomorphic to an open set of $\mathbb{R}^d$ (see Chapter 2.8 of [6]).

Definition 2.1.6. *If $\mathcal{H}_s^\mu(f)$ is a nonempty stratum of dimension d, then*

- *the **relative interior** of $\mathcal{H}_s^\mu(f)$ is the set of polynomials $h \in \mathcal{H}_s^\mu(f)$ for which there exists an open neighbourhood of h which is homeomorphic to an open set in $\mathbb{R}^d$ and*
- *the **relative boundary** of $\mathcal{H}_s^\mu(f)$ is the set of polynomials in $\mathcal{H}_s^\mu(f)$ which does not lie in the relative interior.*

We can use Proposition 2.1.5 to give a description of the relative interior and relative boundary of our strata and also determine their dimension. But the first consequence of the proposition that we need is the following:

Lemma 2.1.7. *If $\ell(\mu) \leq s$, then $\mathcal{H}_s^\mu(f)$ contains at most one polynomial.*

Proof. Suppose $h \in \mathcal{H}_s^\mu(f)$ has the distinct roots $a = (a_1, ..., a_k)$ and composition $\nu = (\nu_1, ..., \nu_k)$, then $k \leq \ell(\mu) \leq s$. If $k = 1$, then $\nu = (n)$ and there is only one solution to the equation

$$-E_1^\nu(x) = -nx_1 = f_1.$$

And so we have $\mathcal{H}_s^\nu(f) = \mathcal{H}_k^\nu(f) = \{h\}$. If $k > 1$, then as previously mentioned, $\mathcal{V}_k^\nu(f)$ can be defined as

$$\{x \in \mathbb{R}^k \mid P_1^\mu(x) = c_1, ..., P_k^\mu(x) = c_k\},$$

where $c_1, ..., c_k \in \mathbb{R}$ are obtained from $f_1, ..., f_k$ using Newton's identities.

The map $P : \mathbb{R}^k \to \mathbb{R}^k$, where $P(x) = (P_1(x_v), ..., P_k(x_v))$, is a continuously differentiable function whose Jacobian matrix is $(i\nu_j x_j^{i-1})_{i,j\leq k}$ and so its determinant is

$$\prod_{i=1}^k i\nu_i \prod_{1\leq j<r\leq k} (x_j - x_r).$$

Since all the a_i's are distinct, the determinant is nonzero at a. Thus the Jacobian matrix is invertible and by the Inverse Function Theorem, P is invertible on some neighbourhood U of $P(a) =$

$(c_1, ..., c_k)$. By Proposition 2.1.5, $V_k^\nu(f) \cap \mathcal{W}_k$ is contractible and since a is isolated in this set, it must be the only point there. Therefore we have $\mathcal{H}_s^\nu(f) = \mathcal{H}_k^\nu(f) = \{h\}$.

So for any composition $\gamma \leq \mu$, that occurs in $\mathcal{H}_s^\mu(f)$, we have that $\mathcal{H}_s^\gamma(f)$ contains a single polynomial. Since there are finitely many compositions smaller than or equal to μ, $\mathcal{H}_s^\mu(f)$ contains finitely many polynomials. But since $\mathcal{H}_s^\mu(f)$ is contractible it can contain at most one polynomial. $\square$

We will let

$$\pi^r : \mathbb{R}^k \to \mathbb{R}^{k-r}$$

denote projection that forgets the last r coordinates. This map will be very useful, firstly for helping us describe the relative interior of the strata.

Lemma 2.1.8. *If $l = \ell(\mu) > s$, then the map*

$$\pi^{n-l} : \mathcal{H}_s^\mu(f) \to \mathbb{R}^{l-s}$$

is a homeomorphism onto its image and the image is closed in $\mathbb{R}^{l-s}$.

Proof. Firstly we consider the case when $l = 1$. Then $\mu = (n)$ and $s = 0$ so for any $a \in \mathbb{R}$ we have that $(t-a)^n = t^n - nat^{n-1} + ... + (-a)^n \in \mathcal{H}_0^\mu(f)$. Thus $\pi^{n-1}((t-a)^n) = -na$ and so the map

$$\pi^{n-l} \circ \mathcal{E}^\mu : \mathbb{R} \to \mathbb{R}$$

is essentially just mapping a to $-na$. This is naturally a homeomorphism and since, by Lemma 2.1.4, $\mathcal{E}^\mu$ is a homeomorphism, then so is π^{n-l}. Lastly, since the image of π^{n-l} is all of $\mathbb{R}$, the image is closed in $\mathbb{R}$.

Next suppose $l \geq 2$. By Lemma 2.1.7, the polynomials of $\mathcal{H}_s^\mu(f)$ are uniquely determined by their first l coefficients, thus π^{n-l} is a bijection between $\mathcal{H}_s^\mu(f)$ and $\pi^{n-l}(\mathcal{H}_s^\mu(f))$. Also, the topology on $\mathcal{H}_s^\mu(f)$ is the subspace topology of the product topology on $\mathbb{R}^{n-s}$ with respect to the projections on each coordinate. Thus the map π^{n-l} is by definition continuous.

To see that the inverse is continuous and that $\pi^{n-l}(\mathcal{H}_s^\mu(f))$ is closed we will show that the image of closed subsets of $\mathcal{H}_s^\mu(f)$ are closed in $\mathbb{R}^{l-s}$. So let S be a closed subset of $\mathcal{H}_s^\mu(f)$ and let g be a point in the closure of $\pi^{n-l}(S)$. Then for any $\epsilon > 0$, the closed ball $\overline{B}_\epsilon(g)$ meets $\pi^{n-l}(S)$ and since $\pi^{n-l} \circ \mathcal{E}^\mu$ is continuous by Lemma 2.1.4, the inverse image

$$M = (\mathcal{E}^\mu \circ \pi^{n-l})^{-1}(\overline{B}_\epsilon(g) \cap \pi^{n-l}(S))$$

is nonempty and closed. Since $\mathcal{V}_s^\mu(f) \cap \mathcal{W}_l$ is closed in $\mathbb{R}^l$, then so is M. We also have that

$$M \subseteq (\mathcal{E}^\mu)^{-1}(S) \cap \{x \in \mathcal{W}_l | g_i - \epsilon \leq (-1)^i E_i^\mu(x) \leq g_i + \epsilon \; \forall \; i \in [l]\}.$$

So if $a \in M$, then $E_1^\mu(a) \leq g_1 - \epsilon$ and $E_2^\mu(a) \geq g_2 - \epsilon$ since $l \geq 2$. Thus, by Newton's identities, we have

$$P_2^\mu(a) = (E_1^\mu(a))^2 - 2E_2^\mu(a) \leq g_1^2 - 2g_1\epsilon + \epsilon^2 - 2g_2 + 2\epsilon$$

and so M is bounded. Since M is closed and bounded, it is compact.

Since $\pi^{n-l} \circ \mathcal{E}^\mu$ is continuous and the continuous image of a compact set is compact, we have that $(\pi^{n-l} \circ \mathcal{E}^\mu)(M)$ is compact. Thus

$$g \in (\pi^{n-l} \circ \mathcal{E}^\mu)(M) \subseteq \pi^{n-l}(S)$$

and so $\pi^{n-l}(S)$ is closed in $\mathbb{R}^{l-s}$. Therefore π^{n-l} is a closed map and thus π^{n-l} is a homeomorphism. Lastly, by setting $S = \mathcal{H}_s^\mu(f)$, we see that $\pi^{n-l}(\mathcal{H}_s^\mu(f))$ is closed in $\mathbb{R}^{l-s}$. $\square$

We see from Lemma 2.1.8 that when $l \geq s$, the largest dimension that $\mathcal{H}_s^\mu(f)$ can have is $l - s$. Therefore we say that the **maximal dimension** of $\mathcal{H}_s^\mu(f)$ is $\max\{l - s, 0\}$.

Theorem 2.1.9. *If $\mathcal{H}_s^\mu(f)$ contains a polynomial with composition μ, then $\mathcal{H}_s^\mu(f)$ is maximal dimensional and its relative interior consists of the polynomials with composition μ.*

Proof. If $s = 0$, the map $\mathcal{E}^\mu : \mathcal{W}_l \to \mathcal{H}_0^\mu(f)$ is a homeomorphism by Lemma 2.1.4 and since the dimension of $\mathcal{W}_l$ is l and its interior are the points with no repeated coordinates, then the dimension of $\mathcal{H}_0^\mu(f)$ is l and its relative interior are the polynomials with composition μ.

Next suppose $s > 0$ and let $A_{f_i}^r$ denote the affine hyperplane of $\mathbb{R}^r$ defined by fixing the i^{th} coordinate to be equal to f_i. Then

$$\pi^{n-l}(\mathcal{H}_s^\mu(f)) = \pi^{n-l}(\mathcal{H}_0^\mu(f) \cap A_{f_1}^n \cap ... \cap A_{f_s}^n) =$$

$$\pi^{n-l}(\mathcal{H}_0^\mu(f)) \cap A_{f_1}^l \cap ... \cap A_{f_s}^l.$$

Thus if there is a polynomial $h \in \mathcal{H}_s^\mu(f)$ with composition μ then by the first paragraph and Lemma 2.1.8, $\pi^{n-l}(h)$ lies in the interior of $\pi^{n-l}(\mathcal{H}_0^\mu(f))$ and therefore also in the interior of $\pi^{n-l}(\mathcal{H}_s^\mu(f))$ and $\pi^{n-l}(\mathcal{H}_s^\mu(f))$ must be of dimension $\max\{l-s, 0\}$. Since π^{l-s} is a homeomorphism, $\mathcal{H}_s^\mu(f)$ is maximal dimensional and h lies in its relative interior.

For the reverse inclusion, suppose $l > s$ so that $\mathcal{H}_s^\mu(f)$ is at least one-dimensional. If its relative interior contains a polynomial g with $c(g) < \mu$, then $\pi^{n-l}(g)$ lies in the interior of the $(l-s)$-dimensional set $\pi^{n-l}(\mathcal{H}_s^\mu(f)) \subseteq \mathbb{R}^{l-s}$. Thus $\pi^{n-l}(g)$ lies in the interior of the one-dimensional set $\pi^{n-l}(\mathcal{H}_{l-1}^\mu(g)) = \pi^{n-l}(\mathcal{H}_s^\mu(f)) \cap A_{g_{s+1}}^l \cap ... \cap A_{g_{l-1}}^l \subseteq \mathbb{R}$.

By Lemma 2.1.7, there are finitely many polynomials in $\mathcal{H}_{l-1}^\mu(g)$ with a smaller composition than μ. Thus there are two polynomials p_- and p_+ in $\mathcal{H}_{l-1}^\mu(g)$, with composition μ, and a $\delta > 0$ such that

$$\pi^{n-l}(p_-) = \pi^{n-l}(g) - \delta \text{ and } \pi^{n-l}(p_+) = \pi^{n-l}(g) + \delta.$$

Since $\pi^{n-l}(p_-)$ and $\pi^{n-l}(p_+)$ are in the interior of $\pi^{n-l}(\mathcal{H}_0^\mu(f))$, there is an $\epsilon > 0$ such that $B_\epsilon(\pi^{n-l}(p_-))$ and $B_\epsilon(\pi^{n-l}(p_+))$ are contained in the interior of $\pi^{n-l}(\mathcal{H}_0^\mu(f))$. And since $\pi^{n-l}(g)$ is in the boundary of $\pi^{n-l}(\mathcal{H}_0^\mu(f))$, the ball $B_\epsilon(\pi^{n-l}(g))$ must contain a point $q = (q_1, ..., q_{n-l})$ that is not in $\pi^{n-l}(\mathcal{H}_0^\mu(f))$.

Thus $A_{q_1}^l \cap ... \cap A_{q_{l-1}}^l$ is a line that passes through q and the two balls $B_\epsilon(\pi^{n-l}(p_-))$ and $B_\epsilon(\pi^{n-l}(p_+))$. But if q separates the nonempty sets

$$B_\epsilon(\pi^{n-l}(p_-)) \cap A_{q_1}^l \cap ... \cap A_{q_{l-1}}^l$$

and

$$B_\epsilon(\pi^{n-l}(p_+)) \cap A_{q_1}^l \cap ... \cap A_{q_{l-1}}^l,$$

then

$$\pi^{n-l}(\mathcal{H}_0^\mu(f)) \cap A_{q_1}^l \cap ... \cap A_{q_{l-1}}^l = \pi^{n-l}(\mathcal{H}_{l-1}^\mu(p_+)) \subset \mathbb{R}$$

is nonempty but not contractible. This contradicts Proposition 2.1.5 and g can therefore not be in the relative interior of $\mathcal{H}_s^\mu(f)$. $\square$

Thus, if $l > s$, the stratum $\mathcal{H}_s^\mu(f)$ is of maximal dimension if and only if it contains a polynomial with composition μ. We can use this observation to determine all the possibilities for the dimension of $\mathcal{H}_s^\mu(f)$. It is worth mentioning that a similar observation to the following can be found in [5], Proposition 5.

Theorem 2.1.10. *If $\ell(\mu) > s$ and $\mathcal{H}_s^\mu(f)$ contains a polynomial with at least s distinct roots, then $\mathcal{H}_s^\mu(f)$ is maximal dimensional. If not, then $\mathcal{H}_s^\mu(f)$ is either empty or a single polynomial.*

Proof. Any composition occurs in $\mathcal{H}_0(f)$ and so by Theorem 2.1.9, any stratum is maximal dimensional. Similarly, if $s = 1$, then for any composition μ, the polynomial $-E_1^\mu(x) - f_1$ has a real zero with l distinct coordinates ordered increasingly. To see this pick l real numbers $a_1, ..., a_l$ such that $a_1/\mu_1 < .... < a_l/\mu_l$ and let

$$a = -\frac{f_1}{\sum_i a_i}\left(\frac{a_1}{\mu_1}, ..., \frac{a_l}{\mu_l}\right),$$

then

$$-E_1^\mu(a) = \sum_j \mu_j \frac{f_1}{\sum_i a_i}\frac{a_j}{\mu_j} = \frac{f_1}{\sum_i a_i}\sum_j a_j = f_1.$$

Thus the composition μ occurs in $\mathcal{H}_1(f)$ and so by Theorem 2.1.9, $\mathcal{H}_1^\mu(f)$ is maximal dimensional.

Next we suppose $s \geq 2$. If $l \leq s$ or $\mathcal{H}_s^\mu(f)$ does not contain a polynomial with at least s distinct roots, then $\mathcal{H}_s^\mu(f) = \cup_{\gamma\leq\mu|\ell(\gamma)=s-1}\mathcal{H}_s^\gamma(f)$. By Lemma 2.1.7, $\mathcal{H}_s^\gamma(f)$ contains at most one polynomial when $\ell(\gamma) = s - 1$. Since there are finitely many compositions of length $s - 1$, then $\mathcal{H}_s^\mu(f)$ contains finitely many polynomials and since $\mathcal{H}_s^\mu(f)$ is contractible it contains at most one polynomial.

So suppose $h \in \mathcal{H}_s^\mu(f)$ has $k \geq s$ distinct roots and $c(h) < \mu$. Let $\nu \leq \mu$ be a composition that covers $c(h)$. Then $\ell(\nu) = k + 1$ and so by Lemma 2.1.8, $\mathcal{H}_k^\nu(h)$ is at most one-dimensional. Since $c(h) < \nu$ we can write h as $\prod_{i=1}^{k+1}(t - b_i)^{\nu_i}$ and without loss of generality we may assume that $b_1 < ... < b_k = b_{k+1}$.

Since $\mathcal{V}_k^\nu(h)$ equals $\{x \in \mathbb{R}^{k+1} | P_1^\nu(x) = c_1, ..., P_k^\nu(x) = c_k\}$, the Jacobian matrix of the defining polynomials is $(i\nu_j x_j^{i-1})_{i\leq k, j\leq k+1}$.

Thus the determinant of the leftmost $k \times k$ submatrix is

$$\prod_{i=1}^{k} i\nu_i \prod_{1 \le j < r \le k} (x_j - x_r).$$

Since the first k coordinates of $b = (b_1, ..., b_{k+1})$ are distinct, the determinant does not vanish at $b \in \mathcal{V}_k^\nu(h)$. So by Proposition 3.3.10 in [6], b is a nonsingular point of a one-dimensional irreducible component, V, of $\mathcal{V}_k^\nu(h)$. Thus b lies in an open neighbourhood U of V where U is a one-dimensional manifold.

By Lemma 2.1.7, the one-dimensional manifold U only intersects the hyperplane $H = \{x \in \mathbb{R}^{k+1} | x_k = x_{k+1}\}$ once. So U must meet the open halfspace $H^+ := \{x \in \mathbb{R}^{k+1} | x_k < x_{k+1}\}$ and thus there is a point in $\mathcal{V}_k^\nu(h) \cap \mathcal{W}_{k+1}$ with no repeated coordinates. So $\mathcal{H}_k^\nu(h) \subseteq \mathcal{H}_s^\nu(f)$ contains a polynomial with composition ν. So by induction we can find a polynomial with composition μ and by Theorem 2.1.9, $\mathcal{H}_s^\mu(f)$ is therefore maximal dimensional. □

Corollary 2.1.11. *Any hyperbolic stratum equals the closure of its relative interior.*

Proof. By Theorem 2.1.10 we may suppose $\mathcal{H}_s^\mu(f)$ is maximal dimensional and at least one-dimensional. We will prove the statement by induction on the dimension of the strata. If $\mathcal{H}_s^\mu(f)$ is one-dimensional, then it is connected by Proposition 2.1.5. Thus the relative boundary contains at most two polynomials and any open ball about a polynomial of $\mathcal{H}_s^\mu(f)$ contains infinitely many polynomials from the relative interior. Thus $\mathcal{H}_s^\mu(f)$ is the closure of its relative interior.

Suppose the statement is true for all $(m-1)$-dimensional strata, where $m - 1 \ge 1$. Suppose $\mathcal{H}_s^\mu(f)$ is n-dimensional and $h \in \mathcal{H}_s^\mu(f)$. By Proposition 2.1.5, $\mathcal{H}_s^\mu(f)$ is connected, so for any $\epsilon > 0$, $B_\epsilon(h)$ must contain infinitely many polynomials from $\mathcal{H}_s^\mu(f)$. Since there are finitely many polynomials in $\mathcal{H}_s^\mu(f)$ with at most s distinct roots, there is a $g \in B_\epsilon(h) \cap \mathcal{H}_s^\mu(f)$ with at least $s+1$ distinct roots.

Then by Theorem 2.1.10, $\mathcal{H}_{s+1}^\mu(g)$ is $(m-1)$-dimensional and by the induction hypothesis, g is in the closure of its relative interior. So by Theorem 2.1.9, any open ball about g contains a polynomial with composition μ. Thus $B_\epsilon(h)$ contains a polynomial with composition μ and so h is in the closure of the relative interior of $\mathcal{H}_s^\mu(f)$. $\square$

2.2 Escaping hyperbolic strata

In this subchapter, we look at the question of which polynomials in a stratum of a hyperbolic slice have a minimal or maximal first free coefficient. This question was asked for hyperbolic slices in [23] and it turned out that the question could be fully answered by looking at the compositions of the polynomials. Thus they classified which polynomials in $\mathcal{H}_s(f)$ have the maximal first free coefficient and which have the minimal (when such polynomials exist). We shall give an analogous classification, except that we will restrict the domain to be any of the strata of $\mathcal{H}_s(f)$.

To state the result, we first need some terminology.

Definition 2.2.1. *If $s < n$ we call $h \in \mathcal{H}_s^\mu(f)$ a **minimal** (resp. **maximal**) polynomial of $\mathcal{H}_s^\mu(f)$ if $h_{s+1} \leq g_{s+1}$ (resp. $h_{s+1} \geq g_{s+1}$) for all $g \in \mathcal{H}_s^\mu(f)$.*

Note that if $h \in \mathcal{H}_s^\mu(f)$, then $h = \prod_{i=1}^{l}(t - b_i)^{\mu_i}$ for some $b_1 \leq ... \leq b_l$, so b_i must have multiplicity at least μ_i for any $i \in [l]$, thus we can mod out this information. Also, note that if a composition λ is less than or equal to μ, there is a unique composition ν such that $\lambda = (\mu_1 + \cdots + \mu_{\nu_1}, \ldots, \mu_{l-\nu_{\ell(\lambda)}+1} + \cdots + \mu_l)$.

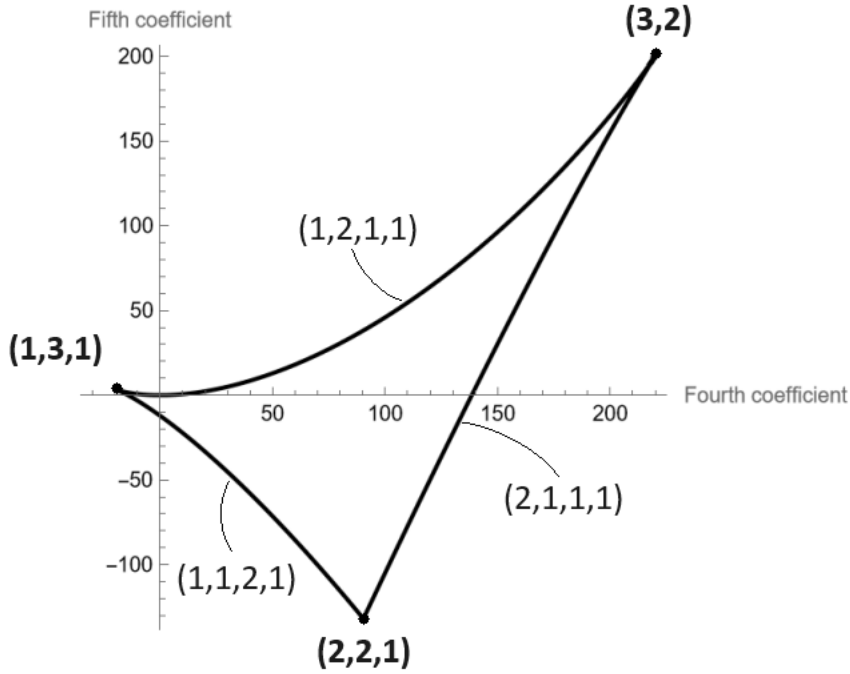
Definition 2.2.2. *If $\lambda \leq \mu$, let λ/μ denote the composition ν such that $\lambda = (\mu_1 + \cdots + \mu_{\nu_1}, \ldots, \mu_{l-\nu_{\ell(\nu)}+1} + \cdots + \mu_l)$.*

So λ/μ can be thought of as "modding out" μ, not dividing by μ. Lastly, we call the composition μ **alternate odd** if $\mu_l = \mu_{l-2} = \cdots = 1$ and **alternate even** if $\mu_{l-1} = \mu_{l-3} = \cdots = 1$.

Theorem 2.2.3. *Let λ be the composition of $h \in \mathcal{H}_s^\mu(f)$ and let $s \geq 2$, then*

1. *there is a unique minimal (resp. maximal) polynomial in $\mathcal{H}_s^\mu(f)$ and*
2. *the polynomial h is minimal (resp. maximal) if and only if λ/μ is less than or equal to an alternate odd (resp. even) composition of length s.*

Example 2.2.4. *Let $d = 5$ and $s = 3$ and let $f = (t+2)^3(t-5)^2$, then if we map the last two coefficients of the polynomials in $\mathcal{H}_3(f)$ we get the following picture:*



.

We see that f is maximal in the stratum $\mathcal{H}_3^{\mu}(f)$, where $\mu = (2,1,1,1)$, and that $c(f) = (3,2)$. This fits with Theorem 2.2.3 since $c(f)/\mu = (2,2)$ and this is smaller than the alternate even composition $(1,1,2)$ of length 3. Similarly, we see that $(2,2,1)/\mu = (1,2,1)$, which is alternate odd and of length 3, so the polynomial with composition $(2,2,1)$ is minimal in $\mathcal{H}_3^{\mu}(f)$.

When $s = 1$ there is also a maximal polynomial for all strata, but no minimal polynomial for any strata other than $\mathcal{H}_1^{\mu}(f)$, where $\mu = (n)$. The maximal polynomial for all the strata is the unique polynomial with only one distinct root. This follows from [23] and therefore we only focus on the cases when $s \geq 2$.

Note that in the generic case, one can replace λ/μ being "less than or equal" by "equal" in the above theorem since no two compositions of the same length are comparable. It should be said that the proof of Theorem 2.2.3 is based on many of the same ideas as in [3] and [23], however some of their techniques do not work in this general setting and others need to be adjusted. We start by proving the first item and we will let $l = \ell(\mu) > s$ for this subchapter as $\mathcal{H}_s^{\mu}(f)$ is either empty or a point if $l \leq s$ according to Theorem 2.1.10.

Proof of Item 1 from Theorem 2.2.3. The statement is clear when $\mathcal{H}_s^\mu(f)$ is just a point so we will assume $\mathcal{H}_s^\mu(f)$ is $(l-s)$-dimensional. Since $\mathcal{V}_s^\mu(f)$ is given by the first s powersums and $s \geq 2$, then $\mathcal{V}_s^\mu(f)$ lies on a sphere and is therefore compact. Since $\mathcal{E}^\mu$ is continuous, then $\mathcal{H}_s^\mu(f)$ is also compact so the existence of minimal and maximal polynomials is clear.

Let $h \in \mathcal{H}_s^\mu(f)$ be a minimal polynomial. To show uniqueness, we assume that $\mathcal{H}_{s+1}^\mu(h)$ contains another polynomial which by Theorem 2.1.10 means that it is of dimension $l-s-1 > 0$. By Theorem 2.1.9, $\mathcal{H}_{s+1}^\mu(h)$ therefore contains a polynomial g with composition μ. By Lemma 2.1.8 and Theorem 2.1.9, $\pi^{n-l}(\mathcal{H}_s^\mu(f))$ is full-dimensional and its interior is the image of the polynomials in $\mathcal{H}_s^\mu(f)$ with composition μ. This contradicts g being minimal in $\mathcal{H}_s^\mu(f)$ as only the boundary of $\pi^{n-l}(\mathcal{H}_s^\mu(f))$ can have minimal coordinates. The argument for maximal polynomials is analogous. □

The proof of the second part of Theorem 2.2.3 will require a lot more work and so we start with some useful tools. For the proof we will use some local arguments so we will need a local definition of minimality and maximality.

Definition 2.2.5. *If $s < n$ we call $h \in \mathcal{H}_s^\mu(f)$ a **locally minimal** (resp. **locally maximal**) polynomial of $\mathcal{H}_s^\mu(f)$ if $h_{s+1} \leq g_{s+1}$ (resp. $h_{s+1} \geq g_{s+1}$) for all $g \in N$, where $N \subset \mathcal{H}_s^\mu(f)$ is some open neighbourhood of h.*

Lemma 2.2.6. *Any locally minimal or locally maximal polynomial in $\mathcal{H}_s^\mu(f)$ has at most s distinct roots.*

Proof. Assume $\mathcal{H}_s^\mu(f)$ is at least one-dimensional since otherwise it follows from Theorem 2.1.10. By Lemma 2.1.8, $\pi^{n-l} : \mathcal{H}_s^\mu(f) \to \mathbb{R}^{l-s}$ is a homeomorphism onto its image which is closed in $\mathbb{R}^{l-s}$. So by Theorem 2.1.9, the image of the polynomials whose composition is strictly smaller than μ make up the boundary of $\pi^{n-l}(\mathcal{H}_s^\mu(f))$. Thus a locally minimal or locally maximal polynomial lies in the relative boundary and therefore has strictly less than l roots and so the statement follows inductively. □

Note that due to the following lemma, we can just work with the local definition of minimal and maximal polynomials:

Lemma 2.2.7. *A polynomial $h \in \mathcal{H}_s^\mu(f)$ is locally minimal (resp. locally maximal) if and only if it is minimal (resp. maximal).*

Proof. One implication is clear, so suppose $h \in \mathcal{H}_s^\mu(f)$ is locally minimal but not minimal. If $\mathcal{H}_{s+1}^\mu(h)$ is at least one-dimensional then by Corollary 2.1.11, for any $\epsilon > 0$ there is a polynomial $g \in \mathcal{H}_{s+1}^\mu(h) \cap B_\epsilon^{n-s-1}(h)$ with composition μ. Thus, by Lemma 2.1.8, there is a δ with $0 < \delta < \epsilon$ such that $\pi^{n-l}(\mathcal{H}_s^\mu(f)) \cap B_\delta^{l-s}(\pi^{n-l}(g))$ lies in the interior of $\pi^{n-l}(\mathcal{H}_s^\mu(f))$. So there is a polynomial in $\mathcal{H}_s^\mu(f) \cap B_\epsilon^{n-s}(h)$ whose first free coefficient is smaller than the first free coefficient of h contradicting the local minimality of h.

Thus, by Theorem 2.1.10, $H_{s+1}^\mu(h)$ must be a point. Since $H_s^\mu(f)$ is contractible, there is a path, $\Phi : [0,1] \to \mathcal{H}_s^\mu(f)$, where $[0,1]$ is the unit interval, from h to the minimal polynomial. Since $H_{s+1}^\mu(h)$ is a point we may assume that the first free coefficient of $\Phi(y)$ is strictly smaller than the first free coefficient of h for all $y \in (0,1]$. But this is a contradiction since h was assumed to be locally minimal. Thus if h is locally minimal, it must also be minimal. The proof for locally maximal polynomials works analogously. □

To prove the second part of Theorem 2.2.3 we will first consider the generic case and do an induction on the dimension of the strata, then we extend the statement to the general case. For the initial step of the induction we consider a generic one-dimensional stratum $\mathcal{H}_s^\mu(f)$. Since $s \geq 2$, the one-dimensional stratum, $\mathcal{H}_s^\mu(f)$, is compact and therefore has two polynomials, h and g, in the boundary. Due to Lemma 2.2.6, one of those is the minimal polynomial and the other one is the maximal polynomial and since $\mathcal{H}_s^\mu(f)$ is generic both $c(h)/\mu$ and $c(g)/\mu$ are alternate. We will use Lagrange multipliers to determine which is which and we should point out that this part of the argument (from here up to and including Proposition 2.2.9) we get from [3] we just go through it for completeness.

Firstly, for the Lagrange multiplier argument it will be useful to work with power sums instead of elementary symmetric polynomials.

Lemma 2.2.8. *Let $a, b \in \mathbb{R}^n$ and suppose $E_i(a) = E_i(b)$ for all $i \in [s]$, then $P_{s+1}(a) > P_{s+1}(b)$ if and only if $(-1)^{s+1}E_{s+1}(a) < (-1)^{s+1}E_{s+1}(b)$.*

Proof. This is straightforward to show using Newtons identities. □

Due to Lemma 2.2.8, instead of looking at the minimisers (resp. maximisers) of $(-1)E^{\mu}_{s+1}$, we will look at the maximisers (resp. minimisers) of P^{μ}_{s+1}. Recall that the set $\mathcal{V}^{\mu}_{s}(f)$ can be written as $\mathcal{V}^{\mu}_{s}(f) = \{x \in \mathbb{R}^{s+1} \mid P^{\mu}_{i}(x) = c_i \ \forall\ i \in [s]\}$ and that the Jacobian matrix of $(P^{\mu}_{1}(x), \ldots, P^{\mu}_{s}(x))$, where $x = (x_1, ..., x_{s+1})$, equals

$$J(x) = (i\mu_j x_j^{i-1})_{i\leq s, j\leq s+1}.$$

Let $x \in \mathcal{V}^{\mu}_{s}(f) \cap \mathcal{W}_{s+1}$ be one of the two points with s distinct coordinates. Suppose $x_{j_1}, .., x_{j_s}$ are the distinct coordinates of x, then as we have seen before the $s \times s$ submatrix of $J(x)$ consisting of the rows $j_1, ..., j_s$ has the determinant $c\prod_{j_i<j_k}(x_{j_i} - x_{j_k})$ for some positive constant c. Since the x_{j_i}'s are distinct, the determinant does not vanish and $\nabla P^{\mu}_{1}(x), \ldots, \nabla P^{\mu}_{s}(x)$ are linearly independent.

Similarly, the Jacobian of $P^{\mu}_{1}(x), \ldots, P^{\mu}_{s}(x), P^{\mu}_{s+1}(x)$ has a vanishing determinant since x only has s distinct roots. Therefore $\nabla P^{\mu}_{1}(x), \ldots, \nabla P^{\mu}_{s+1}(x)$ are linearly dependent and so there exist scalars $a_1, \ldots, a_s$ such that $\nabla L(x) = 0$, where

$$L(x) = P^{\mu}_{s+1}(x) - \sum_{i=1}^{s} a_i P^{\mu}_{i}(x).$$

The gradient of L at x is

$$\nabla L(x) = \nabla P^{\mu}_{s+1}(x) - \sum_{i=1}^{s} a_i \nabla P^{\mu}_{i}(x) = (\mu_1 Q(x_1), \ldots, \mu_{s+1} Q(x_{s+1}))),$$

where $Q(t) = (s+1)t^s - \sum_{i=1}^{s} a_i i t^{i-1}$. The univariate polynomial $Q(t)$ is of degree s and since Q vanishes at x_j for any j, then Q have s distinct roots. Thus we have the following:

Proposition 2.2.9. *Let $\mathcal{H}^{\mu}_{s}(f)$ be generic and one-dimensional, then $h \in \mathcal{H}^{\mu}_{s}(f)$ is the minimal (resp. maximal) polynomial if and only if $\ell(c(h)) = s$ and $c(h)/\mu$ is alternate odd (resp. even).*

Proof. We continue with the notation above and let $x = (x_1, ..., x_{s+1})$, where $x_1 \leq .. \leq x_{s+1}$, be the roots of a polynomial, h, in the relative boundary of $\mathcal{H}^{\mu}_{s}(f)$ and let $a_1, .., a_s$ and Q be as above. From Lemma 2.2.6, h has at most s distinct roots and since $\mathcal{H}^{\mu}_{s}(f)$ is generic h have exactly s distinct roots.

By Theorem 5.4 in [34], x is a local maximiser of P^{μ}_{s+1} (resp. minimiser) if $v^t H(x) v < 0$ (resp. $v^t H(x) v > 0$) for all nonzero vectors $v \in \mathbb{R}^{s+1}$ in the kernel of $J(x)$ where

$$H(x) := \nabla^2 L(x) = \begin{pmatrix} \mu_1 Q'(x_1) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mu_{s+1} Q'(x_{s+1}) \end{pmatrix}.$$

Let $x_k = x_{k+1}$ be the repeated coordinate of x. If $v_k + v_{k+1} = 0$ and all other coordinates of v are zero, then v lies in the kernel of $J(x)$. Also, since the set of such vectors is a one-dimensional subspace of $\mathbb{R}^{s+1}$, they make up the kernel of $J(x)$. So we have that

$$v^t H(x) v = \sum_j \mu_j Q'(x_j) v_j^2 = Q'(x_k)(\mu_k v_k^2 + \mu_{k+1} v_{k+1}^2)$$

is negative (resp. positive) for all $v \neq (0, ...0)$ in the kernel of $J(x)$ if and only if $Q'(x_k)$ is positive (resp. negative).

The univariate polynomial Q has only the simple roots $x_1 < ... < x_k < x_{k+2} < ... < x_{s+1}$, so by Rolle's Theorem the roots of Q' strictly interlace the roots of Q. Also, since the leading coefficient of Q is positive, $Q'(x_{s+1})$ is positive and thus $Q'(x_s) < 0$, $Q'(x_{s-1}) > 0, ...$ Thus x is a maximiser (resp. minimiser) of P^{μ}_{s+1} if and only if $k = s + 1 - 2m$ (resp. $k = s - 2m$) for some nonnegative integer m. That is, x is a minimiser (resp. maximiser) of $(-1)^{s+1} E^{\mu}_{s+1}$ if and only if $c(h)/\mu$ is alternate odd (resp. even). □

Remark 2.2.10. *The reason we will not use the Lagrangian argument above for higher-dimensional strata is that it is not a priori clear that the tuple of roots of a locally minimal (resp. maximal) polynomial in an stratum is a local minimiser (resp. maximiser) of $(-1)^{s+1} E^{\mu}_{s+1}$ over $\mathcal{V}^{\mu}_s(f)$. Other than that, the argument would work similarly as above for the higher-dimensional strata. So it might be possible to use a Lagrange multiplier argument with the additional inequalities, $x_i \leq x_{i+1}$ $\forall\ i \in [\ell(\mu)]$, for the higher-dimensional strata. However, the main tools for the inductive step in the following proof is needed later on anyway so it makes sense for us not to look any further into other methods.*

Having settled the initial step of our induction, we need to establish some tools for the inductive step. For the inductive step we need to show that if a polynomial is minimal (resp. maximal) in all strict substrata of a stratum $\mathcal{H}_s^\mu(f)$, it is also minimal (resp. maximal) in $\mathcal{H}_s^\mu(f)$. For that we need to look closer at the projection π. It should be noted that the following discussion and lemma is analogous to the approach in [18], where the image of the power sums are studied instead of the elementary symmetric polynomials. Before we begin recall that $l > s$ for this subchapter.

By Lemma 2.1.8, $\mathcal{H}_s^\mu(f)$ is homeomorphic to $\pi^{n-l}(\mathcal{H}_s^\mu(f)) \subset \mathbb{R}^{l-s}$ and thus by Theorem 2.1.10, $M := \pi^{n-l}(\mathcal{H}_s^\mu(f))$ is full-dimensional when $\mathcal{H}_s^\mu(f)$ is neither empty nor a single polynomial. If we apply the projection to M, that is, we let $\pi : M \to \mathbb{R}^{l-s-1}$ be the projection given by $(x_1, \ldots, x_{l-s}) \mapsto (x_1, \ldots, x_{l-s-1})$, then for $h \in \mathcal{H}_s^\mu(f)$, the fibre $\pi^{-1}(\pi(\pi^{n-l}(h)))$ is equal to $\pi^{n-l}(\mathcal{H}_{l-1}^\mu(h))$. This fibre is by Theorem 2.1.10, either the point $\pi^{n-l}(h)$, in which case it must lie on the boundary of M, or it is an interval. And if it is an interval, then its endpoints must lie on the boundary of M and its relative interior must lie in the interior of M.

Thus the boundary of M can be written as the union of a "lower" and an "upper" part, $L \cup U$, where

$$L = \{(x_1, \ldots, x_{l-s}) \in M \mid x_{l-s} \le y_{l-s} \ \forall \ (y_1, \ldots, y_{l-s}) \in \pi^{-1}(\pi(x))\},$$

and

$$U = \{(x_1, \ldots, x_{l-s}) \in M \mid x_{l-s} \ge y_{l-s} \ \forall \ (y_1, \ldots, y_{l-s}) \in \pi^{-1}(\pi(x))\}.$$

Lemma 2.2.11. *The sets L and U are closed.*

Proof. We just show that U is closed since the proof for L is analogous. So suppose $\pi^{n-l}(q)$ is in the closure of U but not in U. By Lemma 2.1.8, the boundary of M is closed and thus $\pi^{n-l}(q) \in L$. The fibre $\pi^{-1}(\pi(\pi^{n-l}(q)))$ is an interval whose relative interior lies in the interior of M. Let $\pi^{n-l}(g)$ be one of those relative interior points and let $\epsilon > 0$ be such that $B_\epsilon(\pi^{n-l}(g)) \subset M$.

For any $\pi^{n-l}(h) \in B_\epsilon(\pi^{n-l}(g))$, the point $\pi^{-1}(\pi(\pi^{n-l}(h))) \cap L$ lies below $B_\epsilon(\pi^{n-l}(g))$. Thus the distance between $\pi^{n-l}(q)$ and any point in U is at least as large as $\epsilon/2$. Thus $\pi^{n-l}(q)$ cannot be in the closure of U which is a contradiction and so $\pi^{n-l}(q)$ must lie in U. $\square$

Lemma 2.2.12. *Let $l \geq s+2$, then the polynomial $h \in \mathcal{H}_s^\mu(f)$ is minimal (resp. maximal) if and only if it is minimal (resp. maximal) for all strata that contain h and that are strictly contained in $\mathcal{H}_s^\mu(f)$.*

Proof. One implication is clear, so we just have to show that if for all compositions ν with $h \in \mathcal{H}_s^\nu(f) \subset \mathcal{H}_s^\mu(f)$ we have that h is minimal in $\mathcal{H}_s^\nu(f)$, then h is minimal in $\mathcal{H}_s^\mu(f)$. We assume $\mathcal{H}_s^\mu(f)$ is $(l-s)$-dimensional since the statement is clear when it is just a single polynomial. Also, the argument for maximal polynomials is analogous so we just prove it for minimal polynomials.

Suppose h is not minimal in $\mathcal{H}_s^\mu(f)$, then by Lemma 2.2.7 it is not locally minimal. So for any $i \in \mathbb{N}$, $B_{1/i}(h) \cap \mathcal{H}_s^\mu(f)$ contains a polynomial g_i whose first free coefficient is smaller than the first free coefficient of h. Without loss of generality assume $\pi^{n-l}(h)$ lies in the upper part of the boundary of $M = \pi^{n-l}(\mathcal{H}_s^\mu(f))$. Then for each fibre $\pi^{-1}(\pi(\pi^{n-l}(g_i)))$, let $\pi^{n-l}(q_i)$ be the point in the upper part of the boundary of M. Since the upper part is compact, $(\pi^{n-l}(q_i))$ converges to a point in the upper part which by design equals $\pi^{n-l}(h)$.

As there are finitely many compositions, there is an infinite subsequence of $(\pi^{n-l}(q_i))$, where all the q_i's have the same composition $\lambda \neq \mu$, that converges to $\pi^{n-l}(h)$. By Lemma 2.1.8, Theorem 2.1.9 and Corollary 2.1.11, the image $\pi^{n-l}(\mathcal{H}_s^\lambda(f))$ is the closure of its relative interior which consists of the images of the polynomials with composition λ. Thus $h \in \mathcal{H}_s^\lambda(f)$ and it is by construction not the minimal polynomial. This is a contradiction and so h must be minimal in $\mathcal{H}_s^\mu(f)$. $\square$

For the inductive step we need something analogous to Lemma 2.2.12 on the combinatorial side:

Lemma 2.2.13. *Let $\lambda, \gamma < \mu$ be compositions of n, then $\lambda < \gamma$ if and only if $\lambda/\mu < \gamma/\mu$ and in this case we have $\lambda/\gamma = \frac{\lambda/\mu}{\gamma/\mu}$.*

Proof. Let $\nu = \lambda/\mu$, $\tau = \gamma/\mu$ and let $r = \ell(\lambda) = \ell(\nu)$ and $k = \ell(\gamma) = \ell(\tau)$. Then

$$\lambda = (\mu_1 + ... + \mu_{\nu_1}, ..., \mu_{l-\nu_r+1} + ... + \mu_l)$$

and

$$\gamma = (\mu_1 + ... + \mu_{\tau_1}, ..., \mu_{l-\tau_k+1} + ... + \mu_l).$$

Thus $\lambda < \gamma$ if and only if there is a composition, ρ, of k of length r with

$$\lambda = (\gamma_1 + ... + \gamma_{\rho_1}, ..., \gamma_{k-\rho_r+1} + ... + \gamma_k) =$$

$$\left(\sum_{i \leq \tau_1 + ... + \tau_{\rho_1}} \mu_i, ..., \sum_{i > \tau_1 + ... + \tau_{k-\rho_r}} \mu_i \right)$$

which equals $(\mu_1 + ... + \mu_{\nu_1}, ..., \mu_{l-\nu_r+1} + ... + \mu_l)$ if and only if

$$\nu_1 = \sum_{j=1}^{\rho_1} \tau_j, ..., \nu_r = \sum_{j=k-\rho_r+1}^{k} \tau_j,$$

that is, if and only if $\nu < \tau$. In particular, we see that in this case $\lambda/\gamma = \rho = \nu/\tau$. □

Now we just need one small lemma before we are ready to prove Theorem 2.2.3 for the generic case:

Lemma 2.2.14. *Let $l = \ell(\mu) \geq s + 2$ and let $h \in \mathcal{H}_s^\mu(f)$ have s distinct roots. Then there are two polynomials with distinct compositions, γ and ν, in $\mathcal{H}_s^\mu(f)$ of length $\ell(\mu) - 1$ and with $c(h) < \gamma, \nu$.*

Proof. Let $\lambda = c(h)$, then since $l \geq s + 2$, $\ell(\lambda) = s$ and $\lambda < \mu$ one must replace at least two of the commas in μ with plus signs to obtain λ. So let $j \neq i$ be two indices such that

$$\gamma = (\mu_1, \ldots, \mu_{j-1}, \mu_j + \mu_{j+1}, \mu_{j+2}, \ldots, \mu_l)$$

and

$$\nu = (\mu_1, \ldots, \mu_{i-1}, \mu_i + \mu_{i+1}, \mu_{i+2}, \ldots, \mu_l)$$

are both greater than λ. By Theorem 2.1.9 and Theorem 2.1.10, both of these compositions must occur in $\mathcal{H}_s^\mu(f)$. □

Proposition 2.2.15. *Let $\mathcal{H}_s^\mu(f)$ be of $(l - s)$-dimensional and generic. Then $h \in \mathcal{H}_s^\mu(f)$ is the minimal (resp. maximal) polynomial if and only if $\ell(c(h)) = s$ and $c(h)/\mu$ is alternate odd (resp. even).*

Proof. We prove this by induction in the poset of strata of $\mathcal{H}_s^\mu(f)$. The initial step is when $l = s+1$ and is covered by Proposition 2.2.9. Next, we assume the statement is true for the strata of dimension

$l - s - 1 \geq 1$ and we show that it is true when the stratum is $(l-s)$-dimensional. We will just show the proof for minimal polynomials as the proof for maximal polynomials is analogous.

Let $\lambda = c(h)$ and suppose λ/μ is alternate odd and that $\ell(\lambda) = s$. Let γ be any composition with $\lambda < \gamma < \mu$ such that $\mathcal{H}_s^{\gamma}(f)$ is at least one-dimensional. By Lemma 2.2.13 we have that $\lambda/\gamma = \frac{\lambda/\mu}{\gamma/\mu}$. Note that the i^{th} part of λ/γ is equal to the i^{th} part of λ/μ minus some integer, thus λ/γ is alternate odd since λ/μ is. So by the induction hypothesis, h is the minimal polynomial of $\mathcal{H}_s^{\gamma}(f)$ and thus by Lemma 2.2.12, h is the minimal polynomial of $\mathcal{H}_s^{\mu}(f)$.

For the reverse statement, let h be the minimal polynomial. Then by Lemma 2.2.6, h has s distinct roots. Since $\mathcal{H}_s^{\mu}(f)$ is at least two-dimensional, then by Lemma 2.2.14, there are at least two distinct $(l-s-1)$-dimensional strata $\mathcal{H}_s^{\gamma}(f)$ and $\mathcal{H}_s^{\nu}(f)$ in $\mathcal{H}_s^{\mu}(f)$ containing h and thus $\ell(\gamma) = \ell(\nu) = l - 1$. By Lemma 2.2.12 and the induction hypothesis λ/γ and λ/ν are alternate odd compositions. And since γ and ν are of length $l-1$, there are two indices $j \neq i$ such that

$$\lambda = (\mu_1, \ldots, \mu_{j-1}, \mu_j + \mu_{j+1}, \mu_{j+2}, \ldots, \mu_l)$$

and

$$\nu = (\mu_1, \ldots, \mu_{i-1}, \mu_i + \mu_{i+1}, \mu_{i+2}, \ldots, \mu_l).$$

Thus $\gamma/\mu = (1, \ldots, 1, 2, 1, \ldots, 1)$, where the index 2 is in the j^{th} position and $\nu/u = (1, \ldots, 1, 2, 1, \ldots, 1)$, where the index 2 is in the i^{th} position. Since $\lambda/\gamma = \frac{\lambda/\mu}{\gamma/\mu}$ and $\lambda/\nu = \frac{\lambda/\mu}{\nu/\mu}$, we have

$$\lambda/\gamma = ((\lambda/\mu)_1, \ldots, (\lambda/\mu)_{j-1}, (\lambda/\mu)_j - 1, (\lambda/\mu)_{j+1}, \ldots, (\lambda/\mu)_s)$$

and

$$\lambda/\nu = ((\lambda/\mu)_1, \ldots, (\lambda/\mu)_{i-1}, (\lambda/\mu)_i - 1, (\lambda/\mu)_{i+1}, \ldots, (\lambda/\mu)_s).$$

Since $j \neq i$ then $\lambda/\gamma \neq \lambda/\nu$ and since both compositions are alternate odd then so must λ/μ be. $\square$

Now that we have established the second part of Theorem 2.2.3 for the generic case we will extend it to the non-generic cases. Firstly recall that $l > s$, then we need the following lemma:

Lemma 2.2.16. *If $\mathcal{H}_s^\mu(f)$ is $(l-s)$-dimensional, then there is a $\delta > 0$ such that for any ϵ, with $0 < \epsilon < \delta$, there is a monic polynomial p of degree $n - s$ such that $\mathcal{H}_s^\mu(f \pm \epsilon p)$ is generic and nonempty.*

Proof. By Theorem 2.1.9, there is an $h \in \mathcal{H}_s^\mu(f)$ with composition μ and by Theorem 2.1.10, $\mathcal{H}_{s-1}^\mu(f)$ is of dimension $l - s + 1 > 0$. Thus h is in the relative interior of $\mathcal{H}_s^\mu(f)$ and by Lemma 2.1.8 we can choose a $\delta > 0$ such that $B_\delta(\pi^{n-l}(h)) \subset \pi^{n-l}(\mathcal{H}_{s-1}^\mu(f))$. Since there are finitely many polynomials in $\mathcal{H}_{s-1}(f)$ with at most $s-1$ distinct roots we can choose δ such that for any $\pi^{n-l}(g) \in B_\delta(\pi^{n-l}(h))$ with $g_s \neq f_s$, $\mathcal{H}_s^\mu(g)$ is generic. By Theorem 2.1.9, $c(g) = \mu$ and we can choose g such that $g_s = f_s \pm \epsilon$ for any $0 < \epsilon < \delta$. $\square$

Lemma 2.2.17. *If $h \in \mathcal{H}_s^\mu(f)$ and $c(h) < \gamma$ for some γ of length s where γ/μ is alternate odd (resp. even), then h is minimal (resp. maximal).*

Proof. We will let γ/μ be alternate odd as the proof is analogous when γ/μ is alternate even. If $\mathcal{H}_s^\mu(f)$ is just a point, the statement is clear so by Theorem 2.1.10, we may assume it is $(l-s)$-dimensional.

Suppose h is not minimal, then by Lemma 2.2.7, h is not locally minimal. Thus for any $\delta > 0$, there is a $q \in B_\delta^{n-s}(h) \cap \mathcal{H}_s^\mu(f)$ with $h_{s+1} - q_{s+1} = r > 0$. By Corollary 2.1.11, $\mathcal{H}_s^\mu(f)$ is the closure of its relative interior, so we may assume $c(q) = \mu$. Thus by Theorem 2.1.9 and Lemma 2.1.8, $\pi^{n-l}(q)$ is in the interior of $\pi^{n-l}(\mathcal{H}_0^\mu(f))$ and so $B_\epsilon^l(\pi^{n-l}(q)) \subset \pi^{n-l}(\mathcal{H}_0^\mu(f))$ for some ϵ with $0 < \epsilon < r/2$.

Next, note that all compositions occur in $\mathcal{H}_0(f)$ and since $\mathcal{H}_0^\gamma(f)$ is the closure of its relative interior then $B_\epsilon^l(\pi^{n-l}(h)) \cap \pi^{n-l}(\mathcal{H}_0^\gamma(f))$ contains a point, $\pi^{n-l}(g)$, where $c(g) = \gamma$. The intersection

$$\pi^{n-l}(\mathcal{H}_s^\mu(g)) \cap B_\epsilon(\pi^{n-l}(q)) = A_{g_1}^l \cap ... \cap A_{g_s}^l \cap B_\epsilon^l(\pi^{n-l}(q))$$

is nonempty since $q_i = h_i \; \forall \; i \in [s]$ and $A_{g_1}^l \cap ... \cap A_{g_s}^l \cap B_\epsilon^l(\pi^{n-l}(h))$ is nonempty. Thus there is a polynomial from $B_\epsilon^n(q)$ in $\mathcal{H}_s^\mu(g)$.

By Lemma 2.2.16, we may pick g such that $\mathcal{H}_s^\mu(g)$ is generic so by Proposition 2.2.15, g must be the minimal polynomial in $\mathcal{H}_s^\mu(g)$. However the first free coefficient of any polynomial from $B_\epsilon^n(q)$ is smaller than $h_{s+1} - r/2$ and g_{s+1} is greater than $h_{s+1} - r/2$. This is a contradiction and so h must be minimal in $\mathcal{H}_s^\mu(f)$. $\square$

Lemma 2.2.18. *If $h \in \mathcal{H}_s^\mu(f)$ and $c(h) \not\leq \nu$ for any ν of length s where ν/μ is alternate odd (resp. even), then h is not minimal (resp. maximal).*

Proof. Again we just show the statement for alternate odd compositions. If $s = 2$, then by the main theorem in [23] either f has only one distinct root and $\mathcal{H}_2(f) = \{f\}$ or $\mathcal{H}_2(f)$ contains no polynomials with strictly less than two distinct roots. Since $h \in \mathcal{H}_s^\mu(f)$ does not have composition (n), $\mathcal{H}_2(f)$ is generic and the statement follows from Proposition 2.2.15.

Next we let $s \geq 3$ and by the previous paragraph we have that $\mathcal{H}_2(f)$ is generic and all but the composition (n) occurs. By Corollary 2.1.11, $\mathcal{H}_2^\mu(f)$ is the closure of its relative interior, so for any integer $i \geq 1$ there is a polynomial $g_i \in B_{1/i}(h) \cap \mathcal{H}_2^\mu(f)$ with composition μ. Due to Lemma 2.2.16, we may pick the g_i's such that $\mathcal{H}_s^\mu(g_i)$ is generic. Thus, by Proposition 2.2.15, the composition, ν, of the minimal polynomial in $\mathcal{H}_s^\mu(g_i)$ is such that ν/μ is alternate odd.

Since there are finitely many compositions with this property, there is a composition ν such that for infinitely many i, the minimal polynomial of $\mathcal{H}_s^\mu(g_i)$ has composition ν. For notation's sake we will assume that for all $i \geq 1$, the minimal polynomial, q_i, of $\mathcal{H}_s^\mu(g_i)$ has the same composition ν. Since $(1/i)_{i\geq 1}$ converges to zero and $\mathcal{H}_2^\mu(f)$ is compact, the sequence $(g_i)_{i\geq 1}$ converges. Similarly, since $\mathcal{H}_2^\mu(f)$ is sequentially compact, an infinite subsequence of $(q_i)_{i\geq 1}$ converges and so for notation's sake we will assume this is the sequence $(q_i)_{i\geq 1}$.

The limit of $(g_i)_{i\geq 1}$ is h and since the first $s+1$ coefficients of q_i is equal to the first coefficients of g_i, the limit, q, of $(q_i)_{i\geq 1}$ also lies in $\mathcal{H}_s^\mu(f)$. Since $\mathcal{H}_2^\nu(f)$ is the closure of its relative interior and $c(q_i) = \nu$ for all i, then $c(q) \leq \nu$ and thus by Lemma 2.2.17, q is the minimal polynomial of $\mathcal{H}_s^\mu(f)$. Since $c(h)$ is not smaller than a composition γ such that γ/μ is alternate odd, then $c(h) \not< \nu$ and thus $h \neq q$. So h is not the minimal polynomial of $\mathcal{H}_s^\mu(f)$. $\square$

Proposition 2.2.15, covers the second part of Theorem 2.2.3 for the generic cases and the combination of proves it for the non-generic cases. And since we proved the first part of Theorem 2.2.3 for all cases in the beginning of this subchapter, we are done.

2.3 Contractible interior

An interesting consequence of Theorem 2.2.3 is that we can, without too much work, argue that the relative interior of any hyperbolic stratum is connected. From that point on we only need to slightly strengthen the argument in [18] to show that the relative interior is also contractible. Before we do that however, note that in Theorem 2.1.10 we saw that a stratum is either empty, a single polynomial or maximal dimensional. Thus Theorem 2.2.3 immediately gives a condition to determine when a nonempty stratum is maximal dimensional and when it is just a single polynomial:

Corollary 2.3.1. *Let $s \geq 2$ and let $\mathcal{H}_s^\mu(f)$ be nonempty. Then $\mathcal{H}_s^\mu(f)$ is a single polynomial if and only if for any $h \in \mathcal{H}_s^\mu(f)$, $c(h)/\mu$ is smaller than an alternate odd and an alternate even composition of length s.*

Remark 2.3.2. *The condition in Corollary 2.3.1 can also be phrased as in [23]: that is, let m be the number of odd sequences of consecutive 1's lying between integers greater than 1 in the composition $\nu = c(h)/\mu$ and let*

$$w = m + \sum_{\nu_i > 2} (\nu_i - 2).$$

If $\nu < (1, 1, ..., 1)$, then ν is smaller than or equal to an alternate odd (resp. even) composition of length s and no alternate even (resp. odd) composition of length s if and only if $w = l - s - 1$ and ν ends in an odd (resp. even) number of 1's. Similarly, ν is smaller than or equal to both an alternate odd and an alternate even composition of length s if $w > l - s - 1$.

Now we are ready to show that the relative interior of the hyperbolic strata are connected and the argument is roughly as follows: by arguing inductively, starting from $s = l - 1$ and then let $s = l - 2$ and so on, we can see that by the inductive hypothesis a stratum is like a string of sausages and then we can use Corollary 2.3.1 to say that there can be at most one sausage in this string.

Lemma 2.3.3. *[Sausage Lemma] The relative interior of $\mathcal{H}_s^\mu(f)$ is either empty or connected.*

Proof. We prove the statement with induction starting at the case when $\mathcal{H}_s^\mu(f)$ is one-dimensional and going up in dimension. So suppose $\mathcal{H}_s^\mu(f)$ is one-dimensional. Then by Proposition 2.1.5, $\pi^{n-l}(\mathcal{H}_s(f)) \subset \mathbb{R}$ is an interval and so the interior is connected. By Lemma 2.1.8, $\mathcal{H}_s(f)$ is homeomorphic to $\pi^{n-l}(\mathcal{H}_s(f))$, so the relative interior of $\mathcal{H}_s^\mu(f)$ is also connected.

Assume the statement is true whenever $\mathcal{H}_s^\mu(f)$ is d-dimensional for $d \geq 1$, then we will show that it is true if $\mathcal{H}_s^\mu(f)$ is $(d+1)$-dimensional. So let $\mathcal{H}_s^\mu(f)$ be $(d+1)$-dimensional. By induction we have that the relative interior of $\mathcal{H}_{s+1}^\mu(h)$ is connected for any $h \in \mathcal{H}_s^\mu(f)$. That is any affine hypersurface, $A_a^{n-s} \subset \mathbb{R}^{n-s}$, defined by fixing the first free coordinate to be a, meets only one connected component of the relative interior of $\mathcal{H}_s^\mu(f)$.

Suppose the relative interior of $\mathcal{H}_s^\mu(f)$ has more than one connected component. By Proposition 2.1.5 and Corollary 2.1.11, $\mathcal{H}_s^\mu(f)$ is connected and the closure of its relative interior. So if B is one connected component of the relative interior then there is another connected component, C, of the relative interior with

$$\operatorname{dist}(B, C) := \inf\{\operatorname{dist}(g, h) : g \in B \text{ and } h \in C\} = 0,$$

where $\operatorname{dist}(g, h)$ denotes the Euclidean distance.

Since $\operatorname{dist}(B, C) = 0$ and any hypersurface A_a^{n-s} meets only one connected component of the relative interior of $\mathcal{H}_s^\mu(f)$, then there is a value $b \in \mathbb{R}$ and $\delta \in \mathbb{R}$, with $\delta > 0$, such that $A_{b\pm\epsilon}^{n-s}$ meets B and $A_{b\mp\epsilon}^{n-s}$ meets C for all $0 < \epsilon < \delta$ and where

$$A_b^{n-s} \cap B = A_b^{n-s} \cap C = \emptyset.$$

Without loss of generality, assume that $A_{b+\epsilon}^{n-s}$ meets B for some $\epsilon > 0$. Then B contains only polynomials where the first free coefficient is greater than b and C contains only polynomials where the first free coefficient is smaller than b. But by Proposition 2.1.5 $\mathcal{H}_s^\mu(f)$ is connected, thus there is a polynomial $h \in \mathcal{H}_s^\mu(f)$ with $h_{s+1} = b$. By Theorem 2.1.10, $\mathcal{H}_{s+1}^\mu(h)$ is either of dimension $d \geq 1$ or a single polynomial.

If it is d-dimensional, then by Theorem 2.1.9, it contains a polynomial, g, with composition μ. Therefore g is a relative interior point of $\mathcal{H}_s^\mu(f)$ and sits in a connected component, D, of the relative interior. By Lemma 2.1.8, $\pi^{n-l}(g)$ sits in the connected component $\pi^{n-l}(D)$ of the interior of $\pi^{n-l}(\mathcal{H}_s^\mu(f)$. Thus there is a $0 < \epsilon' < \delta$ such that $B_{\epsilon'}(\pi^{n-l}(g)) \subseteq \pi^{n-l}(D)$ and thus the hyperplane $A_{b+\epsilon'/2}^{l-s} \subset \mathbb{R}^{l-s}$ meets both $\pi^{n-l}(D)$ and $\pi^{n-l}(B)$. But that means $D = B$, since the relative interior of $\mathcal{H}_s^\mu(f) \cap A_{b+\epsilon'/2}^{n-s}$ is connected and $\mathcal{H}_s^\mu(f)$ and $\pi^{n-l}(\mathcal{H}_s^\mu(f))$ are homeomorphic. This is a contradiction and thus $\mathcal{H}_{s+1}^\mu(h)$ must be a point.

By Corollary 2.3.1 and Remark 2.3.2, we therefore have that

$$m + \sum_{\nu_i > 2} (\nu_i - 2) > l - s - 2,$$

where m is the number of odd sequences of consecutive 1's lying between integers greater than 1 in $\nu = c(h)/\mu$. Thus by Remark 2.3.2 and Theorem 2.2.3, h is either the minimal or maximal polynomial of $\mathcal{H}_s^\mu(f)$. But this is a contradiction since any polynomial in B have a first free coefficient greater than h_{s+1} and any polynomial in C have a first free coefficient smaller than h_{s+1}. Thus the relative interior of $\mathcal{H}_s^\mu(f)$ is connected. □

Theorem 2.3.4. *The relative interior of $\mathcal{H}_s^\mu(f)$ is either empty or contractible.*

Proving that the relative interior of $\mathcal{H}_s^\mu(f)$ is contractible is by Lemma 2.1.8 equivalent to proving that the interior of $M := \pi^{n-l}(\mathcal{H}_s^\mu(f)) \subset \mathbb{R}^{l-s}$ is contractible, so we will do this instead. This will be done by an inductive argument starting with the one-dimensional set $\pi^{l-s-1}(M)$ and then the two-dimensional set $\pi^{l-s-2}(M)$ and so on. Thus we will need to study the projection $\pi : \pi^k(M) \to \mathbb{R}^{l-s-k-1}$ that forgets the last coordinate. We will follow the line of argument in section 2.4 of Kostov's article [18] (see statements 2.7 to 2.12), but we need to strengthen a technical part in Kostov's argument. However the main idea can be found in his article so we will be a little less rigid with the proof.

We will assume that M is $(l-s)$-dimensional and we let $l > s$ and $k < l - s - 1$ for the remainder of this subchapter. Also, we will let $s \geq 2$ since the theorem follows from Lemma 2.1.4 when $s \leq 1$. Note that as $s \geq 2$, any closed subset $K \subseteq M$ is compact and since π is continuous, then $\pi(K)$ is compact. Thus π maps open sets to open sets and similar for π^k. The first step is to generalise Lemma 2.2.11 and show that the boundary of $\pi^k(M)$ can be split into a "lower" and an "upper" part, both of which are closed.

Lemma 2.3.5. *Every fibre of $\pi : \pi^k(M) \to \mathbb{R}^{l-s-k-1}$ is either a point or a compact interval.*

Proof. If $p \in \pi^{k+1}(M)$, then $p = \pi^{n-l+k}(h)$ for some $h \in \mathcal{H}_s^\mu(f)$ and $\pi^{-1}(p) = \pi^{n-l+k}(\mathcal{H}_{l-k-1}^\mu(h))$. As usual we view $\pi^{n-l}(\mathcal{H}_{l-k-1}^\mu(h))$ as a subset of $\mathbb{R}^{k+1}$ and by Lemma 2.1.8 and Proposition 2.1.5, $\pi^{n-l}(\mathcal{H}_{l-k-1}^\mu(h))$ is connected and either a single point or of dimension $k+1$. Thus $\pi^{-1}(p) \subset \mathbb{R}$ is a point or an interval and since π^k maps compact sets to compact sets $\pi^{n-l+k}(\mathcal{H}_{l-k-1}^\mu(h))$ is compact. $\square$

Lemma 2.3.6. *Every fibre of $\pi : \pi^k(M) \to \mathbb{R}^{l-s-k-1}$ that is an interval contains exactly two points from $\partial(\pi^k(M))$, these points are the endpoints of the interval.*

Proof. We saw that if $p \in \pi^{k+1}(M)$, then $\pi^{-1}(p) = \pi^{n-l+k}(\mathcal{H}_{l-k-1}^\mu(h))$ for some $h \in \mathcal{H}_s^\mu(f)$, thus $\pi^{-1}(p)$ is the intersection of $\pi^k(M)$ and a line in $\mathbb{R}^{l-s-k}$. Therefore a boundary point of $\pi^{-1}(p)$ is a boundary point of $\pi^k(M)$, so we just need to show that these are the only points in $\pi^{-1}(p)$ from $\partial(\pi^k(M))$.

First we show that there are finitely many points from $\partial(\pi^k(M))$ in $\pi^{-1}(p)$. To see this note that as π^k maps open sets to open sets, then for any composition ν with $\ell(\nu) > l-k-1$, it maps the relative interior of $\mathcal{H}_s^\nu(f)$ into the relative interior of $\pi(\mathcal{H}_s^\nu(f))$ and so the boundary of $\pi^k(M)$ consists of some of the points $\pi^k(q)$, where $q \in \mathcal{H}_s^\mu(f)$ has at most $l-k-1$ distinct roots. So any point in $\pi^{-1}(p)$ from $\partial(\pi^k(M))$ are of the form $\pi^k(g)$, where $g \in \mathcal{H}_{l-k-1}^\mu(h)$ has a composition of length at most $l-k-1$. But for each composition $\nu < \mu$ of length $l-k-1$, $\mathcal{H}_{l-k-1}^\nu(h)$ is at most a single polynomial, thus since there are finitely many compositions, there must be finitely many points from $\partial(\pi^k(M))$ in $\pi^{-1}(p)$.

Lastly, we can use this to argue just as in the proof of Theorem 2.1.9 so we just sketch the argument here. We assume that $\pi^{-1}(p)$ contains a point q in its interior, where $q \in \partial(\pi^k(M))$. Then since there are finitely many points from $\partial(\pi^k(M))$ in $\pi^{-1}(p) = \operatorname{conv}(a,b)$, there is a $q_- \in \operatorname{Int}(\pi^k(M))$ lying between q and a and a point,

$q_+ \in \operatorname{Int}(\pi^k(M))$ lying between q and b. Thus there is an $\epsilon > 0$ such that the balls $B_\epsilon^{l-s-k}(q_-)$ and $B_\epsilon^{l-s-k}(q_+)$ lie in $\pi^k(M)$ and $B_\epsilon^{l-s-k}(q)$ does not. By translating the line spanned by q_- and q_+ slightly, it will meet all three balls but contain a point from $B_\epsilon^{l-s-k}(q)$ that lies outside $\pi^k(M)$. Thus the intersection of the line with $\pi^k(M)$ is a disconnected set of the form $\pi^{n-l+k}(\mathcal{H}_{l-k-1}^{\mu}(r))$, for some $r \in \mathcal{H}_s^{\mu}(f)$, which contradicts Proposition 2.1.5. $\square$

Due to Lemma 2.3.6, we can separate the boundary of $\pi^k(M)$ into a lower and an upper part. That is $\partial(\pi^k(M)) = L_k \cup U_k$, where

$$\begin{aligned} L_k = \{x = (x_1, ..., x_{l-s-k}) \in \pi^k(M)| \\ x_{l-s-k} \leq y_{l-s-k} \ \forall\ (y_1, ..., y_{l-s-k}) \in \pi^{-1}(\pi(x))\}, \end{aligned}$$

and

$$\begin{aligned} U_k = \{x = (x_1, ..., x_{l-s-k}) \in \pi^k(M)| \\ x_{l-s-k} \geq y_{l-s-k} \ \forall\ (y_1, ..., y_{l-s-k}) \in \pi^{-1}(\pi(x))\}. \end{aligned}$$

The following is proven just like Lemma 2.2.11 so we will skip the proof:

Lemma 2.3.7. *The sets L_k and U_k are closed.*

Since π maps open balls in $\mathbb{R}^{l-s-k}$ to open balls in $\mathbb{R}^{l-s-k-1}$, the fibre of a point in $\partial(\pi^{k+1}(M))$ must contain only points from $\partial(\pi^k(M))$. Thus by Lemma 2.3.5 and Lemma 2.3.6, the fibre is a single point. However, due to the connectedness of the interior of M we get the following strengthening of statement 2.11 in [18]:

Lemma 2.3.8. *The fibre $\pi^{-1}(p)$ of $\pi : \pi^k(M) \to \mathbb{R}^{l-s-k-1}$ is a point if and only if $p \in \partial(\pi^{k+1}(M))$.*

Proof. We will do a contrapositive proof, but firstly note that as π^k maps open balls to open balls and by Lemma 2.1.8 and Corollary 2.1.11, M is the closure of its interior so is $\pi^k(M)$. Also, by Lemma 2.3.3 $\operatorname{Int}(M)$ is connected and since π^k is continuous the interior of $\pi^k(M)$ is connected.

Let p lie in the interior of $\pi^{k+1}(M)$ and suppose the fibre $\pi^{-1}(p) = \pi^{n-l+k}(\mathcal{H}_{l-k-1}^{\mu}(h))$ just contains the point $\pi^{n-l+k}(h)$. Since $p = \pi^{n-l+k+1}(h)$ lies in the interior of $\pi^{k+1}(M)$, which is of dimension $l-s-k-1 \geq 1$, then $\pi^{n-l+k+1}(\mathcal{H}_{l-k-2}^{\mu}(h))$ is one-dimensional and so

$\pi^{n-l+k}(\mathcal{H}^{\mu}_{l-k-2}(h))$ is two-dimensional. Thus for notation's sake we will assume $\pi^k(M)$ is already two-dimensional.

Since $p = \pi^{n-l+k+1}(h)$ lies in the interior of $\pi^{k+1}(M)$ there is a point in $\pi^k(M)$ that have a first coordinate greater than the first coordinate of $\pi^{n-l+k}(h)$ and a point whose first coordinate is smaller than the first coordinate of $\pi^{n-l+k}(h)$. We can chose those points to be interior points because $\pi^k(M)$ is the closure of its interior. Thus $\pi^{n-l+k}(h)$ separates the interior of $\pi^k(M)$ which is a contradiction. □

Due to Lemma 2.3.8 we see that $\pi(L_k \cap U_k) = \partial(\pi^{k+1}(M))$ and we are now in a position to prove Theorem 2.3.4:

Proof of Theorem 2.3.4. We will do an induction on the dimension of $\partial(\pi^k(M))$ and the interior of $\partial(\pi^k(M))$ is clearly contractible when $\partial(\pi^k(M))$ is one-dimensional. So let the dimension of $\partial(\pi^k(M))$ be greater than one, that is, we have that $k < l - s - 1$.

The restrictions $\pi|_{L_k} : L_k \to \pi^{k+1}(M)$ and $\pi|_{U_k} : U_k \to \pi^{k+1}(M)$ are continuous and bijective. Also, due to the Closed Graph Theorem (see exercise 8 in paragraph 26, chapter 3 of [25]), their inverses are continuous since L_k and U_k are closed by Lemma 2.3.7. Therefore $\pi|_{L_k}$ and $\pi|_{U_k}$ are homeomorphisms and thus so is the map

$$\phi : \pi^{k+1}(M) \to \pi^k(M),$$

given by

$$\phi(p) = \frac{1}{2}(\pi|_{U_k}^{-1}(p) - \pi|_{L_k}^{-1}(p)).$$

Combined with the induction hypothesis, there is therefore a deformation retract

$$\theta' : \phi(\mathrm{Int}(\pi^{k+1}(M))) \times [0, 1] \to \phi(\mathrm{Int}(\pi^{k+1}(M)))$$

of $\phi(\mathrm{Int}(\pi^{k+1}(M)))$ to some point $q \in \phi(\mathrm{Int}(\pi^{k+1}(M)))$.

The projection $P := \phi \circ \pi : \pi^k(M) \to \pi^k(M)$ is continuous and maps the interior of $\pi^k(M)$ onto $\phi(\mathrm{Int}(\pi^{k+1}(M)))$, so the map

$$\psi : \mathrm{Int}(\pi^k(M)) \times [0, 1] \to \mathrm{Int}(\pi^k(M)),$$

given by

$$\psi(a, t) = (1 - t)a + tP(a),$$

is a deformation retract of the interior of $\pi^k(M)$ onto the subset $\phi(\mathrm{Int}(\pi^{k+1}(M)))$. Thus the map

$$\Psi : \mathrm{Int}(\pi^k(M)) \times [0,1] \to \mathrm{Int}(\pi^k(M)),$$

given by

$$\Psi(a,t) = \begin{cases} \psi(a,2t), & \text{if } t \in [0,1/2], \\ \theta(a,2t-1), & \text{if } t \in [1/2,1], \end{cases}$$

is a deformation retract of $\mathrm{Int}(\pi^k(M))$ to the point $q \in \mathrm{Int}(\pi^k(M))$.

$\square$

Chapter 3

... and combinatorics

In this chapter we use the results from the previous chapter to establish several combinatorial properties of “hyperbolic lattices”, that is, posets of hyperbolic strata. In the first part we ask which compositions occur in any given hyperbolic slice. To answer this we show that similarly to the face lattice of polytopes, hyperbolic lattices are graded, atomic and coatomic lattices. In particular, this gives us a combinatorial algorithm to compute which compositions occur in a hyperbolic slice assuming we know which compositions of length at most s occur. This enables us to compute more examples of hyperbolic lattices than we otherwise would be able to.

In the second part we imitate a line shelling of polytopes and show that in the generic case the boundary complex of the dual of hyperbolic lattices are shellable simplicial complexes. In particular that means that generically the boundary complex of the dual of hyperbolic lattices are combinatorial spheres. This gives us bounds and relations on the number of i-dimensional strata in hyperbolic slices since, according to [2], combinatorial spheres satisfy the *g-theorem*. Next we show that the boundary complex of the dual of hyperbolic lattices are generically like “subdivisions” of the boundary complex of some non-generic dual hyperbolic lattice. This allows us to show that the dual lattices satisfy the *Upper Bound Theorem* (UBT) in general.

3.1 The hyperbolic lattice

In this subchapter we establish that the poset of strata of $\mathcal{H}_s(f)$ is a graded, atomic and coatomic lattice and we use this to construct a combinatorial algorithm to compute which compositions occur hyperbolic slices. We end by giving a restriction to which compositions can occur in hyperbolic slices which can improve the algorithm further. We begin by establishing that the poset of compositions is a simplex before we move on to the posets of strata. So let $\mathcal{C}(n)$ denote the poset of compositions of n.

Lemma 3.1.1. *The poset of compositions of n is isomorphic to the face lattice of an $(n-2)$-dimensional simplex.*

Proof. Let $D(n-1)$ denote the subsets of $[n-1]$ partially ordered by inclusion and let $\phi : \mathcal{C}(n) \to D(n-1)$ be the map given by

$$(\mu_1, ..., \mu_l) \mapsto \{\mu_1, \mu_1 + \mu_2, ..., \mu_1 + ... + \mu_{l-1}\}.$$

Also, let $\theta : D(n-1) \to \mathcal{C}(n)$ be the map given by

$$\{m_1, ..., m_k\} \mapsto (m_1, m_2 - m_1, ..., m_k - m_{k-1}, n - m_k),$$

where the m_i's are ordered increasingly. Then it is clear that both ϕ and θ are injective and we see that $\theta \circ \phi$ is the identity map on $\mathcal{C}(n)$ so θ is the inverse of ϕ. Secondly, we can see that the composition ν is smaller than μ if and only if $\phi(\nu) \subseteq \phi(\mu)$, thus ϕ is a poset isomorphism. Since $D(n-1)$ is isomorphic to the face lattice of any $(n-2)$-dimensional simplex then so is $\mathcal{C}(n)$. $\square$

Note that in particular the poset of compositions is a lattice and the isomorphism in the preceding proof lets us compute the join and meet of any two compositions quite easily. That is, if μ and ν are two compositions of n, then their join, $\mu \vee \nu$, equals $\phi^{-1}(M)$, where M is the smallest subset of $[n-1]$ containing both $\phi(\mu)$ and $\phi(\nu)$. Similarly, the meet of μ and ν equals the preimage of the largest subset contained in both $\phi(\mu)$ and $\phi(\nu)$.

From Lemma 3.1.1 we also get that the set of strata of $\mathcal{H}_s(f)$, partially ordered by inclusion, form a lattice. To see this let us determine the meet of two faces of $\mathcal{H}_s(f)$, $\mathcal{H}_s^\mu(f)$ and $\mathcal{H}_s^\nu(f)$: The meet of two strata must be contained in their intersection since the partial order is given by inclusion. Also, by definition the of the strata we have

$$\mathcal{H}_s^{\mu}(f) \cap \mathcal{H}_s^{\nu}(f) = \mathcal{H}_s^{\mu\wedge\nu}(f),$$

so the intersection is a stratum of $\mathcal{H}_s(f)$. Thus we have that

$$\mathcal{H}_s^{\mu}(f) \wedge \mathcal{H}_s^{\nu}(f) = \mathcal{H}_s^{\mu\wedge\nu}(f)$$

and we have shown that the poset of strata of $\mathcal{H}_s(f)$ is a lattice. This gives rise to the following definition:

Definition 3.1.2. *We let $\mathcal{L}_s(f)$ denote the lattice of strata of $\mathcal{H}_s(f)$ and we call $\mathcal{L}_s(f)$ a **hyperbolic lattice**.*

Note that by Theorem 2.1.9 and Theorem 2.1.10, $\mathcal{L}_s(f)\backslash\emptyset$ is isomorphic to the poset of compositions occurring in $\mathcal{H}_s(f)$. Thus we will identify $\mathcal{L}_s(f)$ with

$$\{c(h)|h \in \mathcal{H}_s(f)\} \cup \{(n)\}$$

when it makes the notation easier.

It is worth pointing out that just because the meet of $\mathcal{H}_s^{\mu}(f)$ and $\mathcal{H}_s^{\nu}(f)$ equals $\mathcal{H}_s^{\mu\wedge\nu}(f)$, this does not a priori mean that $\mu \wedge \nu$ is the only composition, γ, such that $\mathcal{H}_s^{\mu}(f) \wedge \mathcal{H}_s^{\nu}(f) = \mathcal{H}_s^{\gamma}(f)$. For instance if $\mathcal{H}_s^{\mu}(f)$ is empty, then $\mathcal{H}_s^{\mu}(f) \wedge \mathcal{H}_s^{\nu}(f) = \mathcal{H}_s^{\mu}(f)$ even when $\mu \wedge \nu \neq \mu$, so different compositions may label the same stratum. Thus $\{c(h)|h \in \mathcal{H}_s(f)\} \cup \{(n)\}$ is a subposet of the compositions of n and a lattice since $\mathcal{L}_s(f)$ is a lattice, but it need not be a sublattice of the compositions of n.

We shall get started on studying the lattice properties of hyperbolic posets. First we will shortly discuss what happens when $s = 0$ and $s = 1$ and then we move on to the more interesting case of $s \geq 2$. When $s = 0$ naturally all compositions occur in $\mathcal{H}_0(f)$ so $\mathcal{L}_0(f)$ is isomorphic to the set of compositions of n. Also, note that there are no maximal or minimal polynomials in $\mathcal{H}_0(f)$, so all compositions occur in $\mathcal{H}_1(f)$ and $\mathcal{L}_1(f)$ is isomorphic to the set of compositions of n. Thus by Lemma 3.1.1, $\mathcal{L}_0(f)$ and $\mathcal{L}_1(f)$ are isomorphic to the face lattice of an $(n-2)$-dimensional simplex which by Theorem 2.7 in [37] means that these lattices are graded, atomic and coatomic.

For the remainder of this subchapter we will let $s \geq 2$ and recall that when $s \geq 2$ the strata are compact. Also note that by the main theorem in [23] there is one maximal polynomial $h \in \mathcal{H}_1(f)$. It is the unique polynomial in $\mathcal{H}_1(f)$ with only one distinct root and h is maximal for all the strata of $\mathcal{H}_1(f)$. Thus if $s \geq 2$ and $\mathcal{H}_s(f)$ does not equal $\mathcal{H}_s^{(n)}(f)$, then the stratum $\mathcal{H}_s^{(n)}(f)$ must be empty and since the composition (n) is smaller than all compositions, then every stratum contains the empty set. Thus we will also assume $\mathcal{H}_s(f) \neq \mathcal{H}_s^{(n)}(f)$ for the remainder of this subchapter.

Proposition 3.1.3. *When $s \geq 2$ the lattice $\mathcal{L}_s(f)$ is graded and the rank of a stratum is one more than its dimension.*

Proof. The statement is clear when $\mathcal{H}_s(f)$ contains only one polynomial so suppose it contains more than one polynomial. Suppose a stratum $\mathcal{H}_s^{\mu}(f)$ is strictly contained in $\mathcal{H}_s^{\nu}(f)$, then by Theorem 2.1.10, $\dim(\mathcal{H}_s^{\mu}(f)) < \dim(\mathcal{H}_s^{\nu}(f))$. Also, since $\mathcal{L}_s(f)$ contains the empty set as its minimal element, any maximal chain in the lattice of strata has length at most $\dim(\mathcal{H}_s(f)) + 1 = n - s + 1$.

Conversely, suppose we have that $\mathcal{H}_s^{\mu}(f)$ is strictly contained in $\mathcal{H}_s^{\nu}(f)$, where $\dim(\mathcal{H}_s^{\mu}(f)) < \dim(\mathcal{H}_s^{\nu}(f)) - 1$, let us show that there is a stratum, $\mathcal{H}_s^{\gamma}(f)$, with $\mathcal{H}_s^{\mu}(f) \subset \mathcal{H}_s^{\gamma}(f) \subset \mathcal{H}_s^{\nu}(f)$. Firstly, if $\mathcal{H}_s^{\mu}(f)$ is empty, $\mathcal{H}_s^{\nu}(f)$ is at least one-dimensional and by Theorem 2.1.9 its relative interior are the polynomials with composition ν. But since it is compact it must contain a nonempty stratum, $\mathcal{H}_s^{\gamma}(f)$, in its relative boundary. Since $\mathcal{H}_s^{\gamma}(f)$ contains the empty stratum $\mathcal{H}_s^{\mu}(f)$ we have $\mathcal{H}_s^{\mu}(f) \subset \mathcal{H}_s^{\gamma}(f) \subset \mathcal{H}_s^{\nu}(f)$.

Next, if $\mathcal{H}_s^{\mu}(f)$ is nonempty and contains no polynomials with at least s distinct roots, then by Theorem 2.1.10, $\mathcal{H}_s^{\mu}(f)$ is a single polynomial. Since $\mathcal{H}_s^{\mu}(f)$ is zero-dimensional, $\mathcal{H}_s^{\nu}(f)$ is at least two-dimensional and by Proposition 2.1.5, it is contractible. Since it is compact and contractible, its relative boundary is at least one-dimensional and connected. If $\mathcal{H}_s^{\mu}(f)$ is not contained in a one-dimensional stratum of $\mathcal{H}_s^{\nu}(f)$, it must be an isolated part of the relative boundary of $\mathcal{H}_s^{\nu}(f)$. But since the relative boundary is connected, $\mathcal{H}_s^{\mu}(f)$ must be the whole relative boundary which is impossible since $\mathcal{H}_s^{\mu}(f)$ is zero-dimensional. Thus there is a one-dimensional stratum, $\mathcal{H}_s^{\gamma}(f) \subset \mathcal{H}_s^{\nu}(f)$, which strictly contains $\mathcal{H}_s^{\mu}(f)$.

Lastly, if there is a $h \in \mathcal{H}_s^{\mu}(f)$ with at least s distinct roots, then we may assume $\mu = c(h)$ by Theorem 2.1.9 and Theorem 2.1.10.

Since all compositions greater than μ occurs in $\mathcal{H}_s(f)$ and

$$\ell(\mu) - s = \dim(\mathcal{H}_s^\mu(f)) < \dim(\mathcal{H}_s^\nu(f)) - 1 = \ell(\nu) - s - 1,$$

then $\ell(\mu) < \ell(\nu) - 1$ and so there is a composition γ, with $\mu < \gamma < \nu$. By Theorem 2.1.10, $\mathcal{H}_s^\gamma(f)$ is of dimension $(\ell(\gamma) - s)$ and therefore $\mathcal{H}_s^\mu(f) \subset \mathcal{H}_s^\gamma(f) \subset \mathcal{H}_s^\nu(f)$.

Thus any maximal chain will be at least of length $n-s+1$ and so any maximal chain has length $n-s+1$. Also, by the above argument any stratum of dimension $m \geq 0$ covers a stratum of dimension $m-1$, thus its rank must be $m+1$. □

Next up, we will show that hyperbolic lattices are atomic.

Lemma 3.1.4. *If $s \geq 2$ and $m > 0$, any m-dimensional hyperbolic stratum contains at least two distinct $(m-1)$-dimensional strata.*

Proof. By Proposition 3.1.3, an m-dimensional stratum, $\mathcal{H}_s^\mu(f)$, contains an $(m-1)$-dimensional stratum $\mathcal{H}_s^\nu(f)$ which, by Proposition 2.1.5, is contractible. But since $\mathcal{H}_s^\mu(f)$ is compact its relative boundary is nonempty and not contractible so $\mathcal{H}_s^\nu(f)$ cannot be the whole relative boundary of $\mathcal{H}_s^\mu(f)$. Also, since $\mathcal{H}_s^\nu(f)$ is closed, then $\mathrm{RelBd}(\mathcal{H}_s^\mu(f)) \backslash \mathcal{H}_s^\nu(f)$ is relatively open in $\mathrm{RelBd}(\mathcal{H}_s^\mu(f))$. Thus by Corollary 2.1.11, $\mathrm{RelBd}(\mathcal{H}_s^\mu(f)) \backslash \mathcal{H}_s^\nu(f)$ is $(m-1)$-dimensional and so there must be another $(m-1)$-dimensional stratum in $\mathcal{H}_s^\mu(f)$ since, by Theorem 2.1.9, the polynomials with composition μ does not lie in the relative boundary of $\mathcal{H}_s^\mu(f)$. □

Proposition 3.1.5. *The lattice $\mathcal{L}_s(f)$ is atomic.*

Proof. By convention the empty set is the join of an empty set of atoms and an atom is naturally the join of itself. Also, by Proposition 3.1.3, the lattice is graded and a stratum's rank is its dimension plus one, so the atoms are the zero-dimensional strata.

If $\mathcal{H}_s^\mu(f)$ is an m-dimensional stratum, where $m > 0$, then by Lemma 3.1.4, there are two distinct $(m-1)$-dimensional strata, $\mathcal{H}_s^\nu(f)$ and $\mathcal{H}_s^\gamma(f)$, contained in $\mathcal{H}_s^\mu(f)$. Since $\mathcal{H}_s^\nu(f)$ and $\mathcal{H}_s^\gamma(f)$ are distinct, then by Proposition 3.1.3 any stratum that contains both strata must be at least m-dimensional. Since $\mathcal{H}_s^\mu(f)$ is m-dimensional and contains both $\mathcal{H}_s^\nu(f)$ and $\mathcal{H}_s^\gamma(f)$, it must be the join of $\mathcal{H}_s^\nu(f)$

and $\mathcal{H}_s^\gamma(f)$. By induction, both $\mathcal{H}_s^\nu(f)$ and $\mathcal{H}_s^\gamma(f)$ are joins of atoms and since $\mathcal{H}_s^\mu(f)$ is the join of $\mathcal{H}_s^\nu(f)$ and $\mathcal{H}_s^\gamma(f)$, it must also be a join of atoms. □

Similar to the property that hyperbolic lattices are atomic we will show that hyperbolic lattices are also coatomic.

Lemma 3.1.6. *If $\mathcal{H}_s(f)$ contains at least two polynomials and $s \geq 2$, then for $m < n - s - 1$ any m-dimensional hyperbolic stratum is contained in at least two distinct $(m+1)$-dimensional strata.*

Proof. Let $\mathcal{H}_s^\mu(f)$ be an m-dimensional stratum. If $m > 0$ the statement follows from Lemma 2.2.14. If $m = 0$, then $\mathcal{H}_s^\mu(f) = \{h\}$ and since $n - s > 1$, the set $\mathcal{H}_s(f)$ is at least two-dimensional. By Proposition 3.1.3, h is contained in a two-dimensional stratum, $\mathcal{H}_s^\gamma(f)$, and a one-dimensional stratum, $\mathcal{H}_s^\nu(f) \subset \mathcal{H}_s^\gamma(f)$. By Theorem 2.1.9, $\mathcal{H}_s^\nu(f)$ is in the one-dimensional relative boundary of $\mathcal{H}_s^\gamma(f)$ and h is in the relative boundary of $\mathcal{H}_s^\nu(f)$.

According to Corollary 2.1.11, $\mathcal{H}_s^\gamma(f)$ is the closure of its relative interior which by Lemma 2.3.3 is connected. Thus, starting from h, we can traverse its boundary clockwise or counter-clockwise. But since h is one of the relative boundary points of $\mathcal{H}_s^\nu(f)$, at most one of the directions consists immediately of polynomials whose composition is ν. Thus there must be some other one-dimensional stratum for which h is a boundary point.

Lastly, if $m = -1$, then $\mathcal{H}_s^\mu(f)$ is empty. Since $\mathcal{H}_s(f)$ is at least one-dimensional and, by Proposition 3.1.5, the lattice of strata is atomic, it must contain at least two atoms. Thus the empty stratum is contained in at least two zero-dimensional strata. □

All together we therefore have:

Theorem 3.1.7. *When $s \geq 2$ the lattice $\mathcal{L}_s(f)$ is graded, atomic and coatomic and it is ranked by* $\dim(\mathcal{H}_s^\mu(f)) + 1$.

Proof. The argument for coatomicity is analogous to the proof of atomicity, just start the induction from the $(n - s - 1)$-dimensional strata and use Lemma 3.1.6 instead of Lemma 3.1.4 for the induction step. □

3.1.1 Computing lattices

We will use Theorem 3.1.7 to construct an algorithm to compute which compositions occur in $\mathcal{H}_s(f)$ based on the compositions of length at most s that occurs. Next, we will use what we know about minimal and maximal polynomials to bound which compositions can occur in a generic slice and thus improve the computation of which compositions occur in $\mathcal{H}_s(f)$.

Algorithm 3.1.8. *Let $s \geq 2$, $\mathcal{H}_s(f)$ be $(n-s)$-dimensional and let U denote the set of compositions in $\mathcal{H}_s(f)$ of length at most s.*

Step 1: Compute the join of every pair of compositions in U:

$$V := \{\mu \vee \nu | \mu, \nu \in U \ \& \ \mu \neq \nu\}.$$

Step 2: Compute the upward closure of V:

$$\overline{V} := \{\gamma | \exists \ \nu \in V \ \textit{with} \ \nu \leq \gamma\}.$$

Then $U \cup \overline{V}$ is the set of all compositions occurring in $\mathcal{H}_s(f)$.

Proof. Let $\gamma \in W$, then $\gamma \geq \mu \vee \nu$ for some $\mu, \nu \in U$. Thus both μ and ν occur in $\mathcal{H}_s(f)$ and so $\mathcal{H}_s^\gamma(f)$ contain at least two polynomials. So by Theorem 2.1.10, $\mathcal{H}_s^\gamma(f)$ is maximal dimensional and therefore there is a polynomial with composition γ. Thus all compositions computed in the algorithm occurs in $\mathcal{H}_s(f)$.

Suppose a composition μ, with $\ell(\mu) > s$, occurs in $\mathcal{H}_s(f)$. Then by Theorem 2.1.10, $\mathcal{H}_s^\mu(f)$ is at least one-dimensional and by Theorem 3.1.7, $\mathcal{H}_s^\mu(f)$ is the join of at least two distinct atoms $\mathcal{H}_s^\nu(f) = \{h\}$ and $\mathcal{H}_s^\gamma(f) = \{g\}$. We may assume ν and γ are the compositions of h and g respectively. Then $\nu \vee \gamma \in V$ and $\nu \vee \gamma \leq \mu$, thus $\mu \in W$ so it was not left out by the algorithm. □

Remark 3.1.9. *We can see from Theorem 2.1.10 and Proposition 3.1.3 that if $\mathcal{H}_s(f)$ is generic, step 1 in Algorithm 3.1.8 can be skipped and one can just compute the upward closure of the set U to compute all the compositions occurring in $\mathcal{H}_s(f)$.*

Remark 3.1.10. *Step 1 in Algorithm 3.1.8 can be accomplished using the method described after Lemma 3.1.1. That is, the join of μ and ν can be computed by first constructing the set*

$$M = \{\mu_1, \mu_1 + \mu_2, ..., n, \nu_1, \nu_1 + \nu_2, ..., n\}.$$

Next, let $m_1, ..., m_k$ be distinct, increasingly ordered and such that $\{m_1, ..., m_k\} = M$. Then the join of μ and ν is the composition

$$(m_1, m_2 - m_1, m_3 - m_2, ..., m_l - m_{l-1}).$$

However, compositions and our partial order are both implemented in Sage ([30]), so the algorithm can easily be implemented there.

We finish by discussing how to find the compositions of length at most s that occurs in $\mathcal{H}_s(f)$. At first glance this requires one to check which of the sets $\mathcal{V}_s^{\mu}(f) \cap \mathcal{W}_s$ are nonempty for all compositions μ of length s, or equivalently, check which compositions the points in $\mathcal{V}_s^{\lambda}(f)$ gives rise to for all partitions λ of length s. However for generic slices we can improve this approach.

Proposition 3.1.11. *Suppose μ and ν are the minimal and maximal compositions, respectively, of some generic slice $\mathcal{H}_s(f)$. If $\ell(\gamma) = s$ and γ occurs in $\mathcal{H}_s(f)$, then $\gamma_i \leq \max\{\mu_i, \nu_i\}$ for all $i \in [s]$.*

Proof. By Remark 3.1.9 and Lemma 2.2.7 if $\gamma \neq \nu$ there is a composition of the form $\gamma' = (\gamma_1, ..., \gamma_{s-j-1}, \gamma_{s-j} - 1, 1, \gamma_{s-j+1}, ..., \gamma_s)$ in $\mathcal{H}_s(f)$ such that γ is the minimal composition in the stratum $\mathcal{H}_s^{\gamma'}(f)$. Thus by Theorem 2.2.3 j is in the set $\{1, 3, 5, ...\}$. Similarly, the maximal composition in $\mathcal{H}_s^{\gamma'}(f)$ must be of the form $(\gamma'_1, ..., \gamma'_{s-i+1}, \gamma'_{s-i} + \gamma'_{s-i-1}, \gamma'_{s-i-2}, ..., \gamma'_s)$ with $i \in \{2, 4, 6, ...\}$ and thus has a strictly larger $(s-i)^{th}$ part than γ.

That is, for every composition other than the maximal composition, there is a composition with a strictly larger $(s-i)^t h$ part for $i \in \{2, 4, 6, ...\}$. So unless ν has the maximal $(s-i)^{th}$ part, for every $i \in \{2, 4, 6, ...\}$, then there are infinitely many compositions in $\mathcal{H}_s(f)$ which is impossible. Thus ν has the maximal $(s-i)^{th}$ part for every $i \in \{2, 4, 6, ...\}$. The argument for minimal compositions is analogous so we conclude that $\gamma_i \leq \max\{\mu_i, \nu_i\}$ for all i. $\square$

Remark 3.1.12. *When $s = 3$, and $\mathcal{H}_s(f)$ is generic, it can be shown that the compositions in Proposition 3.1.11 will all occur in $\mathcal{H}_s(f)$. This happens since when $s = 3$ and $\mathcal{H}_s(f)$ is generic, there is only one choice for a minimal composition of any stratum. However, when $s = 4$, there are more choices of minimal and maximal compositions and one can find a counterexample.*

Thus if $\mathcal{H}_s(f)$ is generic, we can start by looking for the minimal and maximal compositions. That is we can check which of the sets $\mathcal{V}_s^{\mu}(f) \cap \mathcal{W}_s$ is nonempty for all alternate odd and alternate even compositions, μ, of length s. Next, we check which of the sets $\mathcal{V}_s^{\gamma}(f) \cap \mathcal{W}_s$ are nonempty for all compositions, γ, of length s, whose parts are bounded by the parts of the minimal and maximal composition. For non-generic slices we can do something similar by considering a generic slice "close by", see Proposition 3.2.8 in the following subchapter.

3.2 Sphericity of the dual

In this subchapter we focus on the dual of $\mathcal{L}_s(f)$ and we will first see that the boundary complex of this lattice is generically a simplicial complex. This will make it easier to show that the boundary complex of the dual lattice is shellable in the generic case. This implies that the complex is generically a combinatorial sphere and we can in particular make use of the g-theorem for spheres to bound the number of i-dimensional strata. Next we show that the boundary complex of the dual of a non-generic hyperbolic lattice is like a weld of some generic complex, that is, the boundary complex of a dual generic lattice is like a subdivision of the boundary complex of some dual non-generic lattice. Thus we can use the Upper Bound Theorem for spheres to give an upper bound on the number of i-dimensional strata in general.

We denote the dual lattice of $\mathcal{L}_s(f)$ by $\mathcal{L}_s^{\Delta}(f)$ and the boundary complex of $\mathcal{L}_s^{\Delta}(f)$ by $\partial(\mathcal{L}_s^{\Delta}(f)) := \mathcal{L}_s^{\Delta}(f)\backslash\emptyset$. As we saw in the beginning of the previous subchapter, $\mathcal{L}_s(f)$ is a simplex when $s \leq 1$ and therefore the dual lattice is also a simplex. Thus for this subchapter we will restrict to the cases when $s \geq 2$.

Lemma 3.2.1. *The boundary complex of $\mathcal{L}_s^{\Delta}(f)$ is generically a simplicial complex of dimension $(n-s-1)$.*

Proof. Since $\mathcal{H}_s(f)$ is generic, then by Remark 3.1.9, $\mathcal{L}_s(f)\backslash\emptyset$ is the upward closure in $\mathcal{C}(n)$ of the compositions of length s that occurs in $\mathcal{H}_s(f)$. So by Lemma 3.1.1, $\partial(\mathcal{L}_s^{\Delta}(f))$ is a collection of simplices and by Proposition 3.1.3 these simplices have dimension $n-s-1$. $\square$

Remark 3.2.2. *The restriction to the generic case in Lemma 3.2.1 is sufficient, but not necessary. That is, there are examples of non-generic slices where the boundary complex, $\partial(\mathcal{L}_s^{\Delta}(f))$, is a simplicial complex and examples where it is not (see for instance Example 2.2.4 in the previous chapter and Example 3.3.5 in the next subchapter). However, the same kind of argument as in Lemma 3.2.1 can be used to show that if we remove the empty set and the zero-dimensional strata from $\mathcal{L}_s(f)$, then the dual poset is a simplicial complex even for non-generic cases.*

We will consider a generic slice $\mathcal{H}_s(f)$ and construct a shelling of the pure simplicial complex $\partial(\mathcal{L}_s^{\Delta}(f))$. To do so we shall use Theorem 2.2.3 to define a partial order on the zero-dimensional strata of $\mathcal{H}_s(f)$. This will allow us to imitate dual line shellings of polytopes and give a shelling order for the dual lattice. So let $V_1, \ldots, V_k$ be the zero-dimensional strata of $\mathcal{H}_s(f)$, then $V_1, ..., V_k$ are also the facets of $\partial(\mathcal{L}_s^{\Delta}(f))$.

Definition 3.2.3. *Let "$\leq_p$" denote the partial order on $V_1, \ldots, V_k$ that is defined by $V_i \leq V_j$ if there are indices $i = m_1, m_2, ..., m_d = j$ such that for any $r \in [d-1]$, V_{m_r} and $V_{m_{r+1}}$ are contained in a one-dimensional stratum for which the polynomial in V_{m_r} is minimal and the polynomial in $V_{m_{r+1}}$ is maximal.*

Lemma 3.2.4. *Let S be a stratum of $\mathcal{H}_s(f)$ and let $h \in V_j$ be the minimal (resp. maximal) polynomial of the stratum S. If $V_i \subseteq S$, then $V_j \leq_p V_i$ (resp. $V_j \geq_p V_i$).*

Proof. Since h is minimal in S, then either $V_i = V_j$ or S contains a one-dimensional stratum, S_1, for which $g \in V_i$ is maximal. Otherwise g would be minimal in S by Lemma 2.2.12. By Theorem 2.2.3, the stratum S_1 also contains a minimal polynomial $q \in V_m$ for some m and therefore $V_m <_p V_i$.

And by the same argument as above, either $q = h$ or there must be a one-dimensional stratum $S_2 \subseteq S$, for which q is maximal. We see that by continuing this process we must eventually end up at h and so $V_j \leq_p V_i$. The argument for maximal polynomials is analogous. □

Definition 3.2.5. *Let $\leq$ and $\leq^*$ be partial orders on a set P. Then $\leq$ is* ***finer*** *than $\leq^*$ if for any $a, b \in P$ with $a \leq^* b$ we have $a \leq b$.*

Theorem 3.2.6. *Let $\mathcal{H}_s(f)$ be generic and let $\leq$ be a total order on the zero-dimensional strata $\{V_1, \ldots, V_k\}$. If $\leq$ is finer than $\leq_p$, then the total order (and its reverse) induces a shelling of $\partial(\mathcal{L}_s^\Delta(f))$.*

Proof. We may assume by relabelling that $V_1 < \cdots < V_k$. As we are shelling the boundary complex of the dual lattice we will first rephrase the definition of a shelling to suit our setting:

$V_1, \ldots, V_k$ is a shelling of $\partial(\mathcal{L}_s^\Delta(F))$ if for any $i \in \{2, \ldots, k\}$ and any $j \in [i-1]$, there is an $r \in [i-1]$ such that the minimal stratum containing both V_i and V_j also contains a one-dimensional stratum, R, which contains both V_i and V_r. Note that this guarantees that in $\partial(\mathcal{L}_s^\Delta(f))$, the intersection of the facets V_i and V_j is contained in the ridge R, which again is contained in the facets V_i and V_r.

So let S be the smallest stratum containing both V_i and V_j. The polynomial $h \in V_i$ cannot be the minimal polynomial of S, otherwise $V_i <_p V_j$ by Lemma 3.2.4, which would contradict $\leq$ being finer than $\leq_p$. So by Lemma 2.2.12, h is maximal for a one-dimensional stratum $R \subset S$. Let $g \in V_r$ be the minimal polynomial of R. then $V_r <_p V_i$ and therefore $V_r < V_i$ since $\leq$ refines $\leq_p$ and so $r \in [i-1]$. □

Corollary 3.2.7. *The boundary complex of $\mathcal{L}_s^\Delta(f)$ is generically a combinatorial $(n-s-1)$-sphere.*

Proof. Any ridge of $\partial(\mathcal{L}_s^\Delta(f))$ corresponds to a one-dimensional stratum $\mathcal{H}_s^\mu(f) \in \mathcal{L}_s(f)$. As we have seen before, since $s \geq 2$ $\mathcal{H}_s^\mu(f)$ is compact and thus there are exactly two distinct zero-dimensional strata in $\mathcal{H}_s^\mu(f)$ making up its relative interior. That is, $\partial(\mathcal{L}_s^\Delta(f))$ is an $(n-s-1)$-dimensional shellable simplicial complex where every ridge is contained in exactly two facets. So by Proposition 1.2.12, $\partial(\mathcal{L}_s^\Delta(f))$ is a combinatorial $(n-s-1)$-sphere. □

Next let us look at how the non-generic lattices differ from generic lattices. The main difficulty lies in the fact that $\partial(\mathcal{L}_s^\Delta(f))$ is not always a simplicial complex and thus it is not so easy to see if refinements of $\leq_p$ will induce a shelling of $\partial(\mathcal{L}_s^\Delta(f))$. An even bigger problem is that since $\partial(\mathcal{L}_s^\Delta(f))$ is not always a simplicial complex (for instance the pyramid in Example 3.3.5), it is not clear if $\partial(\mathcal{L}_s^\Delta(f))$ is even a polytope complex in general. However we can use Theorem 2.2.3 to compare non-generic slices to generic slices "close by".

Proposition 3.2.8. *If f has no repeated roots and $n-s>0$, then there is a $\delta>0$ such that for all ϵ with $0<\epsilon<\delta$,*

1. *$\mathcal{H}_s(f+\epsilon t^{n-s})$ is nonempty and generic,*
2. *$\lambda \in \mathcal{L}_s(f+\epsilon t^{n-s}) \implies \lambda \geq \mu$ for some $\mu \in \mathcal{L}_s(f)$,*
3. *$\mu \in \mathcal{L}_s(f)$ & $\ell(\mu) \geq s \implies \mu \in \mathcal{L}_s(f+\epsilon t^{n-s})$ and*
4. *for any $\mu \in \mathcal{L}_s(f)$ with $\ell(\mu)<s$, there is a $\lambda \in \mathcal{L}_s(f+\epsilon t^{n-s})$ of length s such that $\lambda \geq \mu$ and λ is incomparable with all other compositions of length at most s in $\mathcal{L}_s(f)$.*

Proof. Note that as f has no repeated roots and $n-s>0$, then $\dim(\mathcal{H}_s(f)) \geq 1$ by Theorem 2.1.9. The first statement follows from Lemma 2.2.16. For the second statement, let $\lambda \in \mathcal{L}_s(f+\epsilon t^{n-s})$ and let h be the minimal polynomial of $\mathcal{H}_{s-1}^{\lambda}(f)$. By Theorem 2.2.3, h has at most $s-1$ distinct roots. Thus since $\mathcal{H}_s(f+\epsilon' t^{n-s})$ is nonempty and generic for any $0<\epsilon' \leq \epsilon$, h does not lie in $\mathcal{H}_s(f+\epsilon' t^{n-s})$. So we either have $h \in \mathcal{H}_s(f)$ and $c(h)<\lambda$ or $h \notin \mathcal{H}_s(f)$ and $\lambda \in \mathcal{L}_s(f)$.

For the third statement, let q be a polynomial in $\mathcal{H}_s(f)$ with at least s distinct roots and composition μ. By Theorem 2.1.10, $\mathcal{H}_{s-1}^{\mu}(f)$ is of dimension $\ell(\mu)-s+1>0$. By Theorem 2.2.3, $\mathcal{H}_{s-1}^{\mu}(f)$ has a maximal polynomial, g, with at most $s-1$ distinct roots. Thus the first free coefficient of $g \in \mathcal{H}_{s-1}(f)$ is at least as large as $f_s+\delta$. Since $\mathcal{H}_{s-1}^{\mu}(f)$ is contractible the intersection of $\mathcal{H}_{s-1}^{\mu}(f)$ and either of the hyperplanes $A_{f_s+\delta} \subset \mathbb{R}^{n-s+1}$ and A_{f_s} is nonempty, thus so is the intersection with the hyperplane $A_{f_s+\epsilon}$. So $\mathcal{H}_s^{\mu}(f+\epsilon t^{n-s})$ is nonempty and contains no polynomial with strictly less than s distinct roots. Thus $\mathcal{H}_s^{\mu}(f+\epsilon t^{n-s})$ contains a polynomial with composition μ.

For the last statement, suppose $p \in \mathcal{H}_s(f)$ is a polynomial with at most $s-1$ distinct roots. By Theorem 2.2.3, p is neither the minimal nor the maximal polynomial of $\mathcal{H}_{s-1}(f)$. Therefore $s-1>1$ by the main theorem in [23] and so by Lemma 2.2.12, there is a one-dimensional stratum $\mathcal{H}_{s-1}^{\lambda}(f)$ for which p is the minimal polynomial. Similar to the argument above, $\mathcal{H}_s^{\lambda}(f+\epsilon t^{n-s})$ must therefore contain a polynomial with composition λ. Also, by Theorem 2.1.10, the length of λ is s since $\mathcal{H}_s^{\lambda}(f+\epsilon t^{n-s})$ is generic and zero-dimensional. Lastly, by Theorem 2.2.3, p is the unique minimal polynomial of $\mathcal{H}_{s-1}^{\lambda}(f)$, thus $c(p)$ is the only composition in $\mathcal{L}_s(f)$ with $c(p) \leq \lambda$. $\square$

Remark 3.2.9. *We see in Proposition 3.2.8 that a non-generic slice $\mathcal{H}_s(f)$ can be obtained from some generic slice $\mathcal{H}_s(h)$ by "contracting" some of the strata of $\mathcal{H}_s(f)$ to points. This corresponds to merging some of the faces of $\partial(\mathcal{L}_s^\Delta(h))$. In other words if $\partial(\mathcal{L}_s^\Delta(f))$ is a polytopal complex, then the simplicial complex $\partial(\mathcal{L}_s^\Delta(h))$ is a simplicial subdivision of $\partial(\mathcal{L}_s^\Delta(f))$. Thus, in particular, whenever $\partial(\mathcal{L}_s^\Delta(f))$ is a polytopal complex it is also a combinatorial sphere.*

Conjecture 3.2.10. *The boundary complex $\partial(\mathcal{L}_s^\Delta(f))$ is a polytope complex and thus by Remark 3.2.9, a combinatorial sphere.*

Proving the conjecture above includes proving that the upward closure of a zero-dimensional stratum is a polytopal lattice. It is clearly a lattice, and as we saw in Remark 3.2.2, the boundary complex of the dual lattice of the upward closure of a zero-dimensional stratum is a simplicial complex even in the general case and thus realisable. This is not enough to prove Conjecture 3.2.10, however it may be a good place to start.

3.2.1 Bounding f-vectors

Due to Corollary 3.2.7, we can make use of some previously established results for simplicial spheres to say something about the number of i-dimensional strata in $L_s(f)$. Namely we get a "g-theorem" for generic slices and an "Upper Bound Theorem" for general slices.

Definition 3.2.11. *Let $d = \dim(\mathcal{H}_s(f))$ and for $i \in \{0, 1, \ldots, d\}$, let α_i denote the number of i-dimensional strata of $\mathcal{H}_s(f)$. Then $(\alpha_0, \ldots, \alpha_d)$ is the* ***f-vector*** *of $\mathcal{L}_s(f)$.*

Note that the dimension of a stratum is one less than its rank in $\mathcal{L}_s(f)$, so the f-vector depends only on the isomorphism type of $\mathcal{L}_s(f)$. And as we are looking at the dual poset of $\mathcal{L}_s(f)$, generically α_i is the number of $(d-i-1)$-dimensional simplices in $\mathcal{L}_s^\Delta(f)$. Thus $(\alpha_d, \ldots, \alpha_0)$ is the **f-vector** of the simplicial complex $\partial(\mathcal{L}_s^\Delta(f))$ (see Definition 8.16 in [37]). Although the f-vector has an easy interpretation, it is often more convenient to work with the **h-vector**, $(\beta_0, \ldots, \beta_d)$, of $\partial(\mathcal{L}_s^\Delta(f))$, where

$$\beta_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{i-j} \alpha_{d-j}.$$

Conversely, we can pass from the h-vector to the f-vector by using the following relations (see page 249 in [37]):

$$\alpha_{d-i} = \sum_{j=0}^{i} \binom{d-j}{i-j} \beta_j.$$

For generic hyperbolic slices the h-vector has another interpretation:

Lemma 3.2.12. *Let $\mathcal{H}_s(f)$ be generic and $(\beta_0, \ldots, \beta_d)$ be the h-vector of $\partial(\mathcal{L}_s^{\Delta}(f))$. Then β_i is the number of polynomials in $\mathcal{H}_s(f)$ that are maximal in exactly i one-dimensional strata. Also, β_i is the number of polynomials in $\mathcal{H}_s(f)$ that are minimal in exactly i one-dimensional strata.*

Proof. Let again $\leq$ be a total order on the zero-dimensional strata $\{V_1, \ldots, V_k\}$ that is finer than $\leq_p$ and assume that $V_1 < \cdots < V_k$, then by Theorem 3.2.6, $V_1, \ldots, V_k$ is a shelling of $\partial(\mathcal{L}_s^{\Delta}(F))$. We will denote by U_j the set of vertices of the facet V_j of the simplicial complex $\partial(\mathcal{L}_s^{\Delta}(F))$. Also we denote by $R_j \subseteq U_j$ the **restriction** of V_j, which is defined as the subset of vertices of V_j, such that for every $v \in R_j$ the set $U_j \setminus \{v\}$ lies in V_m for some $m < j$. Then from the first part of section 8.3 in [37] we have that β_i is equal to

$$|\{j : |R_j| = i\}|.$$

Let $v \in \mathbb{R}_j$ and let $m < j$, such that $U_j \setminus \{v\} \subset V_m$. Then V_m and V_j lie in a one-dimensional stratum E of $\mathcal{H}_s(f)$ and since $V_m < V_j$, then $h \in V_j$ is maximal in E. Conversely, for any one-dimensional stratum E' of $\mathcal{H}_s(f)$ such that the polynomial in V_j is maximal and the polynomial in V_r is minimal in E', we have that $V_r < V_j$ and $U_j \setminus \{v\} \subset V_r$ for some $v \in U_j$.

Thus $|R_j|$ counts the number of one-dimensional strata of $\mathcal{H}_s(f)$ for which $h \in V_j$ is maximal. And so β_i counts the number of zero-dimensional strata that are maximal in exactly i one-dimensional strata. If we now take the reverse order (which by Theorem 3.2.6 is also a shelling), then by an analogous argument, β_i is equal to the number of zero-dimensional strata that are minimal in exactly i one-dimensional strata. □

If a polynomial is maximal for i one-dimensional strata, it must be minimal for the other $n - s - i$ one-dimensional strata that contain it. Thus Lemma 3.2.12 implies that the h-vector of $\partial(\mathcal{L}_s^\Delta(F))$ is palindromic, that is, it satisfies the **Dehn-Sommerville equations**:

$$\beta_i = \beta_{n-s-i} \text{ for all } i \in \lfloor (n-s)/2 \rfloor.$$

Moreover, since $\partial(\mathcal{L}_s^\Delta(f))$ is a combinatorial sphere, we can obtain further properties of its h-vector from the g-conjecture for simplicial spheres that was recently proven in [2]. In order to state those results, we have to introduce some notation.

Firstly, for $k, i \in \mathcal{N}$ there are unique integers $a_i \geq \cdots \geq a_1 \geq 0$ such that

$$k = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_1}{1} \text{ (see page 265 in [37]).}$$

Definition 3.2.13. *We say that $g = (g_0, \ldots, g_r) \in \mathbb{N}_0^r$ is a **Macaulay (or M-) vector**, if $g_0 = 1$ and for any $i \in [r-1]$*

$$g_{i+1} \leq \binom{a_i+1}{i+1} + \binom{a_{i-1}+1}{i} + \cdots + \binom{a_1+1}{1+1},$$

where

$$g_i = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_1}{1}$$

is the unique representation of g_i introduced above.

Corollary 3.2.14 ("g-theorem")**.** *Let $\mathcal{H}_s(f)$ be generic, then the h-vector $(\beta_0, \ldots, \beta_{n-s})$ of $\partial(\mathcal{L}_s^\Delta(f))$ satisfies*

1. *$\beta_i = \beta_{n-s-i}$ for all $i \leq \lfloor (n-s)/2 \rfloor$ (Dehn-Sommerville),*
2. *$\beta_i \geq \beta_{i-1}$ for all $i \leq \lfloor (n-s)/2 \rfloor$ (lower bound) and*
3. *$(\beta_0, \beta_1 - \beta_0, \ldots, \beta_{\lfloor (n-s)/2 \rfloor} - \beta_{\lfloor (n-s)/2 \rfloor - 1})$ is a Macaulay vector.*

Since we have situations where $\mathcal{L}_s^\Delta(f)$ is isomorphic to non-simplicial polytopes where the g-theorem does not hold, we cannot extend the theorem in its entirety to the general setting. See for instance the pyramid in Example 3.3.5, where the h-vector is not palindromic. However, the third condition in Corollary 3.2.14 can be used to deduce the Upper Bound Theorem for (see Section 3 in [22]) and this is a bound that we can extend to the general case. To state the bound for the general case we need one more definition:

Definition 3.2.15. *We call the map $\phi_d : \mathbb{R} \to \mathbb{R}^d$ given by*

$$x \mapsto (x, x^2, \ldots, x^d)$$

the ***$\mathbf{d}^{th}$ moment curve****. If $x_1, \ldots, x_m \in \mathbb{R}$ are distinct, we say that the convex hull of $\phi_d(x_1), \ldots, \phi_d(x_m)$ is the d-dimensional* ***cyclic polytope*** *on m vertices.*

Corollary 3.2.16 (Upper Bound Theorem)**.** *Let $(\alpha_0, \ldots, \alpha_{n-s})$ be the f-vector of $\mathcal{L}_s(f)$. If c_i is the number of i-dimensional faces of the $(n-s)$-dimensional cyclic polytope with α_{n-s-1} vertices then*

$$\alpha_{n-s-i} \leq c_{i-1} \ \forall \ i \in [n-s].$$

Proof. By Theorem 2.1.10, we may assume $\mathcal{H}_s(f)$ is $(n-s)$-dimensional where $n-s > 0$ and we may assume f has no repeated roots. Then, by Proposition 3.2.8, there is an $\epsilon > 0$ such that $\mathcal{H}_s(f + \epsilon t^{n-s})$ is generic and whose f-vector is component-wise an upper bound on the f-vector of $\mathcal{H}_s(f)$. Thus we can reduce to the case when $\mathcal{H}_s(f)$ is generic.

When $\mathcal{H}_s(f)$ is generic we know that the h-vector of $\partial(\mathcal{L}_s^{\Delta}(f))$ is palindromic. From this, it can be shown that the upper bound on the f-vector of $\partial(\mathcal{L}_s^{\Delta}(f))$ is obtained by establishing the following upper bound on the h-vector of $\partial(\mathcal{L}_s^{\Delta}(f))$ (see chapter 8.4 in [37]):

$$\beta_i \leq \binom{\alpha_{n-s-1} - n + s - 1 + i}{i}.$$

The claim now follows directly from the Upper Bound Theorem for simplicial spheres (Corollary 5.3 in [33]) since $\partial(\mathcal{L}_s^{\Delta}(f))$ is a simplicial complex and a combinatorial sphere for generic $\mathcal{H}_s(f)$ by Lemma 3.2.1 and Corollary 3.2.7. □

Remark 3.2.17. *In [27] (Theorem 4.2) it was shown that the extremal points of the convex hull of $\mathcal{H}_s(F)$ are contained in the set of polynomials in $\mathcal{H}_s(f)$ with at most s distinct roots. Thus an upper bound on the number of zero-dimensional strata is also an upper bound on the number of extremal points. Corollary 3.2.16 together with Exercise 0.9 in [37] gives us an upper bound which improves the* *bound in [28] (Theorem 2.14 and Remark 2.15) to the following:*

$$\alpha_0 \leq \begin{cases} \binom{n-1-(n-s)/2}{(n-s)/2} + \binom{n-2-(n-s)/2}{(n-s)/2-1} & \textit{if } n-s \textit{ is even and} \\ 2\binom{n-2-(n-s-1)/2}{(n-s-1)/2} & \textit{if } n-s \textit{ is odd.} \end{cases}$$
$$= \begin{cases} \binom{(n+s)/2-1}{s-1} + \binom{(n+s)/2-2}{s-1} & \textit{if } n-s \textit{ is even and} \\ 2\binom{(n+s-3)/2}{s-1} & \textit{if } n-s \textit{ is odd.} \end{cases}$$

We have computationally verified that the bound in Remark 3.2.17 can be attained when $n \leq 8$ and $s \leq n$ and one can also use Theorem 2.1.10 to argue that the bound is attained when $s \leq 2$ and when $s \geq n-1$. Therefore we have the following conjecture:

Conjecture 3.2.18. *The bound stated in Remark 3.2.17 is sharp.*

If one were to show that for any degree n and for $s = n-2$ there is a slice $\mathcal{H}_s(f)$ where all compositions of the form $(1, ..., 1, 2, 1, ..., 1)$ occurs then one would have maximised the number of vertices of the dual lattice $\mathcal{L}_s^{\Delta}(f)$ for all $s \leq n-2$. Thus to prove the above conjecture it might be a good idea to focus on the case when $s = n-2$.

Remark 3.2.19. *As in [3], [13] and [18] we could have studied the intersection of the Weyl chamber $\mathcal{W}$ and Vandermonde varieties with positive real weights. That is, we could consider the set*

$$M = \{x \in \mathbb{R}^n | w_1 x_1^i + w_2 x_2^i + ... + w_n x_n^i = c_i \ \forall \ i \in [s]\} \cap \mathcal{W}$$

were $w_1, ..., w_n \in \mathbb{R}$ are positive and $c_1, ..., c_s \in \mathbb{R}$. If $x \in M$ is of the form $x_1 = ... = x_{v_1} < x_{\nu_1+1} = ... = x_{\nu_1+\nu_2} < ... < x_{n-\nu_l+1} = ... = x_n$, we associate to it the composition $c(x) = (\nu_1, \nu_2, ..., \nu_l)$ and for a composition μ we may define a stratum of M as $M^\mu = \{y \in M | c(y) \leq \mu\}$.

When the weights $w_1, ..., w_n$ are integers (and by extension rational numbers) then M^μ is equal to $\iota_\mu(\mathcal{V}_s^\mu(f) \cap \mathcal{W}_l)$ (see the proof of Lemma 2.1.4) for some monic hyperbolic polynomial f of degree n. However, if the weighs are irrational we do not see how to interpret the set M and so we did not consider such cases. But it can be shown that for any positive real weights Theorem 2.1.9, Theorem 2.1.10 and Theorem 2.2.3 also hold for M^μ. Thus the arguments for Theorem 3.1.7 and Corollary 3.2.7 should follow through the same way on thus hold for the poset of strata of M.

3.3 Polytopality of the hyperbolic lattice

We have seen quite a few properties of hyperbolic lattices so far and we have seen examples where hyperbolic slices looks like polytopes, except with a kind of "concave" faces. And although we do not know if hyperbolic lattices are always polytopal, we will discuss the instances when we can answer this question.

We saw just before Proposition 3.1.3 that $\mathcal{H}_0(f)$ and $\mathcal{H}_1(f)$ has a polynomial with composition μ, for any composition, μ, of n. Similarly, either $\mathcal{H}_2(f)$ contains just a polynomial with composition (n) or all other compositions occur. Thus by Lemma 3.1.1, $\mathcal{L}_0(f)$ and $\mathcal{L}_1(f)$ are are $(n-2)$-dimensional simplices and $\mathcal{L}_2(f)$ is either a point or an $(n-2)$-dimensional simplex. Thus we will assume $s > 2$ and $\dim(\mathcal{H}_s(f)) = n - s > 0$ for this subchapter.

We can push this a little bit further and consider the case when $s = 3$. Then $\mathcal{L}_2(f)$ is a simplex and by the following lemma $\mathcal{L}_3(f)$ is polytopal:

Lemma 3.3.1. *When $\mathcal{L}_{s-1}(f)$ is simplex $\mathcal{L}_s(f)$ is polytopal.*

Proof. Since $\mathcal{L}_{s-1}(f)$ is a simplex then by Theorem 3.1.7, it is a $(n-s+1)$-dimensional simplex. Thus there are $n-s+2$ polynomials in $\mathcal{H}_{s-1}(f)$ with at most $s-1$ distinct roots.

Let $A_{f_s} \subset \mathbb{R}^{n-s+1}$ be the hyperplane such that $\mathcal{H}_s(f) = \mathcal{H}_{s-1}(f) \cap A_{f_s}$. If $h, g \in \mathcal{H}_{s-1}(f)$ have at most $s-1$ distinct roots then there is a one-dimensional stratum, $\mathcal{H}^{\mu}_{s-1}(f)$, containing both of them since $\mathcal{L}_{s-1}(f)$ is a simplex. By Theorem 2.1.10 $\mathcal{H}^{\mu}_{s}(f)$ contains at most one polynomial, thus h and g cannot both lie in $\mathcal{H}_s(f)$ so $\mathcal{H}_s(f)$ contains at most one of the polynomials of $\mathcal{H}_{s-1}(f)$ with at most $s-1$ distinct roots.

Next, note that by Theorem 2.2.3, Proposition 2.1.5 and Theorem 2.1.9 A_{f_s} meets the relative interior of a stratum, $\mathcal{H}^{\nu}_{s}(f)$, of $\mathcal{H}_{s-1}(f)$ if and only if one of the open halfspaces given by A_{f_s} contains the minimal polynomial of $\mathcal{H}^{\nu}_{s-1}(f)$ and the other one contains the maximal polynomial of $\mathcal{H}^{\nu}_{s-1}(f)$. Thus $\mathcal{L}_s(f)$ is determined by how A_{f_s} separates the polynomials of $\mathcal{H}_{s-1}(f)$ with at most $s-1$ distinct roots.

Let $P \subset \mathbb{R}^{n-s+1}$ be a simplex whose face lattice is isomorphic to $\mathcal{L}_{s-1}(f)$. Since P is a simplex, for any partition of the vertices into two nonempty disjoint sets, there is a hyperplane strictly separating them. Similarly, there is a hyperplane containing any one of the vertices and strictly separating the rest. So if we choose a hyperplane $A \subset \mathbb{R}^{n-s+1}$ that separates (or strictly separates) the same number of vertices as A_{f_s} separates polynomials with at most $s-1$ distinct roots, then the face lattice of $P \cap A$ is isomorphic to $\mathcal{L}_s(f)$. $\square$

The Lemma above is also useful when $s \neq 3$ as there are cases other than $s = 3$ when $\mathcal{L}_{s-1}(f)$ is a simplex. To see this suppose $\mathcal{H}_{s-1}(f)$ is generic and contains the minimal polynomial h with composition ν. Then we know from Theorem 2.2.3, that $\mathcal{H}_s(h)$ is a point and from Theorem 2.1.10 we know that each composition strictly greater than ν occurs in $\mathcal{H}_s(f)$. By Proposition 2.1.5 and Corollary 2.1.11, the strata are contractible and the closure of their relative interior. Thus if $\epsilon > 0$ is smaller than the distance between h and any other polynomial with $s-1$ distinct roots in $\mathcal{H}_{s-1}(f)$, there is a monic polynomial p of degree $n-s$ such that $\mathcal{H}_s(h+\epsilon p)$ is generic and all compositions strictly greater than ν are all the compositions occurring in $\mathcal{H}_s(h+\epsilon p)$. Since by Lemma 3.1.1 the lattice of compositions is a simplex, we have by Lemma 3.2.1 that $\mathcal{L}_s(h+\epsilon p)$ is a simplex. So by slicing a generic slice $\mathcal{H}_{s-1}(f)$ close to the minimal polynomial we obtain a slice $\mathcal{H}_s(h+\epsilon p)$ whose poset of strata is a simplex. We can also slice $\mathcal{H}_{s-1}(f)$ close to the maximal polynomial and obtain an analogous result. Thus we have:

Lemma 3.3.2. *If $\mathcal{H}_{s-1}(f)$ is generic and $g \in \mathcal{H}_{s-1}(f)$ is "sufficiently close" (see above) to the minimal or maximal polynomial of $\mathcal{H}_{s-1}(f)$, then $\mathcal{L}_s(g)$ is a simplex.*

We have seen that hyperbolic lattices are polytopal when s is small, so let us see what happens when s is large. We know that $\mathcal{L}_n(f)$ is a point and that $\mathcal{L}_{n-1}(f)$ is either a point or the one-dimensional polytope. Also, if $\mathcal{L}_{n-2}(f)$ is not a point, then due to Theorem 2.1.10 and Theorem 3.1.7, it is a graded, atomic and coatomic lattice of rank 3. Therefore $\mathcal{L}_{n-2}(f)$ is a two-dimensional ball and thus a polytope. All told we therefore have the following cases of polytopality:

Theorem 3.3.3. *The lattice $\mathcal{L}_s(f)$ is polytopal when*

1. *either $s \leq 3$ or $s \geq n-2$,*
2. *$\mathcal{H}_s(f)$ is generic and either $s \leq 4$ or $s \geq n-3$,*
3. *$\mathcal{L}_{s-1}(f)$ is a simplex and*
4. *$\mathcal{H}_{s-1}(f)$ is generic and f is "sufficiently close" to the minimal or maximal polynomial of $\mathcal{H}_{s-1}(f)$.*

Proof. Item 1 was shown for $s \leq 3$ in the beginning of this subchapter and for $s \geq n-2$ in the preceding paragraph. The last two items was already shown in Lemma 3.3.1 and Lemma 3.3.2 so let us consider item 2.

For item 2 then due to item 1 we need to consider the cases when $s = 4$ and when $s = n-3$. For $s = 4$ note that since the atoms of $\mathcal{L}_s^{\Delta}(f)$ are labeled by compositions of length $n-1$, there can be at most $n-1$ of them. So by Corollary 3.2.7 $\partial(\mathcal{L}_s^{\Delta}(f))$ is a $(n-s-1)$-sphere with at most $n-1$ vertices. By Theorem 4.12 in Chapter 18 in [15], a d-sphere with at most $d+4$ vertices is the boundary complex of a polytope. So when $s \leq 4$, $\mathcal{L}_s^{\Delta}(f)$ and its dual $\mathcal{L}_s(f)$ is polytopal.

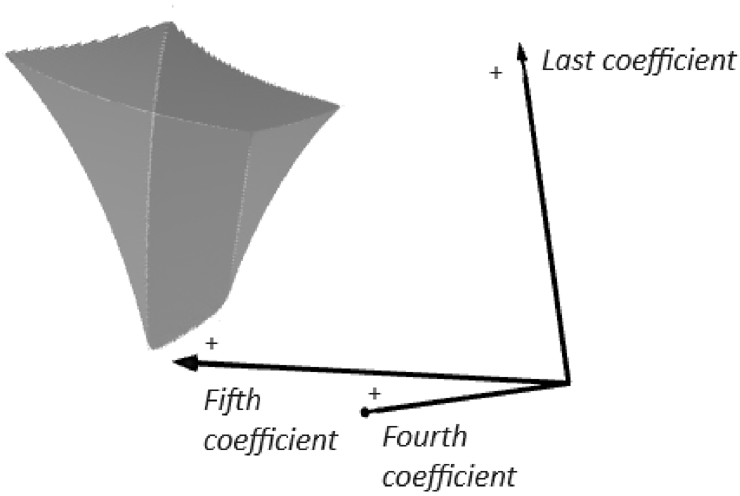
Secondly, for $s = n-3$ then due to the formulation of Steinitz Theorem in Chapter 18 in [15] (Theorem 4.3), a two-dimensional polytope complex is isomorphic to the boundary complex of a polytope if and only if it is a sphere. By Corollary 3.2.7, $\partial(\mathcal{L}_s^{\Delta}(f))$ is generically a sphere and thus $\mathcal{L}_s^{\Delta}(f)$ and its dual $\mathcal{L}_s(f)$ is generically polytopal when $s = n-3$. □

Based on the results of this chapter we have the following natural conjecture:

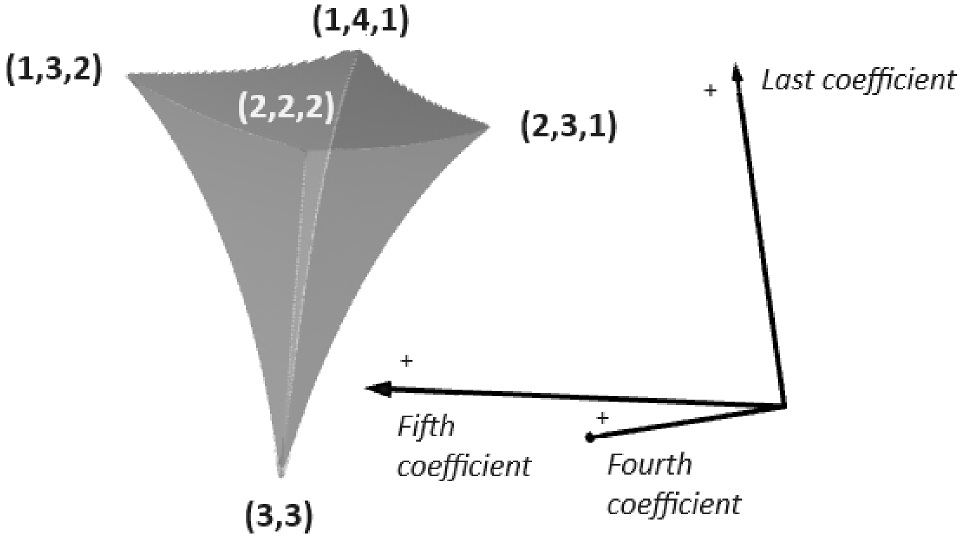
Conjecture 3.3.4. *The hyperbolic lattices are polytopal.*

To prove the conjecture above it would be helpful to have a good guess as to what polytope to identify a given hyperbolic lattice with and a natural guess would perhaps be the convex hull of $\mathcal{H}_s(f)$. However, as the following example shows, this will not always work:

Example 3.3.5. *Let $n = 6$, $s = 3$ and let $g = t^6 - 21/4t^4 + t^3 + 21/4t^2 - 1$, then the slice $\mathcal{H}_s(g)$ is generic and $\mathcal{L}_s(g)$ is isomorphic to the face lattice of a triangular prism (a toblerone):*



We can also perturb the polynomial g slightly to get a non-generic slice. So if we let $h = t^6 - 21/4t^4 + 21/4t^2 - 1$, *then* $\mathcal{H}_s(h)$ *is non-generic but full-dimensional and* $\mathcal{L}_s(h)$ *is isomorphic to the face lattice of a pyramid.*



The convex hull of $\mathcal{H}_s(g)$ *and* $\mathcal{H}_s(h)$ *are not a triangular prism or a pyramid respectively. For instance the polynomials in* $\mathcal{H}_s(h)$ *with compositions* $(1, 4, 1), (1, 3, 2), (2, 2, 2)$ *and* $(2, 3, 1)$ *does not lie in one hyperplane and therefore are not the vertices of a square but rather the vertices of two triangles in the boundary of conv(*$\mathcal{H}_s(h)$*) and so conv(*$\mathcal{H}_s(h)$*) is a triangular bipyramid (two tetrahedra glued together along a facet) instead of a pyramid.*

On the other hand, if one wants to look for counterexamples of polytopality then due to the previous corollary, the first non-generic case where we may find non-polytopal hyperbolic lattices is when $n = 7$ and $s = 4$ and the first generic case is when $n = 9$ and $s = 5$. Thus we could in theory go through all the possible subsets of compositions that may arise from a slice in degree 7 with $s = 4$ or go through all the possible subsets of compositions that may arise from a generic slice in degree 9 with $s = 5$ and check if these are polytopal. However computing all the possible hyperbolic lattices is already getting quite time consuming for the generic case when the degree is 7, so we have not done this. But the topic of "potential hyperbolic lattices" in the following chapter could be helpful to anyone wishing to give it a try.

Chapter 4

The degree principle

Having established quite a few geometric properties of hyperbolic slices and combinatorial properties of hyperbolic lattices we return to our original motivation for studying hyperbolic slices. So in this chapter we will return to Timofte's degree principle and use our results to see how much we can improve upon this principle. However, as symmetric algebraic sets are usually not lying in a single Weyl chamber, this will implicitly involve passing from strata defined by compositions to strata defined by partitions. And as partitions can be viewed as equivalence classes of compositions some of the properties of hyperbolic lattices follows.

Since the degree principle is a reduction of dimension for showing nonemptyness of real symmetric algebraic sets, we improve on this by additionally reducing the number of orbit types needed to check. The orbit types of the points in $\mathbb{R}^n$ are characterised by partitions, so we improve the degree principle by considering test sets of partitions of a given length instead of having to go through all the partitions of a given length. In the first subchapter we give a lower and an upper bound on the size of optimal test sets and in the second we outline a computational approach on how to get better test sets for real algebraic set given by symmetric polynomials in few variables and low degrees.

We start by establishing a variation of the degree principle and as we will be working a lot with partitions we start with some useful notation. We will denote by $\mathcal{P}(n)$ the set of all partitions of n and we partially order $\mathcal{P}(n)$ by the induced partial order on $\mathcal{C}(n)$:

Definition 4.0.1. *For $\lambda, \gamma \in \mathcal{P}(n)$, $\lambda \leq \gamma$ if there are permutations $\sigma \in \mathcal{S}(\ell(\lambda))$ and $\tau \in \mathcal{S}(\ell(\gamma))$, such that $\sigma(\lambda) \leq \tau(\gamma)$ as compositions.*

In other words, $\lambda \leq \gamma$ if λ can be obtained from γ by summing up some of the parts in γ and then ordering the parts decreasingly.

Similarly to the composition of a hyperbolic polynomial, we will denote by $p(f)$ the **partition** of f and we define it as the partition $p(f) := (\lambda_1, ..., \lambda_k)$, where $c(f) = (\lambda_{i_1}, ..., \lambda_{i_k})$ and $\lambda_1 \geq ... \geq \lambda_k$.

Definition 4.0.2. *Let $\mathcal{C}(n, s)$ and $\mathcal{P}(n, s)$ denote the set of all compositions and partitions, respectively, of n into s parts and let*

1. *$\mathcal{C}_{\min}(n, s) := \{\mu \in \mathcal{C}(n, s) | \mu$ is alternate odd$\}$,*
2. *$\mathcal{C}_{\max}(n, s) := \{\mu \in \mathcal{C}(n, s) | \mu$ is alternate even$\}$,*
3. *$\mathcal{P}_{\min}(n, s) := \{\lambda \in \mathcal{P}(n, s) | \lambda_{\lfloor \frac{s}{2} \rfloor + 1} = ... = \lambda_s = 1\}$ and*
4. *$\mathcal{P}_{\max}(n, s) := \{\lambda \in \mathcal{P}(n, s) | \lambda_{\lceil \frac{s}{2} \rceil + 1} = ... = \lambda_s = 1\}$.*

Note that if h is the minimal polynomial in a generic slice $\mathcal{H}_s(f)$ for some $f \in \mathcal{H}$, then by Theorem 2.2.3 for the generic case (see the end of page 24), $c(h) \in \mathcal{C}_{\min}(n, s)$ and $p(h) \in \mathcal{P}_{\min}(n, s)$. Similarly, $\mathcal{C}_{\max}(n, s)$ and $\mathcal{P}_{\max}(n, s)$ contain $c(h)$ and $p(h)$, respectively, if h is instead the maximal polynomial in the generic slice $\mathcal{H}_s(f)$. Additionally, if $g \in \mathcal{H}$ has composition $\mu \in \mathcal{C}_{\min}(n, s)$ (resp. $\mu \in \mathcal{C}_{\max}(n, s)$), then by Theorem 2.2.3, g is the minimal (resp. maximal) polynomial of $\mathcal{H}_s(g)$.

Timofte introduced the degree principle in [35], namely that symmetric polynomials of degree at most s have a common real root if and only if they have a common real root with at most s distinct coordinates. We will improve on this result by considering subsets of the set of points with at most s distinct coordinates. To this end, we introduce some new terminology:

Definition 4.0.3. *Let $P \subseteq \mathcal{P}(n, s)$. We say that P is a (n, s)-**Vandermonde covering**, if for every hyperbolic slice $\mathcal{H}_s(f)$ there is a partition $\lambda \in P$ and a polynomial $h \in \mathcal{H}_s(f)$ with $\lambda \geq p(h)$.*

Instead of considering all points in a symmetric real algebraic set with at most s distinct coordinates in the degree principle, we want to consider only some of the points with orbit types corresponding to a partition in a Vandermonde covering. Thus we define the subsets:

Definition 4.0.4. *If $P \subseteq \mathcal{P}(n,s)$ then we let*

$$O_P := \left\{ (\underbrace{x_1,\ldots,x_1}_{\lambda_1-times}, \underbrace{x_2,\ldots,x_2}_{\lambda_2-times}, \ldots, \underbrace{x_s,\ldots,x_s}_{\lambda_s-times}) \in \mathbb{R}^n \;\middle|\; \lambda \in P \right\}.$$

The following theorem motivates the name "Vandermonde covering" and can also be seen as a first step in strengthening the degree principle that we saw in Theorem 1.1.4.

Theorem 4.0.5. *Let $P \subseteq \mathcal{P}(n,s)$, then the following are equivalent:*

1. *$P \subseteq \mathcal{P}(n,s)$ is a (n,s)-Vandermonde covering.*
2. *For any $F_1,\ldots,F_k \in \mathbb{R}[x]^{\mathcal{S}(n)}$ of degree at most s we have*
$$V_{\mathbb{R}}(F_1,\ldots,F_k) \neq \emptyset \Leftrightarrow V_{\mathbb{R}}(F_1,\ldots,F_k) \cap O_P \neq \emptyset.$$
3. *For all $c \in \mathbb{R}^s$ and with $H_i = E_i + c_i \;\forall\; i \in [s]$ we have*
$$V_{\mathbb{R}}(H_1,\ldots,H_s) \neq \emptyset \Leftrightarrow V_{\mathbb{R}}(H_1,\ldots,H_s) \cap O_P \neq \emptyset.$$

Proof. (1)⇒(2): Let $P \subseteq \mathcal{P}(n,s)$ be a (n,s)-Vandermonde covering and let $a = (a_1,...,a_n) \in V_{\mathbb{R}}(F_1,\ldots,F_k)$. Consider the univariate polynomial

$$f := t^n - E_1(a)t^{n-1} + \cdots + (-1)^n E_n(a)$$

with roots $a_1,\ldots,a_n$. Then there is a partition $\lambda \in P$ and a polynomial $g \in \mathcal{H}_s(f)$ with corresponding partition $p(g) \leq \lambda$ and roots

$$b = (b_1,\ldots,b_n) \in O_P,$$

because P is a (n,s)-Vandermonde covering. Since $F_1,\ldots,F_k$ are polynomials of degree at most s, we can write

$$F_1 = G_1(E_1,\ldots,E_s),\ldots,F_k = G_k(E_1,\ldots,E_s)$$

for some $G_1,\ldots,G_k \in \mathbb{R}[y_1,\ldots,y_s]$ by Theorem 1.1.3. We have

$$0 = F_i(a) = G_i(E_1(a),\ldots,E_s(a)) = G_i(E_1(b),\ldots,E_s(b)) = F_i(b)$$

and therefore $b \in V_{\mathbb{R}}(F_1,\ldots,F_k)$.
(2)⇒(3): Is clear since $E_i + c_i$ is symmetric of degree $i \leq s$.
(3)⇒(1): Assume (3) holds. Let $f = t^n + f_1t^{n-1} + \cdots + f_n$ be a

hyperbolic polynomial with roots $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$. Then the Vandermonde variety $V_{\mathbb{R}}(E_1(x) + f_1, ..., E_s(x) + (-1)^{s-1} f_s)$ contains a by construction and is therefore nonempty. By (3) there is a $b \in V_{\mathbb{R}}(E_1(x) + f_1, ..., E_s(x) + (-1)^{s-1} f_s) \cap O_P$, that is

$$-E_1(b) = f_1, \ldots, (-1)^s E_s(b) = f_s.$$

Thus

$$g := t^n - E_1(b) t^{n-1} + \cdots + (-1)^n E_n(b)$$

is a polynomial in $\mathcal{H}_s(f)$ with corresponding partition $p(g) \leq \lambda$ for some $\lambda \in P$. □

4.1 Bounds for the degree principle

Since every generic hyperbolic slice has a unique minimal polynomial with a corresponding alternate odd composition, we have the following Vandermonde covering:

Theorem 4.1.1. *The set $\mathcal{P}_{\min}(n, s)$ is an (n, s)-Vandermonde covering of size $\left|\mathcal{P}\left(n - \lceil \frac{s}{2} \rceil, \lfloor \frac{s}{2} \rfloor\right)\right|$.*

Proof. Follows directly from Theorem 2.2.3 or from the less general version presented in [23]. □

We will see in Proposition 4.2.7 that $\mathcal{P}_{\min}(n, s)$ is not generally the smallest Vandermonde covering. In order to estimate how good this Vandermonde covering is, we will construct lower bounds on the size of Vandermonde coverings. To this end, we need some properties of the set of minimal and maximal partitions.

Lemma 4.1.2.

1. *$\mathcal{P}_{\min}(n, s) \subseteq \mathcal{P}_{\max}(n, s)$.*
2. *$|\mathcal{P}_{\min}(n, s)| = |\mathcal{P}_{\max}(n-1, s-1)|$.*
3. *Let $P \subseteq \mathcal{P}(n, s)$ be a (n, s)-Vandermonde covering, then for any $\lambda \in \mathcal{P}_{\max}(n, s-1)$ there must be a $\gamma \in P$ with $\gamma > \lambda$.*
4. *Every partition in $\mathcal{P}(n, s)$ covers at most*

$$\frac{\lceil \frac{s-1}{2} \rceil^2 + \lceil \frac{s-1}{2} \rceil}{2} = \frac{\lceil \frac{s-1}{2} \rceil \lceil \frac{s+1}{2} \rceil}{2}$$

partitions in $\mathcal{P}_{\max}(n, s-1)$.

Proof. 1. This is clear from the definition.

2. Follows from the bijection

$$\begin{array}{rcc} \phi: \mathcal{C}_{\max}(n-1,s-1) & \longrightarrow & \mathcal{C}_{\min}(n,s) \\ (\nu_1,\ldots,\nu_{s-1}) & \longmapsto & (\nu_1,\ldots,\nu_{s-1},1) \end{array}.$$

3. Let $\nu \in \mathcal{C}_{\max}(n,s-1)$ and let f be a polynomial whose composition is ν. By the argument preceding Lemma 3.3.2 there is an $\epsilon > 0$ and a monic polynomial, p, of degree $n-s$, such that the zero-dimensional strata of $\mathcal{H}_s(f-\epsilon p)$ corresponds to all compositions that cover ν. Since P is a (n,s)-Vandermonde covering, there has to be a $\gamma \in P$ such that $\gamma \geq p(g)$ for some $g \in \mathcal{H}_s(f-\epsilon p)$ and so we have $\gamma \geq p(g) > p(f)$.

4. In order for $\lambda \in \mathcal{P}(n,s)$ to cover a partition in $\mathcal{P}_{\max}(n,s-1)$ there can be at most $\left\lceil \frac{s-1}{2} \right\rceil + 1$ entries different from 1 in λ. Additionally, all partitions in $\mathcal{P}_{\max}(n,s-1)$ that are covered by λ can be obtained by adding two of the first $\left\lceil \frac{s-1}{2} \right\rceil + 1$ entries in λ and reordering. So λ covers at most

$$\binom{\left\lceil \frac{s-1}{2} \right\rceil + 1}{2} = \frac{\left\lceil \frac{s-1}{2} \right\rceil^2 + \left\lceil \frac{s-1}{2} \right\rceil}{2} = \frac{\left\lceil \frac{s-1}{2} \right\rceil \left\lceil \frac{s+1}{2} \right\rceil}{2}$$

partitions in $\mathcal{P}_{\max}(n,s-1)$.

$\square$

Note that the inclusion in item 1 above is an equality when s is even. From this lemma, we get the following lower bound on the size of any Vandermonde covering:

Proposition 4.1.3. *Let $P \subseteq \mathcal{P}(n,s)$ be a (n,s)-Vandermonde covering, then*

$$|P| \geq \left\lceil \frac{2\left|\mathcal{P}\left(n+1-\left\lceil \frac{s}{2} \right\rceil, \left\lfloor \frac{s}{2} \right\rfloor\right)\right|}{\left\lceil \frac{s-1}{2} \right\rceil \left\lceil \frac{s+1}{2} \right\rceil} \right\rceil.$$

Proof. By Lemma 4.1.2 (3), every partition in $\mathcal{P}_{\max}(n,s-1)$ is covered by a partition in P. Any partition in $\mathcal{P}(n,s)$ covers at most

$$\frac{\left\lceil \frac{s-1}{2} \right\rceil \left\lceil \frac{s+1}{2} \right\rceil}{2}$$

partitions in $\mathcal{P}_{\max}(n, s-1)$ by Lemma 4.1.2 (4). To have at least one partition from every generic slice, then by the pigeonhole principle we need at least

$$\left\lceil \frac{2|\mathcal{P}_{\max}(n, s-1)|}{\left\lceil \frac{s-1}{2} \right\rceil \left\lceil \frac{s+1}{2} \right\rceil} \right\rceil$$

partitions in P. By Lemma 4.1.2 (2) and Theorem 4.1.1 this equals

$$\left\lceil \frac{2|\mathcal{P}_{\min}(n+1, s)|}{\left\lceil \frac{s-1}{2} \right\rceil \left\lceil \frac{s+1}{2} \right\rceil} \right\rceil = \left\lceil \frac{2\left|\mathcal{P}\left(n+1-\left\lceil \frac{s}{2} \right\rceil, \left\lfloor \frac{s}{2} \right\rfloor\right)\right|}{\left\lceil \frac{s-1}{2} \right\rceil \left\lceil \frac{s+1}{2} \right\rceil} \right\rceil.$$

□

We can improve this lower bound by using a more detailed recursive argument.

Theorem 4.1.4. *Let $P \subseteq \mathcal{P}(n, s)$ be a (n, s)-Vandermonde covering. Then*

$$|P| \geq \sum_{i=0}^{\left\lfloor \frac{s}{2} \right\rfloor} B_i,$$

where $B_0 := 0$, $B_1 := 1$ and

$$B_i := \left\lceil 2\frac{|\mathcal{P}(n-s+1, i)| - iB_{i-1} - B_{i-2}}{i^2+i} \right\rceil$$

for all $i \in \left\{2, \ldots, \left\lfloor \frac{s}{2} \right\rfloor\right\}$.

Proof. Denote by

$$P_i := \left\{\lambda \in \mathcal{P}_{\max}(n, s-1) \;\middle|\; |\{j \in [n] \mid \lambda_j \neq 1\}| = i\right\}$$

the partitions in $\mathcal{P}_{\max}(n, s-1)$ that have exactly i entries different from 1. Note the following:

1. $|P_i| = |\mathcal{P}(n-s+1, i)|$.
2. Every partition in $\mathcal{P}(n, s)$ covers at most $\binom{i+1}{2} = \frac{i^2+i}{2}$ partitions in P_i by a similar argument as in the proof of Lemma 4.1.2 (4).
3. Any partition in $\mathcal{P}(n, s)$ that covers a partition in P_i, covers at most $i+1$ partitions in P_{i+1} and at most one partition in P_{i+2}.

In order to cover all partitions in $\mathcal{P}_{\max}(n, s-1)$, we have to cover all partitions in P_i for all $i \in \{1, 2, ..., \left\lceil \frac{s-1}{2} \right\rceil = \left\lfloor \frac{s}{2} \right\rfloor\}$. Combining (1), (2) and (3) we get recursively: We need $B_1 = 1$ partition in $\mathcal{P}(n, s)$

to cover the partition in P_1. It covers at most $(1+1)B_1$ partitions in P_2 and at most B_1 partitions in P_3 by (3). To cover the at least $P_2 - 2B_1$ remaining many partitions in P_2 we need by the pigeonhole principle and (1) at least

$$B_2 = \left\lceil \frac{|P_2| - 2B_1 - B_0}{(2^2+2)/2} \right\rceil = \left\lceil 2\frac{|\mathcal{P}(n-s+1,2)| - 2B_1 - B_0}{2^2+2} \right\rceil$$

additional partitions in $\mathcal{P}(n,s)$. Those partitions cover again at most $(2+1)B_2$ partitions in P_3 and at most B_2 partitions in P_4. To cover at least the $P_3 - 3B_2 - B_1$ remaining partitions in P_3 we need at least

$$B_3 = \left\lceil \frac{|P_3| - 3B_2 - B_1}{(3^2+3)/2} \right\rceil = \left\lceil 2\frac{|\mathcal{P}(n-s+1,3)| - 3B_2 - B_1}{3^2+3} \right\rceil$$

additional partitions in $\mathcal{P}(n,s)$. In general, if B_i denotes the number of additional partitions needed to cover the remaining partitions in P_i, then

$$B_i := \left\lceil 2\frac{|\mathcal{P}(n-s+1,i)| - iB_{i-1} - B_{i-2}}{i^2+i} \right\rceil.$$

In total, we need at least $\sum_{i=0}^{\lfloor \frac{s}{2} \rfloor} B_i$ partitions in $\mathcal{P}(n,s)$ to cover all partitions in $\mathcal{P}_{\max}(n,s-1)$. $\square$

We will see in Example 4.2.8, that this lower bound is also not generally attainable.

4.2 Algorithmic improvements

In the following we present an algorithmic approach on how to obtain smaller and possibly optimal Vandermonde coverings for small s and n. To this end, we try to characterise if a set of compositions $S \subset \mathcal{C}(n,s)$ corresponds to the set of zero-dimensional strata of some hyperbolic slice.

Definition 4.2.1. *Let $S \subseteq \mathcal{C}(n,s)$ let*

$$\mathcal{L}(S) := \{\nu \mid \text{there is a } \mu \in S \text{ with } \mu \leq \nu\}$$

*denote the upward closure of S. We say that $\mathcal{L}(S)$ is a **potential hyperbolic poset**, if for every $\nu \in \mathcal{L}(S)$ there are unique $\mu_{\min}, \mu_{\max} \in S$, such that*

1. *$\mu_{\min}/\nu$ is alternate odd and*
2. *$\mu_{\max}/\nu$ is alternate even.*

*Furthermore, we say that $\mathcal{L}(S)$ is a **realisable hyperbolic poset**, if there is a hyperbolic slice $\mathcal{H}_s(f)$ where exactly the compositions in $\mathcal{L}(S)$ occurs.*

Note that Theorem 2.2.3, for the generic case (see the end of page 24), states that every realisable hyperbolic poset is a potential hyperbolic poset.

Remark 4.2.2. *We can also consider more general potential hyperbolic posets, where S is a set of compositions of n with at most s parts. For this we construct $\mathcal{L}(S)$ analogous to Algorithm 3.1.8, but for simplicity we will focus on the generic cases.*

Since the poset of compositions is a simplex, the upward closure of a composition ν in some potential hyperbolic poset $\mathcal{L}(S)$ is also a simplex. Thus, by the argument in Proposition 2.2.15, analogously to Lemma 2.2.12 potential hyperbolic posets have the following property:

Lemma 4.2.3. *Let $\mathcal{L}(S)$ be a potential hyperbolic poset and let $\nu \in S$ and $\nu < \mu$ for some $\mu \in \mathcal{L}(S)$ with $\ell(\mu) \geq s+2$. Then ν/μ is alternate odd (resp. even) if and only if ν/λ is alternate odd (resp. even) for all $\lambda \in \mathcal{L}(S)$ with $\nu \leq \lambda < \mu$.*

One can see that the arguments in the proof of shellability in subchapter 3.2 only uses the fact that the boundary complex of the dual poset is a pure simplicial complex along with Theorem 2.2.3 and Lemma 2.2.12. By the argument in the proof of Lemma 3.2.1, the dual $\mathcal{L}^\Delta(S)$ of a potential hyperbolic poset $\mathcal{L}(S)$ is a pure simplicial complex of dimension $n-s-1$. So by the defining property of potential hyperbolic posets and Lemma 4.2.3, we get the following:

Theorem 4.2.4. *Let $\mathcal{L}(S)$ be a potential hyperbolic poset, then*

1. *$\mathcal{L}^\Delta(S)$ is a shellable simplicial complex and therefore a combinatorial sphere.*
2. *the h-vector of $\mathcal{L}^\Delta(S)$ satisfies the "g-theorem", that is, the inequalities stated in Corollary 3.2.14, in particular*

$$|S| \leq \begin{cases} \binom{(n+s)/2-1}{s-1} + \binom{(n+s)/2-2}{s-1}, & \text{if } n-s \text{ is even} \\ 2\binom{(n+s-3)/2}{s-1}, & \text{if } n-s \text{ is odd.} \end{cases}$$

Since all the known combinatorial properties of generic $\mathcal{L}_s(f)\backslash\{\emptyset\}$ hold for all potential hyperbolic posets, we do not know any combinatorial way to distinguish potential from realisable hyperbolic posets. Moreover, after computationally realising all potential hyperbolic posets for $s \leq n \leq 6$, we state the following conjecture:

Conjecture 4.2.5. *Every potential hyperbolic poset is realisable.*

Since it is easy to check if the upward closure of a set of compositions in $\mathcal{C}(n, s)$ is a potential hyperbolic poset, one can compute better Vandermonde coverings for small n and s. We illustrate this with an example:

Example 4.2.6. *For $n = 6$ and $s = 4$ there are 10 compositions of 6 into 4 parts. One can check that out of the 2^{10} subsets only in 17 cases are the upward closures potential hyperbolic posets. Up to symmetry (we identify S with $\tilde{S} := \{(\mu_4, \ldots, \mu_1) \mid \mu \in S\}$) we get the 11 subsets*

$$\begin{gathered}
\{(1,1,1,3),(1,1,2,2),(1,1,3,1)\},\\
\{(1,1,3,1),(1,2,2,1),(1,3,1,1)\},\\
\{(1,1,1,3),(2,1,1,2),(2,1,2,1)\},\\
\{(1,1,2,2),(1,1,3,1),(1,2,1,2),(1,2,2,1)\},\\
\{(1,2,1,2),(1,2,2,1),(2,1,1,2),(2,1,2,1)\},\\
\{(1,1,1,3),(1,2,2,1),(2,1,1,2),(3,1,1,1)\},\\
\{(1,1,1,3),(1,1,2,2),(2,1,2,1),(2,2,1,1)\},\\
\{(1,1,1,3),(1,1,3,1),(2,1,1,2),(2,2,1,1)\},\\
\{(1,1,2,2),(1,2,1,2),(1,2,2,1),(2,1,2,1),(2,2,1,1)\},\\
\{(1,1,1,3),(1,1,2,2),(1,2,2,1),(2,2,1,1),(3,1,1,1)\} \textit{ and}\\
\{(1,1,3,1),(1,2,1,2),(1,2,2,1),(2,1,1,2),(2,2,1,1)\}.
\end{gathered}$$

From this we see that $\{(2,2,1,1)\}$ is a $(6,4)$-Vandermonde covering, which naturally is optimal in this case.

Example 4.2.6 generalises in the following way:

Proposition 4.2.7. *$\{(2,2,1,\ldots,1)\}$ is a $(n, n-2)$-Vandermonde covering.*

Proof. Suppose it is not a Vandermonde covering. Then there is a hyperbolic slice $\mathcal{H}_s(f)$ where all the compositions of length at most s

are smaller than or equal to compositions of length s with one entry equal to 3 and the other entries equal to 1. By Theorem 2.2.3 all of these compositions correspond to minimal or maximal polynomials in $\mathcal{H}_s(f)$ and therefore $\mathcal{H}_s(f)$ contains at most two zero-dimensional strata. But by Theorem 2.1.10, $\mathcal{H}_s(f)$ is two-dimensional and thus have at least three extremal points. But by Theorem 2.8 in [28], the extremal points of $\mathcal{H}_s(f)$ have at most s distinct roots. This is a contradiction to $\mathcal{H}_s(f)$ containing at most two zero-dimensional strata. $\square$

Since there are $k = \binom{n-1}{s-1}$ compositions of n into s parts, the procedure in Example 4.2.6 becomes too computationally expensive to apply directly when n and s are large since it involves considering 2^k subsets. However, we can use some weaker conditions to cut down this big set into a more manageable set and that makes it easier to apply our previous method.

Firstly, by the argument in the proof of Theorem 4.2.7 we need at least $n - s + 1$ compositions of length s to construct a potential hyperbolic poset. Secondly, by Theorem 4.2.4 we can have at most m compositions of length s, where

$$m \leq \begin{cases} \binom{(n+s)/2-1}{s-1} + \binom{(n+s)/2-2}{s-1}, & \text{if } n-s \text{ is even} \\ 2\binom{(n+s-3)/2}{s-1}, & \text{if } n-s \text{ is odd.} \end{cases}$$

Thus we (only) need to check $\sum_{i=n-s+1}^{m} \binom{k}{i}$ subsets of $\mathcal{C}(n,s)$.

Additionally we can cut down this set further since we know that a potential hyperbolic has exactly one alternate even and one alternate odd composition. This can be taken further as for every $\nu \in \mathcal{L}(S)$ we have exactly one composition in S with μ/ν alternate odd and exactly one composition in S with μ/ν alternate even. However by just using the aforementioned reductions we can compute a set containing all potential hyperbolic posets up to $s, n \leq 8$ on a standard computer with no more than a few hours running time.

Example 4.2.8. *For $n = 8$ and $s = 4$, we get from Theorem 4.1.1 that there is a Vandermonde covering with 3 partitions and from Theorem 4.1.4 we know that we need at least 1 partition. By computing all the potential hyperbolic posets we get several*

Vandermonde coverings with two elements, for instance the covering

$$\{(3,2,2,1),(4,2,1,1)\}.$$

Additionally one can show that there is no Vandermonde covering with only one partition by realizing appropriate potential hyperbolic posets.

Chapter 5

Even-hyperbolic slices

In this chapter we look at the subset of hyperbolic slices consisting of even-hyperbolic polynomials. We will see that many of the properties of the strata of hyperbolic slices transfer to the strata of even-hyperbolic slices when we consider even-hyperbolic strata as subsets of hyperbolic strata. Firstly we characterise the polynomials in a hyperbolic stratum with a minimal or maximal smallest root which gives us a way to determine which hyperbolic strata contain even-hyperbolic polynomials.

In particular we can use this to show that also in the even-hyperbolic case, any stratum has a unique polynomial with a maximal first free coefficient and a unique polynomial with a minimal first free coefficient. We will also see that just as with hyperbolic strata, the even-hyperbolic strata are connected and either empty, a single polynomial or of maximal possible dimension. These results allow us to show that the poset of even-hyperbolic strata is a lattice that can be computed combinatorially from its atoms. We also show that the boundary complex of the dual lattice is generically shellable and thus a combinatorial sphere. Lastly, we provide some improvements on the degree principle for the hyperoctahedral group.

We start by defining our stratification of even-hyperbolic slices and look at an example of a stratified even-hyperbolic slice. So for the remainder of the thesis we will let $h := t^n + h_1 t^{n-1} + \cdots + h_n \in \mathbb{R}[t]$ be a monic **even-hyperbolic** polynomial. That is, h is a monic polynomial of degree n with only real nonnegative roots. Then we will study the following sets of even-hyperbolic polynomials:

Definition 5.0.1. *Let $\mathcal{N}$ denote the set of all monic even-hyperbolic polynomials of degree n. Then for $s \in [n]$, we call the subset*

$$\mathcal{N}_s(h) := \{t^n + g_1 t^{n-1} + ... + g_n \in \mathcal{N} \mid g_i = h_i \ \forall \ i \in [s]\}$$

a (canonical) ***even-hyperbolic slice****.*

To stratify this set we will extend the notion of compositions and their partial order.

Definition 5.0.2. *A* ***signed composition****, ν, is a composition multiplied with plus or minus one. We call $|\nu| = (\nu_1, ..., \nu_l)$ its composition and $sgn(\nu) \in \{-1, 1\}$, given by $\nu = sgn(\nu)|\nu|$, its* ***signature****. The* ***parts*** *and* ***length*** *of a signed composition are the parts and length of its corresponding composition.*

We will let ν be a signed composition of n of length l for the remainder of the thesis. If $g \in \mathcal{N}$ has the distinct roots $b_1 < ... < b_l$ and respective multiplicities $m_1, ..., m_l$, then the **signed composition** of g is

$$sc(g) := \begin{cases} (m_1, ..., m_l) \text{ if } b_1 > 0 \text{ and} \\ -(m_1, ..., m_l) \text{ if } b_1 = 0. \end{cases}$$

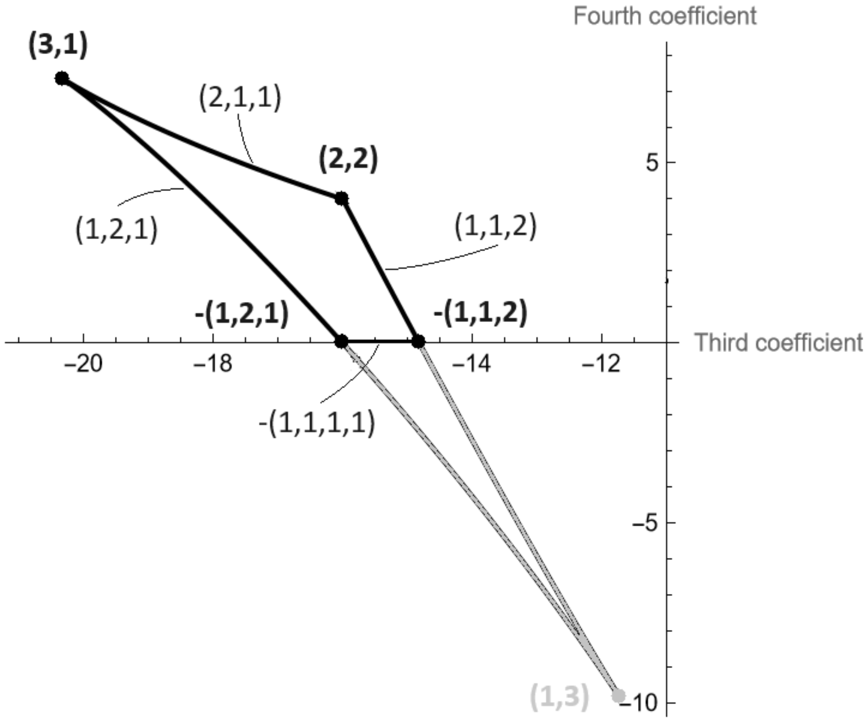
Definition 5.0.3. *If ν and λ are two signed compositions of n we let $\nu \leq \lambda$ if $sgn(\nu) \leq sgn(\lambda)$ and $|\nu| \leq |\lambda|$ with respect to the partial order on compositions.*

Thus we define

$$\mathcal{N}_s^{\nu}(h) := \{g \in \mathcal{N}_s(h) \mid sc(g) \leq \nu\}$$

to be a stratum of $\mathcal{N}_s(h)$ and we see that $\mathcal{N}_s^{\nu}(h) \subseteq \mathcal{H}_s^{|\nu|}(h) \cap \mathcal{N}$ with equality if $\text{sgn}(\nu) = 1$.

Example 5.0.4. *Let $h = t(t-2)^2(t-4) = t^4 - 8t^3 + 20t^2 - 16t$, then $\mathcal{N}_2(h)$ is a strict subset of $\mathcal{H}_2(h)$ as we can see from the following picture:*



5.1 Geometry...

Before we study the strata of even-hyperbolic slices we will explore one final topic for hyperbolic strata. Namely we characterise the polynomials $g \in \mathcal{H}_s^\mu(f)$ with the property that the smallest root of g is greater than the smallest root of any other polynomial in $\mathcal{H}_s^\mu(f)$. Since an even-hyperbolic stratum $\mathcal{N}_s^\nu(h)$ contain at most the polynomials from $\mathcal{H}_s^{|\nu|}(h)$ whose smallest root is at least zero, this allows us to quickly test if $\mathcal{H}_s^{|\nu|}(h)$ contain any even-hyperbolic polynomials or not by checking if the smallest root of g is nonnegative.

5.1.1 Extremal roots of hyperbolic strata

For a polynomial $g \in \mathcal{H}_s^\mu(f)$, whose roots of multiplicities $\mu_1, ..., \mu_l$ are $b_1 \leq ... \leq b_l$ respectively, we call b_k, where $k \in [l]$, the $\boldsymbol{k^{th}}$ **root** of g in the stratum $\mathcal{H}_s^\mu(f)$. Then a natural question to ask is which polynomials in $\mathcal{H}_s^\mu(f)$ have the minimal or maximal k^{th} root? We will first see that polynomials with a minimal or maximal k^{th} root have at most s distinct roots and then we characterise the polynomials with a minimal and maximal first root.

Lemma 5.1.1. *Let $s \geq 2$ and let μ be a composition of length $s+1$. Then if $k \in [s+1]$, the projection $\phi_k : \mathcal{V}_s^{\mu}(f) \cap \mathcal{W}_{s+1} \to \mathbb{R}$ given by*

$$a = (a_1, ..., a_{s+1}) \mapsto a_k$$

is a homeomorphism onto its image.

Proof. If $g, p \in \mathcal{H}_s^{\mu}(f)$ has the same k^{th} root $b \in \mathbb{R}$, then $g = (t-b)^{\mu_k} g^*$ and $p = (t-b)^{\mu_k} p^*$, where $g^* \in \mathcal{H}_s^{\mu^*}(p^*)$ for $\mu^* = (\mu_1, ..., \mu_{k-1}, \mu_{k+1}, ..., , \mu_l)$. But $\mathcal{H}_s^{\mu^*}(p^*)$ is at most a point by Theorem 2.1.10 and thus $g = p$ and ϕ_k is injective.

The projection is surjective onto its image and continuous since the topology on $\mathcal{V}_s^{\mu}(f) \cap \mathcal{W}_{s+1}$ is the subspace topology of the product topology on $\mathbb{R}^{s+1}$. Since $s \geq 2$, $\mathcal{V}_s^{\mu}(f) \cap \mathcal{W}_{s+1}$ is compact so any closed subset $S \subseteq \mathcal{V}_s^{\mu}(f) \cap \mathcal{W}_{s+1}$ is a compact subset of $\mathbb{R}^{s+1}$ and thus $\phi_k(S)$ is a compact subset of $\mathbb{R}$. Thus ϕ_k maps closed sets to closed sets and its inverse is therefore continuous. Thus ϕ_k is a homeomorphism onto its image. $\square$

Proposition 5.1.2. *Suppose $g \in \mathcal{H}_s^{\mu}(f)$ has a minimal or maximal k^{th} root for some $k \in [l]$, then g has at most s distinct roots.*

Proof. When $s = 0$ there are clearly no polynomials in $\mathcal{H}_s^{\mu}(f)$ with a minimal or maximal k^{th} root. When $s = 1$, $(t + f_1/n)^n \in \mathcal{H}_s^{\mu}(f)$ has the maximal k^{th} root for $k \in [\mu_1]$ and the minimal k^{th} root for $k \in \{n - \mu_l + 1 ..., n\}$. Also, for $k \in [n - \mu_l]$ we see that there cannot be any polynomial with a minimal k^{th} root and for $k \in \{\mu_1 + 1 ..., n\}$ there cannot be any polynomial with a maximal k^{th} root. So let us consider the case when $s \geq 2$ and since the statement is clear when $l = \ell(\mu) \leq s$, let $l > s$.

Suppose μ is a composition of length $s+1$, then by Lemma 5.1.1, g lies in the relative boundary of $\mathcal{H}_s^{\mu}(f)$ and thus by Theorem 2.1.9 g has at most s distinct roots. So let $\ell(\mu) > s+1$ and suppose g has more than s distinct roots. Then g has a minimal or maximal k^{th} root in $\mathcal{H}_{s+1}^{\mu}(g)$. By Theorem 2.1.10 $\mathcal{H}_{s+1}^{\mu}(g)$ is of dimension $l - s - 1 > 0$ so by induction g has $s+1$ distinct roots and $\mathcal{H}_s^{c(g)}(f)$ must be one-dimensional. But then g cannot have a minimal or maximal k^{th} root in $\mathcal{H}_s^{c(g)}(f)$ by the induction start. This is a contradiction since g has a minimal or maximal k^{th} root in $\mathcal{H}_s^{\mu}(f) \supset \mathcal{H}_s^{c(g)}(f)$. $\square$

We will proceed to characterise the polynomials in $\mathcal{H}_s^\mu(f)$ with a minimal or maximal first root.

Theorem 5.1.3.

- *If s is odd, $h \in \mathcal{H}_s^\mu(f)$ is the maximal polynomial if and only if it has the maximal first root and the minimal polynomial if and only if it has the minimal first root.*
- *If s is even, $h \in \mathcal{H}_s^\mu(f)$ is the minimal polynomial if and only if it has the maximal first root and the maximal polynomial if and only if it has the minimal first root.*

For the proof of Theorem 5.1.3 we follow the same inductive approach as for Theorem 2.2.3, but as we have already set up quite a few of the tools needed for the inductive step we mostly just have to start the induction. Thus we will do the inductive step first and afterwards cover the start of the induction, which is the one-dimensional strata. Note that we covered the cases $s = 0$ and $s = 1$ in the proof of Proposition 5.1.2 so we will let $s \geq 2$ for the remainder of this subchapter. Since $\mathcal{V}_s^\mu(f) \cap \mathcal{W}_l$ is compact when $s \geq 2$, the existence of polynomials with minimal and maximal first roots is guaranteed.

Proof. For the inductive step, let s be odd, let $\mathcal{H}_s^\mu(f)$ be at least two-dimensional and assume the theorem is true for all proper substrata of $\mathcal{H}_s^\mu(f)$. Suppose $g \in \mathcal{H}_s^\mu(f)$ is a polynomial with a minimal first root, then for any nonempty stratum $\mathcal{H}_s^\gamma(f) \subset \mathcal{H}_s^\mu(f)$, g has the minimal first root in $\mathcal{H}_s^\gamma(f)$. Thus by the induction hypothesis, g is the minimal polynomial of $\mathcal{H}_s^\gamma(f)$. By Lemma 2.2.12, g is therefore the minimal polynomial of $\mathcal{H}_s^\mu(f)$. The argument for a polynomial with a maximal first root or when s is even is analogous. $\square$

Next we let $\mathcal{H}_s^\mu(f)$ be one-dimensional and generic. Then as with Theorem 2.2.3 we use a Lagrange-based argument. We know from Lemma 5.1.1, that one of the two polynomials in the boundary of $\mathcal{H}_s^\mu(f)$ has the minimal first root and the other one has the maximal first root so we need to determine which is which. So let $x \in \mathcal{V}_s^\mu(f)$ have s distinct coordinates, then as we saw before Proposition 2.2.9 the Jacobian of $(P_1^\mu(x), ..., P_s^\mu(x))$ is invertible and so the vectors $\nabla P_1^\mu(x), ..., \nabla P_s^\mu(x)$ are linearly independent.

On the other hand, the Jacobian of $(P_1^\mu(x), ..., P_s^\mu(x), x_1)$ equals

$$J(x) := \begin{pmatrix} \mu_1 & \mu_2 & \cdots & \mu_{s+1} \\ 2\mu_1 x_1 & 2\mu_2 x_2 & \cdots & 2\mu_{s+1} x_{s+1} \\ \vdots & \vdots & \vdots & \vdots \\ s\mu_1 x_1^{s-1} & s\mu_2 x_2^{s-1} & \cdots & s\mu_{s+1} x_{s+1}^{s-1} \\ 1 & 0 & \cdots & 0 \end{pmatrix}$$

and by using cofactor expansion we see that the determinant of $J(x)$ equals the determinant of the upper right $s \times s$ matrix which is

$$c \prod_{2 \le i < j \le s+1} (x_i - x_j),$$

for some positive constant $c \in \mathbb{R}$ (see the discussion before Proposition 2.2.9). Thus $\det(J(x)) = 0$ if and only if $|\{x_2, ..., x_{s+1}\}| < s$, that is, when x_1 is a unique coordinate of x.

Let $p = \prod_{i=1}^{s+1}(t-a_i)^{\mu_i}$ and $q = \prod_{i=1}^{s+1}(t-b_i)^{\mu_i}$, where $a_1 \le ... \le a_{s+1}$ and $b_1 \le ... \le b_{s+1}$, be the two polynomials in the relative boundary of $\mathcal{H}_s^\mu(f)$. Since p and q does not have the same composition, either $a_1 \ne a_2$ or $b_1 \ne b_2$ or both. So without loss of generality we can assume $a_1 < a_2 \le ... \le a_{s+1}$. Thus the determinant of $J(a)$ is zero and there are therefore scalars $c_1, ..., c_s$ such that $\nabla L(a) = 0$, where

$$L = x_1 - \sum_{i=1}^{s} c_i P_i^\mu(x).$$

We have that

$$\nabla L(a) = (1 + \mu_1 Q(a_1), \mu_2 Q(a_2), ..., \mu_{s+1} Q(a_{s+1})) = 0,$$

so the univariate polynomial $Q(t) := -\sum_{j=1}^{s} c_j j t^{j-1}$ has the roots $a_2, ..., a_{s+1}$. Also, only $s-1$ of these roots are distinct since a_1 is not the repeated coordinate of a, thus $Q(t)$ is of degree $s-1$ and so $c_s \ne 0$. For later use we will need to determine the sign of c_s.

Lemma 5.1.4. *The sign of c_s is $(-1)^{s-1}$.*

Proof. Since $1 + \mu_1 Q(a_1) = 0$ we have $Q(a_1) = -\frac{1}{\mu_1}$. Since a_1 is strictly smaller than all the roots of Q, we have $\lim_{z \to -\infty} \operatorname{sgn}(Q(z)) = \operatorname{sgn}(Q(a_1)) = -1$. So since the polynomial Q has degree $s-1$ we have $\operatorname{sgn}(c_s) = (-1)^{s-1}$. $\square$

We can now prove Theorem 5.1.3 for one-dimensional generic strata:

Proof. As before we let $p = \prod_{i=1}^{s+1}(t-a_i)^{\mu_i}$ be a polynomial in the relative boundary of $\mathcal{H}_s^\mu(f)$ with $a_1 < a_2 \leq ... \leq a_{s+1}$. Then by Theorem 5.4 in [34], a is a local maximiser of the polynomial x_1 (resp. minimiser) if for all nonzero vectors $v \in \mathbb{R}^{s+1}$ in the kernel of $J(a)$ we have $v^t H(a) v < 0$ (resp. $v^t H(a) v > 0$), where

$$H(a) := \nabla^2 L(a) = \begin{pmatrix} \mu_1 Q'(a_1) & 0 & \cdots & & 0 \\ 0 & & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & \cdots & 0 & \mu_{s+1} Q'(a_{s+1}) \end{pmatrix}.$$

Just as in the proof of Proposition 2.2.9, $v \in \mathbb{R}^{s+1}$ lies in the kernel of $J(a)$ if and only if $v_k + v_{k+1} = 0$, where k is the repeated coordinate of a, and if all other coordinates of v are zero. So we have that

$$v^t H(a) v = \sum_j \mu_j Q'(a_j) v_j^2 = Q'(a_k)(\mu_k v_k^2 + \mu_{k+1} v_{k+1}^2)$$

is negative (resp. positive) for all $v \neq (0, ...0)$ in the kernel of $J(a)$ if and only if $Q'(a_k)$ is positive (resp. negative). The polynomial Q has no repeated roots and by Lemma 5.1.4, the sign of the leading coefficient of Q is $(-1)^{s-1}$. So by Rolle's Theorem, $Q(a_k)$ is positive (resp. negative) if $k = s+1-2m$ (resp. $k = s-2m$) for some nonnegative integer m and if s is odd. The opposite is true if s is even as in that case the sign of c_s is negative.

Thus if s is odd, then a_1 is the maximal first root in $\mathcal{H}_s^\mu(f)$ if $c(p)/\mu$ is alternate even and a_1 is the minimal first root in $\mathcal{H}_s^\mu(f)$ if $c(p)/\mu$ is alternate odd. Again, we get the opposite statement if s is even. If p has the maximal first root, then by Proposition 5.1.2, the other polynomial in the relative boundary of $\mathcal{H}_s^\mu(f)$ must have the minimal first root and vice versa. Lastly, by Theorem 2.2.3, a polynomial $g \in \mathcal{H}_s^\mu(f)$ is the maximal polynomial if $c(g)/\mu$ is alternate even and the minimal polynomial if $c(g)/\mu$ is alternate odd. □

We finish with a perturbation argument to deal with the non-generic one-dimensional strata:

Proof. As in the proof of Lemma 2.2.18 we can let $s \geq 3$ as there are no non-generic strata when $s = 2$. Let $\mathcal{H}_s^{\mu}(f)$ be a one-dimensional non-generic strata and let s be odd as the argument is analogous when s is even. And by Lemma 2.2.16, we can perturb the last $n - s + 1$ coefficients of f slightly and get a monic hyperbolic polynomial g such that $\mathcal{H}_s^{\mu}(g)$ is generic and nonempty. If q_{min} is the minimal polynomial of $\mathcal{H}_s^{\mu}(f)$ and p_{min}, the is the minimal polynomial of $\mathcal{H}_s^{\mu}(g)$, then as in the proof of Lemma 2.2.18, by minimising $\text{dist}(f, g)$ we can make the distance between the minimal polynomial $\mathcal{H}_s^{\mu}(f)$ and the minimal polynomial of $\mathcal{H}_s^{\mu}(g)$ is smaller than δ, for any $\delta > 0$.

By Lemma 2.1.4 that means that we can make sure the distance between the first root of q_{min} and the first root of p_{min} is smaller than ϵ for any $\epsilon > 0$. Similarly, we can make sure the distance between the maximal polynomial, q_{max}, of $\mathcal{H}_s^{\mu}(f)$ and the maximal polynomial, p_{max}, of $\mathcal{H}_s^{\mu}(g)$ and the distance between their first roots are smaller than δ and ϵ respectively. Thus by choosing small enough δ and ϵ we have that the first root of q_{min} is smaller than the first root of q_{max}. By Proposition 5.1.2 we then have that q_{min} has the minimal first root and q_{max} has the maximal first root. □

Remark 5.1.5. *Note that the same argument works to classify the polynomials in $\mathcal{H}_s^{\mu}(f)$ with a minimal and maximal k^{th} root. All that needs to change is that the sign of c_s is $(-1)^{s-k}$ and thus the classification depends on whether or not $s - k$ is even or odd.*

5.1.2 Dimension, connectedness and escaping strata

In this subchapter we show that even-hyperbolic strata have many properties similar to hyperbolic strata. We determine what possible dimensions the strata may have and we see that even-hyperbolic strata are also connected. Lastly, we use the Theorem 5.1.3 to characterise the "escapes" from even-hyperbolic strata.

Firstly, note that if $\text{sgn}(\nu) = 1$, then the largest possible dimension of $\mathcal{N}_s^{\nu}(h)$ is $l - s$ since that is the maximal dimension of $\mathcal{H}_s^{\nu}(h)$ and $\mathcal{N}_s^{\nu}(h) \subseteq \mathcal{H}_s^{\nu}(h)$. Similarly, if $\text{sgn}(\nu) = -1$, then the largest possible dimension of $\mathcal{N}_s^{\nu}(h)$ is $l - s - 1$ since that is the maximal dimension of $\mathcal{H}_s^{(\nu_2,\dots,\nu_l)}(g/t^{\nu_1})$, where $g \in \mathcal{N}_s^{\nu}(h)$, and we have $\mathcal{N}_s^{\nu}(h) = t^{\nu_1}\mathcal{N}_s^{(\nu_2,\dots,\nu_l)}(g/t^{\nu_1}) \subseteq t^{\nu_1}\mathcal{H}_s^{(\nu_2,\dots,\nu_l)}(g/t^{\nu_1})$.

Definition 5.1.6. *We say that the **maximal dimension** of* $\mathcal{N}_s^\nu(h)$ *is* $\max\{\ell(\nu) - s, 0\}$ *if* $sgn(\nu) = 1$ *and* $\max\{\ell(\nu) - s - 1, 0\}$ *if* $sgn(\nu) = -1$.

Theorem 5.1.7. *If* $\mathcal{N}_s^\nu(h)$ *contains a polynomial with at least* s *distinct positive roots, then it is maximal dimensional. Otherwise* $\mathcal{N}_s^\nu(h)$ *is either empty or a single polynomial.*

Proof. If $l \leq s$, then $\mathcal{H}_s^{|\nu|}(h)$ is either empty or a point by Theorem 2.1.10, thus we will let $l > s$ and we assume $\mathcal{N}_s^\nu(h) \neq \emptyset$.

Suppose $\operatorname{sgn}(\nu) = 1$ and let $g \in \mathcal{H}_s^\nu(h)$ be the polynomial with the maximal first root. By Theorem 2.1.10, $\mathcal{H}_s^\nu(h)$ is maximal dimensional and by Theorem 2.1.9 and Corollary 2.1.11, any open ball around g contains a polynomial with composition ν. If $g(0) \neq 0$, then by Lemma 2.1.4 we can make the ball small enough so that all its polynomials have only positive roots. Thus if $g(0) \neq 0$, $\mathcal{N}_s^\nu(h)$ contains a polynomial with $l > s$ distinct positive roots and there is an $\epsilon > 0$ such that

$$B_\epsilon(g) \cap \mathcal{N}_s^\nu(h) = B_\epsilon(g) \cap \mathcal{H}_s^\nu(h)$$

and thus $\dim(\mathcal{N}_s^\nu(h)) = \dim(\mathcal{H}_s^\nu(h)) = l - s$. If $g(0) = 0$, then by Theorem 5.1.3, g is the only polynomial in $\mathcal{H}_s^\nu(h)$ with all nonnegative roots and so $\mathcal{N}_s^\nu(h) = \{g\}$. Also, since g has at most s distinct roots by Proposition 5.1.2, then it has at most $s-1$ distinct positive roots.

Secondly, if $\operatorname{sgn}(\nu) = -1$ then $\mathcal{N}_s^\nu(h) = t^{\nu_1}\mathcal{N}_s^{\nu'}(p/t^{\nu_1})$ for any $p \in \mathcal{N}_s^\nu(h)$. If $\mathcal{N}_s^\nu(h)$ contains a polynomial with at least s distinct positive roots, then so does $\mathcal{N}_s^{\nu'}(p/t^{\nu_1})$ and we can argue as above that $\mathcal{N}_s^{\nu'}(p/t^{\nu_1})$ is maximal dimensional. If $\mathcal{N}_s^\nu(h)$ does not contain a polynomial with at least s distinct positive roots, then neither does $\mathcal{N}_s^{\nu'}(p/t^{\nu_1})$ and again we can argue as above that $\mathcal{N}_s^{\nu'}(g/t^{\nu_1})$ contains a single polynomial. □

Next we show how the connectedness of the even-hyperbolic strata follows quite easily from our knowledge of hyperbolic strata and we start by looking at the one-dimensional strata. Note however that in [13] they proved that the strata are generically contractible so this is not really a new result, just a different approach.

Lemma 5.1.8. *If $\mathcal{N}_s^\nu(h)$ is one-dimensional it is contractible.*

Proof. Similarly to the proof of Theorem 5.1.7 we may assume $\operatorname{sgn}(\nu) = 1$. Thus by Theorem 5.1.7, $l = s + 1$ since $\mathcal{N}_s^\nu(h)$ is one-dimensional. If $s = 0$, $\mathcal{N}_0^\nu(h)$ is homeomorphic to $\mathbb{R}_{\geq 0}$ by Lemma 2.1.4 and thus the statement is true, so let $s \geq 1$. Note that when $s \geq 1$, then $\mathcal{N}_s^\nu(h)$ is compact since $P_2^\nu(x) = -h_1$ defines a sphere and the semialgebraic set

$$\{x \in \mathbb{R}^{s+1} | - E_1^\nu(x) = h_1 \text{ and } 0 \leq x_1\} \cap \mathcal{W}_{s+1}$$

equals

$$\{(x_1^2, ..., x_{s+1}^2) \in \mathbb{R}^{s+1} | - P_2^\nu(x) = h_1\} \cap \mathcal{W}_{s+1}.$$

Since $\mathcal{N}_s^\nu(h)$ is compact it can be argued as in Lemma 5.1.1 that it is homeomorphic to the set of the polynomials' first roots, $R \subset \mathbb{R}$. Since $\mathcal{V}_s^\nu(h) \cap \mathcal{W}_{s+1}$ is contractible by Proposition 2.1.5, then the set of first roots, M, of the polynomials in $\mathcal{H}_s^\nu(h)$ is contractible. Thus the set $M \cap \mathbb{R}_{\geq 0} = R$ is contractible and thus so must $\mathcal{N}_s^\nu(h)$ be. $\square$

Theorem 5.1.9. *If $\mathcal{N}_s^\nu(h)$ is nonempty, then it is connected.*

Proof. If $s = 0$ the statement follows from Lemma 2.1.4, so let $s \geq 1$. We will do an induction on the dimension of $\mathcal{N}_s^\nu(h)$. By Theorem 5.1.7, the zero-dimensional strata are connected and by Lemma 5.1.8 the one-dimensional strata are connected.

As in the proof of Theorem 5.1.7 we may assume $\operatorname{sgn}(\nu) = 1$. So let $\mathcal{N}_s^\nu(h)$ be at least two-dimensional, then by Theorem 5.1.7 $\mathcal{N}_s^\nu(h)$ is $(l-s)$-dimensional and $\mathcal{N}_s^{-\nu}(h)$ is at most $(l-s-1)$-dimensional. So let $\{C_i\}_i$ be the nonempty connected components of $\mathcal{N}_s^\nu(h)$, then by the induction hypothesis if $\mathcal{N}_s^{-\nu}(h)$ is nonempty it is connected. So at most one of the components have a nonempty intersection with $\mathcal{N}_s^{-\nu}(h)$.

By the argument Lemma 5.1.8, $\mathcal{N}_s^\nu(h)$ is compact and thus so are its connected components. If C_i is a component that does not contain a minimal polynomial of $\mathcal{N}_s^\nu(h)$ then let $g \in C_i$ be a polynomial with minimal $(s+1)^{th}$ coefficient in C_i. If g has strictly positive roots, then by Lemma 2.1.4, there is an open ball about g containing only polynomials with strictly positive roots. So by Lemma 2.2.7, g cannot be locally minimal in C_i without being minimal in $\mathcal{H}_s^\nu(h)$.

Therefore g cannot be globally minimal in C_i which is a contradiction and so g must have a zero root. Similarly, a polynomial in C_i with a maximal $(s+1)^{th}$ coefficient in C_i must either be the maximal polynomial of $\mathcal{H}_s^\nu(h)$ or have zero as a root.

Thus there are at most two connected components C_1 and C_2 of $\mathcal{N}_s^\nu(h)$, one with a nonempty intersection with $\mathcal{N}_s^{-\nu}(h)$ and one containing both the minimal and maximal polynomial of $\mathcal{H}_s^\nu(h)$. But by Theorem 5.1.3, if both the minimal and maximal polynomial of $\mathcal{H}_s^\nu(h)$ lies in $\mathcal{N}_s^\nu(h)$, then $\mathcal{N}_s^\nu(h) = \mathcal{H}_s^\nu(h)$ and is connected by Proposition 2.1.5. $\square$

We will now use Theorem 5.1.3 to classify the polynomials in even-hyperbolic strata with a minimal or maximal first free coefficient.

Definition 5.1.10. *We say that $g \in \mathcal{N}_s^\nu(h)$ is a **minimal** (resp. **maximal**) polynomial of $\mathcal{N}_s^\nu(h)$ if $g_{s+1} \leq p_{s+1}$ (resp. $g_{s+1} \geq p_{s+1}$) for all $p \in \mathcal{N}_s^\nu(h)$.*

Note that the following theorem can be more compactly written and we will do so in the following subchapter (Lemma 5.2.4)

Theorem 5.1.11 (Escaping even-hyperbolic strata)**.** *Let $sgn(\nu) = 1$, $s \geq 1$ and $\mathcal{N}_s^\nu(h) \neq \emptyset$, then there is a unique minimal and maximal polynomial in $\mathcal{N}_s^\nu(f)$. They are determined by the following: Let λ be the signed composition of $g \in \mathcal{N}_s^\nu(h)$ and let $\gamma = (\gamma_1, ..., \gamma_k) = |\lambda|/\nu$, then*

1. *if s is odd*
   - *g is minimal if and only if $sgn(\lambda) = 1$ (resp. $sgn(\lambda) = -1$) and γ (resp. $(\gamma_2, ..., \gamma_k)$) is less than or equal to an alternate odd composition of length s and*
   - *g is maximal if and only if γ is less than or equal to an alternate even composition of length s.*
2. *if s is even*
   - *g is minimal if and only if γ is less than or equal to an alternate odd composition of length s and*
   - *g is maximal if and only if $sgn(\lambda) = 1$ (resp. $sgn(\lambda) = -1$) and γ (resp. $(\gamma_2, ..., \gamma_k)$) is less than or equal to an alternate even composition of length s*

Proof. We will assume s is odd as the proof is analogous when s is even. Note that by the argument in the proof of Lemma 5.1.8 $\mathcal{N}_s^\nu(h)$ is compact and thus it has a minimal and a maximal polynomial. Also, since $\mathcal{N}_s^\nu(h)$ is nonempty and a subset of $\mathcal{H}_s^\nu(h)$ it follows from Theorem 5.1.3 that the maximal polynomial $\mathcal{H}_s^\nu(h)$ also lies in $\mathcal{N}_s^\nu(h)$. Thus it follows from Theorem 2.2.3 (if $s \geq 2$) and the main theorem of [23] (if $s = 1$) that the maximal polynomial of $\mathcal{N}_s^\nu(h)$ is unique and that $g \in \mathcal{N}_s^\nu(h)$ is maximal if and only if γ is less than or equal to an alternate even composition of length s. The statement for the minimal polynomial also follows from Theorem 2.2.3 if $\mathcal{N}_s^\nu(h) = \mathcal{H}_s^\nu(h)$, so suppose $\mathcal{N}_s^\nu(h) \subset \mathcal{H}_s^\nu(h)$. Then for the minimal polynomial we will do an induction on the dimension of $\mathcal{N}_s^\nu(h)$ starting with the one-dimensional strata.

If $\mathcal{N}_s^\nu(h)$ is one-dimensional then by Theorem 5.1.7, $\mathcal{N}_s^\nu(h)$ is maximal dimensional. Since $\mathcal{N}_s^\nu(h) \subset \mathcal{H}_s^\nu(h)$, by Theorem 5.1.3, a minimal polynomial of $\mathcal{N}_s^\nu(h)$ is not the minimal polynomial of $\mathcal{H}_s^\nu(h)$. A minimal polynomial of $\mathcal{N}_s^\nu(h)$ is of course also not the maximal polynomial of $\mathcal{H}_s^\nu(h)$ so it lies in the relative interior of $\mathcal{H}_s^\nu(h)$ and by Theorem 2.1.9 it has the composition ν. And similarly to the proof of Theorem 5.1.9, a minimal polynomial of $\mathcal{N}_s^\nu(h)$ has a zero root when it is not the minimal polynomial of $\mathcal{H}_s^\nu(h)$. Thus if g is a minimal polynomial, $\operatorname{sgn}(\lambda) = -1$ and $(\gamma_2, ..., \gamma_{s+1}) = (1, ..., 1)$ is of length s. By the argument in Lemma 5.1.8 there is at most one polynomial in $\mathcal{N}_s^\nu(h)$ with a zero root, so a minimal polynomial is unique. If g is not the minimal polynomial, then as we just saw $\operatorname{sgn}(\lambda) = 1$ and by Theorem 2.2.3, γ is not less than or equal to an alternate odd composition of length s, thus the equivalence follows.

Next let $\dim(\mathcal{N}_s^\nu(h)) \geq 2$. If g has only positive roots then by Theorem 2.2.3, γ is not less than or equal to an alternate odd composition of length s. By Theorem 5.1.3, the minimal polynomial of $\mathcal{H}_s^\nu(h)$ is not in $\mathcal{N}_s^\nu(h)$ when $\mathcal{N}_s^\nu(h) \subset \mathcal{H}_s^\nu(h)$. Thus, similarly to the proof of Theorem 5.1.9, a minimal polynomial of $\mathcal{N}_s^\nu(h)$ must lie in $\mathcal{N}_s^{-\nu}(h)$. So suppose $g(0) = 0$, then $g = t^{\nu_1} g^*$ has the same $(s+1)^{th}$ coefficient as g^* and is therefore minimal in $\mathcal{N}_s^\nu(h)$ if and only if g^* is minimal in $\mathcal{N}_s^{(\nu_2,\ldots,\nu_l)}(g^*)$. So by induction the minimal polynomial of $\mathcal{N}_s^\nu(h)$ is unique and g is minimal if and only if $(\gamma_2, ..., \gamma_k)$ is less than or equal to an alternate odd composition of length s. $\square$

Note that Theorem 5.1.11 also applies for the stratum $\mathcal{N}_s^\nu(h)$ when $\operatorname{sgn}(\nu) = -1$. We can simply rewrite $\mathcal{N}_s^\nu(h)$ as in the proof of Theorem 5.1.7 and instead work with the stratum $\mathcal{N}_s^{(\nu_2,\dots,\nu_l)}(g/t^{\nu_1})$, where $g \in \mathcal{N}_s^\nu(h)$.

Remark 5.1.12. *It is worth noting that by using Theorem 2.1.9, it is not too much work to show that the relative interior of $\mathcal{N}_s^\nu(h)$ consists of the polynomials with signed compositions equal to ν. Similarly, by using Corollary 2.1.11, one can conclude that the even-hyperbolic strata equals the closure of their relative interior. We also have all the tools we need to show that the relative interior of an even-hyperbolic stratum is connected and we could follow the method in [18] and in subchapter 2.3 to show that the strata and their relative interior are contractible. But we have focused on the most central properties as we will not be needing the others.*

5.2 ... and combinatorics

In this subchapter we delve into the main combinatorial properties of the poset of even-hyperbolic strata. We will see that it is a lattice which can be computed from the signed compositions of the polynomials in $\mathcal{N}_s^\nu(h)$ with at most s distinct positive roots. We will also show that the boundary complex of the dual lattice is generically a dual potential hyperbolic poset and thus shellable simplicial complex and a combinatorial sphere. Then we show we therefore get an improvement of the degree principle for polynomials invariant under the natural action of the hyperoctahedral group.

We start by introducing a map that will let us avoid having to establish everything from scratch and instead take advantage of the work on hyperbolic lattices. So for the remainder of the thesis let $\psi : \mathcal{SC}(n) \to \mathcal{C}(n+1)$ denote the map from the signed compositions of n, $\mathcal{SC}(n)$, to the compositions of $n+1$ given by

$$\psi(\nu) = \begin{cases} (1, \nu_1, \dots, \nu_l), & \text{if } \operatorname{sgn}(\nu) = 1, \\ (\nu_1 + 1, \nu_2, \dots, \nu_l), & \text{if } \operatorname{sgn}(\nu) = -1. \end{cases}$$

Lemma 5.2.1. *The mapping ψ is a poset isomorphism.*

Proof. It is easy to see that ψ is a bijection whose inverse is given by

$$\mu \mapsto \begin{cases} (\mu_2, ..., \mu_l), \text{ if } \mu_1 = 1, \\ (\mu_1 - 1, \mu_2, ..., \mu_l), \text{ if } \mu_1 > 1. \end{cases}$$

To see that it is an order-preserving map let γ and ν be two signed compositions with $\nu < \gamma$. Then $\mathrm{sgn}(\nu) \leq \mathrm{sgn}(\gamma)$ and there is a composition, λ, of $\ell(\gamma)$ with $\ell(\lambda) = l$ such that

$$|\nu| = (\gamma_1 + ... + \gamma_{\lambda_1}, ..., \gamma_{\ell(\gamma)-\lambda_l+1} + ... + \gamma_{\ell(\gamma)}).$$

Thus either

$$\psi(\nu) = (1, \gamma_1 + ... + \gamma_{\lambda_1}, ..., \gamma_{\ell(\gamma)-\lambda_l+1} + ... + \gamma_{\ell(\gamma)}) <$$

$$(1, \gamma_1, ..., \gamma_{\ell(\gamma)}) = \psi(\gamma),$$

if $\mathrm{sgn}(\nu) = \mathrm{sgn}(\gamma) = 1$, or

$$\psi(\nu) = (1 + \gamma_1 + ... + \gamma_{\lambda_1}, ..., \gamma_{\ell(\gamma)-\lambda_l+1} + ... + \gamma_{\ell(\gamma)})$$

is smaller than both the options for $\psi(\gamma)$:

$$(1 + \gamma_1, ..., \gamma_{\ell(\gamma)}) < (1, \gamma_1, ..., \gamma_{\ell(\gamma)}).$$

Similarly it is straightforward to check that the inverse is order-preserving thus ψ is a poset isomorphism. □

An immediate consequence of Lemma 5.2.1 and Lemma 3.1.1 is that the poset of signed compositions of n is isomorphic to the face lattice of an $(n-1)$-dimensional simplex. And as with the poset of hyperbolic strata this means that the set of hyperbolic strata, partially ordered by inclusion, is a lattice. We also see that the meet of two even-hyperbolic strata is given by

$$\mathcal{N}_s^{\nu}(h) \wedge \mathcal{N}_s^{\gamma}(h) = \mathcal{N}_s^{\nu \wedge \gamma}(h),$$

where γ is another signed composition and $\nu \wedge \gamma$ is the meet of ν and γ in the lattice of signed compositions.

Definition 5.2.2. *We let $\mathcal{K}_s(h)$ denote the lattice of strata of $\mathcal{N}_s(h)$ and we call $\mathcal{K}_s(h)$ an **even-hyperbolic lattice.***

Note that by Theorem 5.1.7, if the stratum $\mathcal{N}_s^{\nu}(h)$ is at least one-dimensional, it is maximal dimensional and for any $\gamma < \nu$, $\mathcal{N}_s^{\gamma}(h)$ is at most $(\dim(\mathcal{N}_s^{\nu}(h))-1)$-dimensional. Since there are finitely many signed compositions smaller than ν we have

$$\dim(\cup_{\gamma<\nu}\mathcal{N}_s^{\gamma}(h)) \leq (\dim(\mathcal{N}_s^{\nu}(h)) - 1).$$

Thus there must be a polynomial in $\mathcal{N}_s^{\nu}(h)$ with composition ν. Also, if $\mathcal{N}_s^{\nu}(h)$ is zero-dimensional then it contains only one polynomial. Thus we may abuse notation and identify the lattice $\mathcal{K}_s(h)$ with the union of the signed compositions that occur in $\mathcal{N}_s(h)$ and $-(n)$:

$$\mathcal{K}_s(h) = \{sc(g)|g \in \mathcal{N}_s(h)\} \cup \{-(n)\}.$$

Let us have a look at what kind of even-hyperbolic lattices we can have. Firstly, when $s = 0$ then all signed compositions occur and thus $\mathcal{K}_0(h)$ is an $(n-1)$-dimensional simplex. Thus we will focus on the case when $s \geq 1$. Secondly, note that if $h_1 = 0$, then since the roots of any polynomial in $\mathcal{N}_1(h)$ must satisfy $\sum_{i=1}^{n} x_i = 0$, we have that $\mathcal{N}_1(h) = \{h\} = \{t^n\}$. Also if $h_1 < 0$ we see that the signed composition $-(n)$ cannot occur in $\mathcal{N}_s(h)$ and thus $\mathcal{K}_s(h)$ contains the empty set. However, it is clear that any other signed composition occurs and thus $\mathcal{K}_1(h)$ is an $(n-1)$-dimensional simplex which by Theorem 5.1.7 is ranked by the dimension of the strata.

For larger s things get more complicated, however we can compute an even-hyperbolic lattice from its zero-dimensional strata:

Algorithm 5.2.3. *Let $s \geq 1$, $\mathcal{N}_s(h)$ be $(n-s)$-dimensional and let U contain the signed compositions, γ, occurring in $\mathcal{N}_s(h)$ with either $\ell(\gamma) \leq s$ or $\ell(\gamma) = s+1$ and $sgn(\gamma) = -1$.*

Step 1: Compute the join of every pair in U:

$$V := \{\gamma \vee \lambda | \gamma, \lambda \in U \;\&\; \gamma \neq \lambda\}.$$

Step 2: Compute the upward closure of V:

$$\overline{V} := \{\lambda | \exists\, \gamma \in V \text{ with } \gamma \leq \lambda\}.$$

Then $U \cup \overline{V}$ is the set of all signed compositions occurring in the slice $\mathcal{N}_s(h)$.

Proof. Let $\nu \in \overline{V}$, then $\mathcal{N}_s^\nu(h)$ contains at least two polynomials so by Theorem 5.1.7 and the discussion after Definition 5.2.2, it is maximal dimensional and contains a polynomial with signed composition ν. Thus all the signed compositions computed by the algorithm occurs in $\mathcal{N}_s(h)$.

For the reverse statement note that if ν occurs in $\mathcal{N}_s(h)$ and either $l > s+1$ or $\operatorname{sgn}(\nu) = 1$ and $l = s+1$, then $\mathcal{N}_s^\nu(h)$ is at least one-dimensional by Theorem 5.1.7. By Theorem 5.1.11 there are two distinct polynomials $g, p \in \mathcal{N}_s^\nu(h)$ (the minimal and maximal polynomial), whose signed compositions lie in U. By Theorem 5.1.7 we have $sc(g) \neq sc(p)$, thus $\nu \geq sc(g) \vee sc(p) \in V$ and $\nu \in \overline{V}$ and so all the signed compositions that occur in $\mathcal{N}_s(h)$ is computed by the algorithm. $\square$

To show that $\partial(\mathcal{K}_s^\Delta(h))$ is generically a shellable simplicial complex we first use the isomorphism ψ to rephrase the classification in Theorem 5.1.11.

Lemma 5.2.4. *Let $s \geq 1$ and $\mathcal{N}_s^\nu(h) \neq \emptyset$, then $g \in \mathcal{N}_s^\nu(h)$ is the minimal (resp. maximal) polynomial if and only if $\psi(sc(g))/\psi(\nu)$ is less than or equal to an alternate odd (resp. even) composition of length $s+1$.*

Proof. Let $\lambda = sc(g)$ and $\gamma = |\lambda|/|\nu|$ and suppose s is odd as the argument is analogous when s is even. Also, let $\mathcal{N}_s^\nu(h)$ be generic as the general statement follows from the generic case by Lemma 5.2.1.

If $\operatorname{sgn}(\nu) = 1$, then by Theorem 5.1.11, g is the minimal polynomial if and only if $\operatorname{sgn}(\lambda) = 1$ (resp. $\operatorname{sgn}(\lambda) = -1$) and γ (resp. $(\gamma_2, ..., \gamma_{\ell(\gamma)})$) is an alternate odd composition of length s. We have that $\operatorname{sgn}(\lambda) = 1$ and γ is an alternate odd composition of length s if and only if $\psi(\lambda)/\psi(\nu) = (1, \gamma_1, ..., \gamma_{\ell(\gamma)})$ is an alternate odd composition of length $s+1$. Also we have that $\operatorname{sgn}(\lambda) = -1$ and $(\gamma_2, ..., \gamma_{\ell(\gamma)})$ is an alternate odd composition of length s if and only if $\psi(\lambda)/\psi(\nu) = (1+\gamma_1, \gamma_2, ..., \gamma_{\ell(\gamma)})$ is an alternate odd composition of length $s+1$. This is because when s is odd then $\ell(\gamma) = s+1$ is even and thus $\gamma_2 = \gamma_4 = ... = \gamma_{s+1} = 1$.

For the maximal polynomial, suppose $\mathcal{N}_s^\nu(h)$ is zero dimensional then by Theorem 5.1.7 it contains only g. Then $sc(g) = \nu$ and thus $\psi(\lambda)/\psi(\nu) = (1^{s+1}) = \gamma$ and so the statement follows. Next suppose $\mathcal{N}_s^{\nu}(h)$ is at least one-dimensional, then by Theorem 5.1.3 the maximal polynomial does not have a zero root. Thus by Theorem 5.1.11, g is the maximal polynomial if and only if $\mathrm{sgn}(\lambda) = 1$ and $\gamma = \lambda/\nu$ is an alternate even composition of length s. This is equivalent to $\psi(\lambda)/\psi(\nu) = (1, \gamma_1, ..., \gamma_{\ell(\gamma)})$ being an alternate even composition of length $s+1$.

Lastly, if $\mathrm{sgn}(\nu) = -1$, then g is the minimal polynomial if and only if g^*, where $g = t^{\nu_1} g^*$, is the minimal polynomial in $\mathcal{N}_s^{\nu^*}(g^*)$, where $\nu^* = (\nu_2, ..., \nu_l)$. Since $\mathrm{sgn}(\nu^*) = 1$, the statement follows from the argument above if $\psi(sc(g^*))/\psi(\nu^*) = \psi(sc(g))/\psi(\nu)$, so let us show this. We have $\psi(\nu^*) = (1, \nu_2, ..., \nu_l)$ and

$$\psi(sc(g^*)) = \begin{cases} (1, \lambda_2, ..., \lambda_{\ell(\lambda)}) \text{ if } \lambda_1 = \nu_1 \\ (1 + \lambda_1 - \nu_1, \lambda_2, ..., \lambda_{\ell(\lambda)}) \text{ if } \lambda_1 > \nu_1 \end{cases},$$

thus $\psi(sc(g^*))/\psi(\nu^*) = \gamma$. We also have $\psi(\lambda) = (1 + \lambda_1, \lambda_2, ..., \lambda_{\ell(\lambda)})$ thus $\psi(\lambda)/\psi(\nu) = \gamma$ and so $\psi(\lambda)/\psi(\nu) = \psi(sc(g^*))/\psi(\nu^*)$. □

Theorem 5.2.5. *The poset $\psi(\mathcal{K}_s(h)\backslash\emptyset)$ is generically a potential hyperbolic poset. Thus $\partial(\mathcal{K}_s^{\Delta}(h))$ is an $(n-s-1)$-dimensional shellable simplicial complex and a combinatorial $(n-s-1)$-sphere.*

Proof. Since $\mathcal{N}_s(h)$ is generic the atoms of $\mathcal{K}_s(h)$ correspond to the positive compositions of length s and the negative compositions of length $s+1$. If $g \in \mathcal{N}_s(h)$ is a polynomial with such a signed composition, then by Theorem 5.1.7, $\mathcal{N}_s^{\nu}(h)$ is maximal dimensional for any $\nu \geq sc(g)$. And as in the discussion after Definition 5.2.2, we see that the signed composition ν occurs in $\mathcal{N}_s(h)$. Thus a maximal element of $\partial(\mathcal{K}_s^{\Delta}(h))$ is the dual of the upward closure of $sc(g)$ in the poset of signed compositions.

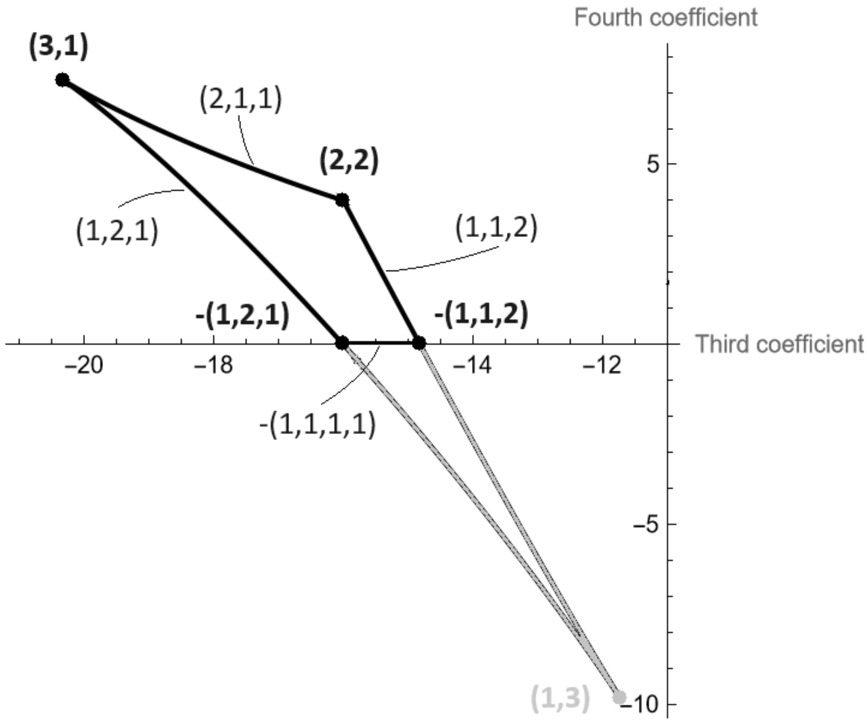
By Lemma 5.2.1, a maximal element of $\partial(\mathcal{K}_s^{\Delta}(h))$ is therefore isomorphic to the dual of the upward closure of $\psi(sc(g))$ in the poset of compositions of $n+1$. Since the length of $\psi(sc(g))$ is $s+1$ then due to Lemma 5.2.4, $\psi(\partial(\mathcal{K}_s(h)))$ is a potential hyperbolic poset. Thus by Lemma 5.2.1 and Theorem 4.2.4, $\partial(\mathcal{K}_s^{\Delta}(h))$ is an $(n-s-1)$-dimensional shellable simplicial complex and in particular a combinatorial $(n-s-1)$-sphere. □

5.2.1 Improving the hyperoctahedral degree principle

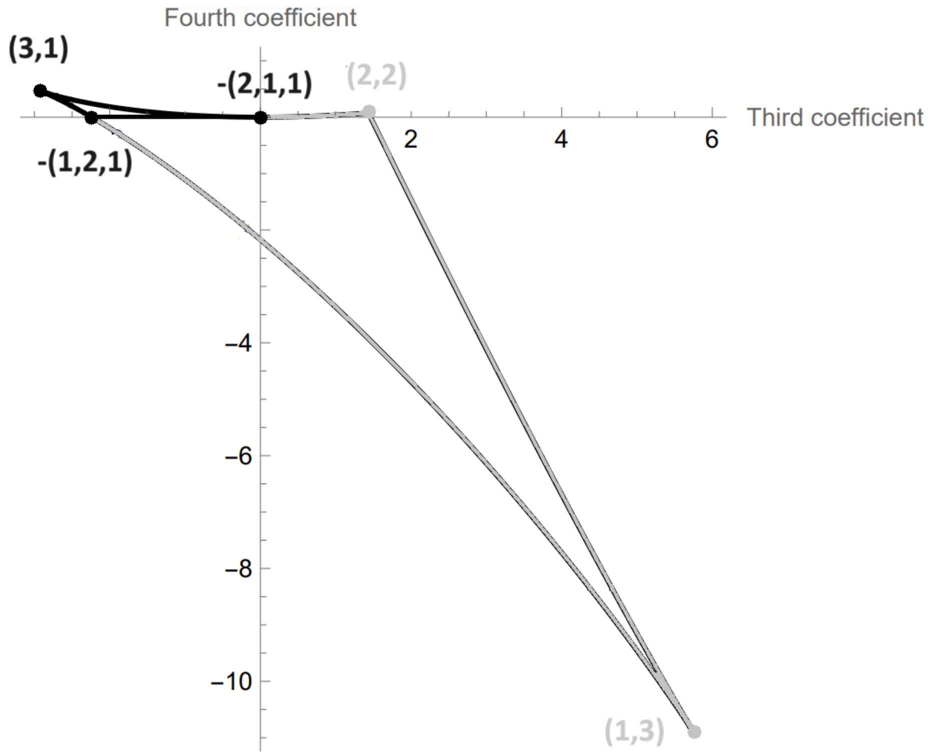
In this subchapter we look at what kind of improvements we can make on the degree principle for the hyperoctahedral group. We see that the possible improvements cannot be better than for the symmetric group $\mathcal{S}(n)$, but on the other hand we can do at least as well as for the symmetric group $\mathcal{S}(n+1)$. Furthermore, it looks like any improvements for $\mathcal{S}(n+1)$ can be applied to $\mathcal{B}(n)$ and so one might as well focus on the symmetric group and hyperbolic slices.

To see that we cannot in general get anything better than for $\mathcal{S}(n)$, consider the following example:

Example 5.2.6. *We will revisit the example from the introduction of this chapter. So let $h = t(t-2)^2(t-4) = t^4 - 8t^3 + 20t^2 - 16t$, then recall that $\mathcal{N}_2(h)$ is the following set:*



The polynomial from $\mathcal{N}_2(h)$ with composition $(2,2)$ is $g = (t+\sqrt{2}-2)^2(t-\sqrt{2}-2)^2 = t^4 - 8t^3 + 20t^2 - 16t + 4$ and so its first root is $-\sqrt{2}+2$. By Theorem 5.1.3 we can translate all the roots of all the polynomials in $\mathcal{H}_s(h)$ by $\sqrt{2}-2-\epsilon$, for some $\epsilon > 0$ and thus get rid of the signed compositions $(2,2), (1,1,2)$ and $-(1,1,2)$ from the even-hyperbolic lattice. Just consider the following picture of $\mathcal{N}_2(h(t-\sqrt{2}+2+1/10))$:



.

In general we see that if $p = h(t+a)$, for some $a \in \mathbb{R}$, then $\mathcal{L}_s(p) = \mathcal{L}_s(h)$ but the following are the possibilities for $\mathcal{K}_s(p)\backslash\emptyset$:

$$\{-(3,1)\},$$
$$\{(1^4), -(1^4), (2,1^2), (1,2,1), -(2,1^2), -(1,2,1), (3,1)\},$$
$$\{(1^4), -(1^4), (2,1^2), (1,2,1), -(2^2), -(1,2,1), (3,1)\},$$
$$\{(1^4), -(1^4), (2,1^2), (1,2,1), (1^2,2), -(1,2,1), -(1^2,2), (2^2), (3,1)\},$$
$$\{(1^4), (2,1,1), (1,2,1), (1,1,2), (2,2), (3,1), -(1,3)\},$$
$$\mathcal{L}_s(h)\backslash\emptyset.$$

As in the example above, it is clear that when $s \geq 2$, then since both $\mathcal{N}_s(h)$ and $\mathcal{H}_s(h)$ are compact, then we can translate the roots of $\mathcal{H}_s(h)$ such that $\mathcal{K}_s(h) = \mathcal{L}_s(h)$. Thus we cannot make better improvements for the group $\mathcal{B}(n)$ compared to $\mathcal{S}(n)$. However due to Theorem 5.1.3 or Theorem 5.2.5 we can make improvements comparable to $\mathcal{S}(n+1)$. So let us look at how this works.

Similarly to signed compositions of even-hyperbolic polynomials we let $sp(h) := m \times \lambda$, where m is the multiplicity of zero as a root of h and $\lambda = p(h/t^m)$. We denote by $\mathcal{SP}(n)$ the set

$$\cup_{i=0}^{n}(n-i) \times \mathcal{P}(i)$$

and by $\mathcal{SP}(n,s)$ the set

$$\{m \times \lambda \in \mathcal{SP}(n) | \ell(\lambda) = s\}.$$

Furthermore we say that $m \times \lambda \leq k \times \gamma$ if there are permutations σ and τ such that $(m, \sigma(\lambda)) \leq (k, \tau(\gamma))$ as compositions. Then as in the symmetric group we define an even Vandermonde covering as:

Definition 5.2.7. *Let $P \subseteq \mathcal{SP}(n,s)$. We say that P is an* ***even*** (n,s)***-Vandermonde covering****, if for every even-hyperbolic slice $\mathcal{N}_s(h)$ there is a $m \times \lambda \in P$ and a polynomial $g \in \mathcal{N}_s(h)$ with $sp(g) \leq m \times \lambda$.*

Just as with the symmetric group, $\mathcal{SP}(n)$ correspond to the possible stabilizer subgroups of $\mathcal{B}(n)$ of points in $\mathbb{R}^n$.

Definition 5.2.8. *If $P \subseteq \mathcal{SP}(n,s)$ then we let $O_P :=$*

$$\left\{ (\underbrace{0,\ldots,0}_{m-times}, \underbrace{x_1,\ldots,x_1}_{\lambda_1-times}, \ldots, \underbrace{x_s,\ldots,x_s}_{\lambda_s-times}) \in \mathbb{R}^n \;\middle|\; m \times \lambda \in P \right\}.$$

Theorem 5.2.9. *Let $P \subseteq \mathcal{P}(n,s)$, then the following are equivalent:*

1. *$P \subseteq \mathcal{SP}(n,s)$ is an even (n,s)-Vandermonde covering.*
2. *For any $F_1, ..., F_k \in \mathbb{R}[x]^{\mathcal{B}(n)}$ with degree at most $2s$ we have*

$$V_{\mathbb{R}}(F_1, ..., F_k) \neq \emptyset \Leftrightarrow V_{\mathbb{R}}(F_1, ..., F_k) \cap O_P \neq \emptyset.$$

3. *For all $c \in \mathbb{R}^s$ we have*

$$V_{\mathbb{R}}(E_1(x^2) + c_1, ..., E_s(x^2) + (-1)^{s-1}c_s) \neq \emptyset \Leftrightarrow$$

$$V_{\mathbb{R}}(E_1(x^2) + c_1, ..., E_s(x^2) + (-1)^{s-1}c_s) \cap O_P \neq \emptyset.$$

Proof. By Theorem 1.1.5, we have $F_i = G_i(E_1(x^2), ..., E_s(x^2))$ for some $G_i \in \mathbb{R}[y_1, ..., y_s]$ and if $a \in V_{\mathbb{R}}(F_1, ..., F_k)$ we associate it to the even-hyperbolic polynomial $t^n - E_1(a^2)t^{n-1} + ... + (-1)^n E_n(a^2)$. Other than those adjustments the proof is analogous to the proof of Theorem 4.0.5. $\square$

Theorem 5.2.10. *The set $0 \times \mathcal{P}_{\max}(n,s)$ is an even (n,s)-Vandermonde covering of size $\left|\mathcal{P}\left(n + 1 - \lceil \frac{s+1}{2} \rceil, \lfloor \frac{s+1}{2} \rfloor\right)\right|$.*

Proof. If s is odd then Theorem 5.1.3 and Theorem 2.2.3 says that for any nonempty even-hyperbolic slice $\mathcal{N}_s(h)$ the maximal polynomial, g, of the hyperbolic slice $\mathcal{H}_s(h)$ is such that $p(g) \leq \lambda$ for some $\lambda \in \mathcal{P}_{\max}(n,s)$ and g has the maximal first root in $\mathcal{H}_s(h)$. Thus $sp(g) \leq 0 \times \lambda$ and $g \in \mathcal{N}_s(h) \subseteq \mathcal{H}_s(h)$. By Lemma 4.1.2 the size of $0 \times \mathcal{P}_{\max}(n,s)$ is the same as the size of $\mathcal{P}_{\min}(n+1, s+1)$ which by Theorem 4.1.1 is $\left|\mathcal{P}\left(n+1-\left\lceil\frac{s+1}{2}\right\rceil, \left\lfloor\frac{s+1}{2}\right\rfloor\right)\right|$. When s is even it is easy to see that $\mathcal{P}_{\min}(n,s) = \mathcal{P}_{\max}(n,s)$ so the argument works similarly. □

For further improvements and bounds on the size of even Vandermonde coverings we can, by Theorem 5.2.5, translate the results from potential hyperbolic posets via the mapping ψ introduced on page 89. Thus if we assume Conjecture 4.2.5 is correct then we may just focus on the symmetric case if we want to improve on the degree principle further for the hyperoctahedral case.

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