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Local classification of 2 dimensional solvable Lie algebra actions on the plane

Abstract

In the thesis the local classification of 2-dimensional solvable Lie algebra action on the plane is given. Normal forms of such actions are found. The classification applied to classification of 2^{nd} order differential equations that are solvable in quadratures.

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Contents

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Chapter 1

Introduction

In this thesis we investigate equivalence classes of 2 dimensional solvable Lie Algebras acting on the plane.

Definition 1. *Let* $\mathfrak{g}$ *be a solvable Lie algebra over* $\mathbb{R}$, $\dim \mathfrak{g} = 2$ *and* $[\mathfrak{g}, \mathfrak{g}] \neq 0$. *Let* $\rho : \mathfrak{g} \to \mathcal{D}(\mathbb{R}^2)$ *be a representation of this algebra into the Lie algebra of vector fields on* $\mathbb{R}^2$ *such that* $\ker \mathfrak{g} = 0$. *We say that two representations* ρ_1, ρ_2 *are* locally equivalent *at a point* a*, if there exist a local diffeomorphism* $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ *where* $\phi(a) = a$*, such that* $\rho_2 = \phi_* \circ \rho_1$.
Representatives of equivalences classes are called normal forms

Sophus Lie in [5] described finite dimensional Lie algebras acting on the plane. From this classification one can extract (See below Theorem 3) the normal form of the locally transitive action of the 2-dimensional solvable Lie algebra on the plane. Singular, or non-transitive actions of Lie algebras(and Lie groups) is still "Terra incognita". Mainly, they concern to actions of Lie algebras and groups with fixed point. Thus, for actions of compact Lie groups, we have the following result

Theorem 1 (E.Cartan)**.** *Let* G *be a compact Lie group, which acts on the* *manifold M in such a way that $g(a) = a$ for all $g \in G$. Then there exists local coordinates such that the action will be linear.*

R. Hermann [2] proved that actions of semi-simple Lie algebras in a neighborhood of a fixed point can be linearized on the formal level. Later S.Sternberg and V. Guillem [7] proved that this result can not be extended to the analytical case.
For non semi-simple Lie algebras there is the classical S. Sternberg theorem on linearization [6]:

Definition 2. *Let λ_i be the eigenvalues of the linear part of X_i at zero, i.e. the matrix $||\frac{\partial X_i}{\partial x_j}(0)||$. We say our system has resonance ,if there exist an eigenvalue λ_i such that:*

$$\lambda_i = \sum_j m_{ij}\lambda_j$$

Where m_{ij} are non-negative integers and $\sum_j m_{ij} \geq 2$.

Theorem 2 (Sternberg)**.** *Let X be a vector field, of the form:*

$$X = \sum_i X_i(x_1, \ldots, x_i)\partial_{x_i}.$$

Where $X_i(0) = 0$ for all i. If the vector field X has no resonances then there are local coordinates in which X has the linear form

$$X = \sum_i \lambda_i x_i \partial_{x_i}$$

For several Lie algebras V. Lychagin proposed in [3], [4] some spectral sequences which give formal classifiaction, and formal normal forms in a neighborhood of a singular orbit.

In this thesis we analyze in details the case of 2 dimensional non-abelian solvable algebras.
We pick a X, Y in the Lie algebra such that

$$[X, Y] = X$$
$$\mathfrak{g} = \langle X, Y \rangle$$

Let $\mathcal{O}(a)$ denote the $\mathfrak{g}$-orbit of the point $a \in \mathbb{R}^2$, and by $\mathcal{O}^{(1)}(a)$ denote the orbit of the derived subalgebra $\mathfrak{g}^{(1)}$. We split our consideration into three cases:

1. $\dim \mathcal{O}(a) = 2$.

   This is the classical case where the action is transitive.

2. $\dim \mathcal{O}(a) = 1$.

   In this case we will call the action weak singular.

3. $\dim \mathcal{O}(a) = 0$.

   In this case we will call the action singular.

Throughout this thesis we pick coordinates in such a way that the point under consideration is at the origin.
Chapters 2,3, and 4 contain detailed description of the normal forms of $\mathfrak{g}$-actions. In the last chapter 5 we find differential invariant algebras for these actions, which then apply to find ordinary differential equations solveable by quadratures.

Chapter 2

Transitive Action

Let $\mathfrak{g}$ be a 2-dimensional non-abelian Lie algebra over $\mathbb{R}$, and let X, Y be a basis in $\mathfrak{g}$ such that

$$[X, Y] = X$$

We will use the same notations X, Y for the images $\rho(X)$ and $\rho(Y)$.
Assume that the action is transitive at the point $a \in \mathbb{R}^2$, or in other words, assume that the vectors $X_0, Y_0 \in T_0\mathbb{R}^2$ are linear independent.
The following result is due to Sophus Lie [5]

Theorem 3 (Sophus Lie)**.** *Let the solvable non-abelian Lie algebra* $\mathfrak{g}$*,* $\dim \mathfrak{g} =$ 2 *, act transitively in a neighborhood of* $O \subset \mathbb{R}^2$*. Then there are local coordinates* (x, y) *such that*

$$X = \partial_x,$$
$$Y = x\partial_x + \partial_y.$$

Proof. First choose coordinates such $X = \partial_x$. Let

$$Y = \alpha(x, y)\partial_x + \beta(x, y)\partial_y.$$

In these coordinates, the commutator relation gives

$$\alpha_x\partial_x + \beta_x\partial_y = \partial_x.$$

Therefore there is a local diffeomorphism $\Theta : (x, y) \to (x, f(y))$ such that

$$\Theta_*(Y) = x\partial_x + \partial_y$$

$\square$

Chapter 3

Weak singular action

In this chapter we investigate the case when $\dim \mathcal{O}(a) = 1$. Let $\mathcal{O}^{(1)}(a)$ denote the $\mathfrak{g}^{(1)}$-orbit of the point a. We split our classification into two cases:

1. The orbit of the derived subalgebra is singular i.e. $\mathcal{O}^{(1)}(a) = 1$,

2. The orbit of the derived subalgebra is a curve i.e. $\mathcal{O}^{(1)}(a) = 0$.

Since $\dim \mathcal{O}(a) = 1$, on of the vectors $X_a, Y_a \in T_a\mathbb{R}^2$ is nonzero. Therefore we can choose coordinates (x, y) in a neighborhood of $a \in \mathbb{R}^2$, such that the corresponding vector field equals to ∂_x.
We need the following lemma which describes the local behavior of vector fields on the line $\mathbb{R}$.

Lemma 1. *Let $X = b(x)\partial_x$ be a vector field on $\mathbb{R}$, such that $X(0) = 0$. Then there exists a local diffeomorphism $\phi : \mathbb{R} \to \mathbb{R}$, $\phi(0) = 0$, such that $\phi_*(X)$ has one of the following forms*

- $\lambda x \partial_x$,
  if $b(x)$ has a zero of order 1 at 0.

- $x^k\partial_x$ *if* $b(x)$,

  has a zero of order k *at 0 where* k *is even.*

- $\pm x^k\partial_x$

  if $b(x)$, *has a zero of orderk at 0 where* k *is odd.*

- $b(x)\partial_x$,

  if $b(x)$, *is a flat at 0.*

Proof. See for example, [1]. □

3.1 Non singular orbit of the derived subalgebra

In this section we assume, that $\dim \mathcal{O}^{(1)}(a) = 1$, or that $X_a \neq 0$. Then we can choose local coordinates (x, y) in such a way that $X = \partial_x$, in a neighborhood of the point $a \in \mathbb{R}^2$. Then

$$Y = \alpha\partial_x + \beta\partial_y,$$

in these coordinates, and the commutator relation $[X, Y] = X$ gives the system of differential equations on the functions α and β:

$$\alpha_x = 1 \quad \beta_x = 0.$$

Therefore, we can assume that in these coordinates:

$$\alpha = x \quad \beta_x = b(y).$$

Note that transformations of the form

$$(x, y) \to (x, Y(y)),$$

do not change the form of X, and in Y they act on the vector field $b(y)\partial_y$. Therefore, applying lemma 1 we get the theorem:

Theorem 4. *Let $\mathfrak{g}$ act in such a way, that*$\dim \mathcal{O}(a) = 1 \dim \mathcal{O}^{(1)}(a) = 1$*. then there are local coordinates (x, y) at a neighborhood of the point $a \in \mathbb{R}^2$, such that*

$$X = \partial_x.$$

And the vector field Y has one of the following forms:

$$\begin{aligned} &1. \quad x\partial_x + \lambda y \partial_y, \\ &2. \quad x\partial_x + y^p \partial_y, \\ &3. \quad x\partial_x \pm y^q \partial_y, \\ &4. \quad x\partial_x + b(y)\partial_y, \end{aligned}$$

where p, q are natural numbers, $p \geq 2, q \geq 3$, and $b(y)$ is a flat function at the point 0.

3.2 Singular orbit of the derived subalgebra

Consider now the case, when $\dim \mathcal{O}(a) = 1$, and $\dim \mathcal{O}^{(1)}(a) = 0$ i.e. the case when $Y_a \neq 0$, but $X_a = 0$.

Then there are local coordinates (x, y) such that $Y = \partial_x$. The commutator relation $[X, Y] = X$ can be rewritten for the functions α, β, when

$$X = \alpha(x, y)\partial_x + \beta(x, y)\partial_y,$$

as the following system of differential equations:

$$\alpha_x = -\alpha \quad \beta_x = -\beta.$$

Solving these equations we get

$$X = e^{-x}\left(\alpha(y)\partial_x + \beta(y)\partial_y\right)$$

Where $\alpha(0) = \beta(0) = 0$.
Again we apply lemma 1 on the vector field $\beta(y)$ and arrive at the theorem:

Theorem 5. *Let* $\dim \mathcal{O}(a) = 1$ *and* $\dim \mathcal{O}^{(1)}(a) = 0$. *Then there are local coordinates* (x, y) *in a neighborhood of the point* $a \in \mathbb{R}^2$, *such that*

$$Y = \partial_x,$$

and the vector field X *has one of the following forms:*

1. $e^{-x}(\alpha(y)\partial_x + \lambda y\partial_y)$,
2. $e^{-x}(\alpha(y)\partial_x + y^p\partial_y)$,
3. $e^{-x}(\alpha(y)\partial_x \pm y^q\partial_y)$,
4. $e^{-x}(\alpha(y)\partial_x + \beta(y)\partial_y)$.

Where $\alpha(y)$ *is an arbitrary function,*$\alpha(0) = 0$, $\lambda \neq 0$ p, q *are natural numbers* $p \geq 2, q \geq 3$ *and* β *is a flat function at 0.*

Chapter 4

Singular action

In this chapter we investigate the case when $\mathcal{O}(a) = a$, or when $X_a = Y_a = 0$. The general procedure divided on the three steps:

1. Find restrictions on the linear term of a representation from the commutator relation $[X, Y] = X$.

2. The commutator relation gives us a differential equation on the coefficients of the vector fields X, Y, which we investigate formally, under the condition that the vector field Y has no resonances.

3. Investigate when the formal solution can be extended to a smooth solution.

Let A, B be the linear parts of X, Y at the point a respectively i.e. $A = [X]_a^1, B = [Y]_a^1$ the 1-st jets of ad$X$ and ad$Y$ at a.
The commutator relations $[X, Y] = X$ of the vector fields, gives the following commutation relation of opertors $[A, B] = A$.
We assume that $B \neq 0$ and split our investigation into the two cases:

1. The representation of X has a non-vanishing first jet at a i.e.$A \neq 0$.

2. The representation of X has vanishing first jet at a i.e. $A = 0$.

4.1 The vector field X has a non-vanishing first jet

We need the following version of the Lie Theorem on representations of solvable Lie algebras.

Proposition 1. *Let $A = [X]_a^1 \neq 0$, $B = [Y]_a^1 \neq 0$. Then there is a basis of $T_a\mathbb{R}^2$, such that*

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \lambda - 1 & 0 \\ 0 & \lambda \end{bmatrix}.$$

Where $\lambda \in \mathbb{R}$

Proof. Since B is a real operator, we have three possibilities for B:

1. Eigenvectors of the operator B form a basis of $T_a\mathbb{R}^2$.
2. The operator B has complex eigenvalues.
3. The operator B has one real eigenvalue.

Consider the first case.
Choose a basis e_1, e_2 that are eigenvectors for the operator B i.e.

$$Be_1 = \lambda_1 e_1 \quad Be_2 = \lambda_2 e_2.$$

The commutator relation $[A, B] = A$ acting on e_1 gives us:

$$[A, B]e_1 = \lambda_1 Ae_1 - BAe_1 = Ae_1,$$
$$BAe_1 = A(\lambda_1 - 1)e_1.$$

For e_2 we have:

$$[A,B]e_2 = \lambda_2 Ae_2 - BAe_2 = Ae_2,$$
$$BAe_2 = A(\lambda_2 - 1)e_2.$$

Therefore, if Ae_1 and Ae_2 are non-zero vectors, they are eigenvectors for the operator B, with eigenvalues $(\lambda_1 - 1)$ and $(\lambda_2 - 1)$ respectively.
This implies that $\lambda_1 = \lambda_2 - 1$ and $\lambda_2 = \lambda_1 - 1$. This is a contradiction and the condition that $A \neq 0$, show that either $Ae_1 = 0$, or $Ae_2 = 0$.

Let us say that $Ae_1 = 0$. The commutator relation shows that $tr(A) = 0$. Therefore $Ae_2 = e_1$ and

$$\lambda_1 - \lambda_2 = 1.$$

Consider the second case.

Then the complexification $B^{\mathbb{C}}$ of the operator B has the eigenvector basis $e, \overline{e} \in T_a^{\mathbb{C}\mathbb{R}^2}$. Then we have that

$$B^{\mathbb{C}}(e) = \lambda e \quad B^{\mathbb{C}}(\overline{e}) = \overline{\lambda}\overline{e}.$$

Then, similarly to the case above, we get that $Ae = \overline{e}$ and $A\overline{e} = 0$, but $A\overline{e} = \overline{Ae} = e$. This contradiction shows that the eigenvalues of B are real.

Finally, Assume that the operator B has a Jordan matrix and let B act on e_1 and e_2 in the following way:

$$Be_1 = \lambda e_1, \quad Be_2 = \lambda e_2 + e_1.$$

The commutator relation $[A, B] = A$ acting on e_1 gives that

$$\begin{aligned} BAe_1 &= ABe_1 - Ae_1, \\ &= (\lambda - 1)Ae_1, \end{aligned}$$

and

$$\begin{aligned} BAe_2 &= ABe_2 - Ae_2 = A(\lambda e_2 + e_1) - Ae_2, \\ &= (\lambda - 1)Ae_2. \end{aligned}$$

But $\lambda - 1 \neq \lambda$ and one of the vectors Ae_1 or Ae_2 is nonzero. This contradiction proves the proposition $\square$

Remark. *Operator B has two eigenvalues λ_1 and λ_2, where $\lambda_1 - \lambda_2 = \pm 1$. In the previous proposition we let λ denote the eigenvalue of the vector which does not belong to* $\ker A$

From this we get the corollary:

Corrolary 1. *There are local coordinates (x, y) in the neighborhood of the point $a \in \mathbb{R}^{\nvDash}$, in which the vector fields X and Y have the following form:*

$$\begin{aligned} X &= x\partial_y + \alpha(x, y)\partial_x + \beta(x, y)\partial_y \\ Y &= (\lambda - 1)x\partial_x + \lambda y\partial_y + \tilde{\alpha}(x, y)\partial_x + \tilde{\beta}(x, y)\partial_y, \end{aligned}$$

where functions $\alpha, \tilde{\alpha}, \beta, \tilde{\beta}$ have zeros of second order at 0 i.e.

$$\alpha(0) = \tilde{\alpha}(0) = \beta(0) = \tilde{\beta}(0) = 0,$$

and

$$d_0\alpha(0) = d_0\tilde{\alpha}(0) = d_0\beta(0) = d_0\tilde{\beta}(0) = 0,$$

4.1.1 Resonance conditions

In this section, we investigate the resonance conditions for the vector field Y, in order to apply the S.Sternber linearization theorem.

Theorem 6. *Let $\lambda - 1$ and λ be the eigenvalues of the linear part of the operator $B = [Y]_a^1$, and let*

$$\lambda \notin \mathbb{Q}[0,1] \cup \{1 + \frac{1}{n} | n \in \mathbb{N}\} \cup \{-\frac{1}{q} | q \in \mathbb{N}\}$$

Then there are local coordinates (x, y) in a neighborhood of the point $a \in \mathbb{R}^2$, such that

$$Y = (\lambda - 1)x\partial_x + \lambda y \partial_y$$

Proof. The result will follow form the Sternber linearization theorem. We show that the conditions on λ are exactly conditions under which the operator B has no resonances. Thus, we analyse the resonance conditions for B.
Assume that

$$m_1\lambda + m_2(\lambda - 1) = \lambda,$$

Where $m_1, m_2 \in \mathbb{Z}_+$ and $m_1 + m_2 \geq 2$.
From this equation we get that

$$\lambda = \frac{m_2}{m_1 + m_2 - 1}.$$

Therefore, λ should be a ration number. Let us put $\lambda = \frac{p}{q}$, where $p, q \in \mathbb{Z}$ are coprime, and $q > 0$.

Then,

$$
\begin{aligned}
m_2 &= kp,\\
m_1 + m_2 - 1 &= kq,
\end{aligned}
$$

for some $k \in \mathbb{Z}$.

Therefore the resonance conditions on λ are equivalent to the inequalities:

$$kp \geq 0, \quad kq \geq 1, \quad k(q-p) \geq -1.$$

The relations $kq \geq 1$ and $q > 0$ imply that $k > 0$, and therefore $p \geq 0$. Therefore we analyst the final inequality

$$k(q-p) \geq -1.$$

We have the following cases:

$$
\begin{aligned}
k(q-p) = -1 &\Longrightarrow k = 1, q - p = -1\\
k(q-p) \geq 0 &\Longrightarrow q \geq p.
\end{aligned}
$$

This lead us to the following resonance set for λ:

- When $k = 1$, $\lambda = \frac{p}{q} = \frac{q+1}{q} = 1 + \frac{1}{q}$, and
- when $q \geq p$, then $0 \leq \lambda = \frac{p}{q} \leq 1$.

We now consider the second resonance condition

$$n_1\lambda + n_2(\lambda - 1) = \lambda - 1,$$

Where $n_1, n_2 \in \mathbb{Z}_+$ and $n_1 + n_2 \geq 2$.

From this equation we get that

$$\lambda = \frac{n_2 - 1}{n_1 + n_2 - 1}.$$

And the resonance conditions can be written as

$$n_2 = lp,$$
$$n_1 + n_2 - 1 = lq,$$

for some $l \in \mathbb{Z}$.

Finally, the resonance conditions are equivalent to the following system of inequalities

$$n_2 = lp + 1 \geq 0,$$
$$n_1 = l(q - p) \geq 0,$$
$$n_1 + n_2 = lq + 1 \geq 2.$$

Alternatively

$$lq \geq 1, \quad lp \geq -1, \quad l(q-p) \geq 0.$$

Conditions, $lq \geq 1$ and $q > 0$ gives that $l > 0$, and therefore

$$lp \geq -1, \qquad q - p \geq 0$$

They give the following cases:

- $lp = -1$, or $l = 1, p = -1$, then $\lambda = \frac{p}{q} = -\frac{1}{q}$.
- $lp \geq 0$, or $l \geq 0, q \geq p$, then $0 \leq \lambda = \frac{p}{q} \leq 1$.

Therefore, the resonance values of λ for the second resonance condition, are

rational number from the interval $[0, 1]$, or $\lambda = -\frac{1}{q}$, where $q \geq 2$.
The union of all these set, will give us all resonant values of λ. Discarding this, Sternberg's linearization conditions proves the theorem. $\square$

4.1.2 Formal Solution

From now on we assume that the conditions in Theorem 6 hold, and therefore there are coordinates (x, y) in a neighborhood such that the vector field Y is linear.

Lemma 2. *In the coordinates (x, y), the vector field X has the following form.*

$$X = ky^{2q}\partial_x + x\partial_y,$$

where q is a natural number, $k \in \mathbb{R}$ and $\lambda = -\frac{2}{2q-1}$

Proof. Let

$$X = \alpha(x, y)\partial_x + \beta(x, y)\partial_y$$

where function α has a 2nd order zero at the point, and $\beta_x = 1, \beta_y = 0$, and

$$Y = (\lambda - 1)x\partial_x + \lambda y\partial_y.$$

Then the commutator relation relation $[X, Y] = X$ gives the system of differential equation:

$$Y(\alpha) = (\lambda - 2)\alpha, \tag{4.1.1}$$
$$Y(\beta) = (\lambda - 1)\beta.$$

The formal series for function α and β will have the form:

$$\alpha = \sum_{ij} a_{ij} x^i y^j$$
$$\beta = x + \sum_{kl} b_{kl} x^k y^l,$$

where $i, j, k, l \in \mathbb{Z}_+$ and $i + j \geq 2, k + l \geq 2$.
From (4.1.1) we get the following linear system of equations

$$((\lambda - 1)i + \lambda j - \lambda + 2 = 0)a_{ij} = 0,$$
$$((\lambda - 1)k + \lambda l - \lambda + 1 = 0)b_{kl} = 0.$$

Assume that $a_{ij} \neq 0$ and $b_{kl} \neq 0$, some pairs (i, j) and (k, l). Then we get the equations on λ:

$$i(\lambda - 1) + j\lambda - \lambda + 2 = 0,$$
$$k(\lambda - 1) + \lambda l - \lambda + 1 = 0.$$

Solving these equations gives

$$\lambda = \frac{p}{q} = \frac{i - 2}{i_1 + j - 1} = \frac{k - 1}{k + l - 1}, \tag{4.1.2}$$

and we should discard the solutions which gives a resonant λ.
Consider the equation of k, l. Here we have that

$$\lambda = \frac{k - 1}{k + l - 1} = 1 - \frac{l}{k + l - 1}.$$

Therefore, $\lambda \in \mathbb{Q}[0, 1]$ if $k \geq 1$, and $\lambda = -\frac{l}{l-1}$, when $k = 0$. Thus, we have no nontrivial solutions for non-resonant λ.

Consider the equation of i, j.

We have

$$\lambda = \frac{i-2}{i+j-1} = 1 - \frac{j+1}{i+j-1}.$$

Therefore $\lambda \in \mathbb{Q}[0,1]$ if $i \geq 2$.

For the case $i = 1$, we have

$$\lambda = -\frac{1}{j}, \quad j \geq 1,$$

and for the case $i = 0$

$$\lambda = -\frac{2}{j-1}, \quad j \geq 2.$$

Therefore, the only non-resonant λ, correspond to the case $i = 0$, $j = 2q$ and $\lambda = -\frac{2}{2q-1}$ where $q \in \mathbb{N}$. □

$$X = ky^{2q}\partial_x + x\partial_y,$$
$$Y = \frac{1+2q}{1-2q}x\partial_x + \frac{2}{1-2q}y\partial_y.$$

This lemma shows that on the formal level, we can transform vector fields X and Y to the following form:

$$X = ky^{2q}\partial_x + x\partial_y,$$
$$Y = \frac{1+2q}{1-2q}x\partial_x + \frac{2}{1-2q}y\partial_y.$$

Thus we have proved the following theorem:

Theorem 7. *Let the eigenvalues $\lambda, \lambda - 1$ be non-resonant. Then there are* *local coordinates (x, y) in a neighborhood of the point $a \in \mathbb{R}^2$, such that ∞-jets of X and Y have the canonical form:*

1. $$X = x\partial_y, \quad Y = (\lambda - 1)x\partial_x + \lambda y\partial_y.$$

2. $$X = x\partial_x + y^{2q}\partial_x, \quad Y = \frac{1+2q}{1-2q}x\partial_x + \frac{2}{1-2q}y\partial_y.$$

Where $q \in \mathbb{N}$.

Remark. *Assume that $k \neq 0$. We can then take a diffeomorphism of the form*

$$\phi(x, y) \longrightarrow (tx, ty),$$

where $t \neq 0$. This diffeomorphism preserves the vector field Y and will act on X in the following way:

$$\phi_*(X) = \frac{k}{t^{2q-1}}y^{2q} + x\partial_x.$$

By choosing $t = k^{\frac{1}{2q-1}}$, we can assume that $k = 1$.

4.2 The vector field X has no first jet

In this section we start to investigate the case, when $A = [X]_a^1 = 0$, but $B = [Y]_a^1 \neq 0$. Then for the operator B we the following options

- Operator B is diagonalizable and has real eigenvalues λ_1, λ_2.
- Operator B has complex eigenvalues $\lambda, \overline{\lambda}$.
- Operator B has eigenvalue λ with multiplicity 2.

We now treat all these cases separately and use the same methods as applied in the previous chapter.

4.2.1 Operator B is a Jordan form

Let us choose local coordinates (x, y) in such a way, that the operator B takes the Jordan form

$$B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

in the basis ∂_x, ∂_y of $T_a\mathbb{R}^2$.

The resonance conditions for vector field Y has the form:

$$\lambda(m_1 + m_2 - 1) = 0.$$

Since $m_1 + m_2 \geq 2$, we have resonance when $\lambda = 0$ only.

Assuming $\lambda \neq 0$, we can apply the Sternberg theorem to vector field Y and get the following representations of vector fields X and Y:

$$X = \alpha(x, y)\partial_x + \beta(x, y)\partial_y,$$
$$Y = \lambda(x\partial_x + y\partial_y) + x\partial_y.$$

Here α, β have second order zeroes at 0.

The commutator relation $[X, Y] = X$ gives us the following system of equations for functions α, β:

$$Y(\alpha) = (\lambda - 1)\alpha, \tag{4.2.1}$$
$$Y(\beta) = (\lambda - 1)\beta + \alpha.$$

Consider first formal solutions of the system.

Let

$$\alpha = \sum_{ij} a_{ij} x^i y^j,$$
$$\beta = \sum_{kl} b_{kl} x^k y^l,$$

be the formal series with $i + j \geq 2$ and $k + l \geq 2$.

It is easy to check that:

$$Y(x^i y^j) = \lambda(i + j)x^i y^j + jx^{i+1} y^{j-1}.$$

Therefore ,

$$Y(\alpha) - (\lambda - 1)\alpha = \sum_{i+j\geq 2} [\lambda(i + j - 1)a_{ij} + a_{ij} + (j + 1)a_{i-1\, j+1}]\, x^i y^j,$$

and

$$Y(\beta) - (\lambda - 1)\beta - \alpha = \sum_{k+l\geq 2} [\lambda(k + l - 1)b_{kl} + b_{kl} + (j + 1)b_{k-1\, l+1} - a_{kl}]\, x^k y^l.$$

Thus, the system of differential equation (4.2.1), on the formal level, is equivalent to the following system of linear equations for coefficients a_{ij}, b_{kl}:

$$(\lambda(i + j - 1) + 1)a_{ij} + (j + 1)a_{i-1\, j+1} = 0, \tag{4.2.2}$$
$$(\lambda(k + l - 1) + 1)b_{kl} + (l + 1)b_{k-1\, l+1} - a_{kl} = 0, \tag{4.2.3}$$

where i, j, k, l are natural numbers such that $i + j \geq 2$ and $k + l \geq 2$.

Let $i + j = n + 1$, where $n \geq 1$ is fixed.

Then the first part of the system (4.2.2) gives the following linear system for vector $||a_{ij}||, i+j=n+1$:

$$(n\lambda+1)a_{ij}+(j+1)a_{i-1\,j+1}=0. \tag{4.2.4}$$

Therefore, if $n\lambda \neq 0$ we will only have a trivial solution of (4.2.4). Then taking $a_{ij}=0$, we get only a trivial solution for b_{kl}.
Now let λ be a rational number of the form

$$\lambda=-\frac{1}{n},$$

where $n \geq 1$.
Then equation (4.2.4) has solution

$$a_{ij}=0,$$

where $j \geq 1$ and $a_{i\,0}$ is arbitrary.
Thus we have a non-trivial solution for α. If α is trivial, investigation of β will be analogous to this case.
Assume α non trivial.

We have that $j=0$ and i is arbitrary. Equations for $b_{i,j}$, where $i+j=n+1$, take the form

$$(j+1)b_{i-1\,j+1}=a_{ij}.$$

Therefore,

$$b_{i-1\,j+1}=0,$$

if $i \neq 2$ and $b_{1\,n}$ is arbitrary.

Thus, we have found the solutions

$$\alpha = k_1 y^{n+1} \quad \beta = k_2 x^n y + k_3 x^{n+1},$$

in the case when $n\lambda + 1 = 0$.

Summarizing we get the following theorem.

Theorem 8. *Let $[X]_a^1 = 0$, and $B = [Y]_a^1$ has nonzero eigenvalues of multiplicity 2 and corresponds to the Jordan form. Then ∞-jets of X and Y at the point $a \in \mathbb{R}^2$ has the following form*

1. $$Y = \lambda(x\partial_x + y\partial_y) + x\partial_y,$$
   $$X = 0,$$

   if $n\lambda + 1 \neq 0$, for all $n \in \mathbb{N}$.

2. $$Y = \lambda(x\partial_x + y\partial_y) + x\partial_y,$$
   $$X = k_1 y^{n+1}\partial_x + (k_1 x^n y + k_2 x^{n+1})\partial_y$$

   where $k_1, k_2, k_3 \in \mathbb{R}$ and $n\lambda + 1 = 0$ $n \in \mathbb{N}$.

Remark. *Take a diffeomorphism of the form:*

$$\phi : (x, y) \longrightarrow (tx, ty).$$

This will preserve the normal form of Y and in the normal form for X it *will transform the coefficients in the following way:*

$$(k_1, k_2) \longrightarrow (t^n k_1, t^n k_2).$$

Therefore, for odd n, we have the following options for (k_1, k_2):

1. $(k_1, k_2) = (1, 0)$,
2. $(k_1, k_2) = (0, 1)$,
3. $(k_1, k_2) = (1, k)$,

where $k \neq 0$.
If n is even we get the following list

1. $(k_1, k_2) = (\pm 1, 0)$,
2. $(k_1, k_2) = (0, \pm 1)$,
3. $(k_1, k_2) = (\pm 1, k)$,

Where $k \neq 0$.

4.2.2 Operator B has complex eigenvalues

The resonance conditions for the vector field Y, will then be:

$$\Re(\lambda)(m_1 + m_2 - 1) = 0,$$
$$\Im(\lambda)(m_1 - m_2 - 1) = 0.$$

From this we see immediately that we have resonance, iff $\Re(\lambda) = 0$.
Consider the action of operator B on the complexification of $T_a\mathbb{R}^2$, and choose

coordinates x, y such that the vectors

$$\partial_z = \frac{1}{2}\partial_x - i\partial_y,$$
$$\partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y),$$

are eigenvectors for B.

In these coordinates the vector fields X and Y have the form

$$X = \alpha(z, \overline{z})\partial_z + \overline{\alpha(z, \overline{z})}\partial_{\overline{z}},$$
$$Y = \lambda z\partial_z + \overline{\lambda}\overline{z}\partial_{\overline{z}}.$$

Here α is a complex function of second order at 0.

Viewing the commutator relation $[X, Y] = X$ as a differential equation on the function α we get the following equation:

$$Y(\alpha) = (\lambda - 1)\alpha. \tag{4.2.5}$$

Now we expand α through the formal series

$$\alpha = \sum_{kl} a_{kl} z^k \overline{z}^l,$$

where $k + l \geq 2$.

Then the equation (4.2.5) takes the form

$$\sum_{k+l\geq 2} (k\lambda + l\overline{\lambda}) a_{kl} z^k \overline{z}^l = (\lambda - 1) \sum_{k+l\geq 2} a_{kl} z^k \overline{z}^l.$$

Therefore,

$$(k\lambda + l\overline{\lambda} - (\lambda - 1)a_{kl}) = 0,$$

and we get nontrivial solution iff

$$k\lambda + l\overline{\lambda} - (\lambda - 1) = 0 \tag{4.2.6}$$

for some natural numbers $k, l,\ k + l \geq 2$.

Let $\lambda = \lambda_0 + i\lambda_1$, where $\lambda_0 = \Re\lambda$, $\lambda_1 = \Im\lambda \neq 0$. Taking the real and imaginary parts of equation (4.2.6), we get the system

$$\begin{aligned} k\lambda_0 + l\lambda_0 + 1 &= 0, \\ k\lambda_1 - l\lambda_1 - \lambda_1 &= 0. \end{aligned} \tag{4.2.7}$$

Since we consider the complex case, $\lambda_1 \neq 0$, we have that

$$k = l + 1.$$

Putting this relation into the first equation of the system (4.2.6), we get

$$2l\lambda + 1 = 0.$$

Summarizing, we get the following result;

Theorem 9. *Let $[X]_a^1 = 0$, and $B = [Y]_a^1$ have complex eigenvalues $(\lambda, \overline{\lambda})$ where $\Im(\lambda) \neq 0$, $\Re(\lambda) \neq 0$. Then ∞- jets of X and Y can be written in one of the following normal forms:*

- $$Y = \lambda z\partial_z + \overline{\lambda}\overline{z}\partial_{\overline{z}},$$
  $$X = 0,$$

  If $2l\Re(\lambda) + 1 \neq 0$*, for all* $l \geq 1$*.*

- $$Y = \lambda z\partial_z + \overline{\lambda}\overline{z}\partial_{\overline{z}},$$
  $$X = a(z,\overline{z})\partial_z + \overline{a(z,\overline{z})}\partial_{\overline{z}},$$

  where
  $$a(z,\overline{z}) = \alpha z|z|^{2l},$$
  $\alpha \in \mathbb{C}/0$*, and* $2l\Re(\lambda) + 1 = 0$*.*

Remark. *A diffeomorphism*

$$\phi : (x,y) \longrightarrow (tx,ty)$$

will preserve the normal form for vector field Y*, and acts on* X *in the following way:*

$$\alpha \longrightarrow \alpha|t|^{2l}.$$

Therefore, in the normal form 2., we can take a parameter t*, so that*

$$|\alpha| = 1.$$

4.2.3 Operator B has real eigenvalues and is diagonalizeable

Let (λ_1, λ_2) be eigenvalues of operator B, and assume that this pair is not resonant. Then due to Sternberg linearization theorem, we can choose local coordinates (x, y) in a neighborhood of the point $a \in \mathbb{R}^2$ in such a way that:

$$X = \alpha(x,y)\partial_x + \beta(x,y)\partial_y,$$
$$Y = \lambda_1 x\partial_x + \lambda_2 y\partial_y.$$

Where α and β are functions of second order.
Viewing the commutator relation $[X, Y] = X$ as a differential equation, we get the following system of equations:

$$Y(\alpha) = (\lambda_1 - 1)\alpha, \tag{4.2.8}$$
$$Y(\beta) = (\lambda_2 - 1)\beta, \tag{4.2.9}$$

on the functions α and β.
Writing the formal series

$$\alpha = \sum_{i+j\geq 2} a_{ij}x^i y^j,$$
$$\beta = \sum_{k+l\geq 2} b_{kl}x^k y^l,$$

Of these equation we get the following system of linear equation on coefficients a_{ij}, b_{kl}

$$(i\lambda_1 + j\lambda_2 - \lambda_1 + 1)a_{ij} = 0, \tag{4.2.10}$$

$$(k\lambda_1 + l\lambda_2 - \lambda_2 + 1)b_{kl} = 0, \tag{4.2.11}$$

where $i + j \geq 2$ and $k + l \geq 2$.

Define the following sets

$$\Sigma_1 = \{(i,j)|(i-1)\lambda_1 + j\lambda_2 = -1, \quad \text{where } i + j \geq 2\},$$
$$\Sigma_2 = \{(k,l)|k\lambda_1 + (l-1)\lambda_2 = -1, \quad \text{where } k + l \geq 2\}.$$

Then the following result holds:

Theorem 10. *Let $[X]^1_a = 0$ and operator $B = [Y]^1_a$ has is diagonalizable with real eigenvalues which are not resonant.*
Then there is a local system coordinates (x, y) in a neighborhood of the point $a \in \mathbb{R}^2$, such that ∞ jet of X and Y at the point have the following normal forms;

$$Y = \lambda_1 x\partial_x + \lambda_2 y\partial_y,$$
$$X = (\sum_{(i,j)\in\Sigma_1} a_{ij}x^i y^j)\partial_x + (\sum_{(k,l)\in\Sigma_1} b_{kl}x^k y^l)\partial_y.$$

4.2.3.1 The function α has a non-vanishing second jet

In this section we investigate the case when the function α in the representation:

$$X = \alpha(x, y)\partial_x + \beta(x, y)\partial_y,$$
$$Y = \lambda_1 x\partial_x + \lambda_2 y\partial_y,$$

has non trivial second jet.

Theorem 11. *Let the linear terms of Y be diagonalizable in, and let the function α have a nontrivial 2nd jet. Then there exists local coordinates (x, y) such that ∞-jets of X and Y are one of the 6 following forms:*

1.

$$Y = -x\partial_x - y\partial_y,$$

and X has one of the forms

$$X = \left(x^2 + k_1xy + k_2y^2\right)\partial_x + \left(k_3x^2 + k_4xy + k_5y^2\right)\partial_y,$$
$$X = \left(k_1x^2 + xy + k_2y^2\right)\partial_x + \left(k_3x^2 + k_4xy + k_5y^2\right)\partial_y,$$
$$X = \left(k_1x^2 + k_2xy + y^2\right)\partial_x + \left(k_3x^2 + k_4xy + k_5y^2\right)\partial_y.$$

Where k_i are arbitrary real numbers.

2.

$$Y = -x\partial_x + \gamma_1 y\partial_y,$$
$$X = x^2\partial_x + kxy\partial_y,$$

where k is an arbitrary real number and $\gamma_1 \in \mathbb{Q}^- - \mathbb{Q}_1^-$, $\mathbb{Q}_1^- = \{-q|q \geq 2\} \cup \{-\frac{1}{q}|q \geq 2\}$.

3.

$$Y = -\frac{2}{n}x\partial_x - y\partial_y,$$

and X has one of the forms

$$X = xy\partial_x + (ky^2 + x^n)\partial_y,$$
$$X = xy\partial_x + ky^2\partial_y,$$

where k is an arbitrary number and $n \geq 3$ is a odd number.

4.

$$Y = \gamma_2 x\partial_x - y\partial_y,$$
$$X = xy\partial_x + ky^2\partial_y,$$

where k is an arbitrary real number and $\gamma_2 \in \mathbb{Q}^- - \mathbb{Q}_2^-$, $\mathbb{Q}_2^- = \{-q|q \geq 2\} \cup \{-\frac{2}{q}|q \geq 3\}$.

5.

$$Y = -\frac{1}{n}(x\partial_x + \frac{1+n}{2}y\partial_y),$$

and X has one of the forms

$$
\begin{aligned}
X &= \pm y^2\partial_x \pm x^n y\partial_y,\\
X &= y^2\partial_x,
\end{aligned}
$$

Where n is an even number.

6.

$$
Y = \frac{1}{2n-1}(3x\partial_x + (n+1)y\partial_y),
$$

and X has one of the following forms

$$
\begin{aligned}
X &= \pm y^2\partial_x + x^n\partial_y,\\
X &= y^2\partial_x,
\end{aligned}
$$

where k is an arbitrary real number and $n \in \mathbb{N} - (\mathbb{N}_3$. $\mathbb{N}_3 = \{k|k = 3p-1, p \geq 2\}$ and $n \geq 2$.

Proof. Given that:

$$
\begin{aligned}
X &= \alpha(x,y)\partial_x + \beta(x,y)\partial_y,\\
Y &= \lambda_1 x\partial_x + \lambda_2 y\partial_y.
\end{aligned}
$$

We expand the functions α and β through the formal series

$$
\begin{aligned}
\alpha &= \sum_{i+j=2} a_{ij}x^i y^j,\\
\beta &= \sum_{i+k\geq 2} b_{kl}x^k y^l.
\end{aligned}
$$

In order to find all the possibilities for normal forms we will split our investigation into 3 cases

1. $i = 2$ and $j = 0$,
2. $i = j = 1$,
3. $i = 0$ and $j = 2$.

4.2.3.1.1 Case 1 Given that $i = 2, j = 0$. From (4.2.10) we find that

$$\lambda_1 = -1,$$
$$\lambda_2 = \frac{k-1}{l-1}.$$

if $l \neq 1$. $l = 1$ is a special case, and will be investigated later. We now get the following resonance conditions

$$m_1 \frac{k-1}{l-1} - m_2 = -1.$$

Where m_1, m_2 are non negative and $m_1 + m_2 \geq 2$. We always have a solution of this equation by setting $m_1 = l - 1$ and $m_2 = k$ except for two cases

- $k = 0, l = 2$,
- $k = 2, l = 0$.

Both these cases give that $\lambda_1 = \lambda_2 = -1$ which does not have any resonances. Therefore we have the normal form

$$X = k_1 x^2 \partial_x + (k_2 x^2 + k_3 y^2)\partial_y,$$
$$Y = -x\partial_x - y\partial_y.$$

Where $k_1 \neq 0$, k_2 and k_3 are arbitrary.

We now investigate the case where $l = 1$.
From (4.2.11) we get that $k = 1$ and λ_2 is arbitrary. We now investigate the resonance conditions

$$-m_1 + \lambda_2 m_2 = -1,$$
$$-n_2 + \lambda_2 n_2 = \lambda_2.$$

If either of these equations are satisfied, we must discard this value for λ_2. We solve the equation for λ_2 to get the expressions

$$\lambda_2 = \frac{m_1 - 1}{m_2},$$
$$\lambda_2 = \frac{n_1}{n_2 - 1}.$$

Where $m_1 + m_2 \geq 2$ and $n_1 + n_2 \geq 2$. We see immediately that, for any non-negative rational we may find m_1, m_2 or n_1, n_2 that satisfy this equation. However if we fix $m_1 = 0$ we find that $\lambda_2 = -\frac{1}{q}$ where $q \geq 2$ gives resonance. Also if fixing $n_2 = 0$ gives us that if λ_2 is a negative integer less than or equal to -2, gives us resonance. Discarding these cases we arrive at the normal forms:

$$X = k_1 x^2 \partial_x + k_2 xy \partial_y,$$
$$Y = -x\partial_x + \gamma y \partial_y.$$

Where $\gamma \in \mathbb{Q}^- - \mathbb{Q}_2^-$, where $\mathbb{Q}_2^- = \{-q | q \geq 2\} \cup \{-\frac{1}{q} | q \geq 2\}$

4.2.3.1.2 Case 2 Given that $i = j = 1$ we find from (4.2.10) that

$$\lambda_2 = -1,$$
$$\lambda_1 = \frac{l-2}{k}.$$

The case when $k = 0$ is a special case and will be investigated later. We note that λ_2 is either a non negative rational or it is equal to

$$\lambda_1 = -\frac{2}{q}.$$

Where $q \geq 2$ We write the resonance conditions, and solve them for λ_1 and arrive to the equations

$$\lambda_1 = \frac{m_2}{m_1 - 1},$$
$$\lambda_1 = \frac{n_2 - 1}{n_1}.$$

If either of these equations can be satisfied for fixed values of l and k, λ_1, λ_2 will be resonant.

We see immediately that if λ_1 is a non-negative rational number it will be resonant, so this case must be discarded.

The case when $n_2 = 0$ will give that

$$\lambda_1 = -\frac{1}{q}.$$

When $q \geq 2$ will be resonant. Looking back at our possibilities for λ_1, we find that the only the case where

$$\lambda_1 = -\frac{2}{n}$$

where n is odd and $n \neq 1$, will remain.
This shows us that $l = 0$ and k is an odd number greater than 1. We then arrive at the normal form

$$X = k_1 xy\partial_x + k_2 x^n \partial_y,$$
$$Y = -\frac{2}{n} x\partial_x - y\partial_y.$$

where n is a odd number larger than 1.

When $l = 1$ we have a special case where $\lambda_1 = \lambda_2 = -1$ which does not have resonance, and the vector fields are of the form

$$X = xy(k_1\partial_x + k_2\partial_y),$$
$$Y = -x\partial_x - y\partial_y.$$

Where k_1 and k_2 are arbitrary reals.

We now treat the case when $k = 0$. From (4.2.11) we get that λ_1 may be arbitrary and $l = 2$. The restriction on λ_1 will be analogous to the case when $i = 2$ $j = 0$ and $l = 1$. We arrive at the vector fields

$$X = k_1 xy\partial_x + k_2 y^2 \partial_y,$$
$$Y = \gamma x\partial_x - y\partial_y.$$

Where $\gamma \in \mathbb{Q}^- - \mathbb{Q}_2^-$, where $\mathbb{Q}_2^-$ is the same as in the earlier case.

4.2.3.1.3 Case 3 We have that $i = 0$ and $j = 2$, we find from (4.2.10) that $\lambda_1 = 2\lambda_2 + 1$. From this (4.2.11) gives us that

$$\lambda_2 = \frac{1+k}{1-2k-l}. \tag{4.2.12}$$

We note that this must always be negative and solve the resonance conditions for λ_2. We have resonance if one of the following equalities hold.

$$\lambda_2 = \frac{1-m_1}{2m_1+m_2-2},$$
$$\lambda_2 = \frac{n_1}{1-2n_1-n_2}.$$

We rewrite these equations to find some solutions

$$\lambda_2 = \frac{(m_1-1)+1}{2(m_1-2)+(m_1+3)-1}, \tag{4.2.13}$$
$$\lambda_2 = \frac{(n_1-1)+1}{2(n_1-1)+(n_2+2)-1}. \tag{4.2.14}$$

We look (4.2.12) and find resonances by the following equalities:

$$m_1 = 2+k, \qquad n_1 = 1+k,$$
$$m_2 = l-3, \qquad n_2 = l-2.$$

This shows us that this method will always find n_1 and n_2 that give us resonance given l and k, expect for the cases

1. $l = 2$ and $k = 0$,
2. $k \geq 1$ and $l = 1$,
3. $l = 0$ and $k \geq 2$.

All the trivial cases for m_1 and m_2 will be contained within our possibilities for n_2 and n_2.
However we may have other solutions to these cases where this approach does not work, and they must therefore be investigated in detail.

Case 1 We have that $l = 2$ and $k = 0$ and look to (4.2.12) to see that $\lambda_1 = \lambda_2 = -1$. When the eigenvalues are equal, we will never have resonance. We therefore arrive at the representations

$$X = y^2(k_1\partial_x + k_2\partial_y),$$
$$Y = -x\partial_x - y\partial_y.$$

Where k_1 and k_2 are arbitrary reals.

4.2.3.1.3.1 Case 2 Recall that $l = 1$ and $k \geq 1$. We begin by looking to (4.2.12) to see that

$$\lambda_2 = -\frac{1+k}{2k}.$$

We now investigate which values of k that have resonance.
First we investigate for m_1 m_2 by looking to the equation:

$$-\frac{1+k}{2k} = -\frac{m_1 - 1}{2m_1 + m_2 - 2}.$$

Where $k \geq 1$, $m_1, m_2 \in \mathbb{N}$ and $m_1 + m_2 \geq 2$.
Solving this equation gives us:

$$2m_1 + (1+k)m_2 = 2.$$

Due to our restrictions on m_1, m_2 and k this equation will never have a solution.

Now we investigate n_1 and n_2

$$-\frac{1+k}{2k} = -\frac{n_1}{2n_1 + n_2 - 1}.$$

Where $n_1, n_2 \in \mathbb{N}$ and $n_1 + n_2 \geq 2$. Solving this we arrive to the equation

$$2n_1 + (n_2 - 1)(1 + k) = 0.$$

Due to our restrictions on n_1, n_2 and k, the only possibility we have to satisfy this equation is when $n_2 = 0$. This gives us that we will have resonance when:

$$k = 2n_1 - 1.$$

where $n_1 \geq 2$.

We then arrive at the vector fields

$$X = k_1 y^2 \partial_x + k_2 x^n y \partial_y,$$
$$Y = -\frac{1}{n}(x\partial_x + \frac{(1+n)}{2} y\partial_y).$$

Where $n \in (\mathbb{N}_{\text{even}} \cup \{1\})$.

Case 3 Recall that $l = 0$ and $k \geq 2$.We look to (4.2.12) and see that

$$\lambda_2 = -\frac{1+k}{2k-1}.$$

We now investigate which values of k that have resonance.

First we investigate for m_1 m_2 by looking to the equation:

$$-\frac{1+k}{2k-1} = -\frac{m_1-1}{2m_1+m_2-2}.$$

Where $k \geq 2$, $m_1, m_2 \in \mathbb{N}$ and $m_1 + m_2 \geq 2$.Solving this equation gives us

$$3m_1 + (1+k)m_2 = 1.$$

Due to our restriction on k, m_1 and m_2, this equation will never have a solution.

We now investigate n_1, n_2

$$-\frac{1+k}{2k} = -\frac{n_1}{2n_1+n_2-1}.$$

Where $n_1, n_2 \in \mathbb{N}$ and $n_1 + n_2 \geq 2$. Solving this we arrive to the equation

$$3n_1 + (k+1)(n_2-1) = 0.$$

Due to our restriction on k, n_1 and n_2 we only have the possibility of $n_2 = 0$. This shows that we have resonance when

$$k = 3n_1 - 1.$$

where $n_1 \geq 2$. We arrive at the vector fields

$$\begin{aligned} X &= k_1 y^2 \partial_x + k_2 x^n \partial_y, \\ Y &= \frac{1}{2n-1}(3x\partial_x + (n+1)y\partial_y). \end{aligned}$$

Where $n \in \mathbb{N} - (\{1\} \cup \mathbb{N}_3.\ \mathbb{N}_3) = \{k | k = 3n-1, n \geq 2\}$.

4.2.3.1.4 Superpositions Having found all these representation, we must take a superposition whenever they have the same eigenvalues. Finally we will argue when it is possible to remove arbitrary coefficients. We gather all these results in the table below

Table 4.1: Normal forms

Case	(λ_1, λ_2)	Normal form of X	restrictions
1	$(-1,-1)$	$k_1x^2\partial_x + (k_2x^2 + k_3y^2)\partial_y$	none
2	$(-1,\gamma)$	$k_1x^2\partial_x + k_2xy\partial_y$	$\gamma \in (\mathbb{Q}^- - \mathbb{Q}_2^-)$
3	$(-\frac{2}{n},-1)$	$k_1xy\partial_x + k_2x^n\partial_y$	$n \geq 3$ odd
4	$(-1,-1)$	$xy(k_1\partial_x + k_2\partial_y)$	none
5	$(\gamma,-1)$	$k_1xy\partial_x + k_2y^2\partial_y$	$\gamma \in (\mathbb{Q}^- - \mathbb{Q}_2^-)$
6	$(-1,-1)$	$y^2(k_1\partial_x + k_2\partial_y)$	none
7	$(-\frac{1}{n},-\frac{n+1}{2n})$	$k_1y^2\partial_x + k_2x^ny\partial_y$	n even or equal 1
8	$(\frac{3}{2n-1},\frac{n+1}{2n-1})$	$k_1y^2\partial_x + k_2x^n\partial_y$	$n \in \mathbb{N} - (\mathbb{N}_3 \cup \{1\})$

4.2.3.1.4.1 $\lambda_1 = \lambda_2 = -1$**.** Here all 8 cases will have some solution. We therefore take a superposition of every possibility to arrive at the normal form

$$Y = -x\partial_x - y\partial_y,$$
$$X = \left(k_1x^2 + k_2xy + k_3y^2\right)\partial_x + \left(k_4x^2 + k_5xy + k_6y^2\right)\partial_y.$$

Where k_i are arbitrary reals.
If one of these constants are non-zero, it is possible to pick a diffeomorphism that brings it to 1. We now investigate the intersections.

4.2.3.1.4.2 Case 2 With the exception of $\gamma = -1$, there are no intersection of these eigenvalues with any of the other cases. Therefore we have the normal form

$$Y = -x\partial_x + \gamma y\partial_y,$$
$$X = k_1x^2\partial_x + k_2xy\partial_y.$$

Where k_1 and k_2 are arbitrary reals.
Since $k_1 \neq 0$ we can take the diffeomorphism $\phi : x \to k_1x$, to remove the constant k_1.

4.2.3.1.4.3 Case 3 Here we have an intersection with case 5 when

$$\gamma = -\frac{2}{n}.$$

$$Y = -\frac{2}{n}x\partial_x - y\partial_y,$$
$$X = k_1xy\partial_x + (k_2y^2 + k_3x^n)\partial_y.$$

Where n is an odd number greater than 1.

If we have that $k_3 \neq 0$ we can take the diffeomorphism

$$\phi : x \to \sqrt[n]{k_1k_3}x,$$
$$: y \to k_1y.$$

This maps both constants $k_1 = k_=13$. The case when $k_3 = 0$ will be treated later.

4.2.3.1.4.4 Case 5 When $\gamma \neq -1$ and $\gamma \neq -\frac{2}{n}$, we have the normal form

$$Y = \gamma x\partial_x - y\partial_y,$$
$$X = k_1xy\partial_x + k_2y^2\partial_y.$$

If we take the diffeomorphism $\phi : y \to k_1y$ we have that $k_1 = 1$.

4.2.3.1.4.5 Case 7 If $n \neq 1$ this does not intersect with any of the other cases. We have that

$$X = k_1y^2\partial_x + k_2x^ny\partial_y,$$
$$Y = -\frac{1}{n}(x\partial_x + \frac{1}{2}(1+n)y\partial_y).$$

Where n is an even number and k_1, k_2 arbitrary.

When $k_2 \neq 0$ we take a diffeomorphism which is a scales x and y in the way $\phi : x \to tx, \quad y \to sy$. We now find values for s and t such that the constants k_1 and k_2 be mapped to 1.

$$\frac{k_1 t}{s^2} = 1,$$
$$\frac{k_2}{t^n} = 1.$$

Since n is an even number we cannot remove the sign of k_1 and k_2. Analogously, if $k_2 = 0$ we cannot remove the sign of k_1.

4.2.3.1.4.6 Case 8 If $n \neq 2$ this gives us a unique case, where the normal form is

$$X = k_1 y^2 \partial_x + k_2 x^n \partial_y,$$
$$Y = \frac{1}{2n-1}(3x\partial_x + (n+1)y\partial_y).$$

If $k_2 \neq 0$ we again remove the arbitrary constant by a diffeomorphism $\phi : x \to tx, y \to sy$ and get the equation

$$\frac{k_1 t}{s^2} = 1,$$
$$\frac{sk_2}{t^n} = 1.$$

Investigating this equation we find that we can remove the sign of k_2 but not k_1. If $k_2 = 0$ we can always set $k_1 = 1$.

□

4.3 Smooth classification

Assume that we take coordinates (x, y) in which vector field Y is linearizable

$$Y = \lambda_1 x\partial_x + \lambda_2 y\partial_y,$$

with the condition $\lambda_1\lambda_2 > 0$.

Let then

$$X = \alpha(x, y)\partial_x + \beta(x, y)\partial_y,$$

be a representation of X.

Theorem 12. *Let*

$$X = \alpha(x, y)\partial_x + \beta(x, y)\partial_y,$$

and

$$\tilde{X} = \alpha(\tilde{x, y})\partial_x + \beta(\tilde{x, y})\partial_y,$$

be two vector fields such that the commutator relations $[X, Y] = X$ *and* $[\tilde{X}, Y] = \tilde{X}$ *hold for both.*

Assume the ∞*-jet of functions* (α, β) *coincide:*

$$[\alpha]_a^\infty = [\tilde{\alpha}]_a^\infty, \quad [\beta]_a^\infty = [\tilde{\beta}]_a^\infty$$

Then $\alpha = \tilde{\alpha}$, $\beta = \tilde{\beta}$ *in a neighborhood of the point* $a \in \mathbb{R}^2$

Proof. Assume α and $\tilde{\alpha}$ satisfy the differential equation

$$Y(\alpha) = (\lambda_1 - 1)\alpha,$$

$$Y(\tilde{\alpha}) = (\lambda_1 - 1)\tilde{\alpha}.$$

Therefore the difference

$$\epsilon = \alpha - \tilde{\alpha},$$

which is a flat function at the point $a \in \mathbb{R}^2$, also satisfies this equation. Consider a trajectory

$$(x(t), y(t)) = \left(e^{\lambda_1 t} x_0, e^{\lambda_2 t} y_0\right).$$

of the vector field Y, and let

$$\phi(t) = \epsilon(x(t), y(t)),$$

be the restriction of the function ϵ on this trajectory.

Then we have

$$\dot{\phi} = (\lambda_1 - 1),$$

and therefore

$$\phi(t) = e^{(\lambda_1 - 1)t}\phi(0) = x(t)^{1-\frac{1}{\lambda_1}}\phi(0).$$

Since $\lambda_1\lambda_2 > 0$ we can always approach the origin by letting $t \to \pm\infty$. We see that ϕ behaves as a power of x as we approach the origin, which contradicts that ϵ is flat. The case for β is analogous to this. $\square$

Remark. *If $\lambda_1\lambda_2 < 0$, the trajectories of the representation, never approach the singularity. We see immediately that $x^{\lambda_1}y^{-\lambda_2}$ (or $x^{-\lambda_2}y^{\lambda_1}$) is an invariant to this action. If we take a flat function f, we can always have a superposition of a flat solution $f(x^{\lambda_2}y^{-\lambda_1})$ as a flat solution to the commutator relation.*

Chapter 5

Applications to differential equations

In this section we apply the results of the previous sections to differential equations. By letting these Lie algebras act on $J^0(x, y)$, the representation of $\mathfrak{g}$ will be the symmetry algebra of a family of second order differential equations. Since our collection of normal forms is rather large, we only investigate some selected representations. We wish to restrict our investigation to finding kth order differential equations of the form

$$F(x, y, y', \ldots, y^{(k)}) = C.$$

In order to find the first order differential equations, we take the first prolongations of X and Y, and find their basic invariant f_1 through the formula

$$X^{(1)}(f_1) = Y^{(1)}(f_1) = 0.$$

Finally we take the second prolongation of the vector fields and find the their common invariance f_2.

We can now find any higher order invariance through the Tresse derivatives:

$$f_i = \frac{\frac{df_{i-1}}{dx}}{\frac{df_1}{dx}}.$$

Where $\frac{df}{dx}$ is the total derivative of f.
In order to illustrate how this invariance is computed, we examine the transitive case thoroughly, and list the results for some of the normal forms found in the previous sections.

5.1 Transitive Action

Corrolary 2. *The class of first order differential equations with a 2 dimensional symmetry algebra with a transitive action is*

$$y'e^y = C.$$

And the class of second order differential equations is

$$F(y'e^y, y''e^{2y}) = C.$$

The class of k-th order differential equation is

$$F(f_1, \ldots, f_k) = C.$$

Where $f_1 = y'e^y, f_2 = y''e^{2y}$ *and* f_k *is given by*

$$f_k = \frac{\frac{df_{k-1}}{dx}}{\frac{df_1}{dx}}.$$

Proof. From 2 we have that

$$X = \partial_x,$$
$$Y = x\partial_x + \partial_y.$$

Taking the first prolongation of these vector fields we get

$$X^{(1)} = X = \partial_x,$$
$$Y^{(1)} = x\partial_x + \partial_y - y_1\partial_{y_1}.$$

The invariant of the vector field $X^{(1)}$ is any arbitrary function of y, y_1. The vector field Y however will have invariance, satisfying the partial differential equation

$$\frac{\partial f_1}{\partial_y} = y_1 \frac{\partial f_1}{\partial_{y_1}}.$$

Which has solution

$$f_1 = y_1 e^y.$$

We now wish to find second order invariance

$$Y^{(2)} = x\partial_x + \partial_y - y_1\partial_{y_1} - 2y_2\partial_{y_2}.$$

Solving the differential equation $Y^{(2)}(f_2) = 0$ gives us finally that.

$$f_2 = y_2 e^{2y_0}.$$

Any higher order invariance are Tresse derivatives of these functions, for example 3rd order invariance will be:

$$f_3 = \frac{\frac{df_2}{dx}}{\frac{df_1}{dx}} = \frac{e^y(2y_2y_1 + y_3)}{y_1^2 + y_2}.$$

□

5.2 Weak Singular Action

In this section we find the differential invariance of some of the vector fields we found in 3

Corrolary 3. *The class of first order differential equations which have the symmetry algebra*

$$X = \partial_x,$$
$$Y = x\partial_x + ky\partial_y,$$

is

$$(y')^k y^{1-k} = C.$$

And the second order differential equations is given by

$$F\left((y')^k y^{k-1}, (y'')^k y^{2-k}\right) = C.$$

Any higher order differential equations, are given by Tresse derivatives as described in 2

Corrolary 4. *The class of first order differential equations which have the* *symmetry algebra*

$$X = \partial_x,$$
$$Y = x\partial_x + y^2\partial_y,$$

is

$$\frac{y'}{y^2}e^{-\frac{1}{y}} = C.$$

And the second order differential equations is given by

$$F\left(\frac{y'}{y^2}e^{-\frac{1}{y}}, \frac{(y''y - 2(y')^2)}{y^3}e^{-\frac{2}{y}}\right) = C.$$

Any higher order differential equations, are given by Tresse derivatives as described in 2

Corrolary 5. *The class of first order differential equations which have the symmetry algebra*

$$X = \partial_x,$$
$$Y = x\partial_x \pm y^3\partial_y.$$

Is given by

$$\frac{y'}{y^3}e^{\mp\frac{1}{2y^2}} = C.$$

And the second order differential equations is given by

$$F\left(\frac{y'}{y^3}e^{\mp\frac{1}{2y^2}}, \frac{(y''y - 3(y')^2)}{y^4}e^{\mp\frac{1}{y^2}}\right) = C.$$

Any higher order differential equations, are given by Tresse derivatives as described in 2

5.3 Singular Action

Corrolary 6. *The class of first order differential equations which have the symmetry algebra*

$$Y = \frac{3}{2n-1}x\partial_x + \frac{n+1}{2n-1}y\partial_y$$
$$X = y^2\partial_x$$

Is given by

$$\left(\frac{-2xy' + y}{y'}\right)^{n+1} y^{-3} = C.$$

And the second order differential equations is given by

$$F\left(\left(\frac{2(y')^3x + y^2y''}{y(y')^3}\right)^{n+1} y^{-3}, \left(\frac{-2xy' + y}{y'}\right)^{n+1} y^{-3}\right) = C.$$

Any higher order differential equations, are given by Tresse derivatives as described in 2

Corrolary 7. *The class of first order differential equations which have the symmetry algebra*

$$Y = -\frac{1}{n}x\partial_x - \frac{n+1}{2n}y\partial_y$$
$$X = y^2\partial_x$$

Is given by

$$\frac{-2xy' + y}{y'}y^{-\frac{2}{n+1}} = C.$$

And the second order differential equations is given by

$$F\left(\left(\frac{2(y')^3x+y^2y''}{y(y')^3}\right)^{n+1}y^{-2},\left(\frac{-2xy'+y}{y'}\right)^{n+1}y^{-2}\right)=C.$$

Any higher order differential equations, are given by Tresse derivatives as described in 2

Remark. *We can use these results to find higher order differential differential equations solvable by quadratures. Taking the differential equation*

$$F(Y^{(k)},Y^{(k-1)},\ldots)=0.$$

Where the derivative is the Tresse derivative.
If this differential equation has a known symmetry algebra and $Y^{(k)}$ *are Tresse derivatives of invariants of some other symmetry algebra. The differential equation has a symmetry algebra of both the invariants corresponding to* Y *and of* F. *If this is of dimension equal to the order of the differential equation, the Lie-Bianchi theorem will give us that it is solvable by quadratures.*

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