Errata to Almost Complex Homogeneous Spaces with Semi-Simple Isotropy

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Two of the entries in the tables of "Almost Complex Homogeneous Spaces with Semi-Simple Isotropy" are missing some parameters. The purpose of this text is to introduce what was missing. The new parameters allow the almost complex structure J to be deformed such that the Nijenhuis tensor N_J is non-degenerate. The new parameters occur in those cases where $\mathfrak{g}$ has an 8d semi-simple subalgebra and $\mathfrak{h} = \mathfrak{su}(1,1)$ or $\mathfrak{h} = \mathfrak{su}(2)$. The notations used here are explained in the parent text.

$\mathfrak{h} = \mathfrak{su}(1,1)$, $\mathfrak{m} = V^{\mathbb{C}} \oplus \mathbb{C}$

Let V be the tautological representation of $\mathfrak{sl}_2 \simeq \mathfrak{su}(1,1)$. Then the complexification $V^{\mathbb{C}}$ is the tautological representation of $\mathfrak{h} = \mathfrak{su}(1,1)$. Let $\mathfrak{m} = V^{\mathbb{C}} \oplus \mathbb{C}$. We will use the following basis of $\mathfrak{m}$.

$$x, y, ix, iy, z, iz$$

Let $\hat{x}$ be the element in the real dual basis which corresponds to x, etc. The following operators are a basis of $\mathfrak{h}$.

$$\begin{aligned} A &= \hat{y} \otimes x - \hat{x} \otimes y + i\hat{y} \otimes ix - i\hat{x} \otimes iy \\ B &= \hat{y} \otimes x + \hat{x} \otimes y + i\hat{y} \otimes ix + i\hat{x} \otimes iy \\ C &= \hat{x} \otimes x - \hat{y} \otimes y + i\hat{x} \otimes ix - i\hat{y} \otimes iy \end{aligned}$$

Thus $\langle x, y \rangle$ and $\langle ix, iy \rangle$ are submodules and A, B, C satisfy the following relations

$$\begin{aligned} [A, B] &= 2C \\ [A, C] &= -2B \\ [B, C] &= -2A \end{aligned}$$

We are interested in the case when the bracket component $\Lambda^2\mathfrak{m} \to \mathfrak{h}$ is non-zero. This gives the following Lie Brackets on $\mathfrak{m}$.

$$
\begin{aligned}
[x,y] &= \alpha z\\
[ix,iy] &= \beta z\\
[x,ix] &= (A+B)\\
[x,iy] &= -C\\
[ix,y] &= C\\
[y,iy] &= (A-B)\\
[z,x] &= (-3/\beta)ix\\
[z,ix] &= (3/\alpha)x\\
[z,y] &= (-3/\beta)iy\\
[z,iy] &= (3/\alpha)y
\end{aligned}
$$

If $\alpha\beta > 0$ then $\mathfrak{g} = \mathfrak{u}(2,1)$, and if $\alpha\beta < 0$ then $\mathfrak{g} = \mathfrak{gl}_3$. $\alpha\beta = 0$ is not allowed. The Nijenhuis tensor is

$$
\begin{aligned}
N_J(x,y) &= (\beta-\alpha)z\\
N_J(x,z) &= -3\frac{\alpha-\beta}{\alpha\beta}ix\\
N_J(y,z) &= -3\frac{\alpha-\beta}{\alpha\beta}iy
\end{aligned}
$$

$\mathfrak{h} = \mathfrak{su}(2)$, $\mathfrak{m} = W \oplus \mathbb{C}$

Let W be the tautological representation of $\mathfrak{su}(2)$. Let $\mathfrak{m} = W \oplus \mathbb{C}$. We will use the following basis of $\mathfrak{m}$.

$$x, y, ix, iy, z, iz$$

Let $\hat{x}$ be the element in the real dual basis which corresponds to x, etc. The following operators are a basis of $\mathfrak{h}$.

$$
\begin{aligned}
u &= \hat{x}\otimes ix - \hat{y}\otimes iy - i\hat{x}\otimes x + i\hat{y}\otimes y\\
k &= \hat{y}\otimes x - \hat{x}\otimes y + i\hat{y}\otimes ix - i\hat{x}\otimes iy\\
m &= \hat{x}\otimes iy + \hat{y}\otimes ix - i\hat{x}\otimes y - i\hat{y}\otimes x
\end{aligned}
$$

u, k, m satisfy the following relations.

$$
\begin{aligned}
[u,k] &= 2m\\
[u,m] &= -2k\\
[k,m] &= 2u
\end{aligned}
$$

We are interested in the case when the bracket component $\Lambda^2\mathfrak{m} \to \mathfrak{h}$ is non-zero. Let $\alpha^2 + \beta^2 + \gamma^2 = 1$. This gives the following Lie brackets on $\mathfrak{m}$.

$$
\begin{aligned}
[x,y] &= -\delta k + \beta z \\
[x,ix] &= \delta u + \gamma z \\
[x,iy] &= \delta m + \alpha z \\
[ix,y] &= -\delta m + \alpha z \\
[ix,iy] &= -\delta k - \beta z \\
[y,iy] &= -\delta u + \gamma z \\
[x,z] &= 3\delta(-\gamma ix - \beta y - \alpha iy) \\
[ix,z] &= 3\delta(\gamma x - \alpha y + \beta iy) \\
[y,z] &= 3\delta(\beta x + \alpha ix - \gamma iy) \\
[iy,z] &= 3\delta(\alpha x + \gamma y - \beta ix)
\end{aligned}
$$

If $\delta > 0$ then $\mathfrak{g} = \mathfrak{u}(3)$, and if $\delta < 0$ then $\mathfrak{g} = \mathfrak{u}(1,2)$. $\delta = 0$ is not allowed. The Nijenhuis tensor is

$$
\begin{aligned}
N_J(x,y) &= -2(\beta + \alpha i)z \\
N_J(x,z) &= 6\delta(\beta + \alpha i)x \\
N_J(y,z) &= -6\delta(\beta + \alpha i)y.
\end{aligned}
$$