Journal of Symbolic Computation 128 (2025) 102393

Contents lists available at ScienceDirect

Journal of Symbolic Computation

journal homepage: www.elsevier.com/locate/jsc

On nonnegative invariant quartics in type A

Sebastian Debus [a], Charu Goel [b], Salma Kuhlmann [c], Cordian Riener [d]

[a] Faculty of Mathematics, Technische Universität Chemnitz, Germany
[b] Department of Basic Sciences, Indian Institute of Information Technology, Nagpur, India
[c] Department of Mathematics and Statistics, University of Konstanz, Germany
[d] Department of Mathematics and Statistics, UIT The Arctic University of Norway, Norway

ARTICLE INFO

Article history:
Received 22 February 2024
Received in revised form 30 September 2024
Accepted 1 October 2024
Available online 9 October 2024

Keywords:
Positive polynomials
Sums of squares
Symmetric forms

ABSTRACT

The equivariant nonnegativity versus sums of squares question has been solved for any infinite series of essential reflection groups but type A. As a first step to a classification, we analyse A_n-invariant quartics. We prove that the cones of invariant sums of squares and nonnegative forms are equal if and only if the number of variables is at most 3 or odd.

© 2024 The Author(s). Published by Elsevier Ltd. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/).

1. Introduction

The study of *nonnegative* real polynomials, i.e. polynomials whose evaluation on any point is nonnegative, is a topic of interest from many perspectives, e.g. verification of polynomial inequalities and polynomial optimization. From a complexity theoretical view the verification is NP-hard (Blum et al., 1998). If one can write a real polynomial as a *sum of squares* of real polynomials, then the polynomial is clearly nonnegative. It was shown by Hilbert (1888) in his celebrated theorem from 1888 that there are basically three cases where any nonnegative polynomial is a sum of squares. We formulate Hilbert's theorem in terms of *forms*, i.e. homogeneous polynomials, since any polynomial is nonnegative if and only if its homogenization is nonnegative and a sum of squares if and only if its homogenization is a sum of squares (Marshall, 2008). Hilbert showed that the cones of

E-mail addresses: sebastian.debus@mathematik.tu-chemnitz.de (S. Debus), charugoel@iiitn.ac.in (C. Goel), salma.kuhlmann@uni-konstanz.de (S. Kuhlmann), cordian.riener@uit.no (C. Riener).

https://doi.org/10.1016/j.jsc.2024.102393
0747-7171/© 2024 The Author(s). Published by Elsevier Ltd. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/).

nonnegative forms and that of sums of squares of degree $2d$ in n variables are equal if and only if $(n, 2d) \in \{(2, 2d'), (n', 2), (3, 4) \mid n', d' \in \mathbb{N}\}$. Hilbert's proof was non-constructive and it took almost 80 years until the first example of a nonnegative polynomial which is not a sum of squares was given (this is the Motzkin polynomial (Motzkin, 1967)). It was then asked by Hilbert whether any nonnegative polynomial is a sum of squares of rational functions. This is known as Hilbert's 17th problem. E. Artin proved that this is true, thereby lying the cornerstone of the field of *real algebraic geometry*.

Motivated by Hilbert's 1888 theorem, several authors investigated the equivariant setting. For a group G acting on the real polynomial ring one restricts to *invariant* forms, i.e. forms which are fixed under the action of G. Choi, Lam and Reznick investigated the question for the symmetric group S_n which was completed by Goel et al. (2016). The signed symmetric group B_n acting on the polynomial ring via permutation of variables and switching of signs was also considered (Goel et al., 2017). Recently, Debus and Riener considered D_n-invariant forms where D_n is the subgroup of B_n of even number of sign changes. All these groups have in common that they are *reflection groups*.

A finite group G is a *real reflection group* if $G \subset \mathrm{GL}_n(\mathbb{R}^n)$ is such that the matrix group is generated by *reflections*, i.e. isometries $\mathbb{R}^n \to \mathbb{R}^n$ with a hyperplane as the set of fixed points. We usually just say that an abstract group G is a real reflection group and the representation of G is implicitly known. A real reflection group is called *essential* if no non-trivial subspace of $\mathbb{R}^n$ is pointwise fixed. It is a classical result that any real reflection group can be decomposed into a direct product of essential reflection groups. The essential real reflection groups were fully classified by Coxeter (1934, 1935). There are four infinite families A_n, B_n, D_n and $I_2(m)$ and six exceptional real reflection groups H_3, H_4, F_4, E_6, E_7, and E_8.

For B_n, D_n and trivially $I_2(m)$ the equivariant classification of nonnegativity versus sums of squares was completed in Debus and Riener (2023). It is a natural question to consider the remaining infinite series of essential reflection groups A_n and to study the equivariant nonnegativity versus sums of squares question. In this paper we initiate a study of A_n-invariant quartics. Although the vector space dimension of A_n-invariant quartics is only 2, we will see that the understanding is challenging. A reason for the complexity involved here is that we do not consider nonnegativity of a polynomial globally. We consider nonnegativity on a hyperplane and do consider sums of squares modulo an ideal which is in general a very difficult problem.

The paper is structured as follows. Section 2 explains the action of the group A_n on an n-dimensional vector space and the induced action on the polynomial ring. Following this, we examine the sets of nonnegative and sums of squares A_n-invariant quartics in Section 3. We begin in Subsection 3.1 to elaborate on the difference between global nonnegativity of quartics and nonnegativity of A_n-invariant quartics. In Subsection 3.2 we provide the extremal elements of the cone of A_n-invariant nonnegative quartics before we analyse the A_n-invariant sums of squares quartics in Subsection 3.3. Finally, we present a proof of our main theorem, Theorem 3.2 in Subsection 3.4.

2. The reflection groups of type A and A_n-invariant polynomials

The real reflection group A_n is, as a group, isomorphic to the symmetric group S_{n+1}. Recall that the reflection group S_{n+1} is acting on $\mathbb{R}^{n+1}$ via permutation of coordinates in all possible ways. We call this action the *permutation action* of the symmetric group. There is a non-trivial fixed subspace which is spanned by the vector $(1, \dots, 1)$ under the permutation action and thus the permutation action does not define an *essential* real reflection group. The action of S_{n+1} on the invariant subspace $U_n := \{a \in \mathbb{R}^{n+1} : \sum_{i=1}^{n} a_i = 0\}$ via permutation of coordinates defines an essential real reflection group called A_n. We also say that it is the reflection group of *type A*.

Recall that any group G acting on $\mathbb{R}^n$ induces an action of G on the polynomial ring $\mathbb{R}[\mathbf{x}]$ in n variables. The action is as follows:

$$\sigma \cdot f(\mathbf{x}) := f(\sigma^{-1} \cdot \mathbf{x})$$

where $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_n)$ is a basis of the dual vector space of $\mathbb{R}^n$ and $\sigma \in G$. We refer to Blekherman and Riener (2021, Section 4) for details.

It is a classical result by Chevalley, Sheppard and Todd that the *invariant ring* of real polynomials under the action of a finite matrix group in $\mathrm{GL}_n(\mathbb{R})$ is isomorphic to a polynomial ring if and only if the group is a real reflection group (Chevalley, 1955; Shephard and Todd, 1954).

In order to study A_n invariant forms we consider the restriction of the permutation action of the symmetric group S_{n+1} to the n dimensional real vector space

$$U_n = \left\{ a \in \mathbb{R}^{n+1} : \sum_i a_i = 0 \right\}.$$

Let $e_i \in \mathbb{R}^n$ denote the unit vector with 1 at the i-th coordinate. A linear basis of U_n is

$$u_1 = e_1 - e_2, \ldots, u_n = e_1 - e_{n+1}.$$

The group A_n acts on U_n via permutation of the e_i's in all possible ways. We obtain an induced action on an n-variate polynomial ring $\mathbb{R}[\mathbf{y}]$, where $\mathbf{y}$ is a basis of the dual vector space of U_n and on the quotient of an $(n+1)$-variate polynomial ring $\mathbb{R}[\mathbf{x}]$ modulo the ideal generated by the linear polynomial $\mathbf{x}_1 + \ldots + \mathbf{x}_{n+1}$. While A_n does act on the $(n+1)$-variate quotient ring $\mathbb{R}[\mathbf{x}]/(\mathbf{x}_1 + \ldots + \mathbf{x}_{n+1})$ via permutation of the $\mathbf{x}_i$'s, the reflection group does not permute the $\mathbf{y}_i$'s. We recall that those rings are isomorphic and two equivalent representations of A_n.

For two real representations V, W of a group G we say a linear map $\phi : V \to W$ is *G-equivariant* if $\sigma \cdot \phi(v) = \phi(\sigma \cdot w)$ for any $v \in V, w \in W, \sigma \in G$.

Proposition 2.1. *The ring homomorphism $\mathbb{R}[\mathbf{y}] \to \mathbb{R}[\mathbf{x}]/(\mathbf{x}_1 + \ldots + \mathbf{x}_{n+1})$ defined by $\mathbf{y}_i \mapsto \mathbf{x}_1 - \mathbf{x}_{i+1}$, for all $1 \le i \le n$, is a A_n-equivariant isomorphism.*

Proof. Recall that A_n fixes the subspace defined by $\mathbf{x}_1 + \ldots + \mathbf{x}_{n+1} = 0$. A basis of this subspace is $\mathbf{x}_1 - \mathbf{x}_{i+1}$ for $1 \le i \le n$. The basis elements and $\mathbf{x}_1 + \ldots + \mathbf{x}_{n+1}$ form a basis of the degree 1 part of $\mathbb{R}[\mathbf{x}]$. We have

$$\mathbb{R}[\mathbf{x}] \quad \cong \quad \mathbb{R}[\mathbf{x}_1 - \mathbf{x}_2, \ldots, \mathbf{x}_1 - \mathbf{x}_{n+1}][\mathbf{x}_1 + \ldots + \mathbf{x}_{n+1}]$$

and

$$\mathbb{R}[\mathbf{x}]/(\mathbf{x}_1 + \ldots + \mathbf{x}_{n+1}) \quad \cong \quad \mathbb{R}[\mathbf{y}]$$

With the discussion above the induced linear isomorphism is A_n-equivariant. □

Since we have a ring isomorphism we have that being a sum of squares is equivalent for the image and preimage. Moreover nonnegativity of the preimage is equivalent to nonnegativity of the image on the subspace U_n of $\mathbb{R}^{n+1}$.

We denote by p_k the *power sum* polynomial of degree k in the $(n+1)$-variables $\mathbf{x}$, i.e. $p_k = \sum_{i=1}^{n+1} \mathbf{x}_i^k$. It is classically known that the power sum polynomials $p_2, \ldots, p_{n+1}$ generate the A_n-invariant ring as $\mathbb{R}$-algebra modulo the ideal (p_1).

Theorem 2.2. *The invariant ring of A_n is isomorphic to a polynomial ring. The invariant ring of A_n acting via permutation of the variables $\mathbf{x}$ on the $(n+1)$-variate quotient ring $\mathbb{R}[\mathbf{x}]/(p_1)$ is $\mathbb{R}[\mathbf{x}]^{A_n} \cong \mathbb{R}[p_2, \ldots, p_{n+1}]$.*

3. SOS versus PSD for A_n-invariant quartics

In this Section we prove our main result Theorem 3.2. We mainly restrict our notation and definitions to quartics. Since the invariant ring is generated by the power sums $p_2, \ldots, p_{n+1}$ the vector space of *A_n-invariant quartics* is 2 dimensional and is spanned by the quotient classes of p_2^2 and p_4.

Definition 3.1. We call a A_n-invariant quartic in $\mathbb{R}[\mathbf{x}]/(p_1)$ *nonnegative* or *psd* if and only if any element in its quotient class in $\mathbb{R}[\mathbf{x}]$ is nonnegative on U_n. We denote the set of psd A_n-invariant quartics by P^{A_n}. We call a A_n-invariant quartic in $\mathbb{R}[\mathbf{x}]/(p_1)$ a *sum of squares* or *sos* if and only if an element in its quotient class in $\mathbb{R}[\mathbf{x}]$ is of the form $g_1^2+\ldots+g_m^2+p_1\cdot g$ for some $g_1,\ldots,g_m,g\in\mathbb{R}[\mathbf{x}]$. We denote the set of all A_n-invariant sos quartics by Σ^{A_n}.

Suppose $f_1 = ap_2^2 + bp_4 + p_1\cdot g_1$ and $f_2 = ap_2^2 + bp_4 + p_1\cdot g_2$ are two equivalent A_n-invariant quartics. Then nonnegativity of the quotient class $f_1 \mod (p_1)$ is well defined since $p_1 = 0$ on U_n.

The sets P^{A_n}, Σ^{A_n} are pointed closed convex cones in the vector space $\mathbb{R}[\mathbf{x}]/(p_1)$.

The main result is the following.

Theorem 3.2. *For $n \geq 3$ we have $P^{A_n} = \Sigma^{A_n}$ if and only if n is odd.*

Note, we have $P^{A_n} = \Sigma^{A_n}$ by Hilbert's classification for all $n \leq 3$. We will provide a proof of Theorem 3.2 in Subsection 3.4. Our strategy is as follows. First, we calculate the extremal rays of the two-dimensional cone P^{A_n}. Second, we give a description of Σ^{A_n} using symmetry reduction. Third, we show that when n is even then one of the extremal rays is not a sum of squares, while for odd n both extremal rays are sum of squares.

To motivate the fundamental difference between S_n-invariant and A_n-invariant nonnegative quartics we start with an overview on nonnegativity in Subsection 3.1.

3.1. Global nonnegativity versus nonnegativity on U_n

We motivate the subtle difference between globally nonnegative forms and forms nonnegative on U_n in the vector space $\langle p_2^2, p_4\rangle_{\mathbb{R}}$. For $n \geq 3$, the vector space of symmetric $(n+1)$-variate quartics is five dimensional and is spanned by the following products of power sum polynomials

$$p_1^4,\, p_2p_1^2,\, p_3p_1,\, p_2^2,\, p_4\,.$$

For any $n \geq 3$, there exist $(n+1)$-variate symmetric quartic psd forms that are not sums of squares (Goel et al., 2016). For instance, there exists the following uniform example

$$\mathfrak{f}_n := 4p_1^4 - 5p_2p_1^2 - \frac{139}{20}p_3p_1 + 4p_2^2 + 4p_4$$

which is always nonnegative but never a sum of squares for any number of variables ≥ 4 (Acevedo et al., 2024, Theorem 3.6). Note however, that restricting to the subspace U_n gives $4(p_2^2 + p_4)$. Thus $\mathfrak{f}_n$ is a sum of squares modulo the ideal (p_1). The form $\mathfrak{f}_n$ can therefore not be used as a counter example for the reflection groups of type A. It was shown in Blekherman and Riener (2021, Example 5.4) that the form

$$\frac{1}{n}p_4 - \frac{2.6}{n^2}p_3p_1 + \frac{1.79}{n^3}p_2p_1^2 - \frac{0.1275}{n^4}p_1^4$$

lies on the boundary of the cone of symmetric sums of squares quartics and in the interior of the cone of symmetric psd quartics for any $n \geq 4$. Moreover in Blekherman and Riener (2021, Theorem 5.3) the authors gave spectrahedral shadow representations of the cones of n-ary symmetric sums of squares quartics. Again, restricting to $p_1 = 0$ leaves a form $\frac{1}{n}p_4$ which is clearly a sum of squares.

We show that any psd form in the vector space $\langle p_2^2, p_4\rangle_{\mathbb{R}}$ is a sum of squares. The proposition follows also from the nonnegativity versus sums of squares classification in type B (Goel et al., 2017). The quartics result for type B was first observed by Choi, Lam and Reznick.

Proposition 3.3. *Let $f = ap_2^2 + bp_4$ be a nonnegative $(n+1)$-ary symmetric form, where $a, b \in \mathbb{R}$. Then f is a sum of squares.*

Proof. We restrict to testing nonnegativity of f on the unit sphere $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : p_2(x) = 1\}$ which we can do without loss of generality by the homogeneity of f. We claim $\frac{1}{n+1} \le p_4 \le 1$ on $\mathbb{S}^n$ and that the lower and upper bounds are attained.

The upper bound is evident and we can evaluate, e.g. at the point $(1, 0, \dots, 0) \in \mathbb{S}^n$. Note that the claimed lower bound is attained, e.g. at the point $\left(\frac{1}{\sqrt{n+1}}, \dots, \frac{1}{\sqrt{n+1}}\right)$. We need to show that it is the minimum of p_4 on $\mathbb{S}^n$. Instead of looking at the minimum of p_4 subject to $p_2 = 1$ we can consider the minimum of p_2 on the simplex $\Delta_n = \{x \in \mathbb{R}^{n+1}_{\ge 0} : \sum_{i=1}^{n+1} x_i = 1\}$, since $p_4(a_1, \dots, a_{n+1}) = p_2(a_1^2, \dots, a_{n+1}^2)$ and $p_2(a_1, \dots, a_{n+1}) = p_1(a_1^2, \dots, a_{n+1}^2)$ for all points $a \in \mathbb{R}^n_{\ge 0}$, and since $a \in \mathbb{S}^n$ is equivalent to $(a_1^2, \dots, a_{n+1}^2) \in \Delta_n$. Note that the function p_2 is convex on Δ_n. Its Hessian is a diagonal matrix with diagonal $(2, \dots, 2)$. For any point $a \in \Delta_n$ we have $p_2(\sigma \cdot a) = p_2(a)$ for any permutation $\sigma \in S_{n+1}$. The point $z := \frac{1}{(n+1)!} \sum_{\sigma \in S_{n+1}} \sigma \cdot a$ lies on the diagonal of $\mathbb{R}^{n+1}$ and in Δ_n. Therefore, we have $z = (\frac{1}{n+1}, \dots, \frac{1}{n+1})$ and

$$\frac{1}{n+1} = p_2(z) = p_2\left(\frac{1}{(n+1)!} \sum_{\sigma \in S_{n+1}} \sigma \cdot a\right) \le \frac{1}{(n+1)!} \sum_{\sigma \in S_{n+1}} p_2(\sigma \cdot a) = p_2(a)$$

for any point $a \in \Delta_n$. This shows the lower bound.

In the following case distinction we suppose $p_2 = 1$ and $\frac{1}{n+1} \le p_4 \le 1$. We need to distinguish three cases depending on the sign of a.

(1) If $a = 0$ we have bp_4 nonnegative implies $b \ge 0$ and thus we have a sum of squares.
(2) If $a > 0$ we suppose without loss of generality that $a = 1$ and we have $1 + bp_4 \ge 0$ on $\mathbb{S}^n$ which implies $b \ge -1$. However, the form

$$p_2^2 - p_4 = p_4 + 2\sum_{i<j} \mathbf{x}_i^2 \mathbf{x}_j^2 - p_4 = 2\sum_{i<j} \mathbf{x}_i^2 \mathbf{x}_j^2$$

on the boundary of the psd cone is also sos.
(3) If $a < 0$ we suppose $a = -1$ and have $-1 + bp_4 \ge 0$ on $\mathbb{S}^n$ implies $b \ge n+1$. The form $(n+1)p_4 - p_2^2$ on the boundary of the psd cone is a sum of squares since

$$(n+1)p_4 - p_2^2 = np_4 - 2\sum_{i<j} \mathbf{x}_i^2 \mathbf{x}_j^2 = \sum_{i<j} (\mathbf{x}_i^2 - \mathbf{x}_j^2)^2. \quad \square$$

This subtle but important difference of nonnegativity on $\mathbb{R}^{n+1}$ and on U_n has important structural consequences regarding A_n-invariant sums of squares.

3.2. PSD A_n invariant quartics

A symmetric $(n+1)$-variate polynomial which is nonnegative on the linear subspace U_n must not necessarily be globally nonnegative (see e.g. the polynomial G_n for any n and F_n for any even n in Lemma 3.5). Since we are considering homogeneous invariant polynomials we have by biduality of convex cones (Blekherman et al., 2012, Lemma 4.18) the following Lemma.

Lemma 3.4. *The boundary of P^{A_n} consists of the forms $f = a \cdot p_2^2 + b \cdot p_4$ for which there exists $0 \ne z \in U_n$ such that $f(z) = 0$.*

In analogy to the proof of Proposition 3.3 we will analyse the maximum and the minimum of p_4 on the semialgebraic set defined by $p_2 = 1$ and $p_1 = 0$.

Lemma 3.5. *For $n \geq 3$ the extremal $(n+1)$-ary A_n-invariant psd quartics are*

$$G_n := p_2^2 - \frac{1}{\beta} p_4 \text{ and } F_n := -p_2^2 + \frac{1}{\alpha} p_4,$$

where $\beta = \dfrac{1-n+n^2}{n+n^2}$ *and* $\alpha = \begin{cases} \dfrac{1}{n+1} & \text{if } n \text{ is odd,} \\ \dfrac{4+2n+n^2}{2n+3n^2+n^3} & \text{if } n \text{ is even.} \end{cases}$

Moreover the polynomials G_n and F_{2n} are not globally nonnegative, but F_{2n+1} is globally nonnegative.

Proof of Lemma 3.5. We begin by verifying the claims on the global nonnegativity of F_n and G_n. Since $F_{2n+1} = (2n+2)p_4 - p_2^2$ and $\frac{1}{2n+2} \leq p_4 \leq 1$ on $\mathbb{S}^{2n+1}$, as argued in the proof of Proposition 3.3, we have F_{2n+1} is globally nonnegative. Moreover, for $p_2 = 1$, since $\frac{1}{n+1} \leq p_4 \leq 1$ on $\mathbb{S}^n$, we have $-1 + \frac{1}{n+1} \cdot (\frac{4+2n+n^2}{2n+3n^2+n^3})^{-1} = -\frac{4}{n^2+2n+4}$, thus the form F_{2n} attains negative values on $\mathbb{S}^{2n}$ and hence is not globally non-negative. Further since $p_2 = 1$ and $p_4 = 1$ has a solution in $\mathbb{S}^n$ we have $G_n(1,0,\ldots,0) = 1 - \frac{n+n^2}{1-n+n^2} = \frac{1-2n}{1-n+n^2} < 0$ which shows that the polynomial G_n cannot be globally nonnegative.

Next, we verify that the polynomials G_n and F_n are indeed extremal A_n-invariant psd quartics. Since the quartics are homogeneous it is sufficient to analyse the minimum and maximum value of p_4 on $\mathbb{S}^n \cap U_n$. We have $p_1 = 0$ and $p_2 = 1$. This translates to the polynomial optimization problem

$$\begin{aligned} &\min_{x \in \mathbb{R}^{n+1}} \pm p_4 \\ &\text{s.t. } p_1 = 0 \\ &\qquad p_2 = 1 \end{aligned}$$

By a variant of Timofte's half degree principle (Riener, 2012, Theorem 1.1) the extremes are attained at a point with at most 2 different coordinates. The equality constraints transfer to the two equations

$$\begin{aligned} lt + (n+1-l)s &= 0 \\ lt^2 + (n+1-l)s^2 &= 1 \end{aligned}$$

where $0 \leq l \leq n+1$ is an integer and $s, t \in \mathbb{R}$ are real numbers. We observe that $l \notin \{0, n+1\}$ which implies $1 \leq l \leq n$. For given integers l and n the equations provide unique solutions for s and t up to sign. However, inserting the solution in p_4 is independent of the signs of the coordinates and we have

$$p_4(\underbrace{t,\ldots,t}_{l \text{ times}}, \underbrace{s,\ldots,s}_{n+1-l \text{ times}}) = \frac{(n+1)^2 - 3l(n+1) + 3l^2}{(n+1)(n+1-l)l}$$

For the claim on the extremality of F_n we are left with verifying

$$\min_{1 \leq l \leq n} \frac{(n+1)^2 - 3l(n+1) + 3l^2}{(n+1)(n+1-l)l} = \begin{cases} \dfrac{1}{n+1} & \text{if } n \text{ is odd} \\ \dfrac{4+2n+n^2}{2n+3n^2+n^3} & \text{if } n \text{ is even} \end{cases}$$

which we do in Lemmas 3.7 and 3.8, and verify

$$\max_{1 \leq l \leq n} \frac{(n+1)^2 - 3l(n+1) + 3l^2}{(n+1)(n+1-l)l} = \frac{1-n+n^2}{n+n^2}$$

to prove that G_n is extremal. This is Lemma 3.6. $\square$

3.2.1. Verification of the extremality of G_n

Lemma 3.6. *For all $n \geq 3$ we have $\max_{1 \leq l \leq n} \frac{(n+1)^2 - 3l(n+1) + 3l^2}{(n+1)(n+1-l)l}$ is attained at $l = 1$ and $l = n$, and equals $\frac{1 - n + n^2}{n + n^2}$.*

Proof. We calculate

$$\begin{aligned} \frac{(n+1)^2 - 3(n+1) + 3}{(n+1)n} &\geq \frac{(n+1)^2 - 3l(n+1) + 3l^2}{(n+1)(n+1-l)l} \\ \Leftrightarrow (-1+l)(n-l)(n+1)^2 &\geq 0. \end{aligned}$$

Note, for $1 \leq l \leq n$ the inequality is tight when $l \in \{1, n\}$ and otherwise strict. □

3.2.2. Verification of the extremality of F_n

Lemma 3.7. *We have $\min_{1 \leq l \leq n} \frac{(n+1)^2 - 3l(n+1) + 3l^2}{(n+1)(n+1-l)l} \geq \frac{1}{n+1}$ and equality holds if and only if n is odd.*

Proof. For $1 \leq l \leq n$ we have

$$\begin{aligned} &\frac{(n+1)^2 - 3l(n+1) + 3l^2}{(n+1)(n+1-l)l} - \frac{1}{n+1} && \geq 0 \\ \Longleftrightarrow\ &\frac{(n+1)^2 - 3l(n+1) + 3l^2}{(n+1)(n+1-l)l} - \frac{(n+1-l)l}{(n+1)(n+1-l)l} && \geq 0 \\ \Longleftrightarrow\ &(n+1)^2 - 3l(n+1) + 3l^2 - (n+1-l)l && \geq 0 \\ \Longleftrightarrow\ &(n+1-2l)^2 && \geq 0 \end{aligned}$$

The last inequality is tight on integer values $1 \leq l \leq n$ if and only if $n+1$ is even. □

Lemma 3.8. *If n is even, then $\min_{1 \leq l \leq n} \frac{(n+1)^2 - 3l(n+1) + 3l^2}{(n+1)(n+1-l)l}$ is attained at $l = \frac{n}{2}$ and $l = \frac{n}{2} + 1$, and equals $\frac{4 + 2n + n^2}{2n + 3n^2 + n^3}$.*

Proof. Evaluating $\frac{(n+1)^2 - 3l(n+1) + 3l^2}{(n+1)(n+1-l)l}$ at $l = \frac{n}{2}$ and $l = \frac{n+2}{2}$ gives $\frac{4 + 2n + n^2}{2n + 3n^2 + n^3}$. Moreover the denominator of

$$\frac{(n+1)^2 - 3l(n+1) + 3l^2}{(n+1)(n+1-l)l} - \frac{4 + 2n + n^2}{2n + 3n^2 + n^3} = \frac{(n+1)(4l^2 - 4l(n+1) + n(n+2))}{l(n+1-l)n(n+2)}$$

is strictly positive for all $1 \leq l \leq n$. The numerator is also nonnegative since

$$4l^2 - 4l(n+1) + n(n+2) = (2l - (n+1))^2 - 1 \geq 0$$

because $n+1$ is odd. □

3.3. SOS A_n-invariant quartics

Given the action of a reflection group, representation theory and invariant theory can be applied to effectively describe the invariant sums of squares cone. We briefly sketch the symmetry reduction for sums of squares invariant by a reflection group. More details can be found in Blekherman and Riener (2021); Debus and Riener (2023); Gatermann and Parrilo (2004); Heaton et al. (2021). A reflection group G acts on the vector space $\mathbb{R}[\mathbf{x}]_d$ of all $(n+1)$-variate forms of degree d giving it the structure of a G-module. We can decompose every G-module into a direct sum of its irreducible sub-modules to obtain its *isotypic decomposition*. Given an isotypic decomposition one constructs a *symmetry adapted basis*, which can be used to understand the invariant sums of squares of elements in $\mathbb{R}[\mathbf{x}]_{2d}$. We outline this in the following.

First, we note that there is a natural projection onto the invariant part of $\mathbb{R}[\mathbf{x}]_d$ via the so called *Reynolds-Operator* of the group G:

$$\mathcal{R}_G : \mathbb{R}[\mathbf{x}]_d \to \mathbb{R}[\mathbf{x}]_d^G,\ f \mapsto \frac{1}{|G|}\sum_{\sigma\in G} \sigma \cdot f.$$

Suppose that we have

$$\mathbb{R}[\mathbf{x}]_d \cong \bigoplus_{j=1}^{\ell} \mathcal{V}_j^{\oplus \eta_j}$$

is the isotypic decomposition of the G action on $\mathbb{R}[\mathbf{x}]_d$, i.e. $\mathcal{V}_j$ are pairwise non-isomorphic irreducible G-modules and each occurs with multiplicity $\eta_j \in \mathbb{N}$ in $\mathbb{R}[\mathbf{x}]_d$. A symmetry adapted basis is a list

$$\{f_{11}, \dots, f_{1\eta_1}, f_{21}, \dots, f_{\ell\eta_\ell}\}$$

with the property that for every j there are G-equivariant homomorphisms ϕ_{ji} which map f_{j1} to f_{ji} for all $1 \le i \le \eta_j$, and furthermore that the orbit of each f_{ji} spans an irreducible G-module isomorphic to $\mathcal{V}_j$ and the set of all orbits of all f_{ji} spans $\mathbb{R}[\mathbf{x}]_d$. Given a symmetry adapted basis we can construct matrix polynomials

$$B_j := \left(\mathcal{R}_G(f_{ji_1} f_{ji_2})\right)_{1\le i_1, i_2 \le \eta_j} \text{ for } 1 \le j \le \ell.$$

With these notations we have the following (see Debus and Riener 2023, Theorem 2.6):

Proposition 3.9. *Let $f \in \mathbb{R}[\mathbf{x}]_{2d}^G$ be an invariant form. Then f is a sum of squares if and only if there exist positive semidefinite matrices $A_1, \dots, A_\ell$ such that*

$$f = \sum_{j=1}^{\ell} \mathrm{Tr}(A_j B_j),$$

where the matrix polynomials B_j are constructed from a symmetry adapted bases of $\mathbb{R}[\mathbf{x}]_d$ as defined above.

Note that calculating an isotypic decomposition of $\mathbb{R}[\mathbf{x}]_d$ and a symmetry adapted basis can in principle be done with linear algebra (see Serre, 1977). For the case of finite groups Hubert and Bazan (2022) constructed an algorithm to calculate equivariants which allows for an effective determination of symmetry adapted basis for all degrees. In the case when $G \in \{A_{n-1}, S_n, B_n, D_n\}$ so-called *higher Specht polynomials* can be used and their construction is completely combinatorial (Debus and Riener, 2023; Morita and Yamada, 1998). An implantation for the symmetric group was developed by Niño Cortés (2019) in Macaulay2 (Grayson and Stillman, no date).

We denote by $\mathbb{S}^\lambda$ the Specht module associated with a partition λ.

Lemma 3.10. *For $n \geq 3$, the S_{n+1}-isotypic decomposition of $\mathbb{R}[\mathbf{x}]_2$ equals*

$$\begin{aligned}\mathbb{R}[\mathbf{x}]_2 \quad &= \quad \langle p_2\rangle_{\mathbb{R}} \oplus \langle p_1^2\rangle_{\mathbb{R}} \oplus \langle p_1(\mathbf{x}_i - \mathbf{x}_j) : i < j\rangle_{\mathbb{R}} \\ &\quad\quad \oplus \langle \mathbf{x}_i^2 - \mathbf{x}_j^2 : i < j\rangle_{\mathbb{R}} \oplus \langle (\mathbf{x}_i - \mathbf{x}_j)(\mathbf{x}_k - \mathbf{x}_l) : \#\{i, j, k, l\} = 4\rangle \\ &= \quad 2 \cdot \mathbb{S}^{(n+1)} \oplus 2 \cdot \mathbb{S}^{(n,1)} \oplus \mathbb{S}^{(n-1,2)}.\end{aligned}$$

Moreover, a symmetry adapted basis of the S_{n+1}-module $\mathbb{R}[\mathbf{x}]_2$ is

$$\{p_1^2, p_2, p_1(\mathbf{x}_1 - \mathbf{x}_2), \mathbf{x}_1^2 - \mathbf{x}_2^2, (\mathbf{x}_1 - \mathbf{x}_2)(\mathbf{x}_3 - \mathbf{x}_4)\}.$$

We point out that the isotypic decomposition of the S_{n+1}-module $\mathbb{R}[\mathbf{x}]_2$ and the same symmetry adapted basis was used in the proof of Theorem 5.1 in Blekherman and Riener (2021).

Proof. Let $n \geq 3$ be an integer. We verify that the claimed isotypic decomposition and symmetry adapted basis of the S_{n+1}-module of $\mathbb{R}[\mathbf{x}]_2$ holds. Therefore, we consider the coinvariant algebra $\mathbb{R}[\mathbf{x}]/(p_1, \dots, p_{n+1})$ which is a S_{n+1}-module and actually isomorphic, as S_{n+1}-module, to the regular representation of S_{n+1} (Bergeron, 2009, Section 8.1). Since $\mathbb{R}[\mathbf{x}] \cong \mathbb{R}[p_1, \dots, p_{n+1}] \otimes_{\mathbb{R}} \mathbb{R}[\mathbf{x}]/(p_1, \dots, p_{n+1})$ is an isomorphism of graded S_n-modules, one can obtain a symmetry adapted basis of $\mathbb{R}[\mathbf{x}]_2$ by multiplying the polynomials in a subset of the symmetry adapted basis of $\mathbb{R}[\mathbf{x}]/(p_1, \dots, p_{n+1})$ (consisting of the polynomials of degrees $0, 1, 2$) with products of power sums p_1 and p_2. The minimal degree d in which a Specht module $\mathbb{S}^\lambda$ in $\mathbb{R}[\mathbf{x}]/(p_1, \dots, p_{n+1})$ occurs is the degree of the associated Specht polynomial and the multiplicity of the Specht module $\mathbb{S}^\lambda$ in this degree component is one (Niño Cortés, 2019, Theorem 10.2). The only Specht polynomials of degree at most 2 are the constant polynomial 1 (for $\lambda = (n+1)$), $\mathbf{x}_i - \mathbf{x}_j$ (for $\lambda = (n, 1)$) and $(\mathbf{x}_i - \mathbf{x}_j)(\mathbf{x}_k - \mathbf{x}_l)$ (for $\lambda = (n-1, 2)$), where i, j, k, l are pairwise distinct integers in $[n+1]$. This is, since if the length of λ is at least three, then $(\mathbf{x}_i - \mathbf{x}_j)(\mathbf{x}_i - \mathbf{x}_k)(\mathbf{x}_j - \mathbf{x}_k)$ divides the Specht polynomial, for some pairwise distinct integers i, j and k. If the length of $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ is at most two and $\lambda_2 \geq 3$ then the degree of any Specht polynomial is also at least 3, since $(\mathbf{x}_{i_1} - \mathbf{x}_{i_2})(\mathbf{x}_{j_1} - \mathbf{x}_{j_2})(\mathbf{x}_{k_1} - \mathbf{x}_{k_2})$ divides the Specht polynomial for some pairwise distinct integers $i_1, i_2, j_1, j_2, k_1, k_2$.

Therefore, $\mathbb{S}^{(n-1,2)}$ has multiplicity one in $\mathbb{R}[\mathbf{x}]_2$ and since p_1^2, p_2 span the subspace of symmetric forms of degree 2, we have $\mathbb{S}^{(n+1)}$ has multiplicity two in $\mathbb{R}[\mathbf{x}]_2$. By the hook length formula we have $\dim_{\mathbb{R}} \mathbb{S}^{(n,1)} = n$ and $\dim_{\mathbb{R}} \mathbb{S}^{(n-1,2)} = \frac{(n-2)(n+1)}{2}$. Since $\dim_{\mathbb{R}} \mathbb{S}^{(n+1)} = 1$ we have

$$\begin{aligned}2\dim_{\mathbb{R}} \mathbb{S}^{(n+1)} + 2\dim_{\mathbb{R}} \mathbb{S}^{(n,1)} + \dim_{\mathbb{R}} \mathbb{S}^{(n-1,2)} &= 2 + 2n + \frac{(n-2)(n+1)}{2} = \binom{n+2}{n} \\ &= \dim_{\mathbb{R}} \mathbb{R}[\mathbf{x}]_2\end{aligned}$$

which shows that the claimed isotypic decomposition is true.

We are left with verifying that the claimed symmetry adapted basis is also true. The forms p_1^2, p_2 are linearly independent symmetric polynomials in $\mathbb{R}[\mathbf{x}]_2$ and therefore belong to the isotypic component of $\mathbb{S}^{(n+1)}$ in $\mathbb{R}[\mathbf{x}]_2$. The Specht polynomial $(\mathbf{x}_1 - \mathbf{x}_2)(\mathbf{x}_3 - \mathbf{x}_4)$ spans, as S_n-module, a S_n-submodule of $\mathbb{R}[\mathbf{x}]_2$ isomorphic to $\mathbb{S}^{(n+1,2)}$ and can be chosen as the element from the isotypic component $\mathbb{S}^{(n-1,2)}$. The Specht polynomial $\mathbf{x}_1 - \mathbf{x}_2$ can be multiplied by p_1 to obtain an element of the symmetry adapted basis belonging to $\mathbb{S}^{(n,1)}$. Note that $p_1(\mathbf{x}_i - \mathbf{x}_j) \mapsto (\mathbf{x}_i^2 - \mathbf{x}_j^2)$ defines an S_{n+1}-equivariant isomorphism and since $\langle p_1(\mathbf{x}_i - \mathbf{x}_j) : i < j\rangle_{\mathbb{R}}$ and $\langle \mathbf{x}_i^2 - \mathbf{x}_j^2 : i < j\rangle_{\mathbb{R}}$ intersect trivially, the direct sum of these S_{n+1}-modules must be the isotypic part of $\mathbb{S}^{(n-1,2)}$ in $\mathbb{R}[\mathbf{x}]_2$. $\square$

We apply the Reynolds-Operator of the symmetric group S_{n+1} to pairwise products of equivariants of the isotypic decomposition which do not use p_1, since we consider sum of squares in $\mathbb{R}[\mathbf{x}]$ modulo the ideal (p_1).

Lemma 3.11. *For $n \geq 3$, we have*

$$\begin{aligned}
\mathcal{R}_{S_{n+1}}(p_2^2) &= p_2^2\,, \\
\mathcal{R}_{S_{n+1}}((\mathbf{x}_1^2-\mathbf{x}_2^2)^2) &= \frac{2}{n}p_4 - \frac{2}{(n+1)n}p_2^2, \text{ and} \\
\mathcal{R}_{S_{n+1}}((\mathbf{x}_1-\mathbf{x}_2)^2(\mathbf{x}_3-\mathbf{x}_4)^2) &= \\
&\frac{4(p_1^4-2np_1^2p_2+n^2p_2^2-2p_1^2p_2-np_2^2+4np_1p_3-n^2p_4+p_2^2-np_4)}{(n+1)n(n-1)(n-2)}.
\end{aligned}$$

For a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ of length $l \leq n+1$ we write m_λ for the *monomial symmetric* polynomial in $(n+1)$-variables with respect to the partition λ, i.e., m_λ is the sum over all monomials $\mathbf{x}_{i_1}^{\lambda_1}\mathbf{x}_{i_2}^{\lambda_2}\cdots\mathbf{x}_{i_l}^{\lambda_l}$ where $1 \leq i_1, i_2, \dots, i_l \leq n+1$ are pairwise distinct integers. For a partition $\lambda = (\lambda_1, \dots, \lambda_l)$ and a monomial $\mathbf{x}^\lambda = \mathbf{x}_1^{\lambda_1}\mathbf{x}_2^{\lambda_2}\cdots\mathbf{x}_l^{\lambda_l}$ we have $\mathcal{R}_{S_{n+1}}(\mathbf{x}^\lambda) = \frac{(n+1-l)!\nu_1!\cdots\nu_k!}{(n+1)!}m_\lambda$, where $\nu_1 \geq \nu_2 \geq \ldots \geq \nu_k$ are the multiplicities of all pairwise distinct parts of the partition λ. For instance, for $\lambda = (4,4,3,2)$ we have $\nu_1 = 2, \nu_2 = 1, \nu_3 = 1$ and $\mathcal{R}_{S_{n+1}}(x^{(4,4,3,2)}) = \frac{(n-3)!2!1!1!}{(n+1)!}m_{(4,4,3,2)}$ for all $n \geq 3$.

Proof. Since p_2^2 is symmetric, we have $\mathcal{R}_{S_{n+1}}(p_2^2) = p_2^2$. Moreover by the linearity of the Reynolds-Operator and since $\mathcal{R}_{S_{n+1}}(f) = \mathcal{R}_{S_{n+1}}(\sigma \cdot f)$ for any permutation $\sigma \in S_{n+1}$ and any polynomial $f \in \mathbb{R}[\mathbf{x}]$, we have

$$\begin{aligned}
\mathcal{R}_{S_{n+1}}((\mathbf{x}_1^2-\mathbf{x}_2^2)^2) &= 2\mathcal{R}_{S_{n+1}}(\mathbf{x}_1^4) - 2\mathcal{R}_{S_{n+1}}(\mathbf{x}_1^2\mathbf{x}_2^2) \\
&= 2\frac{n!}{(n+1)!}m_4 - 2\frac{2(n-1)!}{(n+1)!}m_{(2,2)},
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
\mathcal{R}_{S_{n+1}}((\mathbf{x}_1-\mathbf{x}_2)^2(\mathbf{x}_3-\mathbf{x}_4)^2) &= 4\mathcal{R}_{S_{n+1}}(\mathbf{x}_1^2\mathbf{x}_2^2) - 8\mathcal{R}_{S_{n+1}}(\mathbf{x}_1^2\mathbf{x}_2\mathbf{x}_3) + 4\mathcal{R}_{S_{n+1}}(\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\mathbf{x}_4) \\
&= 4\frac{2!(n-1)!}{(n+1)!}m_{(2,2)} - 8\frac{2!(n-2)!}{(n+1)!}m_{(2,1,1)} \\
&\quad + 4\frac{4!(n-3)!}{(n+1)!}m_{(1,1,1,1)}.
\end{aligned} \tag{3.2}$$

Using Sage (The Sage Developers, 2024) we transform from monomial symmetric polynomials to power sum polynomials and insert the power sums in equations (3.1) and (3.2).

```
Sym = SymmetricFunctions(QQ)
pp = Sym.power()
mm = Sym.monomial()
print(pp(mm[4]))
print(pp(2*mm[2,2]))
print(pp(2*mm[2,1,1]))
print(pp(24*mm[1,1,1,1]))
p[4]
p[2, 2] - p[4]
p[2, 1, 1] - p[2, 2] - 2*p[3, 1] + 2*p[4]
p[1, 1, 1, 1] - 6*p[2, 1, 1] + 3*p[2, 2] + 8*p[3, 1] - 6*p[4]
R.<n,p1,p2,p3,p4> = LaurentPolynomialRing(QQ)
print(2/(n+1)*p4-4/((n+1)*n)*(p2^2-p4)/2)
print(4/((n+1)*n)*(p2^2-p4)-8/((n+1)*n*(n-1))*(p2*p1^2-p2^2-2*p3*p1+2*p4
    )+4/((n+1)*n*(n-1)*(n-2))*(p1^4-6*p2*p1^2+3*p2^2+8*p3*p1-6*p4))
(-4*p2^2 + 4*n*p4 + 4*p4)/(2*n^2 + 2*n)
(4*p1^4 - 8*n*p1^2*p2 + 4*n^2*p2^2 - 8*p1^2*p2 - 4*n*p2^2 + 16*n*p1*p3 -
     4*n^2*p4 + 4*p2^2 - 4*n*p4)/(n^4 - 2*n^3 - n^2 + 2*n)
```

These are precisely the claimed images of the Reynolds-Operator. $\square$

Lemma 3.12. *If $f \in \mathbb{R}[\mathbf{x}]$ is a A_n-invariant sum of squares quartic modulo the ideal (p_1) then*

$$f = a\left(p_4 - \frac{1}{n+1}p_2^2\right) + b((1-n+n^2)p_2^2 - n(1+n)p_4) + p_1 \cdot g$$

for some $a, b \geq 0$ and $g \in \mathbb{R}[x]$.

Proof. Since $f \in \mathbb{R}[\mathbf{x}]$ is A_n-invariant, we can apply the Reynolds-Operator $\mathcal{R}_{A_n} = \mathcal{R}_{S_{n+1}}$ to $g_1^2 + \ldots + g_m^2$ and consider $\mathcal{R}_{S_{n+1}}(g_1^2 + \ldots + g_m^2) \mod p_1$ which has to be of the form

$$\lambda_1 p_2^2 + \lambda_2(p_4 - \frac{1}{n+1}p_2^2) + \lambda_3((1-n+n^2)p_2^2 - n(1+n)p_4) \tag{3.3}$$

for some scalars $\lambda_1, \lambda_2, \lambda_3 \geq 0$, by Lemma 3.11 and the discussion above. This is, since the symmetry adapted basis of the A_n-module $\mathbb{R}[\mathbf{x}]_2$ consists of just one polynomial per isotypic component. The polynomials in the conical combination (3.3) are perfect squares of the elements from the symmetry adapted basis. The Reynolds-Operator applied to a product of two elements from distinct isotypic components is zero. Since

$$(1-n)^2 p_2^2 \quad = \quad n(n+1)(p_4 - \frac{1}{n+1}p_2^2) + ((1-n+n^2)p_2^2 - n(1+n)p_4)$$

we do not have to use the perfect square p_2^2 in the characterization of all symmetric sum of squares quartics modulo (p_1) which proves the claim. $\square$

3.4. Proof of Theorem 3.2

We are ready to prove Theorem 3.2

Proof of Theorem 3.2. There are three statements that we want to show. First, the polynomial $G_n \in \mathbb{R}[\mathbf{x}]$ is a sum of squares modulo (p_1) for all $n \geq 3$. Second, for $n \geq 4$ odd the polynomial $F_n \in \mathbb{R}[\mathbf{x}]$ is a sum of squares modulo (p_1). Third, for $n \geq 3$ even the polynomial $F_n \in \mathbb{R}[\mathbf{x}]$ is not a sum of squares modulo (p_1)

(1) We have

$$\begin{aligned} G_n &= p_2^2 - \frac{n+n^2}{1-n+n^2}p_4 \\ &= \frac{1}{1-n+n^2}((1-n+n^2)p_2^2 - n(1+n)p_4) \end{aligned}$$

which shows that G_n is a sum of squares modulo (p_1).

(2) If $n \geq 4$ is odd we have F_n is a sum of squares by Proposition 3.3. This is, because α equals the global minimum of p_4 on $p_2 = 1$ and we have seen that the corresponding polynomial is a sum of squares.

(3) For even $n \geq 4$ we have $F_n = -p_2^2 + \frac{2n+3n^2+n^3}{4+2n+n^2}p_4$. We suppose that F_n is a sum of squares modulo (p_1). We must have

$$-p_2^2 + \frac{2n+3n^2+n^3}{4+2n+n^2}p_4 \quad = \quad a(p_4 - \frac{1}{n+1}p_2^2) + b((1-n+n^2)p_2^2 + (-n-n^2)p_4)$$

for some $a, b \geq 0$. Comparing the coefficients implies

$$b \quad = \quad -\frac{4}{4-6n+n^2+n^4}$$

which is a contradiction. $\square$

CRediT authorship contribution statement

Sebastian Debus: Conceptualization, Investigation, Methodology, Software, Writing – original draft, Writing – review & editing. **Charu Goel:** Conceptualization, Investigation, Methodology, Writing – original draft, Writing – review & editing. **Salma Kuhlmann:** Conceptualization, Investigation, Methodology, Writing – original draft, Writing – review & editing. **Cordian Riener:** Conceptualization, Investigation, Methodology, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

The authors are thankful to Mathematisches Forschungsinstitut Oberwolfach for its hospitality during Workshops in March 2020 and March 2023, where this collaboration was initiated and progressed. We are also grateful to the referees for their thoughtful suggestions which helped us to substantially improve the article.

Data availability

No data was used for the research described in the article.

References

Acevedo, J., Blekherman, G., Debus, S., Riener, C., 2024. Symmetric nonnegative functions, the tropical Vandermonde cell and superdominance of power sums. ArXiv preprint. arXiv:2408.04616.

Bergeron, F., 2009. Algebraic Combinatorics and Coinvariant Spaces. A. K. Peters/Canadian Mathematical Society, Wellesley, MA/Ottawa.

Blum, L., Cucker, F., Shub, M., Smale, S., 1998. Complexity and Real Computation. Springer-Verlag, New York.

Blekherman, G., Parrilo, P.A., Thomas, R.R., 2012. Semidefinite Optimization and Convex Algebraic Geometry. SIAM.

Blekherman, G., Riener, C., 2021. Symmetric non-negative forms and sums of squares. Discrete Comput. Geom. 65, 764–799.

Chevalley, C., 1955. Invariants of finite groups generated by reflections. Am. J. Math. 77, 778–782.

Coxeter, H.S.M., 1934. Discrete groups generated by reflections. Ann. Math. (2) 35 (3), 588–621.

Coxeter, H.S.M., 1935. The complete enumeration of finite groups of the form $R_i^2 = (R_i R_j)^{k_{ij}} = 1$. J. Lond. Math. Soc. 10 (1), 21–25.

Debus, S., Riener, C., 2023. Reflection groups and cones of sums of squares. J. Symb. Comput. 119, 112–144.

Goel, C., Kuhlmann, S., Reznick, B., 2016. On the Choi–Lam analogue of Hilbert's 1888 theorem for symmetric forms. Linear Algebra Appl. 496, 114–120.

Goel, C., Kuhlmann, S., Reznick, B., 2017. The analogue of Hilbert's 1888 theorem for even symmetric forms. J. Pure Appl. Algebra 221 (6), 1438–1448.

Gatermann, K., Parrilo, P.A., 2004. Symmetry groups, semidefinite programs, and sums of squares. J. Pure Appl. Algebra 192 (1–3), 95–128.

Grayson, D.R., Stillman, M.E., no date. Macaulay2, a software system for research in algebraic geometry. Available at http://www2.macaulay2.com.

Heaton, A., Hosten, S., Shankar, I., 2021. Symmetry adapted Gram spectrahedra. SIAM J. Appl. Algebra Geom. 5 (1), 140–164.

Hilbert, D., 1888. Über die Darstellung definiter Formen als Summe von Formenquadraten. Math. Ann. 32 (3), 342–350.

Hubert, E., Bazan, E. Rodriguez, 2022. Algorithms for fundamental invariants and equivariants of finite groups. Math. Comput. 91 (337), 2459–2488.

Marshall, M., 2008. Positive Polynomials and Sums of Squares. Mathematical Surveys and Monographs, vol. 146. American Mathematical Society, Providence, RI.

Motzkin, T.S., 1967. The arithmetic-geometric inequality. In: Shisha, Oved (Ed.), Inequalities. Academic Press, pp. 205–224.

Morita, H., Yamada, H.-F., 1998. Higher Specht polynomials for the complex reflection group G (r, p, n). Hokkaido Math. J. 27 (3), 505–515.

Niño Cortés, J.A., 2019. Algorithmic calculation of invariant rings under the action of finite sub-(groups): an implementation in Macaulay2. MA thesis. Universidad de Los Andes.

Riener, C., 2012. On the degree and half-degree principle for symmetric polynomials. J. Pure Appl. Algebra 216 (4), 850–856.

Serre, J.-P., 1977. Linear Representations of Finite Groups, vol. 42. Springer.

Shephard, G.C., Todd, J.A., 1954. Finite unitary reflection groups. Can. J. Math. 6, 274–304.

The Sage Developers, 2024. SageMath, the sage mathematics software system (version 10.3). https://www.sagemath.org.