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Almost Complex Homogeneous Spaces with Semi-Simple Isotropy

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Abstract

We classify the almost complex structures on homogeneous spaces $M = G/H$ of $\mathrm{Dim}_{\mathbb{R}}(M) \leq 6$ with semi-simple isotropy group H.

Contents

1 Introduction

1.1 Almost complex structures

An almost complex structure J on a manifold M is a vector bundle isomorphism $TM \to TM$ such that $J^2 = -\mathbb{1}$. This means that J induces the structure of a complex vector space on the tangent space T_xM for every point $x \in M$ (and so $\dim(M)$ must be even). The most basic example of such a structure is the following

Example: almost complex structure on $\mathbb{C}^n$. *If z^α are complex coordinates on $\mathbb{C}^n$, the holomorphic tangent space $T_x\mathbb{C}^n$ has basis ∂_{z^α}, and an almost complex structure is given by $J\partial_{z^\alpha} = i\partial_{z^\alpha}$. Alternatively one might identify the tangent space $T_x\mathbb{C}^n$ with $\mathbb{C}^n$ and let J be multiplication by i. These two approaches result in the same structure J.*

The main invariant of almost complex structures is called the Nijenhuis tensor. The Nijenhuis tensor N_J of an almost complex structure J is given by

$$N_J(X,Y) = -[X,Y] - J[JX,Y] - J[X,JY] + [JX,JY]$$

where $X, Y \in T_xM$, and the brackets on the right hand side denote Lie brackets of arbitrary extensions of X, Y to vector fields which are then evaluated at the point x. All other invariants of J arise as jets of N_J [6], and it is a major theorem by Newlander-Nirenberg that if and only if $N_J = 0$ then J is induced by local holomorphic coordinates [9]. The almost complex structure J called integrable in this case.

The Nijenhuis tensor N_J can be considered as a map $\Lambda^2_{\mathbb{C}}T_xM \to T_xM$ which is anti linear with respect to J. The case when $\dim(M) = 2\dim_{\mathbb{C}}(M) = 6$ is particularly interesting because $\dim(\Lambda^2_{\mathbb{C}}T_xM) = \dim(T_xM)$, so it is possible for N_J to be a linear isomorphism. When this is the case, we call both N_J and J non-degenerate.

Example: Calabi structure on S^6. *Let $\mathbb{O}$ denote the normed non-associative algebra of octonions, and $\Im(\mathbb{O})$ the subspace of imaginary octonions. Identify the set of imaginary octonions of unit length with the sphere S^6. The tangent space T_xS^6 for $x \in S^6$ is then identified with the orthogonal complement of x in $\Im(\mathbb{O})$, denoted $x^\perp$. Multiplication by x preserves $x^\perp$ so for each $x \in S^6$*

define $J : T_xS^6 \to T_xS^6, y \mapsto xy$. *This* J *is an almost complex structure because* $x^2 = -1$. *There exists a complex basis* x_1, x_2, x_3 *of* T_xS^6 *such that* $N_J(x_1, x_2) = x_3, N_J(x_1, x_3) = -x_2, N_J(x_2, x_3) = x_1$, *and since* N_J *is complex anti-linear this means that* $Ker(N_J) = 0$ *so* J *is non-degenerate.*

A symmetry of J is a diffeomorphism of M which leaves J invariant. The space of symmetries is a Lie group.

1.2 Homogeneous spaces

A homogeneous space for a Lie group G is a manifold M such that G has a smooth and transitive action on M. Every homogeneous space is equivalent to a coset space G/H where H is the stabilizer, also called the isotropy subgroup, of some point $x \in M$. Moreover G acts smoothly and transitively on G/H for any Lie subgroup H [10]. Therefore the classification of homogeneous spaces is equivalent to classifying Lie subgroups H of G. This can be done on the Lie algebra level by considering Lie subalgebras $\mathfrak{h}$ of $\mathfrak{g}$, and the Lie algebra $\mathfrak{h}$ has a natural representation on $T_xM = \mathfrak{m} = \mathfrak{g}/\mathfrak{h}$ called the isotropy representation. The homogeneous space M has a G-invariant almost complex structure J if and only if the isotropy representation is a complex representation. If this is the case, then G is contained in the symmetry group of J.

1.3 Motivation and goals

According to the Erlangen program of F.Klein, a geometry is specified by a transitive Lie group action [3]. Though this was generalized and relaxed by E.Cartan, we would like to approach almost complex geometry from this classical viewpoint. The problem is that not many non-integrable almost complex manifolds are known which have transitive symmetry group. In the literature the most well known example is the Calabi structure on S^6 [2] [5] [12], and except for this the non-integrable examples are almost all left invariant on Lie groups. Thus it is important to find examples of highly symmetric almost complex structures. It was shown in [8] that if J is non-degenerate, the maximal symmetry group is 14d, and the almost complex structure which achieves this is unique in the sense that all such structures J are locally equivalent to the Calabi structure. We call the non-degenerate structure J, and also its symmetry group, sub-maximal when the symmetry group is of the second highest possible dimension. The sub-maximal symmetry group was expected to be 8d, but in this text we exhibit many examples of non-degenerate J with 9d symmetry algebra. Our means of producing such symmetric structures is to provide a complete classification of almost complex homogeneous spaces M with semi-simple isotropy group H of $\dim(M) \leq 6$.

1.4 Previous work

All homogeneous spaces with irreducible isotropy representation were classified by J.Wolf in 1968 [11]. A portion of this work is devoted to almost complex homogeneous spaces. However this is purely algebraic, and Wolf does not provide neither geometric information about integrability nor concise examples in his classification. Moreover many interesting homogeneous spaces are not isotropy irreducible, and thus not contained in his list.

1.5 Methods

The direct approach to finding almost complex homogeneous spaces would be to first classify Lie algebras $\mathfrak{g}$ and also their subalgebras $\mathfrak{h}$, and then checking which geometric structures are preserved by the isotropy representation of $\mathfrak{h}$. The advantage of this approach would be that it is complete, any almost complex homogeneous space would show up on the list. However brute force classification of Lie algebras has been performed only up to dimension 6 [1] and improving upon this would require a disproportionally great effort compared to our goals, especially considering that we already know about an interesting example with 14d symmetry algebra (the Calabi structure). In addition, there are homogeneous spaces $M = G/H$ with unrestricted $\dim(G)$, so this approach works with a priori unbounded data. See [4] for examples of 2d homogeneous spaces with arbitrary $\dim(G)$. We will therefore explore an another, more restrictive but also more realistic, approach.

Given a Lie algebra $\mathfrak{h}$ and representation $\mathfrak{m}$, define the Lie bracket on $\mathfrak{h}$ to be the given one and let the bracket between $\mathfrak{h}, \mathfrak{m}$ be given by the module structure,

$$[h, m] = hm$$

for $h \in \mathfrak{h}$ and $m \in \mathfrak{m}$. We may then look for maps

$$[,] : \Lambda^2\mathfrak{m} \to \mathfrak{h} \oplus \mathfrak{m}$$

such that the Jacobi identity is satisfied. This is a Lie algebra structure on $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. We get the Jacobi identity between elements h_1, h_2, m for free - it is equivalent to saying that $\mathfrak{m}$ is an $\mathfrak{h}$-module. The Jacobi identity between elements h, m_1, m_2 however imposes a restriction

$$[h, [m_1, m_2]] + [m_2, [h, m_1]] + [m_1, [m_2, h]] = 0$$

which rewrites as

$$[hm_1, m_2] + [m_1, hm_2] = h[m_1, m_2]$$

This means that the map $[,]$ must be equivariant with respect to the natural $\mathfrak{h}$-module structure on $\Lambda^2\mathfrak{m}$ (we consider $\mathfrak{h}$ as a module over itself). One way to find such equivariant maps is to decompose both $\mathfrak{m}$ and $\Lambda^2\mathfrak{m}$ into irreducible submodules, and to make this feasible we will restrict our considerations to semi-simple Lie algebras $\mathfrak{h}$. See [10] for details about representations of semi-simple Lie algebras.

Our plan of attack is now to systematically treat each pair $(\mathfrak{h}, \mathfrak{m})$ where $\mathfrak{h}$ is semi-simple and $\mathfrak{m}$ is a representation with an $\mathfrak{h}$-invariant complex structure and $\dim(\mathfrak{m}) \leq 6$. The set of such pairs is finite. We will decompose $\Lambda^2\mathfrak{m}$ to look for submodules which either appear in $\mathfrak{m}$ or the adjoint representation of

$\mathfrak{h}$. Provided that such submodules exist, we parameterize the equivariant maps and compute the only Jacobi identity left which is between 3 elements of $\mathfrak{m}$. This yields equations for the parameters of the maps, and the solution sets are the parameters for the desired Lie algebra structures on $\mathfrak{g}$.

Note that taking the bracket on $\mathfrak{m}$ to be the zero map is always possible and it satisfies the Jacobi identity, but this is not interesting because it corresponds to a vector space $\mathbb{C}^n$ with the given linear action of H which is the Lie group of $\mathfrak{h}$. The complex structure is the standard one, which is integrable. We will refer to the zero map as the flat case for this reason, and they are excluded from our table of results.

Each Lie algebra $\mathfrak{g}$ corresponds to at least one Lie group G such that $\mathfrak{h}$ corresponds to a Lie subgroup H. We may thus create the coset space $M = G/H$, and as a homogeneous space M has a G-invariant almost complex structure J. This J is defined by left translation of the complex structure on $\mathfrak{m} = T_eG/T_eH = \mathfrak{g}/\mathfrak{h}$ by G, and the left translation is well defined because J commutes with $\mathfrak{h}$.

1.6 Computing the Nijenhuis tensor

It is easy to compute the Nijenhuis tensor of J in the case when $\mathfrak{m}$ is a Lie subalgebra of $\mathfrak{g}$. For elements $X, Y \in \mathfrak{m}$, we may simply use the formula

$$N_J(X,Y) = -[X,Y] - J[JX,Y] - J[X,JY] + [JX,JY]$$

where the brackets are the ones we defined. This corresponds to extending X, JX, Y, JY to their respective left invariant vector fields on M and taking the commutator, which is then evaluated at $\mathfrak{m}$. Since M is homogeneous, N_J at any other point is the same. If $\mathfrak{m}$ is not a subalgebra, ie. the bracket has some $\mathfrak{h}$-component, we may use the same formula, but now projecting to $\mathfrak{m}$ after taking each bracket.

2 Table of results

Theorem 1. *If $M = G/H$ is a homogeneous space of dim ≤ 6 equipped with a G-invariant almost complex structure J and the isotropy group H is semisimple, then $\mathfrak{g}$ is isomorphic to one of the Lie algebras in the following section and the isomorphism preserves both isotropy algebra $\mathfrak{h}$ and complex structure.*

Note that for $\mathfrak{h} = \mathfrak{su}(2,1), \mathfrak{sl}_3, \mathfrak{sl}_3(\mathbb{C})$ only the flat case is realized so these are exempt from the tables. The complex dimension of the kernel of the Nijenhuis tensor is given in the "Notes" column, with the notation DG_k meaning a $k-$dim kernel. For more information about this notation and the possible types of Nijenhuis tensors see [7].

2.1 $\mathfrak{h} = \mathfrak{su}(2)$

2.1.1 $\mathfrak{m} = W$, the tautological representation

Only the flat case of $\mathbb{C}^2$ with the standard $\mathfrak{su}(2)$-action is realized.

2.1.2 $\mathfrak{m} = Ad^{\mathbb{C}} = Ad \oplus Ad$

In this section u, k, m are the generators of $\mathfrak{su}(2) = \mathfrak{h}$. They satisfy

$$[u,k] = 2m$$
$$[u,m] = -2k$$
$$[k,m] = 2u$$

We use the real basis $u_1 = (u,0), k_1 = (k,0), m_1 = (m,0), u_2 = (0,u), k_2 = (0,k), m_2 = (0,m)$ for $\mathfrak{m}$. The action of $\mathfrak{h}$ is the obvious one. The complex structure J is given by

$$J(h,0) = (-rh, th)$$

for some real r and real $t \neq 0$.

$[,]$	N_J	Notes
$[u_1,k_1] = 2m_1$ $[u_1,m_1] = -2k_1$ $[k_1,m_1] = 2u_1$ $[u_2,k_2] = 2m_2$ $[u_2,m_2] = -2k_2$ $[k_2,m_2] = 2u_2$	$N_J(u_1,k_1) = -2(r^2+1)m_1 + 2(t^2+2rt)m_2$ $N_J(u_1,m_1) = 2(r^2+1)k_1 - 2(t^2+2rt)k_2$ $N_J(k_1,m_1) = -2(r^2+1)u_1 + 2(t^2+2rt)u_2$	$\mathfrak{m} \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ Non-degenerate $\mathfrak{h}$ is the diagonal subalgebra in $\mathfrak{g} = \mathfrak{su}(2)^3$.
$[u_1,k_1] = 2m_1$ $[u_1,m_1] = -2k_1$ $[k_1,m_1] = 2u_1$ $[u_2,k_2] = -2m_1$ $[u_2,m_2] = 2k_1$ $[k_2,m_2] = -2u_1$ $[u_1,k_2] = 2m_2$ $[u_1,m_2] = -2k_2$ $[k_1,m_2] = 2u_2$ $[u_2,k_1] = 2m_2$ $[u_2,m_1] = -2k_2$ $[k_2,m_1] = 2u_2$	$N_J(u_1,k_1) = 2(r^2+1-t^2)m_1 - 4trm_2$ $N_J(u_1,m_1) = -2(r^2+1-t^2)k_1 + 4trk_2$ $N_J(k_1,m_1) = 2(r^2+1-t^2)u_1 - 4tru_2$	$\mathfrak{m} \simeq \mathfrak{sl}_2(\mathbb{C})$ N_J vanishes for $r = 0, t = \pm 1$, non-degenerate else $\mathfrak{h}$ is a diagonal subalgebra in $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{sl}_2(\mathbb{C})$.
$[u_1,k_1] = 2m_1$ $[u_1,m_1] = -2k_1$ $[k_1,m_1] = 2u_1$	$N_J(u_1,k_1) = -2(r^2+1)m_1 + 4rtm_2$ $N_J(u_1,m_1) = 2(r^2+1)k_1 - 4rtk_2$ $N_J(k_1,m_1) = -2(r^2+1)u_1 + 4rtu_2$	$\mathfrak{m} \simeq \mathfrak{su}(2) \oplus \mathbb{R}^3$ Non-degenerate
$[u_1,k_1] = 2m_2$ $[u_1,m_1] = -2k_2$ $[k_1,m_1] = 2u_2$	$N_J(u_1,k_1) = -\frac{4(r^3+r)}{t}m_1 + 2(3r^2-1)m_2$ $N_J(u_1,m_1) = \frac{4(r^3+r)}{t}k_1 - 2(3r^2-1)k_2$ $N_J(k_1,m_1) = -\frac{4(r^3+r)}{t}u_1 + 2(3r^2-1)u_2$	Non-degenerate

2.1.3 $\mathfrak{m} = W \oplus \mathbb{C}$

We use the real basis x, ix, y, iy for W and z, iz for $\mathbb{C}$. A basis of $\mathfrak{h}$ is given by

$$u = \hat{x} \otimes ix - \hat{y} \otimes iy - i\hat{x} \otimes x + i\hat{y} \otimes y$$
$$k = \hat{y} \otimes x - \hat{x} \otimes y + i\hat{y} \otimes ix - i\hat{x} \otimes iy$$
$$m = \hat{x} \otimes iy + \hat{y} \otimes ix - i\hat{x} \otimes y - i\hat{y} \otimes x$$

where $\hat{x}, i\hat{x}, \hat{y}, i\hat{y}$ is the dual basis. The complex structure J acts in the obvious manner.

$[,]$	N_J	Notes
$[x,ix]=[y,iy]=\lambda_1 z$ $[x,y]=-[ix,iy]=\lambda_2 z$ $[ix,y]=[x,iy]=\lambda_3 z$ $[x,iz]=\frac{\delta}{2}x+\beta(\lambda_1 ix+\lambda_2 y+\lambda_3 iy)$ $[ix,iz]=\frac{\delta}{2}ix-\beta(\lambda_1 x-\lambda_3 y+\lambda_2 iy)$ $[y,iz]=\frac{\delta}{2}y-\beta(\lambda_2 x+\lambda_3 ix-\lambda_1 iy)$ $[iy,iz]=\frac{\delta}{2}iy-\beta(\lambda_3 x-\lambda_2 ix+\lambda_1 y)$ $[z,iz]=\delta z$	$N_J(x,y)=-2\lambda_2 z-2\lambda_3 iz$ $N_J(x,z)=2\beta(\lambda_3 y-\lambda_2 iy)$ $N_J(y,z)=2\beta(\lambda_3 x-\lambda_2 ix)$	N_J vanishes for $\lambda_2=\lambda_3$=0, DG_2 for $\beta=0$, otherwise non-degenerate. $\delta=0$ or $\delta=1$
$[x,ix]=[y,iy]=\lambda_1 z+L_1 iz$ $[x,y]=-[ix,iy]=\lambda_2 z+L_2 iz$ $[ix,y]=[x,iy]=\lambda_3 z+L_3 iz$	$N_J(x,y)=2(L_3-\lambda_2-iL_2-i\lambda_3)z$ $N_J(x,z)=0$ $N_J(y,z)=0$	N_J vanishes for $\lambda_2=\lambda_3$=0, DG_2 else.
$[x,z]=\beta x$ $[ix,z]=\beta ix$ $[y,z]=\beta y$ $[iy,z]=\beta iy$ $[x,iz]=\alpha x+(\lambda_1 ix+\lambda_2 y+\lambda_3 iy)$ $[ix,iz]=\alpha ix-(\lambda_1 x-\lambda_3 y+\lambda_2 iy)$ $[y,iz]=\alpha y-(\lambda_2 x+\lambda_3 ix-\lambda_1 iy)$ $[iy,iz]=\alpha iy-(\lambda_3 x-\lambda_2 ix+\lambda_1 y)$	$N_J(x,y)=0$ $N_J(x,z)=2(\lambda_3 y-\lambda_2 iy)$ $N_J(y,z)=2(\lambda_3 x-\lambda_2 ix)$	semi-direct product $W\rtimes\mathbb{C}$ N_J vanishes for $\lambda_2=\lambda_3$=0, DG_1 else.
$[x,iz]=\alpha x+(\lambda_1 ix+\lambda_2 y+\lambda_3 iy)$ $[ix,iz]=\alpha ix-(\lambda_1 x-\lambda_3 y+\lambda_2 iy)$ $[y,iz]=\alpha y-(\lambda_2 x+\lambda_3 ix-\lambda_1 iy)$ $[iy,iz]=\alpha iy-(\lambda_3 x-\lambda_2 ix+\lambda_1 y)$ $[z,iz]=z$	$N_J(x,y)=0$ $N_J(x,z)=2(\lambda_3 y-\lambda_2 iy)$ $N_J(y,z)=2(\lambda_3 x-\lambda_2 ix)$	semi-direct product $W\rtimes\mathbb{C}$ N_J vanishes for $\lambda_2=\lambda_3$=0, DG_1 else.
$[x,ix]=\beta(u+3z)$ $[x,y]=-\beta k$ $[x,iy]=\beta m$ $[ix,y]=-\beta m$ $[ix,iy]=-\beta k$ $[y,iy]=\beta(3z-u)$ $[z,x]=ix$ $[z,ix]=-x$ $[z,y]=iy$ $[z,iy]=-y$	$N_J=0$	

2.2 $\mathfrak{h}=\mathfrak{su}(1,1)$

2.2.1 $\mathfrak{m}=V^{\mathbb{C}}$, the tautological representation

Only the flat case of $\mathbb{C}^2$ with the standard $\mathfrak{su}(1,1)$-action is realized.

2.2.2 $\mathfrak{m}=Ad^{\mathbb{C}}=Ad\oplus Ad$

In this section A,B,C are the generators of $\mathfrak{su}(1,1)=\mathfrak{h}$. They satisfy

$$[A, B] = 2C$$
$$[A, C] = -2B$$
$$[B, C] = -2A$$

We use the real basis $A_1 = (A, 0), B_1 = (B, 0), C_1 = (C, 0), A_2 = (0, A), B_2 = (0, B), C_2 = (0, C)$ of $\mathfrak{m}$. The complex structure J is given by

$$J(h, 0) = (-rh, th)$$

for some real r and real $t \neq 0$.

$[,]$	N_J	Notes
$[A_1, B_1] = 2C_1$ $[A_1, C_1] = -2B_1$ $[B_1, C_1] = -2A_1$ $[A_2, B_2] = 2C_2$ $[A_2, C_2] = -2B_2$ $[B_2, C_2] = -2A_2$	$N_J(A_1, B_1) = -2(r^2 + 1)C_1 + 2(t^2 + 2rt)C_2$ $N_J(A_1, C_1) = 2(r^2 + 1)B_1 - 2(t^2 + 2rt)B_2$ $N_J(B_1, C_1) = 2(r^2 + 1)A_1 - 2(t^2 + 2rt)A_2$	$\mathfrak{m} \simeq \mathfrak{su}(1, 1) \oplus \mathfrak{su}(1, 1)$ Non-degenerate $\mathfrak{h}$ is the diagonal subalgebra in $\mathfrak{g} = \mathfrak{su}(1, 1)^3$.
$[A_1, B_1] = 2C_1$ $[A_1, C_1] = -2B_1$ $[B_1, C_1] = -2A_1$ $[A_2, B_2] = -2C_1$ $[A_2, C_2] = 2B_1$ $[B_2, C_2] = 2A_1$ $[A_1, B_2] = 2C_2$ $[A_1, C_2] = -2B_2$ $[B_1, C_2] = -2A_2$ $[A_2, B_1] = 2C_2$ $[A_2, C_1] = -2B_2$ $[B_2, C_1] = -2A_2$	$N_J(A_1, B_1) = 2(r^2 + 1 - t^2)C_1 - 4trC_2$ $N_J(A_1, C_1) = -2(r^2 + 1 - t^2)B_1 + 4trB_2$ $N_J(B_1, C_1) = -2(r^2 + 1 - t^2)A_1 + 4trA_2$	$\mathfrak{m} \simeq \mathfrak{sl}_2(\mathbb{C})$ N_J vanishes for $r = 0, t = \pm 1$, non-degenerate else $\mathfrak{h}$ is the diagonal subalgebra in $\mathfrak{g} = \mathfrak{su}(1, 1) \oplus \mathfrak{sl}_2(\mathbb{C})$.
$[A_1, B_1] = 2C_1$ $[A_1, C_1] = -2B_1$ $[B_1, C_1] = -2A_1$	$N_J(A_1, B_1) = -2(r^2 + 1)C_1 + 4rtC_2$ $N_J(A_1, C_1) = 2(r^2 + 1)B_1 - 4rtB_2$ $N_J(B_1, C_1) = 2(r^2 + 1)A_1 - 4rtA_2$	$\mathfrak{m} \simeq \mathfrak{su}(1, 1) \oplus \mathbb{R}^3$ Non-degenerate
$[A_1, B_1] = 2C_2$ $[A_1, C_1] = -2B_2$ $[B_1, C_1] = -2A_2$	$N_J(A_1, B_1) = -\frac{4(r^3+r)}{t}C_1 + 2(3r^2 - 1)C_2$ $N_J(A_1, C_1) = \frac{4(r^3+r)}{t}B_1 - 2(3r^2 - 1)B_2$ $N_J(B_1, C_1) = \frac{4(r^3+r)}{t}A_1 - 2(3r^2 - 1)A_2$	Non-degenerate

2.2.3 $\mathfrak{m} = V^{\mathbb{C}} \oplus \mathbb{C}$

We use a real basis x, ix, y, iy for $V^{\mathbb{C}}$ such that x, y and ix, iy are submodules, and z, iz for $\mathbb{C}$. A basis of $\mathfrak{h}$ is given by

$$A = \hat{y} \otimes x - \hat{x} \otimes y + i\hat{y} \otimes ix - i\hat{x} \otimes iy$$
$$B = \hat{x} \otimes y + \hat{y} \otimes x + i\hat{x} \otimes iy + i\hat{y} \otimes ix$$
$$C = \hat{x} \otimes x - \hat{y} \otimes y + i\hat{x} \otimes ix - i\hat{y} \otimes iy$$

Here $\hat{x}, \hat{y}, i\hat{x}, i\hat{y}$ means the real dual basis. The complex structure J acts in the obvious manner.

$[,]$	N_J	Notes
$[x,y] = \lambda_1 z$ $[x,iy] = [ix,y] = \lambda_2 z$ $[ix,iy] = \lambda_3 z$ $[x,iz] = r(\lambda_2 x - \lambda_1 ix)$ $[ix,iz] = r(\lambda_3 x - \lambda_2 ix)$ $[y,iz] = r(\lambda_2 y - \lambda_1 iy)$ $[iy,iz] = r(\lambda_3 y - \lambda_2 iy)$	$N_J(x,y) = (\lambda_3 - \lambda_1)z - 2\lambda_2 iz$ $N_J(x,z) = r(\lambda_3 - \lambda_1)x - 2r\lambda_2 ix$ $N_J(y,z) = r(\lambda_3 - \lambda_1)y - 2r\lambda_2 iy$	$\lambda_1\lambda_3 \neq \lambda_2^2$, N_J vanishes if $\lambda_1 = \lambda_3, \lambda_2 = 0$, non-degenerate else.
$[x,y] = -\beta^2 z$ $[x,iy] = [ix,y] = \alpha\beta z$ $[ix,iy] = -\alpha^2 z$ $[x,iz] = k(\alpha x + \beta ix)$ $[ix,iz] = l(\alpha x + \beta ix)$ $[y,iz] = k(\alpha y + \beta iy)$ $[iy,iz] = l(\alpha y + \beta iy)$	$N_J(x,y) = (\beta^2 - \alpha^2)z - 2\alpha\beta iz$ $N_J(x,z) = (l - ik)(\alpha x + \beta ix)$ $N_J(y,z) = (l - ik)(\alpha y + \beta iy)$	Non-degenerate
$[x,y] = -\beta^2(\gamma z + iz)$ $[x,iy] = \alpha\beta(\gamma z + iz)$ $[ix,y] = \alpha\beta(\gamma z + iz)$ $[ix,iy] = -\alpha^2(\gamma z + iz)$ $[x,iz] = r(\beta\alpha x + \beta^2 ix)$ $[ix,iz] = r(-\alpha^2 x - \alpha\beta ix)$ $[y,iz] = r(\beta\alpha y + \beta^2 iy)$ $[iy,iz] = r(-\alpha^2 y - \alpha\beta iy)$	$N_J(x,y) = -(\alpha + i\beta)^2(i + \gamma)z$ $N_J(x,z) = r(-\alpha - i\beta)(\alpha x + \beta ix)$ $N_J(y,z) = r(-\alpha - i\beta)(\alpha y + \beta iy)$	Non-degenerate
$[x,z] = \gamma x$ $[ix,z] = \gamma ix$ $[y,z] = \gamma y$ $[iy,z] = \gamma iy$ $[x,iz] = \lambda_1 x + \lambda_2 ix$ $[ix,iz] = \lambda_3 x + \lambda_4 ix$ $[y,iz] = \lambda_1 y + \lambda_2 iy$ $[iy,iz] = \lambda_3 y + \lambda_4 iy$	$N_J(x,y) = 0$ $N_J(x,z) = (\lambda_2 + \lambda_3 + i\lambda_4 - i\lambda_1)x$ $N_J(y,z) = (\lambda_2 + \lambda_3 + i\lambda_4 - i\lambda_1)y$	Semi-direct product $V^{\mathbb{C}} \rtimes \mathbb{C}$ DG_1 unless $\lambda_2 = -\lambda_3, \lambda_1 = \lambda_4$ in which case $N_J = 0$.
$[x,y] = \lambda_1 z + L_1 iz$ $[ix,iy] = \lambda_3 z + L_3 iz$ $[x,iy] = \lambda_2 z + L_2 iz$ $[ix,y] = \lambda_2 z + L_2 iz$	$N_J(x,y) = (\lambda_3 - \lambda_1 + 2L_2)z + (L_3 - L_1 - 2\lambda_2)iz$ $N_J(x,z) = 0$ $N_J(y,z) = 0$	DG_2 but $N_J = 0$ for $\lambda_1 - \lambda_3 = 2L_2$, $L_3 + L_1 = 2\lambda_2$

$[,]$	N_J	Notes
$[x,y]=\lambda_1 z$ $[x,iy]=\lambda_2 z$ $[ix,y]=\lambda_2 z$ $[ix,iy]=\lambda_3 z$ $[x,iz]=\frac{1}{2}x+r(\lambda_2 x-\lambda_1 ix)$ $[ix,iz]=\frac{1}{2}ix+r(\lambda_3 x-\lambda_2 ix)$ $[y,iz]=\frac{1}{2}y+r(\lambda_2 y-\lambda_1 iy)$ $[iy,iz]=\frac{1}{2}iy+r(\lambda_3 y-\lambda_2 iy)$ $[z,iz]=z$	$N_J(x,y)=(\lambda_3-\lambda_1)z-2\lambda_2 iz$ $N_J(x,z)=r(\lambda_3-\lambda_1)x-2\lambda_2 ix$ $N_J(y,z)=r(\lambda_3-\lambda_1)y-2\lambda_2 iy$	$\lambda_1\lambda_3\neq\lambda_2^2$, non-degenerate unless $\lambda_1=\lambda_3,\lambda_2=0$, which gives $N_J=0$.
$[x,y]=-\beta^2 z$ $[x,iy]=\alpha\beta z$ $[ix,y]=\alpha\beta z$ $[ix,iy]=-\alpha^2 z$ $[x,iz]=\frac{1}{2}x+k(\alpha x+\beta ix)$ $[ix,iz]=\frac{1}{2}ix+l(\alpha x+\beta ix)$ $[y,iz]=\frac{1}{2}y+k(\alpha y+\beta iy)$ $[iy,iz]=\frac{1}{2}iy+l(\alpha y+\beta iy)$ $[z,iz]=z$	$N_J(x,y)=-(\alpha+i\beta)^2 z$ $N_J(x,z)=(-i\alpha+\beta)(k+il)x$ $N_J(y,z)=(-i\alpha+\beta)(k+il)y$	Non-degenerate
$[x,y]=-\beta^2 z$ $[x,iy]=\alpha\beta z$ $[ix,y]=\alpha\beta z$ $[ix,iy]=-\alpha^2 z$ $[x,z]=r(\alpha\beta x+\beta^2 ix)$ $[ix,z]=r(-\alpha^2 x-\alpha\beta ix)$ $[y,z]=r(\alpha\beta y+\beta^2 iy)$ $[iy,z]=r(-\alpha^2 y-\alpha\beta iy)$ $[x,iz]=\frac{1}{2}x+k(\alpha x+\beta ix)$ $[ix,iz]=\frac{1}{2}ix+l(\alpha x+\beta ix)$ $[y,iz]=\frac{1}{2}y+k(\alpha y+\beta iy)$ $[iy,iz]=\frac{1}{2}iy+l(\alpha y+\beta iy)$ $[z,iz]=z$	$N_J(x,y)=-(\alpha+i\beta)^2 z$ $N_J(x,z)=(\alpha+i\beta)(l-i(k-r(a-b))x$ $N_J(x,z)=(\alpha+i\beta)(l-i(k-r(a-b))y$	$k\alpha+l\beta=1$ non-degenerate unless $l=0,k=r(a-b)$, which gives DG_2
$[x,z]=r(\alpha x-\frac{\alpha^2}{\beta}ix)$ $[ix,z]=r(\beta x-\alpha ix)$ $[y,z]=r(\alpha y-\frac{\alpha^2}{\beta}iy)$ $[iy,z]=r(\beta y-\alpha iy)$ $[x,iz]=(\gamma+\frac{1}{2})x-\frac{\alpha}{\beta}ix$ $[ix,iz]=(\gamma-\frac{1}{2})ix$ $[y,iz]=(\gamma+\frac{1}{2})y-\frac{\alpha}{\beta}iy$ $[iy,iz]=(\gamma-\frac{1}{2})iy$ $[z,iz]=z$	$N_J(x,y)=0$ $N_J(x,z)=(\frac{\alpha}{\beta}+i(1+2r\alpha))x$ $N_J(x,z)=(\frac{\alpha}{\beta}+i(1+2r\alpha))y$	$\beta\neq 0$ Semi-direct product $V^{\mathbb{C}}\rtimes\mathbb{C}$ DG_1

$[,]$	N_J	Notes
$[x,iz]=\lambda_1 x+\lambda_2 ix$ $[ix,iz]=\lambda_3 x+\lambda_4 ix$ $[y,iz]=\lambda_1 y+\lambda_2 iy$ $[iy,iz]=\lambda_3 y+\lambda_4 iy$ $[z,iz]=z$	$N_J(x,y)=0$ $N_J(x,z)=-(\lambda_2+\lambda_3)x+(\lambda_1-\lambda_4)ix$ $N_J(y,z)=-(\lambda_2+\lambda_3)y+(\lambda_1-\lambda_4)iy$	Semi-direct product $V^{\mathbb{C}}\rtimes\mathbb{C}$ DG_1 unless $\lambda_2=-\lambda_3,\lambda_1=\lambda_4$ which gives N_J=0
$[x,y]=-3\alpha z$ $[ix,iy]=-3\alpha z$ $[x,ix]=\alpha(A+B)$ $[y,iy]=\alpha(A-B)$ $[x,iy]=-\alpha C$ $[ix,y]=\alpha C$ $[z,x]=ix$ $[z,ix]=-x$ $[z,y]=iy$ $[z,iy]=-y$	$N_J=0$	

2.3 $\mathfrak{h}=\mathfrak{sl}_2(\mathbb{C})$

2.3.1 $\mathfrak{m}=W$, the tautological representation

Only the flat case is realized.

2.3.2 $\mathfrak{m}=Ad$

We use the basis u,iu,k,ik,m,im of $\mathfrak{m}$, which are copies of the basis of $\mathfrak{h}$ satisfying

$$\begin{aligned}[u,k]&=2m\\ [u,m]&=-2k\\ [k,m]&=2u\end{aligned}$$

and the brackets are complex linear. The complex structure on $\mathfrak{m}$ is the one inherited from the complex Lie algebra structure, and it acts in the obvious manner.

$[,]$	N_J	Notes
$[u,k]=-[iu,ik]=2m$ $[u,m]=-[iu,im]=-2k$ $[k,m]=-[ik,im]=2u$ $[u,ik]=[iu,k]=2im$ $[u,im]=[iu,m]=-2ik$ $[k,im]=[ik,m]=2iu$	$N_J=0$	$\mathfrak{g}=\mathfrak{sl}_2(\mathbb{C})\oplus\mathfrak{sl}_2(\mathbb{C})$

2.3.3 $\mathfrak{m}=W\oplus\mathbb{C}$

We use the basis x,ix,y,iy with x,y the standard complex basis of $\mathbb{C}^2$.

$[,]$	N_J	Notes
$[x,z]=\beta x$ $[ix,z]=\beta ix$ $[y,z]=\beta y$ $[iy,z]=\beta iy$ $[x,iz]=\alpha x+\gamma ix$ $[ix,iz]=\alpha ix-\gamma x$ $[y,iz]=\alpha y+\gamma iy$ $[iy,iz]=\alpha iy-\gamma y$	$N_J=0$	
$[x,iz]=\alpha x+\gamma ix$ $[ix,iz]=\alpha ix-\gamma x$ $[y,iz]=\alpha y+\gamma iy$ $[iy,iz]=\alpha iy-\gamma y$ $[z,iz]=z$	$N_J=0$	
$[x,y]=\alpha z+\beta iz$ $[ix,iy]=-\alpha z-\beta iz$ $[x,iy]=\gamma z+\eta iz$ $[ix,y]=\gamma z+\eta iz$	$N_J(x,y)=-2(\alpha+\eta)z+2(\gamma-\beta)iz$ $N_J(x,z)=0$ $N_J(y,z)=0$	DG_2 unless $\alpha=-\eta,\gamma=\beta$ which gives $N_J=0$.

2.4 $\mathfrak{h}=\mathfrak{su}(3)$

2.4.1 $\mathfrak{m}=W$, the tautological representation

We use the basis $x_1, ix_1, x_2, ix_2, x_3, ix_3$ of $\mathfrak{m}$. A basis of $\mathfrak{su}(3)=\mathfrak{h}$ is given by

$$\begin{aligned}
u_{i11} &= (3\hat{x}_1\otimes ix_1 - 3i\hat{x}_1\otimes x_1 - J)\\
u_{i22} &= (3\hat{x}_2\otimes ix_2 - 3i\hat{x}_2\otimes x_2 - J)\\
u_{12} &= \hat{x}_1\otimes x_2 - \hat{x}_2\otimes x_1 + i\hat{x}_1\otimes ix_2 - i\hat{x}_2\otimes ix_1\\
u_{13} &= \hat{x}_1\otimes x_3 - \hat{x}_3\otimes x_1 + i\hat{x}_1\otimes ix_3 - i\hat{x}_3\otimes ix_1\\
u_{23} &= \hat{x}_2\otimes x_3 - \hat{x}_3\otimes x_2 + i\hat{x}_2\otimes ix_3 - i\hat{x}_3\otimes ix_2\\
u_{i12} &= \hat{x}_1\otimes ix_2 + \hat{x}_2\otimes ix_1 - i\hat{x}_1\otimes x_2 - i\hat{x}_2\otimes x_1\\
u_{i13} &= \hat{x}_1\otimes ix_3 + \hat{x}_3\otimes ix_1 - i\hat{x}_1\otimes x_3 - i\hat{x}_3\otimes x_1\\
u_{i23} &= \hat{x}_2\otimes ix_3 + \hat{x}_3\otimes ix_2 - i\hat{x}_2\otimes x_3 - i\hat{x}_3\otimes x_2
\end{aligned}$$

Here $\hat{x}_1, i\hat{x}_1, \hat{x}_2, i\hat{x}_2, \hat{x}_3, i\hat{x}_3$ denotes the dual basis. The complex structure J acts in the obvious manner.

$[,]$	N_J	Notes
$[x_1, ix_1] = 2u_{i11}$ $[x_2, ix_2] = 2u_{i22}$ $[x_3, ix_3] = -2(u_{i11} + u_{i22})$ $[x_1, x_2] = 3u_{12} + 2x_3$ $[x_1, x_3] = 3u_{13} - 2x_2$ $[x_2, x_3] = 3u_{23} + 2x_1$ $[ix_1, x_2] = -3u_{i12} - 2ix_3$ $[x_1, ix_2] = 3u_{i12} - 2ix_3$ $[ix_1, x_3] = -3u_{i13} + 2ix_2$ $[x_1, ix_3] = 3u_{i13} + 2ix_2$ $[ix_2, x_3] = -3u_{i23} - 2ix_1$ $[x_2, ix_3] = 3u_{i23} - 2ix_1$ $[ix_1, ix_2] = 3u_{12} - 2x_3$ $[ix_1, ix_3] = 3u_{13} + 2x_2$ $[ix_2, ix_3] = 3u_{23} - 2x_1$	$N_J(x_1, x_2) = -8x_3$ $N_J(x_1, x_3) = 8x_2$ $N_J(x_2, x_3) = -8x_1$	$\mathfrak{g} = \mathfrak{g}_2$, the compact form of the exceptional Lie algebra. Non-degenerate.

3 Possible Isotropy Algebras

We are only interested in those representations on which the isotropy subgroup acts effectively, ie. we exclude sub-algebras acting trivially. To ensure an almost complex structure on the homogenous space we consider only real modules with compatible complex structures. Moreover we can restrict ourselves to those isotropy algebras $\mathfrak{h}$ which have such modules $\mathfrak{m}$ of dimension $\mathrm{Dim}(\mathfrak{m}) \leq 6$. These algebras have semi-simple part $\mathfrak{su}(2)$, $\mathfrak{su}(1,1) \simeq \mathfrak{sl}_2$, $\mathfrak{sl}_2(\mathbb{C})$, $\mathfrak{su}(2,1)$, $\mathfrak{su}(3)$, $\mathfrak{sl}_3$ or $\mathfrak{sl}_3(\mathbb{C})$. Sometimes we will also augment these semi-simple algebras by allowing a one dimensional center $\mathbb{R} \subset \mathfrak{h}$.

3.1 Modules

3.1.1 $\mathfrak{su}(2)$

The 4 dimensional tautological representation W has a complex structure. It is irreducible over the reals, but not absolutely irreducible as its complexification splits over $\mathbb{C}$ into two submodules each isomorphic to W. W can also be identified as the spinor representation of $\mathfrak{so}(3)$.

We may complexify the adjoint representation Ad. This gives us a 6d module $Ad^{\mathbb{C}}$, which splits into $Ad \oplus Ad$ over the reals. This module is isomorphic to $S^2_{\mathbb{C}}W$, another obvious candidate.

The last module we consider is $W \oplus \mathbb{C}$, where $\mathbb{C}$ is considered as a 2d trivial module.

3.1.2 $\mathfrak{su}(1,1)$

In contrast with the previous case the 4d tautological representation is not irreducible over the reals. It can be identified with $V^{\mathbb{C}} \simeq V \oplus V$, where V is the 2d irreducible representation of $\mathfrak{sl}_2$.

We may add a trivial 2d module to this to obtain $V^{\mathbb{C}} \oplus \mathbb{C}$.

The adjoint representation Ad can be identified with S^2V, the symmetric tensor square of V, and this module can be complexified to obtain $S^2V^{\mathbb{C}}$ which splits into $S^2V \oplus S^2V$ over the reals.

3.1.3 $\mathfrak{sl}_2(\mathbb{C})$

The representations are the same as for $\mathfrak{su}(2)$, except Ad is irreducible, complex and 6 dimensional in this case.

3.1.4 $\mathfrak{sl}_3$

The tautological representation V is of real dimension 3, so the complexification $V^{\mathbb{C}}$ is 6 dimensional. It splits into $V^{\mathbb{C}} \simeq V \oplus V$ over the reals. The complexification of the dual representation V^* is also eligible.

3.1.5 $\mathfrak{su}(2,1)$

This algebras have a natural complex structure on the tautological representation, which is of real dimension 6 and irreducible. The dual representation is equivalent to the tautological rep. over the reals, but not over complex numbers.

3.1.6 $\mathfrak{su}(3)$

We have a natural complex structure on the tautological representation, which is of real dimension 6 and irreducible. The dual representation is equivalent to the tautological rep. over the reals, but not over complex numbers.

3.1.7 $\mathfrak{sl}_3(\mathbb{C})$

We have a natural action on the complexification of the tautological representation V of $\mathfrak{sl}_3$ and its dual V^*.they are irreducible over the reals in this case because we enlarged the algebra.

4 Homogeneous spaces and calculation of Nijenhuis Tensors

4.1 $\mathfrak{su}(2)$

4.1.1 $\mathfrak{m} = W$

The skew symmetric real tensor product $\Lambda^2 W$ can be computed easily by first noting that the complexified Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ has a natural representation on W (the representation is not absolutely irreducible), ie. we do not need to complexify the representation as would usually be the case. We then make use of the natural embedding of $\mathfrak{su}(1,1)$ in $\mathfrak{sl}_2(\mathbb{C})$ to represent the former on W as well. This identifies $W \simeq V \oplus V$ (under different algebras). The right hand side consists of a direct sum of highest weight representations, which allows us to compute the decomposition of the tensor product easily.

$$\Lambda^2(V \oplus V) = \varepsilon \oplus \mathbb{C} \oplus S^2V$$

The decomposition of W should consist of irreducible components of the same dimensions, and we get

$$\Lambda^2 W = \varepsilon \oplus \mathbb{C} \oplus Ad$$

The standard complex basis x, y of W extends to a real basis x, y, ix, iy. We can write the decomposition concretely in terms of this basis

$$\begin{aligned}
&\langle x \wedge ix + y \wedge iy \rangle \simeq \varepsilon \\
&\langle x \wedge y - ix \wedge iy, x \wedge iy + ix \wedge y \rangle \simeq \Lambda^2_{\mathbb{C}} W = \mathbb{C} \\
&\langle x \wedge iy - ix \wedge y, x \wedge y + ix \wedge iy, x \wedge ix - y \wedge iy \rangle \simeq Ad
\end{aligned}$$

This allows for an equivariant map $\Lambda^2 \mathfrak{m} \to \mathfrak{h}$, and hence opens up the possibility of an algebra structure on $\mathfrak{h} \oplus \mathfrak{m}$. Write $\hat{x}, \hat{y}, i\hat{x}, i\hat{y}$ for the dual basis of x, y, ix, iy. The following set of complex operators

$$\begin{aligned}
u &= \hat{x} \otimes ix - \hat{y} \otimes iy - i\hat{x} \otimes x + i\hat{y} \otimes y \\
k &= \hat{y} \otimes x - \hat{x} \otimes y + i\hat{y} \otimes ix - i\hat{x} \otimes iy \\
m &= \hat{x} \otimes iy + \hat{y} \otimes ix - i\hat{x} \otimes y - i\hat{y} \otimes x
\end{aligned}$$

is a basis of $\mathfrak{h}$ with commutation relations

$$\begin{aligned}
[u, k] &= 2m \\
[u, m] &= -2k \\
[k, m] &= 2u
\end{aligned}$$

We solve for the brackets

$$\begin{aligned}
[x, ix] &= \alpha u \\
[x, y] &= -\alpha k \\
[x, iy] &= \alpha m \\
[ix, y] &= -\alpha m \\
[ix, iy] &= -\alpha k \\
[y, iy] &= -\alpha u
\end{aligned}$$

Attempting to compute the Jacobi identity for elements x, ix, y shows that it fails unless $\mathfrak{g}$ is flat:

$$[x, [ix, y]] + [y, [x, ix]] + [ix, [y, x]] = 3\alpha iy$$

Here α is the free parameter corresponding to choice of equivariant map. This calculation implies a trivial algebra structure on $\mathfrak{m}$ and hence a vanishing Nijenhuis tensor.

We now attempt to add a radical $\mathbb{R}$ to the Lie algebra, yielding $\mathfrak{h} = \mathfrak{su}(2) \oplus \mathbb{R}$. The radical term could in principle be represented by any subalgebra of

$\mathrm{End}_{\mathfrak{h}}(W)$, the operators that commute with $\mathfrak{h}$. $\mathrm{End}_{\mathfrak{h}}(W)$ is isomorphic to the quaternions as an algebra. We demand that our isotropy algebra is $\mathbb{C}$-linear, ie. it must commute with the complex structure. The space of $\mathbb{C}$-linear operators form a subalgebra of $\mathrm{End}_{\mathfrak{h}}(W)$. This subalgebra is isomorphic to $\mathbb{C}$ and is generated by $\mathbb{1}, J$, Suppose r is a basis of $\mathbb{R}$ represented by $\beta J + \gamma$. Note that by the previous Jacobi identity, we need to have β non-zero so we can cancel out the iy-term. If β is nonzero this makes the $\mathbb{C}$ term in the decomposition of $\Lambda^2 W$ irreducible, so the only option is to map $\varepsilon \to \mathbb{R}$. Solving for the brackets and computing the Jacobi identity from before we obtain

$$[x,[ix,y]] + [y,[x,ix]] + [ix,[y,x]] = (3\alpha - \beta)iy - \gamma y$$

Checking that the other identities are verified as well gives a non-flat homogenous space for $\gamma = 0, \beta = 3\alpha$. Since $\mathfrak{su}(2)$ is embedded in $\mathfrak{g}$, $\mathfrak{g}$ must be either $\mathfrak{su}(3)$ or $\mathfrak{su}(2,1)$. The Nijenhuis tensor is zero since the bracket has no $\mathfrak{m}$-component.

4.1.2 $\mathfrak{m} = Ad^{\mathbb{C}}$

Now let $\mathfrak{m} = Ad^{\mathbb{C}}$. Abstractly the possible semi-simple algebra structures on $\mathfrak{h} \oplus \mathfrak{m}$ are $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{sl}_2(\mathbb{C})$.

In the case $\mathfrak{g} = \mathfrak{su}(2)^3$, we know from the nontrivial module decomposition that $\mathfrak{h}$ corresponds to the diagonal subalgebra (It acts nontrivially on each copy of $\mathfrak{su}(2)$). Each term in the abstract description is also an ideal, so we can find two submodules $Ad_1, Ad_2 \subset \mathfrak{m}$ such that

$$\begin{aligned}
[Ad_1, Ad_2] &= 0 \\
[Ad_1, Ad_1] &= Ad_1 \\
[Ad_2, Ad_2] &= Ad_2
\end{aligned}$$

which corresponds to the equivariant map

$$\begin{aligned}
Ad_1 \otimes Ad_2 &\to 0 \\
\Lambda^2 Ad_1 &\to Ad_1 \\
\Lambda^2 Ad_2 &\to Ad_2
\end{aligned}$$

Now consider the complex structure on $Ad^{\mathbb{C}} \simeq Ad \otimes_{\mathbb{R}} \mathbb{C} \simeq Ad \oplus Ad$. The splitting of $\mathfrak{m}$ into $Ad_1 \oplus Ad_2$ may not be compatible with the splitting $Ad \oplus iAd$. If we use the first splitting, J can look like any equivariant map $\mathfrak{m} \to \mathfrak{m}$ with square $J^2 = -1$,

$$\begin{aligned}
J(Ad_1) &= \{(-rh, th)\} \\
J(h, 0) &= (-rh, th)
\end{aligned}$$

for some real r and real $t \neq 0$. Now we have enough information to compute the Nijenhuis tensor.

$$N_J((h,0),(v,0)) = (-(r^2+1)[h,v],(t^2+2rt)[h,v])$$

This is non-degenerate because the bracket $[h,v]$ vanishes only when h,v are proportional.

The second case is $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{sl}_2(\mathbb{C})$. In this case $\mathfrak{h}$ must also correspond to a diagonal subalgebra, with the component in the $\mathfrak{sl}_2(\mathbb{C})$ term consisting of a choice of real $\mathfrak{su}(2)$ (but all choices are equivalent). In this case the bracket on $\mathfrak{m} = Ad \oplus Ad$ is given by

$$[(h_1,v_1),(h_2,v_2)] = ([h_1,h_2]-[v_1,v_2],[h_1,v_2]+[v_1,h_2])$$

and J is given by

$$J(h,0) = (-rh,th)$$

which gives the Nijenhuis tensor

$$N_J((h,0),(v,0)) = ((r^2+1-t^2)[h,v],-2tr[h,v])$$

The coefficients have $r=0, t=\pm 1$ as a common root, which corresponds to the natural complex structure on $\mathfrak{sl}_2(\mathbb{C})$ regarded as a complex Lie algebra. All $r \neq 0$ give a non-degenerate N_J.

There are two possible cases where $\mathfrak{g}$ is not semi-simple but the semi-simple part $Q \subset \mathfrak{g}$ strictly contains $\mathfrak{h}$. We may attempt to extend to either $Q = \mathfrak{sl}_2(\mathbb{C})$ or $Q = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, both with an abelian radical $I = \mathbb{R}^3$. Since we know that I must be a Q-module, the former case is easily disqualified due to the lack of nontrivial $\mathfrak{su}_2(\mathbb{C})$-actions that extend the known $\mathfrak{su}(2)$-action on I.

In the latter case $Q = \mathfrak{su}(2)^2$ we must also have a Q-action on I, and the kernel of this action will be an ideal and hence also a submodule. Denote this submodule by $Ad_1 = \{(h,0)\}$. It is $\mathfrak{su}_2$ as a Lie algebra. As usual we have

$$J(Ad_1) = \{(-rh,th)\}$$
$$J(h,0) = (-rh,th)$$

and we can compute the Nijenhuis tensor for $(h,0),(v,0) \in Ad_1$

$$N_J((h,0),(v,0)) = (-(r^2+1)[h,v],2tr[h,v])$$

This is non-degenerate since the polynomial coefficients do not vanish simultaneously.

Finally we have the case of $\mathfrak{m}$ being a solvable radical. Split $Ad^{\mathbb{C}} \simeq Ad_1 \oplus Ad_2$, and let the Lie bracket be given by

$$[(h_1, v_1), (h_2, v_2)] = (0, [h_1, h_2])$$

This is the only non-flat solvable structure on $\mathfrak{m}$, as the other possibilities violate the Jacobi identity. As before the almost complex structure will in general be given by

$$J(Ad_1) = \{(-rh, th)\}$$
$$J(h, 0) = (-rh, th)$$

for some real r and real $t \neq 0$. We compute the Nijenhuis tensor

$$N_J((h, 0), (v, 0)) = (-\frac{2(r^3 + r)}{t}[h, v], (3r^2 - 1)[h, v])$$

It is always non-degenerate because the polynomial coefficients have no common zeroes.

The case $r = 0, t = \pm 1$ can be identified with the following construction: Identify $\mathfrak{su}(2) \simeq \mathfrak{so}(3)$ and $Ad \simeq V$, where V denotes the 3d tautological representation of $\mathfrak{so}(3)$. We can identify $V \simeq \Lambda^2 V$ as modules. Define $\mathfrak{m} = V \oplus \Lambda^2 V$, and let the Lie bracket be given by the wedge product

$$[v, w] = v \wedge w$$

Now the Hodge-$*$ operator

$$V \to \Lambda^2 V$$
$$\Lambda^2 V \to V$$

is precisely the equivariant almost complex structure we wanted.

It is interesting to note that all of the non-zero Nijenhuis tensors on $\mathfrak{m} = Ad \oplus Ad$ have the same pointwise type, ie. all of them can be transformed to the form of any other at a given point. Of course, this transformation does not necessarily respect the homogeneous structure.

4.1.3 $\mathfrak{m} = W \oplus \mathbb{C}$

The last case to consider is $\mathfrak{m} = W \oplus \mathbb{C}$. We have

$$\Lambda^2 \mathfrak{m} = \Lambda^2 W \oplus W \otimes_{\mathbb{R}} \mathbb{C} \oplus \Lambda^2 \mathbb{C} = \mathbb{C} \oplus Ad \oplus \mathbb{R} \oplus W^{\mathbb{C}} \oplus \mathbb{R}$$

and we know that W is not absolutely irreducible, which means that $W^{\mathbb{C}} \simeq W \oplus W$ (over $\mathbb{C}$). The decomposition of $\Lambda^2 W$ is given by

$$
\begin{aligned}
&\langle x \wedge ix + y \wedge iy \rangle \simeq \varepsilon \\
&\langle x \wedge y - ix \wedge iy, x \wedge iy + ix \wedge y \rangle \simeq \Lambda^2_{\mathbb{C}} W = \mathbb{C} \\
&\langle x \wedge iy - ix \wedge y, x \wedge y + ix \wedge iy, x \wedge ix - y \wedge iy \rangle \simeq Ad
\end{aligned}
$$

Let $\mathrm{End}_{\mathfrak{su}(2)} W$ denote the space of equivariant maps $W \to W$. As an operator ring it is naturally isomorphic to the quaternions $\mathbb{H}$. $A \in \mathrm{End}_{\mathfrak{su}(2)} W$ can be written as $A = \alpha \mathbb{1} + \beta B$ with $B^2 = -\mathbb{1}$, so B is an equivariant complex structure on W. The space of such structures is a 2-sphere in $\mathrm{End}_{\mathfrak{su}(2)} W$. They are all equivalent, so we fix our complex structure to be the standard one. Note that since $\mathbb{H}$ is a division algebra, every non-zero operator in $\mathrm{End}_{\mathfrak{su}(2)} W$ is invertible. Traceless operators correspond to imaginary quaternions under our isomorphism.

To begin we assume that the bracket has no $\mathfrak{h}$-component. The bracket $\Lambda^2 W \to \mathbb{C}$ is given by a $\mathbb{C}$-valued $\mathfrak{h}$-invariant 2-form σ. The bracket $W \otimes \mathbb{C} \to W$ is given by a map $A : \mathbb{C} \to \mathrm{End}_{\mathfrak{su}(2)} W$ such that

$$
\begin{aligned}
&z \mapsto A_z \\
&[X, z] = A_z X
\end{aligned}
$$

In addition to this $\mathbb{C}$ can have a 2d Lie algebra structure, either abelian or non-abelian solvable. We can write all our brackets

$$
\begin{aligned}
&[X, Y] = \sigma(X, Y) \\
&[X, z] = A_z X \\
&z \wedge iz \mapsto [z, iz]
\end{aligned}
$$

for $X, Y \in W, z \in \mathbb{C}$. We compute the Jacobi identities

$$
\begin{aligned}
&A_{\sigma(Y,Z)} X + A_{\sigma(X,Y)} Z + A_{\sigma(Z,X)} Y = 0 \\
&[A_z, A_{iz}] = -A_{[z,iz]} \\
&[\sigma(X, Y), z] = \sigma(A_z X, Y) + \sigma(X, A_z Y)
\end{aligned}
$$

for $X, Y, Z \in W, z \in \mathbb{C}$. The third identity suggests that we look for equivariant symmetries of σ, which will be useful in at least the case of abelian $\mathbb{C}$ (and also in the non-abelian case). We can write our vector valued form as the sum of its components in some basis

$$
\sigma(X, Y) = \sigma_z(X, Y) z + \sigma_{iz}(X, Y) iz
$$

such that σ_z, σ_{iz} are $\mathfrak{h}$-invariant real valued forms. The symmetries of σ will then be the intersection of the symmetries of σ_z, σ_{iz}, so we start by computing the symmetry algebra of the scalar valued forms.

The algebra $\mathfrak{h}$ preserves a hermitian form H on W. The real part of this form is an invariant real inner product. Denote it by $g = \Re H$. Any invariant scalar form σ can be written as $\sigma(X,Y) = g(BX,Y)$ for some operator B. Since both σ and g are invariant, $B \in \mathrm{End}_{\mathfrak{su}(2)}W$. If B contains a non-zero identity component, σ will not be skew symmetric. On the other hand, σ is skew symmetric if B is an imaginary quaternion, so we have a bijection between imaginary quaternions and skew symmetric invariant 2-forms. Therefore we will denote

$$\omega_B(X,Y) = g(BX,Y)$$

for imaginary B. Since non-zero B is an invertible operator ω_B is non-degenerate, and therefore the symmetries are traceless. We easily see that B is contained in the symmetry algebra of ω_B, because

$$\omega_B(Bx,y) + \omega_B(x,By) = \beta^2(g(x,y) - g(x,y)) = 0$$

for β such that $B^2 = -\beta^2$. On the other hand, if we take $A \in \mathrm{End}_{\mathfrak{su}(2)}W$ imaginary and not proportional to B, we get

$$\omega_B(Ax,y) + \omega_B(x,Ay) = 2g(\Im(BA)x,y) = 2\omega_{\Im(BA)}(x,y)$$

which is non-zero if A, B are non-zero. We have shown that for non-zero B

$$\mathrm{Sym}_{\mathfrak{h}}(\omega_B) = \langle B \rangle$$

Let's apply this to compute the algebra structures. Suppose first that both $A \neq 0, \sigma \neq 0$. From the identity

$$[A_z, A_{iz}] = -A_{[z,iz]}$$

we see that A takes values in a 2d Lie subalgebra with the same structure as $\mathbb{C}$. However all 2d Lie subalgebras of $\mathrm{End}_{\mathfrak{h}}W$ are isomorphic to the complex numbers as associative algebras and are therefore abelian Lie algebras. Therefore we can find $z \in \mathbb{C}$ such that A_z is proportional to the identity (possibly $A_z = 0$) and

$$[z, iz] = \delta z$$

Here $\delta = 0$ if $\mathbb{C}$ is abelian (in which case z is not fixed), $\delta = 1$ if $\mathbb{C}$ is non-abelian (z is fixed up to scaling in this case). σ splits into components

$$\sigma(X,Y) = \omega_B(X,Y)z + \omega_C(X,Y)iz$$

(we have changed our notation from the last time this decomposition was used, to indicate symmetries) and the third Jacobi identity becomes

$$\omega_C(A_z X, Y) + \omega_C(X, A_z Y) = 0$$
$$\omega_C(A_{iz} X, Y) + \omega_C(X, A_{iz} Y) = 0$$

which must hold in all cases, and additionally for abelian $\mathbb{C}$ we have

$$\omega_B(A_z X, Y) + \omega_B(X, A_z Y) = 0$$
$$\omega_B(A_{iz} X, Y) + \omega_B(X, A_{iz} Y) = 0$$

or for non-abelian $\mathbb{C}$ we get

$$\omega_B(A_z X, Y) + \omega_B(X, A_z Y) = -\omega_C(X, Y)$$
$$\omega_B(A_{iz} X, Y) + \omega_B(X, A_{iz} Y) = \omega_B(X, Y)$$

If $\mathbb{C}$ is abelian, A_z and A_{iz} must be symmetries of both ω_B and ω_C, so it follows from our discussion of symmetries above that there is some z such that $A_z = 0$, $A_{iz} = B$ and ω_B, ω_C are proportional. If $\mathbb{C}$ is non-abelian, $A_z = 0$ because A_z, A_{iz} is an anti-representation of $\mathbb{C}$. In either case we get $z \in \mathrm{Ker}(A)$. We insert this into the first Jacobi identity, which becomes

$$\omega_C(Y, Z) A_{iz} X + \omega_C(X, Y) A_{iz} Z + \omega_C(Z, X) A_{iz} Y = 0$$

We claim that $\omega_C = 0$. If ω_C is non-zero, it is non-degenerate and for any $Z \in W$we can find non-zero $X, Y \in W$ such that $\omega_C(Y, Z) = \omega_C(X, Z) = 0, \omega_C(X, Y) = 1$, and so we must have $A_{iz} Z = 0$, but non-zero A_{iz} is injective. Now we get

$$\delta \omega_B(X, Y) z = (\omega_B(A_{iz} X, Y) + \omega_B(X, A_{iz} Y)) z$$

which means that

$$A_{iz} = \frac{\delta}{2} \mathbb{1} + \beta B$$

Now all the Jacobi identities are satisfied. The algebra structures are

$$[X, Y] = g(BX, Y) z$$
$$[X, iz] = \frac{\delta}{2} X + \beta B X$$
$$[z, iz] = \delta z$$

Here $X, Y \in W$, $z \in \mathbb{C}$, $\beta \in \mathbb{R}$, $B \in \mathrm{End}_{\mathfrak{su}(2)} W, \mathrm{Tr}(B) = 0$ and δ is 0 or 1. The Nijenhuis tensor is given by

$$
\begin{aligned}
N_J(x,y) &= g(J[J,B]x,y)z + g([J,B]x,y)iz \\
N_J(x,z) &= \beta[B,J]x \\
N_J(y,z) &= \beta[B,J]y
\end{aligned}
$$

in terms of the complex basis x, y, z of $\mathfrak{m}$ we defined earlier. Note that this is a function of $[J, B]$, and therefore vanishes if B is proportional to J. We may also write these brackets and the Nijenhuis tensor in terms real parameters by writing

$$Bx = \lambda_1 ix + \lambda_2 y + \lambda_3 iy$$

and extending it uniquely to $B \in \mathrm{End}_{\mathfrak{su}(2)} W$. This is done by noting that if z a basis element other than x, we can find a unique $h \in \mathfrak{h}$ such that $z = hx$ and so we must have

$$Bz = Bhx = hBx$$

which shows uniqueness of the extension of B. The brackets are then

$$
\begin{aligned}
[x, ix] &= [y, iy] = \lambda_1 z \\
[x, y] &= -[ix, iy] = \lambda_2 z \\
[ix, y] &= [x, iy] = \lambda_3 z \\
[x, iz] &= \frac{\delta}{2}x + \beta(\lambda_1 ix + \lambda_2 y + \lambda_3 iy) \\
[ix, iz] &= \frac{\delta}{2}ix + \beta(-\lambda_1 x + \lambda_3 y - \lambda_2 iy) \\
[y, iz] &= \frac{\delta}{2}y + \beta(-\lambda_2 x - \lambda_3 ix + \lambda_1 iy) \\
[iy, iz] &= \frac{\delta}{2}iy + \beta(-\lambda_3 x + \lambda_2 ix - \lambda_1 y) \\
[z, iz] &= \delta z
\end{aligned}
$$

and the Nijenhuis tensor is

$$
\begin{aligned}
N_J(x,y) &= -2\lambda_2 z - 2\lambda_3 iz \\
N_J(x,z) &= 2\beta(\lambda_3 y - \lambda_2 iy) \\
N_J(y,z) &= 2\beta(\lambda_3 x - \lambda_2 ix)
\end{aligned}
$$

In addition to this we have the cases when $A = 0$ or $\sigma = 0$. If $A = 0$ we can write the brackets

$$
\begin{aligned}
[X,Y] &= \omega_B(X,Y)z + \omega_C(X,Y)iz \\
[z, iz] &= \delta z
\end{aligned}
$$

The only Jacobi identity in this case is

$$[\omega_B(X,Y)z + \omega_C(X,Y)iz, w] = 0$$

for $w \in \mathbb{C}$, so we see that if $\delta = 1$, $B = C = 0$. Otherwise there are no restrictions. The Nijenhuis tensor is

$$\begin{aligned}
&N_J(x,y) = g((J[J,B] - [J,C])x, y)z + g(([J,B] + J[J,C])x, y)iz \\
&N_J(x,z) = 0 \\
&N_J(y,z) = 0
\end{aligned}$$

it vanishes if $\Im(JB - C)$ is proportional to J. If we write

$$\begin{aligned}
Bx &= \lambda_1 ix + \lambda_2 y + \lambda_3 iy \\
Cx &= L_1 x + L_2 y + L_3 iy
\end{aligned}$$

as before this becomes

$$\begin{aligned}
&[x, ix] = [y, iy] = \lambda_1 z + L_1 iz \\
&[x, y] = -[ix, iy] = \lambda_2 z + L_2 iz \\
&[ix, y] = [x, iy] = \lambda_3 z + L_3 iz
\end{aligned}$$

with Nijenhuis tensor

$$\begin{aligned}
&N_J(x,y) = 2(L_3 - \lambda_2)z - 2(\lambda_3 + L_2)iz \\
&N_J(x,z) = 0 \\
&N_J(y,z) = 0
\end{aligned}$$

If $\sigma = 0$ we get that A_z, A_{iz} is any anti-representation of $\mathbb{C}$. We can have abelian $\mathbb{C}$, in which case we get the brackets

$$\begin{aligned}
&[X, z] = \beta X \\
&[X, iz] = \alpha X + BX
\end{aligned}$$

or non-abelian $\mathbb{C}$ with the brackets

$$\begin{aligned}
&[X, iz] = \alpha X + BX \\
&[z, iz] = z
\end{aligned}$$

The Nijenhuis tensor is the same in both cases

$$\begin{aligned}
&N_J(x,y) = 0 \\
&N_J(x,z) = [B,J]x \\
&N_J(y,z) = [B,J]y
\end{aligned}$$

In terms of parameters this can be written

$$\begin{aligned}
[x, z] &= \beta x \\
[ix, z] &= \beta ix \\
[y, z] &= \beta y \\
[iy, z] &= \beta iy \\
[x, iz] &= \alpha x + (\lambda_1 ix + \lambda_2 y + \lambda_3 iy) \\
[ix, iz] &= \alpha ix - (\lambda_1 x - \lambda_3 y + \lambda_2 iy) \\
[y, iz] &= \alpha y - (\lambda_2 x + \lambda_3 ix - \lambda_1 iy) \\
[iy, iz] &= \alpha iy - (\lambda_3 x - \lambda_2 ix + \lambda_1 y)
\end{aligned}$$

in the case of abelian $\mathbb{C}$, and

$$\begin{aligned}
[x, iz] &= \alpha x + (\lambda_1 ix + \lambda_2 y + \lambda_3 iy) \\
[ix, iz] &= \alpha ix - (\lambda_1 x - \lambda_3 y + \lambda_2 iy) \\
[y, iz] &= \alpha y - (\lambda_2 x + \lambda_3 ix - \lambda_1 iy) \\
[iy, iz] &= \alpha iy - (\lambda_3 x - \lambda_2 ix + \lambda_1 y) \\
[z, iz] &= z
\end{aligned}$$

for non-abelian $\mathbb{C}$. The Nijenhuis tensor is

$$\begin{aligned}
N_J(x, y) &= 0 \\
N_J(x, z) &= 2(\lambda_3 y - \lambda_2 iy) \\
N_J(y, z) &= 2(\lambda_3 x - \lambda_2 ix)
\end{aligned}$$

Let's now consider the possibility of $\mathfrak{g}$ with nonzero $\mathfrak{h}$-component. If $\mathfrak{h}$ is not contained in a strictly bigger semi-simple subalgebra, then by Levi decomposition $\mathfrak{m}$ is conjugate to the radical of $\mathfrak{g}$. Since the radical is an ideal, $\mathfrak{h}$ is conjugate in $\mathfrak{g}$ to another subalgebra $\mathfrak{h}' \simeq \mathfrak{h}$ such that the bracket has vanishing $\mathfrak{h}$-component. This means that the homogenous space is equivalent to a space considered previously.

If $\mathfrak{h}$ is strictly contained in a semi-simple subalgebra $Q \subset \mathfrak{g}$, Q must have dimension $6, 8$ or 9. Since W must be contained in Q, it is at least dim 7. All options of dim 9 contain ideals which would necessarily also be submodules, and such submodules are not present. So we are left with dim 8.

This was explored in the section for $\mathfrak{m} = W$, $\mathfrak{h} = \mathfrak{su}(2) \oplus \mathbb{R}$, with a subset $\langle z \rangle = \varepsilon \subset \mathbb{C}$ playing the role that was earlier taken by the center in $\mathfrak{h}$. This process yields Q isomorphic to $\mathfrak{su}(3)$ or $\mathfrak{su}(2,1)$. Since these algebras have no non-trivial 1d representations, $\mathfrak{g} = Q \oplus \mathbb{R}$. $N_J = 0$ because $[x, y] \in \mathfrak{h}$ and the action of z is complex linear. The brackets are

$$
\begin{aligned}
[x, ix] &= \beta(u + 3z) \\
[x, y] &= -\beta k \\
[x, iy] &= \beta m \\
[ix, y] &= -\beta m \\
[ix, iy] &= -\beta k \\
[y, iy] &= \beta(3z - u) \\
[z, x] &= ix \\
[z, ix] &= -x \\
[z, y] &= iy \\
[z, iy] &= -y
\end{aligned}
$$

where u, k, m is the basis for $\mathfrak{h}$ we described earlier. We may rescale our basis to make $\beta = \pm 1$. $\beta = -1$ gives $\mathfrak{g} = \mathfrak{su}(3) \oplus \mathbb{R}$ and $\beta = 1$ gives $\mathfrak{g} = \mathfrak{su}(2,1) \oplus \mathbb{R}$.

4.2 $\mathfrak{su}(1,1)$

4.2.1 $\mathfrak{m} = V^{\mathbb{C}}$

Let $\mathfrak{m} = V^{\mathbb{C}}$, the tautological representation of $\mathfrak{h} = \mathfrak{su}(1,1)$. Choose a Borel subalgebra $\mathfrak{b} \subset \mathfrak{h}$. Pick an element $x \in V^{\mathbb{C}}$ which is annihilated by $\mathfrak{b}$. Then x, ix generates the real splitting $V^{\mathbb{C}} = V \oplus V$. We may pick an element y from the submodule generated by x such that the the following complex operators is a basis of $\mathfrak{h}$, and x, y, ix, iy is a real basis of $V \oplus V$.

$$
\begin{aligned}
A &= \hat{y} \otimes x - \hat{x} \otimes y + i\hat{y} \otimes ix - i\hat{x} \otimes iy \\
B &= \hat{x} \otimes y + \hat{y} \otimes x + i\hat{x} \otimes iy + i\hat{y} \otimes ix \\
C &= \hat{x} \otimes x - \hat{y} \otimes y + i\hat{x} \otimes ix - i\hat{y} \otimes iy
\end{aligned}
$$

Here $\hat{x}, \hat{y}, i\hat{x}, i\hat{y}$ means the real dual basis. The commutation relations are

$$
\begin{aligned}
[A, B] &= 2C \\
[A, C] &= -2B \\
[B, C] &= -2A
\end{aligned}
$$

Note that if $V \oplus V$ is identified with W from the previous section as a vector space, the respective choices of complex basis x, y are different. In particular x, y in the $\mathfrak{su}(1,1)$-sense is not the standard basis, and if we denote the basis from the $\mathfrak{su}(2)$-section by X, Y we can write

$$
\begin{aligned}
x &= JX + Y \\
y &= X + JY
\end{aligned}
$$

For the rest of this section, capital letters will denote arbitrary elements of $V^{\mathbb{C}}$ while x, y means the basis we defined above. Abstractly the decomposition of $\Lambda^2\mathfrak{m}$ is

$$\Lambda^2\mathfrak{m} = \varepsilon \oplus \mathbb{C} \oplus Ad$$

We note immediately that because there is no possible bracket with nonzero $\mathfrak{m}$-component, every Nijenhuis tensor in this section will vanish. The decomposition can be written concretely

$$\begin{aligned}
&\langle x \wedge y + ix \wedge iy \rangle \simeq \varepsilon \\
&\langle x \wedge iy + ix \wedge y, x \wedge y - ix \wedge iy \rangle \simeq \Lambda^2_{\mathbb{C}}\mathfrak{m} = \mathbb{C} \\
&\langle x \wedge ix + y \wedge iy, x \wedge ix - y \wedge iy, x \wedge iy - ix \wedge y \rangle \simeq \langle A, B, C \rangle = Ad
\end{aligned}$$

The identification with $\mathbb{C}$ of the second term is done because the Lie algebra action of J on $\Lambda^2\mathfrak{m}$ maps the two submodules into each other, and so this piece is irreducible with respect to J. It trivial and not irreducible with respect to $\mathfrak{h}$. We solve for the brackets and introduce a parameter α for the map $V \otimes V \to \mathfrak{h}$

$$\begin{aligned}
&[x, y] = 0 \\
&[ix, iy] = 0 \\
&[x, ix] = \alpha(A + B) \\
&[y, iy] = \alpha(A - B) \\
&[x, iy] = -\alpha C \\
&[ix, y] = \alpha C
\end{aligned}$$

Attempting to compute the Jacobi identity for elements x, ix, y shows that it fails unless $\mathfrak{g}$ is flat.

$$[x, [ix, y]] + [y, [x, ix]] + [ix, [y, x]] = -3\alpha x$$

As before we attempt to extend $\mathfrak{h}$ by a 1d center, so now $\mathfrak{h} = \mathfrak{su}(1,1) \oplus \mathbb{R}$. The center can in principle be represented by any element r of $\mathrm{End}_{\mathfrak{su}(1,1)}\mathfrak{m}$, which is isomorphic to $\mathrm{Mat}_{2x2}(\mathbb{R})$ as an algebra. The isomorphism can be manifested by picking an action on the 2d subspace $\langle x, ix \rangle \subset \mathfrak{m}$, and extending by $\mathfrak{su}(1,1)$-equivariance. The result of this is 4x4 matrices containing two identical 2x2 blocks. We demand that the center is complex linear, which reduces the possibilities to complex scalar operators. This means that the center must be represented by

$$r = \gamma \mathbb{1} + \beta J$$

By the Jacobi identity above, the representation of the center needs to map x to something proportional to ix. Therefore we set

$$r = J$$

The new brackets must be equivariant with respect to J, and since $\langle J \rangle$ is a 1d algebra this means we must map the first term in the decomposition of $\Lambda^2\mathfrak{m}$ to J. This gives the brackets

$$[x, y] = \beta r$$
$$[ix, iy] = \beta r$$

which makes our Jacobi identity become

$$[x, [ix, y]] + [y, [x, ix]] + [ix, [y, x]] = -(3\alpha + \beta)x = 0$$

This and all other identities are satisfied if we set

$$\beta = -3\alpha$$

The algebra structure of $\mathfrak{g}$ depends on the sign of α. Since $\mathfrak{su}(1,1)$ is non-compact and the algebra structure we just defined is semi-simple, it must be either $\mathfrak{sl}_3$ or $\mathfrak{su}(2,1)$.

4.2.2 $\mathfrak{m} = Ad^{\mathbb{C}}$

Now consider the $\mathfrak{m} = Ad \oplus Ad$ case. Everything is perfectly analogous to the $\mathfrak{su}(2)$-case, and we proceed by considering sequentially smaller semi-simple extensions of $\mathfrak{h}$. The possible extensions Q are $\mathfrak{su}(1,1) \oplus \mathfrak{su}(1,1) \oplus \mathfrak{su}(1,1)$, $\mathfrak{su}(1,1) \oplus \mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{su}(1,1) \oplus \mathfrak{su}(1,1)$.

The splitting $\mathfrak{m} = Ad_1 \oplus Ad_2$ will depend on the structure of $\mathfrak{g}$, but the possible almost complex structures J depend only on the module decomposition. We can therefore write

$$J(Ad_1) = \{(-rh, th)\}$$
$$J(h, 0) = (-rh, th)$$

for some real r, t with $t \neq 0$.

In the case of maximal Q we know that $\mathfrak{h}$ must be conjugate to a diagonal subalgebra because of the nontrivial $\mathfrak{h}$-module decomposition of $\mathfrak{g}$. For $Q = \mathfrak{su}(1,1)^3$ this yields the brackets

$$[(h_1, v_1), (h_2, v_2)] = ([h_1, h_2], [v_1, v_2])$$

and the Nijenhuis tensor

$$N_J((h, 0), (v, 0)) = (-(r^2 + 1)[h, v], (t^2 + 2rt)[h, v])$$

For $Q = \mathfrak{su}(1,1) \oplus \mathfrak{sl}_2(\mathbb{C})$ we get the brackets

$$[(h_1, v_1), (h_2, v_2)] = ([h_1, h_2] - [v_1, v_2], [h_1, v_2] + [v_1, h_2])$$

with corresponding Nijenhuis tensor

$$N_J((h, 0), (v, 0)) = ((r^2 + 1 - t^2)[h, v], -2tr[h, v])$$

Once again we note that there is precisely one integrable case which is when J is the natural complex structure on $\mathfrak{sl}_2(\mathbb{C})$.

Suppose now that $\mathfrak{g}$ contains the abelian radical $I = Ad_2$. Since $\mathfrak{sl}_2(\mathbb{C})$ lacks a nontrivial action on I, we get $Q = \mathfrak{su}(1,1) \oplus \mathfrak{su}(1,1)$ with the second summand corresponding to the kernel of the Q-action. The brackets are

$$[(h_1, v_1), (h_2, v_2)] = ([h_1, h_2], 0)$$

with corresponding Nijenhuis tensor

$$N_J((h, 0), (v, 0)) = (-(r^2 + 1)[h, v], 2tr[h, v])$$

There is only one algebra structure $\mathfrak{g}$ with $\mathfrak{m}$ solvable, and it is given by the brackets

$$[(h_1, v_1), (h_2, v_2)] = (0, [h_1, h_2])$$

The Nijenhuis tensor is

$$N_J((h, 0), (v, 0)) = (-\frac{2(r^3 + r)}{t}[h, v], (3r^2 - 1)[h, v])$$

It is worth noting that except for the single case that was mentioned, every Nijenhuis tensor in this section was found to be non-degenerate and of the same pointwise type.

4.2.3 $\mathfrak{m} = V^{\mathbb{C}} \oplus \mathbb{C}$

Finally let $\mathfrak{m} = V \oplus V \oplus \mathbb{C}$. We compute

$$\Lambda^2\mathfrak{m} = \varepsilon \oplus Ad \oplus \mathbb{C} \oplus (V \oplus V) \otimes \mathbb{C} \oplus \varepsilon$$

Define

$$\begin{aligned} &\mathfrak{m}_0 = V \oplus V \\ &A \in \mathrm{Hom}(\mathbb{C}, \mathbb{R}) \otimes \mathrm{End}_{\mathfrak{h}}\mathfrak{m}_0 \\ &A_z = \iota_z A \end{aligned}$$

If we suppress the possible $\mathfrak{h}$-component the brackets can be given by

$$\begin{aligned}
&[X,Y] = \sigma(X,Y)\\
&[X,z] = A_z(X)\\
&z_1 \wedge z_2 \mapsto [z_1, z_2]
\end{aligned}$$

with $X, Y \in \mathfrak{m}_0$ and σ takes values in $\mathbb{C}$. There exists a basis z, iz of $\mathbb{C}$ such that the last bracket is given by either

$$\begin{aligned}
&[z, iz] = 0 \text{ or}\\
&[z, iz] = z
\end{aligned}$$

We may write the form σ, given by $e_1, e_2, e_3 \in \mathbb{C}$, in terms of the basis x, y, ix, iy we defined earlier

$$\begin{aligned}
&[x,y] = e_1\\
&[ix,iy] = e_3\\
&[x,iy] = e_2\\
&[ix,y] = e_2
\end{aligned}$$

The brackets give the Jacobi identities

$$\begin{aligned}
&A_{\sigma(Y,Z)}X + A_{\sigma(X,Y)}Z + A_{\sigma(Z,X)}Y = 0\\
&[A_z, A_{iz}] = A_{[iz,z]}\\
&[\sigma(X,Y), b] = \sigma(A_bX, Y) + \sigma(X, A_bY)
\end{aligned}$$

for $X, Y, Z \in \mathfrak{m}_0, b \in \mathbb{C}$. We start by treating the case where $\mathbb{C}$ is abelian. The Jacobi identities change to

$$\begin{aligned}
&A_{\sigma(Y,Z)}X + A_{\sigma(X,Y)}Z + A_{\sigma(Z,X)}Y = 0\\
&[A_z, A_{iz}] = 0\\
&\sigma(A_bX, Y) + \sigma(X, A_bY) = 0
\end{aligned}$$

Note that $\langle A_z, A_{iz}\rangle$ is a commutative subalgebra contained in the intersection $\mathrm{End}_{\mathfrak{h}}\mathfrak{m}_0 \cap \mathrm{sym}(\sigma)$. Let's use the isomorphism $\mathrm{End}_{\mathfrak{h}}\mathfrak{m}_0 \simeq \mathrm{Mat}_{2\times 2}(\mathbb{R}) \simeq \mathbb{R} \oplus \mathfrak{sl}_2$ to compute the symmetry algebra of σ.

First we suppose that σ is a scalar form proportional to a vector $v \in \mathbb{C}$ (which can be taken to be $v = z$), because if $\dim(\mathrm{Im}(\sigma)) = 2$ we can write $\sigma = \sigma_1 z + \sigma_2 iz$, and the symmetry algebra is the intersection of the symmetries of the scalar forms σ_1, σ_2. Write $e_i = \lambda_i z$. σ always has the symmetry A

$$\begin{aligned}
&A(x) = \lambda_2 x - \lambda_1 ix\\
&A(ix) = \lambda_3 x - \lambda_2 ix
\end{aligned}$$

If σ is non-degenerate then all symmetries must be traceless and A is a basis for the symmetry algebra. σ is degenerate if and only if

$$\lambda_1\lambda_3 = \lambda_2^2$$

In this case σ has a kernel, which is $\mathfrak{h}$-invariant and therefore generated by a highest weight vector $\alpha x + \beta ix$. This means that up to sign

$$\begin{aligned}
\lambda_1 &= -\beta^2 \\
\lambda_2 &= \alpha\beta \\
\lambda_3 &= -\alpha^2
\end{aligned}$$

The symmetry algebra is $2d$ and generated by B, C such that

$$\begin{aligned}
B(x) &= \alpha x + \beta ix \\
B(ix) &= 0 \\
C(x) &= 0 \\
C(ix) &= \alpha x + \beta ix
\end{aligned}$$

In particular the symmetry A from above is

$$A = \beta B - \alpha C$$

The basis B, C satisfies the commutation relation

$$[B, C] = \alpha C - \beta B$$

From now on, denote the scalar form σ and the operators A, B, C corresponding to λ_i by $\sigma_\lambda, A_\lambda, B_\lambda, C_\lambda$. Earlier we used A, B, C to denote a basis of $\mathfrak{h}$, but there is no relation.

Let's apply this to our brackets. A maximal abelian symmetry algebra is 1 dimensional. Therefore A_z and A_{iz} are linearly dependent, so we can write

$$A_b = \theta(b)A_0$$

for some fixed $\theta \in \mathrm{Hom}(\mathbb{C}, \mathbb{R}), A_0 \in \mathrm{End}_{\mathfrak{h}}\mathfrak{m}_0$. We assume $\theta \neq 0$, so it must have a 1d kernel. Suppose $z \in \mathbb{C}$ is a basis of $\mathrm{Ker}(\theta)$. Write $\sigma = \sigma_\lambda z + \sigma_L iz$. The first Jacobi identity then becomes

$$A_0(\sigma_L(X, Y)Z + \sigma_L(Z, X)Y + \sigma_L(Y, Z)X) = 0$$

A_0 is a symmetry of σ_L. If σ_L is non-degenerate then its symmetry algebra is generated by A_L, which is invertible. This leads to $A_0 = 0$, so σ_L must be degenerate and if it is non-zero then $\mathrm{Ker}(\sigma_L) = \mathrm{Ker}(A_0)$, which means that

A_0 is proportional to $A_L = \beta B_L - \alpha C_L$. A_L must also be a symmetry of σ_λ so σ_L and σ_λ must be proportional because the traceless part of the symmetry algebra determines L, λ up to scaling. This means that $\mathrm{Im}(\sigma)$ is $1d$. We may also have σ_λ degenerate and non-zero, $\sigma_L = 0$. In that case we lose the restriction $A_0 = rA_\lambda$. Another alternative is that $\sigma_L = 0$, σ_λ is non-degenerate and

$$\theta(\mathrm{Im}(\sigma)) = 0$$

The case of non-degenerate σ_λ has the following brackets

$$\begin{aligned}
&[x, y] = \lambda_1 z \\
&[x, iy] = [ix, y] = \lambda_2 z \\
&[ix, iy] = \lambda_3 z \\
&[x, iz] = r(\lambda_2 x - \lambda_1 ix) \\
&[ix, iz] = r(\lambda_3 x - \lambda_2 ix) \\
&[y, iz] = r(\lambda_2 y - \lambda_1 iy) \\
&[iy, iz] = r(\lambda_3 y - \lambda_2 iy)
\end{aligned}$$

Here $\lambda_1\lambda_3 \neq \lambda_2^2$, and $r \in \mathbb{R}$. The Nijenhuis tensor is

$$\begin{aligned}
N_J(x, y) &= (\lambda_3 - \lambda_1)z - 2\lambda_2 iz \\
N_J(x, z) &= r(\lambda_3 - \lambda_1)x - 2r\lambda_2 ix \\
N_J(y, z) &= r(\lambda_3 - \lambda_1)y - 2r\lambda_2 iy
\end{aligned}$$

and it is non-degenerate unless $\lambda_1 = \lambda_3, \lambda_2 = 0$. In the case when $\sigma_L = 0$ but σ_λ is degenerate, the brackets are

$$\begin{aligned}
&[x, y] = -\beta^2 z \\
&[x, iy] = [ix, y] = \alpha\beta z \\
&[ix, iy] = -\alpha^2 z \\
&[x, iz] = k(\alpha x + \beta ix) \\
&[ix, iz] = l(\alpha x + \beta ix) \\
&[y, iz] = k(\alpha y + \beta iy) \\
&[iy, iz] = l(\alpha y + \beta iy)
\end{aligned}$$

which gives the Nijenhuis tensor

$$\begin{aligned}
N_J(x, y) &= (\beta^2 - \alpha^2)z - 2\alpha\beta iz \\
N_J(x, z) &= (l - ik)(\alpha x + \beta ix) \\
N_J(y, z) &= (l - ik)(\alpha y + \beta iy)
\end{aligned}$$

It is non-degenerate. The brackets when $\sigma_\lambda = \gamma\sigma_L, \sigma_L \neq 0$ are both degenerate are

$$
\begin{aligned}
[x,y] &= -\beta^2(\gamma z + iz) \\
[x,iy] &= \alpha\beta(\gamma z + iz) \\
[ix,y] &= \alpha\beta(\gamma z + iz) \\
[ix,iy] &= -\alpha^2(\gamma z + iz) \\
[x,iz] &= r(\beta\alpha x + \beta^2 ix) \\
[ix,iz] &= r(-\alpha^2 x - \alpha\beta ix) \\
[y,iz] &= r(\beta\alpha y + \beta^2 iy) \\
[iy,iz] &= r(-\alpha^2 y - \alpha\beta iy)
\end{aligned}
$$

which gives the Nijenhuis tensor

$$
\begin{aligned}
N_J(x,y) &= (\gamma(\beta^2-\alpha^2) + 2\alpha\beta)z + (\beta^2 - \alpha^2 - 2\gamma\alpha\beta)iz \\
N_J(x,z) &= r(-\alpha - i\beta)(\alpha x + \beta ix) \\
N_J(y,z) &= r(-\alpha - i\beta)(\alpha y + \beta iy)
\end{aligned}
$$

It is non-degenerate.

We must also consider the two possibilities $\sigma = 0$ and $\theta = 0$. The choice $\sigma = 0$ gives us

$$
\begin{aligned}
[X,z] &= \gamma X \\
[X,iz] &= C(X)
\end{aligned}
$$

such that $C \in \mathrm{End}_{\mathfrak{h}}\mathfrak{m}_0$, $\gamma = 1$ or $\gamma = 0$, with Nijenhuis tensor

$$
\begin{aligned}
N_J(x,y) &= 0 \\
N_J(x,z) &= [C,i]x \\
N_J(y,z) &= [C,i]y
\end{aligned}
$$

The other option $\theta = 0$ gives

$$
\begin{aligned}
[x,y] &= e_1 \\
[ix,iy] &= e_3 \\
[x,iy] &= e_2 \\
[ix,y] &= e_2
\end{aligned}
$$

without any restrictions on $e_1, e_2, e_3 \in \mathbb{C}$, and thus gives Nijenhuis tensor

$$
\begin{aligned}
N_J(x,y) &= e_3 - e_1 - 2ie_2 \\
N_J(x,z) &= 0 \\
N_J(y,z) &= 0
\end{aligned}
$$

Suppose now that $\mathbb{C}$ is non-abelian. The Jacobi identities become

$$\begin{aligned}
&A_{\sigma(Y,Z)}X + A_{\sigma(X,Y)}Z + A_{\sigma(Z,X)}Y = 0\\
&[A_z, A_{iz}] = -A_z\\
&[\sigma(X,Y), b] = \sigma(A_b X, Y) + \sigma(X, A_b Y)
\end{aligned}$$

Let's write $\sigma = \sigma_\lambda z + \sigma_L iz$, where σ_λ, σ_L are scalar forms as defined earlier. The third Jacobi identity becomes

$$[\sigma_\lambda(X,Y)z + \sigma_L(X,Y)iz, b] = (\sigma_\lambda(A_bX,Y) + \sigma_\lambda(X,A_bY))z + (\sigma_L(A_bX,Y) + \sigma_L(X,A_bY))iz$$

Since the iz-component of the right hand side must vanish, we get $A_b \in \mathrm{sym}(\sigma_L)$ for all b. From $[z, iz] = z$ we get

$$\begin{aligned}
&\sigma_\lambda(X,Y) = \sigma_\lambda(A_{iz}X,Y) + \sigma_\lambda(X,A_{iz}Y)\\
&\sigma_\lambda(A_zX,Y) + \sigma_\lambda(X,A_zY) = -\sigma_L(X,Y)
\end{aligned}$$

and we know that A_z is traceless because it is the commutator of A_{iz}, A_z. Since the symmetry algebra of non-degenerate σ_L would be commutative, we have that $A_z = 0$ in this case, a contradiction. Therefore σ_L must be degenerate and A_z is proportional to A_L if $\sigma_L \neq 0$. For degenerate σ_L, $A_L^2 = 0$, so by inserting the second of the two equations above into

$$\sigma_L(A_LX,Y) + \sigma_L(X,A_LY) = 0$$

we obtain

$$\sigma_\lambda(A_LX, A_LY) = 0$$

for all X, Y. This shows that the image of A_L, which is $\mathrm{Ker}(\sigma_L)$, is an isotropic subspace for σ_λ. If the kernels coincide, the forms are proportional and thus share their symmetry algebra. This is not possible, so $\mathrm{Ker}(\sigma_L) \neq \mathrm{Ker}(\sigma_\lambda)$. Thus σ_λ must be non-degenerate, because if it had a kernel distinct from its 2d isotropic submodule, it would be a 2d submodule and their direct sum would be isotropic as well. We also have

$$\sigma_\lambda(X,Y) = \sigma_\lambda(A_{iz}X,Y) + \sigma_\lambda(X,A_{iz}Y)$$

which makes it clear that the kernel of A_{iz} is another isotropic submodule and

$$A_{iz} = \frac{1}{2}\mathbb{1} + S$$

for some symmetry $S \in \mathrm{Sym}(\sigma_\lambda)$. Given a module decomposition $V^{\mathbb{C}} = \mathrm{Ker}(A_z) \oplus \mathrm{Ker}(A_{iz})$ and the knowledge that both of these are isotropic, σ_λ is determined up to scaling as it must be contained in $\mathrm{Ker}(A_z) \otimes \mathrm{Ker}(A_{iz})$ which contains only a 1d trivial module component. A_{iz} can be written as

$$A_{iz} = kB_L + lC_L$$

which means that

$$\mathrm{Ker}(A_{iz}) = \langle -lx + kix \rangle_{\mathfrak{h}}$$

The other expression for A_{iz} above gives the additional condition

$$k\alpha + l\beta = 1$$

because S is traceless so $\mathrm{Tr}(A_{iz}) = \mathrm{Tr}(\frac{1}{2}\mathbb{1})$. We write the equations for isotropy of the kernels of A_z, A_{iz} and the tensor product being non-zero:

$$\begin{aligned}
&\sigma_\lambda(\alpha x + \beta ix, -ly + kiy) = \gamma \\
&\sigma_\lambda(\alpha x + \beta ix, \alpha y + \beta iy) = 0 \\
&\sigma_\lambda(-lx + kix, -ly + kiy) = 0
\end{aligned}$$

where γ is an arbitrary scaling factor. This system is linear in $\lambda_1, \lambda_2, \lambda_3$, and writing it in terms of these yields

$$\begin{aligned}
&-\alpha l\lambda_1 + (\alpha k - \beta l)\lambda_2 + \beta k\lambda_3 = \gamma \\
&\alpha^2\lambda_1 + 2\alpha\beta\lambda_2 + \beta^2\lambda_3 = 0 \\
&l^2\lambda_1 - 2lk\lambda_2 + k^2\lambda_3 = 0
\end{aligned}$$

The solution is

$$\begin{aligned}
\lambda_1 &= -2\gamma\beta k \\
\lambda_2 &= \gamma(\alpha k - \beta l) \\
\lambda_3 &= 2\gamma\alpha l
\end{aligned}$$

We will also write

$$A_z = rA_L = r(\beta B_L - \alpha A_L)$$

and consider again the equation

$$\sigma_\lambda(A_z X, Y) + \sigma_\lambda(X, A_z Y) = -\sigma_L(X, Y)$$

It is clear that the right and left hand side are proportional, because we know from previous considerations that they have the same kernel. We determine the constant of proportionality γ by inserting $X = -lx + kix, Y = -ly + kiy$. This gives

$$
\begin{aligned}
-2r\gamma &= 1 \\
\gamma &= \frac{-1}{2r}
\end{aligned}
$$

The equation

$$A_{iz} = \frac{1}{2}\mathbb{1} + S$$

Is satisfied as well, which we can verify by checking that

$$S = -rA_\lambda$$

We also have the commutation relation

$$[A_z, A_{iz}] = -A_z$$

which holds because $k\alpha + l\beta = 1$. The only Jacobi identity left to be satisfied is then

$$A_{\sigma(Y,Z)}X + A_{\sigma(X,Y)}Z + A_{\sigma(Z,X)}Y = 0$$

which becomes

$$
\begin{aligned}
&A_z(\sigma_\lambda(Y,Z)X + \sigma_\lambda(X,Y)Z + \sigma_\lambda(Z,X)Y)+ \\
&+ A_{iz}(\sigma_L(Y,Z)X + \sigma_L(X,Y)Z + \sigma_L(Z,X)Y) = 0
\end{aligned}
$$

There are two cases to consider: $X, Y \in \mathrm{Ker}(A_z), Z \in \mathrm{Ker}(A_{iz})$ and $Z \in \mathrm{Ker}(A_z), X, Y \in \mathrm{Ker}(A_{iz})$. It is easy to check that the former case vanishes by using kernels and isotropy with respect to σ_λ. The latter case yields

$$A_z(\sigma_\lambda(Y,Z)X + \sigma_\lambda(Z,X)Y) + A_{iz}(\sigma_L(X,Y)Z) = 0$$

and trying for example $X = -lx + kix, Y = -ly + kiy, Z = \alpha x + \beta ix$ gives

$$A_z(\sigma_\lambda(Y,Z)X + \sigma_\lambda(Z,X)Y) + A_{iz}(\sigma_L(X,Y)Z) = -\frac{3}{2}(\alpha x + \beta ix)$$

which shows that it fails unless $\alpha = \beta = 0$, which we take to mean $\sigma_L = 0$. Many of the restrictions that were placed on σ_λ were caused by assuming $\sigma_L \neq 0$, these will no longer be valid. The Jacobi identities are now

$$
\begin{aligned}
&A_z(\sigma_\lambda(Y,Z)X + \sigma_\lambda(X,Y)Z + \sigma_\lambda(Z,X)Y) = 0\\
&[A_z, A_{iz}] = -A_z\\
&\sigma_\lambda(X,Y) = \sigma_\lambda(A_{iz}X,Y) + \sigma_\lambda(X,A_{iz}Y)\\
&\sigma_\lambda(A_zX,Y) + \sigma_\lambda(X,A_zY) = 0
\end{aligned}
$$

The third and fourth identities say that A_z is a symmetry of σ_λ and $A_{iz} = \frac{1}{2}\mathbb{1}+S$ where S is another symmetry. The second identity says that if σ_λ is non-degenerate, then $A_z = 0$ because the symmetry algebra is 1d and commutative in this case. This satisfies all other Jacobi identities as well. The Lie algebra structures in this case is

$$
\begin{aligned}
&[x,y] = \lambda_1 z\\
&[x,iy] = \lambda_2 z\\
&[ix,y] = \lambda_2 z\\
&[ix,iy] = \lambda_3 z\\
&[x,iz] = \frac{1}{2}x + r(\lambda_2 x - \lambda_1 ix)\\
&[ix,iz] = \frac{1}{2}ix + r(\lambda_3 x - \lambda_2 ix)\\
&[y,iz] = \frac{1}{2}y + r(\lambda_2 y - \lambda_1 iy)\\
&[iy,iz] = \frac{1}{2}iy + r(\lambda_3 y - \lambda_2 iy)\\
&[z,iz] = z
\end{aligned}
$$

with $\lambda_1\lambda_3 \neq \lambda_2^2$, the condition for non-degeneracy. The Nijenhuis tensor is

$$
\begin{aligned}
&N_J(x,y) = (\lambda_3 - \lambda_1)z - 2\lambda_2 iz\\
&N_J(x,z) = r(\lambda_3 - \lambda_1)x - 2\lambda_2 ix\\
&N_J(y,z) = r(\lambda_3 - \lambda_1)y - 2\lambda_2 iy
\end{aligned}
$$

If we keep $A_z = 0$, the Jacobi identities are still satisfied for degenerate σ_λ with kernel generated by $\alpha x+\beta ix$. There are now two parameters k,l for determining the symmetry S. The algebra structures are then given by

$$
\begin{aligned}
&[x, y] = -\beta^2 z \\
&[x, iy] = \alpha\beta z \\
&[ix, y] = \alpha\beta z \\
&[ix, iy] = -\alpha^2 z \\
&[x, iz] = \frac{1}{2}x + k(\alpha x + \beta ix) \\
&[ix, iz] = \frac{1}{2}ix + l(\alpha x + \beta ix) \\
&[y, iz] = \frac{1}{2}y + k(\alpha y + \beta iy) \\
&[iy, iz] = \frac{1}{2}iy + l(\alpha y + \beta iy) \\
&[z, iz] = z
\end{aligned}
$$

with corresponding Nijenhuis tensor

$$
\begin{aligned}
&N_J(x, y) = (\beta^2 - \alpha^2)z - 2\alpha\beta iz \\
&N_J(x, z) = (l\alpha + k\beta)x + (l\beta - k\alpha)ix \\
&N_J(y, z) = (l\alpha + k\beta)y + (l\beta - k\alpha)iy
\end{aligned}
$$

Finally we have the case $A_z = rA_\lambda \neq 0$. In this case $A_{iz} \neq 0$ as well, and we get the condition $k\alpha + l\beta = 1$ from their commutation relation. The algebra structures are

$$
\begin{aligned}
&[x, y] = -\beta^2 z \\
&[x, iy] = \alpha\beta z \\
&[ix, y] = \alpha\beta z \\
&[ix, iy] = -\alpha^2 z \\
&[x, z] = r(\alpha\beta x + \beta^2 ix) \\
&[ix, z] = r(-\alpha^2 x - \alpha\beta ix) \\
&[y, z] = r(\alpha\beta y + \beta^2 iy) \\
&[iy, z] = r(-\alpha^2 y - \alpha\beta iy) \\
&[x, iz] = \frac{1}{2}x + k(\alpha x + \beta ix) \\
&[ix, iz] = \frac{1}{2}ix + l(\alpha x + \beta ix) \\
&[y, iz] = \frac{1}{2}y + k(\alpha y + \beta iy) \\
&[iy, iz] = \frac{1}{2}iy + l(\alpha y + \beta iy) \\
&[z, iz] = z
\end{aligned}
$$

with Nijenhuis tensor

$$\begin{aligned}
N_J(x,y) &= (\beta^2-\alpha^2)z - 2\alpha\beta iz \\
N_J(x,z) &= (l\alpha + k\beta - 2r\alpha\beta)x + (l\beta - k\alpha + r(\alpha^2-\beta^2))ix \\
N_J(y,z) &= (l\alpha + k\beta - 2r\alpha\beta)y + (l\beta - k\alpha + r(\alpha^2-\beta^2))iy
\end{aligned}$$

In the case $\sigma_\lambda = 0$ we get

$$\begin{aligned}
[X,z] &= AX \\
[X,iz] &= BX \\
[z,iz] &= z
\end{aligned}$$

such that $A, B \in \mathrm{End}_{\mathfrak{h}}\mathfrak{m}_0$ satisfy $[A,B] = -A$. This yields the Nijenhuis tensor

$$\begin{aligned}
N_J(x,y) &= 0 \\
N_J(x,z) &= -(i[A,i] + [B,i])x \\
N_J(y,z) &= -(i[A,i] + [B,i])y
\end{aligned}$$

Let's now consider the possibility of $\mathfrak{g}$ with nonzero $\mathfrak{h}$-component. If $\mathfrak{h}$ is not contained in a strictly bigger semi-simple subalgebra, then by Levi decomposition $\mathfrak{m}$ is conjugate to the radical of $\mathfrak{g}$. Since the radical is an ideal, $\mathfrak{h}$ is conjugate in $\mathfrak{g}$ to another subalgebra $\mathfrak{h}' \simeq \mathfrak{h}$ such that the bracket has vanishing $\mathfrak{h}'$-component. This means that the homogenous space is equivalent to a space considered previously.

If $\mathfrak{h}$ is strictly contained in a semi-simple subalgebra $Q \subset \mathfrak{g}$, Q must have dimension $6, 8$ or 9. Since only the bracket between two different copies of V can go to $\mathfrak{h}$, both of them must be in Q and this would make Q at least dim 7. All options of dim 9 contain ideals which would necessarily also be submodules, and such submodules are not present. So we are left with dim 8.

This was explored in the section for $\mathfrak{m} = V \oplus V$, $\mathfrak{h} = \mathfrak{su}(1,1) \oplus \mathbb{R}$, with a subset $\varepsilon \subset \mathbb{C}$ playing the role that was earlier taken by the center in $\mathfrak{h}$. This process yields Q isomorphic to $\mathfrak{sl}_3$ or $\mathfrak{su}(2,1)$. Since these algebras have no non-trivial 1d representations, $\mathfrak{g} = Q \oplus \mathbb{R}$. $N_J = 0$ because $[x,y] \in \mathfrak{h}$. The brackets are

$$\begin{aligned}
[x,y] &= -3\alpha z \\
[ix,iy] &= -3\alpha z \\
[x,ix] &= \alpha(A+B) \\
[y,iy] &= \alpha(A-B) \\
[x,iy] &= -\alpha C \\
[ix,y] &= \alpha C \\
[z,x] &= ix \\
[z,ix] &= -x \\
[z,y] &= iy \\
[z,iy] &= -y
\end{aligned}$$

Where A, B, C means the basis of $\mathfrak{h}$ given earlier.

4.3 $\mathfrak{sl}_2(\mathbb{C})$

4.3.1 $\mathfrak{m} = W$

Denote the tautological representation of $\mathfrak{h}$ on $\mathbb{C}^2$ by W, and let $\mathfrak{n} = \langle A, B, C\rangle$ be a real subalgebra of type $\mathfrak{su}(1,1)$. W restricted to $\mathfrak{n}$ is of type $V^{\mathbb{C}}$, so we produce the basis x, y, ix, iy with respect to $\mathfrak{n}$ as described in the section $\mathfrak{h} = \mathfrak{su}(1,1)$. A, B, C, iA, iB, iC is a basis of $\mathfrak{h}$. The last three elements act as the first three composed with J. Note that the subalgebra $\langle A, iB, iC\rangle = \langle u, m, k\rangle$ is of type $\mathfrak{su}(2)$ and precisely the same operators as we described in the section $\mathfrak{h} = \mathfrak{su}(2), \mathfrak{m} = W$. An endomorphism of a $\mathfrak{sl}_2(\mathbb{C})$-module is equivariant if and only if it is equivariant with respect to both the real forms $\mathfrak{su}(1,1), \mathfrak{su}(2)$ that we specified. Now we may decompose

$$\Lambda^2_{\mathbb{R}} W = U \oplus \mathbb{C}$$

U is the tautological representation of $\mathfrak{so}(3,1) \simeq \mathfrak{sl}_2(\mathbb{C})$, and it is irreducible but not isomorphic to W. $\mathbb{C}$ is a trivial module. The concrete decomposition is

$$\begin{aligned}
&\langle x \wedge iy + ix \wedge y, x \wedge y - ix \wedge iy\rangle \simeq \mathbb{C} \\
&\langle x \wedge ix + y \wedge iy, x \wedge ix - y \wedge iy, x \wedge iy - ix \wedge y, x \wedge y + ix \wedge iy\rangle \simeq U
\end{aligned}$$

Since no bracket $\Lambda^2_{\mathbb{R}} W \to \mathfrak{h} \oplus \mathfrak{m}$ is possible $\mathfrak{g}$ must be flat.

4.3.2 $\mathfrak{m} = Ad$

For $\mathfrak{m} = Ad$, the isotropy representation is irreducible. This means that a non-flat algebra structure, if one exists, is unique at least up to a change of sign on all equivariant maps. $\mathfrak{g}$ must therefore either be flat or be of type $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$, which is stable under a sign change. Since the Lie algebra is complex and the complex structure on the module is unique, the non-flat structure is integrable. The bracket is the same as on $\mathfrak{sl}_2(\mathbb{C})$.

4.3.3 $\mathfrak{m} = W \oplus \mathbb{C}$

By the earlier decomposition of $\Lambda^2 W$, there is no possibility for a bracket to $\mathfrak{h}$. Thus $\mathfrak{m}$ must be a solvable ideal. Since $\mathfrak{h}$ contains both $\mathfrak{su}(2)$ and $\mathfrak{su}(1,1)$ and these are represented in the same way as in their respective sections, the algebra structure on $\mathfrak{m}$ must be equivariant with respect to both of these. The endomorphism ring $\mathrm{End}_{\mathfrak{h}} W$ is only complex scalars in this case. Following the development in the section about $\mathfrak{su}(2)$, we get the following assuming that both brackets $\Lambda^2 W \to \mathbb{C}$ and $W \otimes \mathbb{C} \to W$ are non-zero.

$$\begin{aligned}
[X, Y] &= g(BX, Y)z \\
[X, iz] &= \frac{\delta}{2} X + \beta B X \\
[z, iz] &= \delta z
\end{aligned}$$

Here $X, Y \in W$, $z \in \mathbb{C}$, $\beta \in \mathbb{R}$, $B \in \mathrm{End}_{\mathfrak{su}(2)} W, \mathrm{Tr}(B) = 0$ and δ is 0 or 1. g is the real part of the $\mathfrak{su}_2$ invariant hermitian form. B must now be proportional to J because the endomorphism ring is smaller. g is not invariant under $\mathfrak{su}(1,1)$ while J is, so these brackets are not $\mathfrak{sl}_2(\mathbb{C})$ invariant. Therefore one of the brackets must vanish, and we get either a semi-direct product $W \rtimes \mathbb{C}$ or the derived subalgebra of $\mathfrak{m}$ is contained in $\mathbb{C}$. The first case is either

$$\begin{aligned}
[X, z] &= \beta X \\
[X, iz] &= \alpha X + \gamma J X
\end{aligned}$$

or

$$\begin{aligned}
[X, iz] &= \alpha X + \gamma J X \\
[z, iz] &= z
\end{aligned}$$

which corresponds to respectively abelian or non-abelian $\mathbb{C}$. The Nijenhuis tensor vanishes in both cases. Finally we may have an invariant skew-symmetric form on W taking values in $\mathbb{C}$. Following the $\mathfrak{su}(2)$ development again, we must then have abelian $\mathbb{C}$. We use the decomposition of $\Lambda^2 W$ given earlier to write the brackets

$$\begin{aligned}
[x, y] &= -[ix, iy] = v \\
[x, iy] &= [ix, y] = w
\end{aligned}$$

for some $v, w \in \mathbb{C}$. The Nijenhuis tensor is then

$$\begin{aligned}
N_J(x, y) &= -2v + 2iw \\
N_J(x, z) &= 0 \\
N_J(y, z) &= 0
\end{aligned}$$

4.4 $\mathfrak{sl}_3$

4.4.1 $\mathfrak{m} = V^{\mathbb{C}}$

We have that for the tautological representation V, $V^{\mathbb{C}} = V \oplus V$. We compute

$$\begin{aligned}
\Lambda^2 \mathfrak{m} &= \Lambda^2 V \oplus \Lambda^2 V \oplus V \otimes V \\
\Lambda^2 V &= V^* \\
V \otimes V &= U^6 \oplus V^*
\end{aligned}$$

where U^6 is irreducible. Thus there is no bracket to $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$

4.4.2 $\mathfrak{m} = (V^*)^{\mathbb{C}}$

We have that for the dual representation V^*, $(V^*)^{\mathbb{C}} = V^* \oplus V^*$. We compute

$$\begin{aligned}
\Lambda^2 \mathfrak{m} &= \Lambda^2 V^* \oplus \Lambda^2 V^* \oplus V^* \otimes V^* \\
\Lambda^2 V^* &= V \\
V^* \otimes V^* &= U^6 \oplus V
\end{aligned}$$

where U^6 is irreducible. Thus there is no bracket to $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$

4.5 $\mathfrak{su}(2,1)$

The Lie algebra $\mathfrak{h} = \mathfrak{su}(2,1)$ has two non-isomorphic complex representations of dimension 3. These are the tautological representation, which we denote by W, and its dual representation W^*. Since $\mathfrak{h}$ preserves a hermitian form on W, we have an equivariant $\mathbb{C}$-anti-linear map from W to W^*. Therefore they are equivalent as real representations, and since only the real module structure is interesting for determining real Lie algebra structures we need only treat one of them. Our choice will be W.

The endomorphism ring $\mathrm{End}_{\mathfrak{h}}(W)$ is isomorphic to the complex numbers. This is important because any bracket may be post-composed with an endomorphism.

To compute the decomposition of $\Lambda^2\mathfrak{m}$, note that

$$(\Lambda^2_{\mathbb{R}}W)^{\mathbb{C}} = \Lambda^2_{\mathbb{C}}(W^{\mathbb{C}}) = \Lambda^2_{\mathbb{C}}(W \oplus W^*) = (W \oplus W^* \oplus W \otimes_{\mathbb{C}} W^*) = (W \oplus \mathfrak{u}(2,1))^{\mathbb{C}}$$

Here $\mathfrak{u}(2,1)$ means the module $Ad \oplus \varepsilon$. Let $x_k, k = 1,2,3$ be the standard complex basis of W. Then $x_k, ix_k, k = 1,2,3$ is a real basis with dual basis $\hat{x}_k, i\hat{x}_k, k = 1,2,3$. Denote the real part of the invariant hermitian form h by g and the imaginary part by ω. The projection

$$\Lambda^2 W \to \mathfrak{u}(2,1)$$

is given by contraction with the tensor

$$g \otimes \mathbb{1} + \omega \otimes J$$

and removing trace gives the map to $\mathfrak{su}(2,1)$, which is

$$x \wedge y \mapsto \gamma(\iota_x g \otimes y - \iota_y g \otimes x + \iota_x \omega \otimes Jy - \iota_y \omega \otimes Jx - \frac{2}{3}\omega(x,y)J)$$

for arbitrary x, y. This is the $\mathfrak{h}$-component of our bracket, and can be written in terms of our basis as

$$\begin{aligned}
&[x_1, ix_1] = 2\gamma(\hat{x}_1 \otimes ix_1 - i\hat{x}_1 \otimes x_1 - \frac{1}{3}J)\\
&[x_2, ix_2] = 2\gamma(\hat{x}_2 \otimes ix_2 - i\hat{x}_2 \otimes x_2 - \frac{1}{3}J)\\
&[x_3, ix_3] = 2\gamma(-\hat{x}_3 \otimes ix_3 + i\hat{x}_3 \otimes x_3 + \frac{1}{3}J)\\
&[x_1, x_2] = [ix_1, ix_2] = \gamma(\hat{x}_1 \otimes x_2 - \hat{x}_2 \otimes x_1 + i\hat{x}_1 \otimes ix_2 - i\hat{x}_2 \otimes ix_1)\\
&[x_1, x_3] = [ix_1, ix_3] = \gamma(\hat{x}_1 \otimes x_3 + \hat{x}_3 \otimes x_1 + i\hat{x}_1 \otimes ix_3 + i\hat{x}_3 \otimes ix_1)\\
&[x_2, x_3] = [ix_2, ix_3] = \gamma(\hat{x}_2 \otimes x_3 + \hat{x}_3 \otimes x_2 + i\hat{x}_2 \otimes ix_3 + i\hat{x}_3 \otimes ix_2)\\
&[ix_1, x_2] = -[x_1, ix_2] = \gamma(-\hat{x}_1 \otimes ix_2 - \hat{x}_2 \otimes ix_1 + i\hat{x}_1 \otimes x2 + i\hat{x}_2 \otimes x_1)\\
&[ix_1, x_3] = -[x_1, ix_3] = \gamma(-\hat{x}_1 \otimes ix_3 + \hat{x}_3 \otimes ix_1 + i\hat{x}_1 \otimes x3 - i\hat{x}_3 \otimes x_1)\\
&[ix_2, x_3] = -[x_2, ix_3] = \gamma(-\hat{x}_2 \otimes ix_3 + \hat{x}_3 \otimes ix_2 + i\hat{x}_2 \otimes x3 - i\hat{x}_3 \otimes x_2)
\end{aligned}$$

There is a 6d real submodule corresponding to $\Lambda^2_{\mathbb{C}} W$. This must be isomorphic to W as a real module because it has an invariant complex structure.

$$\langle x_m \wedge x_k - ix_m \wedge ix_k, ix_m \wedge x_k + x_m \wedge ix_k \rangle \simeq W$$

for $k, m \in \{1, 2, 3\}, k \neq m$. This means that the $\mathfrak{m}$-components of the brackets are

$$\begin{aligned}
[x_1, x_2] &= -[ix_1, ix_2] = (\alpha + i\beta)x_3 \\
[x_1, x_3] &= -[ix_1, ix_3] = -(\alpha + i\beta)x_2 \\
[x_2, x_3] &= -[ix_2, ix_3] = (\alpha + i\beta)x_1 \\
[x_1, ix_2] &= [ix_1, x_2] = -i(\alpha + i\beta)x_3 \\
[x_1, ix_3] &= [ix_1, x_3] = i(\alpha + i\beta)x_2 \\
[x_2, ix_3] &= [ix_2, x_3] = -i(\alpha + i\beta)x_1
\end{aligned}$$

Note that the $\mathfrak{m}$-component of the bracket is $\mathbb{C}$-anti-linear in both arguments. We compute Jacobi identities

$$\begin{aligned}
[x_1, [ix_1, x_2]] + [x_2, [x_1, ix_1]] + [ix_1, [x_2, x_1]] &= 2(\frac{4\gamma}{3} - \alpha^2 - \beta^2)ix_2 = 0 \\
[x_3, [ix_3, x_1]] + [x_1, [x_3, ix_3]] + [ix_3, [x_1, x_3]] &= 2(-\frac{4\gamma}{3} - \alpha^2 - \beta^2)ix_2 = 0
\end{aligned}$$

The only solution is $\alpha = \beta = \gamma = 0$ so $\mathfrak{g}$ is flat.

4.6 $\mathfrak{su}(3)$

4.6.1 $\mathfrak{m} = W$

The Lie algebra $\mathfrak{h} = \mathfrak{su}(3)$ has two non-isomorphic complex representations of dimension 3. These are the tautological representation, which we denote by W, and its dual representation W^*. Since $\mathfrak{h}$ preserves a hermitian form on W, we have an equivariant $\mathbb{C}$-anti-linear map from W to W^*. Therefore they are equivalent as real representations, and since only the real module structure is interesting for determining real Lie algebra structures we need only treat one of them. Our choice will be W.

The endomorphism ring $\mathrm{End}_{\mathfrak{h}}(W)$ is isomorphic to the complex numbers. This is important because any bracket may be post-composed with an endomorphism.

To compute the decomposition of $\Lambda^2\mathfrak{m}$, note that

$$(\Lambda^2_{\mathbb{R}} W)^{\mathbb{C}} = \Lambda^2_{\mathbb{C}}(W^{\mathbb{C}}) = \Lambda^2_{\mathbb{C}}(W \oplus W^*) = (W \oplus W^* \oplus W \otimes_{\mathbb{C}} W^*) = (W \oplus \mathfrak{u}(3))^{\mathbb{C}}$$

Here $\mathfrak{u}(3)$ means the module $Ad \oplus \varepsilon$. Let $x_k, k = 1, 2, 3$ be the standard complex basis of W. Then $x_k, ix_k, k = 1, 2, 3$ is a real basis with dual basis $\hat{x}_k, i\hat{x}_k, k = 1, 2, 3$. Denote the real part of the invariant hermitian form h by g and the imaginary part by ω. The projection

$$\Lambda^2 W \to \mathfrak{u}(3)$$

is given by contraction with the tensor

$$g \otimes \mathbb{1} + \omega \otimes J$$

such that

$$x_m \wedge x_k \mapsto \hat{x}_m \otimes x_k - \hat{x}_k \otimes x_m + i\hat{x}_m \otimes ix_k - i\hat{x}_k \otimes ix_m$$

and projection to $\mathfrak{su}(3)$ is then only a matter of removing trace. The trivial submodule is then

$$\langle x_1 \wedge ix_1 + x_2 \wedge ix_2 + x_3 \wedge ix_3 \rangle$$

so the full projection to $\mathfrak{su}(3)$ is

$$x \wedge y \mapsto \gamma(\iota_x g \otimes y - \iota_y g \otimes x + \iota_x \omega \otimes Jy - \iota_y \omega \otimes Jx - \frac{2}{3}\omega(x,y)J)$$

for arbitrary x, y. This is the $\mathfrak{h}$-component of our bracket, and can be written in terms of our basis as

$$\begin{aligned}
&[x_1, ix_1] = 2\gamma(\hat{x}_1 \otimes ix_1 - i\hat{x}_1 \otimes x_1 - \frac{1}{3}J) \\
&[x_2, ix_2] = 2\gamma(\hat{x}_2 \otimes ix_2 - i\hat{x}_2 \otimes x_2 - \frac{1}{3}J) \\
&[x_3, ix_3] = 2\gamma(\hat{x}_3 \otimes ix_3 - i\hat{x}_3 \otimes x_3 - \frac{1}{3}J) \\
&[x_1, x_2] = [ix_1, ix_2] = \gamma(\hat{x}_1 \otimes x_2 - \hat{x}_2 \otimes x_1 + i\hat{x}_1 \otimes ix_2 - i\hat{x}_2 \otimes ix_1) \\
&[x_1, x_3] = [ix_1, ix_3] = \gamma(\hat{x}_1 \otimes x_3 - \hat{x}_3 \otimes x_1 + i\hat{x}_1 \otimes ix_3 - i\hat{x}_3 \otimes ix_1) \\
&[x_2, x_3] = [ix_2, ix_3] = \gamma(\hat{x}_2 \otimes x_3 - \hat{x}_3 \otimes x_2 + i\hat{x}_2 \otimes ix_3 - i\hat{x}_3 \otimes ix_2) \\
&[ix_1, x_2] = -[x_1, ix_2] = \gamma(-\hat{x}_1 \otimes ix_2 - \hat{x}_2 \otimes ix_1 + i\hat{x}_1 \otimes x2 + i\hat{x}_2 \otimes x_1) \\
&[ix_1, x_3] = -[x_1, ix_3] = \gamma(-\hat{x}_1 \otimes ix_3 - \hat{x}_3 \otimes ix_1 + i\hat{x}_1 \otimes x3 + i\hat{x}_3 \otimes x_1) \\
&[ix_2, x_3] = -[x_2, ix_3] = \gamma(-\hat{x}_2 \otimes ix_3 - \hat{x}_3 \otimes ix_2 + i\hat{x}_2 \otimes x3 + i\hat{x}_3 \otimes x_2)
\end{aligned}$$

There is a 6d real submodule corresponding to $\Lambda^2_{\mathbb{C}} W$. This must be isomorphic to W as a real module because it has an invariant complex structure.

$$\langle x_m \wedge x_k - ix_m \wedge ix_k, ix_m \wedge x_k + x_m \wedge ix_k \rangle \simeq W$$

for $k, m \in \{1, 2, 3\}, k \neq m$. This means that the $\mathfrak{m}$-components of the brackets are

$$
\begin{aligned}
[x_1, x_2] &= -[ix_1, ix_2] = (\alpha + i\beta)x_3 \\
[x_1, x_3] &= -[ix_1, ix_3] = -(\alpha + i\beta)x_2 \\
[x_2, x_3] &= -[ix_2, ix_3] = (\alpha + i\beta)x_1 \\
[x_1, ix_2] &= [ix_1, x_2] = -i(\alpha + i\beta)x_3 \\
[x_1, ix_3] &= [ix_1, x_3] = i(\alpha + i\beta)x_2 \\
[x_2, ix_3] &= [ix_2, x_3] = -i(\alpha + i\beta)x_1
\end{aligned}
$$

Note that the $\mathfrak{m}$-component of the bracket is $\mathbb{C}$-anti-linear in both arguments. The only non-trivial Jacobi identities are those with 2 out of 3 subscripts equal, for example

$$[x_1, [ix_1, x_2]] + [x_2, [x_1, ix_1]] + [ix_1, [x_2, x_1]] = 2(\frac{4\gamma}{3} - \alpha^2 - \beta^2)ix_2 = 0$$

so we set $\gamma = \frac{3}{4}(\alpha^2 + \beta^2)$. This satisfies the other identities as well. We may make a change of basis by complex multiplication such that

$$
\begin{aligned}
\alpha &= 2 \\
\beta &= 0 \\
\gamma &= 3
\end{aligned}
$$

which makes the Lie algebra structure

$$
\begin{aligned}
[x_1, ix_1] &= (6\hat{x}_1 \otimes ix_1 - 6i\hat{x}_1 \otimes x_1 - 2J) \\
[x_2, ix_2] &= (6\hat{x}_2 \otimes ix_2 - 6i\hat{x}_2 \otimes x_2 - 2J) \\
[x_3, ix_3] &= (6\hat{x}_3 \otimes ix_3 - 6i\hat{x}_3 \otimes x_3 - 2J) \\
[x_1, x_2] &= 3(\hat{x}_1 \otimes x_2 - \hat{x}_2 \otimes x_1 + i\hat{x}_1 \otimes ix_2 - i\hat{x}_2 \otimes ix_1) + 2x_3 \\
[x_1, x_3] &= 3(\hat{x}_1 \otimes x_3 - \hat{x}_3 \otimes x_1 + i\hat{x}_1 \otimes ix_3 - i\hat{x}_3 \otimes ix_1) - 2x_2 \\
[x_2, x_3] &= 3(\hat{x}_2 \otimes x_3 - \hat{x}_3 \otimes x_2 + i\hat{x}_2 \otimes ix_3 - i\hat{x}_3 \otimes ix_2) + 2x_1 \\
[ix_1, x_2] &= 3(-\hat{x}_1 \otimes ix_2 - \hat{x}_2 \otimes ix_1 + i\hat{x}_1 \otimes x_2 + i\hat{x}_2 \otimes x_1) - 2ix_3 \\
[x_1, ix_2] &= -3(-\hat{x}_1 \otimes ix_2 - \hat{x}_2 \otimes ix_1 + i\hat{x}_1 \otimes x_2 + i\hat{x}_2 \otimes x_1) - 2ix_3 \\
[ix_1, x_3] &= 3(-\hat{x}_1 \otimes ix_3 - \hat{x}_3 \otimes ix_1 + i\hat{x}_1 \otimes x_3 + i\hat{x}_3 \otimes x_1) + 2ix_2 \\
[x_1, ix_3] &= -3(-\hat{x}_1 \otimes ix_3 - \hat{x}_3 \otimes ix_1 + i\hat{x}_1 \otimes x_3 + i\hat{x}_3 \otimes x_1) + 2ix_2 \\
[ix_2, x_3] &= 3(-\hat{x}_2 \otimes ix_3 - \hat{x}_3 \otimes ix_2 + i\hat{x}_2 \otimes x_3 + i\hat{x}_3 \otimes x_2) - 2ix_1 \\
[x_2, ix_3] &= -3(-\hat{x}_2 \otimes ix_3 - \hat{x}_3 \otimes ix_2 + i\hat{x}_2 \otimes x_3 + i\hat{x}_3 \otimes x_2) - 2ix_1 \\
[ix_1, ix_2] &= 3(\hat{x}_1 \otimes x_2 - \hat{x}_2 \otimes x_1 + i\hat{x}_1 \otimes ix_2 - i\hat{x}_2 \otimes ix_1) - 2x_3 \\
[ix_1, ix_3] &= 3(\hat{x}_1 \otimes x_3 - \hat{x}_3 \otimes x_1 + i\hat{x}_1 \otimes ix_3 - i\hat{x}_3 \otimes ix_1) + 2x_2 \\
[ix_2, ix_3] &= 3(\hat{x}_2 \otimes x_3 - \hat{x}_3 \otimes x_2 + i\hat{x}_2 \otimes ix_3 - i\hat{x}_3 \otimes ix_2) - 2x_1
\end{aligned}
$$

It is easy to compute the Nijenhuis tensor because it only depends on the $\mathfrak{m}$-component, which is anti-linear. Thus we have

$$\begin{aligned}
N_J(x_1, x_2) &= -8x_3 \\
N_J(x_1, x_3) &= 8x_2 \\
N_J(x_2, x_3) &= -8x_1
\end{aligned}$$

We can identify this Lie algebra as $\mathfrak{g} = \mathfrak{g}_2$, the compact form of the exceptional Lie algebra. The simply connected version of G_2 gives us the homogeneous space $G_2/SU(3) = S^6$, and J is the Calabi structure.

4.7 $\mathfrak{sl}_3(\mathbb{C})$

The representations are the same as for $\mathfrak{sl}_3$, and this is embedded in $\mathfrak{sl}_3(\mathbb{C})$. Only the flat case was realized under the smaller isotropy algebra. Therefore $\mathfrak{sl}_3(\mathbb{C})$ has no non-flat cases as well.

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