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A Two-Component Generalization of the Integrable rdDym Equation*

Oleg I. MOROZOV

Institute of Mathematics and Statistics, University of Tromsø, Tromsø 90-37, Norway
E-mail: *Oleg.Morozov@uit.no*

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Abstract. We find a two-component generalization of the integrable case of rdDym equation. The reductions of this system include the general rdDym equation, the Boyer–Finley equation, and the deformed Boyer–Finley equation. Also we find a Bäcklund transformation between our generalization and Bodganov's two-component generalization of the universal hierarchy equation.

Key words: coverings of differential equations; Bäcklund transformations

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1 Introduction

Recent papers [3, 8, 16] provide two-component generalizations for the hyper-CR Einstein–Weil structure equation [6, 22]

$$s_{yy} = s_{tx} + s_y s_{xx} - s_x s_{xy}, \tag{1.1}$$

Plebański's second heavenly equation [25]

$$s_{xz} = s_{ty} + s_{xx} s_{yy} - s_{xy}^2 \tag{1.2}$$

and the universal hierarchy equation [18, 19, 22]

$$s_{xx} = s_x s_{ty} - s_t s_{xy}. \tag{1.3}$$

Namely, equations (1.1)–(1.3) appear from systems

$$\begin{aligned} s_{yy} &= s_{tx} + (s_y + r)s_{xx} - s_x s_{xy}, \\ r_{yy} &= r_{tx} + (s_y + r)r_{xx} - s_x r_{xy} + r_x^2; \end{aligned} \tag{1.4}$$

$$\begin{aligned} s_{xz} &= s_{ty} + s_{xx} s_{yy} - s_{xy}^2 + r, \\ r_{xz} &= r_{ty} + s_{yy} r_{xx} + s_{xx} r_{yy} - 2 s_{xy} r_{xy}, \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} s_{xx} &= e^r (s_x s_{ty} - s_t s_{xy}), \\ \left(e^{-r}\right)_{xx} &= s_x r_{ty} - s_t r_{xy}, \end{aligned} \tag{1.6}$$

respectively, by substituting for $r = 0$. Other reductions for (1.4) are found in [7, 16]: when $u = 0$, system (1.4) gives the Khokhlov–Zabolotskaya (or dispersionless Kadomtsev–Petviashvili) equation

$$v_{yy} = v_{tx} + v v_{xx} + v_x^2,$$

*This paper is a contribution to the Special Issue "Geometrical Methods in Mathematical Physics". The full collection is available at http://www.emis.de/journals/SIGMA/GMMP2012.html while substituting for $v = u_x$ in (1.4) produces the normal form

$$u_{yy} = u_{tx} + (u_x + u_y)u_{xx} - u_x u_{xy},$$

for the family of equations studied in [7]. Also, we note the reduction $v = u_y$ for system (1.4). This reduction yields equation

$$u_{yy} = u_{tx} - u_x u_{xy}$$

studied in [9, 14, 17, 21].

As it was shown in [3], the reduction $s = x$ for system (1.6) gives the Boyer–Finley equation

$$r_{ty} = \left(e^{-r}\right)_{xx}. \tag{1.7}$$

The purpose of the present paper is to introduce the two-component generalization for equation

$$u_{ty} = u_x u_{xy} - u_y u_{xx}, \tag{1.8}$$

which is integrable in the following sense: it has the differential covering [2, 11, 12, 13]

$$p_t = (u_x - \lambda)p_x, \qquad p_y = \lambda^{-1} u_y p_x \tag{1.9}$$

containing the non-removable parameter $\lambda \neq 0$ [20]. We show that reductions of the generalization include the general r-th dispersionless Dym equation [1]

$$u_{ty} = u_x u_{xy} + \kappa u_y u_{xx}, \tag{1.10}$$

the Boyer–Finley equation (1.7), and the deformed Boyer–Finley equation. Also we find a Bäcklund transformation between our generalization and Bodganov's two-component generalization (1.6) of the universal hierarchy equation (1.3).

2 The two-component generalization

Along with the covering (1.9) equation (1.8) has the covering

$$q_t = (u_x - q)q_x, \qquad q_y = u_y q^{-1} q_x, \tag{2.1}$$

which can be obtained by the method of [20]. While the coverings (1.9) and (2.1) are not equivalent w.r.t. the pseudo-group of contact transformations, (2.1) can be derived from (1.9) by the following procedure, see, e.g., [24]. We consider the function $p = p(t, x, y)$ from (1.9) to be defined implicitly by the equation $q(t, x, y, p(t, x, y)) = \lambda$ with $q_p \neq 0$. Then for $(x^1, x^2, x^3) = (t, x, y)$ we have $q_{x^i} + q_p p_{x^i} = 0$, so $p_{x^i} = -q_{x^i}/q_p$. Substituting these into (1.9) yields (2.1).

Our main observation in this paper is that the covering (2.1) allows the generalization

$$q_t = (u_x - q + v)q_x + v_x q, \qquad q_y = u_y q^{-1} q_x + v_y. \tag{2.2}$$

This system is compatible whenever the two-component system

$$u_{ty} = (u_x + v)u_{xy} - u_y u_{xx}, \tag{2.3}$$

$$v_{ty} = (u_x + v)v_{xy} - u_y v_{xx} + v_x v_y \tag{2.4}$$

holds. In other words, (2.2) is a covering for system (2.3), (2.4).

3 Reductions

By the construction, we have the following reduction for system (2.2):

Reduction A. Substituting for $v = 0$ in equations (2.3), (2.2) gives equations (1.8) and (2.1), while (2.4) becomes an identity.

Also, we have three other reductions.

Reduction B. If we put $v = -(\kappa^{-1} + 1)u_x$, then (2.3) gets the form

$$u_{ty} = -\kappa^{-1} u_x u_{xy} - u_y u_{xx}, \tag{3.1}$$

while (2.4) is its differential consequence. The transformation $u \mapsto -\kappa u$ maps (3.1) to (1.10). The corresponding reduction of (2.2) produces the covering of (1.10) studied in [20, 23].

Reduction C. Taking $v = -u_x$ in (2.3), (2.4), we obtain

$$u_{ty} = -u_y u_{xx}$$

and its differential consequence. Then we divide this equation by u_y, differentiate w.r.t. y and put $u_y = -e^w$. This gives the Boyer–Finley equation [4]

$$w_{ty} = (e^w)_{xx} \tag{3.2}$$

This equation is equation (1.7) in a different notation. Substituting for $q = e^p$ in the corresponding reduction of (2.2), we have the covering [10, 15, 26] for equation (3.2):

$$p_t = w_t - e^p p_x, \qquad p_y = e^{w-p}(w_x - p_x).$$

Reduction D. Finally, when we put $v = u_y - u_x$ into (2.3) and (2.4), we get the equation

$$u_{ty} = u_y \left(u_{xy} - u_{xx}\right)$$

and its differential consequence. Then for $u_y = e^w$ we have the deformed Boyer–Finley equation [5]

$$w_{ty} = (e^w)_{xy} - (e^w)_{xx}\,, \tag{3.3}$$

and the corresponding reduction of equations (2.2) with $q = e^s$ gives the covering

$$s_t = (e^s - e^w) s_x - w_t, \qquad s_y = e^w(s_x - w_x + w_y).$$

for (3.3). This covering in other notations was found in [5, 20].

4 Bäcklund transformations

The substitution

$$u_x = -v + \frac{s_t}{s_x}, \qquad u_y = -\frac{e^{-r}}{s_x}, \qquad v_x = \frac{r_x s_t}{s_x} - r_t, \qquad v_y = -\frac{e^{-r} r_x}{s_x} \tag{4.1}$$

maps system (2.2) to system

$$q_t = \left(\frac{s_t}{s_x} - q\right) q_x + \left(\frac{s_t r_x}{s_x} - r_t\right) q, \qquad q_y = -\frac{e^{-r}}{q s_x}(q_x + r_x q) \tag{4.2}$$

found in [3]. This system is the two-component generalization of the covering

$$q_t = \left(\frac{s_t}{s_x} - q\right) q_x, \qquad q_y = -\frac{q_x}{q s_x}.$$

of equation (1.3). The compatibility conditions for (4.2) coincide with (1.6). Solving (4.1) for s_t, s_x, r_t, r_x yields

$$s_t = -(u_x + v)\frac{e^{-r}}{u_y}, \qquad s_x = -\frac{e^{-r}}{u_y}, \qquad r_t = \frac{v_y}{u_y}, \qquad r_x = \frac{(u_x + v)v_y}{u_y} - v_x. \tag{4.3}$$

This system is compatible whenever equations (2.3), (2.4) are satisfied. Thus equations (4.1) define a Bäcklund transformation from (2.3), (2.4) to (1.6) with the inverse transformation (4.3). In particular, when $v = 0$ and $r = 0$, we have a Bäcklund transformation

$$u_x = \frac{s_t}{s_x}, \qquad u_y = -\frac{1}{s_x},$$

between (1.1) and (1.3) with the inverse transformation

$$s_t = -\frac{u_x}{u_y}, \qquad s_x = -\frac{1}{u_y}.$$

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