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# Weighted estimates of commutators of singular operators in generalized Morrey spaces beyond Muckenhoupt range and applications

**Natasha Samko**[1]

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**Abstract**
For a certain class of radial weights, we prove weighted norm estimates for commutators with BMO coefficients of singular operators in local generalized Morrey spaces. As a consequence of these estimates, we obtain norm inequalities for such commutators in the generalized Stummel-Morrey spaces. We also discuss a.e. well-posedness of singular operators and their commutators on weighted generalized Morrey spaces. The obtained estimates are applied to prove interior regularity for solutions of elliptic PDEs in the frameworks of the corresponding weighted Sobolev spaces based on the local generalized Morrey spaces or Stummel-Morrey spaces. To this end also conditions for the applicability of the representation formula, for the second-order derivatives of solutions to elliptic PDEs, are found for the case of such weighted spaces. In both results, for commutators and applications, we admit weights beyond the Muckenhoupt range.

**Keywords** Non-standard function spaces · Generalized Morrey spaces · Weighted singular integral operators · Weighted commutators and their applications · Elliptic PDE with discontinuous coefficients

**Mathematics Subject Classification** 46E30 · 42B35 · 42B25 · 47B38

## 1 Introduction

We obtain weighted norm estimates, for a certain class of radial weights, in local generalized Morrey spaces $\mathcal{L}^{p,\varphi}(\Omega)$ for commutators of singular operators

$$Tf(x) = \int_{\Omega} \mathcal{T}(x,y) f(y)dy = \lim_{\varepsilon\to 0} \int_{y\in\Omega:\ |x-y|>\varepsilon} \mathcal{T}(x,y) f(y)dy \tag{1.1}$$

✉ Natasha Samko
Natasha.G.Samko@uit.no

1 UiT The Arctic University of Norway, Campus Narvik, Narvik, Norway

over an open bounded set $\Omega \in \mathbb{R}^n$. For interpretation of the operator (1.1) on Morrey spaces, we refer to Sect. 2.2. The general interest to the study of commutators of singular integral operators is due to their use in the investigation of the regularity problems for elliptic PDEs. Our main interest being in application to elliptic PDEs in case of weighted Morrey spaces, in this paper we mainly focus on the case of bounded sets $\Omega$, though some statements are given for unbounded sets.

Commutators of singular operators have been studied in various function spaces. We refer, for instance, to [5] for Lebesgue spaces $L^p(\Omega)$, to [8] for classical Morrey spaces $\mathcal{L}^{p,\lambda}(\Omega)$, and to [4] and [7] for the generalised Morrey spaces. For the theory of Morrey spaces we refer, for instance, to the books [20, 26] and [38] and survey [27], and for the applications to integral operators and PDEs, to the book [38].

Our aim is to obtain weighted estimates for commutators of singular operators in the local generalized Morrey spaces. In [31], in the case of the one-dimensional singular operator (Hilbert transform) there was found an effect of shifting exponents of power weights for the boundedness of this operator. More precisely, the familiar Muckenhoupt interval $-n < \alpha < n(p-1)$, in case of classical Morrey spaces $\mathcal{L}^{p,\lambda}$ is replaced by

$$\lambda - n < \alpha < \lambda + n(p-1)$$

(with n=1 in [31]). This was extended to the multi-dimensional case for the Riesz transforms in [24].

The above shifting cuts off some Muckenhopt weights but on the other hand, admits non-Muckenhoupt weights. Such an effect got the name of “beyond the Muckenhoupt range”-effect see e.g. [10].

We deal with radial weights of a certain class defined in Sect. 2.3.

We also introduce the spaces which we call generalized Stummel-Morrey spaces. For the spaces which might be called by analogy as Stummel-Lebesgue spaces we refer e.g. to [1] and [37]. As a consequence of our weighted estimates for commutators of singular operators in local Morrey spaces, we obtain norm estimates of these commutators in the generalized Stummel-Morrey spaces.

We also give applications of the obtained weighted estimates to regularity problems for solutions to elliptic PDEs. In both the results, for commutators and applications, our preoccupation is to admit weights beyond the Muckenhoupt range.

The study of regularity problems of solutions to elliptic PDEs is based on the so-called representation formula for second order derivatives of solutions. The validity of this representation formula is well known for the Lebesgue spaces $L^s(\Omega)$, $s > 1$, see [5]. Note that weighted Morrey spaces, if not somehow restricted, may be not embedded into any Lebesgue space $L^s(\Omega)$, $s > 1$, and may even contain non-integrable functions, see Sect. 4.1. Consequently, the use of the representation formula in the frameworks of weighted Morrey spaces needs a justification. We focus on such a justification in Sect. 4.1.

We refer the reader to [7] for a comprehensive presentation of application of norm estimates of the singular operators and their commutators in the case of non-weighted generalized Morrey spaces.

The paper is organized as follows. In Sect. 2 we provide necessary definitions, used notation, and recall some known results on norm estimates for commutators of non-weighted singular operators and weighted Hardy operators. The main results for commutators of weighted singular operators are proved in Sect. 3. We start with a certain point-wise estimate for the commutators, for the weights under consideration, which reduces the estimate of the commutator of a weighted singular operator to the estimate of commutators of the following operators: non-weighted singular operator, certain hybrid of potential operator and Hardy operator and weighted Hardy operator. The main result on the weighted norm estimate for the commutator of singular operators in local generalized Morrey spaces is contained in Theorem 3.4. We conclude Sect. 3 by deriving from Theorem 3.4 a similar estimate for Stummel-Morrey spaces. In Sect. 4 we give an application of obtained estimates to interior estimate for solutions of elliptic PDEs.

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## 2 Preliminaries

### 2.1 Defenition of spaces

Let $\Omega$ be an open set in $\mathbb{R}^n$, $\Omega \subseteq \mathbb{R}^n$ and $\ell =$ diam $\Omega$.

The global and local (central) Morrey spaces $\mathcal{L}^{p,\varphi}(\Omega)$ and $\mathcal{L}^{p,\varphi}_{\{x_0\}}(\Omega)$ are defined by the norms

$$\|f\|_{\mathcal{L}^{p,\varphi}(\Omega)} = \sup_{x\in\Omega, r>0} \left( \frac{1}{\varphi(r)} \int\limits_{B(x,r)\cap\Omega} |f(y)|^p \, dy \right)^{\frac{1}{p}} \tag{2.1}$$

and

$$\|f\|_{\mathcal{L}^{p,\varphi}_{\{x_0\}}(\Omega)} = \sup_{r>0} \left( \frac{1}{\varphi(r)} \int\limits_{B(x_{x_0},r)\cap\Omega} |f(y)|^p \, dy \right)^{\frac{1}{p}}, \tag{2.2}$$

respectively, where $x_0 \in \Omega$, $1 \le p < \infty$ and everywhere in the sequel the function $\varphi(r)$ is assumed to satisfy the following *à priori* conditions:

(1) it is a non-negative almost increasing (a.i.) function on $(0, \ell)$,

(2) $\lim_{r\to 0} \varphi(r) = 0$ and $\inf_{\delta<r<\ell} \varphi(r) > 0$ for every $\delta > 0$.

In the case of global spaces we also additionally assume that

(3) $\frac{\varphi(t)}{t^n}$ is almost decreasing (a.d.) on $(0, \ell)$.

Note that the function $\varphi$ satisfies the doubling condition $\varphi(2t) \le C\varphi(t)$, $0 < t < \frac{\ell}{2}$, in view of the assumption 3).

In the case of classic Morrey spaces, i.e. $\varphi(r) = r^\lambda$, we admit $\lambda > 0$ for the local Morrey space and $0 < \lambda \le n$ for the global one.

The weighted version of the space $\mathcal{L}^{p,\varphi}(\Omega, w)$ is defined by the norm

$$\|f\|_{\mathcal{L}^{p,\varphi}(\Omega,w)} = \sup_{x\in\Omega, r>0} \left( \frac{1}{\varphi(r)} \int\limits_{B(x,r)\cap\Omega} |f(y)|^p \, w(y)dy \right)^{\frac{1}{p}},$$

where the weight $w$ and the function $\varphi(r)$ are independent of each other, and similarly for weighted local Morrey space.

In a similar way we interpret the weighted Lebesgue space $L^p(\Omega, w)$.

Everywhere in the sequel, when considering global Morrey spaces, we suppose that in the case $\Omega \neq \mathbb{R}^n$ there holds the so-called condition $\mathcal{A}$ :

$$|\Omega \cap B(x, r)| \geq cr^n \tag{2.3}$$

for all $x \in \bar{\Omega}$ and $0 < r < \ell$.

The following statement is derived from Theorem 3.2 in [1].

**Proposition 2.1** *Let* $1 \leq p < \infty,\ x_0 \in \Omega,\ 0 < \ell \leq \infty,$ *$w$ be a weight on $\Omega$ and $\varphi$ satisfy the condition 1). Then there holds the embedding*

$$\mathcal{L}^{p,\varphi}_{\{x_0\}}(\Omega, w) \hookrightarrow L^p\left(\Omega, \frac{w}{\varphi_{x_0}}\xi_{x_0}\right) \tag{2.4}$$

*with* $\varphi_{x_0}(x) = \varphi(|x - x_0|)$ *and* $\xi_{x_0}(x) = \xi(|x - x_0|)$, *where $\xi$ is any non-negative function on* $(0, \ell)$ *satisfying the conditions:*

$$\int\limits_0^\ell \frac{\xi(t)}{t} dt < \infty, \tag{2.5}$$

*$\xi$ is a.i. on* $(0, \ell)$, *when* $\ell < \infty$, *and a.i. on* $(0, r_0)$ *and a.d. on* $(r_0, \infty)$ *for some* $r_0 > 0$, *when* $\ell = \infty$.

*If additionally $\varphi$ is doubling and* $\frac{\xi(t)}{\varphi(t)}$ *is decreasing on* $(0, \ell)$, *then the inequality*

$$\|f\|_{L^p\left(\Omega, \frac{w}{\varphi_{x_0}}\xi_{x_0}\right)} \leq c\|f\|_{\mathcal{L}^{p,\varphi}_{\{x_0\}}(\Omega,w)}, \tag{2.6}$$

*for norms holds with the constant* $c = c(p, \varphi, w, \xi)$ *not depending on* $x_0$.

The space $\mathrm{BMO}(\mathbb{R}^n)$ is defined by the quasi-norm

$$\|a\|^* = \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|} \int\limits_B |a(z) - a_B| \, dz, \tag{2.7}$$

where $a_B := \frac{1}{|B|} \int_B a(z)dz$.

The space $\mathrm{VMO}(\mathbb{R}^n)$ is defined as the subspace of functions in $\mathrm{BMO}(\mathbb{R}^n)$ such that

$$\eta_a(r) := \sup_{x\in\mathbb{R}^n} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |a(z) - a_{B(x,r)}|\, dz \to 0 \text{ as } r \to 0. \tag{2.8}$$

The spaces $\mathrm{BMO}^{\mathrm{ext}}(\Omega)$ and $\mathrm{VMO}^{\mathrm{ext}}(\Omega)$ are defined as the spaces of restrictions onto $\Omega$ of functions in $\mathrm{BMO}(\mathbb{R}^n)$ and $\mathrm{VMO}(\mathbb{R}^n)$, respectively.

The space $\mathrm{CMO}_{p,x_0}(\mathbb{R}^n)$ is defined by the quasi-norm

$$\|a\|^*_{\mathrm{CMO}_{p,x_0}}(\mathbb{R}^n) := \sup_{r>0} \left( \frac{1}{|B(x_0,r)|} \int\limits_{B(x_0,r)} |a(z) - a_{B(x_0,r)}|^p\, dz \right)^{\frac{1}{p}}, \tag{2.9}$$

where $a_{B(x_0,r)} := \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} a(z)dz$.

Spaces of the type $\mathrm{CMO}_{p,x_0}$ are known as spaces of *central mean oscillation*, see e.g. [2], [15] and [23]. The spaces $\mathrm{CMO}_{p,x_0}(\Omega)$ are defined as the spaces of restrictions onto $\Omega$ of functions in $\mathrm{CMO}_{p,x_0}(\mathbb{R}^n)$, with the quasi-norm

$$\|a\|^*_{\mathrm{CMO}_{p,x_0}}(\Omega) := \inf \|\tilde{a}\|^*_{\mathrm{CMO}_{p,x_0}}(\mathbb{R}^n), \tag{2.10}$$

where inf is taken with respect to all functions $\tilde{a} \in \mathrm{CMO}_{p,x_0}(\mathbb{R}^n)$ coinciding with $a$ on $\Omega$.

Finally, we define generalized Stummel space $\mathfrak{S}^{p,\psi}(\Omega)$, $1 \le p < \infty$, by the norm

$$\|f\|_{\mathfrak{S}^{p,\psi}(\Omega)} := \sup_{x\in\Omega} \left( \int\limits_{\Omega} |f(y)|^p \psi(|x-y|)dy \right)^{\frac{1}{p}}, \tag{2.11}$$

where $\psi$ is a positive function on $(0,\ell)$, see [37, Section 3.1] and references therein. Besides Morrey spaces, spaces of such a type are used in the study of regularity problems for PDEs, see e.g. [21] and [22]. The notion of Stummel spaces goes back to [39], where the case of $\psi(r) = r^{-\lambda}$ and $p = 2$ was considered. In Sect. 3.2 we define spaces which we call Morrey-Stummel spaces.

### 2.2 On interpretation of singular operators on Morrey spaces

We consider singular integral operators (1.1).

In Theorem 3.4 we shall use the class $\mathcal{S}_{\mathrm{CZ}}$ of Calderón-Zygmung kernels $\mathcal{T}(x,y)$, defined as follows. We say that $\mathcal{T}(x,y) \in \mathcal{S}_{\mathrm{CZ}}$, if $\mathcal{T}(x,y) = k(x, x-y)$, where $k(x,z) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfies the conditions:
(i) $k(x,\cdot)$ is homogeneous of degree $-n$ and $k(x,\cdot) \in C^\infty(\mathbb{R}^n \setminus \{0\})$ for almost all $x \in \mathbb{R}^n$;

(ii) $\int_{S^{n-1}} k(x, \sigma) d\sigma(x) = 0$, where $\sigma$ denotes the surface measure;

(iii) $\max_{|\alpha| \le 2n} \| \frac{\partial^\alpha k}{\partial z^\alpha}(x, z) \|_{L^\infty(\mathbb{R}^n \times S^{n-1})} < \infty$.

Singular integral operators are known to be studied in a general setting of so-called *standard kernels*. Recall that the kernel of a singular operator is called standard if it satisfies the size condition

$$|\mathcal{T}(x, y)| \le \frac{C}{|x-y|^n}, \; x \ne y. \tag{2.12}$$

and the conditions

$$|\mathcal{T}(x, y) - \mathcal{T}(x, z)| \le C \frac{|y-z|^\sigma}{|x-y|^{n+\sigma}}, \text{ if } |x-y| > 2|y-z|, \tag{2.13}$$

$$|\mathcal{T}(x, y) - \mathcal{T}(\xi, y)| \le C \frac{|x-\xi|^\sigma}{|x-y|^{n+\sigma}}, \text{ if } |x-y| > 2|x-\xi|, \tag{2.14}$$

for $x, y, z, \xi \in \mathbb{R}^n$ and some $\sigma > 0$, see e.g. [9, p.99].

By $\mathcal{S}_{\text{st}}$ we denote for brevity the class of standard kernels such that the singular operator $T$ generated by them is bounded in $L^2$.

The operator $T$ being well defined on smooth functions is also defined, by extension arguments, on the whole Lebesgue space $L^p(\Omega)$, $1 < p < \infty$, or weighted Lebesgue spaces $L^p(\Omega, w)$ with Muckenhoupt weight. For functions in these spaces a continuous extension from a dense set leads to the representation of singular integrals on the whole space in terms of almost everywhere existence of the principal value.

In the case of Morrey spaces smooth functions are not dense, so that definition by a unique continuous extension proves to be impossible. For discussion of problems of defining singular operators on Morrey spaces, see [11, 16], [38, Vol. I], [28, 41] and references therein. In particular, in [29] it was proved that singular operators admit many continuous extensions from smooth functions to the Morrey space.

Meanwhile, keeping in mind that the singular operators exist almost everywhere for such "bad" functions as functions in $L^p(\Omega)$ may be, one can define the singular operator on the whole Morrey space directly almost everywhere in the principal value sense. Such a definition of singular integrals on Morrey spaces was silently assumed in various papers. Certainly, such an *á priori* assumption needs a justification that the principal value almost everywhere indeed exists for all functions in the Morrey space. When the Morrey space under consideration is embedded into a larger space where the almost everywhere existence of principal value is known, there is no need in such a justification. I.e. the singular operator is defined then in fact in restriction terms.

In particular, the singular operator $T$ is well defined by restriction arguments on Morrey spaces whenever $\varphi \in L^\infty(0, \ell)$, since

$$\mathcal{L}^{p,\varphi}_{\{x_0\}}(\Omega) \hookrightarrow L^p(\Omega),$$

in this case. However, condition $\varphi \in L^{\infty}(0,\ell)$ implies that $\Omega$ should be bounded in the case of the classical Morrey space with $\varphi(r) = r^{\lambda}$.

The situation is more complicated in the case of weighted Morrey spaces, moreover that we admit Morrey spaces with weights beyond the Muckenhoupt range.

We need the following notation for classes of $A_p$-weights. Let $1 < p < \infty$. $A_p(\mathbb{R}^n)$ will stand for the usual Muckenhoupt class, see e.g. [9, p.135] and $A_p^{\text{ext}}(\Omega)$ for restrictions of weights $w \in A_p(\mathbb{R}^n)$ onto $\Omega$. Finally, $A_p(\Omega)$ will denote the class of weights on $\Omega$ defined by the condition

$$\sup_{Q} \left( \frac{1}{|Q|} \int\limits_{Q\cap\Omega} w(x)dx \right) \left( \frac{1}{|Q|} \int\limits_{Q\cap\Omega} w(x)^{1-p'}dx \right)^{p-1} < \infty,$$

where the supremum is taken with respect to all cubes in $\mathbb{R}^n$.

**Proposition 2.2** ([14, p.439, Theorm 5.6]) *Let $w$ be a weight on $\Omega$. Then $w \in A_p^{\text{ext}}(\Omega)$ if and only if there exists $\varepsilon_0$ such that $w^{1+\varepsilon_0} \in A_p(\Omega)$.*

As a justification of definition of singular operators on local Morrey space in restriction terms, we provide the following theorem. Note that the assumption (2.15) may be replaced by an assumption in intrinsic terms of $\Omega$ in view of Proposition 2.2.

**Theorem 2.3** *Let $w$ be a weight on $\Omega$ and $\varphi$ satisfy the condition 1). If*

$$\frac{w(x)}{\varphi(|x-x_0|)} \in A_p^{\text{ext}}(\Omega), \tag{2.15}$$

*then there exists a weight $W \in A_p^{\text{ext}}(\Omega)$ such that*

$$\mathcal{L}^{p,\varphi}_{\{x_0\}}(\Omega, w) \hookrightarrow L^p(\Omega, W). \tag{2.16}$$

***Proof*** By Proposition 2.1 we have the embedding (2.4) with the "correcting" factor $\xi(|x-x_0|)$ in the weight of the larger space. It remains to show that this factor may be chosen so that the condition (2.15) implies the condition $\frac{w(x)}{\varphi(|x-x_0|)}\xi(|x-x_0|) \in A_p^{\text{ext}}(\Omega)$.

Thus, to arrive at the embedding (2.16), we take $W(x) := \frac{w(x)}{\varphi(|x-x_0|)}\xi(|x-x_0|)$, where the function $\xi$ will be appropriately chosen. We have to show that the function $\xi$ may be chosen so that $W \in A_p^{\text{ext}}(\Omega)$.

We assume that $\ell = \infty$ the case $\ell < \infty$ being easier, and choose $\xi(t) = \begin{cases} t^{\varepsilon}, & 0 < t \le 1 \\ t^{-\varepsilon}, & t \ge 1 \end{cases}$, where $\varepsilon > 0$ will be chosen small enough. Note that $\xi(|x-x_0|) \in A_p^{\text{ext}}(\Omega)$ for $\varepsilon < np_-$, where $p_- = \min\{1, p-1\}$. This is easily derived from the fact that the Muckenhoupt condition for radial weights, satisfying the doubling and reverse doubling conditions, may be written in the form

$$\sup_{r>0} \frac{1}{r^n} \int_0^r t^{n-1}\xi(t)dt \left( \frac{1}{r^n} \int_0^r t^{n-1}\xi(t)^{1-p'}dt \right)^{p-1} < \infty.$$

(see [12, p. 2097]), taking also into account that we may take $x_0 = 0$, since the class $A_p(\mathbb{R}^n)$ is invariant with respect to translations.

By Proposition 2.2 there exists an $\varepsilon_0 > 0$ such that

$$\left[ \frac{w(x)}{\varphi(|x-x_0|)} \right]^{1+\varepsilon_0} \in A_p^{\text{ext}}(\Omega). \tag{2.17}$$

We represent the weight $W(x)$ as

$$W(x) = w_1(x)^\lambda w_2(x)^{1-\lambda},$$

where $w_1(x) = \left[ \frac{w(x)}{\varphi(|x-x_0|)} \right]^{1+\varepsilon_0}$, $w_2(x) = \xi(|x-x_0|)^\gamma$, $\lambda = \frac{1}{1+\varepsilon_0} < 1$ and $\gamma = \frac{1}{1-\lambda} = \frac{1+\varepsilon_0}{\varepsilon_0}$. Here $w_1 \in A_p^{\text{ext}}(\Omega)$ by (2.17) and $w_2 \in A_p^{\text{ext}}(\Omega)$ under the choice of $\varepsilon$ sufficiently small: $\varepsilon\gamma < np_-$. It remains to use the well known property of $A_p$-weights: $w_1, w_2 \in A_p^{\text{ext}}(\Omega) \Rightarrow w_1^\lambda w_2^{1-\lambda} \in A_p^{\text{ext}}(\Omega)$. □

**Corollary 2.4** *Let $1 < p < \infty$ and $\varphi$ be almost increasing. Any singular operator $T$ with the kernel $\mathcal{T} \in \mathcal{S}_{st} \cup \mathcal{S}_{CZ}$ and its commutator $C[a, T] = aT - Ta$, $a \in$ $\text{BMO}^{\text{ext}}(\Omega)$, are defined in the a.e. sense (1.1) on every weighted local Morrey space $\mathcal{L}^{p,\varphi}_{\{x_0\}}(\Omega, w)$ satisfying the condition (2.15).*

***Proof*** Recall that singular operators and their commutators are well studied in Lebesgue spaces with $A_p$-weights. Thus, in case of kernels $\mathcal{T} \in \mathcal{S}_{\text{st}}$ we refer to [9] and [6] for the operators $T$ and $C[a, T]$, respectively, and in case of $\mathcal{T} \in \mathcal{S}_{\text{CZ}}$ to [8] for both $T$ and $C[a, T]$. □

Correspondingly, if instead of the operator $T$ in the weighted space $\mathcal{L}^{p,\varphi}_{\{x_0\}}(\Omega, w)$ we consider the weighted operator $wT\frac{1}{w}$ in the non-weighted space $\mathcal{L}^{p,\varphi}_{\{x_0\}}(\Omega)$, then the condition (2.15) is replaced by

$$\frac{w(x)^p}{\varphi(|x-x_0|)} \in A_p^{\text{ext}}(\Omega). \tag{2.18}$$

***Remark 2.5*** In the case of classical Morrey space with $\varphi(r) = r^\lambda$ and radial power weight $w = |x-x_0|^\alpha$, the assumption (2.15) reduces to the familiar condition

$$\lambda - n < \alpha < \lambda + n(p-1).$$

For Muckenhoupt condition in case of radial weights we refer to [12, p.2097]. We say that a weight $v$ on $\mathbb{R}_+$ belongs to the class DRD$(0,\ell)$, $0<\ell\le\infty$, (doubling and reverse doubling condition) if $c_1 v(r)\le v(2r)\le c_2 v(r)$, $0<r<\frac{\ell}{2}$, $c_i>0$, $i=1,2$.

***Remark 2.6*** In the case of radial weights $w(x)=v(|x-x_0|)$ and $\Omega=\mathbb{R}^n$, the condition (2.15) takes the form

$$\sup_{r>0}\frac{1}{r^n}\int\limits_0^r t^{n-1}\frac{v(t)}{\varphi(t)}dt\left(\frac{1}{r^n}\int\limits_0^r t^{n-1}\left[\frac{\varphi(t)}{v(t)}\right]^{p'-1}dt\right)^{p-1}<\infty, \tag{2.19}$$

if

$$\frac{v}{\varphi}\in \text{DRD}(\mathbb{R}_+). \tag{2.20}$$

Note that the condition (2.20) is satisfied for weights considered in this paper, see Lemma 2.12

Finally, we comment the “size condition”

$$|Tf(x)|\le c\int\limits_{\mathbb{R}^n}\frac{|f(y)|}{|x-y|^n}dy,\ x\notin \text{ supp } f, \tag{2.21}$$

which is a formal consequence of the assumption (2.12). As we show in the lemma below, if we only care about existence of the right-hand side of (2.21) for functions in Morrey space, not about definition of the singular operator $T$ in general, then the conditions for such existence may be given in a form milder than (2.15), see conditions (2.23) and (2.26).

**Lemma 2.7** *Let $1<p<\infty$, $w$ be a weight on $\Omega$. Then for all $x\in\Omega$*

$$I(f,x,\delta):=\int\limits_{y+x_0\in\Omega,|x-y|>\delta}\frac{|f(y)|}{|x-y|^n}dy<\infty,\ \delta>0,\ f\in\mathcal{L}^{p,\varphi}_{\{x_0\}}(\Omega,w) \tag{2.22}$$

*for every space $\mathcal{L}^{p,\varphi}_{\{x_0\}}(\Omega,w)$ satisfying the condition that there exists an $\varepsilon$ such that*

$$\int\limits_{|y|>\delta}\left[\frac{\varphi(|y|)(\ln\frac{e}{\delta}|y|)^{1+\varepsilon}}{w(y+x_0)}\right]^{\frac{1}{p-1}}\frac{dy}{|y|^{np'}}<\infty. \tag{2.23}$$

***Proof*** Assume for simplicity that $x_0=0$. Suppose also that $f(y)\equiv 0$ outside $\Omega$ whenever necessary in the proof.

Since $I(f, x, \delta)$ is a decreasing function in $\delta$, it suffices to consider small $\delta$ under which $B(0, \delta) \subset \Omega$. From the embedding of Proposition 2.1 with the choice $\xi(t) = \frac{1}{(\ln \frac{e}{\delta} t)^{1+\varepsilon}}$, $\varepsilon > 0$, for $t > \delta$, and $\xi(t) = 0$ for $t \leq \delta$, we obtain that

$$\int\limits_{|y|>\delta} \frac{|f(y)|^p w(y)}{\varphi(|y|)(\ln \frac{e}{\delta}|y|)^{1+\varepsilon}} dy < \infty \tag{2.24}$$

for every $\varepsilon > 0$, if $f \in \mathcal{L}^{p,\varphi}_{\{x_0\}}(\Omega, w)$. Since $\frac{|y|}{|y-x|} \leq 1 + \frac{|x|}{\delta}$, we have

$$I(f, x, \delta) \leq c(x, \delta) \int\limits_{|y|>\delta} g(y) \left[ \frac{\varphi(|y|)(\ln \frac{e}{\delta}|y|)^{1+\varepsilon}}{w(y)} \right]^{\frac{1}{p}} \frac{dy}{|y|^n},$$

where $g(y) = f(y) \left[ \frac{w(y)}{\varphi(|y|)(\ln \frac{e}{\delta}|y|)^{1+\varepsilon}} \right]^{\frac{1}{p}} \in L^p(\Omega \backslash B(0, \delta))$. It suffices to apply the Hölder inequality, taking into account (2.24) and using the condition (2.23). □

***Remark 2.8*** In the case of radial weights $w(y) = v(|y - x_0|)$, the condition (2.23) reduces to

$$\int\limits_{\delta}^{\ell} \left[ \frac{\varphi(t)(\ln \frac{e}{\delta} t)^{1+\varepsilon}}{v(t) t^n} \right]^{\frac{1}{p-1}} \frac{dt}{t} < \infty. \tag{2.25}$$

If $\ell < \infty$, then the condition (2.25) is trivial when $\frac{\varphi}{v}$ is for instance bounded on $(\delta, \ell)$. In the case of $\ell = \infty$, which is of more interest, the condition (2.25) is fulfilled with $\varepsilon < \varepsilon_0$, if the quotient $\frac{\varphi}{v}$ satisfies the growth condition

$$\frac{\varphi(t)}{v(t)} \leq C \frac{t^n}{(\ln t)^{1+\varepsilon_0}} \text{ as } t \to \infty \tag{2.26}$$

for some $\varepsilon_0 > 0$.

## 2.3 On a class of weights

We deal with radial weights $w(x) = v(|x|)$, where $v$: $(0, \ell) \to (0, \ell)$ belongs to a certain class of functions defined in [31] and reproduced below.

**Definition 2.9** By $\mathbf{V}_{\pm}$, we denote the classes of functions $v$ positive on $(0, \ell)$, defined by the conditions:

$$\mathbf{V}_+ : \quad \frac{|v(t) - v(\tau)|}{|t - \tau|} \leq C \frac{v(t_+)}{t_+}, \tag{2.27}$$

$$\mathbf{V}_- : \quad \frac{|v(t) - v(\tau)|}{|t - \tau|} \leq C \frac{v(t_-)}{t_+}, \tag{2.28}$$

where $t, \tau \in (0, \ell)$, $t \neq \tau$, and $t_+ = \max(t, \tau)$, $t_- = \min(t, \tau)$.

**Lemma 2.10** [31] *Functions $v \in \mathbf{V}_+$ are a.i. and functions $v \in \mathbf{V}_-$ are a.d..*

Recall that a measurable positive function $v$ on $(0, \ell)$, $0 < \ell \leq \infty$, is called *quasi-monotone* if there exist $\alpha, \beta \in \mathbb{R}$ such that $\frac{v(t)}{t^\alpha}$ is a.i. and $\frac{v(t)}{t^\beta}$ is a.d.. Thus, functions in $V_+ \cup V_-$ are quasi-monotone by Lemma 2.10.

For power weights we have

$$t^\gamma \in \mathbf{V}_+ \iff \gamma \geq 0, \qquad t^\gamma \in \mathbf{V}_- \iff \quad \gamma \leq 0.$$

The following lemma provides sufficient conditions for functions to belong to the classes $\mathbf{V}_+$ and $\mathbf{V}_-$.

**Lemma 2.11** ([31, Lemma 2.11 and Example 2.12]) *Let $v$ be a function positive and differentiable on $(0, \ell)$. If*

$$0 \leq v'(t) \leq c\frac{v(t)}{t}, \quad t \in (0, \ell),$$

*for some $c > 0$, then $v \in \mathbf{V}_+$. If*

$$-c\frac{v(t)}{t} \leq v'(t) \leq 0, \ t \in (0, \ell),$$

*for some $c > 0$, then $v \in \mathbf{V}_-$.*

*In particular,*

$$t^\alpha \left(\ln e \max\left\{t, \frac{1}{t}\right\}\right)^\beta \in \begin{cases} \mathbf{V}_+, \ \textit{if}\ \alpha > 0, \beta \in \mathbb{R} \ \textit{or}\ \alpha = 0 \ \textit{and}\ \beta \leq 0 \\ \mathbf{V}_-, \ \textit{if}\ \alpha < 0, \beta \in \mathbb{R} \ \textit{or}\ \alpha = 0 \ \textit{and}\ \beta \geq 0, \end{cases}$$

Note also that for $v \in V_+ \cup V_-$ the following properties hold:

$$t^C v(t) \text{ is increasing and } \frac{v(t)}{t^C} \text{ is decreasing,} \tag{2.29}$$

where $C$ is the constant from (2.27)–(2.28). Indeed, from (2.27)–(2.28) we obtain $-C\frac{\varphi(t)}{t} \leq \varphi'(t) \leq C\frac{\varphi(t)}{t}$, whence $[t^C\varphi(t)]' \geq 0$ and $[t^{-C}\varphi(t)]' \leq 0$.

**Lemma 2.12** *Let $\ell = \infty$ and $\varphi$ satisfy the conditions 1)-3) of Sect. 2.1. The condition (2.20) is satisfied for every weight $v \in V_+ \cup V_-$.*

***Proof*** It suffices to observe that both $v$ an $\varphi$ are in $\mathrm{DRD}(\mathbb{R}_+)$. For the function $v$ this follows from Lemma 2.10 and the properties (2.29), and for the function $\varphi$ from the properties 1)-3) of Sect. 2.1 □

Finally, we shall need Lemma 2.13 given below for quasi-monotone functions. Statements of such a kind may be found dispersed in literature, see e.g. [3, 17], [25, 30] and [32]. For completeness we provide a short straightforward proof of this lemma.

**Lemma 2.13** *Let $g(t)$ be quasi-monotone, $\gamma > 0$ and $0 < \ell \le \infty$. There hold the following equivalences*

$$\int_0^r g(t)\frac{dt}{t} \le Cg(t) \Leftrightarrow \int_0^r g(t)^\gamma \frac{dt}{t} \le Cg(t)^\gamma, \ 0 < r < \ell \tag{2.30}$$

*and*

$$\int_r^\ell g(t)\frac{dt}{t} \le Cg(t) \Leftrightarrow \int_r^\ell g(t)^\gamma \frac{dt}{t} \le Cg(t)^\gamma, \ 0 < r < \ell, \tag{2.31}$$

*and inequalities in (2.30) and (2.31) imply that $g(t)$ is a.i. and a.d., respectively.*

***Proof*** It is known that quasi-monotone functions have finite Matuszewska-Orlicz indices $m(g), M(g) \in (-\infty, \infty)$ and the left-hand side inequalities in (2.30) and (2.31) are equivalent to $m(g) > 0$ and $M(g) < 0$, respectively, see e.g. [17] and [32, Appendix]. Since, $m(g^\gamma) = \gamma m(g)$ and $M(g^\gamma) = \gamma M(g)$ for $\gamma > 0$, the statement of the lemma follows. □

### 2.4 Notation for commutators

Given an operator $T$ and a function $a$, we denote

$$C[a, T] = aT - Ta.$$

In the case $T$ is an integral operator:

$$Tf(x) = \int_\Omega \mathcal{T}(x, y) f(y)dy,$$

we also define

$$\widetilde{C}[a, T]f(x) = \int_\Omega |a(x) - a(y)| \cdot |\mathcal{T}(x, y)| f(y)dy.$$

### 2.5 Norm estimates for commutators of singular and weighted Hardy operators

The following statement is derived from [7, Theorem 3.5] taking into account that its proof given in [7] for global Morrey spaces keeps for local ones as the analyses of the proof shows.

**Proposition 2.14** *Let* $1 < p < \infty$, $\varphi$ *satisfy the conditions 1) and 3),* $\mathcal{T} \in \mathcal{S}_{CZ}$ *and*

$$\int_r^\infty \left(\frac{\varphi(t)}{t^n}\right)^{\frac{1}{p}} \frac{dt}{t} \le c \left(\frac{\varphi(r)}{r^n}\right)^{\frac{1}{p}}. \tag{2.32}$$

*Let* $f \in \mathcal{L}^{p,\varphi}_{\{x_0\}}(\mathbb{R}^n)$ *and* $a \in \mathrm{BMO}$. *Then the limit* (1.1) *and the corresponding limit for the commutator* $C[a, T] = aT - Ta$ *exist almost everywhere and*

$$\|Tf\|_{\mathcal{L}^{p,\varphi}_{\{x_0\}}(\mathbb{R}^n)} \le C\|f\|_{\mathcal{L}^{p,\varphi}_{\{x_0\}}(\mathbb{R}^n)},$$
$$\|C[a, T]f\|_{\mathcal{L}^{p,\varphi}_{\{x_0\}}(\mathbb{R}^n)} \le C\|a\|^*_{\mathrm{BMO}}\|f\|_{\mathcal{L}^{p,\varphi}_{\{x_0\}}(\mathbb{R}^n)}.$$

As regards singular operators with standard kernel, their weighted boundedness in both global and local Morrey spaces is provided by the following proposition derived from [36, Theorem 3.20], where a more general setting of quasi-metric measure spaces was dealt with.

In Proposition 2.15 we impose the following Zygmund-type conditions on the function $\varphi$ :

$$\int_r^\ell \frac{\varphi(t)^{\frac{1}{p}}}{t^{1+\frac{n}{p}}} dt \le c \frac{\varphi(r)^{\frac{1}{p}}}{r^{\frac{n}{p}}} \tag{2.33}$$

and

$$\int_0^r \frac{\varphi(t)}{t} dt \le c\varphi(r), \tag{2.34}$$

where $0 < r < \ell,\ \ell = \mathrm{diam}\, \Omega < \infty$.

**Proposition 2.15** *Let* $1 < p < \infty$, $\varphi$ *satisfy the conditions 1) and 3),* $w_{x_0}(x) = v(|x - x_0|)$, $x_0 \in \Omega$, *where* $v \in V_+ \cup V_-$, *and let* $\varphi$ *satisfy the conditions* (2.33) *and* (2.34). *Let* $T$ *be a singular operator* (1.1) *with the kernel* $\mathcal{T} \in \mathcal{S}_{st}$. *Then the operator* $T$ *is bounded in the spaces* $\mathcal{L}^{p,\varphi}_{\{x_0\}}(\Omega, w)$ *and* $\mathcal{L}^{p,\varphi}(\Omega, w)$ :

$$\|Tf\|_{\mathcal{L}^{p,\varphi}_{\{x_0\}}(\Omega,w)} \le C\|f\|_{\mathcal{L}^{p,\varphi}_{\{x_0\}}(\Omega,w)}$$

*and*

$$\|Tf\|_{\mathcal{L}^{p,\varphi}(\Omega,w)} \le C\|f\|_{\mathcal{L}^{p,\varphi}(\Omega,w)},$$

*where* $C$ *does not depend on* $x_0$, *if*

*(1)* $r^{\frac{n}{p'}}\frac{\varphi(r)^{\frac{1}{p}}}{v(r)}$ *is a.i. and*

$$\int_0^r \left[t^{n(p-1)}\frac{\varphi(t)}{v(t)^p}\right]^{\frac{1}{p}}\frac{dt}{t} \le c\left[r^{n(p-1)}\frac{\varphi(r)}{v(r)^p}\right]^{\frac{1}{p}}, \ r\in(0,\ell), \tag{2.35}$$

*when* $v\in V_+$, *and*

*(2)* $\frac{\varphi(r)^{\frac{1}{p}}}{r^{\frac{n}{p}}v(r)}$ *is a.d. and*

$$\int_r^\ell \left[\frac{\varphi(t)}{t^n v(t)^p}\right]^{\frac{1}{p}}\frac{dt}{t} \le c\left[\frac{\varphi(r)}{r^n v(r)^p}\right]^{\frac{1}{p}}, \ r\in(0,\ell), \tag{2.36}$$

*when* $v\in V_-$.

The following corollary for the classical Morrey space clearly shows the weighted boundedness of singular operators with power weights "beyond the Muckenhoupt range".

**Corollary 2.16** *Let* $p$ *and* $T$ *satisfy the assumptions of Proposition 2.15. Then the operator* $T$ *is bounded, uniformly with respect to* $x_0$, *in the weighted Morrey spaces* $\mathcal{L}^{p,\varphi}_{\{x_0\}}(\Omega,|x-x_0|^\alpha)$ *and* $\mathcal{L}^{p,\varphi}(\Omega,|x-x_0|^\alpha)$ *with* $\varphi(r)=r^\lambda$, *if* $0<\lambda<n$ *and*

$$\lambda-n<\alpha<\lambda+n(p-1). \tag{2.37}$$

***Remark 2.17*** In [31] it was shown that the condition (2.37) is also necessary in the one-dimensional case for the Hilbert transform. This was extended to Riesz transforms in [24].

Norm estimates for commutators of weighted Hardy operators

$$H_w f(x)=\frac{w(x)}{|x|^n}\int\limits_{|y|<|x|}\frac{f(y)}{w(|y|)}dy \ \text{ and } \ \mathcal{H}_w f(x)=w(x)\int\limits_{|y|>|x|}\frac{f(y)}{|y|^n w(|y|)}dy, \tag{2.38}$$

provided in next propositions, are derived from [35, Theorems 3.11 and 3.16]

**Proposition 2.18** *Let* $1<p<\infty$, $\varphi$ *be a.i. and* $\varphi(2r)\le c\varphi(r), r\in\mathbb{R}_+$, $a\in \mathrm{CMO}_{q,x_0}(\mathbb{R}^n)$, *where* $q>p$ *and* $q\ge p'$. *If* $v(t)$ *is quasi-monotone and*

$$\int_0^r \frac{t^{n(p-1)}\varphi(t)}{v(t)^p}\frac{dt}{t}\le c\frac{r^{n(p-1)}\varphi(r)}{v(r)^p}, \tag{2.39}$$

*then*

$$\left\| C\left[a, wH\frac{1}{w}\right] f\right\|_{\mathcal{L}^{p,\varphi}_{\{0\}}(\mathbb{R}^n)} \leq C\|a\|^{*}_{\mathrm{CMO}_{q,x_0}(\mathbb{R}^n)}\|f\|_{\mathcal{L}^{p,\varphi}_{\{0\}}(\mathbb{R}^n)}. \tag{2.40}$$

**Proposition 2.19** *Let* $1 < p < \infty$, $\varphi$ *be a.i. and* $\varphi(2r) \leq c\varphi(r), r \in \mathbb{R}_+$, $a \in \mathrm{CMO}_{q,x_0}(\mathbb{R}^n)$, *where* $q > p$ *and* $q \geq p'$. *If* $v(t)$ *is quasi-monotone and*

$$\int_0^r \frac{\varphi(t)}{t}dt \leq c\varphi(r) \text{ and } \int_r^\infty \frac{\varphi(t)}{t^n v(t)^p}\frac{dt}{t} \leq c\frac{\varphi(r)}{r^n v(r)^p}, \tag{2.41}$$

*then*

$$\left\| C\left[a, w\mathcal{H}\frac{1}{w}\right] f\right\|_{\mathcal{L}^{p,\varphi}_{\{0\}}(\mathbb{R}^n)} \leq C\|a\|^{*}_{\mathrm{CMO}_{q,x_0}(\mathbb{R}^n)}\|f\|_{\mathcal{L}^{p,\varphi}_{\{0\}}(\mathbb{R}^n)}. \tag{2.42}$$

## 3 Main results

Everywhere in this Section we assume that $w(x) = v(|x - x_0|)$, $v \in V_+ \cup V_-$ and according to (2.18) there hold the conditions

$$\frac{1}{\varphi(|x - x_0|)}, \frac{v(|x - x_0|)^p}{\varphi(|x - x_0|)} \in A_p^{\exp}(\Omega). \tag{3.1}$$

By (2.19) the assumption in (3.1) for $\frac{1}{\varphi(t)}$ reduces to

$$\sup_{0<r<\ell} \frac{1}{r^n}\int_0^r \frac{t^{n-1}}{\varphi(t)}dt \left(\frac{1}{r^n}\int_0^r t^{n-1}\varphi(t)^{p'-1}dt\right)^{p-1} < \infty \tag{3.2}$$

and similarly for $\frac{v(t)^p}{\varphi(t)}$

### 3.1 Point-wise estimate for weighted commutators of Singular operators

We consider the weights $w_{x_0}(x) = v(|x - x_0|)$, $x_0 \in \Omega$ and deal with the following "shifted" Hardy operators

$$H_{w_{x_0}} f(x) = \frac{w_{x_0}(x)}{|x - x_0|^n} \int_{\substack{y\in\Omega \\ |y-x_0|<|x-x_0|}} \frac{f(y)}{w_{x_0}(y)}dy,$$

$$\mathcal{H}_{w_{x_0}} f(x) = w_{x_0}(x) \int\limits_{\substack{y\in\Omega \\ |y-x_0|>|x-x_0|}} \frac{f(y)}{|y-x_0|^n w_{x_0}(y)} dy. \tag{3.3}$$

We also need the following "hybrids" of Hardy and potential operators:

$$K_{x_0} f(x) = \frac{1}{|x-x_0|} \int\limits_{\substack{y\in\Omega \\ |y-x_0|<|x-x_0|}} \frac{f(y)dy}{|x-y|^{n-1}} \text{ and } \mathcal{K}_{x_0} f(x) = \int\limits_{\substack{y\in\Omega \\ |y-x_0|>|x-x_0|}} \frac{f(y)dy}{|y-x_0||x-y|^{n-1}}. \tag{3.4}$$

**Theorem 3.1** *Let $v \in V_+(0,\ell) \cup V_-(0,\ell)$, and let $T$ be the operator (1.1) with the size condition (2.12). Then for almost all $x \in \Omega$*

$$\left|C[a, T_{w_{x_0}}]f(x)\right| \le |C[a,T]f(x)| + c\Big(\widetilde{C}[a, H_{w_{x_0}}]|f|(x) + \widetilde{C}[a, K_{x_0}]|f|(x) + \widetilde{C}[a, \mathcal{K}_{x_0}]|f|(x)\Big) \tag{3.5}$$

*if $v \in \mathbf{V}_+$, and*

$$\left|C[a, T_{w_{x_0}}]f(x)\right| \le |C[a,T]f(x)| + c\Big(\widetilde{C}[a, \mathcal{H}_{w_{x_0}}]|f|(x) + \widetilde{C}[a, \mathcal{K}_{x_0}]|f|(x) + \widetilde{C}[a, K_{x_0}]|f|(x)\Big) \tag{3.6}$$

*if $v \in \mathbf{V}_-$, where $c > 0$ does not depend on $f$, $a$ and $x$.*

***Proof*** We have

$$\begin{aligned}
&\left|C[a, T_{w_{x_0}}]f(x)\right| \\
&= \left|\int\limits_{\Omega} [a(x)-a(y)]\left(\frac{w_{x_0}(x)}{w_{x_0}(y)} - 1\right)\mathcal{T}(x,y)f(y)dy + \int\limits_{\Omega}[a(x)-a(y)]\mathcal{T}(x,y)f(y)dy\right| \\
&\le c\int\limits_{\Omega} |a(x)-a(y)|\left(\frac{|w_{x_0}(x)-w_{x_0}(y)|}{w_{x_0}(y)|x-y|^n}\right)|f(y)|dy + |C[a,T]f(x)|,
\end{aligned}$$

after which the estimation of the first term on the right hand side may be made exactly in the same way as in the proof of [36, Theorem 3.11]. Following actions in [36, Page 18], in the case $v \in V_+$ we obtain

$$\left|C[a, T_{w_{x_0}}]f(x)\right| \leq |C[a, T]f(x)| + c\Bigg(\widetilde{C}[a, H_{w_{x_0}}]|f|(x) + \sum_{m=1}^{n-1}\widetilde{C}[a, K_m]|f|(x) + \widetilde{C}[a, \mathcal{K}_{x_0}]|f|(x)\Bigg),$$

where

$$K_m f(x) = \frac{1}{|x - x_0|^m}\int\limits_{\substack{y\in\Omega \\ |y-x_0|<|x-x_0|}} \frac{f(y)dy}{|x-y|^{n-m}},\ K_1 = K_{x_0}$$

and it is assumed that the $\sum_{m=1}^{n-1}$ is omitted in the case $n = 1$. To arrive at (3.5) it remains to observe that $|K_m f(x)| \leq 2K_{m-1}|f|(x),\ m \geq 2$.

Similarly in the case $v \in V_-$, also following arguments on page 18 of [36], we obtain

$$\left|C[a, T_{w_{x_0}}]f(x)\right| \leq |C[a, T]f(x)| + c\Bigg(\widetilde{C}[a, \mathcal{H}_{w_{x_0}}]|f|(x) + \sum_{m=1}^{n-1}\widetilde{C}[a, \mathcal{K}_m]|f|(x) + \widetilde{C}[a, K_{x_0}]|f|(x)\Bigg),$$

where

$$\mathcal{K}_m f(x) = \int\limits_{\substack{y\in\Omega \\ |y-x_0|>|x-x_0|}} \frac{f(y)dy}{|y-x_0|^m|x-y|^{n-m}},\ \mathcal{K}_1 = \mathcal{K}_{x_0}.$$

To get (3.5), note that $|\mathcal{K}_m f(x)| \leq 2\mathcal{K}_{m-1}|f|(x),\ m \geq 2$. □

In the lemma for $\Omega = \mathbb{R}^n$ we consider commutators of operators, slightly more general than the operators $K$ and $\mathcal{K}$ that appeared in Theorem 3.1:

$$K_\alpha f(x) = \frac{1}{|x|^\alpha}\int\limits_{|y|<|x|}\frac{f(y)dy}{|x-y|^{n-\alpha}} \text{ and } \mathcal{K}_\alpha f(x) = \int\limits_{|y|>|x|}\frac{f(y)dy}{|y|^\alpha|x-y|^{n-\alpha}},\ x \in \mathbb{R}^n, \tag{3.7}$$

where $\alpha \in (0, n)$.

**Lemma 3.2** *Let* $1 < p < \infty,\ 0 < \alpha < n$ *and* $b \in \mathrm{BMO}$. *Then*

$$\|C[a, K_\alpha]f\|_{L^p(\mathbb{R}^n)} \leq c\|a\|^*_{\mathrm{BMO}}\|f\|_{L^p(\mathbb{R}^n)}, \tag{3.8}$$

$$\|C[a, \mathcal{K}_\alpha]f\|_{L^p(\mathbb{R}^n)} \leq c\|a\|^*_{\mathrm{BMO}}\|f\|_{L^p(\mathbb{R}^n)}. \tag{3.9}$$

***Proof*** First we note that the operators $K_\alpha$ and $\mathcal{K}_\alpha$, being examples of integral operators with a kernel homogeneous of degree $-n$ and invariant with respect to rotations, are bounded in $L^p(\mathbb{R}^n)$, see the book [19, Section 6.1] or overview [18]. Note that the boundedness of operators of this class in Morrey spaces was studied in [34].

The estimate (3.9) follows from (3.8) by duality arguments. The proof of the estimate (3.8) is standard in the sense that it follows the classical way of estimation of commutators in terms of the maximal operator, see e.g. [40, 418-419]. Following this way in the case of the operator $K_\alpha$, we obtain the point-wise estimate

$$\begin{aligned}|C[a, K_\alpha]f(x)| \ &\leq c\|a\|^*_{\mathrm{BMO}}\left(M\left(|K_\alpha f|\right)(x)^{\frac{1}{s}} + M\left(|f|^s\right)(x)^{\frac{1}{s}}\right.\\ &\left.+K_{\alpha s}\left(|f|^s\right)(x)^{\frac{1}{s}}\right),\ 1 < s < \frac{n}{\alpha},\end{aligned} \tag{3.10}$$

where $M$ is the maximal operator. (We omit details of the proof for (3.10) since this proof is absolutely similar to that of [40, 418-419]). The estimate (3.8) immediately follows from (3.10) in view of the boundedness of the operators $M$ and $K_{\alpha,s}$ in $L^p(\mathbb{R}^n)$. □

### 3.2 Weighted norm estimates for the commutators of singular operators in local Morrey spaces

To prove the main Theorem 3.4 we need an auxiliary estimate given in the following proposition. The statement of this proposition is derived from estimates in the proof of Theorem 3.5 in [7], see the estimates between the formulas (11) and (18) in [7].

**Proposition 3.3** *Assume that $\varphi$ is almost increasing and (2.32) holds and let*

$$A_r f(x) := \chi_{B(x_0,r)}(x) \int\limits_{\mathbb{R}^n\setminus B(x_0,2r)} |a(x)-a(y)|\frac{|f(y)|}{|x-y|^n}dy.$$

*Then*

$$\|A_r f\|_{L^{p,\varphi}_{\{x_0\}}(\mathbb{R}^n)} \leq c\|a\|^*_{\mathrm{BMO}}\|f\|_{L^{p,\varphi}_{\{x_0\}}(\mathbb{R}^n)}, \tag{3.11}$$

**Theorem 3.4** *Let* $1 < p < \infty,\ a \in \mathrm{BMO}^{\mathrm{ext}}(\Omega)$ *and* $w(x) = v(|x-x_0|),\ x_0 \in \Omega,$ *where* $v \in V_+ \cup V_-(0,\ell),\ \ell =$ diam $\Omega,\ 0 < \ell \leq \infty$ *and $T$ be the singular operator* (1.1) *and $\varphi$ satisfy the condition 1) and 3). Let the conditions*

$$\int\limits_0^r \varphi(t)\frac{dt}{t} \leq c\varphi(r), \tag{3.12}$$

$$\int\limits_r^\ell \left[\frac{\varphi(t)}{t^n}\right]^{\frac{1}{p}}\frac{dt}{t} \leq c\left[\frac{\varphi(r)}{r^n}\right]^{\frac{1}{p}}, \tag{3.13}$$

$$\int_0^r \frac{t^{n(p-1)}\varphi(t)}{v(t)^p}\frac{dt}{t} \le c\frac{r^{n(p-1)}\varphi(r)}{v(r)^p}, \tag{3.14}$$

$$\int_r^\ell \frac{\varphi(t)}{t^n v(t)^p}\frac{dt}{t} \le c\frac{\varphi(r)}{r^n v(r)^p} \tag{3.15}$$

*be satisfied. Then the operator* $T_w = wT\frac{1}{w}$ *is bounded in the space* $\mathcal{L}^{p,\varphi}_{\{x_0\}}(\Omega)$ *whenever its kernel* $\mathcal{T}$ *belongs to the class* $\mathcal{S}_{st}$*, as well as its commutator* $C[a, T_w]$ *whenever* $\mathcal{T} \in \mathcal{S}_{CZ}$*, and*

$$\|C[a, T_w]f\|_{\mathcal{L}^{p,\varphi}_{x_0}(\Omega)} \le c\|a\|^*_{\mathrm{BMO}}\|f\|_{\mathcal{L}^{p,\varphi}_{x_0}(\Omega)}, \tag{3.16}$$

*where* $c$ *does not depend on* $f$*,* $a$ *and* $x_0$*.*

***Proof*** The boundedness of the operator $T_w$ follows from Proposition 2.15 if we take into account that the functions $\varphi(t)$ and $v(t)$ are quasi-monotone and consequently conditions (2.35) and (2.36) are equivalent to the corresponding inequalities (3.14) and (3.15) by Lemma 2.13.

Passing to commutators, we write $w = w_{x_0}$ to underline the dependence of weighted operators on the point $x_0$. In view of the estimates (3.5) and (3.6), to prove (3.16) it suffices to have estimates for $\|C[a, T]f\|_{\mathcal{L}^{p,\varphi}_{\{x_0\}}(\Omega)}$, $\|\widetilde{C}[a, H_{w_{x_0}}]f\|_{\mathcal{L}^{p,\varphi}_{\{x_0\}}(\Omega)}$, $\|\widetilde{C}[a, \mathcal{H}_{w_{x_0}}]f\|_{\mathcal{L}^{p,\varphi}_{\{x_0\}}(\Omega)}$, $\|\widetilde{C}[a, K_{x_0}]f\|_{\mathcal{L}^{p,\varphi}_{\{x_0\}}(\Omega)}$, and $\|\widetilde{C}[a, \mathcal{K}_{x_0}]f\|_{\mathcal{L}^{p,\varphi}_{\{x_0\}}(\Omega)}$.

In what follows, we continue the function $f$ outside $\Omega$ by zero, extend the operators $T, H_{w_{x_0}}, \mathcal{H}_{w_{x_0}}, K_{x_0}, \mathcal{K}_{x_0}$ in natural way to $\mathbb{R}^n$ and continue $v(r)$ by any positive constant for $r > \ell$ (in the case $\ell < \infty$). We also extend the function $\varphi(r)$, when $\ell < \infty$, keeping in mind that the conditions (3.12)-(3.15), should be preserved. To this end one can use the extension $\varphi(r) = r^\delta$ for $r > \ell$ with sufficiently small $\delta > 0$. Note that one can take $\delta = 0$ for the preservation of the conditions (3.13)- (3.15), but for the preservation of (3.12) $\delta$ should be positive.

We should take care about uniformness of the constant $c$ with respect to $x_0$.

Estimate for $\|C[a, T]f\|_{\mathcal{L}^{p,\varphi}_{\{x_0\}}(\mathbb{R}^n)}$ is provided by Proposition 2.14 in view of (3.13).

The remaining four commutators $\widetilde{C}[a, H_{w_{x_0}}]$, $\widetilde{C}[a, \mathcal{H}_{w_{x_0}}]$, $\widetilde{C}[a, K_{x_0}]$ and $\widetilde{C}[a, \mathcal{K}_{x_0}]$ depend on $x_0$.

Let $\mathfrak{C}_{x_0}$ denote any of them and let $\tau_{x_0}f(x) = f(x_0 - x)$. We have

$$\mathfrak{C}_{x_0} = \tau_{x_0}\mathfrak{C}_0\tau_{x_0},$$

where $\mathfrak{C}_0 = \mathfrak{C}_{x_0}|_{x_0=0}$. Note that

$$\|\tau_{x_0}f\|_{\mathcal{L}^{p,\varphi}_{\{x_0\}}(\mathbb{R}^n)} = \|f\|_{\mathcal{L}^{p,\varphi}_{\{0\}}(\mathbb{R}^n)}.$$

We obtain

$$\|\mathfrak{C}_{x_0}f\|_{\mathcal{L}^{p,\varphi}_{\{x_0\}}(\mathbb{R}^n)} = \|\mathfrak{C}_0\tau_{x_0}f\|_{\mathcal{L}^{p,\varphi}_{\{0\}}(\mathbb{R}^n)}, \tag{3.17}$$

which insures uniformness with respect to $x_0$ and we can take $x_0 = 0$.

The estimates for $\widetilde{C}[a, H_{w_{x_0}}]$ and $\widetilde{C}[a, \mathcal{H}_{w_{x_0}}]$ follow from Propositions 2.18 and 2.19 in view of (3.12), (3.14) and (3.15).

Let now $\mathfrak{C}_0$ stand for one of the commutators $\widetilde{C}[a, K_{x_0}]$, $\widetilde{C}[a, \mathcal{K}_{x_0}]$. To estimate the norm $\|\mathfrak{C}_0 f\|_{\mathcal{L}^{p,\varphi}_{\{0\}}(\mathbb{R}^n)}$ we split the function $f$ in the standard way:

$$f(y) = \chi_{B(x_0,2r)}(y) + f(y)\chi_{\mathbb{R}^n\setminus B(x_0,2r)}(y) =: f_1(y) + f_2(y).$$

The estimate for $\|\mathfrak{C}_0 f_1\|_{\mathcal{L}^{p,\varphi}_{\{0\}}(\mathbb{R}^n)}$ follows from Lemma 3.2:

$$\|\mathfrak{C}_0 f_1\|_{L^p(B(0,2r)} \le \|\mathfrak{C}_0 f_1\|_{L^p(\mathbb{R}^n)} \le C\|a\|^*_{\mathrm{BMO}}\|f_1\|_{L^p(\mathbb{R}^n)} = C\|a\|^*_{\mathrm{BMO}}\|f\|_{L^p(B(0,2r))}.$$

For $f_2$ observe that $\frac{1}{|x|} < \frac{2}{|x-y|}$ in the case of the operator $K$, and $\frac{1}{|y|} < \frac{2}{|x-y|}$ in the case of the operator $\mathcal{K}$. Hence

$$\chi_{B(0,r)}(x)\ \mathfrak{C}_0 f_2(x) \le 2A_r f_2(x),$$

where $A_r$ is the operator from Proposition 3.3. Then from that proposition

$$\|\mathfrak{C}_0 f_2\|_{\mathcal{L}^{p,\varphi}_{\{0\}}(\mathbb{R}^n)} \le C\|a\|^*_{\mathrm{BMO}}\|f\|_{\mathcal{L}^{p,\varphi}_{\{0\}}(\mathbb{R}^n)}.$$

Gathering the estimates, we arrive at (3.16). □

**Corollary 3.5** *Under the assumptions of Theorem 3.4, there holds the following estimate for the commutator of singular operator $T$ in weighted local Morrey spaces:*

$$\sup_{r>0}\left(\frac{1}{\varphi(r)}\int\limits_{B(x_0,r)\cap\Omega} |C[a,T]f(y)|^p v(|y-x_0|)^p dy\right)^{\frac{1}{p}}$$
$$\le c\|a\|^*_{\mathrm{BMO}}\sup_{r>0}\left(\frac{1}{\varphi(r)}\int\limits_{B(x_0,r)\cap\Omega} |f(y)|^p v(|y-x_0|)^p dy\right)^{\frac{1}{p}}, \qquad (3.18)$$

*where $c$ does not depend on $x_0$.*

In the following corollary we see the "beyond Muckenhoupt range" effect in the estimate for commutators of singular operators in classical Morrey spaces.

**Corollary 3.6** *Let $p$ and $a$ satisfy the assumptions of Theorem 3.4. The estimate (3.16) with $\varphi(r) = r^\lambda$ and $v(r) = r^\alpha$ holds if*

$$0 < \lambda < n \text{ and } -\frac{n}{p} + \frac{\lambda}{p} < \alpha < \frac{n}{p'} + \frac{\lambda}{p}.$$

***Proof*** To derive the statement of the corollary, it suffices to note that the conditions (3.14) and (3.15) are satisfied under the choice $\varphi(r) = r^\lambda$ and $v(r) = r^\alpha$ with $0 < \lambda < n$ and $-\frac{n}{p} + \frac{\lambda}{p} < \alpha < \frac{n}{p'} + \frac{\lambda}{p}$. □

### 3.3 Norm estimates for the commutators of singular operators in Stummel-Morrey spaces

Let $1 \leq p < \infty$ and $\varphi, v : (0, \ell) \to \mathbb{R}_+$. We define Stummel-Morrey space $\mathfrak{S}^{p,\varphi,v}(\Omega)$ by the norm

$$\|f\|_{\mathfrak{S}^{p,\varphi,v}(\Omega)} = \sup_{x\in\Omega, r\in(0,\ell)} \left( \frac{1}{\varphi(r)} \int\limits_{B(x,r)} |f(y)|^p\, v(|x-y|)^p dy \right)^{\frac{1}{p}}. \tag{3.19}$$

Spaces with the norm of the type (3.19) with the power function $\varphi$ appeared in [13].

As a consequence of Corollary 3.5, we arrive at the following statement.

**Theorem 3.7** *Let $p, \varphi$ and $v$ satisfy the assumptions of Theorem 3.4 and $a \in$ $\mathrm{BMO}^{\mathrm{ext}}(\Omega)$. Then*

$$\|C[a, T]f\|_{\mathfrak{S}^{p,\varphi,v}(\Omega)} \leq c\|a\|^*_{\mathrm{BMO}}\|f\|_{\mathfrak{S}^{p,\varphi,v}(\Omega)}, \tag{3.20}$$

***Proof*** It suffices to pass to supremum in (3.18) with respect to $x_0 \in \Omega$, taking into account that the constant $c$ in (3.18) does not depend on $x_0$. □

## 4 Applications to regularity properties of solutions of elliptic PDEs: Interior estimates

Regularity properties of solution to elliptic equation in the non-weighted setting of Lebesgue spaces were studied by Chiarenza etal in [5]. A crucial base in that study was the so-called *representation formula* for second order derivative of solution to elliptic PDEs. This formula, proved in [5] for $C_0^\infty$-functions in case of Lebesgue spaces, is extended by density argument to Sobolev spaces. Such a study of regularity properties in case of Morrey spaces was first made in [8], see also [7] and references therein. Since Morrey spaces on bounded domains are embedded into Lebesgue spaces, application of the representation formula for Morrey spaces on bounded domains does not need a justification.

This is not the case for weighted Morrey spaces: functions in a weighted Morrey space may prove to be non-integrable, see Sect. 4.1. So the use of the representation formula for weighted Morrey spaces needs a justification.

### 4.1 On the representation formula in the case of weighted Morrey spaces

Let $\Omega$ be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$. We study regularity problems for solutions to the elliptic equations

$$Lu := \sum_{i,j=1}^{n} a_{i,j}(x)u_{x_i,x_j} = f, \ x \in \Omega. \tag{4.1}$$

in weighted generalized Morrey spaces, and in Sect. 4.2 provide interior estimates for the second order derivatives of solutions in these spaces.

Everywhere in the sequel, the following conditions of regularity and ellipticity are assumed to be satisfied for the coefficients $a_{i,j}$ :

* $\{a_{i,j}\}_{i,j=1}^n \subset$ VMO$(\Omega) \cap L^\infty(\Omega)$,
* $a_{i,j} = a_{j,i}$ for all $i, j = 1, ..., n$ and for a. e. $x \in \Omega$,
* $\exists m > 0 : m^{-1}|h|^2 \leq \sum_{i,j=1}^n h_i h_j \leq m|h|^2$ for a.e. $x \in \Omega$ and all $x \in \mathbb{R}^n$.

First, following [7] we recall the necessary definitions used in the representation formula proved in [5].

Let

$$\Gamma(x,t) := \frac{1}{(n-2)|B(0,1)|\sqrt{\det a_{i,j}(x)}} \left(A_{i,j}(x)t_i t_j\right)^{(2-n)2}, n \geq 3, \ a.e. \ x \in \Omega,$$

and for all $t \in \mathbb{R}^n \setminus \{0\}$, $A_{i,j}$ denotes the entries of the inverse matrix of the matrix $\{a_{i,j}\}_{i,j=1}^n$;

$$\Gamma_i(x,t) := \frac{\partial}{\partial t_i}\Gamma(x,t), \ \Gamma_{i,j}(x,t) := \frac{\partial^2}{\partial t_i \partial t_j}\Gamma(x,t)$$

and

$$\max_{i,j=1,\dots,n} \max_{|\alpha|\leq 2n} \left\| \frac{\partial^\alpha \Gamma_{i,j}(x,t)}{\partial t^\alpha} \right\|_{L^\infty(\Omega\times\mathbb{S}^{n-1})} =: M.$$

As known $\Gamma_{i,j}(x,t)$ are Calderòn-Zygmund kernels in the $t$ variable and, for any fixed $x_0 \in \Omega$, $\Gamma(x_0,t)$ is a fundamental solution for the operator $L_0u(x) := \sum_{i,j=1}^n a_{i,j}(x_0)u_{x_i,x_j}(x)$.

The representation formula for second order derivatives of a solution to the equation (4.1), proved in [5, Theorem 3.1] for $u \in C_0^\infty(B)$, reads

$$\begin{aligned} u_{x_i,x_j}(x) &= P.V. \int_B \Gamma_{i,j}(x,x-y)\left(\sum_{h,k=1}^n (a_{h,k}(x)-a_{h,k}(y))u_{x_h,x_k}(y) + Lu(y)\right)dy \\ &+ Lu(x)\int_{|t|=1} \Gamma_i(x,t)t_j d\sigma(t), \ \text{for a.e. } x \in B. \end{aligned}$$

As mentioned, by density arguments it is valid for $u \in W_0^{2,p}(B)$, and also for $u \in W_0^2\mathcal{L}^{p,\varphi}(\Omega)$, since Morrey spaces on bounded domains are embedded into Lebesgue spaces.

This is not the case for weighted Morrey spaces. Depending on weight, functions in weighted Morrey spaces may prove to be even non-integrable. Indeed, let e.g. $\Omega = B(0,1)$, $\varphi(r) = r^\lambda$, $0 < \lambda \le n$, and $w = |x|^\alpha$, $\alpha \in \mathbb{R}$. The function

$$f_0(x) = \frac{1}{|x|^{\frac{n-\lambda+\alpha}{p}}}$$

belongs to $\mathcal{L}^{p,\varphi}(\Omega, |x|^\alpha)$ and is not integrable when $\alpha \ge n(p-1) + \lambda$ (note that the value $\alpha = n(p-1) + \lambda$ is "beyond the Muckenhoupt range" borderline value for exponents of power weights, see Corollary 2.16 and Remark 2.17; compare also with Proposition 4.3 ).

Thus, in the case where no á priori information on weights is provided, application of the representation formula for weighted Morrey spaces needs justification. To this end, it suffices to have an embedding of weighted Morrey space into some Lebesgue space $L^s(\Omega)$, $s > 1$.

We make use of the following proposition derived from [33, Theorem 3.2] where it was proved in the general setting of quasi-metric measure spaces.

**Proposition 4.1** *Let $1 < p < \infty$, $\varphi$ satisfy the assumptions 1) - 3) of Sect. 2.1. Suppose that there exists $s \in (1, p)$ such that*

$$t^{n\left(\frac{1}{s}-\frac{1}{p}\right)} \frac{\varphi(t)^{\frac{1}{p}}}{v(t)} \text{ is almost increasing} \tag{4.2}$$

*and*

$$\int_0^\ell t^{n-1} \left[\frac{\varphi(t)^{\frac{1}{p}}}{t^{\frac{n}{p}} v(t)}\right]^s dt < \infty, \ \ell = \text{ diam } \Omega. \tag{4.3}$$

*Then*

$$\mathcal{L}^{p,\varphi}_{\{x_0\}}(\Omega, w^p_{x_0}) \hookrightarrow L^s(\Omega). \tag{4.4}$$

**Corollary 4.2** *Under the assumptions of Proposition 4.1, a similar embedding holds for Stummel-Morrey spaces:*

$$\mathfrak{S}^{p,\varphi,v}(\Omega) \hookrightarrow L^s(\Omega).$$

The next statement of criterion-type for the classical Morrey space, i.e. $\varphi(r) = r^\lambda$, $0 < \lambda \le n$, $0 < r < \ell$, and power-logarithmic weights

$$v(r) = r^\alpha \left(\ln \frac{2\ell}{r}\right)^\beta, \tag{4.5}$$

was proved in [33, Corollary 3.4] in a more general setting of quasimetric measure spaces.

**Proposition 4.3** *Let* $1 < p < \infty$ *Then the embeddings*

$$\mathcal{L}^{p,\varphi}_{\{x_0\}}(\Omega, w_{x_0})|_{\varphi=r^\lambda} \hookrightarrow L^s(\Omega) \text{ and } \mathcal{L}^{p,\varphi}(\Omega, w_{x_0})|_{\varphi=r^\lambda} \hookrightarrow L^s(\Omega), \; \lambda > 0, \tag{4.6}$$

*where* $s \in (1, p)$, *hold, if and only if*

$$\alpha < \lambda + n\left(\frac{p}{s} - 1\right) \text{ and } \beta \in \mathbb{R} \text{ or } \alpha = \lambda + n\left(\frac{p}{s} - 1\right) \text{ and } \beta > \frac{p}{s}, \tag{4.7}$$

**Corollary 4.4** *Let the weight* $w$ *be defined by* (4.5). *The exponent* $s \in (1, p)$ *for the embeddings* (4.6) *exists, if and only if* $\alpha < \lambda + n(p-1)$ *and* $\beta \in \mathbb{R}$, *or* $\alpha = \lambda + n(p-1)$ *and* $\beta > 1$.

### 4.2 Interior estimates

Our main interest being related to weights, for readers' convenience, below we collect all the conditions on the weights arising from the results of Sect. 3 and Proposition 4.1:

$$\int_0^r t^{\frac{n}{p'}} \frac{\varphi(t)^{\frac{1}{p}}}{v(t)} \frac{dt}{t} \leq c r^{\frac{n}{p'}} \frac{\varphi(r)^{\frac{1}{p}}}{v(r)}, \; r \in (0, \ell), \tag{4.8}$$

$$\int_r^\ell \frac{\varphi(t)^{\frac{1}{p}}}{t^{\frac{n}{p}} v(t)} \frac{dt}{t} \leq c \frac{\varphi(r)^{\frac{1}{p}}}{r^{\frac{n}{p}} v(r)}, \; r \in (0, \ell), \tag{4.9}$$

$$\text{There exists } s \in (1, p) \text{ such that } t^{n\left(\frac{1}{s} - \frac{1}{p}\right)} \frac{\varphi(t)^{\frac{1}{p}}}{v(t)} \text{ is almost increasing} \tag{4.10}$$

and

$$\int_0^\ell t^{n-1} \left[\frac{\varphi(t)^{\frac{1}{p}}}{t^{\frac{n}{p}} v(t)}\right]^s dt < \infty, \; \ell = \text{ diam } \Omega. \tag{4.11}$$

In Theorems 4.5 and 4.7 we use some notation for Sobolev-Morrey and Sobolev-Stummel spaces. Denote by $X = X(\Omega)$ any function space on $\Omega$ and let

$$\|f\|_{W^2X} = \|f\|_X + \sum_{j,k=1}^n \left\|\frac{\partial^2 f}{\partial x_j \partial x_k}\right\|_X. \tag{4.12}$$

By $W^2_0 X = W^2_0 X(\Omega)$ we denote the closer, with respect to the norm (4.12), of $C^\infty$-functions with compact support in $\Omega$.

**Theorem 4.5** *Let* $n \geq 3$, $1 < p < \infty$. $a_{i,j} \in \mathrm{VMO}(\Omega) \cap L^\infty(\Omega)$, $q > p$ *and* $q \geq p'$. *Let* $\varphi$ *satisfy the à priory assumptions 1) - 3) of Sect. 2.1, and let the conditions* (2.34) *and* (3.13) *for* $\varphi$ *be satisfied. If the weight* $w(x) = w_{x_0} = v(|x - x_0|)$, $x_0 \in \Omega$, $v \in V_+ \cup V_-$. *satisfies the conditions* (4.8) - (4.11)*, then there exist positive constants* $C = C(n, p, \varphi, w, M)$ *not depending on* $x_0$, *and* $r_0 = r_0(C)$, *such that*

$$\|u_{x_i,x_j}\|_{\mathcal{L}^{p,\varphi}_{\{x_0\}}(B_r, w^p)} \leq C\|f\|_{\mathcal{L}^{p,\varphi}_{\{x_0\}}(B_r, w^p)} \tag{4.13}$$

*for any ball* $B_r \subsetneq \Omega$, $B_r \ni x_0$ *of radius* $r < r_0$, *and all* $u \in W^2_0\mathcal{L}^{p,\varphi}_{\{x_0\}}(\Omega, w^p)$.

***Proof*** The proof follows the known procedure, our main interest being to admit the interior estimate for weighted Morrey spaces, so we omit details. We just have to apply the weighted Morrey norm over $B$, $B \subset \Omega$, to the representation formula termwise, make use of Corollary 3.5 and to pass to small balls $B_r$ using the fact that $a_{i,j} \in \mathrm{VMO}$.

We only mention that the conditions (3.14) and (3.15) of Theorem 3.4 are equivalent to the conditions (4.8) and (4.9), respectively, for quasi-monotone weights, see Lemma 2.13. □

***Remark 4.6*** In Theorem 4.5 one may replace $B_r \ni x_0$ by a ball $B_r$ located anywhere in $\Omega$, the main meaning of the restriction $B_r \ni x_0$ is that our interest concerns weighted Morrey spaces while $\mathcal{L}^{p,\varphi}_{\{x_0\}}(B \backslash B_{x_0,\varepsilon}, w^p_{x_0})$, $\varepsilon > 0$, is a non-weighted space for weights under consideration.

Finally, in the following theorem we extend Theorem 4.5 to Stummel-Morrey spaces, the latter being a kind of replacement of global Morrey spaces.

**Theorem 4.7** *Let* $p$, $\varphi$ *and* $w$ *satisfy the assumptions of Theorem 4.5 and* $a_{i,j} \in \mathrm{VMO}(\Omega) \cap L^\infty(\Omega)$. *Then there exist positive constants* $C = C(n, p, \varphi, w, M)$ *and* $r_0 = r_0(C)$, *such that*

$$\|u_{x_i,x_j}\|_{\mathfrak{S}^{p,\varphi,v}(B_r)} \leq C\|f\|_{\mathfrak{S}^{p,\varphi,v}(B_r)} \tag{4.14}$$

*for any ball* $B_r \subsetneq \Omega$ *of radius* $r < r_0$, *and all* $u \in W^2_0\mathfrak{S}^{p,\varphi,v}(\Omega)$, *where*

$$\|f\|_{\mathfrak{S}^{p,\varphi,v}(B_r)} = \sup_{x_0 \in B_r, 0<t<r} \left( \frac{1}{\varphi(t)} \int\limits_{B(x_0,t)\cap B_r} |f(y)|^p \, v(|x_0 - y|)^p dy \right)^{\frac{1}{p}}.$$

***Proof*** It remains to pass to supremum with respect to $x_0$ in the estimate (4.13), taking into account that the constant $c$ there does not depend on $x_0$. □

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## References

1. Almeida, A., Samko, S.: Embeddings of local generalized Morrey spaces between weighted Lebesgue spaces. Nonlinear Anal. **164**, 67–76 (2017)
2. Alvarez, J., Lakey, J., Guzmán-Partida, M.: Spaces of bounded λ-central mean oscillation, Morrey spaces, and λ-central Carleson measures. Collect. Math. **51**(1), 1–47 (2000)
3. Bari, N.K., Stechkin, S.B.: Best approximations and differential properties of two conjugate functions (in Russian). Proc. Moscow Math. Soc. **5**, 483–522 (1956)
4. Balakishiyev, A.S., Guliyev, V.S., Gurbuz, F., Serbetci, A.: Sublinear operators with rough kernel generated by Calderón-Zygmund operators and their commutators on generalized local Morrey spaces. J. Inequal. Appl. **61**, 18 (2015)
5. Chiarenza, F., Frasca, M., Longo, P.: $w^{2,p}$-solvability of the dirichlet problem for nondivergence elliptic equations with VMO coefficients. Trans. Am. Math. Soc. **336**(2), 841–853 (1993)
6. Cruz-Uribe, D., Pérez, C.: On the two-weight problem for singular integral operators. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **1**(4), 821–849 (2002)
7. Di Fazio, G., Hakim, D., Sawano, Y.: Elliptic equations with discontinuous coefficients in generalized Morrey spaces. Eur. J. Math. **3**(3), 728–762 (2017)
8. Di Fazio, G., Ragusa, M.A.: Interior estimates in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients. J. Funct. Anal. **112**(2), 241–256 (1993)
9. Duoandikoetxea, J.: Fourier analysis, volume 29 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001. Translated and revised from the 1995 Spanish original by David Cruz-Uribe
10. Duoandikoetxea, J., Rosenthal, M.: Boundedness of operators on certain weighted Morrey spaces beyond the Muckenhoupt range. Potential Anal. **53**(4), 1255–1268 (2020)
11. Duoandikoetxea, J., Rosenthal, M.: Singular and fractional integral operators on weighted local Morrey spaces. J. Fourier Anal. Appl. **28**(3), 43 (2022)
12. Dyn'kin, E.M., Osilenker, B.B.: Weighted norm estimates for singular integrals and their applications. J. Sov. Math. **30**, 2094–2154 (1985)
13. Eridani, A., Kokilashvili, V., Meskhi, A.: Morrey spaces and fractional integral operators. Expo. Math. **27**(3), 227–239 (2009)
14. García-Cuerva, J., Rubio de Francia, J.: Weighted norm inequalities and related topics, volume 116 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1985. Notas de Matemática [Mathematical Notes], 104
15. Guzmán-Partida, M.: CLO spaces and central maximal operators. Arch. Math. (Brno) **2**, 119–124 (2013)
16. Ho, K.-P.: Deffinability of singular integral operators on Morrey-Banach spaces. J. Math. Soc. Japan **72**, 155–170 (2020)
17. Karapetiants, N., Samko, N.: Weighted theorems on fractional integrals in the generalized Hölder spaces $H_0^\omega(\rho)$ via the indices $m_\omega$ and $M_\omega$. Fract. Calc. Appl. Anal. **7**(4), 437–458 (2004)
18. Karapetiants, N., Samko, S.: Multidimensional integral operators with homogeneous kernels. Fract. Calculus Appl. Anal. **2**(1), 67–96 (1999)
19. Karapetiants, N., Samko, S.: Equations with Involutive Operators. Birkhäuser, Boston (2001)
20. Kokilashvili, V., Meskhi, A., Rafeiro, H., Samko, S.: Integral Operators in Non-Standard Function Spaces. Volumes I-II, volume 248-249 of *Operator Theory: Advances and Applications*. Birkhäuser Basel, (2016)

21. Leonardi, S.: Remarks on the regularity of solutions of elliptic systems. In *Applied nonlinear analysis*, pp. 325–344. Kluwer/Plenum, New York, (1999)
22. Leonardi, S.: Weighted Miranda-Talenti inequality and applications to equations with discontinuous coefficients. Comment. Math. Univ. Carolin. **43**(1), 43–59 (2002)
23. Lu, S., Yang, D.: The central BMO spaces and Littlewood-Paley operators. Approx. Theory Appl. (N.S.) **11**(3), 72–94 (1995)
24. Nakamura, S., Sawano, Y.: The singular integral operator and its commutator on weighted Morrey spaces. Collect. Math. **68**(2), 145–174 (2017)
25. Nakamura, S., Noi, T., Sawano, Y.: Generalized Morrey spaces and trace operator. Sci. China Math. **59**(2), 281–336 (2016)
26. Pick, L., Kufner, A., John, O., Fučík, S.: Function spaces. Vol. 1. Walter de Gruyter & Co., Berlin, (2013)
27. Rafeiro, H., Samko, N., Samko, S.: Morrey-Campanato spaces: an overview. In: Operator theory, pseudo-differential equations, and mathematical physics, volume 228 of Oper. Theory Adv. Appl., pp. 293–323. Birkhäuser/Springer Basel AG, Basel, (2013)
28. Rosenthal, M., Triebel, H.: Calderon-Zygmund operators in Morrey spaces. Rev. Matem. Complut. **27**(1), 1–11 (2014)
29. Rosenthal, M., Triebel, H.: Morrey spaces, their duals and preduals. Rev. Matem. Complut. **28**(1), 1–30 (2015)
30. Samko, N.: Singular integral operators in weighted spaces with generalized Hölder condition. Proc. A. Razmadze Math. Inst **120**, 107–134 (1999)
31. Samko, N.: Weighted Hardy and singular operators in Morrey spaces. J. Math. Anal. Appl. **350**(1), 56–72 (2009)
32. Samko, N.: Weighted Hardy operators in the local generalized vanishing Morrey spaces. Positivity **17**(3), 683–706 (2013)
33. Samko, N.: Embeddings of weighted generalized Morrey spaces into Lebesgue spaces on fractal sets. Fract. Calc. Appl. Anal. **22**(5), 1203–1224 (2019)
34. Samko, N.: Integral operators commuting with dilations and rotations in generalized Morrey-type spaces. Math. Methods Appl. Sci. **43**(16), 9416–9434 (2020)
35. Samko, N.: Weighted fractional Hardy operators and their commutators on generalized Morrey spaces over quasi-metric measure spaces. Fract. Calc. Appl. Anal. **24**(6), 1643–1669 (2021)
36. Samko, N.: Weighted boundedness of certain sublinear operators in generalized Morrey spaces on quasi-metric measure spaces under the growth condition. J. Fourier Anal. Appl. **28**(2), 27 (2022)
37. Samko, S.: Morrey spaces are closely embedded between vanishing Stummel spaces. Math. Inequal. Appl. **17**(2), 627–639 (2014)
38. Sawano, Y., Di Fazio, G., Hakim, D.I.: Morrey Spaces: Introduction and Applications to Integral Operators and PDE's. Volumes I & II (1st ed.). Chapman and Hall / CRC, New York, (2020)
39. Stummel, F.: Singuläre elliptische Differential-operatoren in Hilbertschen Räumen. Math. Ann. **132**, 150–176 (1956)
40. Torchinsky, A.: Real-variable methods in harmonic analysis. Pure and Applied Mathematics, vol. 123. Academic Press Inc, Orlando, FL (1986)
41. Triebel, H.: Hybrid function spaces, heat and Navier-Stokes equations, volume 24 of EMS Tracts in Mathematics. European Mathematical Society (EMS), Zürich, (2014)