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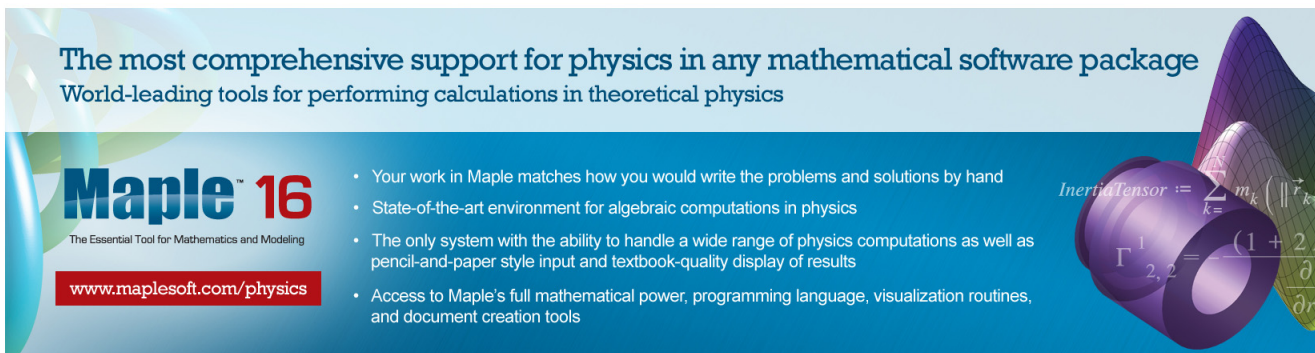
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InertiaTensor := $\sum_{k=1}^n m_k \left(\|\vec{r}_k\|^2 \right)$
 $\Gamma_{2,2}^1 = \frac{(1+2)}{\partial r}$



SDiff(2) and uniqueness of the Plebański equation

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The group of area preserving diffeomorphisms showed importance in the problems of self-dual gravity and integrability theory. We discuss how representations of this infinite-dimensional Lie group can arise in mathematical physics from pure local considerations. Then using Lie algebra extensions and cohomology we derive the second Plebański equation and its geometry. We do not use Kähler or other additional structures but obtain the equation solely from the geometry of area preserving transformations group. We conclude that the Plebański equation is Lie remarkable.

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I. INTRODUCTION

Consider a two-dimensional manifold M equipped with an area 2-form ϖ . This structure can be uniformized according to the genus of M and its total area. In this paper, we would like to assume the simplest possible topology (M is \mathbb{R}^2 or \mathbb{S}^2), concentrating on the geometry of the group SDiff(2) of area preserving transformations.

This group can be already seen in the original Plebański work¹³ on the equations of gravity where he obtained the so-called second heavenly equation

$$u_{ty} - u_{xz} + u_{xx}u_{yy} - u_{xy}^2 = 0, \quad (1)$$

and it played an important role in the subsequent development of the corresponding integrable hierarchies.^{16,17} In this paper, we show how this group arises in relation to the second Plebański equation from a purely local construction; for nonlocal structures such as the Lax pair and the recursion operator see Refs. 4 and 10. The group SDiff(2) is known^{2,10} to be related to the classical symmetries of (1); we shall make this relation two-sided.

For the infinite-dimensional Lie group SDiff(2) the corresponding Lie algebra $\mathcal{D}_0(M)$ consists of divergence free vector fields, which due to trivial topology coincide with Hamiltonian vector fields. This leads to the classical Lie algebras isomorphism $(\mathcal{D}_0(M), [,]) \simeq (C^\infty(M)/\mathbb{R}, \{, \})$, where we use the Lie bracket (commutator) to the left and the Poisson bracket to the right.

The geometry of the Poisson algebra $\mathcal{P} = (C^\infty(M), \{, \})$ is central in our paper. Infinite-dimensional groups are known as an important tool to generate Hamiltonian partial differential equations (PDEs) via Euler-Arnold, Gelfand-Dikij and other methods.¹ Our strategy is to search for differential invariants among simplest possible representations of the Lie algebra sheaf of vector fields. Such invariants determine differential equations that come naturally equipped with a large symmetry algebra.

Since there are no differential invariants in two dimensions for SDiff(2), we have to extend the Lie algebra or its action. It turns out that we need to do both, and that the natural extensions yields the action of 4 copies of \mathcal{P} (this will be shown to have the graded structure) on the space of functions in four dimensions. Then the fundamental invariant is the left-hand side of (1).

To see this we compute the natural differential operators related to the Poisson algebra and calculate the first cohomology of the Lie algebra \mathcal{P} with values in its representations. On this way

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we discover the SDiff(2) analog of the Gelfand-Fuks cocycle, which is central in our method of deriving the Plebański equation (1).

We further explore the symmetry structure of the second Plebański equation and demonstrate that it essentially coincides with our extended graded Lie algebra. This makes (1) the so-called Lie remarkable equation.

At the end of the paper, we shortly discuss the first Plebański equation, the situation with which it happened to be quite similar (the integrability properties for both equations are known to be equivalent, but it is not obvious that the local symmetries and invariants structures are equally reach since the relation between the equations is non-local).

II. SDiff(2) AND ITS COHOMOLOGY

Let \mathcal{P} be the Poisson algebra $(C^\infty(M), \{, \})$, and let $\mathcal{D}(M)$ denote the Lie algebra of all vector fields on M . We denote its subalgebra consisting of Hamiltonian fields by $\mathcal{D}_0(M)$.

The map $h \mapsto X_h$ ($\varpi(X_h, \cdot) = dh$) is an epimorphism of Lie algebras $\mathcal{P} \rightarrow \mathcal{D}_0(M)$ whose kernel is equal to the center $\mathbb{R} \subset \mathcal{P}$ (on \mathbb{S}^2 we can restrict to the space of functions with zero mean).

Since SDiff(2) acts transitively on M , and has open dense orbits in the space of functions on M , to find non-trivial (absolute) invariants we have to consider an extension of the tautological representation $\mathcal{D}_0(M) \subset \text{Der}[C^\infty(M)]$ to a space of bigger dimensions.

Let us start with one-dimensional extension, i.e., we want to find a homomorphism ρ of $\mathcal{D}_0(M)$ to $\mathcal{D}(M \times \mathbb{R})$ such that the vector fields in the image are projectible along \mathbb{R} to our Hamiltonian fields. In other words, if $\pi : M \times \mathbb{R} \rightarrow M$ is the natural projection, then $\pi_* \circ \rho = \mathbf{1}$.

These fields have the form $X_h + \psi(h)\partial_u$, where ∂_u is the unit vector field along the fiber coordinate u of π . The homomorphism condition is equivalent to the claim that ψ is a 1-cocycle on \mathcal{P} with values in $C^\infty(M)$. The action on functions is given by $(h, f) \mapsto X_h(f) = \{h, f\}$ and so is the adjoint representation in \mathcal{P} . Thus, non-trivial extensions are parametrized by the cohomology group $H^1(\mathcal{P}, \mathcal{P})$.

Since all our constructions are required to be local, we will restrict to extensions given by differential operators. This is also guaranteed by our assumption of trivial topology. Thus in what follows all cocycles are expressed via differential operators.

Theorem 1: *The above group is one-dimensional: $H^1(\mathcal{P}, \mathcal{P}) = \mathbb{R}$. In canonical coordinates (t, z) on M such that $\varpi = dt \wedge dz$ the generator is represented by the 1-cocycle*

$$\sigma_1(A) = tA_t + zA_z - 2A.$$

Proof: A linear map $s: C^\infty(M) \rightarrow C^\infty(M)$ is a 1-cocycle if

$$\{s(h), f\} + \{h, s(f)\} = s(\{h, f\}).$$

Let $h_{(i,j)} = \mathcal{D}_t^i \mathcal{D}_z^j(h)$, where \mathcal{D}_t is the total derivative by t , and similar for \mathcal{D}_z . Writing $s(h) = \sum \lambda_{ij} h_{(i,j)}$ in the above relation, with the functions λ_{ij} depending on $(t, z) \in M$, we get an expression $\Psi(h, f) = 0$, where Ψ is a bilinear bi-differential operator.

The coefficients of $h_{(1,0)}f_{(i,j)}$ and $h_{(0,1)}f_{(i,j)}$ with $i + j \neq 1$ give $\lambda_{ij} = \text{const}$, and the coefficients of $h_{(i,j)}f_{(k,l)}$ with $i + j > 1, k + l > 1$, give $\lambda_{ij} = 0$ for $i + j > 1$.

Furthermore, the coefficient of $h_{(1,0)}f_{(0,1)}$ gives the equation

$$\frac{\partial \lambda_{10}}{\partial t} + \frac{\partial \lambda_{01}}{\partial z} = -\lambda_{00} = \text{const}.$$

with the solution $\lambda_{01} = q_t - \frac{1}{2}\lambda_{00}z, \lambda_{10} = -q_z - \frac{1}{2}\lambda_{00}t$, where $q = q(t, z)$ is an arbitrary function. This gives the general formula for the cocycle

$$s(f) = (q_t f_z - q_z f_t) - \frac{1}{2}\lambda_{00}(t f_t + z f_z - 2f).$$

The first expression in parentheses is the trivial 1-cocycle $\{q, f\}$, while the second expression in parentheses is not cohomologous to zero. \square

This cocycle allows us to extend the Hamiltonian vector fields on $M = \mathbb{R}^2 = J^0(\mathbb{R}, \mathbb{R})$ to the contact vector fields on $\hat{M} = M \times \mathbb{R}(u) = J^1(\mathbb{R}, \mathbb{R})$, namely, the new fields are

$$X_h + (th_t + zh_z - 2h)\partial_u = h_z(\partial_t + z\partial_u) - h_t(\partial_z - t\partial_u) - 2h\partial_u$$

and an easy change of coordinates brings this field to the canonical form of the contact Hamiltonian field.

Thus the extension is the standard contact extension, and this algebra can be prolonged to the algebra of Lie (higher contact) fields in $J^k(\mathbb{R}, \mathbb{R})$. This however still acts transitively (no differential invariants) and by this reason in Sec. III we extend the manifold M to a four-dimensional space.

If $M = \mathbb{S}^2$, then we must consider instead the circle bundle $\hat{M} = PT^*M$ over M , and the same arguments work. In Sec. III we use only the case $M = \mathbb{R}^2$ to illustrate the argument in coordinates (the construction is covariant and does not depend on the choice of coordinates, canonical in the sense of Darboux theorem), and we do not discuss the counterpart for the sphere.

III. EXTENSION I: THE TANGENT BUNDLE

In this section we discuss extension of M by two dimensions. There are two natural candidates: the tangent and the cotangent bundles, and they are isomorphic.

The area form $\varpi = dt \wedge dz$ on $M = \mathbb{R}^2(t, z)$ induces the isomorphism $TM \simeq T^*M$, $v \mapsto i_v \varpi$. In the canonical coordinates induced on both bundles by the coordinates on M this correspondence writes as $(x, y) \leftrightarrow (-y, x)$. We are more interested in the tangent bundle. The canonical Liouville form from T^*M writes on it as $\sigma = x dz - y dt \in \Omega^1(TM)$.

The canonical symplectic form

$$\omega = d\sigma = dx \wedge dz + dt \wedge dy \in \Omega^2(TM)$$

is related to the pull-back of ϖ (to four-dimensional TM) by the operator field $2K = \partial_x \otimes dt + \partial_y \otimes dz$, $i_K \omega = \varpi$.

Even more important ingredient is the truncated total field

$$\nabla = x\partial_t + y\partial_z.$$

If we identify $J^1(\mathbb{R}, M) = \mathbb{R} \times TM \simeq \mathbb{R}^1(\tau) \times \mathbb{R}^4(t, z, x, y)$, the total derivative is $\mathcal{D}_\tau = \partial_\tau + x\partial_t + y\partial_z + \dots$ and we quotient $J^1(\mathbb{R}, M)$ by the first factor (consider τ independent functions on this jet-space J^1). The field ∇ relates the two symplectic forms as

$$i_\nabla \varpi = \sigma \Rightarrow L_\nabla \varpi = \omega.$$

In addition we have that $2K(\nabla) = x\partial_x + y\partial_y$ is the Liouville (radial vertical) field on TM .

This ∇ however is not a vector field (as a total differential its tail, namely, the part containing ∂_x, ∂_y is not uniquely defined, see, e.g., Ref. 7), but only a first order differential operator

$$\nabla : C^\infty(M) \rightarrow C^\infty(TM).$$

Proposition 2: A linear differential operator $\tilde{\nabla} : C^\infty(M) \rightarrow C^\infty(TM)$ of order 1 satisfying

$$\tilde{\nabla}\{A, B\}_\varpi = \{\tilde{\nabla}A, \tilde{\nabla}B\}_\omega$$

has the form $\tilde{\nabla} = \nabla + \{q, \cdot\} - k \cdot \sigma_1$, where $q \in C^\infty(M)$ and $k \in \mathbb{R}$ (σ_1 is the cocycle from Theorem 1).

Proof: The general form of first order differential operator is

$$\tilde{\nabla} = a(t, z, x, y)\partial_t + b(t, z, x, y)\partial_z + c(t, z, x, y).$$

In addition, a and b are not simultaneously zero. Thus substituting $B = 1$ we obtain $c = 2k = \text{const}$.

Next it is easy to see that both $a, b \neq 0$. Indeed if, e.g., $b = 0$, then taking $A = z, B = t^2$ we get a contradiction.

Finally consider the coefficients of $A_{tt}B_t, A_{tt}B_z, A_{zz}B_t, A_{zz}B_z$ in the defining relation. They imply the system $a_x = 1, a_y = 0, b_x = 0, b_y = 1$. The coefficient of A_tB_z gives $a_t + b_z = 0$. This gives the following form:

$$\tilde{\nabla} = (x - kt - q_z)\partial_t + (y - kz + q_t)\partial_z + 2k$$

for some function $q = q(t, z)$. In other words $\tilde{\nabla}A = \nabla A + \{q, A\} - k\sigma_1(A)$. \square

Thus we see that though ∇ is natural, the condition that the operator preserves the Poisson brackets leads us to consideration of a pair of independent Hamiltonians on TM : ∇A_0 and A_1 (the later is lifted from M via pull-back), $A_i \in \mathcal{P}$ (we use the fact that the operator $\{q, \cdot\}: C^\infty(M) \rightarrow C^\infty(M)$ is epimorphic; in the case of the sphere with the additional condition that the Hamiltonians have zero mean).

In other words, we have the graded Poisson algebra $\mathcal{H}_1 = \mathcal{P}_0 \oplus \mathcal{P}_1$ (the index refers to the grading), where $\mathcal{P}_i \simeq \mathcal{P}$ and the bracket is given by $\{A_i, B_j\} = \{A, B\}_{i+j}$ (and we assume that the grading 2 is void). This admits the graded Lie algebra homomorphism

$$V : \mathcal{H}_1 \rightarrow \mathcal{D}_1,$$

where $\mathcal{D}_1 = \mathcal{D}(TM)_0 \oplus \mathcal{D}(TM)_1$ is the graded Lie algebra consisting of vector fields in pure gradings, with the bracket being given by the commutator and the same truncation rule as above.

This homomorphism associates to the element $(A, 0) \in \mathcal{H}_1$ the Hamiltonian vector field $X_{\nabla A}$, and to the element $(0, A) \in \mathcal{H}_1$ the field X_A (both with respect to the symplectic structure ω on TM).

In canonical coordinates (t, z, x, y) on TM we can write this homomorphism as $(A_0, A_1) \mapsto (V_0(A_0), V_1(A_1))$ with

$$\begin{aligned} V_0(A) &= A_z\partial_t - A_t\partial_z + (A_{tz}x + A_{zz}y)\partial_x - (A_{tx} + A_{tz}y)\partial_y, \\ V_1(A) &= A_z\partial_x - A_t\partial_y. \end{aligned}$$

Notice that both vector fields are projectible to Hamiltonian vector fields on (M, ϖ) , with the Hamiltonians A and 0 , respectively.

It is natural to ask if the above Lie algebra homomorphism extends to bigger truncated graded Lie algebras $\mathcal{H}_k = \mathcal{P}_0 \oplus \dots \oplus \mathcal{P}_k$ and $\mathcal{D}_k = \mathcal{D}(TM)_0 \oplus \dots \oplus \mathcal{D}(TM)_k$ with the same rule that $\{A_i, B_j\} = \{A, B\}_{i+j}$ if $i + j \leq k$ and $= 0$ if $i + j > k$ (and similar in the case of vector fields: $[V_i, W_j] = [V, W]_{i+j}$ or $= 0$ if $i + j > k$).

Proposition 3: For $k > 1$ the graded Lie algebra homomorphism $V : \mathcal{H}_k \rightarrow \mathcal{D}_k$ extending the one for $k = 1$ vanishes in all gradings > 1 .

Proof: Indeed, $V_2(\{A_1, B_1\}) = [V_1(A_1), V_1(B_1)] = 0$ for all $A_1, B_1 \in \mathcal{P}_1$. Since any element $C_2 \in \mathcal{P}_2$ can be written as $\{A_1, B_1\}$ we get $V|_{\mathcal{P}_2} = 0$. Similarly we conclude that $V|_{\mathcal{P}_i}$ is trivial for all $i > 1$. \square

IV. EXTENSION II: TOWARDS FUNCTIONS IN 4D

Our strategy is to calculate the differential invariants, but for this we need to extend the action from the space TM to the space $J^0 = TM \times \mathbb{R} \simeq \mathbb{R}^4(t, z, x, y) \times \mathbb{R}^1(u)$.

A homomorphism $V : \mathcal{G} \rightarrow \mathcal{D}(TM)$ of a Lie algebra \mathcal{G} extends to a homomorphism $\hat{V} : \mathcal{G} \rightarrow \mathcal{D}(J^0)$, $\hat{V}(\alpha) = V(\alpha) + \Psi(\alpha)\partial_u$, iff Ψ is a 1-cocycle on \mathcal{G} with values in the module $C^\infty(TM)$ (the action is via the representation V). As usual the 1-cocycles that differ by 1-coboundary define isomorphic extensions. Thus, we need to calculate the cohomology group $H^1(\mathcal{G}, C^\infty(TM))$.

In this section we focus on the simplest case, when \mathcal{G} is $\mathcal{H}_0 = \mathcal{P}$.

Theorem 4: We have $H^1(\mathcal{H}_0, C^\infty(TM)) = \mathbb{R}^2$. In the canonical coordinates (t, x, y, z) on TM the cocycles $\sigma_1(A) = tA_t + zA_z - 2A$ and $\sigma_{GF}(A) = \nabla^3(A)$ can be taken as a base for the cohomology group.

Proof: Let $\Psi(A) = \sum \lambda_{ij} A_{(i,j)}$ for some functions $\lambda_{ij} \in C^\infty(TM)$. Then the coefficients of $A_{(i,j)} = \mathcal{D}_t^i \mathcal{D}_z^j(A)$, $i + j > 2$, in the cocycle identity

$$V_0(A)(\Psi(B)) - V_0(B)(\Psi(A)) = \Psi(\{A, B\}) \tag{2}$$

yield $\lambda_{ij} \equiv 0$ for $i + j > 3$. The remaining coefficients of $A_{(i,j)}, B_{(i,j)}$ with $i + j \leq 2$ in (2) provide an over-determined system of PDEs for the functions λ_{ij} , $i + j \leq 3$. Taking into account that the solution $\Psi(A) = \sum_{0 \leq i+j \leq 3} \lambda_{ij} \mathcal{D}_t^i \mathcal{D}_z^j(A)$ of this system is defined up to adding a coboundary $V_0(A) \lrcorner dF$ with an arbitrary $F \in C^\infty(TM)$, we obtain the general solution

$$\Psi(A) = c_1 (tA_t + zA_z - 2A) + c_2 \nabla^3(A) + c_3 (y^{-2}A_{tt} + x^{-2}A_{zz}).$$

The requirement $\lambda_{ij} \in C^\infty(TM)$ implies $c_3 = 0$. □

Remark on Gelfand-Fuks cohomology: Calculations of cohomology of infinite-dimensional algebras have started in 1968 with the Lie algebra $\mathfrak{g} = \mathcal{D}(\mathbb{S}^1)$.⁶ In particular, this work introduced the celebrated Gelfand-Fuks cocycle c_{GF} as the generator of $H^2(\mathfrak{g})$.

Let us notice that the natural morphism $\delta : C^2(\mathfrak{g}) \rightarrow C^1(\mathfrak{g}, \mathfrak{g}')$, where the regular dual $\mathfrak{g}' = \mathcal{F}_2 = \{f(\varphi)d\varphi^2\}$ is the space of quadratic differentials, maps cocycles to cocycles.⁵ It induces an isomorphism in cohomology, and $H^1(\mathfrak{g}, \mathfrak{g}') = \mathbb{R}$ has generator δc_{GF} represented by $f(\varphi)\partial_\varphi \mapsto f'''(\varphi)d\varphi^2$. On the level of Lie groups $H^1(\text{Diff}(\mathbb{S}^1), \mathcal{F}_2)$ is one-dimensional and generated by the Schwarzian derivative.¹²

The higher dimensional versions of Schwarzian derivatives exist, and they are cocycles on $\text{Diff}(M)$ with values in (2,1)-tensor fields. The Lie algebra version in dimension 2 when restricted to the algebra $\mathcal{D}_0(M) \subset \mathcal{D}(M)$ takes values in $\Gamma(S^3 T^*M)$ (notice that $\dim S^3 T_a^*M = 4$ for $\dim M = 2$) and is given by the formula:¹²

$$\hat{c}_{GF}(X_F) = d^3 F.$$

This is clearly the two-dimensional analog of the Gelfand-Fuks cocycle (we think about 1-cocycle given by the morphism δ).

In our case the cocycle $\sigma_{GF} = \nabla^3$ takes values in the space of functions on another four-dimensional space TM (our version gives a lower dimensional representation of elements of the Lie algebra by vector fields). Thus it can be considered as the generalized Gelfand-Fuks 1-cocycle in the case of Lie algebra $\mathcal{D}_0(M)$.

Our construction has some similarity with the one in Ref. 12 which explores the double $\mathfrak{g} \oplus \mathfrak{g}'$ (followed by passing to the current algebra to increase the dimension of the configuration space), but in our case $\mathcal{H}_0 \oplus \mathcal{H}_1$ the second summand is adjoint (not co-adjoint) module and (what is more important) all extensions do satisfy the Lie pseudo-group property: they come with natural representation by (the sheaf of) vector fields and are given by determining differential equations.

V. EXTENSION III: FORMAL SERIES AND NATURAL TRUNCATION

Now we consider the case, when \mathcal{G} is $\mathcal{H}_1 = \mathcal{P}_0 \oplus \mathcal{P}_1 = \{(A_0, A_1)\}$. The same computations as in Theorem 4 give

Theorem 5: $H^1(\mathcal{H}_1, C^\infty(TM)) = \mathbb{R}^2$, and the following two cocycles form its basis: $\sigma_1(A_0) = tA_{0,t} + zA_{0,z} - 2A_0$ and $\sigma_2(A_1) = A_1$.

This result seems to be rather disappointing, since the most interesting cocycle $\nabla^3(A_0)$ disappears after passing from $\mathcal{H}_0 = \mathcal{P}$ to $\mathcal{H}_1 = \mathcal{P} \otimes \mathbb{R}[[\varepsilon]]/\{\varepsilon^2 = 0\}$. The reason is the cut tails in the series.

To overcome the problem, we consider the Lie algebra of formal series $\mathcal{H}_\infty = \mathcal{P}_0 \oplus \mathcal{P}_1 \oplus \dots \oplus \mathcal{P}_k \oplus \dots = \mathcal{P} \otimes \mathbb{R}[[\varepsilon]]$. We want to find an extension of the homomorphism $V : \mathcal{H}_\infty \rightarrow \mathcal{D}(TM)$ to

$$\hat{V} : \mathcal{H}_\infty \rightarrow \mathcal{D}(J^0)$$

(recall that $J^0 = TM \times \mathbb{R}$). This is given by 1-cocycle Ψ on \mathcal{H}_∞ with values in $C^\infty(TM)$ via representation V . By Proposition 3 this latter is equal to

$$V : A = \sum_{k=0}^{\infty} \varepsilon^k A_k \mapsto V_0(A_0) + V_1(A_1).$$

So the defining relation $V(A_i)\Psi(B_j) - V(B_j)\Psi(A_i) = \Psi(\{A, B\}_{i+j})$ implies that the 1-cocycle Ψ on \mathcal{H}_∞ vanishes in grading >3 . The same computations as in Theorems 1 and 4 yield.

Theorem 6: *The following 1-cocycles form a basis in the cohomology group $H^1(\mathcal{H}_\infty, C^\infty(TM))$:*

$$\zeta_1(A) = \frac{1}{6} \nabla^3(A_0) + \frac{1}{2} \nabla^2(A_1) + \nabla(A_2) + A_3,$$

$$\zeta_2(A) = \nabla(A_1) + 2A_2,$$

$$\sigma_1(A) = t A_{0,t} + z A_{0,z} - 2A_0,$$

$$\sigma_2(A) = A_1.$$

The required extension is given by the formula ($c_i = \text{const.}$):

$$\hat{V} : \sum_{k=0}^{\infty} \varepsilon^k A_k \mapsto \hat{V}_0(A_0) + \hat{V}_1(A_1) + \hat{V}_2(A_2) + \hat{V}_3(A_3),$$

$$\hat{V}_0(A_0) = V_0(A_0) + \left(\frac{1}{6} c_1 \nabla^3(A_0) + c_3 (t A_{0,t} + z A_{0,z} - 2A_0) \right) \partial_u,$$

$$\hat{V}_1(A_1) = V_1(A_1) + \left(\frac{1}{2} c_1 \nabla^2(A_1) + c_2 \nabla(A_1) + c_4 A_1 \right) \partial_u,$$

$$\hat{V}_2(A_2) = (c_1 \nabla(A_2) + 2c_2 A_2) \partial_u,$$

$$\hat{V}_3(A_3) = c_1 A_3 \partial_u.$$

Now we shall classify the family $\mathfrak{h}_{(c_1, c_2, c_3, c_4)} = \hat{V}(\mathcal{H}_\infty)$ up to an isomorphism, preserving the filtration $\oplus_{i \geq k} \mathcal{P}_i = \mathcal{P} \otimes \varepsilon^k \mathbb{R}[[\varepsilon]] \subset \mathcal{H}_\infty$.

Theorem 7: *(1) When $c_1 \neq 0$, there exists an isomorphism*

$$\Phi_1 : \mathfrak{h}_{(c_1, c_2, c_3, c_4)} \rightarrow \mathfrak{g}_1 = \mathfrak{h}_{(1, 0, 0, 0)}$$

defined as a superposition of the map (the function σ_1 below is the same as in Theorems 5 and 6)

$$\hat{V}_0(A_0) \mapsto \hat{V}_0(A_0) - \frac{c_3}{c_1} \hat{V}_3 \circ \sigma_1(A_0),$$

$$\hat{V}_1(A_1) \mapsto \hat{V}_1(A_1) - \frac{c_1 c_4 - 2c_2^2}{3c_1} \hat{V}_3(A_1) - \frac{c_2}{c_1} \hat{V}_2(A_1),$$

$$\hat{V}_2(A_2) \mapsto \hat{V}_2(A_2) - \frac{2c_2}{c_1} \hat{V}_3(A_2),$$

$$\hat{V}_3(A_3) \mapsto \hat{V}_3(A_3).$$

and the scaling $u \mapsto -c_1 u, A_2 \mapsto -A_2, A_3 \mapsto -A_3$.

(2) If $c_1 = 0$, $c_2 \neq 0$, then $\hat{V}_3(A_3) = 0$ and the map

$$\begin{aligned}\hat{V}_0(A_0) &\mapsto \hat{V}_0(A_0) - \frac{c_3}{2c_2} \hat{V}_2 \circ \sigma_1(A_0), \\ \hat{V}_1(A_1) &\mapsto \hat{V}_1(A_1) - \frac{c_4}{2c_2} \hat{V}_2(A_1), \\ \hat{V}_2(A_2) &\mapsto \hat{V}_2(A_2)\end{aligned}$$

together with the scaling $u \mapsto c_2 u$ defines an isomorphism

$$\Phi_2 : \mathfrak{h}_{(0, c_2, c_3, c_4)} \rightarrow \mathfrak{g}_2 = \mathfrak{h}_{(0, 1, 0, 0)}.$$

(3) In the case of $c_1 = c_2 = 0$ we have $\hat{V}_2(A_2) = \hat{V}_3(A_3) = 0$. The Lie algebra $\mathfrak{g}_3 = \mathfrak{h}_{(0, 0, c_3, c_4)}$ is defined up scaling of (c_3, c_4) , and it coincides with the extension from Theorem 5.

Let us denote the generators \hat{V}_i in case (1) by W_i ,

$$\begin{aligned}W_0(A_0) &= V_0(A_0) - \frac{1}{6} \nabla^3(A_0) \partial_u, \quad W_2(A_2) = \nabla(A_2) \partial_u, \\ W_1(A_1) &= V_1(A_1) - \frac{1}{2} \nabla^2(A_1) \partial_u, \quad W_3(A_3) = A_3 \partial_u.\end{aligned}$$

The Lie algebra $\mathfrak{g}_1 = \mathfrak{a}_0 \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \mathfrak{a}_3$ is 4-graded, $[\mathfrak{a}_i, \mathfrak{a}_j] = \mathfrak{a}_{i+j}$. Here, $\mathfrak{a}_i = \{W_i(A_i) \mid A_i \in C^\infty(M)\}$, and $\mathfrak{a}_i = 0$ for $i \notin \{0, 1, 2, 3\}$. In addition, Φ_1 is a graded Lie algebra homomorphism.

Similarly, the Lie algebra $\mathfrak{g}_2 = \mathfrak{a}_0 \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_2$ is 3-graded, and Φ_2 is a graded Lie algebra homomorphism.

Finally, the Lie algebra $\mathfrak{g}_3 = \mathfrak{a}_0 \oplus \mathfrak{a}_1$ is 2-graded.

VI. DIFFERENTIAL INVARIANTS OF THE ACTION

Let $\mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{G}_3$ be the Lie pseudo-groups on $J^0 = J^0(TM, \mathbb{R})$ with the Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$, and \mathfrak{g}_3 , respectively. By direct computations, using MAPLE, we find differential invariants of the prolongations of actions of these pseudo-groups on $J^2(TM, \mathbb{R})$,

Theorem 8: *The only differential invariants of the action on $J^2(TM, \mathbb{R})$ are,*

- (1) \mathfrak{G}_1 : $I_1 = u_{ty} - u_{xz} + u_{xx}u_{yy} - u_{xy}^2$.
- (2) \mathfrak{G}_2 : $I_2 = (u_y - x)^2 u_{xx} - 2(u_x + y)(u_y - x)u_{xy} + (u_x + y)^2 u_{yy}$ and $I_3 = u_{xx}u_{yy} - u_{xy}^2$.
- (3) \mathfrak{G}_3 : the above I_3 and $I_4 = u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy}$.

The equation $I_1 = 0$ is the second Plebański equation (1). The equations $I_2 = 0$, $I_3 = 0$, $I_4 = 0$ have only two independent variables and are less interesting. Moreover, they can be linearized by contact transformations. For the Monge-Ampère equation $I_3 = 0$ this result is classical. For two other equations it can be proven by the methods of Ref. 11 or 8:

Theorem 9: *Equations*

$$(u_y - x)^2 u_{xx} - 2(u_x + y)(u_y - x)u_{xy} + (u_x + y)^2 u_{yy} = 0$$

and

$$u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy} = 0$$

are contact-equivalent to the equation

$$\tilde{u}_{\tilde{x}\tilde{x}} = 0.$$

The only differential invariant of the prolongation of action of the Lie pseudo-group \mathfrak{G}_1 on J^3 is the function

$$J_1 = E_2 E_4 - E_1 E_3 - u_{xx} E_3^2 + 2u_{xy} E_2 E_3 - u_{yy} E_2^2,$$

where $E_1 = \mathcal{D}_t(I_1)$, $E_2 = \mathcal{D}_x(I_1)$, $E_3 = \mathcal{D}_y(I_1)$, $E_4 = \mathcal{D}_z(I_1)$. The invariant derivations of the prolongation of \mathfrak{G}_1 on jets of order greater than 2 are

$$\begin{aligned}\mathbb{D}_1 &= E_3 \mathcal{D}_x - E_2 \mathcal{D}_y, \\ \mathbb{D}_2 &= E_3 \mathcal{D}_t + E_4 \mathcal{D}_x - E_1 \mathcal{D}_y - E_2 \mathcal{D}_z, \\ \mathbb{D}_3 &= (u_{yy} E_2 - u_{xy} E_3 - E_4) \mathcal{D}_x - (u_{xy} E_2 - u_{xx} E_3 - E_1) \mathcal{D}_y, \\ \mathbb{D}_4 &= (u_{yy} E_2 - u_{xy} E_3 - E_4) \mathcal{D}_t \\ &\quad - (u_{yy} E_1 + (u_{xx} u_{yy} - u_{xy}^2) E_3 - u_{xy} E_4) \mathcal{D}_x \\ &\quad + (u_{xy} E_1 + (u_{xx} u_{yy} - u_{xy}^2) E_2 - u_{xx} E_4) \mathcal{D}_y \\ &\quad - (u_{xy} E_2 - u_{xx} E_3 - E_1) \mathcal{D}_z.\end{aligned}$$

We have $\mathbb{D}_1(I_1) = \mathbb{D}_2(I_1) = \mathbb{D}_4(I_1) = 0$ and $\mathbb{D}_3(I_1) = -J_1$. Let c_{ij}^k be the structural functions of the frame $\mathbb{D}_1, \dots, \mathbb{D}_4$,

$$[\mathbb{D}_i, \mathbb{D}_j] = c_{ij}^k \mathbb{D}_k, \quad 1 \leq i < j \leq 4.$$

Then c_{ij}^k are rational functions of J_1 and the functions K_1, \dots, K_{11} defined as $K_1 = c_{12}^1$, $K_2 = c_{12}^2$, $K_3 = c_{12}^4$, $K_4 = c_{13}^4$, $K_5 = c_{14}^3$, $K_6 = c_{23}^4$, $K_7 = c_{24}^1$, $K_8 = c_{24}^2$, $K_9 = c_{24}^3$, $K_{10} = c_{24}^4$, and $K_{11} = c_{34}^1$. These together with I_1, J_1 , and $\mathbb{D}_i(J_1)$ form a basis of differential invariants on 4-jets.

As our MAPLE computations indicate, the whole algebra of scalar (absolute) differential invariants of \mathfrak{G}_1 is generated by the fundamental invariant I_1 and the invariant derivations \mathbb{D}_j , i.e., the iterated invariant derivatives of I_1 and functions of them (in particular all K_j are obtained so) yield all the invariants.

However, the generators $\mathbb{D}_1, \dots, \mathbb{D}_4$ vanish on Eq. (1), and thus $\{I_1 = 0\}$ is a singular manifold of the action of \mathfrak{G}_1 on $J^\infty(TM, \mathbb{R})$.

VII. SYMMETRIES OF PLEBAŃSKI II

As we have shown, the second Plebański equation arises naturally from the Lie algebra \mathfrak{g}_1 (which in turn is a natural extension of $\text{SDiff}(2)$). On the other hand, this algebra appears to be an infinite-dimensional part of the algebra of contact symmetries of (1)—the following statement is obtained by a direct computation, cf. Ref. 10

Theorem 10: *The Lie algebra of contact symmetries of Eq. (1) is the graded Lie algebra $\tilde{\mathfrak{g}}_1 = \tilde{\mathfrak{a}}_0 \oplus \tilde{\mathfrak{a}}_1 \oplus \tilde{\mathfrak{a}}_2 \oplus \tilde{\mathfrak{a}}_3$, $[\tilde{\mathfrak{a}}_i, \tilde{\mathfrak{a}}_j] \subset \tilde{\mathfrak{a}}_{i+j}$ with $\tilde{\mathfrak{a}}_0 = \mathfrak{a}_0 \oplus \mathbb{R} \cdot W'_0 \oplus \mathbb{R} \cdot W''_0$, $\tilde{\mathfrak{a}}_1 = \mathfrak{a}_1 \oplus \mathbb{R} \cdot W'_1$, $\tilde{\mathfrak{a}}_2 = \mathfrak{a}_2$, $\tilde{\mathfrak{a}}_3 = \mathfrak{a}_3$ ($\tilde{\mathfrak{a}}_i = 0$ for $i \notin \{0, 1, 2, 3\}$). Here,*

$$\begin{aligned}W'_0 &= t \partial_t + x \partial_x + y \partial_y + z \partial_z + 2u \partial_u, \\ W''_0 &= -x \partial_x - y \partial_y - 3u \partial_u, & W'_1 &= t \partial_x + z \partial_y.\end{aligned}$$

The structure equations of $\tilde{\mathfrak{g}}_1$ are the following (the functions σ_1, σ_2 are the same as in Theorem 5):

$$\begin{aligned}[W_i(A_i), W_j(A_j)] &= W_{i+j}(\{A_i, A_j\}), \\ [W'_0, W_i(A_i)] &= W_i(\sigma_1(A_i)), & [W''_0, W_i(A_i)] &= i W_i(A_i), \\ [W'_1, W_i(A_i)] &= W_{i+1}((\sigma_1 + i \sigma_2)(A_i)), \\ [W'_0, W'_0] &= 0, & [W'_0, W'_1] &= 0, & [W''_0, W'_1] &= W'_1.\end{aligned}$$

Thus, the descending central and derived series of $\tilde{\mathfrak{g}}_1$ are: $[\tilde{\mathfrak{g}}_1, \tilde{\mathfrak{g}}_1] = [\tilde{\mathfrak{g}}_1, \mathfrak{g}_1 \oplus \mathbb{R} \cdot W'_1] = \mathfrak{g}_1 \oplus \mathbb{R} \cdot W'_1$, $[\mathfrak{g}_1 \oplus \mathbb{R} \cdot W'_1, \mathfrak{g}_1 \oplus \mathbb{R} \cdot W'_1] = [\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_1$.

Moreover the algebra $\tilde{\mathfrak{g}}_1$ is restored from $\mathfrak{g}_1 \simeq \mathcal{H}_3$ in two steps. At first we apply the two-dimensional right extension by the derivations $\varepsilon^k A_k \mapsto \varepsilon^k \sigma_1(A_k)$, $\varepsilon^k A_k \mapsto \varepsilon^{k+1}(\sigma_1 + k \sigma_2)(A_k)$ of

degrees 0 and 1. The corresponding cohomology classes in $H^1(\mathcal{H}_3, \mathcal{H}_3)$ are closely related to the fundamental cohomology classes from Theorems 5 and 6.

Then we do one-dimensional extension by the grading element W_0'' .

It is important to stress that if we stop on the first step, we obtain the full symmetry algebra $\mathfrak{g} \oplus \mathbb{R} \cdot W_0' \oplus \mathbb{R} \cdot W_1'$ of the function I_1 . The remaining field does not preserve I_1 – it is a relative differential invariant for W_0'' . Indeed its second prolongation satisfies: $\text{pr}_2(W_0'')(I_1) = -2 I_1$.

Remark on Lie remarkable property. Finite and infinite-dimensional Lie algebras of classical symmetries are important in integration and establishing exact solutions of differential equations.

On the other hand for any Lie pseudo-group of symmetries we can calculate its prolongation to the space of k -jets and consider non-trivial orbits for the smallest k , which can be considered geometrically as differential equations. If these two processes are inverse of each other, the equation is called Lie remarkable (the original paper⁹ deals with point symmetries, i.e., fields on the space of 0-jets $\mathcal{D}(J^0)$, but it extends to the contact fields on the space of 1-jets $\text{cont}(J^1)$).

In the particular case of scalar determined equation (one independent variable and one PDE) we calculate the symmetry group and (if it is non-trivial) look for the lowest order differential invariant $I \in C^\infty(J^k)$. If it is unique (up to the gauge $I \mapsto F(I)$), and $I = 0$ coincides with our PDE, the latter has the above property. Thus, the Plebański equation is Lie remarkable (in general the equation $I = c$ can depend on the value of c , and there can be even regular and non-regular values, but for (1) this constant can be easily absorbed).

Not all equations are Lie remarkable. For instance, the Boyer-Finley equation $u_{z\bar{z}} = (e^u)_{tt}$ (another “heavenly” equation) has 5 differential invariants of order 2 (3 of pure order 2) of its symmetry groups action¹⁵ (in this case the group G is also infinite-dimensional, and it consists of conformal transformations of \mathbb{R}^2 together with a translation and a scaling). This makes possible application of the method of group foliation, but it does not uniquely characterize the equation.

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APPENDIX: PLEBAŃSKI EQUATION

Let us briefly discuss the structure of the contact symmetry algebra of the first Plebański’s heavenly equation¹³

$$u_{tx}u_{yz} - u_{tz}u_{xy} = 1, \quad (\text{A1})$$

which was studied in Refs. 3 and 10. It turns out that the infinite part of this symmetry algebra is also composed of 4 copies of $\text{SDiff}(2)$, but now it is 2-graded, to be more precise it is a copy of two such algebras.

Thus instead of TM we get $M \times M$, with the symplectic form being the product structure. Let $M_1 = \mathbb{R}^2(t, y)$, $M_2 = \mathbb{R}^2(x, z)$ be the two copies of M with $\varpi_1 = dt \wedge dy$, $\varpi_2 = dx \wedge dz$. These generate the Lie sub-algebra

$$\mathfrak{b}_0 = \{X_{A_0} + X_{B_0} \mid A_0 \in C^\infty(M_1), B_0 \in C^\infty(M_2)\} \subset \mathcal{D}(M \times M)$$

consisting of two copies of $\text{SDiff}(2)$. Letting $J^0 = J^0(M \times M) = M \times M \times \mathbb{R}$, with the coordinate u on the last factor, we extend the algebra to include two more copies of $\text{SDiff}(2)$:

$$\mathfrak{b}_1 = \{(A_1 + B_1) \partial_u \mid A_1 \in C^\infty(M_1), B_1 \in C^\infty(M_2)\} \subset \mathcal{D}(J^0).$$

One can easily check that $\mathfrak{g} = \mathfrak{b}_0 \oplus \mathfrak{b}_1$ is a graded Lie algebra.

To indicate the grading we will write the generators of the algebra \mathfrak{g} as $Y_0^\alpha(A_0) = X_{A_0}$, $Y_0^\beta(B_0) = X_{B_0}$, $Y_1^\alpha(A_1) = A_1 \partial_u$, $Y_1^\beta(B_1) = B_1 \partial_u$.

Theorem 11: *The Lie algebra $\tilde{\mathfrak{g}}$ of contact symmetries of Eq. (A1) is equal to $\mathfrak{g} \oplus \mathbb{R}^3 \langle Y'_0, Y''_0, \tilde{Y}_0 \rangle$, where*

$$Y'_0 = t \partial_t - y \partial_y, \quad Y''_0 = x \partial_x - z \partial_z, \quad \tilde{Y}_0 = t \partial_t + y \partial_y - x \partial_x - z \partial_z.$$

It is graded by $\tilde{\mathfrak{b}}_0 = \mathfrak{b}_0 \oplus \mathbb{R}^3 \langle Y'_0, Y''_0, \tilde{Y}_0 \rangle$, $\tilde{\mathfrak{b}}_1 = \mathfrak{b}_1$. The structure equations of $\tilde{\mathfrak{g}}$ are:

$$\begin{aligned} [Y_i^\alpha(A_i), Y_j^\alpha(\bar{A}_j)] &= Y_{i+j}^\alpha(\{A_i, \bar{A}_j\}), \quad [Y_i^\alpha(A_i), Y_j^\beta(B_j)] = 0, \\ [Y_i^\beta(B_i), Y_j^\beta(\bar{B}_j)] &= Y_{i+j}^\beta(\{B_i, \bar{B}_j\}), \\ [Y_i^\alpha(A_i), Y'_0] &= Y_i^\alpha(\mu_-^\alpha(A_i)), \quad [Y_i^\alpha(A_i), Y''_0] = 0, \\ [Y_i^\alpha(A_i), \tilde{Y}_i] &= Y_i^\alpha((2-2i)A_i - \mu_+^\alpha(A_i)), \\ [Y_i^\beta(B_i), Y'_0] &= 0, \quad [Y_i^\beta(B_i), Y''_0] = Y_i^\beta(\mu_-^\beta(B_i)), \\ [Y_i^\beta(B_i), \tilde{Y}_i] &= Y_i^\beta(\mu_+^\beta(B_i) - (2-2i)B_i), \end{aligned}$$

where $\mu_\pm^\alpha(A) = yA_y \pm tA_t$, $\mu_\pm^\beta(B) = zB_z \pm xB_x$.

Both the descending central series and the derived series of $\tilde{\mathfrak{g}}$ stabilize, since $[\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}] = [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. The left-hand side of (A1) is an absolute invariant of the Lie algebra $\tilde{\mathfrak{g}}$, which is a three-dimensional right extension of \mathfrak{g} by $\mu_\pm^\alpha, \mu_\pm^\beta$.

Higher dimensional versions of the second Plebański equation are known.¹⁴ We can produce some analogs via differential invariants.

For instance, taking 6 copies of SDiff(2) and applying the above method for the first Plebański equation, we obtain, modulo the standard copies of four-dimensional equation (A1), the unique six-dimensional equation on $u = u(x_1, p_1, x_2, p_2, x_3, p_3)$,

$$\text{Pf} \begin{bmatrix} 0 & H_{12} & H_{13} \\ -H_{12}^T & 0 & H_{23} \\ -H_{13}^T & -H_{23}^T & 0 \end{bmatrix} = 0,$$

where Pf is the Pfaffian, $H_{ij} = \begin{bmatrix} u_{x_i x_j} & u_{x_i p_j} \\ u_{p_i x_j} & u_{p_i p_j} \end{bmatrix}$ is the 2×2 sub-matrix of Hess(u) and H_{ij}^T its transpose.

This equation is cubic in 2-jets, and the standard integrability methods are not applicable. Still it has a huge local symmetry algebra. The geometry of this equation should be a subject of the further study.

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