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What types of insight do expert students gain during work with ill-structured problems in mathematics?

Eirin Stenberg [a,*], Per Haavold [a], Bharath Sriraman [b]

[a] Department of Education, UiT The Arctic University of Norway, Tromsø, Norway
[b] Department of Mathematical Sciences, University of Montana, Missoula, USA

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ABSTRACT

In our study, we explored how two high-performing mathematics students gained insight while working on ill-structured problems. We followed their problem-solving process through task-based interviews and observed a similar sequence of insights in both participants' work- *(1) Spontaneous insight*, *(2) Passive gradual insight*, *(3) Sudden insight*, and *(4) Active gradual insight*. An impasse occurred in the intersection between the second and third insight and seemed to accelerate the progression toward solution. During this insight sequence, we observed emotional transitions that appeared to impact the process in a useful manner, especially due to the participant's interpretation of uncertainty related to the impasse as a challenge and an inspiration. Future research is needed to study the observed sequence of insights and related affects in a larger data set and in a broader spectrum of problem solvers.

1. Introduction

A much-researched topic within the research on mathematical problem solving has been the conscious and gradual work toward the solution of mathematical problems (Haavold & Sriraman, 2022; Lesh & Zawojewski, 2007; Lester Jr., 2013). A substantial part of this research has tended to emphasize behavioural aspects and metacognitive processes involved in problem solving, regularly framed within stepwise models developed to describe and explain how to solve a problem (Haavold & Sriraman, 2022; Lesh & Zawojewski, 2007; Lester Jr., 2013; Liljedahl & Cai, 2021; Liljedahl et al., 2016; Rott et al., 2021; Schoenfeld, 1992). While such models provide a valuable framework for understanding problem solving, it is important to recognize their limitations. As all models, stepwise problem-solving models are an over-simplification of reality. According to Lester Jr. (2013), these models lack concern for the potential non-linear and unpredictable progress of problem solving. He further suggests that such oversight may comprise the models' accuracy and predictive value, as well as hinder the improvement of mathematics instruction in the classroom (Lester Jr., 2013).

In alignment with this, Rott et al. (2021) highlight the need for more descriptive models of problem solving that capture the dynamic and non-linear nature of real-world problem solving. In this context, "intuition" has emerged as a crucial and overlooked aspect, complementing rational analysis and acknowledging problem solving as also a subtle and unpredictable process with its "non-smooth" events (Lester & Kehle, 2003; Liljedahl et al., 2016; Rott et al., 2021). However, intuition is a concept that is challenging both to comprehend and to describe precisely. This is paradoxically why it mirrors the essence of "real" problem solving so neatly, as this process too is subtle and difficult to capture (Lester Jr., 2013). While intuition can be difficult to define, its essence can be

* Corresponding author.
E-mail addresses: eirin.stenberg@uit.no (E. Stenberg), per.oystein.haavold@uit.no (P. Haavold), sriramab@mso.umt.edu (B. Sriraman).

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summarized as “knowing without being able to explain how we know” (Shirley & Langan-Fox, 1996). More specifically, Fischbein, who did one of the first work on intuition in mathematics education, defines intuitions as “cognitions which appear subjectively to be self-evident, immediate, certain, global, coercive” (Fischbein, 1999, p. 11). In the Merriam-Webster’s Dictionary, intuition is described synonymously to insight. Moreover, intuition is described as “quick and ready insight” (Merriam-Webster, n.d.-b), whilst insight is described as “seeing intuitively” (Merriam-Webster, n.d.-a). This synonymous use of “insight” and “intuition” is also applied in mathematics by e.g. Burton (1999), who argues that “intuitive or insightful leaps” precedes the search for a convincing argumentation while doing mathematics. This sudden insight is reminiscent of the "aha!"- moment that can accompany breakthroughs in problem solving- first systematically described and studied by the Gestaltists almost a hundred years ago. Thus, “intuition” and “insight” are often used synonymously.

Today, there are generally two views on the source of insight in problem solving. On the one hand, insight is described as gradual and as a conscious and analytical “business- as- usual process”, typically related to problem-solving models. On the other hand, insight is described as spontaneous and as an unconscious, “special process” in response to an impasse, typically related to intuition and “aha!”-moments (Haavold & Sriraman, 2022; Ohlsson, 2011; Weisberg, 2015). However, recent work have suggested that a comprehensive understanding of insight will require bringing together aspects of both proposed sources of insight (Weisberg, 2015). More specifically, research has demonstrated that certain problems can often be solved both gradually and spontaneously (Fleck & Weisberg, 2004; Fleck & Weisberg, 2013; Kounios & Beeman, 2014; Weisberg, 2018).

Nevertheless, there are some limitations with this work. First, the research on insight during problem solving have mostly relied on simple “insight problems”, which usually require just a single appropriate mental restructuring. This has limited our understanding of the source of insight and how cognitive processes are involved and interact in more complex and multifaceted problem solving (Robertson, 2017a). The use of deceptive insight problems has also created doubt of the role of impasse. Some studies have for example suggested that spontaneous insight can arise without a preceding impasse. However, it is currently unclear how this observation, based on simple insight problems, is relevant in more complex and multifaceted problem solving (Weisberg, 2015). The second limitation is a lack of research on different sources of insight in subject-specific contexts such as mathematics. The two studies we found, conclude that both gradual conscious work and unconscious and sudden “aha!”- moments can be involved in successful mathematical problem solving, but neither study investigated how the two sources of insight interact or impacts the overall complex and multifaceted problem solving process (Haavold & Sriraman, 2022; Leikin et al., 2016). Thus, more research on the source of insight, i.e. types of insight, have the potential to deepen the understanding of insight in mathematical problem solving.

To bridge these knowledge gaps and gain a more comprehensive understanding of insight, our study aims to delve into the source of insight during students’ engagement with multifaceted and complex mathematical problems. In this context, the term “source” refers to “types” of insight. More specifically, we aim to describe and interpret types of insight in high-performing students’ work with ill-structured mathematical problems. Also included in this investigation is the role of impasse for gaining insight. Thus, we attempted to answer the following research question:

What types of insight do expert students gain to ill-structured problems during mathematical problem solving?

Our study follows a case study design, and from a total sample of 43 students, two were chosen for this in-depth study of insight. We collected data by conducting task-based interviews with high-performing students of mathematics and observed their work with ill-structured problems. Ill-structured problems are particularly well-suited for studying restructuring and insight in problem solving, as they generate uncertainty and facilitate multiple plausible representations of a problem (Webb et al., 2016). The students were considered experts of mathematical problem solving in a school mathematics context, and in this paper, we use the terms “high performing” and “expert” interchangeably. The experts have a high domain specific knowledge (see e.g. Hoffman, 1998), which implies that they have the potential to flexibly restructure (gain insight to) complex problems (see e.g. Ionescu, 2012). Thus, the experts were suitable for studying insight during mathematical problem solving of ill-structured problems.

2. Background

2.1. Insight as restructuring in problem solving

Although insight is commonly understood as apprehending or understanding something in greater depth, the research literature has usually defined insight more precisely as a mental restructuring or reformulation of a problem (Danek, 2018; Haavold & Sriraman, 2022). In other words, insight is a change in the problem solvers’ mental representation of the problem (Ohlsson, 2011). To see why, it is crucial to delineate what a problem is. Schoenfeld (1992) explains that throughout history, the term has had varying interpretations. Generally, two poles of meaning have emerged. On the one hand, *problems as routine exercises* have traditionally been used in mathematics instruction settings. On the other hand, and in this context of problem solving as a research endeavour and an important mathematical enterprise, *problems have to be problematic* (Schoenfeld, 1992). This means that although there is a long line of different problem-solving paradigms, they all seem to consider a problem to be a goal that an individual does not immediately know how to reach (Lester Jr., 2013). Specific definitions of this view are found throughout the research literature. For example, in his influential early work on mathematical problem solving, Schoenfeld (1983) described a problem as “only a problem if you don’t know how to go about solving it” (p. 41). In other words, a task that can be solved comfortably by routine or familiar procedures is not a problem in the context of problem-solving (Carlson & Bloom, 2005). Problem solving in this context is therefore simply what one does to reach this goal and solve the problem (Lester Jr., 2013).

Drawing on concepts from the cognitive science literature, problem solving can be understood in terms of an individual’s *problem* *perception*, which is the mental representation of a problem separated into three components: givens, goal and operations (Ohlsson, 2011; Robertson, 2017b). Givens are the information presented in the problem representation – the "knowns". The goal is the state reached when/if the problem is solved. Operations are the actions the individual engages in to move from givens to the goal. Of course, it is possible that an individual does not possess the necessary knowledge and skills to solve a problem. However, if we assume that an individual can potentially solve a problem, then the underlying cause of what constitutes a problem, is a flawed interpretation or understanding of at least one of the three components of an individual's problem perception (Ohlsson, 2011). In Haavold and Sriraman's (2022) recent study on insight, this is for example seen when the participants erroneously assume that the Roman inheritance problem has a single, correct solution. To solve this, and other problems, a new understanding of the problem is required. This novel understanding is possible through new interpretations of the problem – known as "restructuring" (Weisberg, 2015). More specifically, the individual's representation of the problem changes through a substantial modification in one or more of the three components of problem solving (Chronicle et al., 2004).

The Gestaltists referred to this mental representation as a Gestalt, and to them, a mental restructuring resulted in a more productive and harmonious one (Haavold & Sriraman, 2022; Wertheimer, 1959). On the face of it, and empirically (e.g. Haavold & Sriraman, 2022), it seems wrong to assume that every restructuring leads to an improvement. The modified representation can be successful or unsuccessful (Dominowski & Dallob, 1995; Ohlsson, 1984), depending on whether the problem perception is correct. Insight can therefore more precisely be understood as a mental restructuring of a problem that also moves a problem solver from a state of not knowing how to solve a problem to a state of knowing how to solve it – or at least move closer to a solution when faced with complex multi-stage problems (Danek, 2018; Mayer, 1995). It also follows that there can be several productive restructurings, or insights, during a problem-solving process (Davidson, 2003).

However, the Gestaltists were right in that an individual's problem perception cannot be reduced to isolated interpretations of the problem's different components (Haavold & Sriraman, 2022). The mental representation of a problem is a totality, as each individual component is related to and affects the other components (Ohlsson, 2011). If an individual's interpretation of a single component is restructured, each of the other components are also reinterpreted in such a way that they all fit together in a new, coherent and qualitatively different whole (Ohlsson, 2011). An earlier study by Haavold (2011) illustrates this. In this study, upper secondary students were asked to "find a" in the trigonometric equation $sinx + cosx = a$. After being given a hint, what is a in the equation $sinx = a$, some of the students changed their interpretation of what type of answer was required. This reinterpretation of the goal of the problem also led to a change in the type of operations the students made use of. For example, one student treated the equation as a function, and found an interval of its min and max value through differentiation. Insight can therefore not only be seen as a mental restructuring that moves the problem solver closer to a solution, but it can also more accurately be seen as a productive restructuring in any of the three components of an individual's problem perception (Davidson, 2003; Ohlsson, 2011). In the context of mathematical problem solving, this means that insight, as a mental restructuring, brings about a positive and substantial change in the mathematical concepts, processes, practices, experiences, methods and so on, that becomes accessible in the individual's mind.

2.2. The source of insight

In the modern research literature, there exist mainly two views of the source of insight in problem solving. The first view, commonly referred to *productive thinking*, comes from the neo-Gestaltists', building on the Gestaltists' distinction between insight and analysis. This view is often emphasized in the field of creativity, and from this perspective insight is a result of a sudden "aha!"-experience – a spontaneous realization of a new approach to or understanding of a problem. In other words, a different approach becomes viable through a new interpretation of the problem, suddenly realising what the problem is really about. Insight is perceived to be a result of a special cognitive process of spontaneous and unconscious thinking, which is why this view of insight is also known as the "*special-process view*" (Weisberg, 2015).

The Gestaltists argued that insight was achieved through a sequence of four stages- preparation, incubation, illumination, and verification (Hadamard, 1945; Poincaré, 1948; Wallas, 1926). In the first stage the problem-solver works to understand the problem. In the second stage, the problem is put aside for a while and the mind is occupied by other thoughts. In the third stage, a solution to the problem suddenly appears as a result of unconscious work. In the fourth, stage the solution is verified, making it more precise and possibly extending it (Haavold & Sriraman, 2022). This model has more recently been reformulated by Ohlsson (2011) as an *insight sequence* consisting of the following stages: attempted solutions → consistent failure → impasse → restructuring → "aha!" → solution. The crucial difference between the original four-stage model and the revised insight sequence, is that the former model includes an incubation-stage, while the latter model replaces this with an impasse-stage. An impasse is not the same as an incubation period. During an incubation period, the problem is set aside voluntarily and actively. During an impasse, the individual still works on and thinks about the problem, but there is no progress. The importance and consequences of this modification will be clarified in the next subsection. Here, we will simply point out that although the Gestaltists describe creative processes in general, their view of insight is less informative about the underlying cognitive mechanisms, or sources, of insight (Robertson, 2017a). Furthermore, the insight sequence is more appropriate for analyzing problem solving for two reasons. First, it highlights clear differences between insight as a sudden and unconscious response to an impasse, and insight as a result of gradual analytical thinking. Second, while the Gestaltists considered insight and incubation on timescales up to several years, the revised insight sequence considers insight and impasses on much shorter timescales (Beghetto & Karwowski, 2019; Ohlsson, 2011).

Before probing further, a small clarification is at this point necessary. Although the special-process- view of insight emphasizes the unpredictable, instant and unconscious nature of thinking during problem-solving, this type of restructuring is not "magical" or unexplainable (Weisberg, 2015). A reason for the mystery surrounding sudden and unconscious insight (see for example Sriraman, 2008), is that most of the research literature on productive thinking rarely focuses on the cause of, or what triggers, instant restructurings (Vallée-Tourangeau, 2018). What does seem to be the case, however, is that internalized negative feedback during the problem-solving process, can unconsciously redistribute activation of semantic memory, resulting in a new representation (Ohlsson, 2011). A close reading of the relevant literature suggests that such negative feedback can be triggered by either internal or external factors. In the former case, negative feedback often takes the form of failed solutions to problems. In the latter case, negative feedback is generally related to *chance*. For example, Liljedahl (2009, p. 69), in discussing Hadamard's earlier influential work on mathematical invention, explains that sudden insight can be triggered by "a chance of reading an article, a chance encounter, or some other chance encounter with a piece of mathematical knowledge…". In other words, insight can be the result of a *luck* encounter.

The special-process-view of insight is not only characterized by sudden and unconscious restructuring, but also closely related to strong subjective experiences. Based on an extensive literature review of self-reports of individuals' problem solving experiences, Bowden and Grunewald (2018) argue convincingly that there are also clear affective differences between spontaneous and analytical insight. The results suggest that spontaneous insight is regularly associated with the surprise of the suddenness, confidence in the solution, pleasure or relief of having found the solution, and a new drive or optimism. Opponents of the special-view-process claim that the subjective experiences are irrelevant, and therefore not a valid feature of insight (Weisberg, 1986). However, even if the subjective experiences are merely a byproduct of brain activity, it is still an important marker of spontaneous insight (Bowden & Grunewald, 2018).

Also DeBellis and Goldin (2006) address the role of affect in problem solving, and describe a relation between empowering affect and the actions of taking risks, asking questions and constructing a new plan. Similarly, Liljedahl (2005) found that "aha!"- experiences (spontaneous insight) transform the problem solver's resistance into positive beliefs and attitudes regarding one's own proficiency in mathematics. DeBellis and Goldin (2006) also addresses the role of negative emotions. They argue that disempowering affect hinders understanding and progression with problem solving (DeBellis & Goldin, 2006). Surprise is viewed as a neutral activating emotion that arises in response to unexpected events, as it can have both positive and negative affective impact depending on the situation (Mauss & Robinson, 2009; Muis et al., 2018). More generally, emotions can be categorized as (1) *positive activating* (e.g. curiosity, enjoyment, pride, hope), (2) *positive deactivating* (e.g. relief), (3) *negative activating* (e.g. confusion, frustration), and (4) *negative deactivating* (e.g. boredom) (Linnenbrink, 2007; Pekrun, 2006). Thus, affect may impact the problem-solving process and its diverse insights in several ways.

The second view, also referred to as *reproductive thinking*, posits insight as a result of a gradual, reproductive and analytical process (Weisberg, 2015). Insight is believed to be the result of a conscious process, associated a stepwise progression. This perspective is often called the *"business-as-usual view",* as it describes insight as the result of a thought process no different from the process that underlie ordinary thinking (Bowden et al., 2005; Haavold & Sriraman, 2022; Weisberg, 2015). Even though the phenomenological experience of solving a problem through "aha!"-moments may be different from the experience of an analytical solution of a problem, the proponents of the "business-as-usual" view does not regard this as evidence for the mechanism driving the two insight experiences being different (Weisberg, 2015).

The business-as-usual view, or reproductive thinking, is built on the idea that problems are solved by matching a problem with information in memory and acting on that similarity. Problem-solving is a step-by-step process driven by prior knowledge and conscious evaluation. Initially, after gaining an initial understanding of the problem, or problem perception, the individual would attempt to match the problem with prior knowledge and evaluate whether a solution method could be transferred to the new problem. If this attempt is unsuccessful, the individual would move on to applying heuristic methods. Using heuristics, the problem solver attempts to modify the present state of the problem so that s/he can advance towards the final goal (Weisberg, 2015). Of course, the process is not nearly this simple or linear, but it provides a general overview of the analytic approach to problem solving. Insight, or restructuring of the problem in a new and more productive manner, is gradually gained through a stepwise and conscious process (Robertson, 2017a).

Reproductive thinking is also closely related to what is known in the problem solving research literature as *problem solving by design*, which can be thought of as an algorithmic and deductive approach to problem solving (Liljedahl et al., 2016). According to this approach, problem solving is largely a process of deducing the solution to a problem from existing knowledge and prior experiences. Knowledge and experiences shape both an individual's perception of the problem and the choice of strategies used to solve the problem. Meaning that when someone starts to work on a problem, they rely entirely on what they already know and their past experiences. This acknowledgment has, as we mentioned earlier, led the field of mathematics education to focus on the heuristics of problem solving – often portrayed as gradual and conscious process in the form of stepwise models (Liljedahl et al., 2016; Rott et al., 2021).

No heuristic has been more influential than George Pólya's (1949) pedagogical and easily digestible four step model which he presented in his book *How to Solve It* (1949). According to Pólya, problem solving could be described by four steps: 1) Understanding the problem, 2) Devising a plan, 3) Carrying out a plan, and 4) Looking back. However, despite the revolutionary impact of Pólya's heuristic on problem solving and the teaching of problem solving, one weakness was that it portrayed problem solving as a normative and theoretical process that overlooked practical issues and "managerial skills" required to regulate one's activity during problem solving (Haavold & Sriraman, 2022; Liljedahl et al., 2016). Subsequently, the later and influential work of Alan Schoenfeld (1985) and Frank Lester (1985) was a refinement that also considered problem solving as an emerging and contextually dependent process. In other words, problem solving was seen as a process where an individual's prior knowledge, prior attempts and inner thoughts all came together in a unique way (Liljedahl et al., 2016). The work of Pólya, Schoenfeld, Lester, and others, have contributed to a better understanding of how problems are solved and how problem solving should be taught (Lester & Cai, 2016; Liljedahl et al., 2016). Nevertheless, a common feature in the mentioned models and heuristics, and throughout the field of mathematics education, is that problem solving is laid out as a conscious and incremental process in which problems are solved through experience and conscious evaluation. The meaning and importance of sudden insights is largely ignored.

2.3. Impasse and insight

Although the main focus of this study is the source of insight in mathematical problem solving, the role of impasse is also highly pertinent. Historically, theories of insight have often considered impasse to be necessary for insight to occur in problem solving (Ohlsson, 2011; Petervari & Danek, 2019; Weisberg, 2015). Recently, though, there has been an emerging interest in the possibility of a non-dichotomous view of the source of insight as a growing body of evidence suggest that insight can come about both gradually and spontaneously (Haavold & Sriraman, 2022; Weisberg, 2015). Weisberg (2015), for example, has proposed a four-stage model that attempts to integrate the two views of insight. In this model, both reproductive and productive thinking can bring about a productive mental restructuring, or insight, in problem solving. However, both Weisberg's (2015) model and other theoretical attempts at bridging the two views – e.g. Representation Theory (Ohlsson, 2011) and dual process approaches (Gilhooly & Murphy, 2005)– maintain the divide between the two views or sources of insight as they posit that impasse is needed for spontaneous, sudden and unconscious insight to occur. Thus, much of the research literature still seem to consider the source of insight to be either conscious without an impasse or unconscious with an impasse. Making sense of impasses and their role during problem solving may therefore be a key issue in bringing about a better understanding of the source of insight.

In this regard, there are two important caveats. First, the research literature has established that impasse can occur during problem solving, and that it can be a condition for insight to take place. However, empirical observations also indicate that insight, and even spontaneous insight, can happen without a preceding impasse (Petervari & Danek, 2019; Weisberg, 2015). For example, Fedor et al. (2015) found that impasse was present in only 1/3 of instances of spontaneous insight. In another study, Fleck and Weisberg (2013) observed an impasse in 50 % of instances of what they called "insight solutions". Meaning, in half the occurrences of both gradual and spontaneous insight, there was no preceding impasse. Others, such as Ohlsson (2011), Cranford and Moss (2012) and Weisberg (2015) have concluded similarly. This means that although theoretical explanations of insight highlight the importance of impasses, it is still largely unknown how impasses actually are related to the two sources of insight. Second, how we conceptualize impasse informs wherein more research can add to our understandings of these concepts. To do so, it is important to first compare impasse with incubation, as we did earlier. While impasse and incubation seem related, a crucial difference is that during an impasse an individual's passivity and lack of progress is not voluntary. It is instead "enforced" as the individual is stuck and not able to make progress. Incubation, on the other hand, is a decision to set the problem aside (Weisberg, 2015) – which an individual can decide to do in the face of an impasse. In other words, incubation can be a conscious strategy to overcome an impasse. Although both concepts have been observed in relation to insight, the time demanding nature of incubation makes it challenging to study this aspect of insight (Freiman & Sriraman, 2007; Liljedahl, 2004).

Research on insight and the role of impasse has therefore considered impasse as primarily a qualitative phenomenon, not defined by its lengthiness, but rather as a subjective experience by problem solvers. The most common characteristic of impasse seems to be a feeling of being stuck and out of ideas of how to move forward. It is also associated with hesitation or resistance before making a choice of changing directions (Savic, 2015; Stuyck et al., 2021). More recently, it has also been suggested that impasses that are associated with insight are characterized by an increase in motivation. The feeling of being stuck can also trigger of being challenged and therefore propel the individual towards new ways of tackling the problem (Ross & Arfini, 2024). As such, the proposed insight sequence mentioned earlier may not fully capture the complexity of spontaneous insight and its relationship with impasses.

2.4. Empirical research on the source of insight

Although scarcely researched, diverse sources of insight have been investigated in studies and historical accounts that have documented that professional mathematicians experience moments of insight during their work on mathematical problems, both with and without a preceding impasse (Hadamard, 1945; Poincaré, 1948; Savic, 2015; Sriraman, 2008). Furthermore, professional mathematicians seem to take deliberate actions to overcome experienced impasses. On the one hand, a common approach is taking a break from the problem, which echoes the incubation phase in the gestalt model of creative processes. On the other hand, mathematicians have also reported that as they are about to attack a new problem, they cycle through a repertoire of heuristics and emphasize the unflinching will of never giving up (Sriraman, 2008). As such, their work and source of insight also seem to be aligned with a more analytic and conscious source of insight. Overall, the research on professional mathematicians' source of insight seems to be inconclusive.

As for learners of mathematics, the research is even less clear. Only a few studies have explicitly focused on insight within a mathematics education setting. For example, in a retrospective study of teacher students' mathematical experiences, Liljedahl (2005) found that more than half of the participants had experienced a so-called "aha!" - moment in the context of a recent undergraduate course. Although the study implies that also learners of mathematics do experience moments of insight during problem solving, the retrospective self-report design prevents further understanding of insight with and without impasses – particularly related to the issue of the source of insight. A more recent study by Munzar et al. (2021) overcame some of these issues by employing a think-aloud protocols to investigate how 136 elementary students solved complex mathematical problems. In the study, the authors conclude that not only can young learners of mathematics experience insight during problem solving, but they can also experience and overcome impasses. Furthermore, the emotions the students experience during problem solving seem to have an impact on whether an impasse can be overcome. More specifically, when confusion and uncertainty arise during the problem-solving process, students must realize that this is normal and not necessarily a failure. Unfortunately, the study by Munzar et al. (2021) did not address the issue of the source of insight or clearly delineate insight (and restructuring) with or without an impasse.

To our knowledge, only two studies within mathematics education have explicitly tackled the issue of the source of insight in problem solving. In the first study, Leikin et al. (2016) investigated how general giftedness (G) and school mathematical performance, respectively, was related to insight problems and learning-based mathematics problems. Using an event related potentials methodology, the authors concluded that insight problems was mainly affected by a G-factor, while school mathematical performance was primarily related to the learning-based mathematics problems. This indicates that spontaneous "aha!" – moments are related to general cognitive aptitudes and not learning experiences. In other words, solving insight-based problems seems to be associated with different cognitive mechanisms than learning-based problems. However, the use of insight problems in insight related research have been criticized for limiting opportunities to investigate the nuances of cognitive processes involved in complex problem solving (Robertson, 2017a). Addressing this issue, Haavold and Sriraman (2022) investigated how groups of proficient and non-proficient undergraduate students in mathematics solved complex ill-structured mathematical problems via task-based interviews. A key finding was that insight could be the result of both conscious gradual thinking and spontaneous unconscious thinking. Neither view of insight seems therefore to fully explain a successful problem-solving process, and instead there seems to be a complex intertwined relationship between productive and reproductive thinking.

Haavold and Sriraman's (2022) findings resonate with much of recent literature on insight in general psychology. Although the question of domain specificity of insight in problem solving remains an unanswered question (Plucker & Zabelina, 2008), findings from insight-related research in psychology helps us frame the issue in a larger context. In general, most problems used in insight research, both the more commonly used insight problems and the less commonly used complex problems, can be solved using both analytical reproductive thinking and spontaneous productive thinking (Kounios & Beeman, 2014; Weisberg, 2018). For example, Fleck and Weisberg (2004) and Fleck and Weisberg (2013) found in their studies, using think-aloud protocols, that insight problems designed to require insight solutions, could be solved using both analytical thinking and spontaneous thinking. These findings suggest that insight, as important cognitive leaps or mental restructurings during problem solving, can both be the result of analytical and conscious thinking or spontaneous unconscious thinking. However, these results neither explain in what way, nor if, the two sources of insight are related or distinct.

Further research in the field of psychology provides inconclusive evidence for distinct sources to insight. On the one hand, functional magnetic resonance imaging (fMRI) and self-report studies commonly report both neurocognitive and phenomenological differences between analytical and spontaneous insight (Bowden & Grunewald, 2018; Kounios & Beeman, 2014). The two sources of insight seem therefore to be qualitatively different, indicating that unconscious insight can be thought of as a special-process distinct from normal cognitive and conscious processes involved in problem solving. On the other hand, recent literature reviews have concluded that there is a substantial positive correlation between working memory and insight (Chuderski & Jastrzębski, 2018; Gilhooly & Webb, 2018). The relationship between working memory and insight is a strong prediction of the business-as-usual hypothesis and indicates that insight is not a distinct special cognitive process. Instead, insight is strongly associated and integrated with conscious analytical thinking. The two opposing views of the source of insight, which has previously been thought of as in opposition, are now generally considered to be complementary (Weisberg, 2015). However, the nature of this relationship is still largely unknown.

2.5. Task design for studying insight

As we now know, both reproductive and productive thinking can be involved in successful problem solving (Haavold & Sriraman, 2022). However, research on insight during problem solving have mostly relied on simple insight problems, which usually require just a single appropriate mental restructuring that can come about after either productive or reproductive thinking (Weisberg, 2015). As pointed out in the introduction, this has restricted the understanding of the source of insight and related cognitive processes involved in complex problem solving (Robertson, 2017a).

According to Danek (2018), there are two main criteria for selecting problems for the investigation of insight: 1) The problem should trigger an initial representation that is unlikely to activate the knowledge needed to solve the problem, and 2) solutions to the problems should require one or several insights. This means that problems should be designed so that the problem solver does not fully grasp the problem at first, and that this misrepresentation necessitates one or several productive restructurings so that the problem solver can move towards a solution. One such category of problems are so-called ill-structured problems. While well-structured problems are constrained problems with clear givens, goals and operations, ill-structured problems usually have unclear problem statements, multiple solutions or solution paths, and contain uncertainty about which concepts, rules and principles that are necessary for solving the problem (Jonassen, 1997; Krutetskii, 1976; Pretz et al., 2003; Shin et al., 2003). The term "ill-structured" is often used interchangeably with "ill-defined", as they both refer to problems with missing information in one or more of the components of problem solving, unlike "well-structured", also known as "well-defined", problems (Kim et al., 2013).

Going back to the concept of problem perception, this means that ill-structured problems can more precisely be defined as problems characterised by unknowns or uncertainty in one or more of the givens, goal or operations components of a problem. The lack of clarity not only allows more than one mental representation of the problem, but it is also likely to trigger a flawed initial understanding of the problem (Hardin, 2003; Robertson, 2017b). Ill-structured problems (Kilpatrick, 1987; Schoenfeld, 1985; Simon, 1973) are also described as tasks that "lack a clear formulation, or a specific procedure that will guarantee a solution, and criteria for determining when a solution has been achieved" (Kilpatrick, 1987, p. 134)". Therefore, such problems generally require a change in representation before a path to solution is found. As such, ill-structured problems are particularly suitable for investigating insight (Webb et al., 2016). Two examples of ill-structured problems are found in "The Impossible Squares Problem" and "The Lucky Fractions Problem", presented in Table 1.

The ill-structuredness of the problems in Table 1 follow the definition of ill-structured problems, previously given, in which the presence of uncertainty in one or more of the components of problem solving and the possibility of several restructurings (insights) are central characteristics of such problems. We elaborate and argue for the ill-structuredness of *"The Impossible Squares Problem"* and "*The Lucky Fractions Problem"* in the following two paragraphs.

The Impossible Squares problem was considered an ill-structured problem as there is uncertainty in both givens and operations of the problem that have a high probability of necessitating restructurings. First, the problem statement is straight-forward, and some participants may therefore not realize initially why it is even a problem. This means it may not be immediately clear that certain squares cannot be drawn. The participants need to grasp that there are in fact many squares that can't be drawn, and that these squares have certain mathematical properties. Second, there is no set of procedures or operations that are immediately connected through mental schemas to the problem. Although participants may begin to draw certain squares, only by working on the problem, and making the right insights (restructurings), can participants begin to move closer to the solution.

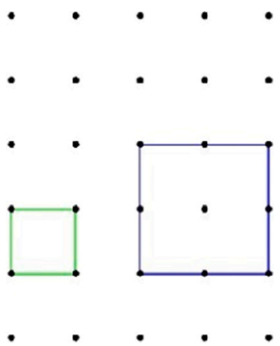
The Lucky Fractions problem was also considered ill-structured. There are two reasons for this. First, although the problem statement, or the givens component, is formulated clearly, the strangeness – or pathological nature (Haavold & Sriraman, 2022)– means participants must accept counterintuitive and "untrue" properties as a premise. This forces the problem solver to deal with uncertainties regarding existing schemas related to basic arithmetic. Second, even if the participants accept the premises of the problem and have an idea of how a solution might look like, the strangeness of the problem makes it difficult to relate any specific procedures, methods or strategies to the problem. Thus, this yields the potential of several restructurings, which is a central feature of ill-structured problems.

2.6. Problem-solving expertise

Overcoming fixations and being able to change direction appears to be a requirement for recovering from an impasse and gaining insight. According to Ionescu (2012), among others (Hoffman, 1998; Shanteau & Phelps, 1977), *experts* are highly flexible in their representation of and solutions to problems. Some argue that this flexibility is a result of rich knowledge structures (Bilalić et al., 2008; Hoffman, 1998). However, there's been a debate of whether expertise is a domain-specific or domain-general ability (Ericsson & Smith, 1991). According to Ericsson and Lehmann (1996), experts show a consistent superior performance *within* a domain, and expertise as an acquired ability have received broad support (de Bruin et al., 2008; Ericsson et al., 1993; Simon & Chase, 1973).

Definitions of expertise within mathematics have been debated (Elgrably & Leikin, 2021; Leikin, 2021). In accordance with the

Table 1
The two problems of the study.

The Impossible Squares Problem	**The Lucky Fractions Problem**
In the grid, the horizontal and vertical lengths between the dots are identical. It is only possible to create squares by drawing lines between the dots. Two examples with areas 1 and 4 are shown in the grid.  a) Which are the five smallest squares you can draw in the outlined grid? b) Assuming an infinite grid, are there squares which cannot be drawn? Why/why not?	This is an example of a fraction which can be reduced using a faulty method, but which still provides a numerically correct result. $$\frac{16}{64} = \frac{1\cancel{6}}{\cancel{6}4} = \frac{1}{4}$$ a) Can you find other fractions with this property? b) Can you find all two-digit fractions with this property?
Suggested Solution Draw tilted squares and list the ones that are possible. Alternatively, use Pythagorean theorem to generate a list. Then list the ones you couldn't draw or generate by the theorem. E.g.: 3 – 6 – 7 – 11 – 12 – 14 – 15 – 19 – 21 Solve it using number theory: Using the Pythagorean theorem, the problem can be restated to determine which integers can't be written as $n = a^2 + b^2$ An integer n can't be written as a sum of two squares if its prime decomposition contains a factor p^k where (prime) $p \equiv 3 \pmod 4$ and k is odd. E.g.: $11 = 3 \pmod 4$, and $k = 1$.	**Suggested Solution** Express the example algebraically: $(10a + b)/\ (10b + c) = a/c$ Use the expression to generate lucky fractions by inserting values: Solve it using algebra: $(10a + b)\ c = a\ (10b + c)$ which simplifies to: $b\ (10a - c) = 9ac$ There are now three cases: $(9, b) = 1$, $(9, b) = 3$, and $(9, b) = 9$ that can be explored further to discover the "lucky fractions". See full solution method in Appendix.

consensus of flexibility as a characteristic of experts in general, several studies in mathematical problem solving confirm strategic- and representational flexibility in highly gifted students of mathematics in particular (Greer, 2009; Star & K.J., 2009; Star & Seifert, 2006). Further, a widely used explanation for mathematical expertise is creativity (Elgrably & Leikin, 2021; Leikin, 2018, 2021; Leikin et al., 2017). Baer (2015) suggests that expertise may not require creativity, but rather that creativity require expertise. This view is challenged by the bulk of research on mathematical expertise, where creativity is identified as a key characteristic of mathematically gifted individuals (e.g. Elgrably & Leikin, 2021). Moreover, creativity is often used as an explanation for high performance in original, non-algorithmic and insight-based solutions to non-routine problems (Ervynck, 1991). Collectively, both creativity and flexibility are characteristics associated with mathematical expertise, which both are connected to the solution of non-routine problems and deliberate practice.

3. Method

3.1. Study design

For our study of insight during expert students' mathematical problem solving, we chose a case study design. The case of our study was two expert students of mathematics, who both met both criteria for participation (see Section 3.2 for recruitment criteria and operational definition of expertise). As defined by Stake (2005), the current study is an *instrumental case study*, in which we examined specific cases to understand a theory better. More specifically, we studied the two expert students' problem-solving processes to understand the theory of insight in mathematical problem solving better. To accomplish this, we observed and described types of insight that the expert students gained during mathematical problem solving, defined by Yin (2009) as a *descriptive* case study. However, we interpreted the students' problem-solving processes to arrive at these descriptions of types of insight, and according to Merriam's (1998) definition of case studies, our study is therefore *interpretative*. Lastly, our study may be viewed as *exploratory*, as we sought to understand a concept better and our findings may lay the ground for further research (Yin, 2009).

We chose a case study design, as this allow for rich, descriptive material (Adelman et al., 1980). Moreover, the small-scale data have the potential to capture features and details that otherwise might be lost in larger studies (Nisbet & Watt, 1984). Additionally, case studies may serve as a departure for later, larger studies (Yin, 2009). Thus, a case study may be advantageous when researching concepts that are insufficiently understood, such as the process of gaining insight, which is the focus of this report. More specifically, the aim of our study was to understand insight in mathematical problem solving better through studying cases of especially skillful problem solvers.

3.2. Research participants

In our study we investigated how experts gained insight during mathematical problem solving. As expertise is largely defined by high knowledge within a specific domain, the participants in the study were recruited on the background of high domain-specific knowledge in mathematical problem solving. More specifically, the main criteria for participating in the study was grade 5 or 6 (6 being the highest possible) in the most advanced mathematics course, "Mathematics R2", in Norwegian high school. An additional criterion for half of the participants in the study was prior participation in the problem-solving competition *Abel.* The recruitment of participants for the study was conducted by sending an invitation to schools known for having a high number of prior participants in the Abel competition, and by sending an invitation to schools known for holding the R2 course, specifically inviting students with the grade 5 or 6 in this course. 43 high school students aged 17–19 consented to participate in the study. All participants met the main criterion, whereas 24 also met the additional criterion.

From the total sample of 43 participants, we focus on two participants in this report. They both met both criterions for participation. These two participants- P28 and P24- were selected for two reasons. Most importantly, we assumed that the participants that came closest to a full solution to the two ill-structured problems of this study were the ones that produced the most productive restructurings and thus the most insightful solutions to the problems. This assumption was based on prior research that state that when a correct solution has been reached, this is typically the result of successful restructuring (insight) (Danek, 2018), which is necessary for solving ill-structured problems (Ohlsson, 2011; Weisberg, 2015). An additional reason for our selection of participants were that the initial exploration of the full data material suggested that the interviews with P28 and P24 were among the most promising data for an in-depth analysis of insight. Moreover, both participants expressed being comfortable in the situation and they both shared their thoughts and arguments with minimal help to think aloud from the interviewer. This made for rich data and gave the interviewer profound access to their thought processes.

3.3. Research instrument

In lines with Section 2.5, in which we present ill-structured problems as a tool for research on insight, the research instrument for the study consisted of three unique ill-structured problems, each subdivided in tasks a) and b). On an overarching level, ill-structured problems were chosen as they are open, flexible and have a high probability of triggering a flawed initial mental representation (Webb et al., 2016). For the scope of this article, the focus was on two out of the three problems- "*The impossible Squares Problem"* and "*The Lucky Fractions Problem*" (see Table 1). The ill-structuredness of these problems are described in Section 2.5 and is grounded on the theoretical definition of ill-structured problems also presented in Section 2.5.

We decided to use "*The impossible Squares Problem"* and "*The Lucky Fractions Problem*" for two key reasons. First, and more generally, investigations into insight in problem solving should make use of problems that are expected “to have a low probability of activating the knowledge needed to solve the problem” (Ohlsson, 1992, p. 10). Although both problem statements are straightforward, both mislead the problem solver. It is therefore unlikely that the participants will have an appropriate initial mental representation. Second, and more specifically, both problems have a well-defined solution set – that can be intuitively suggested from the start. This means that the participants must move from a likely flawed initial mental representation to a clear solution through mental restructurings. In other words, solutions to both problems require insights (restructurings), which in turn allows us to investigate, and compare, the nature and context of insights during problem solving (Ohlsson, 1992).

An additional important criterion for the choice of tasks was that they should be novel to the participants. The subjective novelty of each problem was validated by asking the participants to check a box confirming that they had not seen or solved the problem prior to the interview. None of the participants had seen or solved either task before the interviews.

3.4. Data collection

We collected data by conducting individual task-based interviews. Each interview lasted for 70 minutes and consisted of three sequential problem-solving tasks, divided into part a) and part b). The participants were free to start with a) or b) in whatever order they pleased. For each task, the participants had a maximum of 20 minutes to try and solve the problem, but they were free to ask the interviewer to stop the clock whenever they felt confident in having reached an answer. The interviews were videotaped, filming both the participants’ faces and their working sheets, and each interview was transcribed afterwards.

Both Goldin (1997) and Hunting (1997) consider clinical interviews useful for gaining access to participants’ cognition. More specifically, this method is fruitful for gaining access to experts’ thought during work on representative tasks (Ericsson, 2018). In accordance with Goldin’s (1997) guidelines for conduction of task-based interviews, the participants were asked to share their thoughts orally and the interviewer’s role was to help the participants accomplish that with minimal impact on their direction of thought. In agreement with Goldin (1997), Hunting (1997) underlines the importance of the interviewer’s role, who should refrain as much as possible from influencing the participants’ thoughts to ensure valid results that truly reflects the participants’ competence. However, Hunting (1997) highlights the role of language, emphasizing the need to clarify meaning. According to Hunting (1997), follow-up questions are a key feature of clinical interviews during which participants solve problems. Thus, in accordance with Hunting (1997), questions were used as a tool for gaining access to the participants’ thoughts and understanding during the task-based interviews. More specifically, the questioning techniques were used for several purposes- (a) to confirm that the interviewer was actually listening to the participants, (b) to help the participants share their thoughts whenever they went silent for a longer period of time (Goldin, 1997), (c) as a way of clarifying utterances that was ambiguous and that needed to be understood correctly in order to conduct a valid analysis post interviews (Maher & Sigley, 2014), and (d) to help the participants to demonstrate their full knowledge (Maher & Sigley, 2014). Thus, the goal was minimal inference from the interviewer, yet the described interventions were viewed acceptable and necessary for gaining full access to the participants’ thoughts, understanding and knowledge. This is in alignment with Goldin (1997), who suggest that hints and prompts may be given after the opportunity for free problem solving is given. He also states that this rule is occasionally broken due to time constraints, but in such cases, it is important to recognize the possibility for information being lost.

3.5. Data analysis

The data analysis followed the procedure of what Braun et al. (2019) identifies as *thematic analysis*. In lines with this procedure, we took a pragmatic approach and blended inductive and deductive analysis. The aim of the analysis was to explore the “source of insight” in the high- performing students’ problem-solving process. As described earlier, we chose the two participants that came closest to a full answer to the problems included in this study, as this reflected the most insightful solutions. A full transcript of the participants’ work was conducted as a preparation for the further analysis.

The first step in the analysis was to identify occurrences of insight. As insight was previously defined as a productive restructuring, or reformulation, of a problem, we looked through the data material for instances of substantial changes in the participants’ work and/or utterances that moved the process closer to a solution. We focused this analysis on identifying changes in the one of the three major components of problem perception. Although we focused our analysis on observable restructurings in the three components, any restructuring would of course also impact the problem-solver’s overall problem perception.

The second step in the analysis was deductive. We started our categorization of the discovered insights by sorting them in two predetermined themes, “spontaneous insight” and “gradual insight”. Spontaneous insight was characterised as productive restructurings that was seemingly detached from the work prior to it, as well as being sudden and related to feelings of surprise, joy and/or determination. Gradual insight was characterised as productive restructurings that built upon prior work and that had a stepwise, incremental progression, as well as indications of metacognitive actions.

The third step was to sort the occurrences of insights as either following an impasse or not. An impasse was defined as a stage, or period of time, during problem solving when the participant ran out of possible ways to solve the problem and no progress was observed (Glatzeder et al., 2010; Weisberg, 2015). We also looked for behaviour and other non-verbal expressions that indicated that the participant was stuck and resistant to changing direction.

4. Results

The results of our study were both in lines with and divergent from previous research. Our findings verify that insight is possible through both gradual restructuring of a problem and through spontaneous restructuring of a problem. However, in contrast to previous research, our research demonstrates that the process of gaining insight is not restricted to either gradual restructuring or spontaneous restructuring. The process rather contains both types of restructurings interchangeably, which also demonstrate that a complete insight to more complex problems might require more than one restructuring. An essential discovery was also the distinction between "sudden" and "spontaneous". In prior literature and research, these terms are used interchangeably, even though they are not semantically identical concepts (see Merriam-Webster, n.d.-c; Merriam-Webster, n.d.-d for defintions of "sudden" and "spontaneous"). Our data demonstrated this difference. Moreover, we noted that spontaneous insights occurred from an inner impulse and was not affected by any observed external stimuli, whereas sudden insights similarly occurred promptly, yet in contrast was affected by external stimuli. Furthermore, in contrast to most of the literature, we found that spontaneous insights were possible without an impasse. In total, we identified four categories of insight present in both participants' work – (1) spontaneous insight; (2) passive gradual insight; (3) sudden insight; and (4) active gradual insight. An overview of these different types of insight is illustrated in Fig. 1.

The following presentation of our findings is structured according to this sequential illustration, as this was the order in which the insights occurred in both the participants' work.

4.1. Spontaneous insight

This category describes a juncture in the participant's problem-solving process during which the participant restructured promptly to a more productive understanding without a preceding impasse and without any known external stimuli. Characteristic affections related to this type of insight was surprise and joy.

An example was found in participant 28's work with Impossible Squares, shown in the following extract:

> "So, this is square numbers, then. I can start out by drawing.
>
> So, we have this one *draws a straight 1×1 square*. This one *draws a straight 2×2 square*. Yeah, so it will be the square numbers up until five, then, which is one, four…
>
> One, four, nine… No, maybe not. Let me see… Sixteen. But after sixteen… It is only one, two, three, four… fi… No, One, two, three, four long. Mmm…
>
> Yeah… Oh, yeah, but you can of course also have the square root of two, maybe, as a side length… But hold on for a second… Okay, this is sneaky *smiles and sits upright*."

The extract illustrates that the restructuring in participant 28's work occurred when s/he suddenly realized s/he could draw tilted squares- the participant recognized that the squares did not need to have an area equal to square numbers. This was interpreted as an insight because the restructuring led to a more productive understanding of the problem as it made the participant able to discover areas between the square numbers, which was a necessary discovery for solving the problem. More specifically, the restructuring occurred in the goal component of the problem-solving process, as the participant found a different way of representing her/his answer to the problem. Furthermore, the insight appeared to be independent of the participant's work prior to it, as the new representation did not build upon her/his previous assumption of the areas needing to be square numbers. Neither was it affected by external stimuli. Additionally, the participant expressed surprise ("But hold on for a second…", "Sneaky!") and joy (*smiling*) when s/he realized that s/he could draw tilted squares. For these reasons, the insight was categorized as spontaneous. This sudden change of representation of the answer found place without any feeling of uncertainty, repetitions or stops prior to the insight. As the participant was not feeling

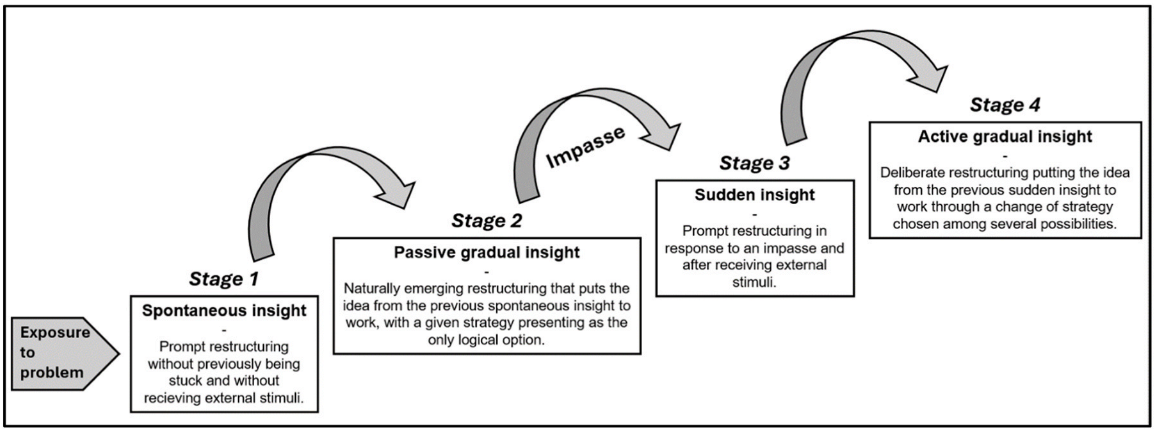


Fig. 1. The observed sequence of different types of insights during the two expert students' mathematical problem- solving process.

stuck prior to the insight, the absence of an impasse was indicated.

A similar example was found in participant 24's work on Lucky Fractions:

> "[…] Other cases [than the example] need to be two-digit numbers, and they must have one common digit… So, for example eleven at the bottom, does not work… Twelve at the bottom *writes down the fraction 12/24 and cancels the twos*, gives us the same… Here, it is also fourths and that is not correct… Thirteen has no similars… So, all above… Or, thirteen could be something above…*Brief silence, scratches her/his back and stares into space for some seconds*
>
> Oh, wait! Looking at numbers, for example thirteen, and multiply it by one, two, three and so on, the last number… Well, the last digit, will only appear as one and three one time each… Because, well… Three modulo ten will, the residual classes… It covers each one if we multiply it up to ten, then we get to zero."

The extract illustrates that the restructuring in participant 24's work occurred when s/he suddenly realized that for the given component (given: "They must have one common digit") to be true, the goal component wouldn't be consistent with investigating every single two-digit fraction. Instead, the last digit of multiples of the numerator must be equal to the first digit of the denominator – which is also a multiple of the numerator. This was interpreted as an insight because the new strategy was more productive than the more extensive search s/he began with, wherein s/he would have to check every two-digit fraction. As this insight led to a narrowing of what the possible answers could be, we interpreted this as a restructuring of the goal component. As such, the restructuring created a starting point for a new strategy for finding Lucky Fractions which would be much more productive for her/his further work. Furthermore, the insight appeared to be independent of the participant's work prior to it, as the representation of the goal component diverged from the one that s/he first started working with, during which s/he considered all two-digit numbers as possible answers. Additionally, the insight was not affected by external stimuli. Just like participant 28, participant 24 also expressed surprise ("Oh, wait!") when s/he realized s/he could proceed more effectively with the problem if s/he thought differently about the representation of the goal component. For these reasons, the insight was categorized as spontaneous. In addition, s/he did not express any feelings of being stuck, nor did s/he repeat her-/himself or pause her/his work, which indicated that there appeared no impasses in her/his work.

4.2. Passive gradual insight

This category of insight contains junctures that followed naturally from the spontaneous insight prior to it without reaching an impasse. In other words, the gradual insight following from the spontaneous one was not a leap of faith or a choice between options per se, rather it emerged as a logical consequence of the prior spontaneous insight and put the idea that emerged from it to work. As such, the process built directly and necessarily upon the work prior to it. Of course, we could not know for certain that other options weren't evaluated, but from what we could observe there wasn't any verbal exploration of other options or any pauses for thinking- the strategy chosen seemed obvious. A characteristic affect related to this type of insight was confidence. The confidence experienced seemed related to the preceding spontaneous insight, as the gradual insight was a direct consequence of that insight.

An example of this type of insight was found in participant 28's work, which built upon her/his spontaneous insight outlined in the previous section, starting out as follows:

> "*Starts drawing a tilted square on dot paper*. So, this is the square root of two. This is also an area *states this as s/he finishes the tilted square with side length 2* It has got… Two, then. Okay. Ehm… Hmm… Okay! *Smiles*."

As reflected in this extract, the participant restructured the operational component by changing her/his strategy, which, importantly, was accompanied by the feeling of confidence ("Okay!" *Smiles*). Fig. 2 illustrates how participant 28's work progressed further:

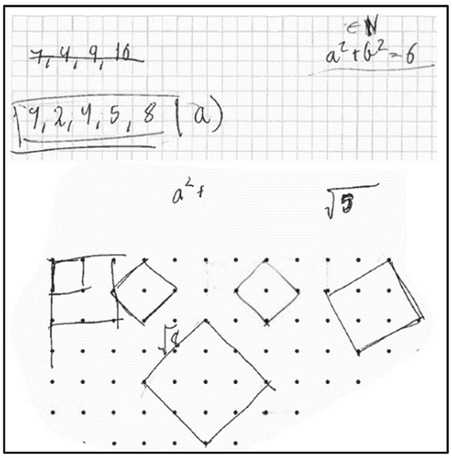


Fig. 2. Passive gradual insight (P28).

This juncture was interpreted as an insight because the restructuring in the operational component led the participant to a more productive strategy of finding the five smallest squares, as it made possible to find the five smallest squares in a 5×6 grid. More, specifically s/he started systematically drawing tilted squares where the baseline is the diagonal of rectangles with different combinations of length and width. Further, the insight was categorized as gradual because it built upon her/his prior work (the spontaneous insight) and was accompanied by a feeling of confidence (“Okay!”). The gradual insight was further categorized as passive because the new strategy was a direct and necessary consequence of the prior spontaneous insight in the goal component. As the participant did not repeat her-/himself, stop or express uncertainty prior to this insight, we interpreted her/him having no impasses.

We found a corresponding example in participant 24’s work, during which s/he based her/his work on her/his prior spontaneous insight, and started out as follows:

> “So *stops spinning her/his pen and writes down the numbers s/he needs to investigate*- the two end numbers we need to look at [for thirteen] is thirty-nine and ninety-one. *Applies this method for all two-digit numbers in the numerator up until 50*”

As illustrated in this brief extract, s/he confidently (“So- “, *stops spinning her/his pen*) concluded what s/he needed to do in her/his further work. Fig. 3 illustrates how s/he executes her/his strategy, which led her/him to discover both trivial (i.e. 10/20 etc., and 11/11 etc.) and special (49/98 etc.) Lucky Fractions.

This juncture was interpreted as an insight because the restructuring in the operational component led the participant to a more productive strategy of solving the problem, as evidenced by her/him effectively finding several Lucky Fractions after the restructuring. Furthermore, the insight was categorized as gradual because the execution of the new strategy built upon the preceding spontaneous insight, and because the participant’s language and targeted execution of the new strategy reflected a feeling of confidence. As the new strategy presented as an inevitable and natural consequence of the prior spontaneous insight, we categorized the gradual insight as passive- there were no other competing strategies which would be meaningful to investigate at this moment in her/his work. As the participant did not repeat her-/himself, stop or express uncertainty prior to this insight, we interpreted her/him having no impasses.

4.3. Sudden insight

We also found instances of sudden insights following an impasse. This category includes junctures during which the participants restructured to a more productive understanding of the problem, following from the feeling of being stuck (reaching an impasse), which in turn was indicated by repetition, stops and/or expressions of uncertainty of how to move forward with the problem. Additionally, the sudden restructuring was influenced by a form of external stimuli from the interviewer, e.g. a question. Importantly, these types of insights appeared promptly and detached from the participants’ work up until this point and presents as intuitive leaps of faith- they involved a sudden and deliberate decision to change strategies. Characteristic affect related to the impasse in this type of insight was the feeling of uncertainty. After the impasse, the participants seemed to have a feeling of determination- an inner drive, a focus or a desire to solve the problem.

An example of an insight in this category was found in participant 28’s work. In this segment, the participant was trying to identify squares that could not be drawn. A key feature was that although s/he was trying to identify the impossible squares, s/he was initally only able to repeatedly express this solution in the form of the squares that are possible to draw (marked in bold):

> “For a square to be possible to draw, it needs to have a side length given by… Eh… A component a *draws a horizontal vector a* and a component b *draws a vertical vector b*, which both only can be integers, so, eh, a and b have to be part of natural integers *writes down “a and $b \in N$” on her/his work sheet*. Eeeh, and that yields this length *stipples a diagonal line between vector a and vector b*. And by a and b the area can be expressed so that… Because **the area is just the square of the length of this line *points to the diagonal line***, which means that ***a* squared plus *b* squared give us the area *writes $a^2 + b^2 = A$*. Eh… Which means that…the… areas which cannot be drawn are the ones that cannot be produced by any value of [a and**

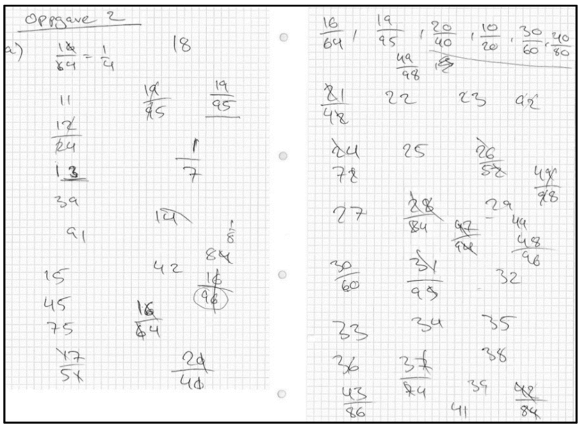


Fig. 3. Passive gradual insight (P24).

> **b] , wherein a or b can equal zero as well.** Yes, so my formula sums it up… […] Some of which cannot be drawn are 3, 6 and 7. 9 is of course possible, 10 is possible. 11 is not. It looks like… maybe I could find a pattern in that then? Eh… **It includes all numbers which two square numbers cannot sum up to.** […] **For a square to be possible to draw, it must have an area that equals the sum of two square numbers**. Eehm… Is there something more to say? *Looks at dot paper, brief silence* Mmm… Yes. ***Writes down "Either be a square number or be the sum of two square numbers"*** Ok, I think I am done now. *Draws a double line under her/his answer*. **That either it must be a square number, or it must be the sum of two square numbers**."

At this point, the participant seemed to think s/he had solved the problem with more time left on the clock. However, the original problem was which squares that could *not* be drawn. In alignment with the interviewer's role in the current study, described in Section 3.4, the interviewer strived to help the participant demonstrate her/his full knowledge. Therefore, the interviewer wanted to create an opportunity for the participant to discover that s/he had answered which squares *can* be drawn, instead of which could not be drawn. The interviewer achieved this by repeating P28's answer as a question: "Yes, so these are the ones you *can* draw then?" The emphasis on "can" was a deliberate way of reflecting to the participant what problem s/he had solved and what s/he had discovered and thus give her/him the opportunity to demonstrate her/his knowledge related to solving the original problem. The participant answered confirmatory to the interviewer's question, which seemed to make her/him uncertain about her/his solution to the problem. S/he continued her/his work by trying to restructure in the goal component and started searching for an explicit expression for the squares that could not be drawn:

> "Those are the ones that are possible to draw. *Rereads problem formulation* Which ones are not possible to draw? Okay, I can try to express that, then. The ones that are not possible to draw… are… Can I put it as simply as that *points to the expression $a^2 + b^2 = A$* plus 3…? *Brief silence, clicking her/his pen* No, maybe not, because 18…*mumbling* Eh… Not possible to draw is when they are not square numbers themselves, nor is the sum of two square numbers…Ehm…"

The participant seemed unsatisfied with her/his own implicit answer of which squares could not be drawn, as s/he started repeating her-/himself again, picking her/his lips, "mmmm"-ing and sitting in silence for a while, seemingly in deep thoughts. After a while, the interviewer asked her/him a question ("What are you thinking?"), to which s/he replied:

> "*Snaps out of her/his own thoughts* Oh, yes… That…Well, I'm just trying to think of… the possibility of expressing my answer without just saying that this *points to her/his written answer* is not satisfied, kind of… Ehm… Because the argumentation is straight forward. It is quite easy to express which ones that can be drawn, but I don't know how to express which… cannot… Ehm… *Leans backwards, pulls her/his fingers through her/his hair, and repeats her/his argumentation orally again and asks for a verification of her/his answer*"

From the two latest extracts it seems clear that the participant was uncertain of her/his solution, as s/he seemed to suspect that the answer could be expressed differently without knowing how to. Following from this, s/he proceeded by asking how the interviewer felt about her/his answer, which added to the interviewer's impression of the presence of an uncertainty. The interviewer responded by reminding the participant that s/he was free to solve the task in whichever way s/he wanted to, as well as asking whether the criteria for which squares could be drawn was her/his answer. The participant replied affirmatively. However, the participant still seemed uncertain and unsatisfied about her/his solution (goal component), as s/he chose to spend more time working on the problem by repeating her/his answer again, preceded by a decision to sit in silence and think for a while. Again, s/he repeated her/his argumentation. From these two latest extracts it is apparent that the participant wished to express her/his answer or solution more explicitly than s/he was capable of in the moment.

The repetitions of her/his answer for impossible squares, the checking of her/his answer with the interviewer and her/his body language, as well as verbally and explicitly expressing hesitation regarding her/his answer, gave an overall impression of uncertainty and a feeling of her/him being stuck. In other words, the participant seemed to have reached an impasse. However, s/he eventually decided to restructure the goal component to a more explicit expression of which squares that could not be drawn. As a consequence, s/

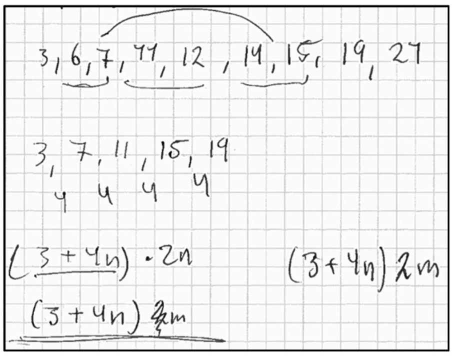


Fig. 4. Active gradual insight (P28).

he restructured in the operational component to a strategy of systematically listing tangible cases of impossible squares and looked for a pattern:

> "[…] But that all… Numbers that are not divisible in the set of N can of course not be drawn. Eeh… I think I will write down some [squares] that cannot be drawn and try to see if I can find a, eeh… System in that."

From this extract it is clear that the participant had reached an insight because s/he found a new, more productive way of solving the problem, which later (see Fig. 4) made her/him able to move forward towards solution. The extract also indicates that the insight was detached from the strategy being used prior to it, because instead of continuing to try to express which squares that was impossible via the ones that were possible, s/he restructured in the goal component and then in operational component by deciding to proceed with pattern investigation of tangible impossible squares. Furthermore, the participant appeared more determined and focused during this restructuring. Based on the influence of the interviewer's question, the newness of the strategy relative to her/his prior work and her/his newfound determination in relation to the restructuring, the insight appeared as a sudden one. As explained previously, prior to this insight s/he had reached an impasse. In summary, the overall episode outlined was categorized as sudden insight following from an impasse.

A similar example, which we will describe more succinctly, was found in participant 24's work. After applying her/his numerical search strategy up until 50 in the denominator, s/he said:

> "Ok, so, my answer for [all two-digit Lucky Fractions], if I haven't overlooked any obvious ones, is… So, I guess you just have to check every possibility. So, it shouldn't be [possible that I have overlooked any] …*Starts explaining her/his method again*"

This utterance was interpreted as an impasse by the interviewer, as the participants' need to say to her-/himself that "It shouldn't be possible that I have overlooked any" seemed more like a question that s/he was asking her-/himself than a statement. In addition, s/he seemed uncertain about her/his method, as s/he felt the need to repeat it for the interviewer. In total, the participant seemed uncertain as to if s/he had in fact overlooked any fractions and as to of her/his method had actually worked. Due to a flaw in her/his strategy, the participant didn't consider the goal component containing a digit different from one in the numerator, and because of this s/he had in fact overlooked a fraction which s/he also would not be able to find with her/his numerical strategy. As the participant's utterance was interpreted as a question by the interviewer and as the participant seemed probable to stop the interview at this point with more time left and not having completed the task, the interviewer answered that the participant had more time. This was in alignment with the interviewer's role in the current study, described in subsection 3.4, regarding helping the participant to demonstrate their full knowledge. After being told s/he had more time, the participant proceeded by discarding her/his numerical search strategy (see Fig. 2), and switched to an algebraic one instead:

> "Ok, if we look at the presentation in… We'll have a look at a and b *writes down $\overline{ab}$ on her/his work sheet*, as a number, then. This gives us ten a plus b… *Writes down $10a + b$ on her/his work sheet*. So, we will end up with… If we have ten c plus d also *writes down $10c + d$ on her/his work sheet*, as the second number… ab divided by cd *writes $\overline{ab}$ / $\overline{cd}$*, should be equal to… *Mumbles under her/his breath*. The options, a over d, a over c, b over c, b over d *writes these as fractions*."

The restructuring was interpreted as a sudden insight because the choice to change strategy (operational component) seemed influenced by the interviewer's intervention, and the strategy was both new and more productive. Furthermore, the participant seemed to gain a new sense of determination. As explained previously, prior to this insight s/he had reached an impasse. Therefore, the overall outlined episode was categorized as sudden insight following from an impasse. However, as time ran out, s/he was never able to find the last fraction.

4.4. Active gradual insight

We also found several instances of what we have called active gradual insight. Unlike the category of passive gradual insight, junctures in this category were categorized by work with a new strategy that was deliberately and actively chosen among several possible strategies, and which put the idea from the previous sudden insight to work. This type of insight was observed in relation to feelings of eagerness and optimism and was not directly related to an impasse.

An example from participant 28 illustrates this. In this example the participant builds on her/his previous sudden insight described in Section 4.3. Although other examples of active gradual insight were observed, we have decided to highlight this segment as it also illustrates how different insights can be related in a complex problem-solving process.

Here, s/he builds on her/his sudden insight described in Section 4.3, during which s/he decided to list some impossible squares. Her/his further work is shown in Fig. 4:

As illustrated in Fig. 4, the participant's listing of the squares that were not possible to draw seemed to make it possible for her/him to investigate potential patterns. S/he started out by observing that her/his list contained several pairs, which s/he underlined. Then s/he observed that there were doubles and decided to remove those from her/his list and make a revised and condensed list- 3–7-11–15-19. At this point s/he started talking faster (eagerness) as s/he observed that there was a jump of 4 units between the areas, which led her/him to the expression $3 + 4\,n$. Afterwards s/he tried out the expression on different tangible cases and tried adjusting it by adding a multiple of $2\,n$, removing it and adding it again. At this point, the time was up, and s/he expressed disappointment of not being able to finish, implying s/he was optimistic to solve the problem with more time at hand.

The juncture from when s/he made observations in her/his list to her/his final expression for impossible squares, was interpreted as a gradual insight, as s/he gained a better understanding of the pattern incrementally and consciously. We also decided to label this category as active, as opposed to passive, as there were no obvious strategies that would allow her/him to identify and determine the pattern. As such, the restructuring occurred in the operational component and did not follow directly from any impasse.

5. Discussion

The aim of the study presented in this paper was to investigate the source of insight and more specifically what types of insight high-performing mathematics students gained when working to solve ill-structured mathematical problems. Overall, we observed instances of both gradual insight and spontaneous insight. Although this is in line with much of the recent literature (e.g. Weisberg, 2015), we also noticed that both gradual and spontaneous insight occurred in the *same* problem-solving process. Meaning, the participants didn't gain insight either gradually or spontaneously, but instead through a complex relationship of interchangeable occurrences.

We also noticed some subtle differences within occurrences of gradual and spontaneous insight. The latter, spontaneous insight, can apparently occur *without* a preceding impasse – which undermines the claim that impasses are necessary for spontaneous insight to occur (Haavold & Sriraman, 2022; Weisberg, 2015). A central finding was also that the influence of external stimuli led to a differentiation between spontaneous insights (no external stimuli) and sudden insights (external stimuli), whereas much of the prior literature do not differentiate semantically between the two and use these terms interchangeably. Thus, the central observation of several insights during a problem-solving process was not restricted to interchangeable gradual and spontaneous insights, but also included what we termed sudden insight. As for the former, gradual insight, there seems to be a difference between gradual insight gained after a spontaneous insight, and gradual insight that is more of an active search through possible mathematical knowledge and heuristics. In the following sections, we will discuss these findings in more detail.

5.1. Insights as a sequence of gradual, spontaneous and sudden restructurings

Distinct for both participants were the interchangeable occurrences of spontaneous, sudden and gradual insights. Interestingly, the different types of insight occurred in similar chronological sequences for both participants. We have decided to label these insight occurrences as stages, as we will in the following section argue that they are to some extent linked.

In the first stage, the insight for both participant 28 and participant 24 was spontaneous without impasse and occurred in the goal component. The spontaneous insight was for both participants related to feelings of surprise and joy. A possible explanation for this being the first insight might be that during the first phase of problem solving, the problem solver works to understand the problems' givens and goal (Ohlsson, 2011; Pólya, 1949; Robertson, 2017b) and if the problem solver experiences that their understanding and its related method doesn't work, this experience will result in negative feedback, inhibiting the original interpretation of the problem (Weisberg, 2015). In turn, this facilitate a new interpretation, resulting in a restructuring of the problem (Weisberg, 2015). This new understanding is in stage 1 the spontaneous insight without impasse. However, much of the literature considers, explicitly or implicitly, impasse as a necessary condition for spontaneous insight to occur (Vallée-Tourangeau, 2018), which is divergent from our results that demonstrate spontaneous insight occurring in the absence of an impasse. Also Ohlsson (1992) consider impasse as a precondition of insight. Thus, our finding illustrates that the much-argued link between spontaneous insight and impasse is not always true.

In the second stage both participants reached what we called passive gradual insight, meaning that the insight was a natural and seemingly inevitable consequence of the prior spontaneous insight. This gradual insight occurred in the operational component, involved a change of strategy for moving from givens to the goal, building upon the insight in stage 1, and was associated with a feeling of confidence. This insight seems straightforward to explain- a new interpretation will necessarily result in a different strategy that builds on this new understanding in one or more of the components of problem solving. Implementing the new strategy then in turn facilitates a further improvement of understanding – or gradual insight.

There may be several explanations as to why we found this connection between a spontaneous insight in stage 1 and a gradual insight in stage 2. We interpreted the passive gradual insight as a consequence and not as part of the spontaneous insight without impasse, which was the reason for categorizing these as two distinct insights. More specifically, the restructuring of the goal component in stage 1 came about promptly and with the feelings of surprise and joy, whereas the restructuring in the operational component in stage 2 was incremental work building on stage 1 and was associated with the feeling of confidence. However, prior research might have interpreted this sequence of restructuring in stage 1 (the spontaneous insight without impasse) and then in stage 2 (the passive gradual insight) as only one insight consisting of two restructurings, as they on the surface appear to be the same occurrence because they result in a single action. Thus, this may explain why prior research have failed to notice the connection between spontaneous and gradual insight and rather have conceived them as distinct processes (Fleck & Weisberg, 2013; Weisberg, 2015).

The third stage for both participants involved a sudden insight after encountering an impasse, meaning that they felt stuck before suddenly performing a restructuring after receiving external stimuli. The restructuring occurred in the goal component, followed by a decision of trying out a new strategy (operational component). There are two important differences that distinguishes this sudden insight from the spontaneous insight in stage 1. Firstly, the sudden insight in stage 3 was a deliberate choice to restructure associated with the feeling of determinism to try something else after feeling uncertain and reaching an impasse, whereas the insight in stage 1 seemed non-deliberate and did not involve an impasse. Secondly, the sudden insight of stage 3 was affected by external stimuli, whereas this was absent in the spontaneous insight in stage 1. Another interesting observation was that P28 had more success of solving the problem after her/his sudden insight than P24 had. A possible explanation for this was that P28 performed two related restructurings- one in the goal component, followed by one in the operational component. We interpreted these as one single sudden insight consisting of two parts, because the first restructuring was a sudden redefinition of how the answer could be expressed related to the feeling of determination, and the second restructuring was a sudden decision to make a change in the operational component, also related to the feeling of determination.

The fourth stage contained an insight that we called active gradual insight, as this restructuring was a conscious, behavioural effort of changing strategies based on the sudden insight in stage 3. Although this insight followed from the insight in stage 3, it appeared as a choice among options. More specifically, several decisions regarding choice of strategy could have followed from the restructuring in the goal component in stage 3. A relevant question in this context is where the boundary should be drawn between sudden and gradual insight. The reason why we chose to make a distinction between the decision to change strategies (restructuring in operational component in stage 3) and the actual *execution* of it (restructuring in stage 4), was twofold- (1) the decision to restructure appeared to be sudden, whereas the execution did not, and (2) the restructuring in the operational component in stage 3 was followed by a feeling of determination to resolve uncertainty, whereas the restructuring in stage 4 was followed by the feelings of eagerness and optimism. Thus, the distinction between the two stages is in lines with our delineation of insight not only involving specific actions, but also its relatedness to different affects.

5.2. *The insight sequence as emotional transitions*

An observation in our findings which helped us differentiate between the different types of insights, was the presence and role of affect. As described previously in Section 5.1, we observed distinct emotions for each of the four types of insights. Spontaneous insight (stage 1) was related to the feelings of joy and surprise, which according to Linnenbrink (2007) and Pekrun (2006) are *positive activating emotions*. These were observed to empower the participants' work with the problems, as they helped them progress in their work. This is in alignment with DeBellis and Goldin (2006), who describe a similar relation between empowerment and actions such as constructing a new plan. Surprise, which can have both positive and negative influence depending on the situation (Mauss & Robinson, 2009; Muis et al., 2018), was viewed as a positive in emotion in the situation of spontaneous insight, as it was associated with progress in the participants' work. In stage 2, in which the participants reached a passive gradual insight, the feeling of confidence was observed. We related this feeling to *positive activating emotions* (Linnenbrink, 2007; Pekrun, 2006), as the participants in this stage worked gradually and made progress in their work on the problem.

A precursor to the sudden insight in stage 3, was the feeling of uncertainty. We relate this feeling to what Linnenbrink (2007) and Pekrun (2006) identify as *negative activating emotions*. We did not view the uncertainty as deactivating, which in a different problem-solving process might have been the case, as the participants appeared almost inspired by this feeling. This was apparent in that they suddenly restructured in response to the feeling, and in that this restructuring was associated with the feeling of determination, which we identified as a *positive activating emotion*. In the fourth stage the participants of our study reached an active gradual insight which related to the feelings of eagerness and optimism, similar to feelings such as pride, hope, enjoyment and curiosity which by Linnenbrink (2007) and Pekrun (2006) are categorized as *positive activating emotions*. The fact that both stage 3 and 4 might be categorized as containing positive activating emotions makes sense in reference to the link we in Section 5.1 described between stage 3 and 4, in which the active gradual insight is a continuation of stage 3, yet a distinct stage.

Thus, there appeared to be a sequence of emotions that can be summarized as *positive activating (stage 1)→ positive activating (stage 2)→ negative activating (stage 3)→ positive activating (stage 3)→ positive activating (stage 4)*. This observation is similar to what Di Leo et al. (2019) found in their emotion-to-emotion transition analyses. More specifically, they found that negative emotions transitioned to positive ones when confusion was resolved. Overall, we argue that parallel to and in reflection of the observed sequence of insights, there existed an affective sequence in which positive activating emotions drove the process up until an impasse, in which the negative activating emotions were experienced. These were finally transitioned to positive activating emotions again as the problem solvers were able to reach a sudden insight, followed by active, gradual insight and a continuation of positive activating emotions.

5.3. *Possible explanations for the sequence of insights*

As apparent from the prior paragraphs, a central aspect of our findings was the observation of several sequential insights. This finding touches on the question of what makes an insight problem and how insight problems are solved – a question that has been and still is being debated (Chronicle et al., 2004). In psychology, insights are traditionally studied with «insight problems", designed such that once you restructure, you easily see how to solve the problem. In other words, the task does not allow for several restructurings to occur. These kinds of insight problems stems from the Gestaltists' particular interest for perceptual restructuring (Webb et al., 2016). Research in the field of psychology has also tended to focus on cognitive restructuring, defined as a sudden change in the way a problem is perceived, which leads to a direct solution of the problem (Webb et al., 2016). A common trait for insight problems seems to be what DeYoung et al. (2008) terms "breaking frame", which involves overcoming cognitive fixation through the ability to recognize anomalous stimuli. Thus, a possible interpretation of our finding is that several restructurings, or insights, were necessary as the tasks were more complex than the classic insight problems, requiring a recognition of more than one anomalous stimulus.

5.4. *Possible explanations for different impacts of the different types of insight*

Also among our findings was the tendency of spontaneous and sudden insight occurring first in the goal component and gradual insights occurring first in the operational component. Related to this tendency, was the impression of the spontaneous and sudden insights as more influential than the gradual ones, as they were related to a new way of thinking about the problem, which in turn was what inspired the gradual changes of strategies. One possible explanation for this tendency might be related to the distinction between cognitive conflicts and restructuring. A cognitive conflict is defined as incongruity between information at hand and a mental representation and must not be confused with response conflict during which a decision of a response must be made. In cases of the cognitive conflict not being resolved, the problem solver reaches an impasse. To break free from the impasse will require a new representation of the problem, which in turn enables the detection of new solution strategies (Danek & Flanagin, 2019). With this in mind, it may be possible that the participants experienced a cognitive conflict prior to sudden insights with an impasse and that the recognition of the existence of this conflict was what presented as the sudden insight. This cognitive conflict was needed to trigger the preceding active gradual insight that presented more as a response conflict during which the problem solver had to choose a new strategy among several possible options.

As for the spontaneous insights without an impasse, there were no observed long-lasting cognitive conflicts, which earlier have been explained by too little time to yet have established a fixated idea about the problem. However, as it is clear that the problem solvers' representation changed in spontaneous insights without an impasse, there was per definition an indication of some form of cognitive conflict having found place. A possible explanation for the perception of the larger profoundness of the sudden insight related to an impasse relative to the spontaneous ones without, might be the size and duration of the cognitive conflict- the lager or more lasting conflict, the more profound effect it may have on the following restructurings (see also Ohlsson, 2011).

Similar to the active gradual insight, the passive gradual insight may be related to a response conflict. However, as the cognitive conflict of the prior spontaneous insight in the absence of an impasse appeared less severe than in the sudden insight with an impasse, the response conflict may by the same token have been less profound in passive gradual insights than in active gradual insight. This in turn my help explain our perception of the passiveness of these gradual insights as these were defined as passive because the choice of strategy seemed so obvious. In the active gradual insights, the choice was less obvious and presented as more of an actual response conflict, which was the reason for our perception of the activeness of these gradual insights. Thus, it may seem like the size of the cognitive conflict and the size of the response conflict could be determining of how profound the insights have the potential to be. This is in turn in accordance with our impression of a higher acceleration towards solution related to the sudden insights with impasse compared to spontaneous insight without impasse. Further, it supports our perception of higher acceleration for spontaneous and sudden insight in general compared to the gradual insights, as the cognitive conflicts were what inspired the response conflicts- no cognitive conflict, no response conflict.

6. Strengths and limitations

There are at least two strengths of our study. Firstly, we studied diverse source of insight in expert students' mathematical problem solving, in contrast to prior studies that have (a) studied this in the work of professional mathematicians (e.g. Savic, 2015; Sriraman, 2008), or (b) merely focused on the occurrence of insights and not its source (e.g. Haavold & Sriraman, 2022; Liljedahl, 2005). Secondly, we used complex ill-structured problems rather than the classical insight problems, priorly used in studies in the field of mathematics education (Leikin et al., 2016), which have been criticized of limiting the opportunity to study nuances of complex problem solving (Robertson, 2017a). The design of our study led us to interesting findings in the study of insight, as it allowed for more complex problem solving to occur.

Furthermore, the study was based on a qualitative definition of both insight and impasse. This has its strengths and limitations. A limitation that follows from such definitions can be that other researcher might analyse the data differently and thus reach different conclusions. However, to accommodate to this limitation, we strived to describe what we observed happening, without drawing too rigid conclusions. A strength of using qualitative definitions and analyse the episodes our self, is that we avoided biases related to a self-report design of insight, which priorly have been used within the field of mathematics education (e.g. Liljedahl, 2005). In addition, our definitions did not limit us to only state whether a problem solver can overcome an impasse or not, as prior studies have already found (e.g. Munzar et al., 2021), but also to study the role of impasse, because "time" was not a criteria in our definition of impasse and thus time was not a limiting factor.

Two additional limitations are (1) the limited data set of two participants, and (2) the time constraint. Acknowledging the first limitation, we nevertheless view our study as one that has added to the understanding of insight in mathematical problem solving and as such might serve as a departure for future research on the subject. The second limitation may have caused the interviewer to intervene earlier than if there was more time, and there is no way of knowing how the process would have unfolded without such interventions and whether some information was lost due to them. However, the interviewer did restrict her/his intervention in accordance with the four functions described in subsection 3.4, which was based on Hunting's (1997) and Goldin's (1997) suggestions of the interviewer's role during task-based interviews.

7. Implications and future research

In summary, full insight to a problem appears to involve several restructurings- spontaneous, sudden and gradual. During this sequence, both negative and positive affects appear important for successful problem solving. As such, it appears that the insight sequence is parallel to emotional transitions during problem solving.

Without having compared our high-performing participants to other students, we hypothesis that the expertise of the high-performing students may in part be due to their interpretation of uncertainty as a challenge or inspiration to try something new, rather than as a sign of failure. This might explain why uncertainty functioned as an activating rather than a deactivating emotion. On the contrary, if negative emotions are interpreted as a sign of doing something wrong rather than as a natural part of learning, students may give up on trying to solve a problem. Although it may seem counter- intuitive, struggling is a productive part of learning which can lead to enhanced understanding (Biccard, 2024). Therefore, in order to gain insight to the solution of a problem, it seems essential that students have knowledge about productive struggle as an important part of learning (Chen, 2022).

An implication for teaching mathematical problem solving is thus the importance of supporting teachers to create opportunities for productive struggles (Bolyard et al., 2024). This may be accomplished through the use of complex, ill-structured problems, which allows for several interpretations and solution methods. As a consequence, it is plausible that students will experience full insight sequences. According to Harel (2008), DNR- based instruction rely on intellectual needs as drivers of students' learning. These needs are inextricably linked to problem solving, and learning in DNR is thought to be driven by exposure to problems that lead to some form of uncertainty, which in the Piagetian sense is thought of as disequilibrium (Harel, 2013). Further, Harel (2013) points out that what stimulate the intellectual needs depends on the learner. Thus, it seems pivotal that students engage in mathematical activities that are real to them, relating intellectual needs and DNR-based instruction to what is known as Realistic Mathematics Education (RME) (see e. g. Artigue & Blomhøj, 2013).

A further implication of our study regards the role of impasse. As previously pointed out, spontaneous insight is possible without a preceding impasse, according to our study. This is in contrast to much of the earlier literature on the subject. Thus, it would be interesting to investigate whether this finding is replicable in future studies of insight. Our choice to apply ill-structured problems instead of the classical insight problems may inspire a different way of studying insight in such future research. Finally, it would be interesting for future research to investigate the sequence of insights and related affects observed in this study in a larger data set, as well as in a broader spectrum of problem solvers.

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CRediT authorship contribution statement

Eirin Stenberg: Writing – review & editing, Writing – original draft, Visualization, Resources, Project administration, Methodology, Investigation, Formal analysis, Data curation. **Per Haavold:** Writing – review & editing, Supervision, Resources, Methodology. **Bharath Sriraman:** Validation, Supervision.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix

Elaboration of solution to "The Lucky Fractions Problem"

Express the example algebraically:

$$(10a + b)/\ (10b + c) = a/c$$

Solve the expression using algebra:

$$(10a + b)\ c = a\ (10b + c)$$

which simplifies to:

$$b\ (10a - c) = 9ac$$

There are now three cases: $(9, b) = 1$, $(9, b) = 9$, and $(9, b) = 3$ that can be explored further to discover the "lucky fractions".

Case 1: $(9, b) = 1$

$$9|(10a - c) \Rightarrow 9|(a - c) \Rightarrow a = c$$

Which means

$$9a^2 = 9ab \Rightarrow a = b = c$$

Which gives the trivial solutions to the problem.

Case 2: $(9, b) = 9$

Which means b = 9.

$$10a - c = ac \Rightarrow (a + 1)\ (10 - c) = 10$$

Which means that we must have either $(a + 1) = 10$, or one of $(a + 1)$ and $(10 - c)$ equal to 2 and the other equal to 5. This gives us $a = 9$ and $c = 9$, or $a = 1$ and $c = 5$, or $a = 4$ and $c = 8$.

Case 3: $(9, b) = 3$

We can assume that $b = 3$ or $b = 6$, as $b = 9$ was covered by case 2.

If $b = 3$, then

$$3(10a - c) = 9ac \Rightarrow 10a - c = 3ac \Rightarrow (3a + 1)(10 - 3c) = 10$$

Which means $a = c = 3$

If $b = 6$, then simplifying and rearranging gives

$$(3a + 2)(20 - 3c) = 40$$

The factors of 40 of the form $3a + 2$ for $1 \leq a \leq 9$ are 5, 8 and 20. This leads to $a = 1$ and $c = 4$, $a = 2$ and $c = 5$, and $a = 6$ and $c = 6$.

Data Availability

The authors do not have permission to share data.

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