Kepler's Analysis of the Dynamics of Planetary Motion

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Introduction

The main disputes among Kepler scholars revolve around physical hypotheses. These disputes concern especially two problems: the first is whether Kepler's analyses of physical causes were sound, and the second is whether his physical hypotheses played any significant role in the development of his laws. One of Kepler's hypotheses that manifest these two problems is the one developed on the basis of magnetism, presented in chapter 57 of *Astronomia nova*. I claim that this physical hypothesis is sound and, furthermore, a decisive and necessary part of Kepler's solution to the correct orbit of Mars. My main claim is that Kepler developed a consistent causal chain modelled after magnetism and thus succeeds in giving a causal explanation for the radial motion.

Kepler's analysis and explanation of the radial motion consists of three parts, which I will present in this paper in three separate sections. In step one I will demonstrate why Kepler thought he had found the correct and "natural" distance between the Sun and planet. In step two, I will demonstrate how he developed a quantitative expression for net power[1] based on a physical principle; the law of the lever. In step three, I will go through his proof that the expression of net power causes the distances presented in part one. However, in step three Kepler met some obstacles that temporarily prevented him from performing a mathematically rigorous proof of the causal chain. The obstacles are connected to a lack of interconnection of the variables involved. Our secondary claim in this paper is that the true orbit was developed partly on the basis of the analysis in chapter 57; and that the true orbit finally settled the causal chain for the radial component of motion.

Step one – Radial motion

In the beginning of the year 1605 Kepler found the distance between the Sun and planet which he later that year showed to be consistent with the orbit being an ellipse. But it was not the data alone that convinced Kepler that this distance was correct. The manner in which this distance varied convinced him that this component of the planetary motion could be explained by a 'natural – or better, corporeal – faculty.'

When Kepler suggested this distance he did not know the exact form of the orbit, but he knew that the orbit had to be some sort of oval. In the chapters 45 to 50 we find several analyses of an oval orbit, all mainly based on a model presented in chapter 45. This orbit is constructed with an epicycle and a concentric circle, where the epicycle has a radius *cN* equal to the eccentricity *AB* (Figure 1). The planet moves with uniform angular motion in the epicycle, while the angular motion of the epicycle centre around

1 We will use the term "power" instead of "force" in order to avoid confusion with Newtonian physics. Kepler himself is not consistent in this chapter, using both the terms *facultas* and *virtus* for the causal power, although *virtus* is applied more consistently in the discussion of the magnetic model. However, it is certain that Kepler by these terms refers to the equivalent of an *efficient* cause. Kepler's discussion on different causes and their importance in astronomical theories are found, for instance, in *Apologia contra Ursum* (Kepler 1984), and in the forewords of As*tronomiae pars optica* (Kepler 1939) and *Epitome astronomiae Copernicanae* (Kepler 1953).

the Sun is governed by the so-called radius rule.[2] But according to observation data, the breadth of the distance between the eccentric circle and the oval of chapter 45, which was 858 units, should be half of that (i.e. 429 units). This meant that the path would have to be between a circle and the oval of chapter 45.

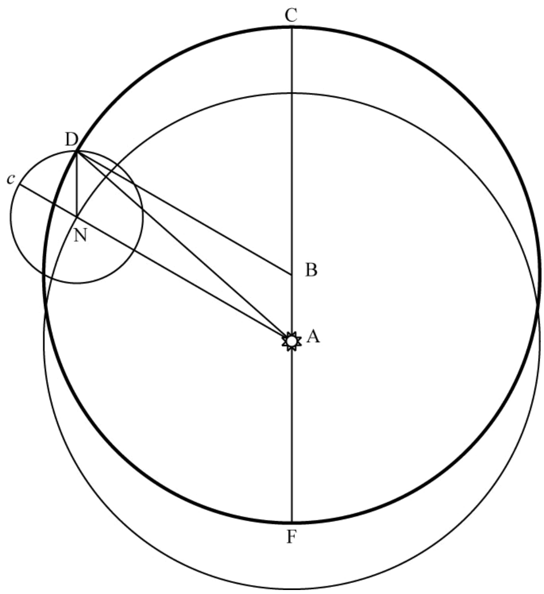

Fig. 1. Concentric circle with epicycle

Eccentric anomaly $\angle CBD = \beta$
True anomaly $\angle CAD = \theta$
$BC = BD = a$
$AB = cN = ae$
$AD = r$

Kepler finds a distance that fulfils this requirement in chapter 56. His argument for this distance is physical, but the way he finds it is by a trigonometric relation. In the epicycle (Figure 2), the line *ca* represents the distance between the Sun and planet when the planet is at aphelion and *cbd*, *dbf*, etc. correspond to equal eccentric anomalies. By chance Kepler found that by projecting the points on the epicycle, *d, f,* etc.*,* onto the line *Ac* the distances at the middle longitudes was reduced by 429 units. In Figure 2 the projection of the points *d* and *f* on *Ac* produces the distances *kA* and *mA*. These distances positioned the planet between the oval of chapter 45 and the perfect circle.

The fact that this distance fits the observations within the limits of accuracy can easily mislead the reader into thinking that Kepler had now justified the distances on empirical grounds. But instead of declaring the correct distance as established empirically, he calls upon the reader to look back at chapter 39:

> *And so the reader should peruse chapter 39 again. He will find that what the observations testify here was already argued there, from natural causes, namely, that it appears reasonable that the planet perform some sort of reciprocation, as if moving on the diameter of the epicycle that is always directed towards the sun. He will also find that there is nothing more at odds with this notion than this: when we proposed to represent a perfect circle, we were forced to make the highest parts* γι [*ci in Figure 2*] *of the reciprocation*[3] *unequal to the lowest* λζ [*lz*], *which parts correspond to equal arcs on the eccentric, the highest being short, and the lowest long. So, now that the planet's circular path is denied, and* κα [*kA*], μα [*mA*], *are taken instead of* δα [*dA*], εα [*fA*], *that is, instead of* ια [*iA*], λα [*lA*], *as was said, it follows further that those parts of the reciprocation, such as* γκ [*ck*], μζ [*mz*], *are equal.*[4]

2 The radius rule, termed the "distance law" by some scholars, was presented in chapter 32 of *Astronomia nova*. It said that the time taken to traverse equal eccentric anomalies is proportional to the distance between the Sun and planet.
3 Donahue has chosen to translate the Latin *libratio* as "reciprocation" throughout the English edition of *Astronomia nova*, whereas I have chosen to use "libration".
4 Kepler and Donahue 1992, p. 544.

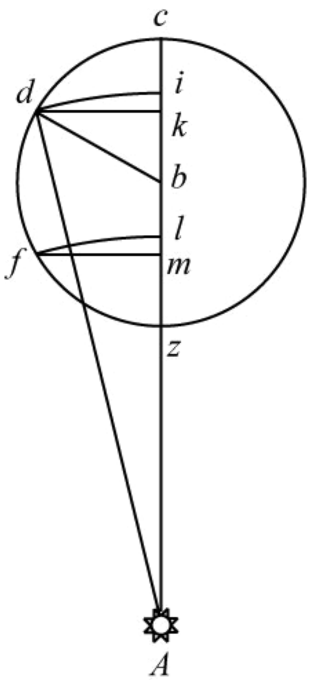

Fig. 2. Radial distance represented by the epicycle

$\angle cbd = \beta$
$a = Ac$
$cb = bd = ae$
$r = Ak$ for the eccentric anomaly $\angle cbd$

If we take Kepler at his word and go back to chapter 39, we will find the foundation for Kepler's conviction that he was on the right track. In this chapter, he pointed out that an epicycle model is not only implausible, but from a causal point of view absurd. The orbit he tries to account for in chapter 39 is an eccentric circle (a circle where the Sun is off centre). The model is basically the same as the one in chapter 45 (Figure 1), but here the line *BD* from the centre of the eccentric to the planet is always parallel to the line *AN* from the sun to the centre of the epicycle. For any of the epicycle-type models Kepler developed he found that it was not possible to give a sensible explanation for all components of motion. He therefore suggested a totally new decomposition of the orbit: the orbit could be a result of one circumsolar motion, and one motion along the radius between the Sun and planet. But this seemed just as unnatural as the epicycle models, since the radial motion was not uniform relative to eccentric anomalies. The parts *cd*, *df*, *fz* (Figure 2) representing equal eccentric anomalies corresponded to unequal parts on the radius where the highest parts, *ci*, are shorter than the lowest, *lz*. Although he had at the time good reasons to think that the epicycle model was physically implausible, the unevenness of this libration along the radius did not seem to be susceptible to a physical explanation either.

Knowing this, we can understand Kepler's excitement when in early 1605 he found a distance that produced a libration that was not chaotic. The motion was still non-uniform, but now the upper and lower parts of the libration were equal. The distance, given as a function of the eccentric anomaly, can be formulated in modern terms as:

$$r = a\,(1+e\cos\beta),$$

where the eccentricity *ae* is the dimension of the epicycle's radius. In modern mathematics the variation of the distance, or speed, is then expressed as:

$$\frac{\mathrm{d}r}{\mathrm{d}\beta} \propto ae\sin\beta.$$

The sinusoidal variation meant that the planet behaved *essentially* differently from the radial motion presented in chapter 39. This was the reason why Kepler believed the distance to be "natural" and why he sets out in the next chapter to explain it causally.

Step two – the cause

The title of chapter 57 is 'By what natural principles the planet may be made to librate as if on the diameter of an epicycle' and in the summary for this chapter he states 'that the libration on the diameter of the epicycle (which supplies distances in agreement with the observations) follows the natural laws of bodies'. Hence, Kepler made it very clear that his objective for this chapter was to find a *principle* that could explain the sinusoidal motion. Nevertheless, this chapter is one of the least understood in the whole of *Astronomia nova*.

It is in this chapter we find the most extensively developed ideas on the magnet as an analogy to the causal relation between the Sun and the motion of the planet. Although many scholars have analysed Kepler's discussion of magnetism, the most decisive element in Kepler's argument seems to have been overlooked. The very title of the chapter points to this element: 'By what natural principles the planet may be made to librate as if on the diameter of an epicycle'.[5] The "natural principle" that he applies is the *law of the lever*. Kepler himself terms it the law of the *balance*. This term is important, as it points to a general principle for a certain class of physical systems where one will find a balance between powers. The role of the magnet analogy is to *provide an example of a natural phenomenon* of push and pull between separated objects. Thus, the analogy provides Kepler with a template for modelling such powers. The law of the lever is *the principle by which Kepler analyses the interaction between powers in such a system*. The resulting model presented in chapter 57 is depicted in Figure 3.

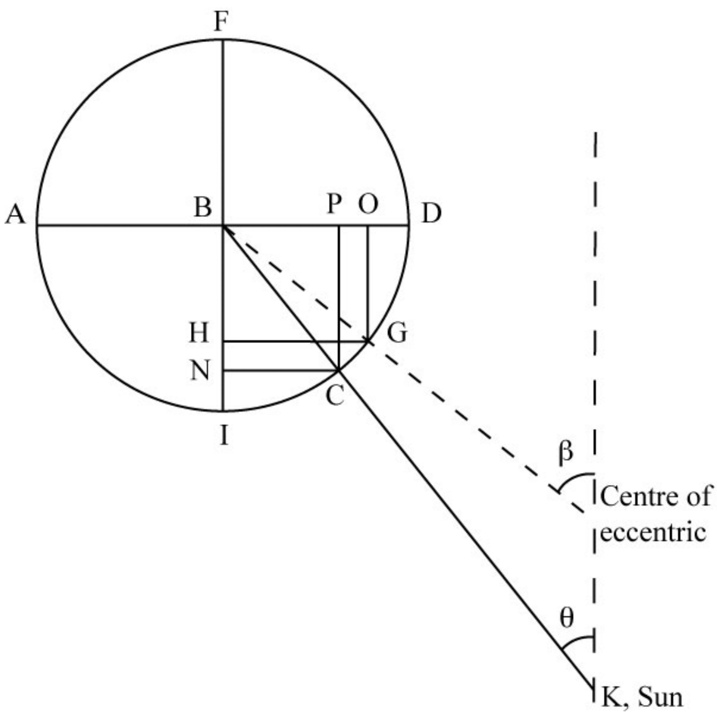

Fig. 3. Causal model of the Sun-planet relation based on the magnet

$\angle GBI = \beta$
$\angle CBI = \theta$
$\sin\beta = GI$
$\sin\theta = CI$
ver sin $\beta = HI$
Magnetic axis *AD*

The planet *FAID* and the Sun *K* were both thought of as magnetic in this model. Since the planet moves closer to the Sun in the first half of the orbit and then recedes, he had to have a mechanism that first attracted the planet and then pushed it away from the Sun. In order to make the model work according to this requirement he could not let the Sun be bipolar. This allowed the bipolar planet to be first attracted to and then repulsed from the Sun. The magnetic fibres of the planet were aligned to

5 Kepler and Donahue 1992, p. 547.

the axis of rotation, as it roughly is with the earth. In the figure above, the line *AD* represents the magnetic fibres and the endpoints *A* and *D* opposite poles. The planet would be attracted when the pole opposite to the Sun's pole was closest to the sun (*D* in the figure), and pulled away when the similar pole (*A* in the figure) was closest to the Sun, thus causing the planet to approach the Sun in the first half period and recede in the second half period. But the attraction and repulsion changes throughout the path according to the angle between the planet's magnetic axis *AD*, and the magnetic fibres from the Sun represented through the line *BK*. The magnetic power "activated" is determined by the angle between the line *BK* and *AD*, i.e. angle *CBD* for the attractive pole *D* and angle *CBA* for the repulsive pole.

However, Kepler lacked a physical principle that enabled him to analyse how such powers behaved, and eventually, to find a quantitative measure for the net power acting on the planet. He needed a mechanism that was both probable and that could be expressed mathematically.[6] Kepler found this in the law of the balance:

> *Now the approach occurs because the seeking pole D is inclined towards the sun K at the angle ABK. And since the strength of this angle is natural, it will follow the same ratio as the balance.*[7]

Kepler was not claiming that the planet considered as a magnet is a lever. He was only demonstrating that the magnetic axis followed the law of the lever. The manner in which the law of the lever applied to the magnet model, was quite different from how the law of the lever applied to an actual lever. Kepler recommended the reader to read *Astronomiae pars optica*[8] in order to get the full analysis on the law of the lever for various phenomena. Here, I will only present the argument as it stands in *Astronomia nova*. However, *Optica* supports our claim that Kepler used the law of the balance as a general principle, applicable to a series of different phenomena. In *Astronomia nova* chapter 57 Kepler applied it to the interaction between the magnetic poles of the planet and the Sun. This can be stated more generally: the governing principle for the interaction of powers for such a set-up is the balance. The law of the lever states that for a lever to be in balance the fulcrum must divide the line between two bodies in the ratio inversely proportional to their weights.

With the physical principle in place, Kepler could proceed to work out the net power acting on the planet. The magnetic axis *AD* is seen as a balancing beam with scales suspended from *A* and *D*. Kepler postulated that this model was equivalent to imagining the beam suspended from the arm *CP* at *P*. The weight at *A* is proportional to *DP* and the weight at *D* is proportional to *AP*, or $A_{weight}/D_{weight} = DP/AP$. In other words the weight at *A* is proportional to the distance *DP* and the weight at *D* is proportional to *DP*. Now, the scales suspended from *A* and *D* are meant to be analogous to the magnetic powers acting on the respective poles. Thus, the magnetic power acting on the repulsive pole *A* is proportional to *DP* and the attracting pole *D* is proportional to *AP*. By subtracting *DP* from *AP* one gets the net power represented by the distance *SP* (*AS* equals *DP*). Since *SP* is twice *BP* and *BP* equals *NC*,

6 Kepler did not know any other behavioural features of this 'power' than that it was repulsive and attractive. Quite a lot of additional information was needed to know how to determine the vector components from the attractive and repulsive pole and subsequently sum these vectors to get the net power.
7 Kepler and Donahue 1992, p. 556.
8 Kepler 1939, pp. 28–30.

NC represents the net power acting on the planet. *NC* is the sine of true (equated) anomaly, and Kepler concludes:

> *So the sine of the equated anomaly is the measure of the strength of the planet's approach towards the sun in this place. And this is the measurement of the increments of power.*[9]

Thus, Kepler managed to develop a quantitative measure for the cause of motion at every instant given by the sine of true anomaly.

Step three – demonstrating the effect from the cause

After establishing the measure of the cause, the next step Kepler needed to take was to demonstrate that the effect followed. The paragraph following Kepler's presentation of "the increments of power" starts thus: 'The measure of the distance of the libration traversed by these continuous increments of power is quite another thing'[10] Kepler's measure of radial displacement was given by the cosine of the eccentric anomaly, or alternatively, by the versed sine of the eccentric anomaly. This is the measure of the radial distance the planet has traversed for a given eccentric anomaly since it was at aphelion.[11] He now needed to demonstrate that the summation of incremental effects of the powers acting on the planet from aphelion to a given eccentric anomaly *GI* would result in the covered distance *HI*.

As Kepler's principle of inertia implies that motion and power are proportional the incremental measure of power is also an incremental measure of speed. Kepler needed therefore to demonstrate that the libration *HI* could be 'deduced from the previously indicated measure of the speed *CN*'.[12] But even though he had expressions for incremental radial speed and the radial distance covered, he did not have the proper mathematical tools for dealing with the problem of linking them determinately. The exact solution of this would require a general differential and integral calculus. But integration was not the only problem he had to face. The distance covered was given as a function of eccentric anomaly, while the new incremental measure for power and velocity was given as a function of true anomaly.

With Kepler's mathematical techniques this meant that he had to prove that the sums of the sines of the true anomaly were equivalent to the versed sines or cosines of the eccentric anomaly, corresponding to the point on the planet's path where the sums end. For the moment he let the true and eccentric anomaly be considered as equal in order to see if it was possible to prove the above suggested equivalence between sums of sines and the versed sine of the total angle: 'Furthermore, letting *IC* and *IG*, though they are unequal elsewhere, be equal here to avoid confusion, the sum of the sines of the arc *IG* is to the sum of the sines over the quadrant, approximately as the versed sine *IH* of that arc *IG* is to the versed sine *IB* of the quadrant.' Lacking the modern calculus needed to prove this with mathematical rigour, Kepler performed a numerical analysis to verify this correlation.[13] If, however, we take Kepler's summation to be a substitution

9 Kepler and Donahue 1992, p. 556.
10 Kepler and Donahue 1992, p. 556.
11 We can substitute the versed sine with the cosine by the equivalence ver sin $\beta = (1 - \cos\beta)$. This is equivalent to the previous measurement of the libration, with the only difference that $\cos\beta$ measures the actual distance, $(1 + e\cos\beta)$, while ver sin β measures the diminishing of the planet's greatest distance (at aphelion).
12 Kepler and Donahue 1992, p. 557.
13 He divided the quadrant in 90 units and compared the sum of these sines with the versed sine of the quadrant, i.e. 90°. He proves the same for 15, 30 and 60 units/degrees.

for modern integration it is easily shown that this assumption is correct since the integration of the sine of an angle is equal to the versed sine of that angle. The integration of sin β from aphelion to β = *IG* (β_{IG}) is given as:

$$\int_0^{\beta_{IG}} \sin\beta \, d\beta = 1 - \cos\beta_{IG} = \operatorname{ver}\sin\beta_{IG} \ .$$

Kepler was correct in assuming this correlation, though it was not precisely what he sought. The mathematical proof of the correlation between increments of power and distance covered depend on the relation between the true and eccentric anomaly. Hence, he still had to deal with the problem that the measure of covered distance involved the eccentric anomaly while the measure of the power involved the true anomaly.

The relevance of the causal hypothesis to the construction of the ellipse

Although handling the integration problem more or less adequately, the expression of power and distance covered still did not correlate, as they are functions of different anomalies. This problem is indicative of the obstacles Kepler faced in the subsequent chapters 58 and 59. In order to get some idea of the complexities that were involved let us summarize what was now known of the orbit. From step one (chapter 56) we had:

1. $r = a\,(1+e \cos \beta)$; radial distance as a function of eccentric anomaly
2. ae ver sin β; radial distance covered since the planet was at aphelion
3. $\frac{dr}{d\beta} \propto ae \sin \beta$; differential notation of radial speed

From step two we had a measure for the net radial power and, consequently, speed:

4. Power $\propto \sin \theta$
5. Incremental speed $\propto \sin \theta$

We have already mentioned the most obvious problem we can see in the above: the differential expression for speed from chapter 56 (3) and the incremental expression for speed in chapter 57 (5), depend on the eccentric and the true anomaly respectively. Both of these expressions for speed might be true for some orbit. The challenge was thus to find a correlation between those anomalies that made both of these expressions for the speed true. A second problem was that time was not explicitly expressed in either of the expressions above. But Kepler realized that time was the independent variable for the expression of power and the corresponding incremental speed: "since the sine measures the strength, and *the strength acts in proportion to time*".[14] The differential for the speed should therefore be written as:

4'. $\frac{dr}{dt} \propto ae \sin \theta$.

At the end of step three Kepler performs a qualitative evaluation of the relation between this measure of force (5) and the distance covered (2) based on this correction.[15]

14 Kepler and Donahue 1992, p. 558.
15 Kepler and Donahue 1992, pp. 558–559.

Kepler was fully aware at that time that in order to find a solution to the orbit, 1 to 5 had to be correlated with regard to the variables involved: time, eccentric anomaly and true anomaly. The main problem was to link sums of corresponding expressions of incremental speed (5) to the expressions for the distance covered along the Sun-planet radius (2). As will be demonstrated in a forthcoming paper on chapter 59, the manner in which Kepler handled this problem was through the correlation of the partition of the eccentric circle into equal "minimal" arcs, with a partition of the postulated planetary orbit into unequal small arcs, by drawing ordinates perpendicular to the line of apsides – a method already well established at this stage in *Astronomia nova*. He then demonstrated that this partition of the orbit of the planet into unequal arcs had to correspond to a partition into *equal* transradial arcs, if the sums of incremental speed expressed in true anomaly (5) should be proportional to the distance covered during the libration along the Sun-planet radius (2). This incremental partitioning of the path is decisive in the construction of the ellipse, and it is only the ellipse that fulfils the physical and dynamical requirements developed in chapter 56 and 57.

Conclusion

I have argued that the physical hypothesis explaining the radial component of motion was well founded in dynamics. By analysing the interaction of powers in a model based on magnetism, but with the law of the lever as the principle determining this interaction, Kepler found a quantitative measure for net instantaneous force. Lacking the relevant mathematical tools, he proceeded to prove both qualitatively and by numerical estimations that the covered distance was a result of the power. With hindsight, we know that the incremental expression for the motion, deduced from the power, is true for the final ellipse. Thus, Kepler achieved his goal of developing a physical explanation for the radial motion.

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