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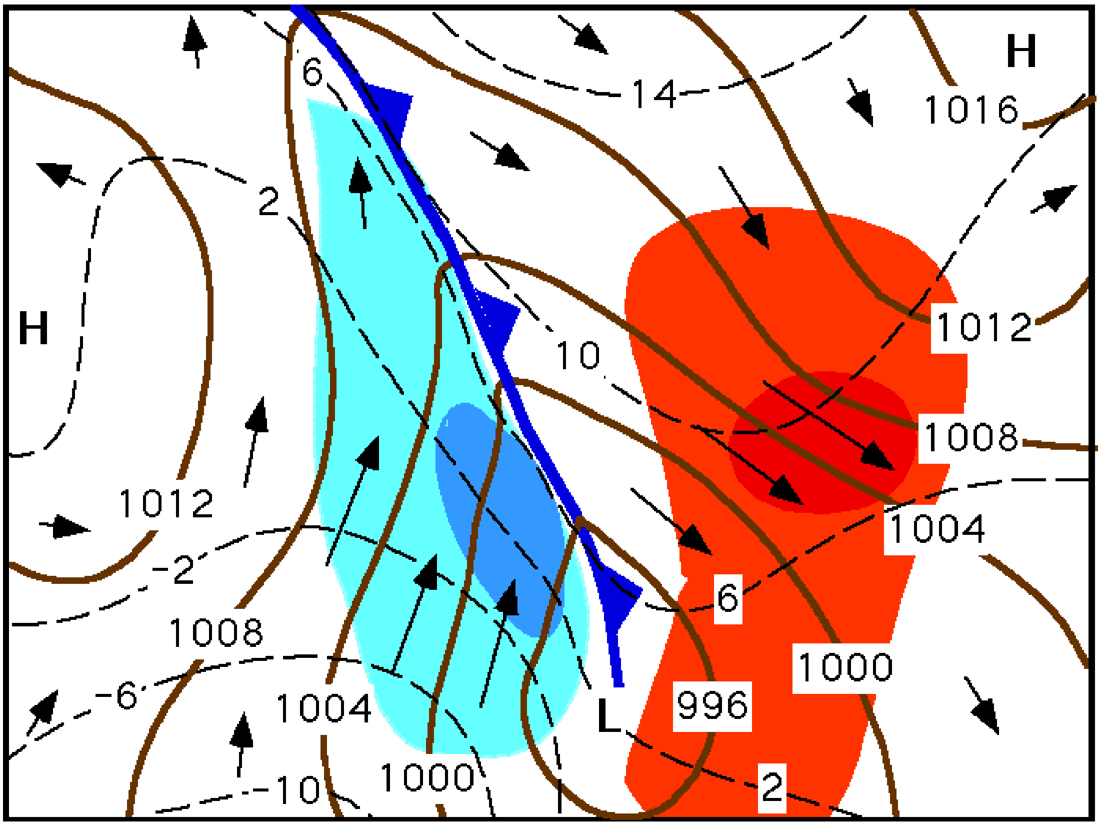
Reduced Models for Geophysical Fluid Dynamics



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Master Thesis
Reduced Models for Geophysical Fluid Dynamics

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Abstract

This thesis present a self-consistent derivation of reduced fluid models for geophysical dynamics confined to the midlatitude region. The reduced model will be derived by use of a regular perturbation method, that gives the same result as the classical models, such as the barotropic and baroclinic quasi-gestrophic potential vorticity model. It will be shown that such a rigorous treatment self-consistently comprises otherwise classic assumptions known as, the Boussinesq approximation, shallow-water approximation, β-plane approximation (slab-approximation) and thin shell approximation. We use the understanding of these reduced models to generalize the baroclinic quasi-gestrophic potential vorticity model to include interaction with global scale. This will be done by using a multi-scale expansion, assosisert with the separation of spatio-temporal scales.

Takk

Da er tiden endelig inne for å takke de som har bidratt til at jeg endelig er ferdig med masteroppgaven. Det har vært et travelt år fylt med mye lesing, skriving og beregninger som har resultert i en endelig masteroppgave. På denne veien har jeg fått utmerket veiledning fra mine to veiledere Odd Erik Garica og Kristoffer Rypdal. Spesielt vil jeg rette en stor takk til Odd Erik som har vært hovedveileder. Til slutt vil jeg takke min familie og mine venner for god støtte gjennom dette arbeidet.

Contents

Chapter 1

Introduction and overview

For centuries, the ocean and the atmosphere has been a source of wonder and curiosity. Phenomena such as buoyancy, transport of mass and heat, weather and waves have given pleasure and rumination to the great philosophers, scientists and artists. Take for instance Archimedes's Eureka, Benjamin Franklin and Timothy Folger's first map of the Gulf stream or Jule Gregory Charney's beautiful model for quasi-geostrophic flow in the midlatitude. Archimedes motivation was to find out if King Hieron II's crown was made of pure gold, while Benjamin Franklin and Timothy Folger would find the fastest path across the Atlantic by skip.

The common feature of the great thinkers were and are simplified models in order to understand why the ocean and the atmosphere behaves as it do. One of the first to do this in a structured and beautiful mathematical way was Jule Gregory Charney, who introduced the use of scaling analysis to find reduced models for large-scale midlatitude atmospheric circulation models [1]. This work has been further developed by many other scientists, where perhaps the one that has contributed most is Joseph Pedlosky who has written one of the most widely used text books in geophysical fluid mechanics [2]. This book has been an inspiration to many other books in the field, f.ex. *Dynamical Oceanography* by Henk A. Dijkstra and *Atmospheric and Oceanic Fluid Dynamics* by Geoffrey K. Vallis.
One problem with all these text books is that they treat the ocean and the atmosphere as a one-component system, so that the thermodynamic description of the ocean is wrong, i.e., this means that they derive their models from the wrong equations, since the ocean is a two-component system. This does not mean that the models are wrong, but the models should be derived from the correct equations. Thus, in Appendix A of this thesis we derive the correct equations for a two-component fluid.

This thesis present a self-consistent derivation of reduced fluid models for geophysical dynamics confined to the midlatitude region. The reduced model will be derived by use of a regular perturbation method, that gives the same result as the classical models, such as the barotropic and baroclinic quasi-gestrophic potential vorticity model. It will be shown that such a rigorous treatment self-consistently comprises otherwise classic assumptions known as, the Boussinesq approximation, shallow-water approximation, β-plane approximation (slab-approximation) and thin shell approximation. We use the understanding of these reduced models to generalize the baroclinic quasi-gestrophic potential vorticity model to include interaction with global scale. This will be done by using a multi-scale expansion, assosisert with the separation of spatio-temporal scales.

The structure of this thesis is as follows: In Appendix A we give a brief derivation of the equations of motion, where the main focus is a detailed derivation of the thermodynamic equations that applies to a two-component, one-phase fluid such as the ocean which consists of fresh water and salt. These equations are the all calculations presented in the thesis. In chapter 2 we introduce normalization of these equations and introduce dimensionless number that will be the key to derive reduced models. In addition, we average these equations to apply on a large scale. This process leads to introduction of turbulent fluxes. Since these equations contains all types of phenomena that are associated by the ocean, we present in chapter 3 an understanding of the various spatio-temporal scales. One of the main focuses of my thesis is to derive a reduced model that describes the interactions between global and local scales in the midlatitude region. To have some models to compare this model with, we will in chapters 4 and 5 derive two classical models for barotropic and baroclinic quasi-geostrophic flow that includes boundary layer theory. The interacting model will be described in detail in Chapter 6.

Chapter 2

The fluid model equations

The main goal of this chapter is to derive the dimensionless equations that describes the dynamics of the ocean on large scale, i.e. on a length scale where rotation, stratification, curvature may be important. The chapter starts with a presentation of the equations of motion, followed by an introduction of scaling analysis and normalization of these equations. In section 2.3, we will perform a Hesselberg averaging of the normalized equations. This averaging leads to that the fast turbulent fluctuations will be filtered out of the system, and the remaining part will describe the large-scale motions. At the end of the chapter we will discuss the background state of the ocean, by then deriving the evolution equations for the mass density, velocity, pressure, temperature and salinity deviations from the background state.

2.1 The equations of motion

The description of the ocean is given by the equations of motion, that is closed by prognostic and diagnostic equations for the thermodynamic variables. Since the ocean consists mainly of salt water that is in the liquid phase, the thermodynamic description of the ocean must be represented by three independent thermodynamic variables which completely determines the thermodynamic properties of the system. This follows directly from the Gibbs phase rule. There are many different independent variables that can be used, but we will use the pressure p, temperature T and salinity S. In Appendix A 8 we have derived all the equations, so we will just give an presentation of them here. The closed set of equations in the

(p, T, S)-representation reads

$$
\begin{aligned}
\frac{d\rho}{dt} &= -\rho\nabla\cdot\mathbf{u}, &(2.1)\\
\rho\frac{d\mathbf{u}}{dt} &= -\nabla p + \nabla\cdot\boldsymbol{\sigma}' - 2\rho\boldsymbol{\Omega}\times\mathbf{u} + \rho\mathbf{g}, &(2.2)\\
\rho\, c_p\left(\frac{dT}{dt} - \Gamma\frac{dp}{dt}\right) &= \boldsymbol{\sigma}' : \mathsf{D} - \nabla\cdot\mathbf{q} - \mathbf{J}_S\cdot\nabla\left(\Delta h\right), &(2.3)\\
\rho\frac{dS}{dt} &= -\nabla\cdot\mathbf{J}_S, &(2.4)\\
\rho &= \rho\left(p, T, S\right), &(2.5)
\end{aligned}
$$

where ρ is the mass density, $\mathbf{u}$ is the fluid velocity, p is the pressure, $\boldsymbol{\sigma}'$ is the viscous stress tensor, D is the deformation tensor, $\boldsymbol{\Omega}$ is the angular velocity of the earth, $\mathbf{r}$ is the position to a fluid element, $\mathbf{g}$ is the gravity of the earth, T is the temperature, c_p is the specific heat capacity at constant pressure, $\mathbf{q}$ is the conductive heat flux, p is the pressure, Γ is the adiabatic temperature gradient, S is the salinity, $\mathbf{J}_S$ is the diffusive salinity flux and Δh is the partial enthalpy difference. It should be noted that

$$
\begin{aligned}
\frac{d}{dt} &= \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla, &(2.6)\\
\mathsf{D} &= \frac{1}{2}\left[\nabla\mathbf{u} + \left(\nabla\mathbf{u}\right)^T\right], &(2.7)
\end{aligned}
$$

is the material derivative and the deformation tensor, respectively. The molecular fluxes in the equations are given by

$$
\begin{aligned}
\mathbf{q} &= -\kappa\nabla T + k_T\left(\frac{\partial\Delta\mu}{\partial S}\right)_{p,T}\mathbf{J}_S, &(2.8)\\
\mathbf{J}_S &= -\rho D\left(\frac{k_T}{T}\nabla T + \frac{k_p}{p}\nabla p + \nabla S\right), &(2.9)\\
\boldsymbol{\sigma}' &= \eta\left[\nabla\mathbf{u} + \left(\nabla\mathbf{u}\right)^T - \frac{2}{3}\left(\nabla\cdot\mathbf{u}\right)\mathsf{I}\right] + \zeta\left(\nabla\cdot\mathbf{u}\right)\mathsf{I}. &(2.10)
\end{aligned}
$$

where κ is the thermal conductivity that specifies heat transfer in the absence of salt flux. D is the salt diffusion coefficient that specifies salinity transfer in the absence of thermal and pressure gradients. k_T is the thermo-salt diffusion coefficient that specifies salinity transfer in the absence of salinity and pressure gradients. k_p is the baro-salt diffusion coefficient that specifies salinity transfer in the absence of salinity and temperature gradients. η is the dynamical shear viscosity and ζ is the bulk viscoisity due to compression and expansion, $\Delta\mu$ is

Coefficient	**Definition**
Thermal expansion coefficient	$\beta_T = -\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_{p,S}$
Compressibility coefficient	$\beta_p = \frac{1}{\rho}\left(\frac{\partial\rho}{\partial p}\right)_{T,S}$
Salinity contraction coefficient	$\beta_S = \frac{1}{\rho}\left(\frac{\partial\rho}{\partial S}\right)_{p,T} = -\rho\left(\frac{\partial\Delta\mu}{\partial p}\right)_{T,S}$
Adiabatic compressibility coefficient	$\widetilde{\kappa} = \beta_p - \Gamma\beta_T$
Adiabatic temperature gradient	$\Gamma = \frac{\beta_T T}{c_p\rho}$
Speed of sound	$c = \sqrt{\frac{1}{\rho\widetilde{\kappa}}}$

Table 2.1: Definition of transport coefficients in fluid model equations

the chemical potential difference between sea salt and freshwater and I is the unit tensor. However, experiments show that a very good approximation for the heat, salinity and viscosity fluxes in seawater are

$$\mathbf{q} \approx -\kappa\nabla T \tag{2.11}$$

$$\mathbf{J}_S \approx -\rho D\left(\nabla S + \frac{k_p}{p}\nabla p\right) = -\kappa_S\nabla S - \kappa_{Sp}\nabla p \tag{2.12}$$

$$\boldsymbol{\sigma}' \approx \eta\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T - \frac{2}{3}(\nabla\cdot\mathbf{u})\,\mathsf{I}\right), \tag{2.13}$$

where $\kappa_S = \rho D$ and $\kappa_{Sp} = \rho\, D\, k_p/p$. See [9, p. 56] . All the thermodynamic coefficients have to be specified as a function of (p, T, S). Other usefull relations are given in table 2.1.

2.2 Geophysical scaling

The full equations of a physical system consist of several terms of different orders of magnitude, which describes the overall behavior of the system. Depending on the magnitude of the terms, some are important and others will be less important to describe the system. In this section we will go through scaling analysis and show how we can simplify the full equations of a physical system by ignoring non-important terms in a consistent manner without changing the basic physics, based on the ratio between the magnitude of the terms. Let

$$\sum_{i=1}^{n} f_i = 0 \tag{2.14}$$

be a hypothetical equation which describes a physical system, for example the momentum equation. The magnitude of each term,

$$|f_i|_m \, , \tag{2.15}$$

is defined in a manner such that the dimensionless term

$$\hat{f}_i = \frac{f_i}{|f_i|_m} \tag{2.16}$$

is of order unity, $\hat{f}_i \sim \mathcal{O}(1)$. According to equation (2.16), the hypothetical equation (2.14) can be written as

$$\sum_{i=1}^{n} |f_i|_m \hat{f}_i = 0. \tag{2.17}$$

In order to compare the magnitude of the terms, we introduce dimensionless characteristic numbers given by the ratio of the magnitude between term i and term j by

$$N_{i,j} = \frac{|f_i|_m}{|f_j|_m}. \tag{2.18}$$

If we are interested in significance of the term j in comparison to the other terms, we can divide $|f_j|_m$ on equation (2.17). This results in a dimensionless equation,

$$\sum_{i=1}^{n} N_{i,j} \hat{f}_i = 0. \tag{2.19}$$

where the characteristic numbers will determine the importance of term j. In the limit where all the characteristic numbers $N_{i,j} \ll 1$, the term j will play a dominant role. In contrast, if all the characteristic numbers $N_{i,j} \gg 1$, the term j has no significant role and may be neglected. It should be noted that the the numbers n of characteristic numbers is unique, but the choice of parameters is not unique.

2.2.1 Dimensionless variables and parameteres

Let us now apply the scaling analysis on equations (2.1)-(2.5). We introduce a typical magnitude of mass density ρ_m, pressure p_m, horizontal velocity $U_{\perp,m}$, vertical velocity $U_{\parallel,m}$, temperature T_m, salinity S_m, dynamical viscosity η_m, horizontal spatial scale $L_{\perp,m}$, vertical spatial scale $L_{\parallel,m}$ and temporal scale t_m, and then define the corresponding dimensionless quantities

$$\hat{\rho} = \frac{\rho}{\rho_m}, \quad \hat{p} \quad = \frac{p}{p_m}, \quad \hat{\mathbf{u}} = \frac{\mathbf{u}}{U_m}, \quad \hat{T} = \frac{T}{T_m}, \quad \hat{S} = \frac{S}{S_m},$$
$$\hat{\mathbf{x}}_\perp = \frac{\mathbf{x}_\perp}{L_{\perp,m}}, \quad \hat{S} = \frac{S}{S_m}, \quad \hat{\mathbf{x}}_\parallel = \frac{\mathbf{x}_\parallel}{L_{\parallel,m}}, \quad \hat{t} = \frac{t}{t_m}, \quad \hat{\eta} = \frac{\eta}{\eta_m}$$

This implies that the dimensionless spatial and temporal differential operators become

$$\hat{\nabla}_\perp = \delta L_{\perp,m} \nabla_\perp, \quad \hat{\nabla}_\parallel = \delta L_{\parallel,m} \nabla_\parallel, \quad \frac{\partial}{\partial \hat{t}} = \delta t_m \frac{\partial}{\partial t}$$

where the δ in front of L_m and t_m represents respectively the characterisic length-scale and temporal-scale for the change of some quantity. For example

$$|\nabla \rho|_m$$

represents the typical magnitude of the change in the density $\delta\rho_m$ on length-scale δL_m. Throughout the discussion, we will assume that the typical magnitude of the change in the velocity is equal to the typical magnitude of velocity, i.e. $\delta U_{\perp,m} = U_{\perp,m}$ and $\delta U_{\parallel,m} = U_{\parallel,m}$. This turns out to be a good assumption for scaling in fluid mechanics. We will also assume that horizontal and vertival advection terms are of the same order, i.e.

$$|\mathbf{u}_\perp \cdot \nabla_\perp|_m = \left|\mathbf{u}_\parallel \cdot \nabla_\parallel\right|_m . \tag{2.20}$$

This means that the relationship between the characteristic value of the horizontal velocity and vertical velocity are

$$U_{\parallel,m} = \frac{\delta L_{\parallel,m}}{\delta L_{\perp,m}} U_{\perp,m}. \tag{2.21}$$

Therefore, its natural to introduce the aspect ratio between the horizontal and vertical motion

$$\gamma \equiv \frac{\delta L_{\parallel,m}}{\delta L_{\perp,m}}. \tag{2.22}$$

We will also define the aspect ratio between large-scale and small-scale motion as

$$\Gamma \equiv \frac{\delta L_{\perp,m}}{r_m}. \tag{2.23}$$

This assumption is acceptable as long as we expect the fluid to be almost incompressible. Either way it will come up later if the assumption is good or not. We will later see that the advection with horizontal velocity is greater than the advection with vertical velocity. Furthermore, we will use the definition of the sound velocity C_s to find the relation between the typical scale for the variation of the mass density and det variation of pressure, given by

$$\delta\rho_m = \frac{\delta p_m}{C_s^2}. \tag{2.24}$$

Before we start on the discussion of the dimensionless form of the equations of motion, we will present some characteristic dimensionless numbers to be used extensively in the following.

$$
\begin{array}{rlll}
\text{the Strouhal number:} & St & = \dfrac{\left|\frac{\partial \mathbf{u}}{\partial t}\right|_m}{|\mathbf{u}\cdot\nabla\mathbf{u}|_m} & \sim \dfrac{\delta L_m}{\delta t_m U_m} \\
\text{the Euler number:} & Eu & = \dfrac{|\nabla p|_m}{|\rho\mathbf{u}\cdot\nabla\mathbf{u}|_m} & \sim \dfrac{\delta p_m}{\rho_m U_m \delta U_m} \\
\text{the Reynolds number:} & Re & = \dfrac{|\rho\mathbf{u}\cdot\nabla\mathbf{u}|_m}{|\nabla\cdot\boldsymbol{\sigma}'|_m} & \sim \dfrac{\rho_m U_m \delta U_m}{\delta\sigma_m} \\
\text{the Rossby number:} & Ro & = \dfrac{|\mathbf{u}\cdot\nabla\mathbf{u}|_m}{|2\boldsymbol{\Omega}\times\mathbf{u}|_m} & \sim \dfrac{\delta U_m}{2\Omega_m \delta L_m} \\
\text{the Centrifugal number:} & Ce & = \dfrac{|\mathbf{u}\cdot\nabla\mathbf{u}|_m}{|\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})|_m} & \sim \dfrac{U_m \delta U_m}{\Omega_m^2 r_m \delta L_m} \\
\text{the Froude number:} & Fr & = \dfrac{|\mathbf{u}\cdot\nabla\mathbf{u}|_m}{|\mathbf{g}|_m} & \sim \dfrac{U_m \delta U_m}{\delta L_m g_m} \\
\text{the Mach number:} & Ma & = \dfrac{|\mathbf{u}|_m}{\left|\left(\frac{\partial p}{\partial \rho}\right)^{1/2}\right|_m} & \sim \dfrac{U_m}{C_s} \\
\text{the heat Peclet number:} & Pe_T & = \dfrac{|\rho\, c_p \mathbf{u}\cdot\nabla T|_m}{|\nabla\cdot\mathbf{q}|_m} & \sim \dfrac{\rho_m c_{p,m} U_m \delta L_m}{\kappa_m} \\
\text{the salinity Peclet number I:} & Pe_S & = & \sim \dfrac{\rho_m U_m \delta L_m}{\kappa_{S,m}} \\
\text{the salinity Peclet number II:} & Pe_{Sp} & = & \sim \dfrac{\rho_m U_m \delta L_m}{\kappa_{Sp,m}\delta\, p/\delta\, S} \\
\text{the Eckert number:} & Ec & = & \sim \dfrac{U_m^2}{c_{p,m}\delta T}
\end{array}
$$

We have not taken into account the anisotropy in the dimensionless numbers. This will be clear in each case. Furthermore, for the viscosity tensor, the heat flux and the salinity flux, the magnitude of the fluxes will be estimated by the maximum contribution. If it turns out that the magnitude is small compared to terms in the equations, then the fluxes can be neglected. If not, then each term have to be carefully normalized to find the contribution of each term.

The dimensionless continuity equation

Due to the anisotropy in the characteristic scale of the horizontal and vertical direction, it would be advantageous to write the continuity equation for mass as

$$\frac{\partial \rho}{\partial t} + \mathbf{u}_\perp \cdot \nabla_\perp \rho + \mathbf{u}_\parallel \cdot \nabla_\parallel \rho + \rho \left(\nabla \cdot \mathbf{u}\right)_\perp + \rho \left(\nabla \cdot \mathbf{u}\right)_\parallel = 0, \tag{2.25}$$

where $(\nabla \cdot \mathbf{u})_\perp$ and $(\nabla \cdot \mathbf{u})_\parallel$ are respectively the horizontal and vertical parts of the divergence given by equation (9.15). By using the definition of the dimensionless quantities, the continuity equation for mass can be written as

$$\begin{aligned}\left|\frac{\partial \rho}{\partial t}\right|_m \frac{\partial \hat{\rho}}{\partial \hat{t}} &+ \left|\mathbf{u}_\perp \cdot \nabla_\perp \rho\right|_m \hat{\mathbf{u}}_\perp \cdot \hat{\nabla}_\perp \hat{\rho} + \left|\mathbf{u}_\parallel \cdot \nabla_\parallel \rho\right|_m \hat{\mathbf{u}}_\parallel \cdot \hat{\nabla}_\parallel \hat{\rho} \\ &+ \left|\rho \left(\nabla \cdot \mathbf{u}\right)_\perp\right|_m \hat{\rho} \left(\hat{\nabla} \cdot \hat{\mathbf{u}}\right)_\perp + \left|\rho \left(\nabla \cdot \mathbf{u}\right)_\parallel\right|_m \hat{\rho} \left(\hat{\nabla} \cdot \hat{\mathbf{u}}\right)_\parallel = 0,\end{aligned}$$

From equation (2.20) and the assumption that the typical magnitude of the change in the velocity is equal to the typical magnitude of velocity, it follows that $\left|\mathbf{u}_\perp \cdot \nabla_\perp \rho\right|_m = \left|\mathbf{u}_\parallel \cdot \nabla_\parallel \rho\right|_m$ and $\left|\rho \left(\nabla \cdot \mathbf{u}\right)_\perp\right|_m = \left|\rho \left(\nabla \cdot \mathbf{u}\right)_\parallel\right|_m$. Hence, the continuity equation takes the form

$$\left|\frac{\partial \rho}{\partial t}\right|_m \frac{\partial \hat{\rho}}{\partial \hat{t}} + \left|\mathbf{u}_\perp \cdot \nabla_\perp \rho\right|_m \hat{\mathbf{u}} \cdot \hat{\nabla} \hat{\rho} + \left|\rho \left(\nabla \cdot \mathbf{u}\right)_\perp\right|_m \hat{\rho} \hat{\nabla} \cdot \hat{\mathbf{u}} = 0 \tag{2.26}$$

Since seawater is highly incompressible, it would be natural to compare all the terms in the continuity equation with the compression term. In this case the dimensionless continuity equation becomes

$$\frac{\left|\frac{\partial \rho}{\partial t}\right|_m}{\left|\rho \left(\nabla \cdot \mathbf{u}\right)_\perp\right|_m} \frac{\partial \hat{\rho}}{\partial \hat{t}} + \frac{\left|\mathbf{u}_\perp \cdot \nabla_\perp \rho\right|_m}{\left|\rho \left(\nabla \cdot \mathbf{u}\right)_\perp\right|_m} \hat{\mathbf{u}} \cdot \hat{\nabla} \hat{\rho} + \hat{\rho} \hat{\nabla} \cdot \hat{\mathbf{u}} = 0,$$

where the dimensionless numbers scales as

$$\begin{aligned}\frac{\left|\frac{\partial \rho}{\partial t}\right|_m}{\left|\rho \left(\nabla \cdot \mathbf{u}\right)_\perp\right|_m} &\sim \frac{\delta \rho_m \delta L_{\perp,m}}{\rho_m \delta U_{\perp,m} \delta t_m} = \frac{\delta p_m}{\rho_m U_{\perp,m} \delta U_{\perp,m}} \frac{U_{\perp,m}^2}{C_s^2} \frac{\delta L_{\perp,m}}{\delta t_m U_{\perp,m}} \sim Eu\, Ma^2\, Sr \\ \frac{\left|\mathbf{u}_\perp \cdot \nabla_\perp \rho\right|_m}{\left|\rho \left(\nabla \cdot \mathbf{u}\right)_\perp\right|_m} &\sim \frac{U_{\perp,m} \delta \rho_m}{\rho_m \delta U_{\perp,m}} = \frac{\delta p_m}{\rho_m U_{\perp,m} \delta U_{\perp,m}} \frac{U_{\perp,m}^2}{C_s^2} \sim Eu\, Ma^2\end{aligned}$$

Therefore, the dimensionless continuity equation is

$$Eu\,Ma^2\left(Sr\,\frac{\partial\hat{\rho}}{\partial\hat{t}}+\hat{\mathbf{u}}\cdot\hat{\nabla}\hat{\rho}\right)+\hat{\rho}\hat{\nabla}\cdot\hat{\mathbf{u}}=0, \tag{2.27}$$

We will later discuss the various limits of the dimensionless numbers and see how this will lead to a reduced equation of continuity.

The dimensionless momentum equation

Due to the anisotropy of the horizontal and vertical length scales, the momentum equation can be split up into one horizontal component and one vertical component as

$$\rho\left(\frac{d\mathbf{u}}{dt}\right)_\perp = -\nabla_\perp p+(\nabla\cdot\boldsymbol{\sigma}')_\perp-2\rho\,(\boldsymbol{\Omega}\times\mathbf{u})_\perp\,, \tag{2.28}$$

$$\rho\left(\frac{d\mathbf{u}}{dt}\right)_\parallel = -\nabla_\parallel p+(\nabla\cdot\boldsymbol{\sigma}')_\parallel-2\rho\,(\boldsymbol{\Omega}\times\mathbf{u})_\parallel+\rho\mathbf{g}, \tag{2.29}$$

where the acceleration terms are given by

$$\left(\frac{d\mathbf{u}}{dt}\right)_\perp = \left(\left.\frac{d\mathbf{u}_\perp}{dt}\right|_{\mathbf{e}_i}\right)+\frac{w}{r}\mathbf{u}_\perp+\frac{u}{r}\tan\theta\,\widehat{\mathbf{r}}\times\mathbf{u}_\perp, \tag{2.30}$$

$$\left(\frac{d\mathbf{u}}{dt}\right)_\parallel = \left(\left.\frac{d\mathbf{u}_\parallel}{dt}\right|_{\mathbf{e}_i}\right)-\frac{\mathbf{u}_\perp}{r}\widehat{\mathbf{r}}. \tag{2.31}$$

Note that the vertical line symbolizes that the unit vectors remain constant during the differentiation. The horizontal and vertical part of the Coriolis force are

$$(2\boldsymbol{\Omega}\times\mathbf{u})_\perp = l\widehat{\boldsymbol{\theta}}\times\mathbf{u}_\parallel+f\widehat{\mathbf{r}}\times\mathbf{u}_\perp, \tag{2.32}$$

$$(2\boldsymbol{\Omega}\times\mathbf{u})_\parallel = l\widehat{\boldsymbol{\theta}}\times\mathbf{u}_\perp, \tag{2.33}$$

where, $f=2\,|\boldsymbol{\Omega}|\,\sin\theta$ and $l=2\,|\boldsymbol{\Omega}|\,\cos\theta$. Let us first look at the scaling for the horizontal part of the momentum equation. This part can be written as

$$\begin{aligned}
&\left|\rho\frac{\partial\mathbf{u}_\perp}{\partial t}\right|_m\hat{\rho}\left.\frac{\partial\hat{\mathbf{u}}_\perp}{\partial\hat{t}}\right|_{\mathbf{e}_i}+|\rho\mathbf{u}_\perp\cdot\nabla_\perp\mathbf{u}_\perp|_m\,\hat{\rho}\,\hat{\mathbf{u}}\cdot\hat{\nabla}\hat{\mathbf{u}}_\perp\Big|_{\mathbf{e}_i}\\
&+\left|\rho\frac{w}{r}\mathbf{u}_\perp\right|_m\hat{\rho}\frac{\hat{w}}{\hat{r}}\hat{\mathbf{u}}_\perp+\left|\rho\frac{u}{r}\widehat{\mathbf{r}}\times\mathbf{u}_\perp\right|_m\hat{\rho}\frac{\hat{u}}{\hat{r}}\tan\theta\,\widehat{\mathbf{r}}\times\hat{\mathbf{u}}_\perp\\
&=-\left|\nabla_\perp p\right|_m\hat{\nabla}_\perp\hat{p}+|(\nabla\cdot\boldsymbol{\sigma}')_\perp|_m\left(\hat{\nabla}\cdot\hat{\boldsymbol{\sigma}}'\right)_\perp\\
&\quad-\left|\rho\widehat{\boldsymbol{\theta}}\times\mathbf{u}_\parallel\right|_m\hat{\rho}l\widehat{\boldsymbol{\theta}}\times\hat{\mathbf{u}}_\parallel-|\rho\widehat{\mathbf{r}}\times\mathbf{u}_\perp|_m\,\hat{\rho}f\widehat{\mathbf{r}}\times\hat{\mathbf{u}}_\perp.
\end{aligned} \tag{2.34}$$

Since there are many currents in the ocean that are determined by the balance between the Coriolis force and the pressure force, it would be natural to compare all the terms in the momentum equation with the inertia force. According to this the momentum equation can be written in dimensionless form as

$$
\begin{aligned}
&\frac{\left|\rho\frac{\partial \mathbf{u}_\perp}{\partial t}\right|_m}{|\rho\mathbf{u}_\perp\cdot\nabla_\perp\mathbf{u}_\perp|_m}\hat{\rho}\left.\frac{\partial\hat{\mathbf{u}}_\perp}{\partial\hat{t}}\right|_{\mathbf{e}_i}+\hat{\rho}\,\hat{\mathbf{u}}\cdot\hat{\nabla}\hat{\mathbf{u}}_\perp\Big|_{\mathbf{e}_i}\\
&\quad+\frac{\left|\rho\frac{w}{r}\mathbf{u}_\perp\right|_m}{|\rho\mathbf{u}_\perp\cdot\nabla_\perp\mathbf{u}_\perp|_m}\hat{\rho}\frac{\hat{w}}{\hat{r}}\hat{\mathbf{u}}_\perp+\frac{\left|\rho\frac{u}{r}\widehat{\mathbf{r}}\times\mathbf{u}_\perp\right|_m}{|\rho\mathbf{u}_\perp\cdot\nabla_\perp\mathbf{u}_\perp|_m}\hat{\rho}\frac{\hat{u}}{\hat{r}}\tan\theta\,\widehat{\mathbf{r}}\times\hat{\mathbf{u}}_\perp\\
&\quad=-\frac{|\nabla_\perp p|_m}{|\rho\mathbf{u}_\perp\cdot\nabla_\perp\mathbf{u}_\perp|_m}\hat{\nabla}_\perp\hat{p}+\frac{|(\nabla\cdot\boldsymbol{\sigma}')_\perp|_m}{|\rho\mathbf{u}_\perp\cdot\nabla_\perp\mathbf{u}_\perp|_m}\left(\hat{\nabla}\cdot\hat{\boldsymbol{\sigma}}'\right)_\perp\\
&\qquad-\frac{\left|\rho\widehat{\boldsymbol{\theta}}\times\mathbf{u}_\parallel\right|_m}{|\rho\mathbf{u}_\perp\cdot\nabla_\perp\mathbf{u}_\perp|_m}\hat{\rho}l\widehat{\boldsymbol{\theta}}\times\hat{\mathbf{u}}_\parallel-\frac{|\rho\widehat{\mathbf{r}}\times\mathbf{u}_\perp|_m}{|\rho\mathbf{u}_\perp\cdot\nabla_\perp\mathbf{u}_\perp|_m}\hat{\rho}f\widehat{\mathbf{r}}\times\hat{\mathbf{u}}_\perp.
\end{aligned}
\tag{2.35}
$$

Hence, by using the definitions of the dimensionless numbers, the horizontal momentum equation can be written as

$$
\begin{aligned}
&\hat{\rho}\left(Sr\left.\frac{\partial\hat{\mathbf{u}}_\perp}{\partial\hat{t}}\right|_{\mathbf{e}_i}+\hat{\mathbf{u}}\cdot\hat{\nabla}\hat{\mathbf{u}}_\perp\Big|_{\mathbf{e}_i}+\gamma\Gamma\frac{\hat{w}}{\hat{r}}\hat{\mathbf{u}}_\perp+\Gamma\frac{\hat{u}}{\hat{r}}\tan\theta\,\widehat{\mathbf{r}}\times\hat{\mathbf{u}}_\perp\right)\\
&\quad=-Eu\,\hat{\nabla}_\perp\hat{p}+\frac{1}{Re}\left(\hat{\nabla}\cdot\hat{\boldsymbol{\sigma}}'\right)_\perp-\frac{\gamma}{Ro}\cos\theta\hat{\rho}\widehat{\boldsymbol{\theta}}\times\hat{\mathbf{u}}_\parallel-\frac{1}{Ro}\sin\theta\hat{\rho}\widehat{\mathbf{r}}\times\hat{\mathbf{u}}_\perp.
\end{aligned}
\tag{2.36}
$$

The vertical momentum equation can be written as

$$
\begin{aligned}
&\left|\rho\frac{\partial\mathbf{u}_\parallel}{\partial t}\right|_m\hat{\rho}\left.\frac{\partial\hat{\mathbf{u}}_\parallel}{\partial\hat{t}}\right|_{\mathbf{e}_i}+\left|\rho\mathbf{u}_\perp\cdot\nabla_\perp\mathbf{u}_\parallel\right|_m\hat{\rho}\,\hat{\mathbf{u}}\cdot\hat{\nabla}\hat{\mathbf{u}}_\parallel\Big|_{\mathbf{e}_i}-\left|\rho\frac{\mathbf{u}_\perp^2}{r}\right|_m\hat{\rho}\frac{\hat{\mathbf{u}}_\perp^2}{\hat{r}}\widehat{\mathbf{r}}\\
&\quad=-\left|\nabla_\parallel p\right|_m\hat{\nabla}_\parallel\hat{p}+\left|(\nabla\cdot\boldsymbol{\sigma}')_\parallel\right|_m\left(\hat{\nabla}\cdot\hat{\boldsymbol{\sigma}}'\right)_\parallel\\
&\qquad-\left|\rho\widehat{\boldsymbol{\theta}}\times\mathbf{u}_\perp\right|_m\hat{\rho}l\widehat{\boldsymbol{\theta}}\times\hat{\mathbf{u}}_\perp+|\rho\mathbf{g}|_m\,\hat{\rho}\hat{\mathbf{g}}.
\end{aligned}
\tag{2.37}
$$

The dimensionless vertical momentum equation reads

$$
\begin{aligned}
&\frac{\left|\rho\frac{\partial\mathbf{u}_\parallel}{\partial t}\right|_m}{\left|\rho\mathbf{u}_\perp\cdot\nabla_\perp\mathbf{u}_\parallel\right|_m}\hat{\rho}\left.\frac{\partial\hat{\mathbf{u}}_\parallel}{\partial\hat{t}}\right|_{\mathbf{e}_i}+\hat{\rho}\,\hat{\mathbf{u}}\cdot\hat{\nabla}\hat{\mathbf{u}}_\parallel\Big|_{\mathbf{e}_i}-\frac{\left|\rho\frac{\mathbf{u}_\perp^2}{r}\right|_m}{\left|\rho\mathbf{u}_\perp\cdot\nabla_\perp\mathbf{u}_\parallel\right|_m}\hat{\rho}\frac{\hat{\mathbf{u}}_\perp^2}{\hat{r}}\widehat{\mathbf{r}}\\
&\quad=-\frac{\left|\nabla_\parallel p\right|_m}{\left|\rho\mathbf{u}_\perp\cdot\nabla_\perp\mathbf{u}_\parallel\right|_m}\hat{\nabla}_\parallel\hat{p}+\frac{\left|(\nabla\cdot\boldsymbol{\sigma}')_\parallel\right|_m}{\left|\rho\mathbf{u}_\perp\cdot\nabla_\perp\mathbf{u}_\parallel\right|_m}\left(\hat{\nabla}\cdot\hat{\boldsymbol{\sigma}}'\right)_\parallel\\
&\qquad-\frac{\left|\rho\widehat{\boldsymbol{\theta}}\times\mathbf{u}_\perp\right|_m}{\left|\rho\mathbf{u}_\perp\cdot\nabla_\perp\mathbf{u}_\parallel\right|_m}\hat{\rho}l\widehat{\boldsymbol{\theta}}\times\hat{\mathbf{u}}_\perp+\frac{|\rho\mathbf{g}|_m}{\left|\rho\mathbf{u}_\perp\cdot\nabla_\perp\mathbf{u}_\parallel\right|_m}\hat{\rho}\hat{\mathbf{g}}.
\end{aligned}
\tag{2.38}
$$

By using the non-dimension numbers it becomes

$$\hat{\rho}\left(Sr\left.\frac{\partial\hat{\mathbf{u}}_{\parallel}}{\partial\hat{t}}\right|_{\mathbf{e}_i}+\hat{\mathbf{u}}\cdot\hat{\nabla}\hat{\mathbf{u}}_{\parallel}\Big|_{\mathbf{e}_i}-\frac{\Gamma}{\gamma}\frac{\hat{\mathbf{u}}_{\perp}^2}{\hat{r}}\widehat{\mathbf{r}}\right)$$
$$=-\frac{Eu}{\gamma^2}\hat{\nabla}_{\parallel}\hat{p}+\frac{1}{Re}\left(\hat{\nabla}\cdot\hat{\boldsymbol{\sigma}}'\right)_{\parallel}-\frac{1}{Ro}\cos\theta\widehat{\boldsymbol{\theta}}\times\hat{\mathbf{u}}_{\perp}+\frac{1}{Fr}\hat{\rho}\hat{\mathbf{g}}, \tag{2.39}$$

where the dimensionless numbers we have used are

$$Sr=\frac{\delta L_{m,\perp}}{U_{\perp}\delta t_m},\quad Eu=\frac{\delta p_m}{\rho_m U_{\perp,m}\delta U_{\perp,m}},\quad Re=\frac{\rho_m U_{\parallel,m}\delta L_{\parallel,m}}{\eta_m},$$
$$Ro=\frac{\delta U_{\perp,m}}{2\Omega\delta L_{\perp,m}},\quad Fr=\frac{U_{\parallel,m}\delta U_{\parallel,m}}{g\delta L_{\parallel,m}}.$$

In the limit where $\gamma=1$ the momentum equation can be written as

$$\hat{\rho}\left(St\frac{\partial\hat{\mathbf{u}}}{\partial\hat{t}}+\hat{\mathbf{u}}\cdot\hat{\nabla}\hat{\mathbf{u}}\right)=-Eu\hat{\nabla}\hat{p}+\frac{1}{Re}\hat{\nabla}\cdot\hat{\boldsymbol{\sigma}}'-\frac{1}{Ro}\hat{\rho}\hat{\boldsymbol{\Omega}}\times\hat{\mathbf{u}}+\frac{1}{Fr}\hat{\rho}\hat{\mathbf{g}},$$

where the dimensionless stress tensor is given by

$$\hat{\boldsymbol{\sigma}}'=\frac{\delta L_m}{\eta_m\delta U_m}\boldsymbol{\sigma}'=\hat{\eta}\left[\hat{\nabla}\hat{\mathbf{u}}+\left(\hat{\nabla}\hat{\mathbf{u}}\right)^T-\frac{2}{3}\left(\hat{\nabla}\cdot\hat{\mathbf{u}}\right)\mathsf{I}\right]. \tag{2.40}$$

We will later discuss the various limits of the dimensionless numbers and see how this will lead to a reduced equation of momentum.

The dimensionless temperature equation

The temperature equation can be written as

$$\left|\rho\, c_p\frac{\partial\, T}{\partial\, t}\right|_m\hat{\rho}\hat{c}_p\frac{\partial\hat{T}}{\partial\hat{t}}+\left|\rho\, c_p\mathbf{u}_{\perp}\cdot\nabla_{\perp}T\right|_m\hat{\rho}\hat{c}_p\hat{\mathbf{u}}\cdot\hat{\nabla}\hat{T}$$
$$=\left|\beta_T T\frac{\partial\, p}{\partial\, t}\right|_m\hat{\beta}_T\hat{T}\frac{\partial\hat{p}}{\partial\hat{t}}+\left|\beta_T T\,\mathbf{u}_{\perp}\cdot\nabla_{\perp}p\right|_m\hat{\beta}_T\hat{T}\hat{\mathbf{u}}\cdot\hat{\nabla}\hat{p}$$
$$+\left|\boldsymbol{\sigma}':\mathsf{D}\right|_m\hat{\boldsymbol{\sigma}}':\hat{\mathsf{D}}-\left|\nabla\cdot\mathbf{q}\right|_m\hat{\nabla}\cdot\hat{\mathbf{q}}-\left|\mathbf{J}_S\cdot\nabla\left(\Delta h\right)\right|_m\hat{\mathbf{J}}_S\cdot\hat{\nabla}\left(\Delta\hat{h}\right). \tag{2.41}$$

It will be natural to compare the temperature equation with the advection term. According to this, the temperature equation can be written as

$$\begin{aligned} &\frac{\left|\rho\, c_p \frac{\partial T}{\partial t}\right|_m}{\left|\rho\, c_p \mathbf{u}_\perp \cdot \nabla_\perp T\right|_m} \hat{\rho}\hat{c}_p \frac{\partial \hat{T}}{\partial \hat{t}} + \hat{\rho}\hat{c}_p \hat{\mathbf{u}} \cdot \hat{\nabla}\hat{T} \\ &= \frac{\left|\beta_T T \frac{\partial p}{\partial t}\right|_m}{\left|\rho\, c_p \mathbf{u}_\perp \cdot \nabla_\perp T\right|_m} \hat{\beta}_T \hat{T} \frac{\partial \hat{p}}{\partial \hat{t}} + \frac{\left|\beta_T T\, \mathbf{u}_\perp \cdot \nabla_\perp p\right|_m}{\left|\rho\, c_p \mathbf{u}_\perp \cdot \nabla_\perp T\right|_m} \hat{\beta}_T \hat{T} \hat{\mathbf{u}} \cdot \hat{\nabla}\hat{p} \\ &\quad + \frac{\left|\boldsymbol{\sigma'} : \mathsf{D}\right|_m}{\left|\rho\, c_p \mathbf{u}_\perp \cdot \nabla_\perp T\right|_m} \hat{\boldsymbol{\sigma}}' : \hat{\mathsf{D}} - \frac{\left|\nabla \cdot \mathbf{q}\right|_m}{\left|\rho\, c_p \mathbf{u}_\perp \cdot \nabla_\perp T\right|_m} \hat{\nabla} \cdot \hat{\mathbf{q}} \\ &\quad - \frac{\left|\mathbf{J}_S \cdot \nabla\left(\Delta h\right)\right|_m}{\left|\rho\, c_p \mathbf{u}_\perp \cdot \nabla_\perp T\right|_m} \hat{\mathbf{J}}_S \cdot \hat{\nabla}\left(\Delta \hat{h}\right). \end{aligned} \tag{2.42}$$

Hence, by using the definitions of the dimensionless numbers, the temperature equation can be written as

$$\begin{aligned} &\hat{\rho}\hat{c}_p \left(Sr \frac{\partial \hat{T}}{\partial \hat{t}} + \hat{\mathbf{u}} \cdot \hat{\nabla}\hat{T} \right) - \beta_{T,m} T_m Eu\, Ec\, \hat{\beta}_T \hat{T} \left(Sr \frac{\partial \hat{p}}{\partial \hat{t}} + \hat{\mathbf{u}} \cdot \hat{\nabla}\hat{p} \right) \\ &= \frac{Ec}{Re} \hat{\boldsymbol{\sigma}}' : \hat{\mathsf{D}} - \frac{1}{Pe} \hat{\nabla} \cdot \hat{\mathbf{q}} - \frac{\left|\mathbf{J}_S \cdot \nabla\left(\Delta\, h\right)\right|_m}{\left|\rho\, c_p \mathbf{u}_\perp \cdot \nabla_\perp T\right|_m} \hat{\mathbf{J}}_S \cdot \hat{\nabla}\left(\Delta \hat{h}\right), \end{aligned} \tag{2.43}$$

where the new dimensionless numbers are

$$Ec = \frac{U_{\perp,m}^2}{c_{p,m}\delta T_m}, \quad Pe_T = \frac{\rho_m c_{p,m} U_{\parallel,m} \delta L_{\parallel,m}}{\kappa_m}$$

In the limit where the horizontal and the vertical length scales are equal, we can find explicit expressions for the dimensionless molecular fluxes. For example the conductive heat flux can be written as

$$\mathbf{q} = -\left|\kappa \nabla T\right|_m \hat{\kappa} \hat{\nabla}\hat{T},$$

where the normalized thermal conductivity is defined by

$$\hat{\kappa} = \frac{\kappa}{\kappa_m}. \tag{2.44}$$

Thus, the normalized heat flux is

$$\hat{\mathbf{q}} = \frac{\delta L_m}{\kappa_m \delta T_m} \mathbf{q} = -\hat{\kappa} \hat{\nabla}\hat{T}. \tag{2.45}$$

The deformation tensor can be written as

$$\mathsf{D} = \frac{1}{2} \left[\left|\nabla \mathbf{u}\right|_m \hat{\nabla}\hat{\mathbf{u}} + \left|(\nabla \mathbf{u})^T\right|_m \left(\hat{\nabla}\hat{\mathbf{u}}\right)^T \right],$$

such that the normalized deformation tensor is

$$\hat{\mathsf{D}} = \frac{\delta L_m}{\delta U_m} \mathsf{D} = \frac{1}{2} \left[\hat{\mathbf{u}} + \left(\hat{\nabla}\hat{\mathbf{u}}\right)^T \right]. \tag{2.46}$$

The dimensionless salinity equation

The salinity equation can be written as

$$\left|\rho\frac{\partial\, S}{\partial\, t}\right|_m \hat{\rho}\frac{\partial\, \hat{S}}{\partial\, \hat{t}} + \left|\rho\mathbf{u}_\perp\cdot\nabla_\perp S\right|_m \hat{\rho}\hat{\mathbf{u}}\cdot\hat{\nabla}\,\hat{S} = -\left|\nabla\cdot\mathbf{J}_S\right|_m \hat{\nabla}\cdot\hat{\mathbf{J}}_S. \tag{2.47}$$

It will be natural to compare all the terms by the advection term. According to this, the salinity equation on dimensionless form is

$$\frac{\left|\rho\frac{\partial S}{\partial t}\right|_m}{\left|\rho\mathbf{u}_\perp\cdot\nabla_\perp S\right|_m}\hat{\rho}\frac{\partial\, \hat{S}}{\partial\, \hat{t}} + \hat{\rho}\hat{\mathbf{u}}\cdot\hat{\nabla}\,\hat{S} = -\frac{\left|\nabla\cdot(\kappa_S\nabla\, S)\right|_m}{\left|\rho\mathbf{u}_\perp\cdot\nabla_\perp S\right|_m}\hat{\nabla}\cdot\hat{\mathbf{J}}_S. \tag{2.48}$$

Hence, by using the definitions of the dimensionless numbers, the dimensionless salinity equation reads

$$\hat{\rho}\left(Sr\,\frac{\partial\, \hat{S}}{\partial\, \hat{t}} + \hat{\mathbf{u}}\cdot\hat{\nabla}\,\hat{S}\right) = -\frac{1}{Pe_S}\hat{\nabla}\cdot\hat{\mathbf{J}}_S. \tag{2.49}$$

where the new dimensionless number is

$$Pe_S = \frac{\rho_m U_{\parallel,m}\delta L_{\parallel,m}}{\kappa_S}. \tag{2.50}$$

2.2.2 Asymptotic reductions of the equations

The dimensionless equations from the last section is

$$Eu\,Ma^2\left(Sr\,\frac{\partial\hat{\rho}}{\partial\hat{t}} + \hat{\mathbf{u}}\cdot\hat{\nabla}\hat{\rho}\right) = -\hat{\rho}\hat{\nabla}\cdot\hat{\mathbf{u}}, \tag{2.51}$$

$$\begin{aligned}\hat{\rho}\left(St\frac{\partial\hat{\mathbf{u}}}{\partial\,\hat{t}} + \hat{\mathbf{u}}\cdot\hat{\nabla}\hat{\mathbf{u}}\right) &= -Eu\hat{\nabla}\,\hat{p} + \frac{1}{Re}\hat{\nabla}\cdot\hat{\boldsymbol{\sigma}'} - \frac{1}{Ro}\hat{\rho}\hat{\boldsymbol{\Omega}}\times\hat{\mathbf{u}} \\ &\quad - \frac{1}{Ce}\hat{\rho}\hat{\boldsymbol{\Omega}}\times\left(\hat{\boldsymbol{\Omega}}\times\hat{\mathbf{r}}\right) + \frac{1}{Fr}\hat{\rho}\hat{\mathbf{g}},\end{aligned} \tag{2.52}$$

$$\begin{aligned}\hat{\rho}\hat{c}_p\left(Sr\frac{\partial\hat{T}}{\partial\hat{t}} + \hat{\mathbf{u}}\cdot\hat{\nabla}\hat{T}\right) &= \beta_{T,m}T_m Eu\,Ec\,\hat{\beta}_T\hat{T}\left(Sr\frac{\partial\hat{p}}{\partial\hat{t}} + \hat{\mathbf{u}}\cdot\hat{\nabla}\hat{p}\right) \\ &\quad + \frac{Ec}{Re}\hat{\boldsymbol{\sigma}'}:\hat{\mathsf{D}} - \frac{1}{Pe}\hat{\nabla}\cdot\hat{\mathbf{q}} \\ &\quad - \frac{\left|\mathbf{J}_S\cdot\nabla\left(\Delta\, h\right)\right|_m}{\left|\rho\, c_p\mathbf{u}\cdot\nabla\, T\right|_m}\hat{\mathbf{J}}_S\cdot\hat{\nabla}\left(\Delta\hat{h}\right),\end{aligned} \tag{2.53}$$

$$\hat{\rho}\left(Sr\,\frac{\partial\,\hat{S}}{\partial\,\hat{t}} + \hat{\mathbf{u}}\cdot\hat{\nabla}\,\hat{S}\right) = -\frac{1}{Pe_S}\hat{\nabla}\cdot\left(\hat{\kappa}_S\hat{\nabla}\,\hat{S}\right) - \frac{1}{Pe_{Sp}}\hat{\nabla}\cdot\left(\hat{\kappa}_{Sp}\hat{\nabla}\,\hat{p}\right). \tag{2.54}$$

These dimensionless equations show that the dynamical behavior of the fluid is determined by the dimensionless numbers. When some of these numbers are very small or large the equations can be reduced to a simplified model, without changing the basic physics. In all the problems we will be discussing there is no external forcing. Therefore, it is natural to assume that the temporal scale, scales as $\delta L_m/U_m$ which correspond to a Strouhal number of order unity, i.e. $Sr = \mathcal{O}(1)$. There are basically two distinct limits that are interesting to study, one limit where the compressible effects are important, i.e., $EuMa^2 = \mathcal{O}(1)$ and one limit where the fluid may be considered incompressible, i.e., $EuMa^2 \ll \mathcal{O}(1)$. When compressible effects are important the pressure must scale in such a way that the pressure variations gives an Euler number that compensates for the size of the Mach number. If the Mach number is small then the Euler number must be large. This is the typical scaling of acoustical dynamics. When compressible effects are not important the pressure must scale in such a way that the pressure act as a reaction force, i.e., that the pressure gradient is mainly balanced by some other forces, for example the Coriolus force or the viscous force.

Incompressible flow

In the case that the Mach number Ma tends to zero and $EuMa^2 \ll \mathcal{O}(1)$ the equations reduce to the equations for incompressible flows,

$$\begin{aligned} \hat{\nabla} \cdot \hat{\mathbf{u}} &= 0, && (2.55) \\ \hat{\rho}\left(\frac{\partial \hat{\mathbf{u}}}{\partial \hat{t}} + \hat{\mathbf{u}} \cdot \hat{\nabla} \hat{\mathbf{u}}\right) &= -Eu \hat{\nabla} \hat{p} + \frac{1}{Re} \hat{\nabla} \cdot \hat{\boldsymbol{\sigma}}' - \frac{1}{Ro} \hat{\rho} \hat{\boldsymbol{\Omega}} \times \hat{\mathbf{u}} \\ & \quad - \frac{1}{Ce} \hat{\rho} \hat{\boldsymbol{\Omega}} \times \left(\hat{\boldsymbol{\Omega}} \times \hat{\mathbf{r}}\right) + \frac{1}{Fr} \hat{\rho} \hat{\mathbf{g}}. && (2.56) \end{aligned}$$

If the Reynolds number tends to zero and the Euler number Eu is of $\mathcal{O}(1/Re)$ the momentum equation reduces to a balance equation between pressure and viscosity. If the Reynolds number tends to infinity and the Euler number Eu is of $\mathcal{O}(1/Ro)$, the momentum equation reduces to

$$0 = -Eu \hat{\nabla} \hat{p} - \frac{1}{Ro} \hat{\rho} \hat{\boldsymbol{\Omega}} \times \hat{\mathbf{u}} + \frac{1}{Fr} \hat{\rho} \hat{\mathbf{g}}, \tag{2.57}$$

where we have used that the centrifugal force is small compared with the other forces. Note that these equations decouple from the thermodynamical equations if the mass density and the viscosity are independent of the thermodynamical variables.

Parameter	**Symbol**	**Value**
Thermal expansion coefficient	$\beta_{T,m}$	$1.0 \times 10^{-4}\,\mathrm{K}^{-1}$
Compresibility coefficient	$\beta_{p,m}$	$4.1 \times 10^{-10}\,\mathrm{Pa}^{-1}$
Salinity contraction coefficient	$\beta_{S,m}$	$7.6 \times 10^{-4}\,\mathrm{ppt}^{-1}$
Refrence mass density	ρ_m	$1.0 \times 10^{3}\,\mathrm{kg\,m}^{-3}$
Refrence temperature	T_m	$279\,\mathrm{K}$
Heat capacity	$c_{p,m}$	$4.2 \times 10^{3}\mathrm{Jkg}^{-1}\mathrm{K}^{-1}$
Speed of sound	$C_{s,m}$	$1.5 \times 10^{3}\mathrm{m\,s}^{-1}$
Kinematic viscosity	$\nu_m = \frac{\eta_m}{\rho_m}$	$1.3 \times 10^{-6}\mathrm{m}^2\mathrm{s}^{-1}$
Salt diffusion coefficient	D_m	$1.2 \times 10^{-9}\mathrm{m}^2\mathrm{s}^{-1}$
Angular velocity of the earth	Ω_m	$7.3 \times 10^{-5}\mathrm{rad\,s}^{-1}$
Gravity of the earth	g_m	$9.8 \times 10^{0}\mathrm{m\,s}^{-2}$
Mean radius of the earth	r_m	$6371 \times 10^{3}\mathrm{m}$
Molecular heat diffusion	κ_T	$1 \times 10^{-7}\mathrm{m}^2/\mathrm{s}$
Molecular salt diffusion	κ_S	$1 \times 10^{-9}\mathrm{m}^2/\mathrm{s}$

Table 2.2: Characteristic values for fluid parameters and transport coefficients for large-scale circulation.

2.2.3 Typical values for the ocean circulation dynamics

In this section we will estimate the typical values of the dimensionless numbers for large-scale flows discussed in section 2.2. We will see that the molecular transport of momentum, heat and salt can be neglected compared to advection. We will also see that the Reynolds number is much greater than the typical critical value for the transition between laminar flow and turbulence. Therefore, geophysical fluid motion is generally highly turbulent. For large-scale ocean circulation, the typical velocity is $U_{\perp,m} = \delta U_{\perp,m} = 5 \times 10^{-2}$ m/s, the typical length-scale is $L_{\perp,m} = \delta L_{\perp,m} = 10^5$ m and $L_{\|,m} = \delta L_{\|,m} = 10^3$ m and the typical temporal-scale is $t_m = \delta t_m = L_{\perp,m}/U_{\perp,m} = 2 \times 10^6$s. According to equation (2.21), the typical vertical velocity $U_{\|,m}$ will be of $\mathcal{O}(\gamma\, U_{\perp,m})$. For these given values, the order of magnitude of the dimensionless numbers relative to the Rossby number are

$$
\begin{aligned}
&Sr \sim \mathcal{O}(1), \qquad Ma \sim \mathcal{O}(Ro^2), \quad \frac{1}{Re} \sim \mathcal{O}(Ro^2), \quad Fr \sim \mathcal{O}(Ro^3), \\
&\frac{1}{Pe_T} \sim \mathcal{O}(Ro^5), \qquad \frac{1}{Pe_S} \sim \mathcal{O}(Ro^4), \qquad \gamma \sim \mathcal{O}(Ro), \qquad \Gamma \sim \mathcal{O}(Ro), \\
&Ec \sim \mathcal{O}(Ro^3) \quad (\beta_{T,m} T_m) \sim \mathcal{O}(Ro)
\end{aligned}
$$

where the Rossby number is $Ro \sim 10^{-3}$. We will show later that the Euler number will be of $\mathcal{O}(Ro^{-1})$, such that the horizontal pressure force will be of the same order as the Coriolis force. The minimum value of the Reynolds number is $Re = 10^7$, which is far above the critical value for the transition between laminar flow and turbulent flow. Therefore, the ocean flow is generally highly turbulent and the molecular viscous effects can be neglected in comparison with the other terms. The same applies for the molecular transport of heat and salinity. In the next section we will derive equations for large-scale motion.

2.3 Averaged equations for large-scale motions

In geophysical fluid dynamics we are particularly interested in the motion which occurs on large spatial and slow temporal scales. The equations we presented the in the previous section are valid for the motion at all scales. Thus, we must performing an averaging of the equations of motion that remove the fast turbulent fluctuations, but retains the variations in the mean large-scale variations.

2.3.1 Hesselberg averaging

Assume that any field variable G can be decomposed into one large-scale mean-field part, denoted with angular brackets, $\langle G \rangle$, and one small-scale fluctuating part, denoted with a tilde, $\widetilde{G}$. Since the fluid is generally compressible it will be appropriate to define the average as a mass-weighted average [9, p.70]

$$
\langle G \rangle \equiv \frac{\overline{\rho\, G}}{\overline{\rho}}, \tag{2.58}
$$

where

$$
\overline{G} = \frac{1}{T} \int_{t-T/2}^{t+T/2} G\left(\mathbf{x}, t'\right) \, \mathrm{d}t', \quad T_1 \ll T \ll T_2 \tag{2.59}
$$

is the time-average and T_1 is the time scale for the turbulent fluctuations and T_2 is the time scale for the mean-field variations. It should be pointed out that (2.59), is not valid if there is not a distinct separation of time scales between the small-scale fluctuations and the mean-field variations, i.e., $T_1 \ll T \ll T_2$. From (2.58) and (2.59) it can be shown that the following rules are valid

$$
\begin{array}{ccc}
\left\langle \rho\widetilde{G} \right\rangle \equiv 0, & \overline{\overline{G}} = \overline{G}, & \overline{\langle G \rangle} = \langle G \rangle \,, \\
\overline{\langle G \rangle \langle G \rangle} = \langle G \rangle \langle G \rangle \,, & \langle \langle G \rangle \rangle = \langle G \rangle \,, & \langle \langle G \rangle F \rangle = \langle G \rangle \langle F \rangle \,, \\
\left\langle \overline{G} \right\rangle = \overline{G}, & \left\langle \overline{G} F \right\rangle = \overline{G} \langle F \rangle \,, & \langle G + F \rangle = \langle G \rangle + \langle F \rangle \,, \\
\overline{G + F} = \overline{G} + \overline{F}, & \overline{\nabla G} = \nabla \overline{G}, & \dfrac{\overline{\partial G}}{\partial t} = \dfrac{\partial \overline{G}}{\partial t}.
\end{array}
$$

If we now apply the mass-weighted average (2.58), the field variabels $(\mathbf{u}, \rho, p, \theta, S)$ can be decomposed as

$$
\mathbf{u} = \langle \mathbf{u} \rangle + \widetilde{\mathbf{u}}, \quad \rho = \langle \rho \rangle + \widetilde{\rho}, \quad p = \langle p \rangle + \widetilde{p}, \quad \theta = \langle \theta \rangle + \widetilde{\theta}, \quad , S = \langle S \rangle + \widetilde{S}, \tag{2.60}
$$

where the mean-field part is given by

$$
\langle \mathbf{u} \rangle = \frac{\overline{\rho \mathbf{u}}}{\overline{\rho}}, \quad \langle \rho \rangle = \overline{\rho}, \quad \langle p \rangle = \frac{\overline{\rho\, p}}{\overline{\rho}}, \quad \langle \theta \rangle = \frac{\overline{\rho \theta}}{\overline{\rho}}, \quad \langle S \rangle = \frac{\overline{\rho\, S}}{\overline{\rho}}. \tag{2.61}
$$

Note that the mass-weighted average of the mass density is equal to the time average of the mass density. Therefore, it follows that the time average of the fluctuating mass density is equal to zero, i.e. $\overline{\widetilde{\rho}} = 0$. This is not the case for the other fluctuating quantities. By substituting the decompositions into the definitions of the mass-weighted average, (2.58), it follows that

$$
\begin{aligned}
\langle G \rangle &= \frac{\overline{\left(\langle \rho \rangle + \widetilde{\rho}\right) \left(\langle G \rangle + \widetilde{G} \right)}}{\overline{\rho}} \\
&= \frac{\overline{\rho} \langle G \rangle + \overline{\widetilde{\rho}} \langle G \rangle + \overline{\rho}\overline{\widetilde{G}} + \overline{\widetilde{\rho}\widetilde{G}}}{\overline{\rho}} \\
&= \langle G \rangle + \overline{\widetilde{G}} + \frac{\overline{\widetilde{\rho}\widetilde{G}}}{\overline{\rho}},
\end{aligned}
$$

which is equivalent to

$$
\overline{\widetilde{G}} = -\frac{\overline{\widetilde{\rho}\widetilde{G}}}{\overline{\rho}}. \tag{2.62}
$$

Hence it follows that

$$
\overline{\widetilde{\mathbf{u}}} = -\frac{\overline{\widetilde{\rho}\widetilde{\mathbf{u}}}}{\overline{\rho}}, \quad \overline{\widetilde{\rho}} = 0, \quad \overline{\widetilde{p}} = -\frac{\overline{\widetilde{\rho}\widetilde{p}}}{\overline{\rho}}, \quad \overline{\widetilde{\theta}} = -\frac{\overline{\widetilde{\rho}\widetilde{\theta}}}{\overline{\rho}}, \quad \overline{\widetilde{S}} = -\frac{\overline{\widetilde{\rho}\widetilde{S}}}{\overline{\rho}}. \tag{2.63}
$$

It should be noted that in the limit where the fluid is incompressible the mass density is approximately constant. Hence the mass-weighted average will reduce to an time average, i.e., $\langle G\rangle = \overline{G}$, which is the same as the Reynolds average. By using equation (2.62), it follows that the time average of the product of the mass density and a dynamic variable G is

$$\begin{aligned}\overline{\rho\, G} &= \overline{\left(\langle\rho\rangle + \widetilde{\rho}\right)\left(\langle G\rangle + \widetilde{G}\right)} \\ &= \langle\rho\rangle\langle G\rangle + \langle\rho\rangle\,\overline{\widetilde{G}} + \overline{\widetilde{\rho}\widetilde{G}} \\ &= \langle\rho\rangle\langle G\rangle, \end{aligned} \tag{2.64}$$

and the product between the mass density and two dynamical variabels G and F is

$$\begin{aligned}\overline{\rho\, G\, F} &= \overline{\left(\langle\rho\rangle + \widetilde{\rho}\right)\left(\langle G\rangle + \widetilde{G}\right)\left(\langle F\rangle + \widetilde{F}\right)} \\ &= \langle\rho\rangle\langle G\rangle\langle F\rangle + \langle\rho\rangle\,\overline{\widetilde{G}\widetilde{F}} + \overline{\widetilde{\rho}\widetilde{G}\widetilde{F}}, \end{aligned} \tag{2.65}$$

For our purposes, the triple correlation can be neglected since the fluctuations in the mass density is always very small compared to the tubulent fluctuations.

2.3.2 Averaged equations

According to equation (2.58), it will be advantageous to write all of the equations in conservative form as

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0, \tag{2.66}$$

$$\frac{\partial\left(\rho\, S\right)}{\partial t} + \nabla\cdot\left(\rho\, S\mathbf{u}\right) = -\nabla\cdot\mathbf{J}_S, \tag{2.67}$$

$$\frac{\partial\left(\rho\mathbf{u}\right)}{\partial t} + \nabla\cdot\left(\rho\mathbf{u}\mathbf{u}\right) = -\nabla p + \nabla\cdot\boldsymbol{\sigma'} - 2\rho\boldsymbol{\Omega}\times\mathbf{u} + \mathbf{f}, \tag{2.68}$$

$$\begin{aligned} c_p\left[\frac{\partial\left(\rho\, T\right)}{\partial\, t} + \nabla\cdot\left(\rho\, T\mathbf{u}\right)\right] &= c_p\Gamma\left[\frac{\partial\left(\rho\, p\right)}{\partial\, t} + \nabla\cdot\left(\rho\, p\mathbf{u}\right)\right] \\ &\quad + \boldsymbol{\sigma'} : \mathsf{D} - \nabla\cdot\mathbf{q} - \mathbf{J}_S\cdot\nabla\left(\Delta h\right) \end{aligned} \tag{2.69}$$

By taking the time average of the equations above, and applying equations (2.64) and (2.65), we obtain equations for the large-scale motion

$$\frac{\partial \langle\rho\rangle}{\partial t} + \nabla\cdot\langle\rho\rangle\langle\mathbf{u}\rangle = 0, \tag{2.70}$$

$$\langle\rho\rangle\frac{\partial \langle S\rangle}{\partial t} + \langle\rho\rangle\langle\mathbf{u}\rangle\cdot\nabla\langle S\rangle = -\nabla\cdot\left(\overline{\mathbf{J}_S} + \mathbf{J}_S^{\mathrm{turb}}\right), \tag{2.71}$$

$$\langle\rho\rangle\frac{\partial \langle\mathbf{u}\rangle}{\partial t} + \langle\rho\rangle\langle\mathbf{u}\rangle\cdot\nabla\langle\mathbf{u}\rangle = -\nabla\langle p\rangle + \nabla\cdot\left(\overline{\boldsymbol{\sigma}'} + \boldsymbol{\sigma}^{\mathrm{turb}}\right) - 2\langle\rho\rangle\,\boldsymbol{\Omega}\times\langle\mathbf{u}\rangle + \overline{\mathbf{f}} \tag{2.72}$$

$$\begin{aligned}\langle\rho\rangle\, c_p\left(\frac{\partial\langle T\rangle}{\partial t} + \langle\mathbf{u}\rangle\cdot\nabla\langle T\rangle\right) &= \beta_T\langle T\rangle\left(\frac{\partial\langle p\rangle}{\partial t} + \langle\mathbf{u}\rangle\cdot\nabla\langle p\rangle\right) + \overline{\boldsymbol{\sigma}' : \mathsf{D}} \\ &\quad - \nabla\cdot\left(\overline{\mathbf{q}} + \mathbf{q}^{\mathrm{turb}}\right) - \overline{\mathbf{J}_S\cdot\nabla(\Delta h)},\end{aligned} \tag{2.73}$$

where we have used that c_p and Γ are slowly varying, such that they can be treated as constant. The turbulent fluxes are given by

$$\mathbf{J}_S^{\mathrm{turb}} = \langle\rho\rangle\,\overline{\widetilde{S}\widetilde{\mathbf{u}}}, \tag{2.74}$$

$$\boldsymbol{\sigma}^{\mathrm{turb}} = -\langle\rho\rangle\,\overline{\widetilde{\mathbf{u}}\widetilde{\mathbf{u}}}, \tag{2.75}$$

$$\mathbf{q}^{\mathrm{turb}} = c_p\langle\rho\rangle\,\overline{\widetilde{T}\widetilde{\mathbf{u}}}, \tag{2.76}$$

where the triple correlations and the turbulent heat flux due to turbulent pressure fluctuations are neglected. The question now is: How to perform an averaging of the equation of state? We will assume that the time-avarage of the fluctuating terms in the equation of state is very small compared to the contributions from the mean-field terms. Therefore, we will perform a Taylor series expansion around the mean-field variabels $(\langle p\rangle, \langle T\rangle, \langle S\rangle)$ and then perform an averaging. The result is

$$\langle\rho\rangle = \langle\rho\rangle(\langle p\rangle, \langle T\rangle, \langle S\rangle) + \mathcal{O}(\overline{\widetilde{p}}, \overline{\widetilde{T}}, \overline{\widetilde{S}}). \tag{2.77}$$

A consequence of averaging the equations of motion is the introduction of new transport term, which has the same structure as the molecular fluxes. These transport terms represent the turbulent transport of small-scale fluxes into the large-scale dynamics. Since we do not know the small-scale fluctuating variables, we get a closure problem for the average equations. In the next section we will try to close the system of the averaged equations by parameterize the turbulent fluxes in respect of the mean-field variables.

2.3.3 The turbulent mixing of momentum, heat and salt

The turbulent flow is characterized by rapid fluctuations which redistribute momentum, heat and salt. The fluctuations are assumed to be distributed randomly

Mixing coefficients	**Values from the deep ocean to the upper ocean**
$A_\perp$	$10 \quad - \quad 10^5\,\mathrm{m^2s^{-1}}$
$A_\parallel$	$10^{-5} \quad - \quad 10^{-1}\,\mathrm{m^2s^{-1}}$
$K_\perp$	$10 \quad - \quad 10^3\,\mathrm{m^2s^{-1}}$
$K_\parallel$	$10^{-5} \quad - \quad 10^{-5}\,\mathrm{m^2s^{-1}}$

Table 2.3: Typical values for the turbulent diffusivity.

and act as the molecular fluxes, but much more efficiently. Therefore, we assume that the turbulent fluxes can be written as

$$\mathbf{\Psi} = -K^{\mathrm{turb}}\nabla\psi, \tag{2.78}$$

where $\mathbf{\Psi}$ is the flux of the quantity ψ with turbulent diffusivity K^{turb}. Since the characteristic length scale in the horizontal and vertical directions are very different, we assume that the fluxes of momentum, heat and salt are of the form [6, p.57]

$$\begin{aligned}
\boldsymbol{\sigma}^{\mathrm{turb}} &= \rho_m A_\perp \left[\nabla_\perp \mathbf{u} + (\nabla_\perp \mathbf{u})^T\right] + \rho_m A_\parallel \left[\nabla_\parallel \mathbf{u} + \left(\nabla_\parallel \mathbf{u}\right)^T\right] && (2.79)\\
\mathbf{q}^{\mathrm{turb}} &= -\rho_m c_p \left(K_\perp \nabla_\perp T + K_\parallel \nabla_\parallel T\right) && (2.80)\\
\mathbf{J}_S^{\mathrm{turb}} &= -\rho_m \left(K_\perp \nabla_\perp S + K_\parallel \nabla_\parallel S\right) && (2.81)
\end{aligned}$$

where $A_\perp$ and $A_\parallel$ are the horizontal and vertical mixing coefficients of momentum, and $K_\perp$ and $K_\parallel$ are the corresponding mixing coefficients of heat and salt.

If one treats the mixing coefficients as constant, the turbulent mixing of momentum in spherical coordinates reads

$$\begin{aligned}
\left(\nabla \cdot \boldsymbol{\sigma}^{\mathrm{turb}}\right)_\perp &= A_\perp \left.\nabla_\perp^2 \mathbf{u}_\perp\right|_{\mathbf{e}_i} + A_\parallel \left.\nabla_\parallel^2 \mathbf{u}_\perp\right|_{\mathbf{e}_i} + A_\perp \nabla_\perp (\nabla \cdot \mathbf{u})\\
&+ \frac{A_\parallel - A_\perp}{r} (\widehat{\mathbf{r}} \cdot \nabla) \left.\mathbf{u}_\perp\right|_{\mathbf{e}_i} + \frac{2A_\perp \tan\theta}{r} \widehat{\mathbf{r}} \times \left(\widehat{\boldsymbol{\phi}} \cdot \nabla\right) \left.\mathbf{u}_\perp\right|_{\mathbf{e}_i}\\
&+ \frac{2A_\perp}{r} \nabla_\perp w - \frac{A_\perp}{r^2 \cos^2\theta} \mathbf{u}_\perp, && (2.82)\\
\left(\nabla \cdot \boldsymbol{\sigma}^{\mathrm{turb}}\right)_\parallel &= A_\perp \left.\nabla_\perp^2 \mathbf{u}_\parallel\right|_{\mathbf{e}_i} + A_\parallel \left.\nabla_\parallel^2 \mathbf{u}_\parallel\right|_{\mathbf{e}_i} + A_\perp \nabla_\parallel (\nabla \cdot \mathbf{u})\\
&+ \frac{A_\parallel - 3A_\perp}{r} (\nabla \cdot \mathbf{u}) \widehat{\mathbf{r}} + \frac{2A_\perp}{r^2} \mathbf{u}_\parallel\\
&- \frac{A_\parallel - 3A_\perp}{r} (\widehat{\mathbf{r}} \cdot \nabla) \mathbf{u}_\parallel, && (2.83)
\end{aligned}$$

and the turbulent mixing of heat and salt reads

$$\begin{aligned}
-\nabla \cdot \mathbf{q}^{\mathrm{turb}} &= \rho_m c_p \left(K_\perp \nabla_\perp^2 T + K_\parallel \nabla_\parallel^2 T\right), && (2.84)\\
-\nabla \cdot \mathbf{J}_S^{\mathrm{turb}} &= \rho_m \left(K_\perp \nabla_\perp^2 S + K_\parallel \nabla_\parallel^2 S\right). && (2.85)
\end{aligned}$$

2.3.4 The dimensionless equations

The dimensionless equations of motion are

$$Eu\,Ma^2 \widehat{\left(\frac{d}{dt}\right)} \hat{\rho} = -\hat{\rho}\hat{\nabla} \cdot \hat{\mathbf{u}}, \tag{2.86}$$

$$\hat{\rho} \widehat{\left(\frac{d\mathbf{u}}{dt}\right)}_{\perp} = -Eu\,\hat{\nabla}_{\perp}\hat{p} + \frac{1}{Re}\left(\hat{\nabla} \cdot \hat{\boldsymbol{\sigma}}'\right)_{\perp} + \hat{\mathbf{F}}^{\text{turb}}_{\perp} - \frac{\gamma}{Ro}\cos\theta\hat{\rho}\widehat{\boldsymbol{\theta}} \times \hat{\mathbf{u}}_{\parallel} - \frac{1}{Ro}\sin\theta\hat{\rho}\widehat{\mathbf{r}} \times \hat{\mathbf{u}}_{\perp} \tag{2.87}$$

$$\hat{\rho} \widehat{\left(\frac{d\mathbf{u}}{dt}\right)}_{\parallel} = -\frac{Eu}{\gamma^2}\hat{\nabla}_{\parallel}\hat{p} + \frac{1}{Re}\left(\hat{\nabla} \cdot \hat{\boldsymbol{\sigma}}'\right)_{\parallel} + \hat{\mathbf{F}}^{\text{turb}}_{\parallel} - \frac{1}{Ro}\cos\theta\widehat{\boldsymbol{\theta}} \times \hat{\mathbf{u}}_{\perp} + \frac{1}{Fr}\hat{\rho}\hat{\mathbf{g}}. \tag{2.88}$$

$$\hat{\rho}\hat{c}_p \widehat{\left(\frac{d}{dt}\right)} \hat{T} = (\beta_{T,m}T_m)\,Eu\,Ec\,\hat{\beta}_T\hat{T}\widehat{\left(\frac{d}{dt}\right)}\hat{p} + \frac{Ec}{Re}\hat{\boldsymbol{\sigma}}' : \hat{\mathsf{D}} - \frac{1}{Pe}\hat{\nabla} \cdot \hat{\mathbf{q}} - \frac{\left|\mathbf{J}_S \cdot \nabla\left(\Delta h\right)\right|_m}{\left|\rho\, c_p \mathbf{u}_{\perp} \cdot \nabla_{\perp} T\right|_m}\hat{\mathbf{J}}_S \cdot \hat{\nabla}\left(\Delta\hat{h}\right) + \hat{Q}^{\text{turb}}_T, \tag{2.89}$$

$$\hat{\rho} \widehat{\left(\frac{d}{dt}\right)} \hat{S} = -\frac{1}{Pe_S}\hat{\nabla} \cdot \hat{\mathbf{J}}_S + \hat{Q}^{\text{turb}}_S. \tag{2.90}$$

where

$$\widehat{\left(\frac{d\mathbf{u}}{dt}\right)}_{\perp} = \left(Sr\,\left.\frac{\partial\hat{\mathbf{u}}_{\perp}}{\partial\hat{t}}\right|_{\mathbf{e}_i} + \hat{\mathbf{u}} \cdot \hat{\nabla}\hat{\mathbf{u}}_{\perp}\Big|_{\mathbf{e}_i} + \gamma\Gamma\frac{\hat{w}}{\hat{r}}\hat{\mathbf{u}}_{\perp} + \Gamma\frac{\hat{u}}{\hat{r}}\tan\theta\,\widehat{\mathbf{r}} \times \hat{\mathbf{u}}_{\perp}\right) \tag{2.91}$$

$$\widehat{\left(\frac{d\mathbf{u}}{dt}\right)}_{\parallel} = \left(Sr\,\left.\frac{\partial\hat{\mathbf{u}}_{\parallel}}{\partial\hat{t}}\right|_{\mathbf{e}_i} + \hat{\mathbf{u}} \cdot \hat{\nabla}\hat{\mathbf{u}}_{\parallel}\Big|_{\mathbf{e}_i} - \frac{\Gamma}{\gamma}\frac{\hat{\mathbf{u}}_{\perp}^2}{\hat{r}}\widehat{\mathbf{r}}\right), \tag{2.92}$$

$$\widehat{\left(\frac{d}{dt}\right)} = \left(Sr\,\frac{\partial}{\partial\,\hat{t}} + \hat{\mathbf{u}} \cdot \hat{\nabla}\right), \tag{2.93}$$

and

$$\hat{\mathbf{F}}^{\text{turb}}_{\perp} = \frac{1}{Re^{\text{t}}_{\perp}}\hat{A}_{\perp}\,\hat{\nabla}^2_{\perp}\hat{\mathbf{u}}_{\perp}\Big|_{\mathbf{e}_i} + \frac{1}{Re^{\text{t}}_{\parallel}}\hat{A}_{\parallel}\,\hat{\nabla}^2_{\parallel}\hat{\mathbf{u}}_{\perp}\Big|_{\mathbf{e}_i} + \frac{1}{Re^{\text{t}}_{\perp}}\hat{A}_{\perp}\hat{\nabla}_{\perp}\left(\hat{\nabla} \cdot \hat{\mathbf{u}}\right) + \frac{\gamma\Gamma}{Re^{\text{t}}_{\parallel}}\frac{\hat{A}_{\parallel}}{\hat{r}}\left(\widehat{\mathbf{r}} \cdot \hat{\nabla}\right)\hat{\mathbf{u}}_{\perp}\big|_{\mathbf{e}_i} - \frac{\Gamma}{\gamma\,Re^{\text{t}}_{\parallel}}\frac{\hat{A}_{\perp}}{\hat{r}}\left(\widehat{\mathbf{r}} \cdot \hat{\nabla}\right)\hat{\mathbf{u}}_{\perp}\big|_{\mathbf{e}_i} + \frac{2\Gamma}{Re^{\text{t}}_{\perp}}\frac{\hat{A}_{\perp}\tan\theta}{\hat{r}}\widehat{\mathbf{r}} \times \left(\widehat{\boldsymbol{\phi}} \cdot \hat{\nabla}\right)\hat{\mathbf{u}}_{\perp}\big|_{\mathbf{e}_i} + \frac{2\gamma\Gamma}{Re^{\text{t}}_{\perp}}\frac{\hat{A}_{\perp}}{\hat{r}}\hat{\nabla}_{\perp}\hat{w} - \frac{\Gamma^2}{Re^{\text{t}}_{\perp}}\frac{\hat{A}_{\perp}}{\hat{r}^2\cos^2\theta}\hat{\mathbf{u}}_{\perp}, \tag{2.94}$$

$$\begin{aligned}
\hat{\mathbf{F}}_{\parallel}^{\text{turb}} = {} & \frac{1}{Re_{\perp}^{\text{t}}}\hat{A}_{\perp}\,\hat{\nabla}_{\perp}^{2}\hat{\mathbf{u}}_{\parallel}\Big|_{\mathbf{e}_i} + \frac{1}{Re_{\parallel}^{\text{t}}}\hat{A}_{\parallel}\,\hat{\nabla}_{\parallel}^{2}\hat{\mathbf{u}}_{\parallel}\Big|_{\mathbf{e}_i} + \frac{1}{Re_{\parallel}^{\text{t}}}\hat{A}_{\perp}\hat{\nabla}_{\parallel}\left(\hat{\nabla}\cdot\hat{\mathbf{u}}\right) \\
& + \frac{\gamma\Gamma}{Re_{\parallel}^{\text{t}}}\frac{\hat{A}_{\parallel}}{\hat{r}}\left(\hat{\nabla}\cdot\hat{\mathbf{u}}\right)\widehat{\mathbf{r}} - \frac{3\Gamma}{\gamma\,Re_{\perp}^{\text{t}}}\frac{\hat{A}_{\perp}}{\hat{r}}\left(\hat{\nabla}\cdot\hat{\mathbf{u}}\right)\widehat{\mathbf{r}} + \frac{2\Gamma^{2}}{Re_{\perp}^{\text{t}}}\frac{\hat{A}_{\perp}}{\hat{r}^{2}}\hat{\mathbf{u}}_{\parallel} \\
& - \frac{\gamma\Gamma}{Re_{\parallel}^{\text{t}}}\frac{\hat{A}_{\parallel}}{\hat{r}}\left(\widehat{\mathbf{r}}\cdot\hat{\nabla}\right)\hat{\mathbf{u}}_{\parallel} + \frac{3\Gamma}{\gamma\,Re_{\perp}^{\text{t}}}\frac{\hat{A}_{\perp}}{\hat{r}}\left(\widehat{\mathbf{r}}\cdot\hat{\nabla}\right)\hat{\mathbf{u}}_{\parallel},
\end{aligned} \tag{2.95}$$

$$\hat{Q}_{T}^{\text{turb}} = \frac{1}{Pe_{\perp}^{\text{t}}}\hat{\nabla}_{\perp}^{2}\hat{T} + \frac{1}{Pe_{\parallel}^{\text{t}}}\hat{\nabla}_{\parallel}^{2}\hat{T} \tag{2.96}$$

$$\hat{Q}_{S}^{\text{turb}} = \frac{1}{Pe_{\perp}^{\text{t}}}\hat{\nabla}_{\perp}^{2}\hat{S} + \frac{1}{Pe_{\parallel}^{\text{t}}}\hat{\nabla}_{\parallel}^{2}\hat{S} \tag{2.97}$$

where the dimensionless numbers that describing turbulent transport are

$$Re_{\perp}^{\text{t}} = \frac{U_{\perp,m}\delta L_{\perp,m}}{A_{\perp,m}} \quad Re_{\parallel}^{\text{t}} = \frac{U_{\parallel,m}\delta L_{\parallel,m}}{A_{\parallel,m}} \quad Pe_{\perp}^{\text{t}} = \frac{U_{\perp,m}\delta L_{\perp,m}}{K_{\perp,m}} \quad Pe_{\parallel}^{\text{t}} = \frac{U_{\parallel,m}\delta L_{\parallel,m}}{K_{\parallel,m}}.$$

These equations form the basis of all phenomena at large-scale. Together with the thermodynamic equation of state these equations form a closed system of equations. In order to have a well-defined problem, the system of equations need complemented boundary value conditions. This will be discussed detailed in next section.

2.4 Boundary conditions

The atmosphere and ocean is bounded by continents, topography and the interface between the atmosphere and ocean. Through the boundary, there will be a transport of mass, momentum and energy. Therefore it is necessary to specify the boundary value conditions, in order to solve the equations of motion. In this section, we derive the boundary value conditions to the equations described in the previous section.

Let the surface and bottom of the fluid be described by the functions $h(\phi,\theta,t)$ and $h_b(\phi,\theta,t)$, and let the average height of the fluid be H_0, so that the deviation from the average height is $\zeta(\phi,\theta,t)$, such that

$$h(\phi,\theta,t) + h_b(\phi,\theta,t) = H_0 + \zeta(\phi,\theta,t)\,. \tag{2.98}$$

The bottom topography is measured relative to the average radius r_0 of the Earth, and specified by the function

$$r - r_0 = h_b(\phi,\theta,t)\,. \tag{2.99}$$

Thus, the interface between the bottom and the fluid will be described by the surface function

$$F_b(\phi, \theta, r, t) = h_b(\phi, \theta, t) - (r - r_0) \tag{2.100}$$

and similarly will the interface between the fluid and the upper surface be described by the surface function

$$F_u(\phi, \theta, r, t) = (r - r_0) - (h_b(\phi, \theta, t) + h(\phi, \theta, t)). \tag{2.101}$$

The unit vector normal to the bottom is

$$\widehat{\mathbf{n}}_b = \frac{\nabla F_b}{|\nabla F_b|} = \frac{1}{|\nabla F_b|}(\nabla_\perp h_b - \widehat{\mathbf{r}}) \tag{2.102}$$

and the unit vector normal to the surface is

$$\widehat{\mathbf{n}}_u = \frac{\nabla F_u}{|\nabla F_u|} = \frac{1}{|\nabla F_u|}(\widehat{\mathbf{r}} - \nabla_\perp \zeta). \tag{2.103}$$

We will assume that the local curvature of interface between the ocean and the atmosphere is very small, i.e.,

$$(\mathsf{I} - \widehat{\mathbf{n}}_u \widehat{\mathbf{n}}_u) \nabla \cdot \widehat{\mathbf{n}}_u \approx \mathbf{0}, \tag{2.104}$$

such that the upper unit vector is approximately equal to the unit vector along the radial direction, $\widehat{\mathbf{n}}_u \approx \widehat{\mathbf{r}}$.

2.4.1 The kinematic boundary conditions

The boundary conditions of the bottom

At the bottom, $r - r_0 = h_b$, we assume that the tangential component of the fluid velocity is equal to the tangential component of the rigid surface velocity. This is the non-slip condition and can be expressed as

$$\mathbf{u} \cdot \widehat{\mathbf{t}} = \mathbf{u}_b \cdot \widehat{\mathbf{t}}, \tag{2.105}$$

where $\widehat{\mathbf{t}}$ is the unit tangent vector to the bottom surface and $\mathbf{u}_b$ is the velocity of the bottom surface. We will only consider the case where the topography is only a function of the space, therefore, the bottom does not have any velocity and the non-slip boundary condition reduces to

$$\mathbf{u} \cdot \widehat{\mathbf{t}} = 0. \tag{2.106}$$

In order to have no mass transfer across the boundary, the normal component of the fluid velocity must be equal to the normal component of the rigid surface velocity, i.e.

$$\mathbf{u} \cdot \widehat{\mathbf{n}}_b = \mathbf{u}_b \cdot \widehat{\mathbf{n}}_b, \tag{2.107}$$

but since the bottom surface is static, the condition reduces to

$$\mathbf{u} \cdot \widehat{\mathbf{n}}_b = 0. \tag{2.108}$$

By using equation (2.102) and splitting the velocity field into a horizontal and a vertical component, $\mathbf{u} = \mathbf{u}_\perp + \mathbf{u}_\|$, equation (2.108) can be written as a boundary condition for the vertical velocity,

$$\mathbf{u}_\| \cdot \widehat{\mathbf{r}} = \mathbf{u}_\perp \cdot \nabla_\perp h_b. \tag{2.109}$$

Equation (2.105) and (2.107) implies that the fluid velocity must match the rigid surface velocity. Since the bottom surface is static, the fluid velocity must be equal to zero at the bottom.

The boundary condition of the upper surface

Simular to the lower boundary value condition, we will assume that there is no mass transfer between the interface $r - r_0 = \zeta + H_0$, i.e. any fluid particle which lies on the interface stays there for all time. Therefore, the normal component of the fluid velocity must be equal to the normal velocity of the interface itself. This is the upper kinematic boundary condition and can be expressed as

$$(\mathbf{u} - \mathbf{u}_u) \cdot \widehat{\mathbf{n}}_u = 0 \tag{2.110}$$

or

$$\frac{dF_u}{dt} = 0. \tag{2.111}$$

By using that the vertical velocity is defined as the material derivative of the radial coordinate, and that the fluid hight is independent of the vertical coordinate, equation (2.111) can be written as

$$\mathbf{u}_\| \cdot \widehat{\mathbf{r}} = \left(\frac{\partial}{\partial t} + \mathbf{u}_\perp \cdot \nabla_\perp \right) (h_b + h) \, . \tag{2.112}$$

Analogous to the no-slip boundary condition, there must be a continuity in the tangential component of the fluid velocity and the tangential interface velocity,

$$(\mathbf{u} - \mathbf{u}_u) \cdot \widehat{\mathbf{t}} = 0. \tag{2.113}$$

Since the boundary value conditions (2.106), (2.109), (2.112) and (2.113) gives the relationship between the kinematics of the bottom, upper surface and the fluid are these boundary value conditions called the kinematic boundary value conditions.

2.4.2 The dynamic boundary conditions

The normal and shear stress conditions for the molecular fluxes

In general the stress tensor $\boldsymbol{\sigma}$ will be discontinuous at the boundary between two fluids, and/or at a surface that is characterized by a surface tension. In this case the fluids are the ocean and the atmosphere. First, we derive general boundary value conditions between two fluids. Then, we make simplifications by assuming that the atmosphere is dynamic neglectable on the boundary, and that the local surface curvature is very small.

Applying the momentum equation to a Gaussian pillbox extending just slightly into the fluids on either side of the boundary, where the pillbox is cylindrical with height 2ϵ and radius ϵ, we obtain the force balance

$$\int_V \rho \frac{d\mathbf{u}}{dt}\,\mathrm{d}V = -\int_S (\boldsymbol{\sigma}_{\mathrm{o}} - \boldsymbol{\sigma}_{\mathrm{a}}) \cdot \widehat{\mathbf{n}}_u\,\mathrm{d}S + \oint_C T\mathbf{s}\,\mathrm{d}l + \int_V \rho\mathbf{g}\,\mathrm{d}V. \qquad (2.114)$$

Where S is the surface bounded by the closed curve C, where C goes around the periphery of the cutting line of the box between the fluids. The subindices o and a refers to the ocean and atmosphere, respectively. Surface tension T is intended to flatten the surface. Therefore, we have assumed that the force $T\mathbf{s}$ associated with surface tension is directed normal to the curve C and tangential to the surface S. This direction is given by the unit tangent vector $\mathbf{s}$. By using that the unit tangetial vector along the curve C is $\widehat{\mathbf{t}}$ and the Stoke's therem, the force balance can be written as

$$\begin{aligned}\int_V \rho \frac{d\mathbf{u}}{dt}\,\mathrm{d}V = &-\int_S (\boldsymbol{\sigma}_{\mathrm{o}} - \boldsymbol{\sigma}_{\mathrm{a}}) \cdot \widehat{\mathbf{n}}_u\,\mathrm{d}S \\ &+ \oint_S \left(\mathsf{P}\cdot\nabla T - T\widehat{\mathbf{n}}_u\left((\mathsf{P}\cdot\nabla)\cdot\widehat{\mathbf{n}}_u\right)\right)\,\mathrm{d}S + \int_V \rho\mathbf{g}\,\mathrm{d}V,\end{aligned} \qquad (2.115)$$

where P is the surface projection operator defined as

$$\mathsf{P} = \mathsf{I} - \widehat{\mathbf{n}}_u\widehat{\mathbf{n}}_u. \qquad (2.116)$$

In the limit where the thickess of the pillbox goes to zero, i.e. $\epsilon \to 0$, the surface forces have to balance, since the body force will be of $\mathcal{O}(\epsilon^3)$ and the surface force will be of order $\mathcal{O}(\epsilon^2)$. The balance is

$$-(\boldsymbol{\sigma}_{\mathrm{o}} - \boldsymbol{\sigma}_{\mathrm{a}}) \cdot \mathbf{n}_u + \mathsf{P}\cdot\nabla T - T\mathbf{n}_u\left[(\mathsf{P}\cdot\nabla)\cdot\mathbf{n}_u\right] = \mathbf{0} \qquad (2.117)$$

If we use the assumption that the local curvature is very small, the surface tension can be neglectet and if we assume that the mass density of the air is much smaller than the mass density of the ocean, the change in the pressure in the air due to the ocean motion is negligible, hence the air is dynamically negligible. This implies zero viscous stress at the surface. In this limit we say that the surface is free. Applying this assumtion on equation (2.117), we find that the pressures must be equal at the surface, i.e.,

$$p_o \widehat{\mathbf{n}}_u = p_a \widehat{\mathbf{n}}_u, \tag{2.118}$$

where the unit vector normal to the surface is approximately the unit vector in the vertical direction, $\widehat{\mathbf{n}}_u \approx \widehat{\mathbf{r}}$.

The normal and shear stress conditions for the turbulent fluxes

Using the same analysis as above on the turbulent flux boundary value conditions, we get

$$\left(\boldsymbol{\sigma}_{\mathrm{o}}^{\mathrm{turb}} - \boldsymbol{\sigma}_{\mathrm{a}}^{\mathrm{turb}}\right) \cdot \mathbf{n}_u = \mathbf{0} \tag{2.119}$$

If we assume that the local curvature is small, the unit vector normal to the surface is approximately the unit vector in the vertical direction. This leads to the balance

$$\left(\sigma_{\phi\, r} - \tau_{\phi\, r}\right) \widehat{\boldsymbol{\phi}} + \left(\sigma_{\theta\, r} - \tau_{\theta\, r}\right) \widehat{\boldsymbol{\theta}} + \left(\sigma_{r\, r} - \tau_{r\, r}\right) \widehat{\mathbf{r}} = \mathbf{0}, \tag{2.120}$$

where σ_{ij} and τ_{ij} are the turbulent stresses in the ith direction acting on an element of surface oriented in the jth direction. $\{\sigma\}_{ij}$ is the turbulent stress in the ocean and $\{\tau\}_{ij}$ is the turbulent stress in the atmosphere. The turbulent stress acting on the interface between the ocean and the atmosphere is due to the fluctuations in the wind. Therefore, we call $\{\tau\}_{ij}$ the wind stress. The wind that blows over the ocean is approximately horizontal. Hence, we neglect the stress in the vertical direction acting on the surface. Consequently, the boundary value condition reduces to

$$\left(\sigma_{\phi\, r} - \tau_{\phi\, r}\right) \widehat{\boldsymbol{\phi}} + \left(\sigma_{\theta\, r} - \tau_{\theta\, r}\right) \widehat{\boldsymbol{\theta}} = \mathbf{0}. \tag{2.121}$$

By using the expression for the turbulent stress tensor, we can write

$$\begin{aligned}
\tau_{\phi\, r} &= \rho_m A_\perp \left(\frac{1}{r\cos\theta}\frac{\partial w}{\partial \phi} - \frac{u}{r}\right) + \rho_m A_\parallel \frac{\partial u}{\partial r} \\
&= \rho_m A_\perp \left(\widehat{\boldsymbol{\phi}} \cdot \nabla\, w - \frac{u}{r}\right) + \rho_m A_\parallel \left(\widehat{\mathbf{r}} \cdot \nabla\right) u \qquad (2.122)\\
\tau_{\theta\, r} &= \rho_m A_\perp \left(\frac{1}{r}\frac{\partial w}{\partial \theta} - \frac{v}{r}\right) + \rho_m A_\parallel \frac{\partial v}{\partial r} \\
&= \rho_m A_\perp \left(\widehat{\boldsymbol{\theta}} \cdot \nabla\, w - \frac{v}{r}\right) + \rho_m A_\parallel \left(\widehat{\mathbf{r}} \cdot \nabla\right) v \qquad (2.123)
\end{aligned}$$

Since the wind stress only contains a meriodinal component and a zonal component, it is natural to define the vector

$$\boldsymbol{\tau} = \tau_{\phi\, r}\widehat{\boldsymbol{\phi}} + \tau_{\theta\, r}\widehat{\boldsymbol{\theta}}. \tag{2.124}$$

We will use this later when we discuss the boundary layer theory.

The boundary conditions for the temperature and the salinity

Since the normal and the tangetial compontents of the velocity is continuous at the bottom, the fluid velocity is zero. Therefore, there can not be any transport of heat and salt on either small-scala and large-scale,

$$\mathbf{q} \cdot \widehat{\mathbf{n}}_b = 0, \tag{2.125}$$
$$\mathbf{q}^{\mathrm{turb}} \cdot \widehat{\mathbf{n}}_b = 0, \tag{2.126}$$
$$\mathbf{J}_S \cdot \widehat{\mathbf{n}}_b = 0, \tag{2.127}$$
$$\mathbf{J}_S^{\mathrm{turb}} \cdot \widehat{\mathbf{n}}_b = 0, \tag{2.128}$$

or equivalently

$$\mathbf{q} \cdot \widehat{\mathbf{n}}_b = 0, \tag{2.129}$$
$$K_\perp \nabla_\perp T \cdot \nabla_\perp h_b = K_\parallel \nabla_\parallel T \cdot \widehat{\mathbf{r}}, \tag{2.130}$$
$$\mathbf{J}_S \cdot \widehat{\mathbf{n}}_b = 0, \tag{2.131}$$
$$K_\perp \nabla_\perp S \cdot \nabla_\perp h_b = K_\parallel \nabla_\parallel S \cdot \widehat{\mathbf{r}}. \tag{2.132}$$

At the free surface we have applied the heat and salinity equation to a Gaussian pillbox. The analysis is simular to the analysis of the boundary condition for the stress. The result is

$$(\mathbf{q}_o - \mathbf{q}_a) \cdot \widehat{\mathbf{n}}_u = 0, \tag{2.133}$$
$$\left(\mathbf{q}_o^{\mathrm{turb}} - \mathbf{q}_a^{\mathrm{turb}}\right) \cdot \widehat{\mathbf{n}}_u = 0, \tag{2.134}$$
$$(\mathbf{J}_{S,o} - \mathbf{J}_{S,a}) \cdot \widehat{\mathbf{n}}_u = 0, \tag{2.135}$$
$$\left(\mathbf{J}_{S,o}^{\mathrm{turb}} - \mathbf{J}_{S,a}^{\mathrm{turb}}\right) \cdot \widehat{\mathbf{n}}_u = 0. \tag{2.136}$$

Since the atmosphere is dynamically negligible and the local surface curvature is small the boundary conditions reduces to

$$\mathbf{q}_o \cdot \widehat{\mathbf{r}} = 0, \tag{2.137}$$
$$\left(\mathbf{q}_o^{\mathrm{turb}} - \mathbf{q}_a^{\mathrm{turb}}\right) \cdot \widehat{\mathbf{r}} = 0, \tag{2.138}$$
$$\mathbf{J}_{S,o} \cdot \widehat{\mathbf{r}} = 0, \tag{2.139}$$
$$\left(\mathbf{J}_{S,o}^{\mathrm{turb}} - \mathbf{J}_{S,a}^{\mathrm{turb}}\right) \cdot \widehat{\mathbf{r}} = 0, \tag{2.140}$$

or equivalently

$$\mathbf{q}_o \cdot \widehat{\mathbf{r}} = 0, \tag{2.141}$$
$$\rho_m c_p K_\parallel \nabla_\parallel T \cdot \widehat{\mathbf{r}} = \mathbf{q}_a^{\mathrm{turb}} \cdot \widehat{\mathbf{r}}, \tag{2.142}$$
$$\mathbf{J}_{S,o} \cdot \widehat{\mathbf{r}} = 0, \tag{2.143}$$
$$\rho_m K_\parallel \nabla_\parallel S \cdot \widehat{\mathbf{r}} = \mathbf{J}_{S,a}^{\mathrm{turb}} \cdot \widehat{\mathbf{r}}. \tag{2.144}$$

Dimensionless boundary conditions

By introducing dimensionless numbers, the boundary value condition at the bottom $\hat{z} = h_{b,m}\,\hat{h}_b/L_{\parallel,m}$ is given by

$$\hat{w} = \frac{h_{b,m}}{\delta L_{\parallel,m}}\hat{\mathbf{u}}_\perp \cdot \hat{\nabla}\hat{h}_b, \tag{2.145}$$

where $h_{b,m}$ is the typical magnitude in the bottom topograpy. The boundary value conditions at the free surface $\hat{z} = 1 + \zeta_m\,\hat{\zeta}/\delta L_{\parallel,m}$ are

$$\begin{aligned}
\hat{w} &= \frac{\zeta_m}{\delta L_{\parallel,m}}\left(Sr\frac{\partial}{\partial \hat{t}} + \hat{\mathbf{u}}_\perp \cdot \hat{\nabla}\right)\hat{\zeta}, && (2.146)\\
\alpha\hat{\tau}_{\phi\,r} &= \frac{A_{\perp,m}}{A_{\parallel,m}}\gamma\hat{A}_\perp\left(\gamma\widehat{\boldsymbol{\phi}}\cdot\hat{\nabla}\hat{w} - \Gamma\frac{\hat{u}}{\hat{r}}\right) + \hat{A}_\parallel\left(\widehat{\mathbf{r}}\cdot\hat{\nabla}\right)\hat{u}, && (2.147)\\
\alpha\hat{\tau}_{\theta\,r} &= \frac{A_{\perp,m}}{A_{\parallel,m}}\gamma\hat{A}_\perp\left(\gamma\widehat{\boldsymbol{\theta}}\cdot\hat{\nabla}\hat{w} - \Gamma\frac{\hat{v}}{\hat{r}}\right) + \hat{A}_\parallel\left(\widehat{\mathbf{r}}\cdot\hat{\nabla}\right)\hat{v}, && (2.148)\\
\hat{p} &= \hat{p}_a, && (2.149)
\end{aligned}$$

where τ_m is the typical magnitude in the wind stress, ζ_m is the typical magnitude of the interface amplitude, $\hat{p}_a$ is the dimensionless pressure at the interface and the new dimensionless number is

$$\alpha = \frac{\tau_m \delta L_{\parallel,m}}{\rho_m A_{\parallel,m} U_{\perp,m}}. \tag{2.150}$$

Note that we have defined the vertical coordinate as $z = r - r_0$ and used that the typical magnitude in the average depth H_0 is $\delta L_{\parallel,m}$.

2.5 Slab coordinates

If one wants to describe phenomena on small scales in relation to the earth radius, it will be advantageous to introduce a local rectangular coordinate system (X, Y, Z) fixed on the Earth's surface. Let the origin to this local coordinate system be given by the position $\phi_0\widehat{\boldsymbol{\phi}}_0 + \theta_0\widehat{\boldsymbol{\theta}}_0 + r_0\widehat{\mathbf{r}}_0$, where the unit vectors which span the coordinate system are defined by

$$\begin{aligned}
\widehat{\mathbf{X}} &\equiv \widehat{\boldsymbol{\phi}}_0 = -\sin\phi_0\widehat{\mathbf{x}} + \cos\phi_0\widehat{\mathbf{y}}, && (2.151)\\
\widehat{\mathbf{Y}} &\equiv \widehat{\boldsymbol{\theta}}_0 = -\sin\theta_0\cos\phi_0\widehat{\mathbf{x}} - \sin\theta_0\sin\phi_0\widehat{\mathbf{y}} + \cos\theta_0\widehat{\mathbf{z}}, && (2.152)\\
\widehat{\mathbf{Z}} &\equiv \widehat{\mathbf{r}}_0 = \cos\theta_0\cos\phi_0\widehat{\mathbf{x}} + \cos\theta_0\sin\phi_0\widehat{\mathbf{y}} + \sin\theta_0\widehat{\mathbf{z}}. && (2.153)
\end{aligned}$$

For small excursions on the plane, the geometry gives that the slab coordinates are related to spherical coordinates by

$$\begin{aligned}\mathbf{X} &= X\,\widehat{\mathbf{X}} + Y\,\widehat{\mathbf{Y}} + Z\,\widehat{\mathbf{Z}}, \\ &= (\phi-\phi_0)\,r_0\cos\theta_0\,\widehat{\mathbf{X}} + (\theta-\theta_0)\,r_0\,\widehat{\mathbf{Y}} + (r-r_0)\,\widehat{\mathbf{Z}}. \end{aligned} \tag{2.154}$$

By using the chain rule it follows that the relation between the derivatives in the local coordinate system and the spherical coordinate system is

$$\frac{\partial}{\partial\phi} = r_0\cos\theta_0\frac{\partial}{\partial X}, \quad \frac{\partial}{\partial\theta} = r_0\frac{\partial}{\partial Y}, \quad \frac{\partial}{\partial r} = \frac{\partial}{\partial Z} \tag{2.155}$$

and

$$\frac{d\phi}{dt} = \frac{1}{r_0\cos\theta_0}\frac{dX}{dt}, \quad \frac{d\theta}{dt} = \frac{1}{r_0}\frac{dY}{dt}, \quad \frac{dr}{dt} = \frac{dZ}{dt}. \tag{2.156}$$

The first order gradient operator in this coordinate system is

$$\nabla = \widehat{\mathbf{X}}\,\frac{r_0\cos\theta_0}{r\cos\theta}\frac{\partial}{\partial X} + \widehat{\mathbf{Y}}\,\frac{r_0}{r}\frac{\partial}{\partial Y} + \widehat{\mathbf{Z}}\,\frac{\partial}{\partial Z}, \tag{2.157}$$

and the second order spatial derivative is

$$\nabla^2 = \frac{r_0^2\cos^2\theta_0}{r^2\cos^2\theta}\frac{\partial^2}{\partial X^2} + \frac{r_0^2}{r^2}\left(\frac{\partial^2}{\partial Y^2} - \frac{\tan\theta}{r_0}\frac{\partial}{\partial Y}\right) + \left(\frac{\partial^2}{\partial Z^2} + \frac{2}{r}\frac{\partial}{\partial Z}\right). \tag{2.158}$$

The velocity is given by

$$\mathbf{u} = \frac{r\cos\theta}{r_0\cos\theta_0}\frac{dX}{dt}\,\widehat{\mathbf{X}} + \frac{r}{r_0}\frac{dY}{dt}\,\widehat{\mathbf{Y}} + \frac{dZ}{dt}\,\widehat{\mathbf{Z}}. \tag{2.159}$$

By introducing non-dimensional variabels, the spatial differential operators in horizontal and vertival directions become

$$\hat{\nabla}_\perp = \widehat{\mathbf{X}}\frac{1}{\hat{r}}\frac{\cos\theta_0}{\cos\theta}\frac{\partial}{\partial\hat{X}} + \widehat{\mathbf{Y}}\frac{1}{\hat{r}}\frac{\partial}{\partial\hat{Y}}, \tag{2.160}$$

$$\hat{\nabla}_\parallel = \widehat{\mathbf{Z}}\frac{\partial}{\partial\hat{Z}}, \tag{2.161}$$

and the Laplacian become

$$\hat{\nabla}^2_\perp = \frac{1}{\hat{r}^2}\frac{\cos^2\theta_0}{\cos^2\theta}\frac{\partial^2}{\partial\hat{X}^2} + \frac{1}{\hat{r}^2}\left(\frac{\partial^2}{\partial\hat{Y}^2} - \Gamma\tan\theta\frac{\partial}{\partial\hat{Y}}\right), \tag{2.162}$$

$$\hat{\nabla}^2_\perp = \left(\frac{\partial^2}{\partial\hat{Z}^2} + \gamma\Gamma\frac{2}{\hat{r}}\frac{\partial}{\partial\hat{Z}}\right). \tag{2.163}$$

It should be noted that the introduction of slab coordinates has not resulted in any loss of information in the equations, i.e., this is not some kind of approximation. The trigonometric functions can be expanded around the origin of the local rectangular system,

$$\sin\theta = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\mathrm{d}^n}{\mathrm{d}\theta^n} (\sin\theta)|_{\theta=\theta_0} \right) (\theta-\theta_0)^n, \tag{2.164}$$

$$\cos\theta = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\mathrm{d}^n}{\mathrm{d}\theta^n} (\cos\theta)|_{\theta=\theta_0} \right) (\theta-\theta_0)^n, \tag{2.165}$$

$$\tan\theta = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\mathrm{d}^n}{\mathrm{d}\theta^n} (\tan\theta)|_{\theta=\theta_0} \right) (\theta-\theta_0)^n. \tag{2.166}$$

Since the dimensionless meridional coordinate $\hat{Y}$ is

$$\hat{Y} = \frac{r_0}{L_\perp} (\theta-\theta_0) = \frac{1}{\Gamma} (\theta-\theta_0), \tag{2.167}$$

the expansions can be written as

$$\sin\theta = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\mathrm{d}^n}{\mathrm{d}\theta^n} (\sin\theta)|_{\theta=\theta_0} \right) \left(\Gamma\hat{Y}\right)^n, \tag{2.168}$$

$$\cos\theta = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\mathrm{d}^n}{\mathrm{d}\theta^n} (\cos\theta)|_{\theta=\theta_0} \right) \left(\Gamma\hat{Y}\right)^n, \tag{2.169}$$

$$\tan\theta = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\mathrm{d}^n}{\mathrm{d}\theta^n} (\tan\theta)|_{\theta=\theta_0} \right) \left(\Gamma\hat{Y}\right)^n. \tag{2.170}$$

From equation (2.154) it follows that the dimensionless radial coordinate is given by

$$\hat{r} = 1 + \gamma\Gamma\hat{Z}. \tag{2.171}$$

Therefore, it follow that $1/\hat{r}$ can be expanded as

$$\frac{1}{\hat{r}} = \left(1+\gamma\Gamma\hat{Z}\right)^{-1} \sim 1 - \gamma\Gamma\hat{Z} + (\gamma\Gamma)^2\,\hat{Z}^2 + \mathcal{O}((\gamma\Gamma)^3). \tag{2.172}$$

In the next three chapters, we will discuss models that are limited to a horizontal length scale which is of the order much less than Earth's radius. In this case the introduction of slab-coordinates cause that the meridional and zonal coordinates will be mapped into a rectangular coordinate system. The expansions of the trigonometric functions will show in a consistent manner which part of the equations of motion where curvature is important. It will turn out that the curvature will be important in the Coriolis force, even where a local rectangular coordinate system will be a good approximasjon. It should also be noted that the radius (2.171) to $\mathcal{O}(\gamma\Gamma)$ will be approximated constant, even without the introduction of slab-coordinates. Therefore, we can in a consistent manner using equation (2.172) to determine to which order the variation of the radius is important. Equation (2.172) is indeed the relation that gives the widely used thin shell approximation for dynamics described in spherical coordinates, in the limit where the fluid is shallow.

2.6 The background state of the ocean

In the absence of motion, $\mathbf{u} = \mathbf{0}$, $\mathrm{d}T/\mathrm{d}t = 0$, $\mathrm{d}p/\mathrm{d}t = 0$ and $\mathrm{d}S/\mathrm{d}t = 0$, the fluid will be in state of mechanical equilbrium, where the pressure is hydrostatically distributed along the vertical direction and there will be no mixing of momentum, heat and salt. The equations for this state is

$$\begin{aligned}
\nabla_{\perp} p_h &= \mathbf{0} && (2.173)\\
\nabla_{\parallel} p_h &= \rho_h \mathbf{g} && (2.174)\\
\nabla \cdot \mathbf{q}_h &= \mathbf{0} && (2.175)\\
\nabla \cdot \mathbf{J}_{S,h} &= \mathbf{0} && (2.176)\\
\rho_h &= \rho\left(T_h, S_h, p_h\right) && (2.177)
\end{aligned}$$

where the h sub-script indicate the hydrostatic state. Since the acceleration of gravity is constant and is pointing in the negative vertical direction and the horizontal pressure gradient is zero, it follows that the pressure is only a function of vertical coordinate, $p_h = p_h(r)$. The rotation of the hydrostatic equation is zero, which implies that

$$\nabla p_h \times \nabla \rho_h = \mathbf{0}, \tag{2.178}$$

hence the density gradient is parallel with the pressure gradient, so that the mass density associated with this mechanical equilibrium is only a function of the vertical coordinate, i.e. $\rho_h = \rho_h(r)$, or it is constant, $\rho_h = \rho_0$. If the mass density is constant, the thermodynamic equations become decoupled from the continuity and momentum equations. If not, consideration of the thermodynamic equations are necessary. In the case where the density is a function of the vertical coordinate, the equation of state implies that the temperature and the salinity also must be a function of vertical coordinate only. Such a fluid is said to be stratified. It should be be noted that stratified fluids can never be in a state of thermodynamic equilibrium, since the temperature is not constant. When the mass density is constant, $\rho_h = \rho_0$, the solution for the hydrostatic equation is

$$p_h(z) = p_a - \rho_0 g\left[z - (H_0 + \zeta)\right], \tag{2.179}$$

where p_a is the atmospheric pressure at the interface between the ocean and the atmosphere.

2.6.1 Stratification

Let a fluid particle move adiabatically by a small amount ζ from its initial position r. Assume that at each instant of time the thermodynamic state of the particle may be assumed to be an equilibrium state. In this case the entropy s and the salinity S are constant and hence the density of the fluid particle at position $r+\zeta$ is

$$\left(\rho_h\left(r+\zeta\right)\right)_{s_h,S_h} = \rho_h\left(r\right) + \left(\frac{\mathrm{d}\rho_h}{\mathrm{d}r}\right)_{s_h,S_h}\zeta. \tag{2.180}$$

The density of the surrounding fluid is

$$\rho_h\left(r+\zeta\right) = \rho_h\left(r\right) + \frac{\mathrm{d}\rho_h}{\mathrm{d}r}\zeta. \tag{2.181}$$

The fluid particle will thus experience a buoyancy force density

$$\begin{aligned} f_b &= g\left(\rho_h\left(r+\zeta\right) - \left(\rho_h\left(r+\zeta\right)\right)_{s_h,S_h}\right) \\ &= g\left(\frac{\mathrm{d}\rho_h}{\mathrm{d}r} - \left(\frac{\mathrm{d}\rho_h}{\mathrm{d}r}\right)_{s_h,S_h}\right)\zeta. \end{aligned} \tag{2.182}$$

If $f_b < 0$, the buoyant force will tend to return the fluid particle to it's initial position and the stability of the equilibrium is stable (stably stratified). Otherwice, if $f_b > 0$, the bouyanct force will accelerate the fluid particle away from it's initial position and the stability of the equilbrium is unstable. The change of the mass density with height when the salinity and the entropy are constant can be written as

$$\left(\frac{\mathrm{d}\rho_h}{\mathrm{d}r}\right)_{s_h,S_h} = \left(\frac{\partial\rho_h}{\partial p_h}\right)_{s_h,S_h}\frac{\mathrm{d}p_h}{\mathrm{d}r}, \tag{2.183}$$

and the change of the mass density of the surrounding fluid is given by the equation of state

$$\frac{\mathrm{d}\rho_h}{\mathrm{d}r} = \left(\frac{\partial\rho_h}{\partial T_h}\right)_{p_h,S_h}\frac{\mathrm{d}T_h}{\mathrm{d}r} + \left(\frac{\partial\rho_h}{\partial p_h}\right)_{T_h,S_h}\frac{\mathrm{d}p_h}{\mathrm{d}r} + \left(\frac{\partial\rho_h}{\partial S_h}\right)_{T_h,p_h}\frac{\mathrm{d}S_h}{\mathrm{d}r}. \tag{2.184}$$

The adiabatic compressibility coefficient to the hydrostatic state is

$$\widetilde{\kappa}_h = \frac{1}{\rho_h}\left(\frac{\partial\rho_h}{\partial p_h}\right)_{s_h,S_h} = \frac{1}{\rho_h}\left(\frac{\partial\rho_h}{\partial p_h}\right)_{T_h,S_h} + \frac{\Gamma_h}{\rho_h}\left(\frac{\partial\rho_h}{\partial T_h}\right)_{p_h,S_h}, \tag{2.185}$$

and the relation between the adiabatic compressibility coefficient and the sound velocity is given by

$$c_h^2 = \frac{1}{\rho_h \widetilde{\kappa}_h}. \tag{2.186}$$

By using (2.185) and (2.186), equation (2.183) and (2.184) can be written as

$$\left(\frac{\mathrm{d}\rho_h}{\mathrm{d}r}\right)_{s,S} = \frac{1}{c_h^2}\frac{\mathrm{d}p_h}{\mathrm{d}r}, \tag{2.187}$$

$$\frac{\mathrm{d}\rho_h}{\mathrm{d}r} = -\rho_h \beta_{T_h}\frac{\mathrm{d}T_h}{\mathrm{d}r} + \left(\frac{1}{c_h^2} + \rho_h \beta_{T_h}\Gamma_h\right)\frac{\mathrm{d}p_h}{\mathrm{d}r} + \rho_h \beta_{S_h}\frac{\mathrm{d}S_h}{\mathrm{d}r}. \tag{2.188}$$

By using these relations and that the pressure is hydrostatically distributed, it follows that the buoyancy force density can be written as

$$f_b = g\left[-\rho_h \beta_{T_h}\left(\frac{\mathrm{d}T_h}{\mathrm{d}r} + \rho_h g \Gamma_h\right) + \rho_h \beta_{S_h}\frac{\mathrm{d}S_h}{\mathrm{d}r}\right]\zeta. \tag{2.189}$$

Therefore, the force balance for the fluid particle is given by

$$\rho_h \frac{\mathrm{d}^2\zeta}{\mathrm{d}t^2} = g\left[-\rho_h \beta_{T_h}\left(\frac{\mathrm{d}T_h}{\mathrm{d}r} + \rho_h g \Gamma_h\right) + \rho_h \beta_{S_h}\frac{\mathrm{d}S_h}{\mathrm{d}r}\right]\zeta. \tag{2.190}$$

This equation has the same structure as for a harmonic oscillator, where the frequency of the oscillator N is given by

$$N^2 = g\left[\beta_{T_h}\left(\frac{\mathrm{d}T_h}{\mathrm{d}r} + \rho_h g \Gamma_h\right) - \beta_{S_h}\frac{\mathrm{d}S_h}{\mathrm{d}r}\right]. \tag{2.191}$$

This frequency is called the buoyancy frequency. The buoyancy frequency can be used as a measure for the stability of the stratification. It follows directly that the stratification is stable if $N^2 > 0$, unstable if $N^2 < 0$ and neutrally stable if $N^2 = 0$. It should be noted that another equivalent form for the buoyancy frequency is given by

$$N^2 = -\frac{g}{\rho_h}\frac{\mathrm{d}\rho_h}{\mathrm{d}r} - \frac{g^2}{c_h^2}. \tag{2.192}$$

This formulation is better to use to estimate the magnitude of the frequency. There are several dimensionless numbers that are associated with the buoyancy frequency. The most common are the stratification Froude number Fs and Burger number Bu, define as

$$Fs = \sqrt{\frac{U_{\perp,m}^2}{N_m^2 \delta L_{\|,m}^2}}, \tag{2.193}$$

$$Bu = \left(\frac{N_m \delta L_{\|,m}}{2\Omega \delta L_{\perp,m}}\right)^2 = \left(\frac{Ro}{Fs}\right)^2, \tag{2.194}$$

where the stratification Froud number is associated with the importance of stratification versus inertia and Burgers number is associated with the importance of rotation versus stratification. Note that N_m is the typical magnitude in the buoyancy frequency. For small values of Fs, stratification is important compared to inertia, and for large values of Fs, stratification is unimportant compared to inertia. When rotation is as important as stratification, then the Burger number is of order one. By introducing dimensionless numbers, the dimensionless buoyancy frequency becomes

$$\hat{N}^2 = \frac{N_{Tp,m}^2}{N_m^2}\hat{N}_{Tp}^2 + \frac{N_{S,m}^2}{N_m^2}\hat{N}_S^2, \tag{2.195}$$

where $\hat{N}_{Tp}$ is the dimensionless buoyancy frequency due to change in the mass density because of the change of temperature and pressure, and $\hat{N}_S$ is the dimensionless buoyancy frequency due to change in the mass density because of the change of salinity. The dimensionless buoyancy frequencies are define as

$$\hat{N}_{Tp}^2 = \hat{\beta}_{T_h}\left(\frac{\delta T_{h,m}}{\delta T_m}\frac{\mathrm{d}\hat{T}_h}{\mathrm{d}\hat{r}} + \left(\beta_{T,m}T_m\right)\frac{Ec}{Fr}\gamma^2\hat{\rho}_h\hat{\Gamma}_h\right), \tag{2.196}$$

$$\hat{N}_S^2 = -\hat{\beta}_{S_h}\left(\frac{\delta S_{h,m}}{\delta S_m}\frac{\mathrm{d}\hat{S}_h}{\mathrm{d}\hat{r}}\right), \tag{2.197}$$

$$\tag{2.198}$$

and their typical magnitude is

$$N_{Tp,m}^2 = \frac{g\beta_{T_h,m}\delta T_m}{\delta L_{\parallel,m}}, \tag{2.199}$$

$$N_{S,m}^2 = \frac{g\beta_{S_h,m}\delta S_m}{\delta L_{\parallel,m}}, \tag{2.200}$$

where δT_m and δS_m are the typical magnitude of the characteristic change in the full temperature and salinity, respectively.

2.6.2 The equations for the deviation from the background state

We denote respectively the mass density, pressure, temperature and salinity deviations from the background state by $\widetilde{\rho}$, $\widetilde{p}$, $\widetilde{T}$ and $\widetilde{S}$, defined as

$$\widetilde{\rho} = \rho - \rho_h, \quad \widetilde{p} = p - p_h, \quad \widetilde{T} = T - T_h, \quad \widetilde{S} = S - S_h, \tag{2.201}$$

such that the equation of state for the deviation is

$$\widetilde{\rho} = \rho\left(T, p, S\right) - \rho_h\left(T_h, p_h, S_h\right). \tag{2.202}$$

The thermodynamic evolution equation for the mass density is

$$\frac{\mathrm{d}\widetilde{\rho}}{\mathrm{d}t} = \frac{\mathrm{d}\rho}{\mathrm{d}t} - \frac{\mathrm{d}\rho_h}{\mathrm{d}t}, \tag{2.203}$$

where

$$\begin{aligned} \frac{\mathrm{d}\rho}{\mathrm{d}t} &= -\rho\beta_T\left(\frac{\mathrm{d}T}{\mathrm{d}t} - \Gamma\frac{\mathrm{d}p}{\mathrm{d}t}\right) + \rho\beta_S\frac{\mathrm{d}S}{\mathrm{d}t} + \frac{1}{c_s^2}\frac{\mathrm{d}p}{\mathrm{d}t}, && (2.204)\\ \frac{\mathrm{d}\rho_h}{\mathrm{d}t} &= -w\frac{\rho_h}{g}N^2 - \frac{w}{c_h^2}\rho_h g. && (2.205) \end{aligned}$$

Hence, the thermodynamic mass density equation can be written as

$$\frac{\mathrm{d}\widetilde{\rho}}{\mathrm{d}t} = -\rho\beta_T\left(\frac{\mathrm{d}T}{\mathrm{d}t} - \Gamma\frac{\mathrm{d}p}{\mathrm{d}t}\right) + \rho\beta_S\frac{\mathrm{d}S}{\mathrm{d}t} + w\frac{\rho_h}{g}N^2 + \frac{1}{c_s^2}\left(\frac{\mathrm{d}p}{\mathrm{d}t} + \frac{c_s^2}{c_h^2}w\rho_h g\right). \tag{2.206}$$

By introducing dimensionless numbers, the equation can be written in dimensionless form as

$$\begin{aligned} \frac{\delta\widetilde{\rho}_m}{\rho_m}\widehat{\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)}\hat{\widetilde{\rho}} &= -\frac{\hat{\beta}_T}{\hat{c}_p}\left(\hat{\rho}\hat{c}_p\widehat{\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)}\hat{T} - (\beta_{T,m}T_m)\, Eu\, Ec\hat{\beta}_T\hat{T}\widehat{\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)}\hat{p}\right)\\ &\quad + (\beta_{S,m}\delta S_m)\,\hat{\rho}\hat{\beta}_S\widehat{\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)}\hat{S} + Eu\, Ma^2\frac{1}{\hat{c}_s^2}\widehat{\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)}\hat{p}\\ &\quad + \frac{\delta\rho_{h,m}}{\rho_m}\left(Bu\, F\hat{\rho}_h\hat{N}^2 + \gamma^2\frac{Ma_h^2}{Fr}\frac{\hat{\rho}_h}{\hat{c}_h^2}\right)\hat{w} && (2.207) \end{aligned}$$

where the new dimensionless numbers are

$$F = \frac{(2\Omega)^2\,\delta L_{\perp,m}^2}{g\delta L_{\|,m}}, \quad Eu_h = \frac{\delta p_{h,m}}{\rho_m U_{\perp,m}\delta U_{\perp,m}}, \quad Ma_h = \frac{U_{\perp,m}}{c_{h,m}}. \tag{2.208}$$

respectively, the rotational Froud number, the hydrostatic Euler number and the hydrostatic Mach number. We will later find the typical magnitude for the deviations in the pressure and the mass density. By using the dimensionless heat equation and the dimensionless salinity equation, the evolution equation above can be written as

$$\begin{aligned} \frac{\delta\widetilde{\rho}_m}{\rho_m}\widehat{\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)}\hat{\widetilde{\rho}} &= \frac{\delta\rho_{h,m}}{\rho_m}\left(Bu\, F\hat{\rho}_h\hat{N}^2 + \gamma^2\frac{Ma_h^2}{Fr}\frac{\hat{\rho}_h}{\hat{c}_h^2}\right)\hat{w}\\ &\quad + Eu\, Ma^2\frac{1}{\hat{c}_s^2}\widehat{\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)}\hat{p} + H, && (2.209) \end{aligned}$$

where H is the heat source function define as

$$
\begin{aligned}
H \;=\; & -\frac{\hat{\beta}_T}{\hat{c}_p}\left[\frac{Ec}{Re}\hat{\boldsymbol{\sigma}}' : \hat{\mathsf{D}} - \frac{1}{Pe}\hat{\nabla}\cdot\hat{\mathbf{q}} - \frac{\left|\mathbf{J}_S\cdot\nabla\left(\Delta h\right)\right|_m}{\left|\rho\, c_p \mathbf{u}_\perp\cdot\nabla_\perp T\right|_m}\hat{\mathbf{J}}_S\cdot\hat{\nabla}\left(\Delta\hat{h}\right) + \hat{Q}_T^{\mathrm{turb}}\right] \\
& + \left(\beta_{S,m}\delta S_m\right)\hat{\beta}_S\left(-\frac{1}{Pe_S}\hat{\nabla}\cdot\hat{\mathbf{J}}_S + \hat{Q}_S^{\mathrm{turb}}\right). \qquad (2.210)
\end{aligned}
$$

In dimensionless form, the vertical pressure force can be written as

$$
\frac{Eu}{\gamma^2}\,\hat{\nabla}_\parallel\hat{p} = \frac{1}{\gamma^2}\left(Eu_h\,\hat{\nabla}_\parallel\hat{p}_h + \widetilde{Eu}\,\hat{\nabla}_\parallel\hat{\widetilde{p}}\right), \qquad (2.211)
$$

where we have introduced a new dimensionless number, the Euler number of the pressure deviation,

$$
\widetilde{Eu} = \frac{\delta\widetilde{p}_m}{\rho_m U_{\perp,m}\delta U_{\perp,m}}. \qquad (2.212)
$$

The body force in dimensionless form is

$$
\frac{1}{Fr}\hat{\rho}\hat{\mathbf{g}} = \frac{\rho_{h,m}}{\rho_m}\frac{1}{Fr}\hat{\rho}_h\hat{\mathbf{g}} + \frac{\widetilde{\rho}_m}{\rho_m}\frac{1}{Fr}\hat{\widetilde{\rho}}\hat{\mathbf{g}}, \qquad (2.213)
$$

such that the mass density in dimensionless form is

$$
\hat{\rho} = \frac{\rho_{h,m}}{\rho_m}\hat{\rho}_h + \frac{\widetilde{\rho}_m}{\rho_m}\hat{\widetilde{\rho}}. \qquad (2.214)
$$

The background state of the ocean satisfies the dimensionless hydrostatic balance,

$$
\mathbf{0} = -\frac{Eu_h}{\gamma^2}\,\hat{\nabla}_\parallel\hat{p}_h + \frac{\rho_{h,m}}{\rho_m}\frac{1}{Fr}\hat{\rho}_h\hat{\mathbf{g}}, \qquad (2.215)
$$

and the background pressure is independent of horizontal coordinates,

$$
\mathbf{0} = Eu_h\,\hat{\nabla}_\perp\hat{p}_h. \qquad (2.216)
$$

From equation (2.215) it follows that

$$
\frac{Eu_h}{\gamma^2}\frac{\rho_m}{\rho_{h,m}}Fr = \mathcal{O}(1), \qquad (2.217)
$$

which is equivalent to say that the characteristic magnitude of change in the hydrostatic pressure is

$$
\delta p_{h,m} = \mathcal{O}(\rho_{h,m} g\,\delta L_\parallel). \qquad (2.218)
$$

If (2.201), (2.211), (2.213), (2.215) and (2.214) are substituded into the horizontal and vertical momentum equations, (2.87) and (2.88), the momentum equations for the deviations from the background state becomes

$$
\begin{aligned}
\hat{\rho}\widehat{\left(\frac{d\mathbf{u}}{dt}\right)}_{\perp} &= -\widetilde{Eu}\,\hat{\nabla}_{\perp}\hat{\tilde{p}} + \frac{1}{Re}\left(\hat{\nabla}\cdot\hat{\boldsymbol{\sigma}}'\right)_{\perp} + \hat{\mathbf{F}}_{\perp}^{\mathrm{turb}} \\
&\quad -\frac{\gamma}{Ro}\cos\theta\hat{\rho}\widehat{\boldsymbol{\theta}}\times\hat{\mathbf{u}}_{\parallel} - \frac{1}{Ro}\sin\theta\hat{\rho}\widehat{\mathbf{r}}\times\hat{\mathbf{u}}_{\perp}, \qquad (2.219)\\
\hat{\rho}\widehat{\left(\frac{d\mathbf{u}}{dt}\right)}_{\parallel} &= -\frac{\widetilde{Eu}}{\gamma^2}\hat{\nabla}_{\parallel}\hat{\tilde{p}} + \frac{1}{Re}\left(\hat{\nabla}\cdot\hat{\boldsymbol{\sigma}}'\right)_{\parallel} + \hat{\mathbf{F}}_{\parallel}^{\mathrm{turb}} \\
&\quad -\frac{1}{Ro}\cos\theta\widehat{\boldsymbol{\theta}}\times\hat{\mathbf{u}}_{\perp} + \frac{\widetilde{\rho}_m}{\rho_m}\frac{1}{Fr}\hat{\tilde{\rho}}\hat{\mathbf{g}}. \qquad (2.220)
\end{aligned}
$$

If (2.201) and (2.215) are substituted into the heat equation, (2.89), the heat equation becomes

$$
\begin{aligned}
\frac{\delta\widetilde{T}_m}{T_m}\hat{\rho}\hat{c}_p\widehat{\left(\frac{d}{dt}\right)}\hat{\tilde{T}} &= -\hat{\rho}\hat{c}_p w\left(\frac{\delta T_{h,m}}{\delta T_m}\frac{d\hat{T}_h}{d\hat{r}} + (\beta_{T,m}T_m)\,\frac{Ec}{Fr}\gamma^2\hat{\rho}_h\hat{\Gamma}_h\right) \\
&\quad + (\beta_{T,m}T_m)\,\widetilde{Eu}\,Ec\,\hat{\beta}_T\hat{T}\widehat{\left(\frac{d}{dt}\right)}\hat{\tilde{p}} + \frac{Ec}{Re}\hat{\boldsymbol{\sigma}}' : \hat{\mathsf{D}} \\
&\quad -\frac{1}{Pe}\hat{\nabla}\cdot\hat{\mathbf{q}} - \frac{\left|\mathbf{J}_S\cdot\nabla\left(\Delta h\right)\right|_m}{\left|\rho\, c_p\mathbf{u}_{\perp}\cdot\nabla_{\perp}T\right|_m}\hat{\mathbf{J}}_S\cdot\hat{\nabla}\left(\Delta\hat{h}\right) + \hat{Q}_T^{\mathrm{turb}},
\end{aligned}
$$

where $\delta\widetilde{T}_m$ is the typical magnitude for the change in the deviation of the temperature from hydrostatic equilibrium. By using the dimensionless buoyancy frequency associated with the temperature and pressure, equation (2.196), the deviation heat equation can be written as

$$
\begin{aligned}
&\hat{\rho}\hat{c}_p\left[\frac{\delta\widetilde{T}_m}{T_m}\widehat{\left(\frac{d}{dt}\right)}\hat{\tilde{T}} + \frac{\hat{N}_{Tp}^2}{\hat{\beta}_{T_h}}w\right] - + (\beta_{T,m}T_m)\,\widetilde{Eu}\,Ec\,\hat{\beta}_T\hat{T}\widehat{\left(\frac{d}{dt}\right)}\hat{\tilde{p}} \\
&= \frac{Ec}{Re}\hat{\boldsymbol{\sigma}}' : \hat{\mathsf{D}} - \frac{1}{Pe}\hat{\nabla}\cdot\hat{\mathbf{q}} - \frac{\left|\mathbf{J}_S\cdot\nabla\left(\Delta h\right)\right|_m}{\left|\rho\, c_p\mathbf{u}_{\perp}\cdot\nabla_{\perp}T\right|_m}\hat{\mathbf{J}}_S\cdot\hat{\nabla}\left(\Delta\hat{h}\right) + \hat{Q}_T^{\mathrm{turb}}, \qquad (2.221)
\end{aligned}
$$

If (2.201) and (2.197) are substituded into the salinity equation, (2.90), the heat equation becomes

$$
\hat{\rho}\left[\frac{\delta\widetilde{S}_m}{S_m}\widehat{\left(\frac{d}{dt}\right)}\hat{S} - \frac{\hat{N}_S^2}{\hat{\beta}_{S_h}}\hat{w}\right] = -\frac{1}{Pe_S}\hat{\nabla}\cdot\hat{\mathbf{J}}_S + \hat{Q}_S^{\mathrm{turb}}, \qquad (2.222)
$$

where $\delta\widetilde{S}_m$ is the typical magnitude for the change in the deviation salinity. If (2.201), (2.215) and (2.195) are substituded into the mass continuity equation, (2.86), the mass continuity equation for the deviation mass density becomes

$$Eu\,Ma^2\left[\frac{\delta\widetilde{\rho}_m}{\rho_m}\widehat{\left(\frac{d}{dt}\right)}\hat{\widetilde{\rho}}-\frac{\delta\rho_{h,m}}{\rho_m}\left(Bu\,F\hat{\rho}_h\hat{N}^2+\gamma^2\frac{Ma_h^2}{Fr}\frac{\hat{\rho}_h}{\hat{c}_h^2}\right)\hat{w}\right]=-\hat{\rho}\hat{\nabla}\cdot\hat{\mathbf{u}}. \quad (2.223)$$

Using the continuity equation, (2.223), and the evolution equation for the equation of state, (2.209), one can find an evolution equation for the pressure;

$$\left(Eu\,Ma^2\right)^2\frac{1}{\hat{c}_s^2}\widehat{\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)}\hat{p}=-\hat{\rho}\hat{\nabla}\cdot\hat{\mathbf{u}}-Eu\,Ma^2\,H. \quad (2.224)$$

The equations derived in this section will be the basis for the remaining chapters in the thesis. Note that all equations are driven by buoyancy. Except the horizontal momentum equation.

2.7 Summary

In this chapter we have derived the general dimensionless equations of motion for a two component, one phase fluid system. The continuity equation for the deviation of the mass density around the background state is

$$Eu\,Ma^2\left[\frac{\delta\widetilde{\rho}_m}{\rho_m}\widehat{\left(\frac{d}{dt}\right)}\hat{\widetilde{\rho}}-\frac{\delta\rho_{h,m}}{\rho_m}\left(Bu\,F\hat{\rho}_h\hat{N}^2+\gamma^2\frac{Ma_h^2}{Fr}\frac{\hat{\rho}_h}{\hat{c}_h^2}\right)\hat{w}\right]=-\hat{\rho}\hat{\nabla}\cdot\hat{\mathbf{u}}, \quad (2.225)$$

the corresponding momentum equations are

$$\begin{aligned}\hat{\rho}\widehat{\left(\frac{d\mathbf{u}}{dt}\right)}_{\perp} &= -\widetilde{Eu}\,\hat{\nabla}_{\perp}\hat{\widetilde{p}}+\frac{1}{Re}\left(\hat{\nabla}\cdot\hat{\boldsymbol{\sigma}}'\right)_{\perp}+\hat{\mathbf{F}}_{\perp}^{\mathrm{turb}}\\ &\quad -\frac{\gamma}{Ro}\cos\theta\hat{\rho}\widehat{\boldsymbol{\theta}}\times\hat{\mathbf{u}}_{\parallel}-\frac{1}{Ro}\sin\theta\hat{\rho}\widehat{\mathbf{r}}\times\hat{\mathbf{u}}_{\perp}, && (2.226)\\ \hat{\rho}\widehat{\left(\frac{d\mathbf{u}}{dt}\right)}_{\parallel} &= -\frac{\widetilde{Eu}}{\gamma^2}\hat{\nabla}_{\parallel}\hat{\widetilde{p}}+\frac{1}{Re}\left(\hat{\nabla}\cdot\hat{\boldsymbol{\sigma}}'\right)_{\parallel}+\hat{\mathbf{F}}_{\parallel}^{\mathrm{turb}}\\ &\quad -\frac{1}{Ro}\cos\theta\widehat{\boldsymbol{\theta}}\times\hat{\mathbf{u}}_{\perp}+\frac{\widetilde{\rho}_m}{\rho_m}\frac{1}{Fr}\hat{\widetilde{\rho}}\hat{\mathbf{g}}, && (2.227)\end{aligned}$$

and the corresponding heat and salinity equations are

$$\hat{\rho}\hat{c}_p\left(\frac{\delta\widetilde{T}_m}{T_m}\widehat{\left(\frac{d}{dt}\right)}\hat{\widetilde{T}}+\frac{\hat{N}^2_{Tp}}{\hat{\beta}_{T_h}}\hat{w}\right)=\beta_{T,m}T_m\widetilde{Eu}\,Ec\,\hat{\beta}_T\hat{T}\widehat{\left(\frac{d}{dt}\right)}\hat{\widetilde{p}}$$
$$+\frac{Ec}{Re}\hat{\boldsymbol{\sigma}}':\hat{\mathsf{D}}-\frac{1}{Pe}\hat{\nabla}\cdot\hat{\mathbf{q}}-\frac{|\mathbf{J}_S\cdot\nabla(\Delta h)|_m}{|\rho\,c_p\mathbf{u}_\perp\cdot\nabla_\perp T|_m}\hat{\mathbf{J}}_S\cdot\hat{\nabla}\left(\Delta\hat{h}\right)+\hat{Q}_T^{\mathrm{turb}}, \tag{2.228}$$

$$\hat{\rho}\left(\frac{\delta\widetilde{S}_m}{S_m}\widehat{\left(\frac{d}{dt}\right)}\hat{S}-\frac{\hat{N}^2_S}{\hat{\beta}_{S_h}}\hat{w}\right)=-\frac{1}{Pe_S}\hat{\nabla}\cdot\hat{\mathbf{J}}_S+\hat{Q}_S^{\mathrm{turb}}, \tag{2.229}$$

where the thermodynamic evolution equation for the mass density is

$$\frac{\delta\widetilde{\rho}_m}{\rho_m}\widehat{\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)}\hat{\widetilde{\rho}} = \frac{\delta\rho_{h,m}}{\rho_m}\left(Bu\,F\hat{\rho}_h\hat{N}^2+\gamma^2\frac{Ma_h^2}{Fr}\frac{\hat{\rho}_h}{\hat{c}_h^2}\right)\hat{w}$$
$$+Eu\,Ma^2\frac{1}{\hat{c}_s^2}\widehat{\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)}\hat{p}+H. \tag{2.230}$$

Note that the mass density is given by equation (2.214). The total system of equations consists of seven equations with seven unknowns. In the next chapter we will consider the value of the dimensionless numbers for flow in the midlatitude. Based on these values, we will use perturbation theory to derive reduced model confined to the midlatitude.

Chapter 3

The dominant balance in the ocean

In this section we want to find the different time scales in the equation of motion. The method we will use is to look at the linearized motion equations, by then finding their eigen modes in the Fourier space associated with the different time scales. By writing the dimensionless numbers as the ratio between the different time and length scales, we can detect the different regimes in time and space that determines the dynamics. At the end of the chapter, we estimate the magnitude of the various dimensionless numbers for the mid-latitude at ocean.

3.1 The local equations of motion

By introducing a local slab coordinate system around the latitude θ_0, (2.154), and expand the trigonometric functions,(2.168), (2.169) and (2.170), and the radius, (2.172), to $\mathcal{O}(\Gamma)$, the momentum equations read

$$\begin{aligned}
\hat{\rho}\widehat{\frac{d\mathbf{u}_\perp}{dt}} &= -\widetilde{Eu}\,\hat{\nabla}_\perp\hat{\widetilde{p}} + \frac{1}{Re}\left(\hat{\nabla}\cdot\hat{\boldsymbol{\sigma}}'\right)_\perp + \hat{\mathbf{F}}_\perp^{\text{turb}} \\
&\quad -\frac{\gamma}{Ro_L}\left(Ro_L\frac{\beta}{\Gamma} - \Gamma\hat{Y}\right)\hat{\rho}\widehat{\mathbf{Y}}\times\hat{\mathbf{u}}_\parallel - \frac{1}{Ro_L}\left(1 + Ro_L\beta\hat{Y}\right)\hat{\rho}\widehat{\mathbf{Z}}\times\hat{\mathbf{u}}_\perp \qquad (3.1)\\
\hat{\rho}\widehat{\frac{d\mathbf{u}_\parallel}{dt}} &= -\frac{\widetilde{Eu}}{\gamma^2}\hat{\nabla}_\parallel\hat{\widetilde{p}} + \frac{1}{Re}\left(\hat{\nabla}\cdot\hat{\boldsymbol{\sigma}}'\right)_\parallel + \hat{\mathbf{F}}_\parallel^{\text{turb}} \\
&\quad -\frac{1}{Ro_L}\left(Ro_L\frac{\beta}{\Gamma} - \Gamma\hat{Y}\right)\widehat{\mathbf{Y}}\times\hat{\mathbf{u}}_\perp + \frac{\widetilde{\rho}_m}{\rho_m}\frac{1}{Fr}\hat{\widetilde{\rho}}\hat{\mathbf{g}}, \qquad (3.2)
\end{aligned}$$

where we have defined the new dimensionless numbers,

$$Ro_L = \frac{Ro}{\sin\theta_0} = \frac{U_{\perp,m}}{f_0 \delta L_{\perp,m}}, \tag{3.3}$$

$$\beta = \frac{\cos\theta_0 \Gamma}{Ro} = \beta_0 \frac{\delta L_{\perp,m}^2}{U_{\perp,m}}, \tag{3.4}$$

respectively the local Rossby and the β-number, where the β-number is associated with importance in the meridional variation of Coriolis acceleration in relation to inertia. In dimensional values, the Coriolis parameter takes the form $f = 2\Omega\sin\theta$. For the slab-approximation to first order, the Coriolis parameter can be approximated as $f = f_0 + \beta_0 y$, where $f_0 = 2\Omega\sin\theta_0$ is the Coriolis parameter evaluated at latitude θ_0 and β_0 is the the meriodional variation of the Coriolis force. The slab-approximation has reduced the inertia terms to

$$\widehat{\frac{d\mathbf{u}_\perp}{dt}} = \left(Sr\,\frac{\partial\hat{\mathbf{u}}_\perp}{\partial\hat{t}} + \hat{\mathbf{u}}\cdot\hat{\nabla}\hat{\mathbf{u}}_\perp + \gamma\Gamma\hat{w}\hat{\mathbf{u}}_\perp + \Gamma\hat{u}\tan\theta_0\,\widehat{\mathbf{Z}}\times\hat{\mathbf{u}}_\perp \right), \tag{3.5}$$

$$\widehat{\frac{d\mathbf{u}_\parallel}{dt}} = \left(Sr\,\frac{\partial\hat{\mathbf{u}}_\parallel}{\partial\hat{t}} + \hat{\mathbf{u}}\cdot\hat{\nabla}\hat{\mathbf{u}}_\parallel - \frac{\Gamma}{\gamma}\hat{\mathbf{u}}_\perp^2\widehat{\mathbf{Z}} \right), \tag{3.6}$$

and the turbulent viscosity terms to

$$\hat{\mathbf{F}}_\perp^{\mathrm{turb}} = \frac{1}{Re_\perp^{\mathrm{t}}}\hat{\rho}\hat{A}_\perp\hat{\nabla}_\perp^2\hat{\mathbf{u}}_\perp + \frac{1}{Re_\parallel^{\mathrm{t}}}\hat{\rho}\hat{A}_\parallel\hat{\nabla}_\parallel^2\hat{\mathbf{u}}_\perp, \tag{3.7}$$

$$\hat{\mathbf{F}}_\parallel^{\mathrm{turb}} = \frac{1}{Re_\perp^{\mathrm{t}}}\hat{\rho}\hat{A}_\perp\hat{\nabla}_\perp^2\hat{\mathbf{u}}_\parallel + \frac{1}{Re_\parallel^{\mathrm{t}}}\hat{\rho}\hat{A}_\perp\hat{\nabla}_\parallel^2\hat{\mathbf{u}}_\parallel, \tag{3.8}$$

where the horizontal and the vertical gradient-operators to $\mathcal{O}(\Gamma)$ are given by

$$\hat{\nabla}_\perp = \widehat{\mathbf{X}}\frac{\partial}{\partial\hat{X}} + \widehat{\mathbf{Y}}\frac{\partial}{\partial\hat{Y}}, \tag{3.9}$$

$$\hat{\nabla}_\parallel = \widehat{\mathbf{Z}}\frac{\partial}{\partial\hat{Z}}. \tag{3.10}$$

Note that the unit vectors in the slab geometry are constant, hence we have removed the vertical line. By introducing the slab-approximation to the boundary conditions (2.145), (2.146), (2.147), (2.148) and (2.149), the lower boundary condition at $\hat{Z} = h_{b,m}\,\hat{h}_b/\delta L_{\parallel,m}$ reads

$$\hat{w} = \frac{h_{b,m}}{\delta L_{\parallel,m}}\hat{\mathbf{u}}_\perp\cdot\hat{\nabla}_\perp\hat{h}_b. \tag{3.11}$$

The boundary value conditions at the free surface $\hat{Z} = 1 + \zeta_m \hat{\zeta}/L_{\|,m}$ reads

$$\begin{aligned}
\hat{w} &= \frac{\zeta_m}{\delta L_{\|,m}} \left(Sr \frac{\partial}{\partial \hat{t}} + \hat{\mathbf{u}}_\perp \cdot \hat{\nabla}_\perp \right) \hat{\zeta}, && (3.12)\\
\alpha \hat{\tau}_{XZ} &= \frac{A_{\perp,m}}{A_{\|,m}} \gamma \hat{A}_\perp \left(\gamma \widehat{\mathbf{X}} \cdot \hat{\nabla}_\perp \hat{w} - \Gamma \hat{u} \right) + \hat{A}_\| \left(\widehat{\mathbf{Z}} \cdot \hat{\nabla}_\| \right) \hat{u}, && (3.13)\\
\alpha \hat{\tau}_{YZ} &= \frac{A_{\perp,m}}{A_{\|,m}} \gamma \hat{A}_\perp \left(\gamma \widehat{\mathbf{Y}} \cdot \hat{\nabla}_\perp \hat{w} - \Gamma \hat{v} \right) + \hat{A}_\| \left(\widehat{\mathbf{Z}} \cdot \hat{\nabla}_\| \right) \hat{v}, && (3.14)\\
\hat{p} &= \hat{p}_a, && (3.15)
\end{aligned}$$

Note that the wind stress has changed subindex from spherical description to a local slab description.

3.2 Typical values for the midlatitude ocean

In this section we will calculate the typical values of the dimensionless numbers for large-scale flows. We will see that the molecular transport of momentum, heat and salt can be neglected compared to advection. We will also see that the Reynolds number is orders of magnitude greater than the typical critical value for the transition between laminar flow and turbulence. Therefore, the motion is turbulent. In table 3.1 and table 3.2 we have calculate the magnitude of the dimensionless numbers given by some typical values for the flow in the midlatitude. Where the differences in the tables are the characteristic scales in the horizontal length and velocity. This means that in table 3.1, the effect of stratification is much less important than rotation, and in table 3.2, the effect of stratification is equally important as rotation. In chapter 4 we will derive a reduced model for the midlatitude based on table3.1 and in chapter 5 we derive a reduced model for the midlatitude based on table 3.2.

For both tables, the Rossby numbers are small, so we can assume that the dominant balance in in the horizontal direction is between the horizontal pressure gradient and Coriolis force, i.e.,

$$\widetilde{Eu} Ro_L = \mathcal{O}(1), \tag{3.16}$$

where $Ro_L = U_{\perp,m}/f_0 \delta L_{\perp,m}$ is the local Rossby number given at the latitude θ_0, and $f_0 = 2\Omega \sin\theta_0$ is the corresponding Coriolis parameter at this latitude. Equation (3.16) is equivalent to say that the typical magnitude of the change in the deviation pressure is

$$\delta \widetilde{p} = \mathcal{O}(\rho_m U_{\perp,m} f_0 \delta L_{\perp,m}). \tag{3.17}$$

Parameter	$\delta L_{\perp,m}$	$\delta L_{\parallel,m}$	$U_{\perp,m}$	$A_{\perp,m}$	$A_{\parallel,m}$
Value	$10^6\,\mathrm{m}$	$10^3\,\mathrm{m}$	$10^{-2}\,\mathrm{m/s}$	$10^2 - 10^4\,\mathrm{m^2/s}$	$10^{-4} - 10^{-2}\,\mathrm{m^2/s}$
Parameter	ρ_m	τ_m	N_m	$K_{\perp,m}$	$K_{\parallel,m}$
Value	$10^3\,\mathrm{kg/m^3}$	$10^{-1}\,\mathrm{Pa}$	$10^{-2}\,\mathrm{s^{-1}}$	$10 - 10^3\,\mathrm{m^2/s}$	$10^{-5} - 10^{-4}\,\mathrm{m^2/s}$
D. number	Γ	γ	Ro	Ro_L	Ma
Value	10^{-1}	10^{-3}	10^{-4}	10^{-4}	10^{-5}
D. number	Fr	Ec	Pe_T	Pe_S	F
Value	10^{-14}	10^{-8}	10^{11}	10^{10}	10^0
D. number	β	Re	$Re^{\mathrm{t}}_{\perp}$	$Re^{\mathrm{t}}_{\parallel}$	Bu
Value	10^2	10^4	$10^2 - 10^0$	$10^2 - 10^0$	10^3
D. number	$Pe^{\mathrm{t}}_{\perp}$	$Pe^{\mathrm{t}}_{\parallel}$	α		
Value	$10^3 - 10^1$	$10^3 - 10^2$	$10^5 - 10^3$		

Table 3.1: Typical values for large-scale flows at latitue $\theta_0 = 45^o$, which correspond to a Coriolis parameter $f_0 = 10^{-4}\,\mathrm{s}^{-1}$ and a meriodional variation to $\beta_0 = 10^{-11}$ $(\mathrm{ms})^{-1}$. Note that D. is an abbreviation for dimensionless

Parameter	$\delta L_{\perp,m}$	$\delta L_{\parallel,m}$	$U_{\perp,m}$	$A_{\perp,m}$	$A_{\parallel,m}$
Value	$10^5\,\mathrm{m}$	$10^3\,\mathrm{m}$	$10^{-1}\,\mathrm{m/s}$	$10^2 - 10^4\,\mathrm{m^2/s}$	$10^{-4} - 10^{-2}\,\mathrm{m^2/s}$
Parameter	ρ_m	τ_m	N_m	$K_{\perp,m}$	$K_{\parallel,m}$
Value	$10^3\,\mathrm{kg/m^3}$	$10^{-1}\,\mathrm{Pa}$	$10^{-2}\,\mathrm{s^{-1}}$	$10 - 10^3\,\mathrm{m^2/s}$	$10^{-5} - 10^{-4}\,\mathrm{m^2/s}$
D.number	Γ	γ	Ro	Ro_L	Ma
Value	10^{-2}	10^{-2}	10^{-2}	10^{-2}	10^{-4}
D. number	Fr	Ec	Pe_T	Pe_S	F
Value	10^{-10}	10^{-6}	10^{13}	10^{12}	10^{-2}
D.	β	Re	$Re^{\mathrm{t}}_{\perp}$	$Re^{\mathrm{t}}_{\parallel}$	Bu
Value	10^0	10^5	$10^2 - 10^0$	$10^5 - 10^2$	10^0
D.	$Pe^{\mathrm{t}}_{\perp}$	$Pe^{\mathrm{t}}_{\parallel}$	α		
Value	$10^3 - 10^1$	$10^5 - 10^4$	$10^4 - 10^2$		

Table 3.2: Typical values for large-scale flows given at latitue $\theta_0 = 45^o$, which correspond to a Coriolis parameter $f_0 = 10^{-4}\,\mathrm{s}^{-1}$ and a meriodional variation to $\beta_0 = 10^{-11}$ $(\mathrm{ms})^{-1}$. Note that D. is an abbreviation for dimensionless

The main balance in the vertical direction must be between the vertical pressure force and the gravitational force, this impies that

$$\frac{\widetilde{Eu}}{\gamma^2}\frac{\rho_m}{\widetilde{\rho_m}}Fr = \mathcal{O}(1). \tag{3.18}$$

This expression gives that the typical magnitude of the change in the deviation pressure is

$$\delta\widetilde{p} = \mathcal{O}(\widetilde{\rho}_m g\delta L_{\|,m}). \tag{3.19}$$

Since (3.17) and (3.19) must be equal, it follows that the typical magnitude in the deviation mass density is

$$\widetilde{\rho}_m = \rho_m\frac{Fr}{\gamma^2 Ro_L} = \rho_m Ro_L F_L, \tag{3.20}$$

where $F_L = f_0^2\delta L_{\perp,m}^2/g\delta L_{\|,m}$ is the local rotational Froud number, which corresponds to the rotational Froud number $F = (2\Omega)^2\,\delta L_{\perp,m}^2/g\delta L_{\|,m}$. The typical magnitude in the mass denisty is the typical magnitude in the hydrostatic mass density, hence the dimensionless mass density can be written as

$$\hat{\rho} = \hat{\rho}_h + Ro_L F_L\hat{\widetilde{\rho}}. \tag{3.21}$$

The only dimensionless number that we have not calculated, is the Mach number associated with the sound speed c_h of the background state of the ocean. Measurements show that [9, p.120]

$$\frac{c_s - c_h}{c_s} \approx 0.05,$$

so that the speed of sound to the background state is approximated given by $c_h \approx 0.95c_s$. Therefore, the Mach numbers are related by

$$Ma_h = \frac{1}{0.95}Ma. \tag{3.22}$$

According to the scaling in the pressure deviation and density deviation, the slab-approximated momentum equations becomes

$$\begin{aligned} Ro_L\hat{\rho}\widehat{\frac{d\mathbf{u}_\perp}{dt}} &= -\hat{\nabla}_\perp\hat{\widetilde{p}} + \frac{Ro_L}{Re}\left(\hat{\nabla}\cdot\hat{\boldsymbol{\sigma}}'\right)_\perp + \frac{Ek_\perp}{2}\hat{\rho}\hat{A}_\perp\hat{\nabla}_\perp^2\hat{\mathbf{u}}_\perp + \frac{Ek_\|}{2}\hat{\rho}\hat{A}_\|\hat{\nabla}_\|^2\hat{\mathbf{u}}_\perp \\ &\quad -\gamma\left(Ro_L\frac{\beta}{\Gamma} - \Gamma\hat{Y}\right)\hat{\rho}\widehat{\boldsymbol{\theta}}\times\hat{\mathbf{u}}_\| - \left(1 + Ro_L\beta\hat{Y}\right)\hat{\rho}\widehat{\mathbf{r}}\times\hat{\mathbf{u}}_\perp, \end{aligned} \tag{3.23}$$

$$\begin{aligned} Ro_L\gamma^2\hat{\rho}\widehat{\frac{d\mathbf{u}_\|}{dt}} &= -\hat{\nabla}_\|\hat{\widetilde{p}} + \gamma^2\frac{Ro_L}{Re}\left(\hat{\nabla}\cdot\hat{\boldsymbol{\sigma}}'\right)_\| + \gamma^2\frac{Ek_\perp}{2}\hat{\rho}\hat{A}_\perp\hat{\nabla}_\perp^2\hat{\mathbf{u}}_\| \\ &\quad +\gamma^2\frac{Ek_\|}{2}\hat{\rho}\hat{A}_\perp\hat{\nabla}_\|^2\hat{\mathbf{u}}_\| - \gamma^2\left(Ro_L\frac{\beta}{\Gamma} - \Gamma\hat{Y}\right)\widehat{\boldsymbol{\theta}}\times\hat{\mathbf{u}}_\perp + \hat{\widetilde{\rho}}\hat{\mathbf{g}}, \end{aligned} \tag{3.24}$$

where we have used equation (3.17) and equation (3.19) and defined two new dimensionless numbers, respectively the horizontal and the vertical Ekman numbers

$$Ek_\perp = 2\frac{Ro_L}{Re_\perp^{\mathrm{t}}}, \tag{3.25}$$

$$Ek_\parallel = 2\frac{Ro_L}{Re_\parallel^{\mathrm{t}}}. \tag{3.26}$$

The Ekman number is the ratio of viscous forces to Coriolis forces. One consequence of equation (3.17) and (3.19) is that we can calculate the typical magnitude for the interface amplitude ζ. By using the dynamical boundary condition at the interface, (2.118), and use that the background pressure is hydrostatically distributed, (2.179), it follows from the scaling in the pressure deviation (3.17) that the dynamical boundary condition is

$$p_a - \rho_m g\left[Z - (H_0 + \zeta)\right] - \rho_m U_{\perp,m} f_0 \delta L_{\perp,m} \hat{\tilde{p}} = p_a.$$

Since the mean depth H_0, and the vertical coordinate Z is of $\mathcal{O}(\delta L_{\parallel,m})$ and the typical magnitude of the interface amplitude ζ is ζ_m it follows that the boundary condition can be written as

$$-\left[\hat{Z} - \left(1 + \frac{\zeta_m}{\delta L_{\parallel,m}}\hat{\zeta}\right)\right] = \frac{U_{\perp,m} f_0 \delta L_{\perp,m}}{g \delta L_{\parallel,m}} \hat{\tilde{p}}.$$

We will assume that the interface amplitude ζ is much less then the mean depth H_0, hence it follows to lowest order that $Z \approx H_0$, or in dimensionless form that $\hat{Z} \approx 1$, which gives that

$$\frac{\zeta_m}{\delta L_{\parallel,m}}\hat{\zeta} = Ro_L F_L \hat{\tilde{p}}, \tag{3.27}$$

and since $\hat{\zeta}$ and $\hat{\tilde{p}}$ are normalized such that they are of order unity it follows that

$$\frac{\zeta_m}{\delta L_{\parallel,m}} = Ro_L F_L, \tag{3.28}$$

$$\hat{\zeta} = \hat{\tilde{p}}, \tag{3.29}$$

at the interface. For midlatitude flows $Ro_L F_L$ has a maximum value of $\mathcal{O}(Ro_L)$. Thus, this is a good approximation. Equation (3.28) implies that the kinematic boundary value condition of the vertical velocity at the surface is

$$\hat{w} = Ro_L F_L \left(Sr\frac{\partial}{\partial \hat{t}} + \hat{\mathbf{u}}_\perp \cdot \hat{\nabla}_\perp\right)\hat{\zeta}. \tag{3.30}$$

The Strouhal number Sr will be assumed to of order $\mathcal{O}(1)$ such that the typical magnitude for the time is $T = \delta L_{\perp,m}/U_{\perp,m}$.

Chapter 4

The midlatitude barotropic ocean circulation model

In this chapter we want to describe an ocean circulation model that includes wind stress and topology, where the model is limited to midlatitudes. The idea is to reduce the geometry to a plane tangential to the Earth, where the origin of the plane is centered in the middle latitude. It will be seen that this follows directly from the scaling analysis. This shows the power of dimensional analysis, namely that the equations reduce to a simplified model, without losing the essential physics. The reduction of the geometry implies that the curvature disappears from the model, except in the Coriolis term. In this model, we also use that the mass density is constant, $\hat{\rho} = 1$ and $\hat{\widetilde{\rho}} = 0$, so that the fluid is not stratified and that the equations of motion disconnect from the thermodynamic equations. This means that the total mass density is only a function of pressure and vice versa. Such fluids are called barotropic. The consequence is that the flow is incompressible to all orders and that the buoyancy force in the vertical momentum equation vanishes. This is equivalent to letting the stratification Froude number Fs go to infinity such that rotation is the important effect. Accordingly, we will use the values in table 3.1 to derive a reduced model describing the barotropic dynamics. Since the local Rossby number is small, the phenomena described at this scales will be contained in the equations of motion to order Ro. Therefore, we will truncate the equations of motion of order $\mathcal{O}(Ro_L)$. According to table 3.1, the truncate momentum equations, (3.23) and (3.24), and the continuity equation, (2.225), reads

$$\begin{aligned} Ro_L \widehat{\frac{d\mathbf{u}_\perp}{dt}} &= -\hat{\nabla}_\perp \hat{\tilde{p}} + \frac{Ek_\perp}{2}\hat{\nabla}_\perp^2 \hat{\mathbf{u}}_\perp + \frac{Ek_\parallel}{2}\hat{\nabla}_\parallel^2 \hat{\mathbf{u}}_\perp \\ &\quad + \gamma\Gamma\hat{Y}\widehat{\mathbf{Y}} \times \hat{\mathbf{u}}_\parallel - \left(1 + Ro_L\beta\hat{Y}\right)\widehat{\mathbf{Z}} \times \hat{\mathbf{u}}_\perp \qquad (4.1) \\ \mathbf{0} &= \hat{\nabla}_\parallel \hat{\tilde{p}}, \qquad (4.2) \\ 0 &= \hat{\nabla}_\perp \cdot \hat{\mathbf{u}}_\perp + \hat{\nabla}_\parallel \cdot \hat{\mathbf{u}}_\parallel, \qquad (4.3) \end{aligned}$$

where we have assumed that the normalized turbulent mixing coefficients are constant and where we have splitted the divergence of the velocity field into a horizontal and vertical divergence. This is done since the unit vectors in the slab-approximation are constant. The truncated horizontal inertia term is

$$\widehat{\frac{d\mathbf{u}_\perp}{dt}} = \left(\frac{\partial \hat{\mathbf{u}}_\perp}{\partial \hat{t}} + \hat{\mathbf{u}} \cdot \hat{\nabla}\hat{\mathbf{u}}_\perp\right). \qquad (4.4)$$

Throughout the chapter we will assume that the topography h_b is of order Ro_L relative to the average depth H_0, i.e., $h_{b,m}/\delta L_{\parallel,m} \sim \mathcal{O}(Ro_L)$, hence the truncated boundary conditions, (3.11), at the lower boundary, $\hat{Z} = Ro_L\hat{h}_b$ reads

$$\hat{w} = Ro_L \hat{\mathbf{u}}_\perp \cdot \hat{\nabla}_\perp \hat{h}_b, \qquad (4.5)$$

and the truncated boundary conditions, (3.30), (3.13), (3.14) and (3.29) at the free surface $\hat{Z} = 1 + Ro_L F_L \hat{\zeta}$ reads

$$\begin{aligned} \hat{w} &= Ro_L F_L \left(\frac{\partial}{\partial \hat{t}} + \hat{\mathbf{u}}_\perp \cdot \hat{\nabla}_\perp\right)\hat{\zeta}, \qquad (4.6) \\ \alpha\hat{\tau}_{XZ} &= \frac{\partial \hat{u}}{\partial \hat{Z}}, \qquad (4.7) \\ \alpha\hat{\tau}_{YZ} &= \frac{\partial \hat{v}}{\partial \hat{Z}}, \qquad (4.8) \\ \hat{\tilde{p}} &= \hat{\zeta}. \qquad (4.9) \end{aligned}$$

Just to clarify, the nabla-operator in the slab-approximation is given by

$$\begin{aligned} \hat{\nabla}_\perp &= \widehat{\mathbf{X}}\frac{\partial}{\partial \hat{X}} + \widehat{\mathbf{Y}}\frac{\partial}{\partial \hat{Y}}, \qquad (4.10) \\ \hat{\nabla}_\parallel &= \widehat{\mathbf{Z}}\frac{\partial}{\partial \hat{Z}}. \qquad (4.11) \end{aligned}$$

In the next sections we will describe the motion of the ocean for small Rossby numbers using a regular pertubation method. This provides a systematic and physically transparent description of the motion to the desired order in the Rossby number. The method is based on an asymptotic solution in a small parameter, in this case the Rossby number. Therefore, we have to determine the order of magnitude of the other dimensionless numbers with respect to the local Rossby number Ro_L. According to table 3.1, the dimensionless numbers are related to the Rossby number by, $F_L \sim \mathcal{O}(1)$, $\beta \sim \mathcal{O}(1)$, $Ek_\perp \sim \mathcal{O}(Ro_L)$, $Ek_\parallel \sim \mathcal{O}(Ro_L)$ and $\alpha \sim \mathcal{O}(1/Ro_L)$, where we have chosen the maximum probable value of the Ekman numbers.

4.1 The asymptotic reduction

The dynamics described by equations (4.1), (4.2) and (4.3) are characterized and determined by the small local Rossby number, $Ro_L \ll 1$, such that the solution of the equations depend on this small dimensionless number, i.e.

$$\hat{\mathbf{u}}_\perp = \hat{\mathbf{u}}_\perp \left(\hat{\mathbf{X}}, \hat{t}; Ro_L\right), \quad \hat{\mathbf{u}}_\parallel = \hat{\mathbf{u}}_\parallel \left(\hat{\mathbf{X}}, \hat{t}; Ro_L\right), \quad \hat{\tilde{p}} = \hat{\tilde{p}} \left(\hat{\mathbf{X}}, \hat{t}; Ro_L\right). \tag{4.12}$$

But exact solutions to the equations cannot be found, but since $Ro_L \ll 1$ one seeks to find approximated solution. We will assume that the solution can be expanded in a regular power series in Ro_L,

$$\hat{\mathbf{u}}_\perp \left(\hat{\mathbf{X}}, \hat{t}; Ro_L\right) = \sum_{i=0}^{\infty} Ro_L^i \hat{\mathbf{u}}_{i,\perp} \left(\hat{\mathbf{X}}, \hat{t}\right), \tag{4.13}$$

$$\hat{\mathbf{u}}_\parallel \left(\hat{\mathbf{X}}, \hat{t}; Ro_L\right) = \sum_{i=0}^{\infty} Ro_L^i \hat{\mathbf{u}}_{i,\parallel} \left(\hat{\mathbf{X}}, \hat{t}\right), \tag{4.14}$$

$$\hat{\tilde{p}} \left(\hat{\mathbf{X}}, \hat{t}; Ro_L\right) = \sum_{i=0}^{\infty} Ro_L^i \hat{\tilde{p}}_i \left(\hat{\mathbf{X}}, \hat{t}\right), \tag{4.15}$$

such that $\left(\hat{\mathbf{u}}_{0,\perp}, \hat{\mathbf{u}}_{0,\parallel}, \hat{\tilde{p}}_0\right)$ are equal asymptotically to $\left(\hat{\mathbf{u}}_\perp, \hat{\mathbf{u}}_\parallel, \hat{\tilde{p}}\right)$ when $Ro_L \to 0$ and where $\left(\hat{\mathbf{u}}_{i,\perp}, \hat{\mathbf{u}}_{i,\parallel}, \hat{\tilde{p}}_i\right)$ are independent of Ro_L for each order i. By substituting the expansions (4.13), (4.14) and (4.15) into the equations (4.1), (4.2) and (4.3) and collecting part of the same order, we obtain equations for determining the dynamics to the desired order. Note that (4.1), (4.2) and (4.3) are truncated equations which are only valid up to $\mathcal{O}(Ro_L)$. If we want to find the equations of $\mathcal{O}(Ro_L^2)$ or higher, we must include higher order correction terms to (4.1), (4.2) and (4.3).

4.1.1 The geostrophic flow

To zeroth order in Ro_L, the horozintal momentum equation reduces to a balance between the zeroth order Coriolis force and the zeroth order pressure gradient,

$$\mathbf{0} = -\hat{\nabla}_{\perp}\hat{\tilde{p}}_0 - \widehat{\mathbf{Z}} \times \hat{\mathbf{u}}_{0,\perp}, \tag{4.16}$$

which is known as the geostrophic balance. By taking the cross product of $\widehat{\mathbf{Z}}$ with the zeroth order balance, (4.16), we obtain a diagnostic equation for the zeroth order horizontal velocity field $\hat{\mathbf{u}}_{0,\perp}$ given by the zeroth order pressure gradient,

$$\hat{\mathbf{u}}_{0,\perp} = \widehat{\mathbf{Z}} \times \hat{\nabla}_{\perp}\hat{\tilde{p}}_0. \tag{4.17}$$

Note that the horozontal divergence of the zeroth order horizontal velocity is divergence free, i.e.

$$\hat{\nabla}_{\perp} \cdot \hat{\mathbf{u}}_{0,\perp} = 0. \tag{4.18}$$

Helmholtz's theorem for vector fields in two dimensions, (**??**), says then that the zeroth order pressure deviation acts as a streamfunction in the horizontal plane. The vertical momentum equation (4.2) to zeroth order gives that the zeroth order pressure deviation is independent of the vertical coordinate Z,

$$\mathbf{0} = \hat{\nabla}_{\parallel}\hat{\tilde{p}}_0. \tag{4.19}$$

This implies that the zeroth order velocity field, (4.17), is also independent of the vertical coordinate. Equation (4.16) and equation (4.19) show that the zeroth order pressure deviation is undetermined to this order. The continuity equation, (4.3), to lowest order in Ro_L reads

$$0 = \hat{\nabla}_{\perp} \cdot \hat{\mathbf{u}}_{0,\perp} + \hat{\nabla}_{\parallel} \cdot \hat{\mathbf{u}}_{0,\parallel}, \tag{4.20}$$

and since the first term on the right side is constrained to be zero according to equation (4.18), it follows that the vertical divergence of the vertical velocity is zero,

$$\hat{\nabla}_{\parallel} \cdot \hat{\mathbf{u}}_{0,\parallel} = 0. \tag{4.21}$$

The lowest order boundary condition, (4.5), at the bottom $\hat{Z}_0 = 0$ is

$$\hat{w}_0 = Ro_L \hat{\mathbf{u}}_{0,\perp} \cdot \hat{\nabla}_{\perp}\hat{h}_b, \tag{4.22}$$

and the lowest order boundary conditions, (4.6), (4.7), ,(4.8) and (4.9), at the free surface $\hat{Z}_0 = 1$ are

$$\begin{aligned} \hat{w}_0 &= 0, && (4.23)\\ \alpha\hat{\tau}_{XZ} &= \frac{\partial \hat{u}_0}{\partial \hat{Z}}, && (4.24)\\ \alpha\hat{\tau}_{YZ} &= \frac{\partial \hat{v}_0}{\partial \hat{Z}}, && (4.25)\\ \hat{\tilde{p}}_0 &= \hat{\zeta}_0. && (4.26) \end{aligned}$$

From equation (4.21) and (4.23) it follows that the zeroth order vertical velocity is equal to zero, $\hat{w}_0 = 0$ everywhere in space and time. However, there is a dilemma in the boundary value conditions (4.22), (4.24) and (4.25). This is because $\hat{h}_b$, $\hat{\tau}_{XZ}$ and $\hat{\tau}_{YZ}$ are prescribed functions of $\mathcal{O}(1)$, which are given as an input to the model, and since α is of $\mathcal{O}(1/Ro_L)$ and in addition, the horizontal velocity field is independent of the vertical coordinate, the order of magnitude on the right hand side is of a different order than the left side. This means that the lowest-order equations, (4.16), (4.19), and (4.20), and expansions, (4.13), (4.14) and (4.15) are not valid to describe the dynamics of the entire interval $0 \leq Z_0 \leq 1$. So there must exist a boundary layer at $\hat{Z}_0 = 0$ and $\hat{Z}_0 = 1$ where the normalization of the vertical coordinate Z is different than initially assumed such that the boundary value conditions are correct. We will discuss this in detail in section 4.2. The fact that there exists boundary layers does not mean that equations (4.1), (4.2) and (4.3) are not valid, it just means that they are valid outside the region where there is a boundary layer. Therefore, the expansions (4.13), (4.14) and (4.15) are called an outer expansion and the solution of equations (4.1), (4.2) and (4.3) are an outer solution which describes the dynamics of the interior. Therefore, the boundary value conditions are connected to the interior via the boundary layers.

4.1.2 The ageostrophic flow

We have seen that to lowest order, the pressure deviation is undetermined. This means that we need to look at the first order equations to see if we can find an evolution equation for $\hat{\tilde{p}}_0$. To first order in Ro_L, the horizontal momentum equation is

$$\left(\frac{\partial}{\partial \hat{t}} + \hat{\mathbf{u}}_{0,\perp} \cdot \hat{\nabla}_\perp\right)\hat{\mathbf{u}}_{0,\perp} = -\hat{\nabla}_\perp \hat{\tilde{p}}_1 + \frac{1}{Re^{\mathrm{t}}_\perp}\hat{\nabla}^2_\perp \hat{\mathbf{u}}_{0\perp} - \beta\hat{Y}\widehat{\mathbf{Z}} \times \hat{\mathbf{u}}_{0,\perp} - \widehat{\mathbf{Z}} \times \hat{\mathbf{u}}_{1,\perp}. \quad (4.27)$$

This equation describes the dynamical deviation from geostrophic flow, often called ageostrofisk flow. It should be noted that, since $\hat{\mathbf{u}}_{0,\perp}$ has no vertical dependency, the advection term has been reduced to a pure advection of the horizontal velocity by itself, and similar the turbulent mixing due to shear in the horizontal velocity disappears. The vertical momentum equation to first order in Ro_L implies that also $\hat{\tilde{p}}_1$ is independent of the vertical coordinate, such as the zeroth order pressure deviation,

$$\mathbf{0} = \hat{\nabla}_\parallel \hat{\tilde{p}}_1. \quad (4.28)$$

Hence the ocean in this model is hydrostatically distributed up to at least first order in Ro_L. The continuity equation to first order in Ro_L is as expected

$$0 = \hat{\nabla}_\perp \cdot \hat{\mathbf{u}}_{1,\perp} + \hat{\nabla}_\parallel \cdot \hat{\mathbf{u}}_{1,\parallel}. \quad (4.29)$$

After we have treated the boundary layer problem, we will show how we can go to the vorticity formalism to find the equation for $\hat{\tilde{p}}_0$ from (4.27). This equation will still contain the first order velocity field in the horizontal direction, but by using the continuity equation (4.29), and the fact that the zeroth order velocity field in the horizontal direction has no vertical dependency, we can integrate the equation to eliminate the first order corrections.

4.2 The boundary layers

As discussed there must exist boundary layers at $\hat{Z}_0 = 0$ and $\hat{Z}_0 = 1$, so that the scaling is consistent. According to the boundary value conditions, (4.24) and (4.25), the vertical length scale must be rescaled in order to have balance on both sides of the equations, i.e., that the characteristic length scale in the vertical direction for the boundary layers are of a different order than the interior domain. The characteristic vertical length scale in the interior is of $\mathcal{O}(\delta L_{\parallel,m})$, hence we will assume that the characteristic vertical length scale in the boundary layer is of $\mathcal{O}(\lambda\, \delta L_{\parallel,m})$, where λ is a scaling parameter that specifies the relationship between the vertical length scale in the boundary layer and the interior. This section will consist of two subsections, the first will cover the lower boundary layer, called the bottom Ekman layer, and the second sub section will cover the upper boundary layer, called the upper Ekman layer.

4.2.1 The bottom Ekman layer

The dimensionless vertical variable in the lower boundary layer is

$$\hat{Z}^* = \frac{Z}{\lambda\, \delta L_{\parallel,m}}, \tag{4.30}$$

while the scaling in the horizontal direction will remain unchanged. Here and in the following we use a single asterisk to denote scaled variables in the bottom Ekman layer. Similar the scaling of the horizontal velocity will also ramain unchanged, but according to (2.21), the new scaling for the vertical velocity is

$$U^*_{\parallel,m} = \lambda \frac{\delta L_{\parallel,m}}{\delta L_{\perp,m}} U_{\perp,m}, \tag{4.31}$$

such that the dimensionless vertical velocity in the lower boundary layer is

$$\hat{\mathbf{u}}^*_{\parallel} = \frac{1}{U^*_{\parallel,m}} \mathbf{u}_{\parallel}. \tag{4.32}$$

Note that the index $*$ symbolizes that the variables are only valid in the boundary layer. In order to rescale (4.1), (4.2) and (4.3) to boundary layer scale, we must find a relationship between $\hat{Z}^*$ and $\hat{Z}$ and between $\hat{\mathbf{u}}_\parallel^*$ and $\hat{\mathbf{u}}_\parallel$. The relations are

$$\hat{Z}^* = \frac{1}{\lambda}\hat{Z}, \tag{4.33}$$

$$\hat{\mathbf{u}}_\parallel^* = \frac{1}{\lambda}\hat{\mathbf{u}}_\parallel, \tag{4.34}$$

where the corresponding scaled vertical nabla-operator is

$$\hat{\nabla}_\parallel^* = \lambda\hat{\nabla}_\parallel \quad \text{or} \quad \frac{\partial}{\partial \hat{Z}^*} = \lambda\frac{\partial}{\partial \hat{Z}}. \tag{4.35}$$

This leads to the rescaled equations of motion in the boundary layers,

$$Ro_L\left(\frac{\partial}{\partial \hat{t}} + \hat{\mathbf{u}}_\perp^* \cdot \hat{\nabla}_\perp + \hat{\mathbf{u}}_\parallel^* \cdot \hat{\nabla}_\parallel^*\right)\hat{\mathbf{u}}_\perp^* = -\hat{\nabla}_\perp\hat{\tilde{p}}^* + \frac{Ek_\perp}{2}\hat{\nabla}_\perp^2\hat{\mathbf{u}}_\perp^* + \frac{Ek_\parallel}{2\lambda^2}\hat{\nabla}_\parallel^{*2}\hat{\mathbf{u}}_\perp^*$$
$$+\gamma\Gamma\hat{Y}\widehat{\mathbf{Y}} \times \hat{\mathbf{u}}_\parallel^* - \left(1 + Ro_L\beta\hat{Y}\right)\widehat{\mathbf{Z}} \times \hat{\mathbf{u}}_\perp^* \tag{4.36}$$

$$\mathbf{0} = \hat{\nabla}_\parallel^*\hat{\tilde{p}}^*, \tag{4.37}$$

$$0 = \hat{\nabla}_\perp \cdot \hat{\mathbf{u}}_\perp^* + \hat{\nabla}_\parallel^* \cdot \hat{\mathbf{u}}_\parallel^*, \tag{4.38}$$

and the rescaled lower boundary condition at $\hat{Z}^* = Ro_L/\lambda\hat{h}_b$,

$$\hat{w} = \frac{Ro_L}{\lambda}\hat{\mathbf{u}}_\perp^* \cdot \hat{\nabla}_\perp\hat{h}_b. \tag{4.39}$$

The question now is: What should λ be in the boundary layer in order to satisfy the boundary condition? To get the boundary condition, (4.39), to be consistent, the horizontal turbulent mixing of momentum due to shear in the horisontal velocity $\hat{\mathbf{u}}_\perp^*$ must be of $\mathcal{O}(1)$. This implies that $Ek_\parallel/2\lambda^2 \sim \mathcal{O}(1)$ or equivalently

$$\lambda = \sqrt{Ek_\parallel}, \tag{4.40}$$

such that the characteristic length scale in vertical direction of the boundary layer is $\sqrt{Ek_\parallel}\,\delta L_{\parallel,m}$. The scaling (4.40) states that even for small values of $Ek_\parallel$ the friction will be of a significant importance and the boundary layers will be characterized as a friction layers. Similar to the outer solution, (4.12), and the outer expansions, (4.13), (4.14) and (4.15), the boundary solution (more commonly known as the inner solution) must be a function of the new boundary layer coordinate, (4.33), and the Rossby number Ro_L;

$$\hat{\mathbf{u}}_\perp^* = \hat{\mathbf{u}}_\perp^* \left(\hat{\mathbf{X}}_\perp, \hat{Z}^*, \hat{t}; Ro_L\right), \quad \hat{\mathbf{u}}_\parallel^* = \hat{\mathbf{u}}_\parallel^* \left(\hat{\mathbf{X}}_\perp, \hat{Z}^*, \hat{t}; Ro_L\right), \quad \hat{\tilde{p}}^* = \hat{\tilde{p}}^* \left(\hat{\mathbf{X}}_\perp, \hat{Z}^*, \hat{t}; Ro_L\right). \tag{4.41}$$

Just as for the outer solution there exists no general exact solution to the equations, but since $Ro_L \ll 1$ one seeks to find an approximated solution. We will assume that the solution can be expanded in a regular power series in Ro_L,

$$\hat{\mathbf{u}}_\perp^* \left(\hat{\mathbf{X}}_\perp, \hat{Z}^*, \hat{t}; Ro_L\right) = \sum_{i=0}^{\infty} Ro_L^i \hat{\mathbf{u}}_{i,\perp}^* \left(\hat{\mathbf{X}}_\perp, \hat{Z}^*, \hat{t}\right), \tag{4.42}$$

$$\hat{\mathbf{u}}_\parallel^* \left(\hat{\mathbf{X}}_\perp, \hat{Z}^*, \hat{t}; Ro_L\right) = \sum_{i=0}^{\infty} Ro_L^i \hat{\mathbf{u}}_{i,\parallel}^* \left(\hat{\mathbf{X}}_\perp, \hat{Z}^*, \hat{t}\right), \tag{4.43}$$

$$\hat{\tilde{p}}^* \left(\hat{\mathbf{X}}_\perp, \hat{Z}^*, \hat{t}; Ro_L\right) = \sum_{i=0}^{\infty} Ro_L^i \hat{\tilde{p}}_i^* \left(\hat{\mathbf{X}}_\perp, \hat{Z}^*, \hat{t}\right), \tag{4.44}$$

such that $(\hat{\mathbf{u}}_{0,\perp}^*, \hat{\mathbf{u}}_{0,\parallel}^*, \hat{\tilde{p}}_0^*)$ are equal asymptotically to $(\hat{\mathbf{u}}_\perp^*, \hat{\mathbf{u}}_\parallel^*, \hat{\tilde{p}}^*)$ when $Ro_L \to 0$, and where $(\hat{\mathbf{u}}_{n,\perp}^*, \hat{\mathbf{u}}_{n,\parallel}^*, \hat{\tilde{p}}_n^*)$ are independent of Ro_L. The asymptotical requirement consequently has the effect of stretching the region near $\hat{Z}_0 = 0$ when $Ro_L \to 0$ ($\sqrt{Ek_\parallel} \to 0$). This stretching ensures that $\hat{Z}^*$ is of $\mathcal{O}(1)$, even though the characteristic vertical length scale of the boundary layer approaches zero. By a simple discussion of the stretched boundary layer coordinate, (4.33), it follows that when $\hat{Z}$ is fixed and $\sqrt{Ek_\parallel} \to 0$, then $\hat{Z}^* \to \infty$. On the other hand, it follows that when $\hat{Z}^*$ is fixed and $\sqrt{Ek_\parallel} \to 0$, then $\hat{Z} \to 0$. By substituting the expansions (4.42), (4.43) and (4.44) into the equations (4.36), (4.37) and (4.38) and collecting terms of the same order, we obtain equations determining the dynamics in the boundary layer. To zeroth order in Ro_L, the horizontal momentum equation in the boundary layer reduces to a balance between the zeroth order Coriolis force, the zeroth order friction force and the zeroth order pressure gradient,

$$\mathbf{0} = -\hat{\nabla}_\perp \hat{\tilde{p}}_0^* + \frac{1}{2} \hat{\nabla}_\parallel^{*2} \hat{\mathbf{u}}_{0,\perp}^* - \widehat{\mathbf{Z}} \times \hat{\mathbf{u}}_{0,\perp}^*. \tag{4.45}$$

The corresponding vertical momentum equation is

$$\mathbf{0} = \hat{\nabla}_\parallel^* \hat{\tilde{p}}_0^*, \tag{4.46}$$

and the corresponding continuity equation is

$$0 = \hat{\nabla}_\perp \cdot \hat{\mathbf{u}}_{0,\perp}^* + \hat{\nabla}_\parallel^* \cdot \hat{\mathbf{u}}_{0,\parallel}^*. \tag{4.47}$$

Just as in the interior domain, the zeroth order pressure is independent of the vertical coordinate $\hat{Z}^*$. There is no obvious link between the equations of motion in the interior and the lower boundary layer, but there must exist a connection. The key to finding this connection lies in the understanding that the inner and outer expansions represent an approximation of the same physical quantities. This means that the physical quantities must be continuous in the transition region between the interior and the boundary layer and in this region the Ekman number $Ek_\parallel \to 0$. Hence, if we write the boundary layer solution in terms of of the interior coordinate $\hat{Z}$, and the interior solution in terms of the boundary layer coordinate $\hat{Z}^*$, the continuity in the transition region requires

$$\lim_{\substack{\text{fixed}\hat{Z} \\ Ek_\parallel \to 0}} \hat{\tilde{p}}\left(\hat{Z}/\sqrt{Ek_\parallel}\right) = \lim_{\substack{\text{fixed}\hat{Z}^* \\ Ek_\parallel \to 0}} \hat{\tilde{p}}^*\left(\sqrt{Ek_\parallel}\hat{Z}^*\right), \tag{4.48}$$

$$\lim_{\substack{\text{fixed}\hat{Z} \\ Ek_\parallel \to 0}} \hat{\mathbf{u}}_\perp\left(\hat{Z}/\sqrt{Ek_\parallel}\right) = \lim_{\substack{\text{fixed}\hat{Z}^* \\ Ek_\parallel \to 0}} \hat{\mathbf{u}}^*_\perp\left(\sqrt{Ek_\parallel}\hat{Z}^*\right), \tag{4.49}$$

$$\lim_{\substack{\text{fixed}\hat{Z} \\ Ek_\parallel \to 0}} \hat{\mathbf{u}}_\parallel\left(\hat{Z}/\sqrt{Ek_\parallel}\right) = \lim_{\substack{\text{fixed}\hat{Z}^* \\ Ek_\parallel \to 0}} \sqrt{Ek_\parallel}\,\hat{\mathbf{u}}^*_\parallel\left(\sqrt{Ek_\parallel}\hat{Z}^*\right). \tag{4.50}$$

To zeroth order, this requires that the pressure $\hat{\tilde{p}}^*_0$ and the horizontal velocity $\hat{\mathbf{u}}^*_{0,\perp}$ when leaving the boundary layer, $\hat{Z}^*_0 \to \infty$, is equal to the pressure $\hat{\tilde{p}}_0$ and the horizontal velocity $\hat{\mathbf{u}}_{0,\perp}$ when leaving the interior, $\hat{Z}_0 \to 0$,

$$\lim_{\hat{Z}^* \to \infty} \hat{\tilde{p}}^*_0 = \lim_{\hat{Z} \to 0} \hat{\tilde{p}}_0, \tag{4.51}$$

$$\lim_{\hat{Z}^* \to \infty} \hat{\mathbf{u}}^*_{0,\perp} = \lim_{\hat{Z} \to 0} \hat{\mathbf{u}}_{0,\perp}. \tag{4.52}$$

Therefore, the connections between the interior and the boundary layer in the transition region is

$$\hat{\nabla}_\perp \hat{\tilde{p}}^*_0 = \hat{\nabla}_\perp \hat{\tilde{p}}_0, \tag{4.53}$$

but since the pressure to lowest order is independent of the vertical coordinate in both the boundary layer and the interior, equation (4.53) is valid for all $\hat{Z}^*$ in the boundary layar. By using equation (4.45) and (4.16) together with equation (4.53), we find that the equation which connects the horizontal velocity fields is

$$\frac{1}{2}\hat{\nabla}^{*2}_\parallel \hat{\mathbf{u}}^*_{0,\perp} - \widehat{\mathbf{Z}} \times \hat{\mathbf{u}}^*_{0,\perp} = -\widehat{\mathbf{Z}} \times \hat{\mathbf{u}}_{0,\perp}. \tag{4.54}$$

This is a system of two second-order ordinary differential equations. By using the vertical Laplace operator $\hat{\nabla}^{*2}_\parallel$ on equation (4.54), the system transfors into two linear fourth-order differential equations

$$\hat{\nabla}^{*4}_\parallel \hat{\mathbf{u}}^*_{0,\perp} + 4\hat{\mathbf{u}}^*_{0,\perp} = 4\hat{\mathbf{u}}_{0,\perp}. \tag{4.55}$$

Assuming that the homogeneous solution is of the form $\mathbf{v}e^{r\hat{Z}^*}$ gives that the eigenvalues are determined by the characteristic polynomial $r^4 + 4 = 0$. The solutions are

$$r_1 = 1 + i, \quad r_2 = 1 - i, \quad r_3 = -(1 + i), \quad r_4 = -(1 - i). \tag{4.56}$$

Therefore, the general solution to (4.55) is

$$\hat{\mathbf{u}}^*_{0,\perp} = \hat{\mathbf{u}}_{0,\perp} + e^{\hat{Z}^*}\left(\mathbf{v}_1 e^{\hat{Z}^* i} + \mathbf{v}_2 e^{-\hat{Z}^* i}\right) + e^{-\hat{Z}^*}\left(\mathbf{v}_3 e^{-\hat{Z}^* i} + \mathbf{v}_4 e^{\hat{Z}^* i}\right), \tag{4.57}$$

where $\mathbf{v}_n$ is the corresponding eigenvector to the eigenvalue r_n which is determined by the boundary value conditions. Using Euler's formula, the solution, (4.57) can be written more compactly as

$$\hat{\mathbf{u}}^*_{0,\perp} = \hat{\mathbf{u}}_{0,\perp} + e^{\hat{Z}^*}\left(\mathbf{A}_1 \cos \hat{Z}^* + \mathbf{A}_2 \sin \hat{Z}^*\right) + e^{-\hat{Z}^*}\left(\mathbf{A}_3 \cos \hat{Z}^* + \mathbf{A}_4 \sin \hat{Z}^*\right), \tag{4.58}$$

where $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, and $\mathbf{A}_4$ are redefined eigenvectors. It can be shown from equation (4.58) and equation (4.54) that the eigenvectors have the structure

$$\mathbf{A}_1 = [A, -B]^{\mathrm{T}}, \quad \mathbf{A}_2 = [B, A]^{\mathrm{T}}, \quad \mathbf{A}_3 = [C, D]^{\mathrm{T}}, \quad \mathbf{A}_4 = [D, -C]^{\mathrm{T}}, \tag{4.59}$$

so the task now is to determine A, B, C and D from the boundary conditions. According to equation (4.52), the boundary layer velocity $\hat{\mathbf{u}}^*_{0,\perp}$ must merge smoothly with the geostrophic velocity $\hat{\mathbf{u}}_{0,\perp}$ in the transition region $\hat{Z}^* \to \infty$. This implies that $A = B = 0$, since these represent growing solutions. At the bottom, $\hat{Z}^* = Ro_L/\sqrt{E_\parallel}\hat{h}_b$, the boundary layer velocity must be zero, in order to satisfy the boundary conditions (2.105) and (2.107). This gives that C and D are determined by

$$\mathbf{0} = \hat{\mathbf{u}}_{0,\perp} + e^{-\frac{Ro_L}{\sqrt{Ek_\parallel}}\hat{h}_b}\left[\mathbf{A}_3 \cos\left(\frac{Ro_L}{\sqrt{Ek_\parallel}}\hat{h}_b\right) + \mathbf{A}_4 \sin\left(\frac{Ro_L}{\sqrt{Ek_\parallel}}\hat{h}_b\right)\right], \tag{4.60}$$

which gives

$$C = \left[\hat{v}_0 \sin\left(\frac{Ro_L}{\sqrt{Ek_\parallel}}\hat{h}_b\right) - \hat{u}_0 \cos\left(\frac{Ro_L}{\sqrt{Ek_\parallel}}\hat{h}_b\right)\right] e^{\frac{Ro_L}{\sqrt{Ek_\parallel}}\hat{h}_b} \tag{4.61}$$

$$D = -\left[\hat{v}_0 \cos\left(\frac{Ro_L}{\sqrt{Ek_\parallel}}\hat{h}_b\right) + \hat{u}_0 \sin\left(\frac{Ro_L}{\sqrt{Ek_\parallel}}\hat{h}_b\right)\right] e^{\frac{Ro_L}{\sqrt{Ek_\parallel}}\hat{h}_b}. \tag{4.62}$$

Thus, by substituting the constants into the general solution, we obtain the specific solution that satisfies the boundary value conditions,

$$\begin{aligned}\hat{\mathbf{u}}^*_{0,\perp} &= \hat{\mathbf{u}}_{0,\perp}\left[1-\cos\left(\hat{Z}^*-\frac{Ro_L}{\sqrt{Ek_\parallel}}\hat{h}_b\right)e^{-\left(\hat{Z}^*-\frac{Ro_L}{\sqrt{Ek_\parallel}}\hat{h}_b\right)}\right]\\ &\quad+\widehat{\mathbf{Z}}\times\hat{\mathbf{u}}_{0,\perp}\sin\left(\hat{Z}^*-\frac{Ro_L}{\sqrt{Ek_\parallel}}\hat{h}_b\right)e^{-\left(\hat{Z}^*-\frac{Ro_L}{\sqrt{Ek_\parallel}}\hat{h}_b\right)},\end{aligned}\tag{4.63}$$

This solution we can used together with (4.47) to determine the vertical velocity field in the boundary layer. By substituting (4.63) into (4.47), we get

$$\begin{aligned}&\hat{\nabla}^*_\parallel\cdot\hat{\mathbf{u}}^*_{0,\parallel}=-\hat{\nabla}_\perp\cdot\hat{\mathbf{u}}^*_{0,\perp}\\ &=\frac{Ro_L}{\sqrt{Ek_\parallel}}\left(\hat{\mathbf{u}}_{0,\perp}\cdot\hat{\nabla}_\perp\hat{h}_b\right)e^{-\left(\hat{Z}^*-\frac{Ro_L}{\sqrt{Ek_\parallel}}\hat{h}_b\right)}\left[\sin\left(\hat{Z}^*-\frac{Ro_L}{\sqrt{E_\parallel}}\hat{h}_b\right)+\cos\left(\hat{Z}^*-\frac{Ro_L}{\sqrt{Ek_\parallel}}\hat{h}_b\right)\right]\\ &+\frac{Ro_L}{\sqrt{Ek_\parallel}}\left(\widehat{\mathbf{Z}}\times\hat{\mathbf{u}}_{0,\perp}\right)\cdot\hat{\nabla}_\perp\hat{h}_b e^{-\left(\hat{Z}^*-\frac{Ro_L}{\sqrt{Ek_\parallel}}\hat{h}_b\right)}\left[\cos\left(\hat{Z}^*-\frac{Ro_L}{\sqrt{Ek_\parallel}}\hat{h}_b\right)-\sin\left(\hat{Z}^*-\frac{Ro_L}{\sqrt{Ek_\parallel}}\hat{h}_b\right)\right]\\ &+\widehat{\mathbf{Z}}\cdot\left(\hat{\nabla}_\perp\times\hat{\mathbf{u}}_{0,\perp}\right)e^{-\left(\hat{Z}^*-\frac{Ro_L}{\sqrt{Ek_\parallel}}\hat{h}_b\right)}\sin\left(\hat{Z}^*-\frac{Ro_L}{\sqrt{Ek_\parallel}}\hat{h}_b\right),\end{aligned}\tag{4.64}$$

where we have used (4.18) and $\hat{\nabla}_\perp\cdot(\widehat{\mathbf{Z}}\times\hat{\mathbf{u}}_{0,\perp})=-\widehat{\mathbf{Z}}\cdot(\hat{\nabla}_\perp\times\hat{\mathbf{u}}_{0,\perp})$. From equation (4.64) we can find the vertical velocity field in the transition region. By integrating equation (4.64) from the bottom $\hat{Z}^*=Ro_L/\sqrt{E_\parallel}\hat{h}_b$ to the transition region $\hat{Z}^*\to\infty$, we get

$$\lim_{\hat{Z}^*\to\infty}\hat{w}^*_0=\frac{Ro_L}{\sqrt{E_\parallel}}\hat{\mathbf{u}}_{0,\perp}\cdot\hat{\nabla}_\perp\hat{h}_b+\frac{1}{2}\widehat{\mathbf{Z}}\cdot\left(\hat{\nabla}_\perp\times\hat{\mathbf{u}}_{0,\perp}\right).\tag{4.65}$$

Here we have used that the velocity field is zero at the bottom. Since the vertical velocity field must be continuous in the transition region, it follows from equation (4.34), that to $\mathcal{O}(Ro_L)$ the interior velocity at $\hat{Z}_0=0$ is

$$\lim_{\hat{Z}\to0}\left(\hat{w}_0+Ro_L\hat{w}_1\right)=\sqrt{Ek_\parallel}\lim_{\hat{Z}^*\to\infty}\hat{w}^*_0\tag{4.66}$$

which implies that

$$\hat{w}_1\left(\hat{\mathbf{X}}_\perp,0,\hat{t}\right)=\hat{\mathbf{u}}_{0,\perp}\cdot\hat{\nabla}_\perp\hat{h}_b+\frac{1}{2}\frac{\sqrt{E_\parallel}}{Ro_L}\widehat{\mathbf{Z}}\cdot\left(\hat{\nabla}_\perp\times\hat{\mathbf{u}}_{0,\perp}\right).\tag{4.67}$$

4.2.2 The upper Ekman layer

Similarly to the bottom boundary layer, there must exist an upper boundary layer at $\hat{Z}_0 = 1$ due to the wind-stess. The analysis of the upper boundary layer will be very similar to the analysis of the lower boundary layer, only with some minor adjustments. In the lower boundary layer, we saw that the boundary layer coordinate, (4.33), had the property to stretch out the region around $\hat{Z}_0 = 0$. Therefore, we need to introduce a boundary layer coordinate which has the property to stretch out the region around $\hat{Z}_0 = 1$. Let $\hat{Z}^{**}$ denote this coordinate, where $**$ indicates that the variables are only valid in the upper boundary layer. Then it follows that the upper boundary layer coordinate, $\hat{Z}^{**}$, must have the following properties; when $\hat{Z}$ is fixed and $\lambda \to 0$, then $\hat{Z}^{**} \to \infty$. On the other hand, when $\hat{Z}^{**}$ is fixed and $\lambda \to 0$, then $\hat{Z} \to 1$. This implies that the upper boundary layer coordinate is given by

$$\hat{Z}^{**} = \frac{1-\hat{Z}}{\lambda}, \tag{4.68}$$

and the corresponding vertical nabla-operator is

$$\hat{\nabla}_{\parallel}^{**} = -\lambda \hat{\nabla}_{\parallel} \quad \text{or} \quad \frac{\partial}{\partial \hat{Z}^{**}} = -\lambda \frac{\partial}{\partial \hat{Z}}. \tag{4.69}$$

This leads to the rescaled equations of motion in the boundary layer,

$$\begin{aligned} Ro_L \left(\frac{\partial}{\partial \hat{t}} + \hat{\mathbf{u}}_{\perp}^{**} \cdot \hat{\nabla}_{\perp} - \hat{\mathbf{u}}_{\parallel}^{**} \cdot \hat{\nabla}_{\parallel}^{**} \right) \hat{\mathbf{u}}_{\perp}^{**} & \\ = -\hat{\nabla}_{\perp} \hat{\tilde{p}}^{**} + \frac{Ek_{\perp}}{2} \hat{\nabla}_{\perp}^2 \hat{\mathbf{u}}_{\perp}^{**} + \frac{Ek_{\parallel}}{2\lambda^2} \hat{\nabla}_{\parallel}^{**2} \hat{\mathbf{u}}_{\perp}^{**} & \\ + \gamma \Gamma \hat{Y} \widehat{\mathbf{Y}} \times \hat{\mathbf{u}}_{\parallel}^{**} - \left(1 + Ro_L \beta \hat{Y} \right) \widehat{\mathbf{Z}} \times \hat{\mathbf{u}}_{\perp}^{**}, & \qquad (4.70) \\ \mathbf{0} = \hat{\nabla}_{\parallel}^{*} \hat{\tilde{p}}^{**}, & \qquad (4.71) \\ 0 = \hat{\nabla}_{\perp} \cdot \hat{\mathbf{u}}_{\perp}^{**} - \hat{\nabla}_{\parallel}^{**} \cdot \hat{\mathbf{u}}_{\parallel}^{**}, & \qquad (4.72) \end{aligned}$$

and the rescaled boundary conditions for the free surface at$\hat{Z}^{**} = -Ro_L F_L / \lambda \hat{\zeta}^{**}$,

$$\begin{aligned} \hat{w}^{**} &= \frac{Ro_L F_L}{\lambda} \left(\frac{\partial}{\partial \hat{t}} + \hat{\mathbf{u}}_{\perp} \cdot \hat{\nabla}_{\perp} \right) \hat{\zeta}^{**}, & (4.73) \\ \alpha^{**} \hat{\tau}_{XZ} &= -\frac{\partial \hat{u}^{**}}{\partial \hat{Z}^{**}}, & (4.74) \\ \alpha^{**} \hat{\tau}_{YZ} &= -\frac{\partial \hat{v}^{**}}{\partial \hat{Z}^{**}}, & (4.75) \\ \hat{\tilde{p}}^{**} &= \hat{\zeta}^{**}, & (4.76) \end{aligned}$$

where the new rescaled dimensionless number is $\alpha^{**} = \alpha\,\lambda$. Similar to the bottom boundary layer, the scaling parameter is $\lambda = \sqrt{Ek_{\parallel}}$, since the horizontal turbulent mixing of momentum due to shear in the horisontal velocity $\hat{\mathbf{u}}_{\perp}^{**}$ must be of $\mathcal{O}(1)$ in order to satisfy the boundary value conditions. If we now apply the same analysis as for the lower boundary layer, and assume that the solution can be expanded in a regular power series in Ro_L, it follows to zeroth order that the dynamics are described by

$$\begin{aligned} \mathbf{0} &= -\hat{\nabla}_{\perp}\hat{\tilde{p}}_0^{**} + \frac{1}{2}\hat{\nabla}_{\parallel}^{**2}\hat{\mathbf{u}}_{0,\perp}^{**} - \widehat{\mathbf{Z}} \times \hat{\mathbf{u}}_{0,\perp}^{**}, & (4.77)\\ \mathbf{0} &= \hat{\nabla}_{\parallel}^{**}\hat{\tilde{p}}_0^{**}, & (4.78)\\ 0 &= \hat{\nabla}_{\perp} \cdot \hat{\mathbf{u}}_{0,\perp}^{**} - \hat{\nabla}_{\parallel}^{**} \cdot \hat{\mathbf{u}}_{0,\parallel}^{**}. & (4.79) \end{aligned}$$

As expected, the equations (4.77), (4.78) and (4.79) are exactly the same as the equations in the lower boundary layer, except the minus sign in the equation of continuity. Thus it follows that the equation that links the velocity fields in the upper boundary layer and the interior is, (4.54), where the solution is given by

$$\hat{\mathbf{u}}_{0,\perp}^{**} = \hat{\mathbf{u}}_{0,\perp} + e^{\hat{Z}^{**}}\left(\mathbf{A}_1 \cos \hat{Z}^{**} + \mathbf{A}_2 \sin \hat{Z}^{**}\right) + e^{-\hat{Z}^{**}}\left(\mathbf{A}_3 \cos \hat{Z}^{**} + \mathbf{A}_4 \sin \hat{Z}^{**}\right), \tag{4.80}$$

where $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, and $\mathbf{A}_4$ are given by the scalar constants

$$\mathbf{A}_1 = [A, -B]^{\mathrm{T}}, \quad \mathbf{A}_2 = [B, A]^{\mathrm{T}}, \quad \mathbf{A}_3 = [C, D]^{\mathrm{T}}, \quad \mathbf{A}_4 = [D, -C]^{\mathrm{T}}, \tag{4.81}$$

which must be determined from the boundary value conditions. According to equation (4.52), the boundary layer velocity $\hat{\mathbf{u}}_{0,\perp}^{**}$ must merge smoothly with the geostrophic velocity $\hat{\mathbf{u}}_{0,\perp}$ in the transition region $\hat{Z}^{**} \to \infty$. This implies that $A = B = 0$, since these represent growing solutions. At the free surface $\hat{Z}^{**} = -Ro_L F_L/\lambda\hat{\zeta}^{**}$, the horizontal boundary layer velocity must satisfy the lowest order boundary conditions,

$$\begin{aligned} \alpha^{**}\hat{\tau}_{XZ} &= -\frac{\partial \hat{u}_0^{**}}{\partial \hat{Z}^{**}}, & (4.82)\\ \alpha^{**}\hat{\tau}_{YZ} &= -\frac{\partial \hat{v}_0^{**}}{\partial \hat{Z}^{**}}, & (4.83) \end{aligned}$$

which provides that C and D are given by

$$\begin{aligned} C &= \frac{\alpha^{**}}{2}\left(\hat{\tau}_{XZ} + \hat{\tau}_{YZ}\right) & (4.84)\\ D &= \frac{\alpha^{**}}{2}\left(\hat{\tau}_{YZ} - \hat{\tau}_{XZ}\right). & (4.85) \end{aligned}$$

Thus, by substituting the constants into the general solution, we obtaine the specific solution that satisfies the boundary value conditions,

$$\hat{\mathbf{u}}_{0,\perp}^{**} = \hat{\mathbf{u}}_{0,\perp} + \frac{\alpha^{**}}{2}\left[\hat{\mathbf{T}}\left(\cos\hat{Z}^{**} - \sin\hat{Z}^{**}\right) - \widehat{\mathbf{Z}}\times\hat{\mathbf{T}}\left(\cos\hat{Z}^{**} + \sin\hat{Z}^{**}\right)\right]e^{-\hat{Z}^{**}}, \tag{4.86}$$

where $\hat{\mathbf{T}} = \hat{\tau}_{XZ}\widehat{\mathbf{X}} + \hat{\tau}_{YZ}\widehat{\mathbf{Y}}$ is the dimensionless wind-stress vector. The vertical divergence of the vertical velocity, (4.79), is given by

$$\begin{aligned}\hat{\nabla}_{\parallel}^{**}\cdot\hat{\mathbf{u}}_{0,\parallel}^{**} &= \hat{\nabla}_{\perp}\cdot\hat{\mathbf{u}}_{0,\perp}^{**}\\ &= \frac{\alpha^{**}}{2}\left[\hat{\nabla}_{\perp}\cdot\hat{\mathbf{T}}\left(\cos\hat{Z}^{**} - \sin\hat{Z}^{**}\right) + \widehat{\mathbf{Z}}\cdot\left(\hat{\nabla}_{\perp}\times\hat{\mathbf{T}}\right)\left(\cos\hat{Z}^{**} + \sin\hat{Z}^{**}\right)\right]e^{-\hat{Z}^{**}},\end{aligned} \tag{4.87}$$

where we have used (4.18) and $\hat{\nabla}_{\perp}\cdot(\widehat{\mathbf{Z}}\times\hat{\mathbf{T}}) = -\widehat{\mathbf{Z}}\cdot(\hat{\nabla}_{\perp}\times\hat{\mathbf{T}})$. From equation (4.87) we can find what the vertical velocity field in the transition region is. By integrating equation (4.87) from the free surface to the transition region $\hat{Z}^{**}\to\infty$, we get

$$\lim_{\hat{Z}^{**}\to\infty}\hat{w}_0^{**} = \frac{Ro_L F_L}{\sqrt{Ek_{\parallel}}}\left(\frac{\partial}{\partial\hat{t}} + \hat{\mathbf{u}}_{\perp}\cdot\hat{\nabla}_{\perp}\right)\hat{\tilde{p}}_0^{**} + \frac{\alpha^{**}}{2}\widehat{\mathbf{Z}}\cdot\left(\hat{\nabla}_{\perp}\times\hat{\mathbf{T}}\right), \tag{4.88}$$

where we have used the velocity field at the free surfaces is given by (4.73). Since the vertical velocity field must be continuous in the transition region, it follows from equation (4.34), that to $\mathcal{O}(Ro_L)$ the interior velocity at $\hat{Z}_0 = 1$ is

$$\lim_{\hat{Z}\to 1}\left(\hat{w}_0 + Ro_L\hat{w}_1\right) = \sqrt{Ek_{\parallel}}\lim_{\hat{Z}^{**}\to\infty}\hat{w}_0^{**}, \tag{4.89}$$

from which follows that

$$\hat{w}_1\left(\hat{\mathbf{X}}_{\perp}, 1, \hat{t}\right) = F_L\left(\frac{\partial}{\partial\hat{t}} + \hat{\mathbf{u}}_{\perp}\cdot\hat{\nabla}_{\perp}\right)\hat{\tilde{p}}_0 + \frac{\sqrt{Ek_{\parallel}}}{Ro_L}\frac{\alpha^{**}}{2}\widehat{\mathbf{Z}}\cdot\left(\hat{\nabla}_{\perp}\times\hat{\mathbf{T}}\right). \tag{4.90}$$

4.3 The barotropic quasi-geostrophic vorticity equation

According to the discussion of the geostrophic and the ageostrophic flow, the zeroth order pressure $\hat{\tilde{p}}_0$ is determined by the first-order dynamics, (4.27), (4.28) and (4.29). The only problem is that the horizontal momentum equation to order $\mathcal{O}(Ro_L)$, (4.27), depends on the first-order pressure $\hat{\tilde{p}}_1$. This means that we must eliminate the pressure $\hat{\tilde{p}}_1$ in order to find an equation to determine $\hat{\tilde{p}}_0$. The way this will be done is to use the vorticity formalism. In general the vorticity $\boldsymbol{\Theta}$ of a flow field is given by the local rotation of its velocity field $\mathbf{u}$, expressed by

$$\boldsymbol{\Theta} = \nabla \times \mathbf{u}. \tag{4.91}$$

This implies that the zeroth order nondimensional vorticity $\hat{\boldsymbol{\Theta}}_0$ associated with the geostrophic flow $\hat{\mathbf{u}}_0$ is given by

$$\hat{\boldsymbol{\Theta}}_0 = \hat{\nabla} \times \hat{\mathbf{u}}_0, \tag{4.92}$$

but since the zero-order velocity field $\hat{\mathbf{u}}_0$ only contains a horizontal component, $\hat{\mathbf{u}}_{0,\perp}$, and is independent of the vertical coordinate $\hat{Z}$, the vorticity reduces to

$$\hat{\boldsymbol{\Theta}}_0 = \hat{\nabla}_\perp \times \hat{\mathbf{u}}_{0,\perp}. \tag{4.93}$$

A consequence of this is that vorticity $\hat{\boldsymbol{\Theta}}_0$ has only a vertical component along $\widehat{\mathbf{Z}}$, which is independent of the vertical coordinate $\hat{Z}$. By using the geostrophic relation (4.17), one can show that the vorticity can be written as the laplacian of the zeroth order pressure,

$$\hat{\boldsymbol{\Theta}}_0 = \hat{\nabla}_\perp^2 \hat{\tilde{p}}_0 \, \widehat{\mathbf{Z}}. \tag{4.94}$$

Note that the advection term in the horizontal momentum equation (4.27) can be rewritten as

$$\left(\hat{\mathbf{u}}_{0,\perp} \cdot \hat{\nabla}_\perp\right) \hat{\mathbf{u}}_{0,\perp} = \hat{\boldsymbol{\Theta}}_0 \times \hat{\mathbf{u}}_{0,\perp} + \hat{\nabla}_\perp \left(\frac{\hat{\mathbf{u}}_{0,\perp} \cdot \hat{\mathbf{u}}_{0,\perp}}{2}\right), \tag{4.95}$$

with help of the vector identity

$$\mathbf{A} \times (\nabla \times \mathbf{B}) = \nabla (\mathbf{A} \cdot \mathbf{B}) - (\mathbf{A} \cdot \nabla) \mathbf{B} - (\mathbf{B} \cdot \nabla) \mathbf{A} - \mathbf{B} \times (\nabla \times \mathbf{A}). \tag{4.96}$$

This leads to an equivalent form of the horizontal momentum equation,

$$\begin{aligned} \frac{\partial \hat{\mathbf{u}}_{0,\perp}}{\partial \hat{t}} + \hat{\boldsymbol{\Theta}}_0 \times \hat{\mathbf{u}}_{0,\perp} = &- \hat{\nabla}_\perp \left(\hat{\tilde{p}}_1 + \frac{\hat{\mathbf{u}}_{0,\perp} \cdot \hat{\mathbf{u}}_{0,\perp}}{2}\right) + \frac{1}{Re_\perp^{\mathrm{t}}} \hat{\nabla}_\perp^2 \hat{\mathbf{u}}_{0\perp} \\ &- \beta \hat{Y} \widehat{\mathbf{Z}} \times \hat{\mathbf{u}}_{0,\perp} - \widehat{\mathbf{Z}} \times \hat{\mathbf{u}}_{1,\perp}, \end{aligned} \tag{4.97}$$

where all gradient fields appear explicitly. By taking the horizontal curl of (4.97) the gradient fields vanish and the remaining part is the vorticity equation,

$$\begin{aligned} \frac{\partial \hat{\boldsymbol{\Theta}}_0}{\partial \hat{t}} + \hat{\nabla}_\perp \times \left(\hat{\boldsymbol{\Theta}}_0 \times \hat{\mathbf{u}}_{0,\perp}\right) = &\frac{1}{Re_\perp^{\mathrm{t}}} \hat{\nabla}_\perp \times \left(\hat{\nabla}_\perp^2 \hat{\mathbf{u}}_{0\perp}\right) - \hat{\nabla}_\perp \times \left(\beta \hat{Y} \widehat{\mathbf{Z}} \times \hat{\mathbf{u}}_{0,\perp}\right) \\ &- \hat{\nabla}_\perp \times \left(\widehat{\mathbf{Z}} \times \hat{\mathbf{u}}_{1,\perp}\right), \end{aligned} \tag{4.98}$$

The second term on the left-hand side of the vorticity equation (4.98) can be written as

$$\hat{\nabla}_\perp \times \left(\hat{\mathbf{\Theta}}_0 \times \hat{\mathbf{u}}_{0,\perp}\right) = \hat{\mathbf{\Theta}}_0 \left(\hat{\nabla}_\perp \cdot \hat{\mathbf{u}}_{0,\perp}\right) + \left(\hat{\mathbf{u}}_{0,\perp} \cdot \hat{\nabla}_\perp\right) \hat{\mathbf{\Theta}}_0 - \hat{\mathbf{u}}_{0,\perp}\left(\hat{\nabla}_\perp \cdot \hat{\mathbf{\Theta}}_0\right) - \left(\hat{\mathbf{\Theta}}_0 \cdot \hat{\nabla}_\perp\right) \hat{\mathbf{u}}_{0,\perp}, \tag{4.99}$$

where the first term represents the stretching due to compressibility, which of course vanishes since the horizontal velocity is divergence free according to equation (4.18). The second term represents the advection of the vorticity with the horizontal velocity, and the third term is always zero since the vorticity is the curl of the velocity field. The fourth term represent the stretching due to the velocity gradient, but since the fluid flow to lowest order is confined to the horizontal plane, and independent of the vertical coordinate, there can be no stretching due to the velocity gradient, so this term also vanishes. Therefore, (4.99) reduces to a pure advection term,

$$\hat{\nabla}_\perp \times \left(\hat{\mathbf{\Theta}}_0 \times \hat{\mathbf{u}}_{0,\perp}\right) = \left(\hat{\mathbf{u}}_{0,\perp} \cdot \hat{\nabla}_\perp\right) \hat{\mathbf{\Theta}}_0. \tag{4.100}$$

The first term of the right-hand side of the vorticity equation (4.98) is a sink term which is associated with damping of vorticity. This is best seen by writing the term as a diffusion of vorticity,

$$\frac{1}{Re^{\mathrm{t}}_\perp}\hat{\nabla}_\perp \times \left(\hat{\nabla}^2_\perp \hat{\mathbf{u}}_{0\perp}\right) = \frac{1}{Re^{\mathrm{t}}_\perp}\hat{\nabla}^2_\perp \hat{\mathbf{\Theta}}_0. \tag{4.101}$$

The last two terms of the vorticity equation (4.98) represent the production of vorticity due to the Earth's rotation. These contributions can be written more compact as

$$\hat{\nabla}_\perp \times \left(\beta \hat{Y} \widehat{\mathbf{Z}} \times \hat{\mathbf{u}}_{0,\perp}\right) = \widehat{\mathbf{Z}} \left[\hat{\mathbf{u}}_{0,\perp} \cdot \hat{\nabla}_\perp \left(\beta \hat{Y}\right)\right], \tag{4.102}$$

$$\hat{\nabla}_\perp \times \left(\widehat{\mathbf{Z}} \times \hat{\mathbf{u}}_{1,\perp}\right) = \widehat{\mathbf{Z}} \left[\hat{\nabla}_\perp \cdot \hat{\mathbf{u}}_{1,\perp}\right]. \tag{4.103}$$

If we now substitute the relations (4.100), (4.101), (4.102) and (4.103) into the vorticity equation (4.98) and use the continuity equation (4.29), the vorticity equation may be written as

$$\left(\frac{\partial}{\partial \hat{t}} + \hat{\mathbf{u}}_{0,\perp} \cdot \hat{\nabla}_\perp\right) \hat{\mathbf{\Theta}}_0 = \frac{1}{Re^{\mathbf{t}}_\perp}\hat{\nabla}^2_\perp \hat{\mathbf{\Theta}}_0 - \widehat{\mathbf{Z}} \left[\hat{\mathbf{u}}_{0,\perp} \cdot \hat{\nabla}_\perp \left(\beta \hat{Y}\right)\right] + \widehat{\mathbf{Z}} \left(\hat{\nabla}_\parallel \cdot \hat{\mathbf{u}}_{1,\parallel}\right). \tag{4.104}$$

This shows that the vorticity equation associated with the zeroth order geostrophic velocity $\hat{\mathbf{u}}_{0,\perp}$, still depends on a first-order quantity, namely the first-order vertical velocity $\hat{\mathbf{u}}_{1,\parallel}$. Fortunately, the zeroth order variables are independent of $\hat{Z}$. Thus by integrating the equation over the interval $0 < \hat{Z} < 1$, everything will remain unchanged, except the integral over the vertical divergence of the vertical velocity field that is

$$\begin{aligned}\int_0^1 \hat{\nabla}_\parallel \cdot \hat{\mathbf{u}}_{1,\parallel}\,\mathrm{d}\hat{Z} &= \hat{w}_1\left(\hat{\mathbf{X}}_\perp, 1, \hat{t}\right) - \hat{w}_1\left(\hat{\mathbf{X}}_\perp, 0, \hat{t}\right) \\ &= F_L\left(\frac{\partial}{\partial\hat{t}} + \hat{\mathbf{u}}_\perp \cdot \hat{\nabla}_\perp\right)\hat{\tilde{p}}_0 + \frac{\sqrt{Ek_\parallel}}{Ro_L}\frac{\alpha^{**}}{2}\widehat{\mathbf{Z}} \cdot \left(\hat{\nabla}_\perp \times \hat{\mathbf{T}}\right) \\ &\quad - \hat{\mathbf{u}}_{0,\perp} \cdot \hat{\nabla}_\perp \hat{h}_b - \frac{1}{2}\frac{\sqrt{Ek_\parallel}}{Ro_L}\widehat{\mathbf{Z}} \cdot \left(\hat{\nabla}_\perp \times \hat{\mathbf{u}}_{0,\perp}\right), \end{aligned} \tag{4.105}$$

where we have used the boundary values conditions (4.67) and (4.90). Therefore, the integrated vorticity equation (4.104) is

$$\begin{aligned}&\left(\frac{\partial}{\partial\hat{t}} + \hat{\mathbf{u}}_{0,\perp} \cdot \hat{\nabla}_\perp\right)\left(\widehat{\mathbf{Z}} \cdot \hat{\mathbf{\Theta}}_0\right) \\ &\quad = \frac{1}{Re_\perp^{\mathrm{t}}}\hat{\nabla}_\perp^2\left(\widehat{\mathbf{Z}} \cdot \hat{\mathbf{\Theta}}_0\right) - \beta\frac{\partial\hat{\tilde{p}}_0}{\partial\hat{X}} + F_L\left(\frac{\partial}{\partial\hat{t}} + \hat{\mathbf{u}}_\perp \cdot \hat{\nabla}_\perp\right)\hat{\tilde{p}}_0 \\ &\quad\quad + \frac{\sqrt{Ek_\parallel}}{Ro_L}\frac{\alpha^{**}}{2}\widehat{\mathbf{Z}} \cdot \left(\hat{\nabla}_\perp \times \hat{\mathbf{T}}\right) - \hat{\mathbf{u}}_{0,\perp} \cdot \hat{\nabla}_\perp \hat{h}_b - \frac{1}{2}\frac{\sqrt{Ek_\parallel}}{Ro_L}\widehat{\mathbf{Z}} \cdot \left(\hat{\nabla}_\perp \times \hat{\mathbf{u}}_{0,\perp}\right), \end{aligned} \tag{4.106}$$

where we have taken the dot product with $\widehat{\mathbf{Z}}$ and used that the vorticity due to the meridional variation in the Coriolis force is given by $\hat{\mathbf{u}}_{0,\perp} \cdot \hat{\nabla}_\perp\left(\beta\hat{Y}\right) = \beta\,\hat{v}_0 = \beta\,\partial\hat{\tilde{p}}_0/\partial\hat{X}$, together with the definition of the meridional velocity $\hat{v}_0 = \left(\partial/\partial\hat{t} + \hat{\mathbf{u}}_\perp \cdot \hat{\nabla}_{0,\perp}\right)\hat{Y}$. This is done without loss of generality, since all the terms only have a component along $\widehat{\mathbf{Z}}$. To show that equation (4.106) is an evolution equation for the lowest order pressure $\hat{\tilde{p}}_0$, we use that the lowest order vorticity is given by equation (4.94) and the lowest order velocity is given by equation (4.17). The resulting equation for the pressure is

$$\begin{aligned}&\left(\frac{\partial}{\partial\hat{t}} + \hat{\mathbf{u}}_{0,\perp} \cdot \hat{\nabla}_\perp\right)\left(\hat{\nabla}_\perp^2\hat{\tilde{p}}_0 - F_L\hat{\tilde{p}}_0 + \hat{h}_b + \beta\hat{Y}\right) \\ &\quad\quad = \frac{1}{Re_\perp^{\mathrm{t}}}\hat{\nabla}_\perp^4\hat{\tilde{p}}_0 - \frac{1}{2}\frac{\sqrt{Ek_\parallel}}{Ro_L}\hat{\nabla}_\perp^2\hat{\tilde{p}}_0 + \frac{\sqrt{Ek_\parallel}}{Ro_L}\frac{\alpha^{**}}{2}\widehat{\mathbf{Z}} \cdot \left(\hat{\nabla}_\perp \times \hat{\mathbf{T}}\right). \end{aligned} \tag{4.107}$$

Since equation (4.107) is a scalar equation and contains more terms than just the vorticity on the left side, this equation is known as the barotropic quasi-geostrophic potential vorticity equation, where the quasi-geostrophic potential vorticity $\hat{q}$ is defined as

$$\hat{q} = \hat{\nabla}_\perp^2 \hat{\tilde{p}}_0 - F_L \hat{\tilde{p}}_0 + \hat{h}_b + \beta \hat{Y}. \tag{4.108}$$

According to the definition of the Poisson bracket (9.48), the advection term can be written as

$$\hat{\mathbf{u}}_{0,\perp} \cdot \hat{\nabla}_\perp = \left(\widehat{\mathbf{Z}} \times \hat{\nabla}_\perp \hat{\tilde{p}}_0 \right) \cdot \hat{\nabla}_\perp = \left\{ \hat{\tilde{p}}_0, \cdot \right\}, \tag{4.109}$$

where the Poisson bracket is given by

$$\left\{ \hat{\tilde{p}}_0, \cdot \right\} = \frac{\partial \hat{\tilde{p}}_0}{\partial \hat{X}} \frac{\partial}{\partial \hat{Y}} - \frac{\partial \hat{\tilde{p}}_0}{\partial \hat{Y}} \frac{\partial}{\partial \hat{X}}. \tag{4.110}$$

Hence, the quasi-geostrophic potential vorticity equation, (4.107) can be written in terms of the quasi-geostrophic potential vorticity and the Poisson bracket as

$$\frac{\partial \hat{q}}{\partial \hat{t}} + \left\{ \hat{\tilde{p}}_0, \hat{q} \right\} = \frac{1}{Re_\perp^{\mathbf{t}}} \hat{\nabla}_\perp^4 \hat{\tilde{p}}_0 - \frac{1}{2} \frac{\sqrt{E_\parallel}}{Ro_L} \hat{\nabla}_\perp^2 \hat{\tilde{p}}_0 + \frac{\sqrt{Ek_\parallel}}{Ro_L} \frac{\alpha^{**}}{2} \widehat{\mathbf{Z}} \cdot \left(\hat{\nabla}_\perp \times \hat{\mathbf{T}} \right). \tag{4.111}$$

4.4 Physical interpretation

In the limit where the fluid is inviscid, the bottom is flat, the meridional variation of the Coriolis force is zero ($\beta = 0$) and the rotational Froud number approaches zero, such that the charachteristic length scale in the horizontal direction is

$$\delta L_{\perp,m} \ll \sqrt{\frac{g \delta L_{\perp,m}}{4\Omega^2}}, \tag{4.112}$$

it follows that the dynamics occurs on a length scale where rotational effects becomes unimportant and the interface amplitude become very small. In this limit the potential vorticity equation, (4.111) reduces to the two dimensional Euler equation

$$\left(\frac{\partial}{\partial \hat{t}} + \hat{\mathbf{u}}_{0,\perp} \cdot \hat{\nabla}_\perp \right) \hat{\nabla}_\perp^2 \hat{\tilde{p}}_0 = 0, \tag{4.113}$$

which describes two-dimensional turbulence in incompressible fluids. Thus, it would be natural to assume that equation (4.111) has many of the same properties as equation (4.113).

Let us consider the case when equation (4.112) is fulfilled and the fluid is inviscid. In this case, the quasi-geostrophic potential vorticity equation, equation (4.111) reduces to

$$\left(\frac{\partial}{\partial \hat{t}} + \hat{\mathbf{u}}_{0,\perp} \cdot \hat{\nabla}_\perp \right) \left(\hat{\nabla}_\perp^2 \hat{\tilde{p}}_0 + \beta \hat{Y} \right) = 0, \tag{4.114}$$

such that the potential vorticity $\hat{\nabla}_{\perp}^{2}\hat{\tilde{p}}_{0} + \beta\hat{Y}$ is a conserved quantity. Thus we can study how the meridional variation of the Coriolis force will influence the vorticity. Suppose there occurs an initial disturbance in the pressure $\hat{\tilde{p}}_{0}$ at a given time along a line of constant latitude in the northern hemisphere. This will give rise to a production of vorticity, since the potential vorticity is conserved. Therefore, all displacements directed in the positive meridional direction (northward) results in a production of negative vorticity (clockwise), in order to compensate for the increase in the vorticity that is associated with the increase in the Coriolis force. On the other hand, all displacements directed in the negative meridional direction (southward) result in a production of positive vorticity (counterclockwise), in order to compensate for the decrease in the vorticity that is associated with the decrease in the Coriolis force. The vorticity produced will thus cause the disturbance to drift westward since the induced velocity field which is associated by the produced vorticity will try to "push"the disturbances against the constant line of latitude. These waves are called Rossby waves.

Let us investigate the wave propagation in more detail when the fluid is unbounded. Just to simplify the notation we set $\hat{\psi} = \hat{\tilde{p}}_{0}$, and write equation (4.114) as

$$\frac{\partial}{\partial\hat{t}}\hat{\nabla}_{\perp}^{2}\hat{\psi} + \left\{\hat{\psi}, \hat{\nabla}_{\perp}^{2}\hat{\psi}\right\} + \beta\frac{\partial\hat{\psi}}{\partial\hat{X}} = 0. \tag{4.115}$$

In order to have the disturbance/perturbation described above, there must be a background. For simplicity, we will let this background be given by a constant flow

$$\hat{\mathbf{U}} = \widehat{\mathbf{Z}} \times \hat{\nabla}_{\perp}\hat{\Psi}, \tag{4.116}$$

such that the total velocity field is given by

$$\hat{\mathbf{u}}_{0,\perp} = \hat{\mathbf{U}} + \hat{\mathbf{u}}', \tag{4.117}$$

where $\hat{\Psi}$ is the background pressure and $\hat{\mathbf{u}}'$ is the perturbed velocity that correspond to the perturbed pressure $\hat{\psi}'$. This means that the total pressure and the perturbed velocity are given respectively as

$$\hat{\psi} = \hat{\Psi} + \epsilon\,\hat{\psi}' \tag{4.118}$$

and

$$\hat{\mathbf{u}} = \epsilon\widehat{\mathbf{Z}} \times \hat{\nabla}_{\perp}\hat{\psi}' \tag{4.119}$$

where ϵ is a formal ordering parameter which representing that the perturbation is small compared with the background. The background pressure $\hat{\Psi}$ must be a linear function in $\hat{X}$ and $\hat{Y}$ to satisfy the condition that the background velocity is constant. Thus, we choose that background pressure is $\hat{\Psi} = -V\hat{Y}$, such that the the background flow is zonal and constant. If we now substitute equation (4.118) into equation (4.115) and linearize to first order in ϵ, we get the evolution equation for the perturbation

$$\frac{\partial}{\partial \hat{t}} \nabla_\perp^2 \hat{\psi}' + V \frac{\partial}{\partial \hat{X}} \nabla_\perp^2 \hat{\psi}' + \beta \frac{\partial \hat{\psi}'}{\partial \hat{X}} = 0. \tag{4.120}$$

By assuming a plane wave solution of the form

$$\hat{\psi}' = \hat{A}\, e^{i\left(\hat{\mathbf{k}}_\perp \cdot \hat{\mathbf{X}}_\perp - \hat{\omega}\, \hat{t}\right)}, \tag{4.121}$$

where $\hat{\mathbf{k}}_\perp = \hat{k}_X \widehat{\mathbf{X}} + \hat{k}_Y \widehat{\mathbf{Y}}$ is the horizontal wave vector and $\hat{\omega}$ is the angular frequency, equation (4.120) reduces to an algebraic equation

$$\left[\left(-\hat{\omega} + V\, \hat{k}_X\right) \left|\hat{\mathbf{k}}_\perp\right|^2 - \beta \hat{k}_X\right] A = 0. \tag{4.122}$$

A non-trivial solution implies that the expression inside the square brackets is zero, hence the dispersion relation for the Rossby waves is

$$\hat{\omega} = V\, \hat{k}_X - \beta \frac{\hat{k}_X}{\left|\hat{\mathbf{k}}_\perp\right|^2}, \tag{4.123}$$

and the zonal phase velocity is

$$\hat{\mathbf{c}}_p = \frac{\hat{\omega}}{\hat{k}_X} \widehat{\mathbf{X}} = \left(V - \beta \frac{1}{\left|\hat{\mathbf{k}}_\perp\right|^2}\right) \widehat{\mathbf{X}}. \tag{4.124}$$

Note that the presence of the zonal flow have introduced a Doppler shift in the frequency and a Galilean transformation in the phase velocity. The Rossby waves will propagates westward as long as $\beta / \left|\hat{\mathbf{k}}_\perp\right|^2 > V$. Of course, in the absence of the zonal flow, all Rossby waves will propagates westward as discussed previously. An interesting case is Rossby waves with wave numbers $\left|\hat{\mathbf{k}}_\perp\right|^2 = \beta / V$, these waves will be stationary Rossby waves.

Let us now consider the case when the interface amplitude contributes to the potential vorticity, then the potential vorticity equation becomes

$$\left(\frac{\partial}{\partial \hat{t}} + \hat{\mathbf{u}}_{0,\perp} \cdot \hat{\nabla}_\perp\right) \left(\hat{\nabla}_\perp^2 \hat{\psi} - F_L \hat{\psi} + \beta \hat{Y}\right) = 0, \tag{4.125}$$

or in terms of Possion bracket notaion

$$\frac{\partial}{\partial \hat{t}}\left(\hat{\nabla}_{\perp}^{2}\hat{\psi} - F_L\hat{\psi}\right) + \left\{\hat{\psi}, \hat{\nabla}_{\perp}^{2}\hat{\psi}\right\} + \beta\frac{\partial \hat{\psi}}{\partial \hat{X}} = 0. \tag{4.126}$$

If we assume the same disturbance that without the contribution of the surface amplitude $\zeta = \hat{\psi}$, it follows that the same mechanism will produce vorticity, but in addition the local increase (or decrease) in the surface hight ζ will give rise to a local increase (or decrease) in the pressure $\hat{\tilde{p}}_0$. Therefore, all displacements directed in the positive meridional direction (northward) will have larger value of pressure than the local surroundings, which will set up a clockwise geostrophic flow. On the other hand, all displacements directed in the negative meridional direction (southward) will have less value in the pressure $\hat{\tilde{p}}_0$ than the the local surroundings, which will set up a counterclockwise geostrophic flow. These two effects provide that $\hat{\nabla}_{\perp}^{2}\hat{\psi} - F_L\hat{\psi} < 0$, where $\hat{\nabla}_{\perp}^{2}\hat{\psi} < 0$ and $\hat{\psi} > 0$ in the positive meridional direction and vice versa in the negative meridional direction. Once again the produced vorticity in addition to the surface amplitude will thus cause the disturbance to drift westward since the induced velocity field will try to "push"the disturbances against the constant line of latitude. Similar as above, we will take a closer look at the wave propagation properties.

If we use the same background as above and perturb the system around this basis state, the linearized version of equation (4.126) to first order in ϵ becomes

$$\frac{\partial}{\partial \hat{t}}\left(\hat{\nabla}_{\perp}^{2}\hat{\psi}' - F_L\hat{\psi}'\right) + V\frac{\partial}{\partial \hat{X}}\hat{\nabla}_{\perp}^{2}\hat{\psi}' + \beta\frac{\partial \hat{\psi}'}{\partial \hat{X}} = 0. \tag{4.127}$$

This equation corresponds to the dispersion relation

$$\hat{\omega} = V\,\hat{k}_X - \hat{k}_X\frac{V\,F_L + \beta}{F_L + \left|\hat{\mathbf{k}}_{\perp}\right|^2}, \tag{4.128}$$

and the zonal phase velocity

$$\hat{\mathbf{c}}_p = \left(V - \frac{V\,F_L + \beta}{F_L + \left|\hat{\mathbf{k}}_{\perp}\right|^2}\right)\widehat{\mathbf{X}}, \tag{4.129}$$

In the absence of the zonal background velocity, equation (4.128) is very similar to the dispersion relation (4.123) except that the denominator has an additional term because of the free surface. The main difference is that the phase shift due to the zonal flow is not longer uniform, which implies that the phase velocity of the

Rossby waves actually depends on the magnitude of the zonal background flow. But the value of the zonal flow V required to provide a stationary wave is the same.

Finally, we look at the effect of bottom topography. In this case the potential vorticity equation becomes

$$\left(\frac{\partial}{\partial \hat{t}} + \hat{\mathbf{u}}_{0,\perp} \cdot \hat{\nabla}_{\perp}\right) \left(\hat{\nabla}_{\perp}^{2} \hat{\psi} - F_L \hat{\psi} + \hat{h}_b + \beta \hat{Y}\right) = 0, \tag{4.130}$$

or equivalent in terms of Possion bracket notation

$$\frac{\partial}{\partial \hat{t}} \left(\hat{\nabla}_{\perp}^{2} \hat{\psi} - F_L \hat{\psi}\right) + \left\{\hat{\psi}, \hat{\nabla}_{\perp}^{2} \hat{\psi} + \hat{h}_b\right\} + \beta \frac{\partial \hat{\psi}}{\partial \hat{X}} = 0. \tag{4.131}$$

Let us for simplicity assume that the bottom only varies in the meridional direction, such that the variation is almost constant over one wavelength, i.e. $\partial \hat{h}_b / \partial \hat{Y}$ will be treated as constant and $\partial \hat{h}_b / \partial \hat{X} = 0$. For this case, the linearized version of equation (4.131) to first order in ϵ becomes

$$\frac{\partial}{\partial \hat{t}} \left(\hat{\nabla}_{\perp}^{2} \hat{\psi}' - F_L \hat{\psi}'\right) + V \frac{\partial}{\partial \hat{X}} \hat{\nabla}_{\perp}^{2} \hat{\psi}' + \left(\beta + \frac{\partial \hat{h}_b}{\partial \hat{Y}}\right) \frac{\partial \hat{\psi}'}{\partial \hat{X}} = 0. \tag{4.132}$$

This equation is almost the same equation as equation (4.127). The only difference is that the last term on the right side of equation (4.132) has a correction due to the variation in the bottom. Thus, the meridional variation of the bottom has the same properties as the meridional variation of the Coriolis force, such that $\hat{h}_b$ will actually give rise to an artificial β-plane effect.

Chapter 5

The midlatitude baroclinic ocean circulation model

In this chapter we want to modify the dynamics described in previous chapter by including the effect of stratification. When we derive this model we will use the length scale that is described in Table 3.2. At this length scale the Burger number Bu is of $\mathcal{O}(1)$. This means that the effect of stratification is equally important as rotation. Similar to the barotropic model, this model will describe dynamics at the midlatitude where the local Rossby number Ro_L is small. Therefore, the dynamics of interest may be of $\mathcal{O}(Ro_L)$ and we will only consider of terms of order $\mathcal{O}(Ro_L)$ in the equations. Since the dynamics are at a length scale that is much smaller than Earth's mean radius, i.e., $\Gamma \sim \mathcal{O}(Ro_L)$, we introduce slab-coordinates where the origin is located in the midlatitude. This implies that to first order, the curvature of the earth will disappear from the model except from the Coriolis force. Where there will be a contribution from the meridional variation of Coriolis force (the β number). The truncated equations of motion to $\mathcal{O}(Ro_L)$ are

$$
\begin{aligned}
Ro_L \widehat{\frac{d}{dt}} \hat{\mathbf{u}}_\perp &= -\frac{1}{\hat{\rho}_h + Ro_L F_L \hat{\tilde{\rho}}} \hat{\nabla}_\perp \hat{\tilde{p}} + \frac{Ek_\perp}{2} \hat{\nabla}_\perp^2 \hat{\mathbf{u}}_\perp + \frac{Ek_\parallel}{2} \hat{\nabla}_\parallel^2 \hat{\mathbf{u}}_\perp \\
&\quad -\frac{\gamma}{\Gamma} Ro_L \hat{Y} \widehat{\mathbf{Y}} \times \hat{\mathbf{u}}_\parallel - \left(1 + Ro_L \beta \hat{Y}\right) \widehat{\mathbf{Z}} \times \hat{\mathbf{u}}_\perp && (5.1) \\
\mathbf{0} &= -\hat{\nabla}_\parallel \hat{\tilde{p}} + \hat{\tilde{\rho}} \hat{\mathbf{g}}, && (5.2) \\
0 &= \hat{\nabla}_\perp \cdot \hat{\mathbf{u}}_\perp + \hat{\nabla}_\parallel \cdot \hat{\mathbf{u}}_\parallel, && (5.3)
\end{aligned}
$$

and the truncated thermodynamic equations are

$$
\begin{aligned}
Ro_L F_L \widehat{\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)} \hat{\tilde{\rho}} &= Bu_L\, F_L \hat{\rho}_h \hat{N}^2 \hat{w} \\
&\quad + Eu\, Ma^2 \frac{1}{\hat{c}_s^2} \widehat{\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)} \hat{\tilde{p}} + H \qquad (5.4)\\
\hat{\rho}\hat{c}_p \left(\frac{\delta \widetilde{T}_m}{T_m} \widehat{\left(\frac{d}{dt}\right)} \hat{\tilde{T}} + \frac{\hat{N}_{Tp}^2}{\hat{\beta}_{T_h}} \hat{w} \right) &= \frac{1}{Pe_\perp^{\mathrm{t}}} \hat{\nabla}_\perp^2 \hat{\tilde{T}} + \frac{1}{Pe_\parallel^{\mathrm{t}}} \hat{\nabla}_\parallel^2 \hat{\tilde{T}}, \qquad (5.5)\\
\hat{\rho} \left(\frac{\delta \widetilde{S}_m}{S_m} \widehat{\left(\frac{d}{dt}\right)} \hat{\tilde{S}} - \frac{\hat{N}_S^2}{\hat{\beta}_{S_h}} \hat{w} \right) &= \frac{1}{Pe_\perp^{\mathrm{t}}} \hat{\nabla}_\perp^2 \hat{\tilde{S}} + \frac{1}{Pe_\parallel^{\mathrm{t}}} \hat{\nabla}_\parallel^2 \hat{\tilde{S}}^{\mathrm{turb}}, \qquad (5.6)
\end{aligned}
$$

where the truncate heat source function to the mass density deviation is

$$
H = -\frac{\hat{\beta}_T}{\hat{c}_p} \left(\frac{1}{Pe_\perp^{\mathrm{t}}} \hat{\nabla}_\perp^2 \hat{\tilde{T}} + \frac{1}{Pe_\parallel^{\mathrm{t}}} \hat{\nabla}_\parallel^2 \hat{\tilde{T}} \right) \qquad (5.7)
$$

Note that we have used $Bu\, F = Bu_L\, F_L$ where Bu_L is the local Burger number given by

$$
Bu_L = \left(\frac{N_m \delta L_{\parallel,m}}{f_0 \delta L_{\perp,m}} \right)^2. \qquad (5.8)
$$

The truncated inertia term is

$$
\widehat{\frac{d}{dt}} = \left(\frac{\partial}{\partial \hat{t}} + \hat{\mathbf{u}} \cdot \hat{\nabla} \right). \qquad (5.9)
$$

Throughout this chapter we will assume that the topography h_b is of order Ro_L relative to the average depth H_0, i.e. $h_{b,m}/\delta L_{\parallel,m} \sim \mathcal{O}(Ro_L)$, hence the truncated boundary conditions, (3.11), at the lower boundary, $\hat{Z} = Ro_L \hat{h}_b$ reads

$$
\hat{w} = Ro_L \hat{\mathbf{u}}_\perp \cdot \hat{\nabla}_\perp \hat{h}_b, \qquad (5.10)
$$

and the truncated boundary conditions, (3.30), (3.13), (3.14) and (3.29) at the free surface $\hat{Z} = 1 + Ro_L F_L \hat{\zeta}$ reads

$$
\begin{aligned}
\hat{w} &= Ro_L F_L \left(\frac{\partial}{\partial \hat{t}} + \hat{\mathbf{u}}_\perp \cdot \hat{\nabla}_\perp \right) \hat{\zeta}, \qquad (5.11)\\
\alpha \hat{\tau}_{XZ} &= \frac{\partial \hat{u}}{\partial \hat{Z}}, \qquad (5.12)\\
\alpha \hat{\tau}_{YZ} &= \frac{\partial \hat{v}}{\partial \hat{Z}}, \qquad (5.13)\\
\hat{\tilde{p}} &= \hat{\tilde{p}}_a. \qquad (5.14)
\end{aligned}
$$

where $\hat{\widetilde{p}}_a$ is the pressure deviation of the atmospheric pressure at the interface. Just to clarify, the nabla-operator in the slab-approximation is given by

$$\hat{\nabla}_\perp = \widehat{\mathbf{X}}\frac{\partial}{\partial \hat{X}} + \widehat{\mathbf{Y}}\frac{\partial}{\partial \hat{Y}}, \tag{5.15}$$

$$\hat{\nabla}_\parallel = \widehat{\mathbf{Z}}\frac{\partial}{\partial \hat{Z}}. \tag{5.16}$$

Similar as in the previous chapter we will use a regular pertubation method to determine the dynamics of the zeroth- and first-order variables based on the small parameter Ro_L. Therefore, we have to determine the order of magnitude of the other dimensionless numbers with respect to the local Rossby number Ro_L. According to Table 3.2, the dimensionless numbers are related to the Rossby number by; $F_L \sim \mathcal{O}(Ro_L)$, $\beta \sim \mathcal{O}(1)$, $Ek_\perp \sim \mathcal{O}(Ro_L)$, $Ek_\parallel \sim \mathcal{O}(Ro_L^2)$, $Bu_L \sim \mathcal{O}(1)$, $Ma \sim \mathcal{O}(Ro_L^2)$, $Eu \sim \mathcal{O}(1/Ro_L^3)$, $Pe_\perp^{\mathrm{t}} \sim \mathcal{O}(1/Ro_L^2)$, $Pe_\parallel^{\mathrm{t}} \sim \mathcal{O}(1/Ro_L^2)$ and $\alpha \sim \mathcal{O}(1/Ro_L)$, but the magnitude of $\delta\widetilde{T}_m/T_m$ and $\delta\widetilde{S}_m/S_m$ are not known. Since the salinity deviation is not included in the thermodynamic equation for the mass density, this equation is disconnected from the description. For the mass density, we have that $\delta\widetilde{\rho}_m/\rho_m = Ro_L F_L$, so we will assume that this also applies for the temperature, i.e., $\delta\widetilde{T}_m/T_m = Ro_L F_L$. We have a dilemma in the equations, since the fluid is incompressible to $\mathcal{O}(Ro_L^2)$, should this imply that sound waves are not included in the description. However, The thermodynamic equation for the mass density, (5.4), contains a pressure term that is associated with compression. This term has to be neglected in order to obtain a consistent model. The beauty of neglecting the pressure term is that the density varies only as a consequence of changes in the temperature and advection with the background density. Since the fluid in addition is shallow, i.e. $\gamma \sim \mathcal{O}(Ro_L)$, we will assume that the background density is constant, $\hat{\rho}_h = 1$, together with the thermodynamic coefficients , e.g. $\hat{c}_p = \hat{\beta}_T = 1$. Hence, the reduced continuity and the momentum equations are given by

$$\begin{aligned} Ro_L \widehat{\frac{d}{dt}}\hat{\mathbf{u}}_\perp &= -\frac{1}{1 + Ro_L F_L \hat{\widetilde{\rho}}}\hat{\nabla}_\perp \hat{\widetilde{p}} + \frac{Ek_\perp}{2}\hat{\nabla}_\perp^2 \hat{\mathbf{u}}_\perp + \frac{Ek_\parallel}{2}\hat{\nabla}_\parallel^2 \hat{\mathbf{u}}_\perp \\ &\quad -\frac{\gamma}{\Gamma} Ro_L \hat{Y}\widehat{\mathbf{Y}} \times \hat{\mathbf{u}}_\parallel - \left(1 + Ro_L \beta \hat{Y}\right)\widehat{\mathbf{Z}} \times \hat{\mathbf{u}}_\perp \end{aligned} \tag{5.17}$$

$$\mathbf{0} = -\hat{\nabla}_\parallel \hat{\widetilde{p}} + \hat{\widetilde{\rho}}\hat{\mathbf{g}}, \tag{5.18}$$

$$0 = \hat{\nabla}_\perp \cdot \hat{\mathbf{u}}_\perp + \hat{\nabla}_\parallel \cdot \hat{\mathbf{u}}_\parallel, \tag{5.19}$$

and the reduced thermodynamical equations are

$$Ro_L F_L \widehat{\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)} \hat{\tilde{\rho}} = Bu_L\, F_L \hat{N}^2 \hat{w} - \frac{1}{Pe^{\mathrm{t}}_{\perp}} \hat{\nabla}^2_{\perp} \hat{\tilde{T}} - \frac{1}{Pe^{\mathrm{t}}_{\parallel}} \hat{\nabla}^2_{\parallel} \hat{\tilde{T}} \tag{5.20}$$

$$Ro_L F_L \widehat{\left(\frac{d}{dt}\right)} \hat{\tilde{T}} + \hat{N}^2_{Tp} \hat{w} = \frac{1}{1 + Ro_L F_L \hat{\tilde{\rho}}} \left(\frac{1}{Pe^{\mathrm{t}}_{\perp}} \hat{\nabla}^2_{\perp} \hat{\tilde{T}} + \frac{1}{Pe^{\mathrm{t}}_{\parallel}} \hat{\nabla}^2_{\parallel} \hat{\tilde{T}} \right). \tag{5.21}$$

Note that the truncated equations indeed satisfy the Boussinesq approximation because the density deviation is so small that it can be neglected to first order, except in the buoyancy term (gravity term). This means that although the Boussinesq approximation has the property that it filters out sound waves, it still exist a compressibility term in the equation of state. Therefore one should have introduced additional requirements that the Boussinesq approximation is only valid when the sound speed is infinitely large.

5.1 The asymptotic reduction

Similar to the previous chapter, we will assume that the solutions of equation (5.17), (5.18), (5.19), (5.20) and (5.21) can be expanded in a regular power series in Ro_L,

$$\hat{\mathbf{u}}_{\perp}\left(\hat{\mathbf{X}}, \hat{t}; Ro_L\right) = \sum_{i=0}^{\infty} Ro_L^i \hat{\mathbf{u}}_{i,\perp}\left(\hat{\mathbf{X}}, \hat{t}\right), \tag{5.22}$$

$$\hat{\mathbf{u}}_{\parallel}\left(\hat{\mathbf{X}}, \hat{t}; Ro_L\right) = \sum_{i=0}^{\infty} Ro_L^i \hat{\mathbf{u}}_{i,\parallel}\left(\hat{\mathbf{X}}, \hat{t}\right), \tag{5.23}$$

$$\hat{\tilde{p}}\left(\hat{\mathbf{X}}, \hat{t}; Ro_L\right) = \sum_{i=0}^{\infty} Ro_L^i \hat{\tilde{p}}_i\left(\hat{\mathbf{X}}, \hat{t}\right), \tag{5.24}$$

$$\hat{\tilde{\rho}}\left(\hat{\mathbf{X}}, \hat{t}; Ro_L\right) = \sum_{i=0}^{\infty} Ro_L^i \hat{\tilde{\rho}}_i\left(\hat{\mathbf{X}}, \hat{t}\right), \tag{5.25}$$

$$\hat{\tilde{T}}\left(\hat{\mathbf{X}}, \hat{t}; Ro_L\right) = \sum_{i=0}^{\infty} Ro_L^i \hat{\tilde{T}}_i\left(\hat{\mathbf{X}}, \hat{t}\right), \tag{5.26}$$

such that $\left(\hat{\mathbf{u}}_{0,\perp}, \hat{\mathbf{u}}_{0,\parallel}, \hat{\tilde{p}}_0, \hat{\tilde{\rho}}_0, \hat{\tilde{T}}_0\right)$ are equal asymptotically to $\left(\hat{\mathbf{u}}_{\perp}, \hat{\mathbf{u}}_{\parallel}, \hat{\tilde{p}}, \hat{\tilde{\rho}}, \hat{\tilde{T}}\right)$ when $Ro_L \to 0$ and where $\left(\hat{\mathbf{u}}_{n,\perp}, \hat{\mathbf{u}}_{n,\parallel}, \hat{\tilde{p}}_n, \hat{\tilde{\rho}}_n, , \hat{\tilde{T}}_n\right)$ are independent of Ro_L. By substituting the expansions (5.22), (5.23), (5.24), (5.25) and (5.26) into the equations (5.17), (5.18), (5.19), (5.20) and (5.21), and collecting parts of the same order, we obtain equations determining the dynamics to the desired order.

5.1.1 The geostrophic flow

To zeroth order in Ro_L, the horozontal momentum equation reduces to a balance between the zeroth order Coriolis force and the zeroth order pressure gradient,

$$\mathbf{0} = -\hat{\nabla}_\perp \hat{\tilde{p}}_0 - \widehat{\mathbf{Z}} \times \hat{\mathbf{u}}_{0,\perp}, \tag{5.27}$$

which is the geostrophic balance. By taking the cross product of $\widehat{\mathbf{Z}}$ with the zeroth order balance, (5.27), we obtain a diagnostic equation for the zeroth order horizontal velocity field $\hat{\mathbf{u}}_{0,\perp}$ given by the zeroth order pressure gradient,

$$\hat{\mathbf{u}}_{0,\perp} = \widehat{\mathbf{Z}} \times \hat{\nabla}_\perp \hat{\tilde{p}}_0. \tag{5.28}$$

Note that the horozontal divergence of the zeroth order horizontal velocity is divergence free, i.e.,

$$\hat{\nabla}_\perp \cdot \hat{\mathbf{u}}_{0,\perp} = 0. \tag{5.29}$$

Helmholtz's theorem for vector fields in two dimensions, (**??**), says then that the zeroth order pressure deviation acts as a streamfunction for the horizontal velocity. The vertical momentum equation (5.18) to zeroth order gives that the zeroth order pressure gradient is balanced with the zeroth order buoyancy force,

$$\mathbf{0} = -\hat{\nabla}_\parallel \hat{\tilde{p}}_0 - \hat{\tilde{\rho}}_0\, \widehat{\mathbf{Z}}, \tag{5.30}$$

where we have used that the unit vector along the gravity is in the negative vertical direction, $\hat{\mathbf{g}} = -\widehat{\mathbf{Z}}$. Equations (5.27) and (5.30) show that the zeroth order pressure deviation is undetermined to this order. The continuity equation (5.19), to lowest order in Ro_L reads

$$0 = \hat{\nabla}_\perp \cdot \hat{\mathbf{u}}_{0,\perp} + \hat{\nabla}_\parallel \cdot \hat{\mathbf{u}}_{0,\parallel}, \tag{5.31}$$

and since the first term on the right hand side is constrained to be zero according to equation (5.29), it follows that the vertical divergence of the vertical velocity is zero,

$$\hat{\nabla}_\parallel \cdot \hat{\mathbf{u}}_{0,\parallel} = 0. \tag{5.32}$$

So far we have seen that the dynamics of the baroclinic model to lowest order is equal to the dynamics of the barotropic model to lowest order, except that the pressure now depends on the vertical coordinate. Thus the horizontal velocity field $\hat{\mathbf{u}}_{0,\perp}$ also depends on the vertical coordinate. This means that there exists a vertical shear in the the horizontal velocity field $\hat{\mathbf{u}}_{0,\perp}$. By differentiating equation (5.28) with respect to the vertical coordinate $\hat{Z}$ and use the lowest order momentum balance in the vertical direction (5.30), it follows that the vertical shear in the geostrophic flow is related to the horizontal gradient of the mass density deviations by

$$\frac{\partial \hat{\mathbf{u}}_{0,\perp}}{\partial \hat{Z}} = -\widehat{\mathbf{Z}} \times \hat{\nabla}_\perp \hat{\tilde{\rho}}_0. \tag{5.33}$$

This equation is often referred as the thermal wind balance. The lowest order boundary condition, (5.10), at the bottom $\hat{Z}_0 = 0$ is

$$\hat{w}_0 = Ro_L \hat{\mathbf{u}}_{0,\perp} \cdot \hat{\nabla}_\perp \hat{h}_b, \tag{5.34}$$

and the lowest order boundary conditions, (5.11), (5.12), (5.13) and (5.14), at the free surface $\hat{Z}_0 = 1$ are

$$\begin{aligned} \hat{w}_0 &= 0, && (5.35) \\ \alpha \hat{\tau}_{XZ} &= \frac{\partial \hat{u}_0}{\partial \hat{Z}}, && (5.36) \\ \alpha \hat{\tau}_{YZ} &= \frac{\partial \hat{v}_0}{\partial \hat{Z}}, && (5.37) \\ \hat{\tilde{p}}_0 &= 0. && (5.38) \end{aligned}$$

From equation (5.32) and (5.35) it follows that the zeroth order vertical velocity is equal to zero, $\hat{w}_0 = 0$ everywhere in space and time. Thus it follows that the thermodynamic equations do not give any contribution to order Ro_L. Similar to the barotropic model there is a contradiction in the boundary value conditions (5.34), (5.36) and (5.37). According to the discussion in section 4.1.1, the consequence of the dilemma is that there must exist boundary layers at $\hat{Z}_0 = 0$ and $\hat{Z}_0 = 1$.

5.1.2 The ageostrophic flow

To first order in Ro_L, the horizontal momentum equation is

$$\left(\frac{\partial}{\partial \hat{t}} + \hat{\mathbf{u}}_{0,\perp} \cdot \hat{\nabla}_\perp \right) \hat{\mathbf{u}}_{0,\perp} = -\hat{\nabla}_\perp \hat{\tilde{p}}_1 + \frac{1}{Re^{\mathrm{t}}_\perp} \hat{\nabla}^2_\perp \hat{\mathbf{u}}_{0\perp} - \beta \hat{Y} \widehat{\mathbf{Z}} \times \hat{\mathbf{u}}_{0,\perp} - \widehat{\mathbf{Z}} \times \hat{\mathbf{u}}_{1,\perp}, . \tag{5.39}$$

which is the same as for the first order horizontal equation in the barotropic model (5.39). This is very useful since we can use some results from the vorticity formalism of the previous chapter later. Note that the advection term is a pure advection of the horizontal velocity by itself since the vertical velocity to lowest order is zero. The vertical momentum equation to first order in Ro_L provides that the vertical gradient of $\hat{\tilde{p}}_1$ is balanced by the first order buoyancy force,

$$\mathbf{0} = -\hat{\nabla}_\parallel \hat{\tilde{p}}_1 - \hat{\tilde{\rho}} \widehat{\mathbf{Z}}. \tag{5.40}$$

The continuity equation to first order in Ro_L is as expected

$$0 = \hat{\nabla}_\perp \cdot \hat{\mathbf{u}}_{1,\perp} + \hat{\nabla}_\parallel \cdot \hat{\mathbf{u}}_{1,\parallel}. \tag{5.41}$$

The thermodynamic equation for the mass density to first order in Ro_L is

$$\left(\frac{\partial}{\partial\hat{t}} + \hat{\mathbf{u}}_{0,\perp}\cdot\hat{\nabla}_{\perp}\right)\hat{\tilde{\rho}}_0 = Bu_L\,\hat{N}^2\hat{w}_1 - \frac{1}{Ro_LF_L}\left(\frac{1}{Pe^{\mathrm{t}}_{\perp}}\hat{\nabla}^2_{\perp} + \frac{1}{Pe^{\mathrm{t}}_{\parallel}}\hat{\nabla}^2_{\parallel}\right)\hat{\tilde{T}}_0 \tag{5.42}$$

and the heat equation (5.21) to first order is

$$\left(\frac{\partial}{\partial\hat{t}} + \hat{\mathbf{u}}_{0,\perp}\cdot\hat{\nabla}_{\perp}\right)\hat{\tilde{T}}_0 = -\frac{1}{F_L}\hat{N}^2_{Tp}\hat{w}_1 + \frac{1}{Ro_LF_L}\left(\frac{1}{Pe^{\mathrm{t}}_{\perp}}\hat{\nabla}^2_{\perp} + \frac{1}{Pe^{\mathrm{t}}_{\parallel}}\hat{\nabla}^2_{\parallel}\right)\hat{\tilde{T}}_0. \tag{5.43}$$

5.2 The boundary layers

The boundary layers of the baroclinic model is quite similar to the boundary layers in the barotropic model, except that we have a stratified fluid. Thus, we will refer to some previous discussions that are valid for both cases, and instead concentrate the discussion about, what is the effect of stratification on the boundary layers. The method will be to introduce a stretched boundary layer coordinate, linking together the variables in the boundary layer with the variables in the interior and use this to determine the boundary value conditions of the interior. From the beginning, we will use that the characteristic length scale of the boundary layers must be of $\mathcal{O}(\sqrt{Ek_{\parallel}}\delta L_{\parallel,m})$, in order to have that the horizontal turbulent mixing of momentum due to shear in the horisontal velocity $\hat{\mathbf{u}}^*_{\perp}$ is of $\mathcal{O}(1)$.

5.2.1 The bottom Ekman layer

As for the barotropic model, the stretched boundary layer coordinate at the lower boundary $\hat{Z}_0 = 0$ is

$$\hat{Z}^* = \frac{1}{\sqrt{Ek_{\parallel}}}\hat{Z}, \tag{5.44}$$

where the corresponding scaled vertical nabla-operator is

$$\hat{\nabla}^*_{\parallel} = \sqrt{Ek_{\parallel}}\hat{\nabla}_{\parallel} \quad \Leftrightarrow \quad \frac{\partial}{\partial\hat{Z}^*} = \sqrt{Ek_{\parallel}}\frac{\partial}{\partial\hat{Z}}. \tag{5.45}$$

All other variables have the same scaling in the boundary layer as for the interior, except for the vertical velocity, which is given by

$$\hat{\mathbf{u}}^*_{\parallel} = \frac{1}{\sqrt{Ek_{\parallel}}}\hat{\mathbf{u}}_{\parallel}. \tag{5.46}$$

Hence it follows that the rescaled equations of motion in the lower boundary layer is

$$\begin{aligned} Ro_L\left(\frac{\partial}{\partial\hat{t}}+\hat{\mathbf{u}}_\perp^*\cdot\hat{\nabla}_\perp+\hat{\mathbf{u}}_\parallel^*\cdot\hat{\nabla}_\parallel^*\right)\hat{\mathbf{u}}_\perp^* =& -\frac{1}{1+Ro_LF_L\hat{\tilde{\rho}}^*}\hat{\nabla}_\perp\hat{\tilde{p}}^*+\frac{Ek_\perp}{2}\hat{\nabla}_\perp^2\hat{\mathbf{u}}_\perp^* \\ &+\frac{1}{2}\hat{\nabla}_\parallel^{*2}\hat{\mathbf{u}}_\perp^*-\frac{\gamma}{\Gamma}Ro_L\sqrt{Ek_\parallel}\hat{Y}\widehat{\mathbf{Y}}\times\hat{\mathbf{u}}_\parallel^* \\ &-\left(1+Ro_L\beta\hat{Y}\right)\widehat{\mathbf{Z}}\times\hat{\mathbf{u}}_\perp^* \end{aligned} \tag{5.47}$$

$$\mathbf{0} = -\hat{\nabla}_\parallel^*\hat{\tilde{p}}^*-\sqrt{Ek_\parallel}\hat{\tilde{\rho}}^*\,\widehat{\mathbf{Z}}, \tag{5.48}$$

$$0 = \hat{\nabla}_\perp\cdot\hat{\mathbf{u}}_\perp^*+\hat{\nabla}_\parallel^*\cdot\hat{\mathbf{u}}_\parallel^*, \tag{5.49}$$

and the rescaled lower boundary condition at $\hat{Z}^*=\frac{Ro_L}{\sqrt{Ek_\parallel}}\hat{h}_b$,

$$\hat{w}=\frac{Ro_L}{\sqrt{Ek_\parallel}}\hat{\mathbf{u}}_\perp^*\cdot\hat{\nabla}_\perp\hat{h}_b. \tag{5.50}$$

If we now use that the solution of $(\hat{\mathbf{u}}_\perp^*,\hat{\mathbf{u}}_\parallel^*,\hat{\tilde{p}}^*)$ can be expanded similar to (4.42), (4.43) and (4.44), and that the solution to the mass density can be expanded as

$$\hat{\tilde{\rho}}^*\left(\hat{\mathbf{X}}_\perp,\hat{Z}^*,\hat{t};Ro_L\right)=\sum_{i=0}^{\infty}Ro_L^i\hat{\tilde{\rho}}_i^*\left(\hat{\mathbf{X}}_\perp,\hat{Z}^*,\hat{t}\right),, \tag{5.51}$$

it follows to zeroth order in Ro_L that the the lowest order dynamics in the lower boundary layer is dominated of a horizontal balance between the zeroth order Coriolis force, the zeroth order friction force and the zeroth order pressure gradient,

$$\mathbf{0}=-\hat{\nabla}_\perp\hat{\tilde{p}}_0^*+\frac{1}{2}\hat{\nabla}_\parallel^{*2}\hat{\mathbf{u}}_{0,\perp}^*-\widehat{\mathbf{Z}}\times\hat{\mathbf{u}}_{0,\perp}^*. \tag{5.52}$$

The corresponding vertical momentum equation gives that the lowest order pressure is independent of the vertical coordinate in the boundary layer

$$\mathbf{0}=\hat{\nabla}_\parallel^*\hat{\tilde{p}}_0^*, \tag{5.53}$$

such that the lowest order density is undetermined to this order. This implise that the lowest order pressure is constant over the boundary layer. The lowest order continuity equation is

$$0=\hat{\nabla}_\perp\cdot\hat{\mathbf{u}}_{0,\perp}^*+\hat{\nabla}_\parallel^*\cdot\hat{\mathbf{u}}_{0,\parallel}^*. \tag{5.54}$$

The lowest order dynamics in the boundary layer in the baroclinic model is actually completely similar to the lowest order dynamics in the boundary layer in the barotropic model. Thus, we expect that the solution of the boundary value conditions will be exactly the same. The only difference so far is that the lowest-order pressure in the interior depends on the vertical coordinate in the baroclinic model, in contrast to the barotropic model. This means that we can not connect the variables in the interior with the variables in the boundary layer in the same way as for the barotropic model. By using the vertical Laplace operator $\hat{\nabla}_{\parallel}^{*2}$ on equation (5.52), the system transforms into two linear fourth-order differential equations

$$\hat{\nabla}_{\parallel}^{*4}\hat{\mathbf{u}}_{0,\perp}^{*} + 4\hat{\mathbf{u}}_{0,\perp}^{*} = 4\widehat{\mathbf{Z}} \times \hat{\nabla}_{\perp}\hat{\tilde{p}}_{0}^{*}. \tag{5.55}$$

The general solution to equation (5.55) is given by

$$\hat{\mathbf{u}}_{0,\perp}^{*} = \widehat{\mathbf{Z}} \times \hat{\nabla}_{\perp}\hat{\tilde{p}}_{0}^{*} + e^{\hat{Z}^*}\left(\mathbf{A}_1 \cos \hat{Z}^* + \mathbf{A}_2 \sin \hat{Z}^*\right) + e^{-\hat{Z}^*}\left(\mathbf{A}_3 \cos \hat{Z}^* + \mathbf{A}_4 \sin \hat{Z}^*\right), \tag{5.56}$$

where the coefficients are given by

$$\mathbf{A}_1 = [A, -B]^{\mathrm{T}}, \quad \mathbf{A}_2 = [B, A]^{\mathrm{T}}, \quad \mathbf{A}_3 = [C, D]^{\mathrm{T}}, \quad \mathbf{A}_4 = [D, -C]^{\mathrm{T}}. \tag{5.57}$$

By using that the boundary layer variables must merge smoothly to the interior variables in the transistion region $\hat{Z}^* \to \infty$, and that the boundary layer velocity is zero at the bottom $\hat{Z}^* = Ro_L/\sqrt{Ek_{\parallel}}\hat{h}_b$ it can be shown (by the same way as previously) by using the continuity equation (5.54) that the first order boundary condition to the vertical velocity in the interior at $\hat{Z}_0 = 0$ is

$$\hat{w}_1\left(\hat{\mathbf{X}}_{\perp}, 0, \hat{t}\right) = \hat{\mathbf{u}}_{0,\perp} \cdot \hat{\nabla}_{\perp}\hat{h}_b + \frac{1}{2}\frac{\sqrt{Ek_{\parallel}}}{Ro_L}\widehat{\mathbf{Z}} \cdot \left(\hat{\nabla}_{\perp} \times \hat{\mathbf{u}}_{0,\perp}\right). \tag{5.58}$$

This is the same result as for the barotropic model, just as we expected.

5.2.2 The upper Ekman layer

As for the barotropic model, the stretched boundary layer coordinate at the upper boundary $\hat{Z}_0 = 1$ is

$$\hat{Z}^{**} = \frac{1 - \hat{Z}}{\sqrt{Ek_{\parallel}}}, \tag{5.59}$$

and the corresponding vertical nabla-operator is

$$\hat{\nabla}_{\parallel}^{**} = -\sqrt{Ek_{\parallel}}\hat{\nabla}_{\parallel} \quad \Leftrightarrow \quad \frac{\partial}{\partial \hat{Z}^{**}} = -\sqrt{Ek_{\parallel}}\frac{\partial}{\partial \hat{Z}}. \tag{5.60}$$

All other variables have the same scaling in the boundary layer as for the interior domain, except for the vertical velocity, which is given by (5.46). Hence it follows that the rescaled equations of motion in the upper boundary layer is

$$
\begin{aligned}
Ro_L\left(\frac{\partial}{\partial\hat{t}}+\hat{\mathbf{u}}_\perp^{**}\cdot\hat{\nabla}_\perp-\hat{\mathbf{u}}_\parallel^{**}\cdot\hat{\nabla}_\parallel^{**}\right)\hat{\mathbf{u}}_\perp^{**} = &-\frac{1}{1+Ro_LF_L\hat{\tilde{\rho}}^{**}}\hat{\nabla}_\perp\hat{\tilde{p}}^{**}+\frac{Ek_\perp}{2}\hat{\nabla}_\perp^2\hat{\mathbf{u}}_\perp^{**} \\
&+\frac{1}{2}\hat{\nabla}_\parallel^{**2}\hat{\mathbf{u}}_\perp^{**}-\frac{\gamma}{\Gamma}Ro_L\sqrt{E_\parallel}\hat{Y}\widehat{\mathbf{Y}}\times\hat{\mathbf{u}}_\parallel^{**} \\
&-\left(1+Ro_L\beta\hat{Y}\right)\widehat{\mathbf{Z}}\times\hat{\mathbf{u}}_\perp^{**} \qquad (5.61)
\end{aligned}
$$

$$
\mathbf{0} = \hat{\nabla}_\parallel^{**}\hat{\tilde{p}}^{**}-\sqrt{E_\parallel}\,\hat{\tilde{\rho}}^{**}\,\hat{\mathbf{Z}}, \qquad (5.62)
$$

$$
0 = \hat{\nabla}_\perp\cdot\hat{\mathbf{u}}_\perp^{**}-\hat{\nabla}_\parallel^{**}\cdot\hat{\mathbf{u}}_\parallel^{**}, \qquad (5.63)
$$

and the rescaled boundary conditions at the free surface $\hat{Z}^{**}=-Ro_LF_L/\sqrt{E_\parallel}\hat{\zeta}^{**}$

$$
\begin{aligned}
\hat{w}^{**} &= \frac{Ro_LF_L}{\sqrt{E_\parallel}}\left(\frac{\partial}{\partial\hat{t}}+\hat{\mathbf{u}}_\perp\cdot\hat{\nabla}_\perp\right)\hat{\zeta}^{**}, && (5.64)\\
\alpha^{**}\hat{\tau}_{XZ} &= -\frac{\partial\hat{u}^{**}}{\partial\hat{Z}^{**}}, && (5.65)\\
\alpha^{**}\hat{\tau}_{YZ} &= -\frac{\partial\hat{v}^{**}}{\partial\hat{Z}^{**}}, && (5.66)
\end{aligned}
$$

where the new rescaled dimensionless number is $\alpha^{**}=\alpha\sqrt{Ek_\parallel}$. By applying that the solution can be expanded in a regular power serie in Ro_L, equations (4.42), (4.43), (4.44) and (5.51), we find that the lowest order dynamics in the upper boundery layer is described by

$$
\begin{aligned}
\mathbf{0} &= -\hat{\nabla}_\perp\hat{\tilde{p}}_0^{**}+\frac{1}{2}\hat{\nabla}_\parallel^{**2}\hat{\mathbf{u}}_{0,\perp}^{**}-\widehat{\mathbf{Z}}\times\hat{\mathbf{u}}_{0,\perp}^{**}, && (5.67)\\
\mathbf{0} &= \hat{\nabla}_\parallel^{**}\hat{\tilde{p}}_0^{**}, && (5.68)\\
0 &= \hat{\nabla}_\perp\cdot\hat{\mathbf{u}}_{0,\perp}^{**}-\hat{\nabla}_\parallel^{**}\cdot\hat{\mathbf{u}}_{0,\parallel}^{**}, && (5.69)
\end{aligned}
$$

with the corresponding boundary value conditions at $\hat{Z}^{**}=0$

$$
\begin{aligned}
\hat{w}_0^{**} &= 0, && (5.70)\\
\alpha^{**}\hat{\tau}_{XZ} &= -\frac{\partial\hat{u}_0^{**}}{\partial\hat{Z}^{**}}, && (5.71)\\
\alpha^{**}\hat{\tau}_{YZ} &= -\frac{\partial\hat{v}_0^{**}}{\partial\hat{Z}^{**}}. && (5.72)
\end{aligned}
$$

The lowest order dynamics in the upper boundary layer for the barotropic model is very similar to the lowest order dynamics in the upper boundary layer in the baroclinic model, the only difference is the boundary value condition (5.70). What this mean is that the deformation of the interface, ζ makes no contribution to the lowest order dynamics since the local rotational Froud number is of $\mathcal{O}(Ro_L)$ and the local Burger number is of $\mathcal{O}(1)$. Similar to the lower boundary layer, the zeroth order mass density $\hat{\tilde{\rho}}_0^{**}$ in the upper boundary layer is undetermined to this order. This means that the stratification will have no effect on the upper boundary layer to the lowest order. Thus, we expect that the solution of equation (5.67) will give the same solution as in the barotropic model, apart from the contribution of vorticity due to the deformation of the interface. By using that the boundary layer variables must merge smoothly to the interior variables in the transistion region $\hat{Z}^{**} \to \infty$, and that the boundary layer velocity is zero at the free surface (5.70) $\hat{Z}^{**} = 0$ in addition to the dynamical boundary conditions (5.71) and (5.72), it can be shown (by the same way as previously) by using the continuity equation (5.69) that the first order boundary condition to the vertical velocity in the interior at $\hat{Z}_0 = 1$ is

$$\hat{w}_1\left(\hat{\mathbf{X}}_\perp, 1, \hat{t}\right) = \frac{\sqrt{Ek_\parallel}}{Ro_L}\frac{\alpha^{**}}{2}\widehat{\mathbf{Z}} \cdot \left(\hat{\nabla}_\perp \times \hat{\mathbf{T}}\right). \tag{5.73}$$

where $\hat{\mathbf{T}}$ is the dimensionless wind-stress vector given by $\hat{\mathbf{T}} = \hat{\tau}_{XZ}\widehat{\mathbf{X}} + \hat{\tau}_{YZ}\widehat{\mathbf{Y}}$.

5.3 The baroclinic quasi-geostrophic potential vorticity

Since the horizontal momentum equation to zero- and first order in Ro_L is the same for both the barotropic model and the baroclinic model, the corresponding vorticity equation associated with the geostrophic velocity $\hat{\mathbf{u}}_{0,\perp}$ is equal. The zeroth order vorticity $\mathbf{\Theta}_0$ is given by equation (4.94), which is described by the vorticity equation (4.104). Taking the dot product of the unit vector in the vertical direction $\widehat{\mathbf{Z}}$ with equation (4.104), the vorticity equation may be written as

$$\left(\frac{\partial}{\partial \hat{t}} + \hat{\mathbf{u}}_{0,\perp} \cdot \hat{\nabla}_\perp\right)\hat{\nabla}_\perp^2 \hat{\tilde{p}}_0 = \frac{1}{Re_\perp^{\mathbf{t}}}\hat{\nabla}_\perp^4 \hat{\tilde{p}}_0 - \hat{\mathbf{u}}_{0,\perp} \cdot \hat{\nabla}_\perp\left(\beta \hat{Y}\right) + \hat{\nabla}_\parallel \cdot \hat{\mathbf{u}}_{1,\parallel}. \tag{5.74}$$

Similar to in the barotropic model, the vorticity equation depends on the first order vertical velocity. In the barotropic vorticity equation we had that all terms was independent of the vertical coordinate, so that we could eliminate $\hat{w}_1$ by integrating the equation over the vertical coordinate. In this case the baroclinic vorticity equation depends on the vertical coordinate, thus we must eliminate $\hat{w}_1$ in a different way. The best way to do this is to use the thermodynamic equation for the mass density (5.42), since we know the relationship between the zeroth order mass density $\hat{\tilde{\rho}}_0$ and pressure $\hat{\tilde{p}}_0$. By substituting equation (5.30) into equation

(5.42), we find an expression for the vertical velocity to first order,

$$\hat{w}_1 = -\left(\frac{\partial}{\partial \hat{t}} + \hat{\mathbf{u}}_{0,\perp}\cdot\hat{\nabla}_\perp\right)\left(\frac{1}{Bu_L}\frac{1}{\hat{N}^2}\frac{\partial \hat{\tilde{p}}_0}{\partial \hat{Z}}\right) + \frac{1}{Ro_L F_L Bu_L}\frac{1}{\hat{N}^2}\left(\frac{1}{Pe^{\mathrm{t}}_\perp}\hat{\nabla}^2_\perp + \frac{1}{Pe^{\mathrm{t}}_\parallel}\hat{\nabla}^2_\parallel\right)\hat{\tilde{T}}_0, \tag{5.75}$$

where we have used that the buoyancy frequency $\hat{N}$ only depends on the vertical coordinate. Thus, by differentiating equation (5.75) with respect of $\hat{Z}$ and substituting the result into equation (5.74), the vorticity equation becomes

$$\left(\frac{\partial}{\partial \hat{t}} + \hat{\mathbf{u}}_{0,\perp}\cdot\hat{\nabla}_\perp\right)\left(\hat{\nabla}^2_\perp\hat{\tilde{p}}_0 + \frac{1}{Bu_L}\frac{\partial}{\partial \hat{Z}}\left(\frac{1}{\hat{N}^2}\frac{\partial \hat{\tilde{p}}_0}{\partial \hat{Z}}\right) + \beta\hat{Y}\right) = \frac{1}{Re^{\mathbf{t}}_\perp}\hat{\nabla}^4_\perp\hat{\tilde{p}}_0 + \frac{1}{Ro_L F_L Bu_L}\frac{\partial}{\partial \hat{Z}}\left(\frac{1}{\hat{N}^2}\left(\frac{1}{Pe^{\mathrm{t}}_\perp}\hat{\nabla}^2_\perp + \frac{1}{Pe^{\mathrm{t}}_\parallel}\hat{\nabla}^2_\parallel\right)\hat{\tilde{T}}_0\right), \tag{5.76}$$

where we have used that the vorticity due to the variation in the Coriolis parameter can be written as $\hat{\mathbf{u}}_{0,\perp}\cdot\hat{\nabla}_\perp\left(\beta\hat{Y}\right) = \beta\,\hat{v}_0 = \beta\,\partial\hat{\tilde{p}}_0/\partial\hat{X}$ together with the definition of the meridional velocity $\hat{v}_0 = \left(\partial/\partial\hat{t} + \hat{\mathbf{u}}_{0,\perp}\cdot\hat{\nabla}_\perp\right)\hat{Y}$. The lower boundary condtion to the quasi-geostrophic potential vorticity equation at $\hat{Z}_0 = 0$ can be for formulated as

$$\left(\frac{\partial}{\partial \hat{t}} + \hat{\mathbf{u}}_{0,\perp}\cdot\hat{\nabla}_\perp\right)\left(\frac{1}{Bu_L}\frac{1}{\hat{N}^2}\frac{\partial \hat{\tilde{p}}_0}{\partial \hat{Z}} + \hat{h}_b\right) = \frac{1}{Ro_L F_L Bu_L}\frac{1}{\hat{N}^2}\left(\frac{1}{Pe^{\mathrm{t}}_\perp}\hat{\nabla}^2_\perp + \frac{1}{Pe^{\mathrm{t}}_\parallel}\hat{\nabla}^2_\parallel\right)\hat{\tilde{T}}_0 - \frac{1}{2}\frac{\sqrt{E_\parallel}}{Ro_L}\hat{\nabla}^2_\perp\hat{\tilde{p}}_0, \tag{5.77}$$

where we have used the lower boundary conditions for the first order verical velocity (5.58) together with (5.75), and the upper boundary condition at $Z_0 = 1$ can be formulated as

$$\left(\frac{\partial}{\partial \hat{t}} + \hat{\mathbf{u}}_{0,\perp}\cdot\hat{\nabla}_\perp\right)\left(\frac{1}{Bu_L}\frac{1}{\hat{N}^2}\frac{\partial \hat{\tilde{p}}_0}{\partial \hat{Z}}\right) = \frac{1}{Ro_L F_L Bu_L}\frac{1}{\hat{N}^2}\left(\frac{1}{Pe^{\mathrm{t}}_\perp}\hat{\nabla}^2_\perp + \frac{1}{Pe^{\mathrm{t}}_\parallel}\hat{\nabla}^2_\parallel\right)\hat{\tilde{T}}_0 - \frac{\sqrt{Ek_\parallel}}{Ro_L}\frac{\alpha^{**}}{2}\hat{\mathbf{Z}}\cdot\left(\hat{\nabla}_\perp\times\hat{\mathbf{T}}\right). \tag{5.78}$$

where we have used (5.73) together with (5.75).

Chapter 6

An interacting baroclinic ocean circulation model

In this chapter we will extend the model from the previous chapter by taking into account the interaction between the global and local scales. The method we will use is a multi-scale expansion which is based on spatial and temporal scale separation. Since the model will depend on the global scale, we can not use slab-coordinates, such that we have to keep the equations in a spherical coordinate system. To illustrate the method in a best possible way, we neglect all dissipation terms, so that the heat equation and salinity equation disconnect from the system. If we use that the dimensionless numbers have the same magnitude as in the previous chapter and filters out sound waves as in the previous chapter, it follows that the truncated continuity equation is

$$0 = \hat{\nabla} \cdot \hat{\mathbf{u}}, \tag{6.1}$$

the truncated momentum equations are

$$Ro \widehat{\left(\frac{d\mathbf{u}}{dt} \right)}_{\perp} = -\frac{1}{1 + Ro\, F \hat{\tilde{\rho}}} \hat{\nabla}_{\perp} \hat{\tilde{p}} - \gamma \cos\theta\, \widehat{\boldsymbol{\theta}} \times \hat{\mathbf{u}}_{\parallel} - \sin\theta\, \widehat{\mathbf{r}} \times \hat{\mathbf{u}}_{\perp}, \tag{6.2}$$

$$\mathbf{0} = -\hat{\nabla}_{\parallel} \hat{\tilde{p}} - \hat{\tilde{\rho}} \hat{\mathbf{r}}, \tag{6.3}$$

and the truncated thermodynamic mass density equation is

$$Ro \widehat{\left(\frac{\mathrm{d}}{\mathrm{d}t} \right)} \hat{\tilde{\rho}} \ = \ Bu\, \hat{N}^2 \hat{w}, \tag{6.4}$$

where the horizontal acceleration term is

$$\widehat{\left(\frac{d\mathbf{u}}{dt} \right)}_{\perp} = \left(\left. \frac{\partial \hat{\mathbf{u}}_{\perp}}{\partial \hat{t}} \right|_{\mathbf{e}_i} + \left. \hat{\mathbf{u}} \cdot \hat{\nabla} \hat{\mathbf{u}}_{\perp} \right|_{\mathbf{e}_i} + \gamma \Gamma \frac{\hat{w}}{\hat{r}} \hat{\mathbf{u}}_{\perp} + \Gamma \frac{\hat{u}}{\hat{r}} \tan\theta\, \widehat{\mathbf{r}} \times \hat{\mathbf{u}}_{\perp} \right), \tag{6.5}$$

and the total time derivative is

$$\widehat{\left(\frac{d}{dt}\right)} = \left(\frac{\partial}{\partial \hat{t}} + \hat{\mathbf{u}} \cdot \hat{\nabla}\right). \tag{6.6}$$

Since the equations are in a spherical coordinate system the nabla-operators are

$$\nabla_\perp = \widehat{\boldsymbol{\phi}} \frac{1}{r_m \hat{r} \cos\theta} \frac{\partial}{\partial \phi} + \widehat{\boldsymbol{\theta}} \frac{1}{r_m \hat{r}} \frac{\partial}{\partial \theta}, \quad \nabla_\parallel = \widehat{\mathbf{r}} \frac{\partial}{\partial Z}, \tag{6.7}$$

where the normalized operators are $\hat{\nabla}_\perp = \delta L_{\perp,m} \nabla_\perp$ and $\hat{\nabla}_\parallel = \delta L_{\parallel,m} \nabla_\parallel$ and the normalized radius $\hat{r}$ is given by equation (2.171), i.e. $\hat{r} = 1 + \gamma \Gamma \hat{Z}$. Where we have redefine the vertical coordinate $Z = r - r_m$. It should be noted that the model will be limited to the midlatitude, such that $\sin\theta, \cos\theta$ will be of $\mathcal{O}(1)$. This means that the local Rossby number Ro_L is replaced by the Rossby number Ro. Therefore, we have used that $Ro\,\widetilde{Eu} \sim \mathcal{O}(1)$ and that the dimensionless mass density is $\hat{\rho} = \hat{\rho}_h + Ro\,F\hat{\widetilde{\rho}}$. Similar to the previous chapter we have used that the fluid is shallow, such that the background mass denisity and the gravity is approximated constant, i.e., $\hat{\rho}_h \approx 1$ and $\hat{\mathbf{g}} = -\widehat{\mathbf{r}}$.

6.1 The local and global equations

For the dynamics in the midlatitudes, there exist two distinct scales. One large-scale assosiated with the planetary scale, which means phenomena comparable to the earth's radius r_m, and then also one scale that is comperable to the deformation radius L_D. From now we will call this small-scale. The deformation radius is given by that length scale where rotational effects becomes equally important as other phenomena, e.g., buoyancy. As previously discussed, we have that the dimensionless number that indicates the importance of stratification versus rotation is Burgers number Bu. From the definition of Bu, equation (2.194), it follows that the length-scale where stratification and rotation are equally important is

$$L_D = \frac{N_m \delta L_{\parallel,m}}{2\Omega}, \tag{6.8}$$

such that the Burgers number can be written as

$$Bu = \left(\frac{L_D}{\delta L_{\perp,m}}\right)^2. \tag{6.9}$$

Since the local dynamics in our case is characterized such that Bu is of $\mathcal{O}(1)$, it follows that the characteristic length scale $\delta L_{\perp,m}$ of the local dynamics is of $\mathcal{O}(L_D)$.

Therefore, we anticipate the existence of small and large spatial scales associated with the local and global motions, respectively. These are formally defined by

$$\boldsymbol{\xi}_\perp = \widehat{\boldsymbol{\phi}}\,\phi + \widehat{\boldsymbol{\theta}}\,\theta, \quad \boldsymbol{X}_\perp = \Gamma\left(\widehat{\boldsymbol{\phi}}\,\phi + \widehat{\boldsymbol{\theta}}\,\theta\right), \tag{6.10}$$

where $\boldsymbol{\xi}_\perp = \widehat{\boldsymbol{\phi}}\,\phi_l + \widehat{\boldsymbol{\theta}}\,\theta_l$ are the local independent horizontal coordinates and $\boldsymbol{X}_\perp = \widehat{\boldsymbol{\phi}}\,\phi_g + \widehat{\boldsymbol{\theta}}\,\theta_g$ are the global independent horizontal coordiantes. In the following, these spatial scales will be treated as distinct independent variables. The separation of scale implies that the horizontal nabla operator transforms as

$$\nabla_\perp = \nabla_{l,\perp} + \Gamma\nabla_{g,\perp} \tag{6.11}$$

where

$$\nabla_{l,\perp} = \widehat{\boldsymbol{\phi}}\frac{1}{r_m\hat{r}\cos\theta_g}\frac{\partial}{\partial\phi_l} + \widehat{\boldsymbol{\theta}}\frac{1}{r_m\hat{r}}\frac{\partial}{\partial\theta_l}, \tag{6.12}$$

$$\nabla_{g,\perp} = \widehat{\boldsymbol{\phi}}\frac{1}{r_m\hat{r}\cos\theta_g}\frac{\partial}{\partial\phi_g} + \widehat{\boldsymbol{\theta}}\frac{1}{r_m\hat{r}}\frac{\partial}{\partial\theta_g}, \tag{6.13}$$

In dimensionless form the new nabla operator is

$$\hat{\nabla}_\perp = \hat{\nabla}_{l,\perp} + \Gamma\hat{\nabla}_{g,\perp}. \tag{6.14}$$

Similary, there exists fast and slow temporal scales associated with the local and global motions, given by

$$\hat{t}_l = \hat{t}, \quad \hat{t}_g = \Gamma\hat{t}, \tag{6.15}$$

such that the partial time derivative transforms as

$$\frac{\partial}{\partial\hat{t}} = \frac{\partial}{\partial\hat{t}_l} + \Gamma\frac{\partial}{\partial\hat{t}_g} \tag{6.16}$$

This means that the local motion occurs on a time scale that is $\mathcal{O}(1)$ and the global motion takes place on a time scales that is $\mathcal{O}(1/\Gamma)$. In the following, these two temporal scales will be treated as distinct independent variables. By introducing the seperation of fast and slow spatio-temporal scales into the equations of motion, (6.1), (6.2), (6.3) and (6.4), we obtain the continuity equation,

$$0 = \left(\hat{\nabla}_{l,\perp} + \Gamma\hat{\nabla}_{g,\perp} + \hat{\nabla}_\parallel\right)\cdot\hat{\mathbf{u}}, \tag{6.17}$$

the horizontal momentum equation

$$\begin{aligned} Ro\Bigg[\left(\frac{\partial\hat{\mathbf{u}}_\perp}{\partial\hat{t}_l}\bigg|_{\mathbf{e}_i} + \hat{\mathbf{u}}\cdot\left(\hat{\nabla}_{l,\perp} + \hat{\nabla}_\parallel\right)\hat{\mathbf{u}}_\perp\Big|_{\mathbf{e}_i}\right) + \Gamma\left(\frac{\partial\hat{\mathbf{u}}_\perp}{\partial\hat{t}_g}\bigg|_{\mathbf{e}_i} + \hat{\mathbf{u}}\cdot\hat{\nabla}_{g,\perp}\hat{\mathbf{u}}_\perp\Big|_{\mathbf{e}_i}\right) \\ +\gamma\Gamma\frac{\hat{w}}{\hat{r}}\hat{\mathbf{u}}_\perp + \Gamma\frac{\hat{u}}{\hat{r}}\tan\theta\,\widehat{\mathbf{r}}\times\hat{\mathbf{u}}_\perp\Bigg] = -\frac{1}{1+Ro\,F\hat{\tilde{\rho}}}\left(\hat{\nabla}_{l,\perp} + \Gamma\hat{\nabla}_{g,\perp}\right)\hat{\tilde{p}} \\ -\gamma\cos\theta\,\widehat{\boldsymbol{\theta}}\times\hat{\mathbf{u}}_\parallel - \sin\theta\,\widehat{\mathbf{r}}\times\hat{\mathbf{u}}_\perp, \end{aligned} \tag{6.18}$$

the vertical momentum equation

$$\mathbf{0} = -\hat{\nabla}_{\|}\hat{\tilde{p}} - \hat{\tilde{\rho}}\hat{\mathbf{r}}, \tag{6.19}$$

and the thermodynamic mass density equation

$$Ro\left[\left(\frac{\partial\hat{\tilde{\rho}}}{\partial\hat{t}_l} + \hat{\mathbf{u}}\cdot\left(\hat{\nabla}_{l,\perp} + \hat{\nabla}_{\|}\right)\hat{\tilde{\rho}}\right) + \Gamma\left(\frac{\partial\hat{\tilde{\rho}}}{\partial\hat{t}_g} + \hat{\mathbf{u}}\cdot\hat{\nabla}_{g,\perp}\hat{\tilde{\rho}}\right)\right] = Bu\,\hat{N}^2\hat{w}. \tag{6.20}$$

Similar to the previous chapters, we will assume that the solutions of equations (6.17), (6.18), (6.19) and (6.20) can be expanded in a regular power series in the Rossby number Ro, but since we have introduced multiple scales in the equations, there must be some small adjustments. From equation (6.20) it follows that the advection of $\hat{\tilde{\rho}}$ by the vertical velocity $\hat{\mathbf{u}}_{\|}$ must be of $\mathcal{O}(1/Ro)$, in order to satisfy the right hand side of the equation. This implies that the regular expansion of the mass density must be

$$\hat{\tilde{\rho}}\left(\hat{x},\hat{t};Ro\right) = \frac{1}{Ro}\sum_{i=0}^{\infty} Ro^i\,\hat{\tilde{\rho}}_i\left(\hat{\boldsymbol{\xi}}_{\perp},\hat{\boldsymbol{X}}_{\perp},\hat{Z},\hat{t}_l,\hat{t}_g\right), \tag{6.21}$$

which in turn implies that the pressure must be scaled similarly in order to to satisfy equation (6.19),

$$\hat{\tilde{p}}\left(\hat{x},\hat{t};Ro\right) = \frac{1}{Ro}\sum_{i=0}^{\infty} Ro^i\,\hat{\tilde{p}}_i\left(\hat{\boldsymbol{\xi}}_{\perp},\hat{\boldsymbol{X}}_{\perp},\hat{Z},\hat{t}_l,\hat{t}_g\right). \tag{6.22}$$

The horizontal momentum equation (6.18) gives that the velocity must be of $\mathcal{O}(1)$ to prevent the velocity field to achieves unnaturally large values. If the velocity field was of $\mathcal{O}(1/Ro)$, this would imply that the Rossby number was of $\mathcal{O}(1)$, which is wrong. Therefore, the expansions in the velocity fields are

$$\hat{\mathbf{u}}_{\perp}\left(\hat{x},\hat{t};Ro\right) = \sum_{i=0}^{\infty} Ro^i\,\hat{\mathbf{u}}_{i,\perp}\left(\hat{\boldsymbol{\xi}}_{\perp},\hat{\boldsymbol{X}}_{\perp},\hat{Z},\hat{t}_l,\hat{t}_g\right), \tag{6.23}$$

$$\hat{\mathbf{u}}_{\|}\left(\hat{x},\hat{t};Ro\right) = \sum_{i=0}^{\infty} Ro^i\,\hat{\mathbf{u}}_{i,\|}\left(\hat{\boldsymbol{\xi}}_{\perp},\hat{\boldsymbol{X}}_{\perp},\hat{Z},\hat{t}_l,\hat{t}_g\right). \tag{6.24}$$

Due to introduction of multiple scales in the equations, we can not guarantee a well defined asymptotic expansion of the solution. Therefore, we must impose a solvability condition to guarantee that the expansion of the solution is well defined. To guarantee that the asymptotic expansion of $\left(\hat{\tilde{\rho}},\hat{\tilde{p}},\hat{\mathbf{u}}_{\perp},\hat{\mathbf{u}}_{\|}\right)$ is well defined, $\left(\hat{\tilde{\rho}}_i,\hat{\tilde{p}}_i,\hat{\mathbf{u}}_{i,\perp},\hat{\mathbf{u}}_{i,\|}\right)$ must have the property that it grows slower than linearly in any of the local coordinates $\left(\hat{\boldsymbol{\xi}}_\perp, \hat{t}_l\right)$. This solvability conditions is known as the sublinear growth condition. Mathematically it can be formulated as follow: Let $\hat{C}_l$ denotes one of the local spatio-temporal coordinates $\left(\hat{\boldsymbol{\xi}}_\perp, \hat{t}_l\right)$, and let $\hat{C}_g$ denotes one of the global spatio-temporal coordinates $\left(\hat{\boldsymbol{X}}_\perp, \hat{t}_g\right)$. Then the sublinear growth condition for $\hat{C}_l$ implies that each variable $\hat{V} = \left(\hat{\tilde{\rho}}_i, \hat{\tilde{p}}_i, \hat{\mathbf{u}}_{i,\perp}, \hat{\mathbf{u}}_{i,\parallel}\right)$ must satisfy the limit

$$\lim_{Ro\to 0} \frac{\hat{V}}{\hat{C}_l + 1} = \lim_{Ro\to 0} \frac{\hat{V}}{\frac{\hat{C}_g}{Ro} + 1} = 0, \tag{6.25}$$

when all coordinates are held fixed with respect to Ro, exept $\hat{C}_l$. Equation (6.25) is not particularly useful, but has the significant consequence that the local average of the derivative of V with respect of C_l must disappear, e.g.

$$\left\langle \frac{\partial V}{\partial C_l} \right\rangle = \lim_{Ro\to 0} \frac{Ro}{2\delta C_{l,m}} \int_{\frac{C_p}{Ro} - \frac{\delta C_{l,m}}{Ro}}^{\frac{C_p}{Ro} + \frac{\delta C_{l,m}}{Ro}} \frac{\partial V}{\partial C_l} \mathrm{d}C_l = 0, \tag{6.26}$$

where $\delta C_{l,m}$ is the characteristic length scale of C_l. Therefore, it is natural to split all variables into a local average part and the deviation from this average. Where the average is a spatio-temporal avarege over the local scale. This means for example that the velocity field can be splitted as

$$\hat{\mathbf{u}}\left(\hat{\boldsymbol{\xi}}_\perp, \hat{\boldsymbol{X}}_\perp, \hat{Z}, \hat{t}_l, \hat{t}_g\right) = \hat{\mathbf{u}}^{\mathrm{g}}\left(\hat{\boldsymbol{X}}_\perp, \hat{Z}, \hat{t}_g\right) + \hat{\mathbf{u}}^{\mathrm{l}}\left(\hat{\boldsymbol{\xi}}_\perp, \hat{\boldsymbol{X}}_\perp, \hat{Z}, \hat{t}_l, \hat{t}_g\right), \tag{6.27}$$

where $\hat{\mathbf{u}}_g$ is the spatio-temporal average of $\hat{\mathbf{u}}$ and $\hat{\mathbf{u}}_l$ is the deviation. We will just use the notation $\langle\rangle_l$ for the spatio-temporal average operator later. Note

$$\langle \hat{\mathbf{u}} \rangle_l = \hat{\mathbf{u}}^{\mathrm{g}},$$
$$\langle \hat{\mathbf{u}}^{\mathrm{l}} \rangle_l = \mathbf{0}.$$

6.2 The reduced equation

In this section we will substitute the expansions (6.21), (6.22), (6.23) and (6.24) into the equations of motion (6.17), (6.18), (6.19) and (6.20), and collect terms of the same order to obtain the dynamics to desired order.

6.2.1 The lowest order dynamics

To lowest order in Ro the continuity equation reduces to

$$0 = \left(\hat{\nabla}_{l,\perp} + \hat{\nabla}_\parallel\right) \cdot \left(\hat{\mathbf{u}}_{0,\perp} + \hat{\mathbf{u}}_{0,\parallel}\right) = \hat{\nabla}_{l,\perp} \cdot \hat{\mathbf{u}}_{0,\perp} + \hat{\nabla}_\parallel \cdot \hat{\mathbf{u}}_{0,\parallel}, \tag{6.28}$$

where we have used that the vertical divergence of the horizontal velocity is zero, i.e., $\hat{\nabla}_{\|} \cdot \hat{\mathbf{u}}_{0,\perp} = 0$, since the horizontal unit vectors in the spherical coordinate system does not change with radius. In addition, it should be noted that $\hat{\nabla}_{l,\perp} \cdot \hat{\mathbf{u}}_{0,\|} = 2\Gamma \frac{\hat{w}_0}{\hat{r}}$ does not give any contribution to lowest order, since this term is of $\mathcal{O}(\Gamma)$. The lowest order horizontal momentum equation states that the lowest order pressure is independent of local spatio-temporal coordinates,

$$\mathbf{0} = -\hat{\nabla}_{l,\perp} \hat{\tilde{p}}_0, \tag{6.29}$$

which means that the deviation from the spatio-temporal avarege of $\hat{\tilde{p}}_0$ over the local scale is zero, i.e. $\hat{\tilde{p}}_0^{\mathrm{g}} = \hat{\tilde{p}}_0$ and $\hat{\tilde{p}}_0^{\mathrm{l}} = 0$. Thus, the zeroth order pressure $\hat{\tilde{p}}_0$ may be associated with the large-scale dynamics. Later we will see that it is this pressure that gives rise to the lowest-order horizontal velocity on a global scale. The vertical momentum equation gives that the zeroth order pressure gradient is balanced with the zeroth order buoyancy force,

$$\mathbf{0} = -\hat{\nabla}_{\|} \hat{\tilde{p}}_0 - \hat{\tilde{\rho}}_0 \hat{\mathbf{r}}, \tag{6.30}$$

which implies that the zeroth order mass density $\hat{\tilde{\rho}}_0$ is also independent of local spatio-temporal coordinates. We will not write down the thermodynamic equation for the mass density to lowest order, since it later will appear that this does not make any contribution because the vertical velocity to lowest order will turn out to be zero.

6.2.2 The geostrophic flow

To first order in Ro the continuity equation reduces to

$$0 = \hat{\nabla}_{l,\perp} \cdot \hat{\mathbf{u}}_{1,\perp} + \hat{\nabla}_{\|} \cdot \hat{\mathbf{u}}_{1,\|} + \frac{\Gamma}{Ro} \hat{\nabla}_{g,\perp} \cdot \hat{\mathbf{u}}_{0,\perp}, \tag{6.31}$$

where $\hat{\nabla}_{l,\perp} \cdot \hat{\mathbf{u}}_{1,\|}$ and $\hat{\nabla}_{g,\perp} \cdot \hat{\mathbf{u}}_{0,\|}$ are not included since these terms are one order of magnitude higher than the other terms. The first order horizontal momentum equation gives that the Coriolis force is balanced by the local pressure gradient to first order and the global pressure gradient to zeroth order

$$\mathbf{0} = -\hat{\nabla}_{l,\perp} \hat{\tilde{p}}_1 - \frac{\Gamma}{Ro} \hat{\nabla}_{g,\perp} \hat{\tilde{p}}_0 - \sin\theta_g \, \hat{\mathbf{r}} \times \hat{\mathbf{u}}_{0,\perp}, \tag{6.32}$$

since $\frac{\Gamma}{Ro}$ is of $\mathcal{O}(1)$. Note that the only contribution to the first term on the right hand side of the horizontal momentum equation is $\hat{\tilde{p}}_1^{\mathrm{l}}$, since $\hat{\tilde{p}}_1^{\mathrm{g}}$ only depend on the global coordiantes. By taking the spatio-temporal average of equation (6.32) over the local scales, it follows that the Coriolis force that is associated with the large-scale part of the horizontal velocity field is balanced by the horizontal pressure gradient which corresponds to the large-scale dynamics,

$$\mathbf{0} = -\frac{\Gamma}{Ro}\hat{\nabla}_{g,\perp}\hat{\tilde{p}}_0 - \sin\theta_g\,\widehat{\mathbf{r}}\times\hat{\mathbf{u}}^{\mathrm{g}}_{0,\perp}. \tag{6.33}$$

Similar, by subtracting the average equation (6.33), from equation (6.32), it follows that the Coriolis force that is associated with the small-scale part of the horizontal velocity field is balanced by the horizontal pressure gradient which correspond to the small-scale dynamics,

$$\mathbf{0} = -\hat{\nabla}_{l,\perp}\hat{\tilde{p}}_1 - \sin\theta_g\,\widehat{\mathbf{r}}\times\hat{\mathbf{u}}^{\mathrm{l}}_{0,\perp}. \tag{6.34}$$

If we take the cross product with $\widehat{\mathbf{r}}$ of equations (6.33) and (6.34), it follows that the horizontal velocity field associated with the large and small scales are respectively given by

$$\hat{\mathbf{u}}^{\mathrm{g}}_{0,\perp} = \frac{\Gamma}{Ro}\frac{1}{\sin\theta_g}\widehat{\mathbf{r}}\times\hat{\nabla}_{g,\perp}\hat{\tilde{p}}_0, \tag{6.35}$$

$$\hat{\mathbf{u}}^{\mathrm{l}}_{0,\perp} = \frac{1}{\sin\theta_g}\widehat{\mathbf{r}}\times\hat{\nabla}_{l,\perp}\hat{\tilde{p}}_1, \tag{6.36}$$

such that the total horizontal velocity field is given by

$$\hat{\mathbf{u}}_{0,\perp} = \hat{\mathbf{u}}^{\mathrm{g}}_{0,\perp} + \hat{\mathbf{u}}^{\mathrm{l}}_{0,\perp}, \tag{6.37}$$

$$= \frac{\Gamma}{Ro}\frac{1}{\sin\theta_g}\widehat{\mathbf{r}}\times\hat{\nabla}_{g,\perp}\hat{\tilde{p}}_0 + \frac{1}{\sin\theta_g}\widehat{\mathbf{r}}\times\hat{\nabla}_{l,\perp}\hat{\tilde{p}}_1. \tag{6.38}$$

Note that the large-scale horizontal velocity $\hat{\mathbf{u}}^{\mathrm{g}}_{0,\perp}$ will be independent of the local spatio-temporal coordinates and that the horizontal divergence of the zeroth order horizontal velocity, (6.37), is divergence free;

$$\hat{\nabla}_{l,\perp}\cdot\hat{\mathbf{u}}_{0,\perp} = 0. \tag{6.39}$$

Therefore, it follows from the lowest order continuity equation, (6.28), that the vertical divergence of the zeroth order vertical velocity vanishes,

$$0 = \hat{\nabla}_{\parallel}\cdot\hat{\mathbf{u}}_{0,\parallel}, \tag{6.40}$$

which implies that the zeroth order vertical velocity vanishes if it is zero at the boundary. According to the boundary value conditions from chapter 5, it will be zero to lowest order at the free surface, i.e.,

$$\hat{w}_0 = 0, \tag{6.41}$$

everywhere in space and time. Because of equation (6.39), it also follows that the horizontal velocity field which is associated with the small-scale dynamics also is divergence free,

$$\hat{\nabla}_{l,\perp} \cdot \hat{\mathbf{u}}^{l}_{0,\perp} = 0, \tag{6.42}$$

hence, it follows from Helmholtz theorem and equation (6.36) that $\frac{\hat{\tilde{p}}_1}{\sin\theta_g}$ acts as a streamfunction for the lowest order local horizontal velocity field $\hat{\mathbf{u}}^{l}_{0,\perp}$.
The vertical momentum equation to first order in Ro gives the same balance as the zero order vertical momentum equation, namely that the first order pressure gradient is balanced with the first order buoyancy force

$$\mathbf{0} = -\hat{\nabla}_{\parallel}\hat{\tilde{p}}_1 - \hat{\tilde{\rho}}_1\hat{\mathbf{r}}, \tag{6.43}$$

but since $\hat{\tilde{p}}_0$ and $\hat{\tilde{\rho}}_0$ are independent of the local spatio-temporal coordinates, equation (6.30) only describes a balance on a large scale, and hence there is a slight difference between the equations. Because equation (6.43) describes a balance on both small and large scale spatio-temporal scales. If we take the spatio-temproal average of equation (6.43) over the local spatio-temporal scale, it follows to first order that there exist a balance on the large scale,

$$\mathbf{0} = -\hat{\nabla}_{\parallel}\hat{\tilde{p}}^{g}_1 - \hat{\tilde{\rho}}^{g}_1\hat{\mathbf{r}}. \tag{6.44}$$

Similar by subtracting equation (6.44) from equation (6.43) it follows that there exist a balance on the small scale,

$$\mathbf{0} = -\hat{\nabla}_{\parallel}\hat{\tilde{p}}^{l}_1 - \hat{\tilde{\rho}}^{l}_1\hat{\mathbf{r}}. \tag{6.45}$$

We see that (6.36), (6.42) and (6.45) are completely similar to the equations in section 5.1.1. The only difference in this section is that we in addition determine the dynamics that occurs on a large scales, and not surprisingly, the dynamics associated with large scale are described by geostrophic balance and hydrostatic equilibrium of the same order as the dynamics of small-scale when $\frac{\Gamma}{Ro}$ is of $\mathcal{O}(1)$.

In section 5.1.1 we saw that the horizontal velocity field to lowest order depend on the vertical coordinate because the mass density was dependent on the vertical coordinate. Therefore there is a vertical shear in the horizontal velocity which was given by the horizontal density gradients, equation (5.33). Hence we expect that the local horizontal velocity $\hat{\mathbf{u}}^{l}_{0,\perp}$ also has a vertical shear that depend on the local gradient of $\hat{\tilde{\rho}}_1$. By differentiating equation (6.36) with respect of the vertical coordinate $\hat{Z}$ and use the first order momentum balance in the vertical direction (6.43), it follows that the vertical shear in the local geostrophic velocity is related to the horizontal gradient of the first order mass density deviations by

$$\frac{\partial \hat{\mathbf{u}}^{\mathrm{l}}_{0,\perp}}{\partial \hat{Z}} = -\frac{1}{\sin\theta_g}\widehat{\mathbf{r}} \times \hat{\nabla}_{l,\perp}\hat{\widetilde{\rho}}_1. \tag{6.46}$$

Similarly, differentiating equation (6.35) with respect of the vertical coordinate $\hat{Z}$ and useing the zeroth order momentum balance in the vertical direction (6.30), it follows that the global geostrophic velocity also has a vertical shear that depend on the horizontal gradient of the zeroth order mass density deviations by

$$\frac{\partial \hat{\mathbf{u}}^{\mathrm{g}}_{0,\perp}}{\partial \hat{Z}} = -\frac{1}{\sin\theta_g}\frac{\Gamma}{Ro}\widehat{\mathbf{r}} \times \hat{\nabla}_{g,\perp}\hat{\widetilde{\rho}}_0. \tag{6.47}$$

The thermodynamic equation for the mass density to first order in Ro is given by

$$\left(\frac{\partial}{\partial \hat{t}_l} + \hat{\mathbf{u}}_{0,\perp}\cdot\hat{\nabla}_{l,\perp}\right)\hat{\widetilde{\rho}}_1 + \hat{\mathbf{u}}_{1,\parallel}\cdot\hat{\nabla}_{\parallel}\hat{\widetilde{\rho}}_0 + \frac{\Gamma}{Ro}\left(\frac{\partial}{\partial \hat{t}_g} + \hat{\mathbf{u}}_{0,\perp}\cdot\hat{\nabla}_{g,\perp}\right)\hat{\widetilde{\rho}}_0 = Bu\,\hat{N}^2\hat{w}_1, \tag{6.48}$$

where we have used that the zeroth order vertical velocity is zero, i.e., not included in the advection terms. Already now we see how the separation of spatio-temporal scales have led to interactions between scales. We see that the first order mass density $\hat{\widetilde{\rho}}_1$ is locally advected by the zeroth order total horizontal velocity (6.37), and similarly the zeroth order mass density $\hat{\widetilde{\rho}}_0$ is globally advected by the same velocity (6.37). According to section 5.3 we know that the vorticity due to buoyancy come from the thermodynamic equation for the mass density via the first order vertical velocity. Thus, we expect that the term $\hat{\mathbf{u}}_{1,\parallel}\cdot\hat{\nabla}_{\parallel}\hat{\widetilde{\rho}}_0$ plays the same role as the stratification term $Bu\,\hat{N}^2\hat{w}_1$ at least at the local scale. This means that $\hat{\mathbf{u}}_{1,\parallel}\cdot\hat{\nabla}_{\parallel}\hat{\widetilde{\rho}}_0$ will give rise to stratification on local scales in the sense that $\hat{\widetilde{\rho}}_0$ will be a background density compared to $\hat{\widetilde{\rho}}_1$. Therefore, inspired by the definition of the buoyancy frequency (2.192), we define a buoyancy frequency N_0 associated with the zero-order mass density by

$$N_0^2 = -\frac{g}{\rho_h}\frac{\partial\widetilde{\rho}_0}{\partial Z}. \tag{6.49}$$

Using dimensionless numbers, it follows that equation (6.49) can be written as

$$Bu_g\,\hat{N}_0^2 = -\frac{\partial\hat{\widetilde{\rho}}_0}{\partial \hat{Z}}, \tag{6.50}$$

where we have used that $\hat{\rho}_h \approx 1$ and defined the new Burger number Bu_g as

$$Bu_g = \left(\frac{N_{0,m}\delta L_{\parallel,m}}{2\Omega\delta L_{\perp,m}}\right)^2. \tag{6.51}$$

Therefore, by substituting equation (6.50) into the thermodynamic mass density equation (6.48) we get

$$\left(\frac{\partial}{\partial \hat{t}_l} + \hat{\mathbf{u}}_{0,\perp} \cdot \hat{\nabla}_{l,\perp}\right) \hat{\tilde{\rho}}_1 + \frac{\Gamma}{Ro} \left(\frac{\partial}{\partial \hat{t}_g} + \hat{\mathbf{u}}_{0,\perp} \cdot \hat{\nabla}_{g,\perp}\right) \hat{\tilde{\rho}}_0 = \left(Bu\, \hat{N}^2 + Bu_g\, \hat{N}_0^2\right) \hat{w}_1. \tag{6.52}$$

By taking the spatio-temporal average of equation (6.52) over the local scales, we can split the equation into an equation that describes the mass density at the local scale and at the global scale. Before we do that, we will try to rewrite the equation in a more friendly form, i.e., so it follows directly which part of the equation which disappears under the averaging process. Since the buoyancy frequency is independent of the local coordinates we can write equation (6.52) as

$$\begin{aligned} \hat{w}_1 = &- \left(\frac{\partial}{\partial \hat{t}_l} + \hat{\mathbf{u}}_{0,\perp} \cdot \hat{\nabla}_{l,\perp}\right) \left[\frac{\sin\theta}{Bu\, \hat{N}^2 + Bu_g\, \hat{N}_0^2} \frac{\partial}{\partial \hat{Z}} \left(\frac{\hat{\tilde{p}}_1}{\sin\theta}\right)\right] \\ &+ \frac{1}{Bu\, \hat{N}^2 + Bu_g\, \hat{N}_0^2} \frac{\Gamma}{Ro} \left(\frac{\partial}{\partial \hat{t}_g} + \hat{\mathbf{u}}_{0,\perp} \cdot \hat{\nabla}_{g,\perp}\right) \hat{\tilde{\rho}}_0. \end{aligned} \tag{6.53}$$

If we now use that the horizontal velocity field $\hat{\mathbf{u}}_{0,\perp}$ can be split into two parts, where the global part $\hat{\mathbf{u}}^{\mathrm{g}}_{0,\perp}$ is independent of the local spatio-temporal coordinates it follows that we can write equation (6.53) in such a way that we can isolate all parts that are only dependent on global spatio-temporal coordinates from the other terms, which depend on both global and local spatio-temporal coordinates. The result is

$$\begin{aligned} \hat{w}_1 = &- \frac{\partial}{\partial \hat{t}_l} \left[\frac{\sin\theta}{Bu\, \hat{N}^2 + Bu_g\, \hat{N}_0^2} \frac{\partial}{\partial \hat{Z}} \left(\frac{\hat{\tilde{p}}_1}{\sin\theta}\right)\right] \\ &- \hat{\nabla}_{l,\perp} \cdot \left(\hat{\mathbf{u}}_{0,\perp} \frac{\sin\theta}{Bu\, \hat{N}^2 + Bu_g\, \hat{N}_0^2} \frac{\partial}{\partial \hat{Z}} \left(\frac{\hat{\tilde{p}}_1}{\sin\theta}\right)\right) \\ &+ \hat{\nabla}_{l,\perp} \cdot \left[\frac{1}{Bu\, \hat{N}^2 + Bu_g\, \hat{N}_0^2} \hat{\tilde{p}}_1 \frac{\partial \hat{\mathbf{u}}^{\mathrm{g}}_{0,\perp}}{\partial \hat{Z}}\right] \\ &+ \frac{1}{Bu\, \hat{N}^2 + Bu_g\, \hat{N}_0^2} \frac{\Gamma}{Ro} \left(\frac{\partial}{\partial \hat{t}_g} + \hat{\mathbf{u}}^{\mathrm{g}}_{0,\perp} \cdot \hat{\nabla}_{g,\perp}\right) \hat{\tilde{\rho}}_0, \end{aligned} \tag{6.54}$$

where the last term on the right side depends only on global spatio-temporal coordinates and the vertical coordinate, the remaining terms depend on both the global- and local spatio-temporal coordinates. Note that in the second term on the right hand side we have used that the zeroth order horizontal velocity is incompressible at the local scale. In the third term on the right hand side we have used the scalar triple product as follows

$$\begin{aligned}
\hat{\mathbf{u}}^{\mathrm{l}}_{0,\perp} \cdot \hat{\nabla}_{g,\perp}\hat{\tilde{\rho}}_0 &= \frac{1}{\sin\theta}\left(\widehat{\mathbf{r}} \times \hat{\nabla}_{l,\perp}\hat{\tilde{p}}_1\right) \cdot \hat{\nabla}_{g,\perp}\hat{\tilde{\rho}}_0 \\
&= -\frac{1}{\sin\theta}\widehat{\mathbf{r}} \cdot \left(\hat{\nabla}_{g,\perp}\hat{\tilde{\rho}}_0 \times \hat{\nabla}_{l,\perp}\hat{\tilde{p}}_1\right) \\
&= -\frac{1}{\sin\theta}\left(\widehat{\mathbf{r}} \times \hat{\nabla}_{g,\perp}\hat{\tilde{\rho}}_0\right) \cdot \hat{\nabla}_{l,\perp}\hat{\tilde{p}}_1 \\
&= \frac{Ro}{\Gamma}\frac{\partial \hat{\mathbf{u}}^{\mathrm{g}}_{0,\perp}}{\partial \hat{Z}} \cdot \hat{\nabla}_{l,\perp}\hat{\tilde{p}}_1 \\
&= \frac{Ro}{\Gamma}\hat{\nabla}_{l,\perp} \cdot \left(\frac{\partial \hat{\mathbf{u}}^{\mathrm{g}}_{0,\perp}}{\partial \hat{Z}}\hat{\tilde{p}}_1\right)
\end{aligned} \tag{6.55}$$

with help of equation (6.47) and equation (6.36), in addition to using that the global velocity $\hat{\mathbf{u}}^{\mathrm{g}}_{0,\perp}$ is independent of local coordinates. If we take the spatio-temporal average of equation (6.54) over the local scale, it follows from the sublinear growth conditions that the only surviving terms are the last term on the right hand side of (6.54) and the global contribution of the vertical velocity $\hat{w}^g_1$ to first order on the left hand side of (6.54), i.e.,

$$\hat{w}^g_1 = \frac{1}{Bu\,\hat{N}^2 + Bu_g\,\hat{N}_0^2}\frac{\Gamma}{Ro}\left(\frac{\partial}{\partial \hat{t}_g} + \hat{\mathbf{u}}^{\mathrm{g}}_{0,\perp} \cdot \hat{\nabla}_{g,\perp}\right)\hat{\tilde{\rho}}_0. \tag{6.56}$$

Since equation (6.56) is an evolution equation for the lowest order mass density $\hat{\tilde{\rho}}_0$ confined to the global scale, it would be better to write (6.56) as

$$\frac{\Gamma}{Ro}\left(\frac{\partial}{\partial \hat{t}_g} + \hat{\mathbf{u}}^{\mathrm{g}}_{0,\perp} \cdot \hat{\nabla}_{g,\perp} + \hat{w}^g_1\frac{\partial}{\partial \hat{Z}}\right)\hat{\tilde{\rho}}_0 = Bu\,\hat{N}^2\hat{w}^g_1. \tag{6.57}$$

where we have used equation (6.50). By subtracting equation (6.56) from (6.54) it follows that the thermodynamic equation on the local scale is given by

$$\begin{aligned}
\hat{w}^{\mathrm{l}}_1 = &-\left(\frac{\partial}{\partial \hat{t}_l} + \hat{\mathbf{u}}_{0,\perp} \cdot \hat{\nabla}_{l,\perp}\right)\left[\frac{\sin\theta}{Bu\,\hat{N}^2 + Bu_g\,\hat{N}_0^2}\frac{\partial}{\partial \hat{Z}}\left(\frac{\hat{\tilde{p}}_1}{\sin\theta}\right)\right] \\
&+ \frac{\sin\theta}{Bu\,\hat{N}^2 + Bu_g\,\hat{N}_0^2}\frac{\partial \hat{\mathbf{u}}^{\mathrm{g}}_{0,\perp}}{\partial \hat{Z}} \cdot \hat{\nabla}_{l,\perp}\left(\frac{\hat{\tilde{p}}_1}{\sin\theta}\right).
\end{aligned} \tag{6.58}$$

If we compare equation (6.58) and (6.57), we see that these equations can be used to solve for the zeroth- and first order pressure and mass density, which can then be used to find the velocity field to lowest order. But to do this we need to know the first order vertical velocity, which is undetermined to this order. However, the main difference between the equations is that the global thermodynamic equation (6.57) does not contain any interaction terms from local scales. This is not surprising since we have the sublinear growth condition which says that the spatio-temporal average of a local divergence of a variable is zero and in addition we know that the zeroth order mass density have to be independent of local coordinates. The local thermodynamic equation (6.58) contains two interaction terms with the global scale on the right hand side. The first term describes local advection of the first order mass density (since the vertical gradient of the first order pressure is equal to the first order mass density) with the zeroth order horizontal global velocity, and the second term describe local advection of the first order pressure with the vertical shear of the zeroth order horizontal global velocity.

6.2.3 The ageostrophic flow

The only equation we have to find to second order in Ro is the horizontal momentum equation, because this equation will close the thermodynamic equations (6.57) and (6.58). This means that we can find a complete equation for the vertical velocity to first order that depends on lower order quantities. The horizontal momentum equation to this order reads

$$\left.\frac{\partial \hat{\mathbf{u}}^{\mathrm{l}}_{0,\perp}}{\partial \hat{t}_l}\right|_{\mathbf{e}_i} + \hat{\mathbf{u}}_{0,\perp} \cdot \hat{\nabla}_{l,\perp} \hat{\mathbf{u}}^{\mathrm{l}}_{0,\perp}\Big|_{\mathbf{e}_i} = -\hat{\nabla}_{l,\perp}\hat{\tilde{p}}_2 - \frac{\Gamma}{Ro}\hat{\nabla}_{g,\perp}\hat{\tilde{p}}_1 - \sin\theta_g\,\widehat{\mathbf{r}} \times \hat{\mathbf{u}}_{1,\perp}. \tag{6.59}$$

where we have used that the global horizontal velocity to zeroth order is independent of global coordinates. Note that the advection term can be written as

$$\hat{\mathbf{u}}_{0,\perp} \cdot \hat{\nabla}_{l,\perp} \hat{\mathbf{u}}^{\mathrm{l}}_{0,\perp}\Big|_{\mathbf{e}_i} = \left(\hat{\nabla}_{l,\perp} \times \hat{\mathbf{u}}^{\mathrm{l}}_{0,\perp}\Big|_{\mathbf{e}_i}\right) \times \hat{\mathbf{u}}_{0,\perp} + \hat{\nabla}_{l,\perp}\left[\hat{\mathbf{u}}^{\mathrm{l}}_{0,\perp} \cdot \left(\frac{1}{2}\hat{\mathbf{u}}^{\mathrm{l}}_{0,\perp} + \hat{\mathbf{u}}^{\mathrm{g}}_{0,\perp}\right)\right]. \tag{6.60}$$

Since equation (6.59) is a nonlinear evolution equation for the local geostrophic velocity, it is difficult to use the sublinear growth condition to separate the equation into a global equation and a local equation. Thus, we will in the following section apply the vorticity formalism, which has a simpler structure according to the application of the sublinear growth condition. In addition the introduction of vorticity will also remove the second order pressure $\hat{\tilde{p}}_2$, which is undetermined to this order.

6.3 The quasi-geostrophic potential vorticity

In the following we will derive the corresponding vorticity equation of the horizontal momentum equation (6.59). Since the unit vectors of the local velocity field

$\hat{\mathbf{u}}^{\mathrm{l}}_{0,\perp}$ is kept constant during the material derivative in the horizontal momentum equation, it will be natural to define the vorticity that is associated with the local velocity field $\hat{\mathbf{u}}^{\mathrm{l}}_{0,\perp}$ as

$$\hat{\mathbf{\Theta}}_0 = \hat{\nabla}_{l,\perp} \times \hat{\mathbf{u}}^{\mathrm{l}}_{0,\perp}\Big|_{\mathbf{e}_i} = \widehat{\mathbf{r}}\,\hat{\nabla}^2_{l,\perp}\left(\frac{\hat{\widetilde{p}}_1}{\sin\theta_g}\right). \tag{6.61}$$

Anyway, the unit vectors are independent of local coordinates, this means that the unit vectors are constant on the local scales. Therefore, we do not need to require that the unit vectors must be held constant during differentiation with respect of local spatio-temporal coordinates. If we take the horizontal curl $\hat{\nabla}_{l,\perp}\times$ of the ageostrophic momentum equation (6.59), we get the vorticity equation

$$\left(\frac{\partial}{\partial\hat{t}_l} + \hat{\mathbf{u}}_{0,\perp}\cdot\hat{\nabla}_{l,\perp}\right)\hat{\mathbf{\Theta}}_0 = -\frac{\Gamma}{Ro}\,\hat{\nabla}_{l,\perp}\times\hat{\nabla}_{g,\perp}\hat{\widetilde{p}}_1\Big|_{\mathbf{e}_i} - \sin\theta_g\,\hat{\nabla}_{l,\perp}\times(\widehat{\mathbf{r}}\times\hat{\mathbf{u}}_{1,\perp})\big|_{\mathbf{e}_i}. \tag{6.62}$$

Just to point out that the unit vectors remain constant during the local differentiation, we use the vertical line to symbolizes that the unit vectors are constant at the local scale. Recognizing that the second term on the right-hand side can be written as

$$\hat{\nabla}_{l,\perp}\times(\widehat{\mathbf{r}}\times\hat{\mathbf{u}}_{1,\perp})\big|_{\mathbf{e}_i} = \widehat{\mathbf{r}}\,\left(\hat{\nabla}_{l,\perp}\cdot\hat{\mathbf{u}}_{1,\perp}\right)\Big|_{\mathbf{e}_i}, \tag{6.63}$$

it follows that all terms have only a vertical component. Thus we can take the dot product of the unit vector along the vertical direction $\widehat{\mathbf{r}}$ with equation (6.62) to obtain a scalar equation

$$\left(\frac{\partial}{\partial\hat{t}_l} + \hat{\mathbf{u}}_{0,\perp}\cdot\hat{\nabla}_{l,\perp}\right)\hat{\nabla}^2_{l,\perp}\frac{\hat{\widetilde{p}}_1}{\sin\theta} = -\frac{\Gamma}{Ro}\widehat{\mathbf{r}}\cdot\left(\hat{\nabla}_{l,\perp}\times\hat{\nabla}_{g,\perp}\hat{\widetilde{p}}_1\Big|_{\mathbf{e}_i}\right) - \sin\theta_g\,\left(\hat{\nabla}_{l,\perp}\cdot\hat{\mathbf{u}}_{1,\perp}\right)\Big|_{\mathbf{e}_i}. \tag{6.64}$$

The first term on the right hand side of equation (6.64) represents the production of vorticity due to local rotation of the global pressure gradient to first order. This term is caused by the global horizontal compression of the fluid to lowest order and the meridional variation of the Coriolis force. This can easily be shown by taking the global horizontal divergence of equation (6.38) when the unit vectors are held constant during the differentiation,

$$\widehat{\mathbf{r}}\cdot\left(\hat{\nabla}_{l,\perp}\times\hat{\nabla}_{g,\perp}\hat{\widetilde{p}}_1\Big|_{\mathbf{e}_i}\right) = \sin\theta_g\,\hat{\nabla}_{g,\perp}\cdot\hat{\mathbf{u}}_{0,\perp}\Big|_{\mathbf{e}_i} + \cos\theta\,\hat{v}_0. \tag{6.65}$$

The second term on the right hand side of (6.64) represents the production of vorticity due to local horizontal compression of the first order horizontal velocity field. This term can be rewritten with help of the first order continuity equation (6.31),

$$\hat{\nabla}_{l,\perp} \cdot \hat{\mathbf{u}}_{1,\perp}\Big|_{\mathbf{e}_i} = -\,\hat{\nabla}_{\|} \cdot \hat{\mathbf{u}}_{1,\|}\Big|_{\mathbf{e}_i} - \frac{\Gamma}{Ro}\,\hat{\nabla}_{g,\perp} \cdot \hat{\mathbf{u}}_{0,\perp}\Big|_{\mathbf{e}_i}. \tag{6.66}$$

With the help of (6.65) and (6.66), the vorticity equation (6.64) can be written as

$$\left(\frac{\partial}{\partial \hat{t}_l} + \hat{\mathbf{u}}_{0,\perp} \cdot \hat{\nabla}_{l,\perp}\right) \hat{\nabla}^2_{l,\perp}\left(\frac{\hat{\tilde{p}}_1}{\sin\theta}\right) = -\frac{\Gamma}{Ro}\cos\theta_g\,\hat{v}_0 + \sin\theta_g\,\frac{\partial \hat{w}_1}{\partial \hat{Z}}. \tag{6.67}$$

Already now we can see that the vorticity equation (6.67) is quite similar to the barotropic vorticity equation (5.74). The only difference is that equation (5.74) only contains variables that are valid at the local scale, while equation (6.67) contains variables that apply on both global and local scales. The second term on the right hand side of equation (5.74) represents the planetary vorticity, which we have shown in section 5.3 can be written as $\beta\,\hat{v}_0$[1], where the dimensionless meridional variation/gradient of the Coriolis force β is given by equation (3.4). The planetary vorticity in equation (6.67) is describe by the first term on the right side. The factor in front of the meridional velocity is the same as β, except that $\frac{\Gamma}{Ro}\cos\theta_g$ is not constant. However $\frac{\Gamma}{Ro}\cos\theta_g$ will be constant on the local scale, since it only depends on the global coordinate of latitude. Thus, we will later use that

$$\beta = \frac{\Gamma}{Ro}\cos\theta_g. \tag{6.68}$$

Similar to the thermodynamic equation for the mass density we want to write the vorticity equation (6.67) in such a way that we can isolate all parts that are only dependent on global spatio-temporal coordinates from the other terms, which depend on both global and local spatio-temporal coordinates. This is done by splitting the velocity fields into a local part and a global part. This results in

$$\begin{aligned}\left(\frac{\partial}{\partial \hat{t}_l} + \hat{\mathbf{u}}_{0,\perp} \cdot \hat{\nabla}_{l,\perp}\right) \hat{\nabla}^2_{l,\perp}\left(\frac{\hat{\tilde{p}}_1}{\sin\theta}\right) + \frac{\Gamma}{Ro}\cos\theta_g\,\hat{v}^{\mathrm{l}}_0 - \sin\theta_g\,\frac{\partial \hat{w}^{\mathrm{l}}_1}{\partial \hat{Z}} \\ = -\frac{\Gamma}{Ro}\cos\theta_g\,\hat{v}^{\mathrm{g}}_0 + \sin\theta_g\,\frac{\partial \hat{w}^{\mathrm{g}}_1}{\partial \hat{Z}},\end{aligned} \tag{6.69}$$

where the right hand side only depends on global spatio-temporal coordinates and vertical coordinates, while the left hand side depend on both global and local spatio-temporal coordinates. By substituting the equation for the local vertical

[1]Note that equation (5.74) is only valid on local scales, and therefore $\hat{v}_0$ is the local meridional velocity which corresponds to $\hat{v}^l_0$ in this chapter.

velocity to first order, equation (6.58), the vorticity equation (6.69) can be written in a very similar way to the potential vorticity equation (5.76),

$$\left(\frac{\partial}{\partial \hat{t}_l} + \hat{\mathbf{u}}_{0,\perp} \cdot \hat{\nabla}_{l,\perp}\right) \hat{q} - \hat{\nabla}_{l,\perp} \cdot \left[\frac{\partial}{\partial \hat{Z}} \left(\frac{\sin^2\theta}{Bu\,\hat{N}^2 + Bu_g\,\hat{N}_0^2} \frac{\hat{\tilde{p}}_1}{\sin\theta_g} \frac{\partial \hat{\mathbf{u}}^{\mathrm{g}}_{0,\perp}}{\partial \hat{Z}}\right)\right]$$
$$= -\frac{\Gamma}{Ro} \cos\theta_g\, \hat{v}^{\mathrm{g}}_0 + \sin\theta_g \frac{\partial \hat{w}^{\mathrm{g}}_1}{\partial \hat{Z}}, \tag{6.70}$$

where we have defined the quasi-geostrophic potential vorticity

$$\hat{q} = \hat{\nabla}^2_{l,\perp}\left(\frac{\hat{\tilde{p}}_1}{\sin\theta_g}\right) + \frac{\partial}{\partial \hat{Z}}\left(\frac{\sin^2\theta_g}{Bu\,\hat{N}^2 + Bu_g\,\hat{N}_0^2} \frac{\partial}{\partial \hat{Z}}\left(\frac{\hat{\tilde{p}}_1}{\sin\theta_g}\right)\right) + \frac{\Gamma}{Ro}\cos\theta_g\,\theta_l. \tag{6.71}$$

Since the local horizontal divergence of the zeroth order horizontal velocity $\hat{\mathbf{u}}_{0,\perp}$ is zero, equation (6.70) may be written in conservative form as

$$\frac{\partial\,\hat{q}}{\partial \hat{t}_l} + \hat{\nabla}_{l,\perp} \cdot (\hat{\mathbf{u}}_{0,\perp}\hat{q}) - \hat{\nabla}_{l,\perp} \cdot \left(\frac{\partial}{\partial \hat{Z}} \left(\frac{\sin^2\theta}{Bu\,\hat{N}^2 + Bu_g\,\hat{N}_0^2} \frac{\hat{\tilde{p}}_1}{\sin\theta} \frac{\partial \hat{\mathbf{u}}^{\mathrm{g}}_{0,\perp}}{\partial \hat{Z}}\right)\right)$$
$$= -\frac{\Gamma}{Ro} \cos\theta_g\, \hat{v}^{\mathrm{g}}_0 + \sin\theta_g \frac{\partial \hat{w}^{\mathrm{g}}_1}{\partial \hat{Z}}. \tag{6.72}$$

The beauty of writing the vorticity equation (6.72) in conservative form is that it is very easy to apply the sublinear growth condition in order to find the vorticity equation on the local scale and the vorticity equation on the global scale.

6.3.1 The global vorticity equation

If we take the spatio-temporal average of equation (6.72) over the local scales, it follows from the sublinear growth condition that all terms on the left hand side vanish and the only surviving terms are those on the right hand side where the variables are independent of local coordinates,

$$0 = -\frac{\Gamma}{Ro} \cos\theta_g\, \hat{v}^{\mathrm{g}}_0 + \sin\theta_g \frac{\partial \hat{w}^{\mathrm{g}}_1}{\partial \hat{Z}}. \tag{6.73}$$

Equation (6.73) is known as the global vorticity equation. To interpret this equation, we take the spatio-temporal average of the first-order continuity equation, equation (6.31) which provides

$$0 = \frac{\partial \hat{w}^{\mathrm{g}}_1}{\partial \hat{Z}} + \frac{\Gamma}{Ro} \hat{\nabla}_{g,\perp} \cdot \hat{\mathbf{u}}^{\mathrm{g}}_{0,\perp}. \tag{6.74}$$

Thos equation states that the global horizontal divergence of the global horizontal velocity field gives rise to vortex-tube stretching. According to equation (6.73), the vorticity production that is associated with vortex-tube stretching is balanced by the planetary vorticity. Just to summarize a bit, the equations that form a model on the global scale are given by the global vorticity equation (6.73), the thermodynamic mass density equation (6.57) and the global horizontal geostrophic velocity (6.35). Some people like to introduce a potential vorticity equation instead of the thermodynamic equation. So if we differentiate equation (6.57) in respect to $\hat{Z}$, we get

$$\frac{\Gamma}{Ro}\left(\frac{\partial}{\partial \hat{t}_g} + \hat{\mathbf{u}}^{\mathrm{g}}_{0,\perp}\cdot\hat{\nabla}_{g,\perp} + \frac{Ro}{\Gamma}\hat{w}^g_1\frac{\partial}{\partial \hat{Z}}\right)\left(\frac{\partial \hat{\tilde{\rho}}_0}{\partial \hat{Z}}\right) = -\frac{\partial \hat{w}^g_1}{\partial \hat{Z}}\frac{\partial \hat{\tilde{\rho}}_0}{\partial \hat{Z}} + \frac{\partial}{\partial \hat{Z}}\left(Bu\,\hat{N}^2\hat{w}^g_1\right), \tag{6.75}$$

or equivalently

$$\left(\frac{\partial}{\partial \hat{t}_g} + \hat{\mathbf{u}}^{\mathrm{g}}_{0,\perp}\cdot\hat{\nabla}_{g,\perp} + \frac{Ro}{\Gamma}\hat{w}^g_1\frac{\partial}{\partial \hat{Z}}\right)\left(\sin\theta_g\frac{\partial \hat{\tilde{\rho}}_0}{\partial \hat{Z}}\right) = \frac{\partial}{\partial \hat{Z}}\left(\frac{Ro\,Bu}{\Gamma}\sin\theta_g\hat{N}^2\hat{w}^g_1\right), \tag{6.76}$$

where we have used equation (6.73). Equation (6.76) is the global potential vorticity equation and according to the buoyancy frequency (6.50), this is an evolution equation for the stratification that is associated with the zeroth order global mass density $\hat{\tilde{\rho}}_0$. Note that none of the equations for the global dynamics not contain interactions terms with local variables.

6.3.2 The local vorticity equation

The equations that determines the dynamics on the local scale is given by the deviation equations from global scale. If we subtract the global vorticity equation (6.73) from the total vorticity equation (6.70), we get the local vorticity equation that determines the vorticity on the local scale, The local vorticity equation reads

$$\begin{aligned}&\left(\frac{\partial}{\partial \hat{t}_l} + \hat{\mathbf{u}}^{\mathrm{l}}_{0,\perp}\cdot\hat{\nabla}_{l,\perp}\right)\hat{q} + \hat{\mathbf{u}}^{\mathrm{g}}_{0,\perp}\cdot\hat{\nabla}_{l,\perp}\,q\\ &-\frac{\partial}{\partial \hat{Z}}\left[\frac{\sin^2\theta_g}{Bu\,\hat{N}^2 + Bu_g\,\hat{N}_0^2}\frac{\partial \hat{\mathbf{u}}^{\mathrm{g}}_{0,\perp}}{\partial \hat{Z}}\cdot\hat{\nabla}_{l,\perp}\left(\frac{\hat{\tilde{\rho}}_1}{\sin_g\theta}\right)\right] = 0,\end{aligned} \tag{6.77}$$

where the first term on the left hand side describes the evolution of the quasi-geostrophic potential vorticity $\hat{q}$ with the local scale. This part of the vorticity equation is completely similar to the baroclinic quasi-geostrophic potential vorticity equation (5.76) in the previous chapter in the limit where the fluid is inviscid.

The additional terms describes interactions with global scale, where the second term represent advection of the quasi-geostrophic potential vorticity $\hat{q}$ with the global velocity, and the third term describes interaction between the first order mass density $\widetilde{\rho}_1$ and the shear in the global velocity. The global flow will aslo contribute to buoyancy dynamics on the local scale. This is because the global flow will be associated with a mass density that will produce the same effect as the real background state of the ocean, i.e., will give rise to stratiffcation on the local scale. Therefore, the quasi-geostrophic potential vorticity $\hat{q}$ depend on $\hat{N}_0$.

Chapter 7

Conclusion

In this thesis we have used a regular perturbation theory and scaling anaylse to derive reduced models confined to the midlatitude region. The main result is the quasi-geostrophic potential vorticity model in chapter 6, which describes local dynamics in the midlatitude region under influence of global scales. This is a modified version of the classical baroclinic quasi-geostrophic vorticity in chapter 5. The modified model included interaction between the local quasi-geostrophic potential vorticity and the global geostrophic velocity. In addition the global flow contribution to buoyancy dynamics on the local scale. This is because the global flow will be associated with a mass density that will produce the same effect as the real background state of the ocean, namely will give rise to stratification on the local scale.

Chapter 8

Appendix A

The equations of motion can be obtained in many differet ways; kinetic theory, Raynholds transport theorem or with a control volum method. The equations describes the evolution of mass density, momentum density, and energy denity. These equations has more unknowns than the number of equations, thus we need thermodynamic equations of state to close the system of equations. In this section we will only present the equations, and it will always be understood that it only applies during an initial system.

8.1 The equations of motion

8.1.1 The equation of continuity

The equation for the mass density ρ without any source and sink, reads in flux form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{u} = 0, \tag{8.1}$$

where $\mathbf{u}$ is the fluid velocity. By introducing the substantial time derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla, \tag{8.2}$$

the continuity equation can be written in parcel form as

$$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{u} = 0. \tag{8.3}$$

The salinity equation

The ocean consists of salt water, where both components, salt and freshwater must satisfy the continuity equation for mass, i.e.

$$\frac{\partial \rho_s}{\partial t} + \nabla \cdot \rho_s \mathbf{u}_s = 0, \tag{8.4}$$
$$\frac{\partial \rho_w}{\partial t} + \nabla \cdot \rho_w \mathbf{u}_w = 0. \tag{8.5}$$

The mass density of the salt ρ_s and the mass density of the freshwater compose the total mass density ρ, i.e. $\rho = \rho_s + \rho_w$. By adding (8.4) and (8.5) together, yields the conervation of the salt water,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{u} = 0, \tag{8.6}$$

where we have used the mass-weighted mean velocity $\mathbf{u} = \frac{\rho_s \mathbf{u}_s + \rho_w \mathbf{u}_w}{\rho}$. The fraction of the salt and freshwater are respectively given by

$$S = \frac{\rho_s}{\rho}, \quad W = \frac{\rho_w}{\rho}. \tag{8.7}$$

Therefor, the (8.4) can be written as an equation for the fraction of salt,

$$\rho \frac{dS}{dt} = -\nabla \cdot \mathbf{J}_S, \tag{8.8}$$

where $\mathbf{J}_S = \rho S \left(\mathbf{u}_s - \mathbf{u}\right)$ is the diffusive salinity flux. The equation for the fraction of freshwater is

$$\rho \frac{dW}{dt} = -\nabla \cdot \mathbf{J}_W, \tag{8.9}$$

where $\mathbf{J}_W = \rho W \left(\mathbf{u}_w - \mathbf{u}\right)$ is the diffusive freshwater flux.

8.1.2 The equation of momentum

The equation for the momentum density $\rho\mathbf{u}$ with external density body forces $\mathbf{f}$, reads in flux form

$$\frac{\partial \left(\rho \mathbf{u}\right)}{\partial t} = -\nabla \cdot \left(\rho \mathbf{u}\mathbf{u} + \boldsymbol{\sigma}\right) + \mathbf{f}, \tag{8.10}$$

where $\boldsymbol{\sigma}$ is the stresstensor. By using the chain rule and the continuity equation, (8.1), the equation of the momentum can be written in parcel form as

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla \cdot \boldsymbol{\sigma} + \mathbf{f}. \tag{8.11}$$

8.1.3 The equation of energy

The equation for the energy density $\frac{1}{2}\rho\mathbf{u}^2 + \rho\epsilon$, reads in flux form

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\mathbf{u}^2 + \rho\epsilon\right) + \nabla\cdot\left[\left(\frac{\rho}{2}\mathbf{u}^2 + \rho\epsilon\right)\mathbf{u} + \mathbf{u}\cdot\boldsymbol{\sigma} + \mathbf{Q}\right] = \mathbf{f}\cdot\mathbf{u}, \tag{8.12}$$

where $\frac{1}{2}\rho\mathbf{u}^2$ is the kinetic energy density, $\rho\epsilon$ is the internal energy density, ϵ is the internal energy per unit mass and $\mathbf{Q}$ is the total heat flux density[1]. By taking the dot product between the momentum equation, (8.11) and the fluid velocity $\mathbf{u}$, we obtain an equation for the kinetic energy in parcel form

$$\rho\frac{d}{dt}\left(\frac{1}{2}\mathbf{u}^2\right) = -\mathbf{u}\cdot(\nabla\cdot\boldsymbol{\sigma}) + \mathbf{f}\cdot\mathbf{u}, \tag{8.13}$$

or equivalent in flux form

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\mathbf{u}^2\right) + \nabla\cdot\left(\frac{1}{2}\rho\mathbf{u}^2\mathbf{u}\right) + \mathbf{u}\cdot(\nabla\cdot\boldsymbol{\sigma}) = \mathbf{f}\cdot\mathbf{u}. \tag{8.14}$$

An equation for the internal energy is obtained by subtracting (8.14) from (8.12), by then using the tensor identity

$$\begin{aligned} \mathbf{u}\cdot(\nabla\cdot\boldsymbol{\sigma}) &= \nabla\cdot(\mathbf{u}\cdot\boldsymbol{\sigma}) - (\boldsymbol{\sigma}\cdot\nabla)\cdot\mathbf{u} \\ &= \nabla\cdot(\mathbf{u}\cdot\boldsymbol{\sigma}) - \boldsymbol{\sigma} : \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right). \end{aligned} \tag{8.15}$$

In flux form the equation read

$$\frac{\partial}{\partial t}(\rho\epsilon) + \nabla\cdot(\rho\epsilon\mathbf{u}) + \nabla\cdot\mathbf{Q} = -\boldsymbol{\sigma} : \mathsf{D}, \tag{8.16}$$

and in parcel form the equation read

$$\rho\frac{d\epsilon}{dt} = -\boldsymbol{\sigma} : \mathsf{D} - \nabla\cdot\mathbf{Q}, \tag{8.17}$$

where

$$\mathsf{D} = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right) \tag{8.18}$$

is the deformation tensor.

[1]The total heat flux density consist of energy transfers to the system that are not mechanical, e.g. thermal conductivity, heat do to chemical reaction and so on.

8.2 Thermodynamic and closure of the equations

8.2.1 The law of thermodynamics

The first law of thermodynamics (T1) states that if a small amount of heat per unit mass Δq is applied to a system, the internal energy per unit mass ϵ is changed simultaneously with the system performing a certain amount of physical work per unit mass Δw on the environment and chemical work $-\sum_i \mu_i \Delta N_i$. Since energy can not be created or destroyed, this means that the energy must be conserved, i.e.

$$\Delta q = \Delta\epsilon + \Delta w - \sum_i \mu_i \Delta N_i, \tag{8.19}$$

where μ_i is the chemical potential of particle type i and N_i is the fraction of particle type i. By using (8.7) and $S + W = 1$, the chemical work can be written as

$$\sum_i \mu_i \Delta N_i = \Delta\mu dS, \tag{8.20}$$

where $\Delta\mu = \mu_s - \mu_w$ is the chemical potential difference between the sea salt and water.
The second law of thermodynamics (T2) states that the entropy per unit mass s of an isolated system which is not in thermodynamic equilibrium increases with time, and will instantant attempt to achieve thermodynamic equilibrium, so that the energy has a minimum and entropy has a maximum. Thus, the inequality

$$\Delta s \geq \frac{\Delta q}{T}, \tag{8.21}$$

must be fulfilled. Where T is the temperature in the system and Δq are supplied amount of heat per unit mass. For processes which are reversible, the inequality sign go over to an equality sign. In the limit where the supplied amount of heat per unit mass is infinitesimal, i.e. $\Delta q \to dq$, all processes will be reversible. In this limit the first and the second law, (T1) and (T2) becomes

$$d\epsilon = Tds - dw + \Delta\mu dS. \tag{8.22}$$

This equation is the fundamental equation of thermodynamics. It can be shown that the work per unit mass dw of a fluid particle on its environment is

$$dw = pd\left(\frac{1}{\rho}\right) = -\frac{p}{\rho^2}d\rho, \tag{8.23}$$

where p is the kinematic pressure. If the volume of the fluid particle increases, dw will be negative so the fluid particle has done work on its environment, and if the volume decreases, the environment have done work on fluid particle. By dividing on the infinitesimal time dt, the fundamental equation of the thermodynamics becomes

$$\frac{d\epsilon}{dt} = T\frac{ds}{dt} + \frac{p}{\rho^2}\frac{d\rho}{dt} + \Delta\mu\frac{dS}{dt}, \tag{8.24}$$

or equivalent

$$\frac{d\epsilon}{dt} = T\frac{ds}{dt} - \frac{p}{\rho}\nabla\cdot\mathbf{u} - \frac{\Delta\mu}{\rho}\nabla\cdot\mathbf{J}_S, \tag{8.25}$$

where we have used the continuity equation $\frac{d\rho}{dt} = -\rho\nabla\cdot\mathbf{u}$ and the salinity equation $\rho\frac{dS}{dt} = -\nabla\cdot\mathbf{J}_S$.

The entropy equation

By substituting (8.25) into (8.17), the equation for the internal energy becomes a general equation of heat transfer

$$\rho T\frac{ds}{dt} = p\nabla\cdot\mathbf{u} - \boldsymbol{\sigma} : \mathsf{D} - \nabla\cdot\mathbf{Q} + \Delta\mu\nabla\cdot\mathbf{J}_S. \tag{8.26}$$

In the absence of heat exchange and energy dissipation due to internal friction. The fluid motion is adiabatic, i.e. that the entropy of any fluid particle remains constant. In this case the fluid is ideal, and it follows from (8.26) that the stress tensor must be equal to the pressure times the unit tensor,

$$\boldsymbol{\sigma} = p\mathsf{I}. \tag{8.27}$$

8.2.2 Thermodynamic state relation

Gibbs' phase rule and thermodynamic potentials

Gibbs' phase rule state the the degree of freedom F, i.e. the number of independent intensive variable that completely determines the thermodynamic properties of the system (such as pressure, temperatur and salinity), is equal to

$$F = C - P + 2, \tag{8.28}$$

where C is the number of components of the system and P is the number of phases in thermodynamic equilibrium with each other. The ocean consist of two components, salt and freshwater, and consist of one phase, liquid. Therefor, the number of independent variables are three. There are many different independent variables that can be used. These form the basis for the representation of the thermodynamic relations describing the physical nature of the fluid. The four most common choices of the intensive variables are

$$\left(\rho^{-1}, s, S\right), \quad (p, s, S), \quad \left(\rho^{-1}, T, S\right), \quad (p, T, S). \tag{8.29}$$

To represent the thermodynamic state of the fluid, there exist a thermodynamic function that determines the thermodynamic properties of the fluid. This function is often called the thermodynamic potential. It follows from (8.22) and (8.23) that the internal energy per unit mass ϵ is the thermodynamic potential when (ρ^{-1}, s, S) are chosen as the independent intensive variables. Therefor, it also follows that the equations of state to this potential are given by

$$T = \left(\frac{\partial \epsilon}{\partial s}\right)_{S,\rho}, \quad p = -\left(\frac{\partial \epsilon}{\partial \rho^{-1}}\right)_{S,s}, \quad \Delta\mu = \left(\frac{\partial \epsilon}{\partial S}\right)_{s,\rho}. \tag{8.30}$$

From Euler's homogeneous function theorem it follows that the internal energy per unit mass can be written as

$$\epsilon = Ts - p\rho^{-1} + \mu_S S + \mu_W (1 - S). \tag{8.31}$$

This equation is often called the Euler's identity. By subtracting the first law of thermodynamics, (8.22), from the total differential of Euler's identity, (8.31), result in the Gibbs-Durham relation

$$\rho^{-1}dp - sdT = Sd\mu_S + (1 - S)\, d\mu_W. \tag{8.32}$$

Instead of using (ρ^{-1}, s, S) as independent variables to describe the thermodynamic properties of the fluid, we will choose (p, T, S) as independent variables. The question is now: Which thermodynamic potential is a function of these variables, and what is the equation of state? By adding the differential $d\left(-Ts + p\rho^{-1}\right)$ on both side of (8.22), and define the free enthalpy per unit mass

$$g = \epsilon - Ts + p\rho^{-1}, \tag{8.33}$$

(8.22) becomes

$$dg = -sdT + \rho^{-1}dp + \Delta\mu dS, \tag{8.34}$$

which states that the free enthalpy per unit mass is the thermodynamic potetial with (p, T, S) as independent variables. From the chain rule it follows that the equations of state is

$$s = -\left(\frac{\partial g}{\partial T}\right)_{S,p}, \quad \rho^{-1} = \left(\frac{\partial g}{\partial p}\right)_{S,T}, \quad \Delta\mu = \left(\frac{\partial g}{\partial S}\right)_{T,p}. \tag{8.35}$$

Since the equations of state will be a function of (p, T, S), these equations can be used to transform the entropy equation, (8.26) to prognostic equations for temperature and mass density. The Euler's identety for the free enthalpy per unit mass is given by substitute the Euler's identety for the internal energy per unit mass, (8.31) in to the expression for the free enthalpy, (8.33),

$$g = \mu_S S + \mu_W (1 - S). \tag{8.36}$$

The temperature equation

From the chain rule it follows that the evolution equation for the entropy $s(p, T, S)$ is

$$T\frac{ds}{dt} = T\left(\frac{\partial s}{\partial T}\right)_{p,S}\frac{dT}{dt} + T\left(\frac{\partial s}{\partial p}\right)_{T,S}\frac{dp}{dt} + T\left(\frac{\partial s}{\partial S}\right)_{p,T}\frac{dS}{dt}, \tag{8.37}$$

where $T\left(\frac{\partial s}{\partial T}\right)_{p,S} = c_p$ is the spesific heat capasity. By using the Maxwell relation

$$\left(\frac{\partial s}{\partial p}\right)_{T,S} = \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial T}\right)_{p,S} = -\frac{1}{\rho}\beta_T, \tag{8.38}$$

here β_T is the thermal expansion coefficient, (8.26) and (8.8) the entropy equation, (8.37) becomes an evolution equation for the temperature,

$$\rho\, c_p\frac{dT}{dt} = T\beta_T\frac{dp}{dt} + p\nabla\cdot\mathbf{u} - \boldsymbol{\sigma} : \mathsf{D} - \nabla\cdot\mathbf{Q} + \left(\Delta\mu + T\left(\frac{\partial s}{\partial S}\right)_{p,T}\right)\nabla\cdot\mathbf{J}_S \tag{8.39}$$

Under adiabitic conditions, the stress tensor reduces to (8.27) and the temperature equation reduces to

$$\rho\, c_p\left(\frac{dT}{dt} - \frac{T\beta_T}{\rho\, c_p}\frac{dp}{dt}\right) = 0. \tag{8.40}$$

This means that the temperature and pressure are related by $dT = \frac{T\beta_T}{\rho\, c_p}dp$. Hence, the temperature is not a conserved quantity under adiabatic conditions, since change in pressure leads to change in temperature. Later we will introduce a potential temperature, which are a conserved quantity under adiabatic conditions. It will be more practical to deal with potential temperature instead of temperature.

The thermodynamic mass density equation

From the chain rule it follows that the evolution equation for the entropy $\rho^{-1}(p, T, S)$ is

$$\begin{aligned}
\frac{d\rho^{-1}}{dt} &= \left(\frac{\partial \rho^{-1}}{\partial T}\right)_{p,S}\frac{dT}{dt} + \left(\frac{\partial \rho^{-1}}{\partial p}\right)_{T,S}\frac{dp}{dt} + \left(\frac{\partial \rho^{-1}}{\partial S}\right)_{p,T}\frac{dS}{dt}, \\
\frac{d\rho}{dt} &= \rho\left(-\beta_T\frac{dT}{dt} + \beta_p\frac{dp}{dt} + \beta_S\frac{dS}{dt}\right),
\end{aligned} \tag{8.41}$$

where β_T, β_p and β_S are the thermal expansion coefficient, the compresibility coeffisient and the salinity contraction coefficient respectively. By using (8.39) and (8.8), (8.41) becomes an evolution equation for the mass density,

$$\frac{d\rho}{dt} = \rho\widetilde{\kappa}\frac{dp}{dt} - \frac{\beta_T}{c_p}\left(p\nabla\cdot\mathbf{u} - \boldsymbol{\sigma} : \mathsf{D} - \nabla\cdot\mathbf{Q}\right) - \left(\frac{\beta_T}{c_p}\left(\Delta\mu + T\left(\frac{\partial s}{\partial S}\right)_{p,T}\right) + \beta_S\right)\nabla\cdot\mathbf{J}_S, \tag{8.42}$$

where $\widetilde{\kappa} = \beta_p - \beta_T\Gamma$ is the adiabatic compressibility coefficient and $\Gamma = \beta_T T / c_p \rho$ is the adiabatic temperature gradient. The adiabatic compressibility coefficient can also be written as

$$\widetilde{\kappa} = \frac{1}{\rho}\left(\frac{\partial\rho}{\partial p}\right)_{s,S} = \frac{1}{\rho}\left(\frac{\partial\rho}{\partial p}\right)_{T,S} + \frac{\Gamma}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_{p,S}. \tag{8.43}$$

The relation between the adiabatic compressibility coefficient and the sound velocity is given by

$$c^2 = \frac{1}{\rho\widetilde{\kappa}} \tag{8.44}$$

It should be notet that (8.42) is not an equation for mass conservation, but an equation for the energy conservation in the fluid.

Heat flux

The total heat flux $\mathbf{Q}$ given in all the energy equations is often decomposed into one heat flux due to conductivity $\mathbf{q}_{cond}$, and one chemical heat flux $\mathbf{q}_{chem}$ due to change in salt and freshwater consentration. It is well known from thermodynamics that the change in enthalpy h is equal to the amount of heat added to the system due to chemical prosesses, when the pressure is constant. By adding the differential $d\left(p\rho^{-1}\right)$ on both side of (8.22), and define the enthalpy per unit mass as

$$h = \epsilon + p\rho^{-1}, \tag{8.45}$$

(8.22) becomes

$$dh = Tds + \rho^{-1}dp + \Delta\mu dS. \tag{8.46}$$

The enthalpy can be spitt into two parts, one part h_S due to the salt, and one part h_W due to the fresh water. The total enthalpy is given by the mass-weighted mean

$$h = h_S S + h_W\left(1 - S\right). \tag{8.47}$$

The chemical heat transport through a surface element $d\mathbf{A}$ is therefor given by

$$\mathbf{q}_{chem}\cdot d\mathbf{A} = \left(h_s\mathbf{J}_S + h_w\mathbf{J}_W\right)\cdot d\mathbf{A}. \tag{8.48}$$

Since the diffusive fluxes satisfies

$$\mathbf{J}_S + \mathbf{J}_W = \mathbf{0}, \tag{8.49}$$

it follows that the chemical heat flux is

$$\mathbf{q}_{chem} = \Delta h \mathbf{J}_S, \tag{8.50}$$

where $\Delta h = h_S - h_W$ is the partial enthalpy difference. From equation (8.47) it follows that the partial enthalpy difference is given by the thermodynamic state

$$\Delta h = \left(\frac{\partial h}{\partial S}\right)_{p,T}. \tag{8.51}$$

In order to describe the chemical heat flux with (p, T, S) as independent variables, we need to find the relation between the free enthalpy and enthalpy. By combinating (8.33) and (8.45), the relation between the free enthalpy and the enthalpy becomes

$$g = h - Ts, \tag{8.52}$$

therefor, it follows that (8.51) can be written in terms of free enthalpy and entropy as

$$\begin{aligned} \left(\frac{\partial h}{\partial S}\right)_{p,T} &= \left(\frac{\partial g}{\partial S}\right)_{p,T} + \left(\frac{\partial s}{\partial S}\right)_{p,T} \\ &= \Delta\mu + T\left(\frac{\partial s}{\partial S}\right)_{p,T}. \end{aligned} \tag{8.53}$$

The chemical heat flux can then be written as

$$\mathbf{q}_{chem} = \left(\Delta\mu + T\left(\frac{\partial s}{\partial S}\right)_{p,T}\right)\mathbf{J}_S. \tag{8.54}$$

8.2.3 Closure of the equations

In thermodynamic equilibrium[2], each thermodynamic flux related to irreversible effects must be zero, i.e. $\mathbf{q} = \mathbf{0}$, $\mathbf{J}_S = \mathbf{0}$ and the part of the stress tensor $\boldsymbol{\sigma}$ which correspond to irreversible effects must be the zero tensor. From (8.27) it is natural to decompose the stress tensor into two contribution,

$$\boldsymbol{\sigma} = p\mathsf{I} - \boldsymbol{\sigma}', \tag{8.55}$$

[2]Thermodynamic equilibrium is achieved when $T = constant$, $\Delta\mu = constant$ and $\mathsf{D} = 0$

where p is the thermodynamic pressure, I is the unit tensor and $\boldsymbol{\sigma}'$ is the viscous stress tensor which gives the irreversible viscous transfer of momentum in the fluid. Therefor, the fluxes we have to determined is the heat flux $\mathbf{q}$, the salt flux $\mathbf{J}_S$ and the viscouse stress tensor $\boldsymbol{\sigma}'$. The methods we will use is based on the second law of thermodynamics, linearization, symmetries and Onsager's law. The equation for the entropy, (8.26) can be rewritten in the form

$$\rho\frac{ds}{dt} + \nabla\cdot\mathbf{S} = \Sigma \tag{8.56}$$

where

$$\mathbf{S} = \frac{1}{T}\left(\mathbf{q} + T\left(\frac{\partial s}{\partial S}\right)_{p,T}\mathbf{J}_S\right), \tag{8.57}$$

is the entropy flux and

$$\Sigma = \mathbf{q}\cdot\nabla\left(\frac{1}{T}\right) - \mathbf{J}_S\cdot\left(\frac{1}{T}\nabla\left(\Delta\mu\right)_T\right) + \boldsymbol{\sigma}' : \left(\frac{1}{T}\mathsf{D}\right) \tag{8.58}$$

is the entropy production rate. According to the second law of thermodynamic the entropy production rate must satisfy $\Sigma \geq 0$. Just to simplify the notation, introduce the flux as

$$\mathbf{J}_1 = \mathbf{q}, \quad \mathbf{J}_2 = \mathbf{J}_S, \quad \mathbf{J}_3 = \boldsymbol{\sigma}', \tag{8.59}$$

and the thermodynamic force as

$$\mathbf{F}_1 = \nabla\left(\frac{1}{T}\right), \quad \mathbf{F}_2 = -\frac{1}{T}\nabla\left(\Delta\mu\right)_T, \quad \mathbf{F}_3 = \frac{1}{T}\mathsf{D}. \tag{8.60}$$

By using these notation, the entropy production can be written as

$$\Sigma = \sum_{i=1}^{3}\mathbf{J}_i \bullet \mathbf{F}_i, \tag{8.61}$$

where $\bullet$ is the dot product $\cdot$ when the fluxes and forces are of rank one, and the double dot product $:$ when fluxes and forces are of rank two. Assuming that the gradients of the pressure p, temperature T, salinity S and velocity $\mathbf{u}$ to be not large, and that the fluxes $\mathbf{J}_i$ are linear functions of the thermodynamic force, i.e.

$$\mathbf{J}_i = \sum_{j=1}^{3}\mathsf{L}_{ij}\cdot\mathbf{F}_j + \mathcal{O}(|\mathbf{F}_j|^2), \tag{8.62}$$

where L_{ij} is the phenomenological coefficients tensor which correspond to flux i and force j. It should be noted that the sub indices is just the "name" of the tensor and have nothing with the tensor indices. The phenomenological coefficients tensor must must fulfill

$$\text{Rank}\left(\mathsf{L}_{ij}\cdot\mathbf{F}_j\right)=\text{Rank}\left(\mathbf{J}_i\right). \tag{8.63}$$

According to (8.62), the fluxes are

$$\mathbf{q} = \mathsf{L}_{11}\cdot\nabla\left(\frac{1}{T}\right)+\mathsf{L}_{12}\cdot\left(-\frac{1}{T}\nabla\left(\Delta\mu\right)_T\right)+\mathsf{L}_{13}\cdot\left(\frac{1}{T}\mathsf{D}\right) \tag{8.64}$$

$$\mathbf{J}_S = \mathsf{L}_{21}\cdot\nabla\left(\frac{1}{T}\right)+\mathsf{L}_{22}\cdot\left(-\frac{1}{T}\nabla\left(\Delta\mu\right)_T\right)+\mathsf{L}_{23}\cdot\left(\frac{1}{T}\mathsf{D}\right) \tag{8.65}$$

$$\boldsymbol{\sigma}' = \mathsf{L}_{31}\cdot\nabla\left(\frac{1}{T}\right)+\mathsf{L}_{32}\cdot\left(-\frac{1}{T}\nabla\left(\Delta\mu\right)_T\right)+\mathsf{L}_{33}\cdot\left(\frac{1}{T}\mathsf{D}\right), \tag{8.66}$$

where L_{11}, L_{12}, L_{21} and L_{22} are tensors of rank two, L_{13}, L_{23}, L_{31} and L_{32} are tensors of rank three and L_{33} is a tensor of rank four. We will assume that the sea water is isotrophic and therefor, the tensors L_{ij} also have to be isotrophic. Isotrophic tensors of rank two depend only on one scalar l and has the form

$$\left\{\mathsf{L}_{Rank2}\right\}_{ij}=l\delta_{ij}. \tag{8.67}$$

Isotrophic tensors of rank three is equal to the zero tensor,

$$\mathsf{L}_{Rank3}=\mathsf{0}, \tag{8.68}$$

and therefor, the heat and salt fluxes due not depend on the deformation tensor D and the viscouse stress tensor $\boldsymbol{\sigma}'$ due not depend on the

$$\nabla\left(\frac{1}{T}\right),\quad -\frac{1}{T}\nabla\left(\Delta\mu\right)_T \tag{8.69}$$

Furthermore, an isotrophic tensors of rank four depend on three scalars l_1, l_2 and l_3 and has the form

$$\left\{\mathsf{L}_{Rank4}\right\}_{ijkl}=l_1\delta_{ij}\delta_{kl}+l_2\delta_{ik}\delta_{jl}+l_3\delta_{il}\delta_{jk}. \tag{8.70}$$

Since the deformation tensor D and the viscouse stress tensor $\boldsymbol{\sigma}'$ are symmetric, L_{Rank4} must be symmetric in the indices (i,j) and (k,l). Therefor, (8.70) reduces to

$$\left\{\mathsf{L}_{Rank4}\right\}_{ijkl}=l_1\delta_{ij}\delta_{kl}+l_2\left(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right). \tag{8.71}$$

Using the rules for isotropic tensors, the thermodynamical fluxes becomes

$$\mathbf{q} = a\nabla\left(\frac{1}{T}\right)+b\left(-\frac{1}{T}\nabla\left(\Delta\mu\right)_T\right) \tag{8.72}$$

$$\mathbf{J}_S = c\nabla\left(\frac{1}{T}\right)+d\left(-\frac{1}{T}\nabla\left(\Delta\mu\right)_T\right) \tag{8.73}$$

$$\boldsymbol{\sigma}' = e\left(\nabla\cdot\mathbf{u}\right)\mathsf{I}+2f\mathsf{D}, \tag{8.74}$$

where a, b, c, d, e and f are molecular diffusion coefficients which depend on temperature T, pressure p and salinty S. It follows from Onsagers principle that the matrix of coefficients must be symmetric, this implies that $b = c$. By using that the gradient to the chemical potential difference is given by

$$\nabla (\Delta\mu)_T = \left(\frac{\partial \Delta\mu}{\partial p}\right)_{T,S} \nabla p + \left(\frac{\partial \Delta\mu}{\partial S}\right)_{T,p} \nabla S \tag{8.75}$$

and rewrite the gradient in the equation of the thermodynamic fluxes. The thermodynamic fluxes can be written as

$$\mathbf{q} = -\left(\frac{ad - b^2}{dT^2}\right) \nabla T + \frac{b}{d}\mathbf{J}_S \tag{8.76}$$

$$\mathbf{J}_S = -\frac{b}{T^2}\nabla T - \frac{d}{T}\left(\left(\frac{\partial \Delta\mu}{\partial p}\right)_{T,S} \nabla p + \left(\frac{\partial \Delta\mu}{\partial S}\right)_{T,p} \nabla S\right) \tag{8.77}$$

$$\boldsymbol{\sigma}' = e\,(\nabla \cdot \mathbf{u})\,\mathsf{I} + 2f\mathsf{D}, \tag{8.78}$$

To make the physical picture more clear, we introduce new coefficient

$$\kappa = \frac{ad - b^2}{dT^2}, \tag{8.79}$$

$$D = \frac{d}{\rho T}\left(\frac{\partial \Delta\mu}{\partial S}\right)_{p,T}, \tag{8.80}$$

$$k_T = \frac{b}{d\left(\frac{\partial \Delta\mu}{\partial S}\right)_{p,T}}, \tag{8.81}$$

$$k_p = p\frac{\left(\frac{\partial \Delta\mu}{\partial p}\right)_{S,T}}{\left(\frac{\partial \Delta\mu}{\partial S}\right)_{p,T}}, \tag{8.82}$$

$$\eta = f, \tag{8.83}$$

$$\zeta = \frac{3e + 2f}{3}, \tag{8.84}$$

and new tensors

$$\mathsf{S} = \mathsf{D} - \mathsf{V}, \tag{8.85}$$

$$\mathsf{V} = \frac{1}{3}\,(\nabla \cdot \mathbf{u})\,\mathsf{I}, \tag{8.86}$$

where κ is the thermal conductivity that specifies heat transfer in the absence of salt flux. D is the salt diffusion coefficient that specifies salinity transfer in the absence of thermal and pressure gradient. k_T is the thermo-salt diffusion coefficient that specifies salinity transfer in the absence of salinity and pressure gradient. k_p is the baro-salt diffusion coefficient that specifies salinity transfer in the absence of salinity and temperature. η is the dynamical shear viscoisity and ζ is the bulk viscoisity due to compression and expansion. By using the new coefficients and decompositions, the fluxes becomes

$$\begin{aligned}
\mathbf{q} &= -\kappa\nabla T + k_T\left(\frac{\partial\Delta\mu}{\partial S}\right)_{p,T}\mathbf{J}_S && (8.87)\\
\mathbf{J}_S &= -\rho D\left(\frac{k_T}{T}\nabla T + \frac{k_p}{p}\nabla p + \nabla S\right) && (8.88)\\
\boldsymbol{\sigma}' &= \eta\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T - \frac{2}{3}(\nabla\cdot\mathbf{u})\,\mathsf{I}\right) + \zeta(\nabla\cdot\mathbf{u})\,\mathsf{I}, && (8.89)
\end{aligned}$$

It is not clear that the heat flux $\mathbf{q}$ depends on gradients of pressure and temperature. This is because the salt flux $\mathbf{J}_S$ carrying a flow of entropy, due to the deviation from the thermodynamic equilibrium state $\nabla S = -\frac{k_p}{p}\nabla p$. The question now is: Which constraints due to the second law of thermodynamics due we have on the coefficients. To answer this, we have to substitute the expressions for the fluxes into the expression for the entropy production rate, (8.58). The result becomes

$$\begin{aligned}
\Sigma &= -\frac{1}{T^2}\mathbf{q}\cdot\nabla T - \frac{1}{T}\left(\frac{\partial\Delta\mu}{\partial S}\right)_{p,T}\mathbf{J}_S\cdot\left(\frac{k_p}{p}\nabla p + \nabla S\right) + \frac{1}{T}\boldsymbol{\sigma}' : (\mathsf{S} + 3\mathsf{V})\\
&= -\frac{1}{T^2}\left(-\kappa\nabla T + k_T\left(\frac{\partial\Delta\mu}{\partial S}\right)_{p,T}\mathbf{J}_S\right)\cdot\nabla T - \frac{1}{T}\left(\frac{\partial\Delta\mu}{\partial S}\right)_{p,T}\mathbf{J}_S\cdot\left(-\frac{\mathbf{J}_S}{\rho D} - \frac{k_T}{T}\nabla T\right)\\
&\quad + \frac{1}{T}(2\eta\mathsf{S} + 2\zeta\mathsf{V}) : (\mathsf{S} + 3\mathsf{V})\\
&= \frac{\kappa}{T^2}\nabla T\cdot\nabla T + \frac{1}{T\rho D}\left(\frac{\partial\Delta\mu}{\partial S}\right)_{p,T}\mathbf{J}_S\cdot\mathbf{J}_S + \frac{1}{T}(2\eta\mathsf{S} : \mathsf{S} + 3\zeta\mathsf{V} : \mathsf{V}), \qquad (8.90)
\end{aligned}$$

By using the thermodynamic inequalities $T > 0$ and $\left(\frac{\partial\Delta\mu}{\partial S}\right)_{p,T} > 0$ and the second law of thermodynamics, it follows that the thermal conductivity κ, the salt diffusion coefficient D, the shear viscoisity η and the bulk viscoisity ζ are positiv. In contrast, the coefficients k_T and k_p have no constraints on the sign. It should be noted that the mechanical pressure p_{mech} is equal to

$$p_{\text{mech}} = \frac{1}{3}\text{Tr}(\boldsymbol{\sigma}). \qquad (8.91)$$

This implies that the mechanical pressure is not equal to the thermodynamical pressure. By taking the trace of the stress tensor, it follows that the difference between the pressures are

$$p - p_{\mathrm{mech}} = \zeta \nabla \cdot \mathbf{u} = -\frac{\zeta}{\rho} \frac{d\rho}{dt}. \tag{8.92}$$

In thermodynamic equilibrium the pressure difference must be zero, hence the bulk viscosity must be zero.

8.2.4 The complete set of equation

(p, T, S)-representation

To get more overview of the equations describing single-phase, two component fluids, we here present an list over the equations of motion by using the pressure, temperature and salinity as independent thermodynamic variables. By using the equation for the chemical heat flux, (8.54), and the equation for the stress tensor, (8.55), the prognostic equations of motion are given by

$$\begin{aligned}
\rho \frac{d\mathbf{u}}{dt} &= -\nabla p + \nabla \cdot \boldsymbol{\sigma}' + \mathbf{f} && (8.93) \\
\rho \widetilde{\kappa} \frac{dp}{dt} &= -\rho \nabla \cdot \mathbf{u} + \frac{\beta_T}{c_p} \left(\boldsymbol{\sigma}' : \mathsf{D} - \nabla \cdot \mathbf{q} - \mathbf{J}_S \cdot \nabla (\Delta h) \right) + \beta_S \nabla \cdot \mathbf{J}_S && (8.94) \\
\rho \, c_p \left(\frac{dT}{dt} - \Gamma \frac{dp}{dt} \right) &= \boldsymbol{\sigma}' : \mathsf{D} - \nabla \cdot \mathbf{q} - \mathbf{J}_S \cdot \nabla (\Delta h) && (8.95) \\
\rho \frac{dS}{dt} &= -\nabla \cdot \mathbf{J}_S && (8.96)
\end{aligned}$$

where we have substitute the the continuity equation, (8.1), into the thermodynamic mass density equation, (8.42), to get an prognostic equation for the pressure. The thermodynamic flux parameterization is given by

$$\begin{aligned}
\mathbf{q} &= -\kappa \nabla T + k_T \left(\frac{\partial \Delta \mu}{\partial S} \right)_{p,T} \mathbf{J}_S && (8.97) \\
\mathbf{J}_S &= -\rho D \left(\frac{k_T}{T} \nabla T + \frac{k_p}{p} \nabla p + \nabla S \right) && (8.98) \\
\boldsymbol{\sigma}' &= \eta \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T - \frac{2}{3} (\nabla \cdot \mathbf{u}) \mathsf{I} \right) + \zeta (\nabla \cdot \mathbf{u}) \mathsf{I}. && (8.99)
\end{aligned}$$

where

$$\begin{aligned}
\kappa &= \kappa\,(p,T,S) & (8.100)\\
k_T &= k_T\,(p,T,S) & (8.101)\\
k_p &= k_p\,(p,T,S) & (8.102)\\
\eta &= \eta\,(p,T,S) & (8.103)\\
\zeta &= \zeta\,(p,T,S)\,. & (8.104)
\end{aligned}$$

To close the system we have to specify the equations of state as a function on the pressure, temperature and salinity

$$\begin{aligned}
\rho &= \rho\,(p,T,S) & (8.105)\\
c_p &= c_p\,(p,T,S) & (8.106)\\
\Delta h &= \Delta h\,(p,T,S) & (8.107)\\
\left(\frac{\partial\Delta\mu}{\partial S}\right)_{p,T} &= \left(\frac{\partial\Delta\mu}{\partial S}\right)_{p,T}(p,T,S)\,, & (8.108)
\end{aligned}$$

where the other thermodynamical coefficients are given by the equations of state as

Name	Definitions
Thermal expansion coefficient	$\beta_T = -\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_{p,S}$
Compresibility coefficient	$\beta_p = \frac{1}{\rho}\left(\frac{\partial\rho}{\partial p}\right)_{T,S}$
Salinity contraction coefficient	$\beta_S = \frac{1}{\rho}\left(\frac{\partial\rho}{\partial S}\right)_{p,T} = -\rho\left(\frac{\partial\Delta\mu}{\partial p}\right)_{T,S}$
Adiabatic compressibility coefficient	$\widetilde{\kappa} = \beta_p - \Gamma\beta_T$
Adiabatic temperature gradient	$\Gamma = \frac{\beta_T T}{c_p\rho}$
Speed of sound	$c = \sqrt{\frac{1}{\rho\widetilde{\kappa}}}$

(p,θ,S)-representation

As discussed earlier, it will be more convenient to introduce a potential temperature which is a conserved quantity under adiabatic conditions. Let us define the potential temperature θ as the temperature a fluid particle of temperature T and pressure p will assume when it is brought adiabatically to a location with pressure p_0. On mathematical form is given by

$$\theta\left(p,T,S\right) = T - \int_{p_0}^{p} dp'\Gamma\left(p, T\left(s,S,p'\right),S\right), \tag{8.109}$$

where

$$s\left(S,T,p\right) = s\left(S,\theta,p_0\right) = s^0\left(S,\theta\right) \tag{8.110}$$

is the entropy. From the chain rule it follows that the evolution equation for the potential equation is

$$\frac{d\theta}{dt} = \left(\frac{\partial\theta}{\partial S}\right)_{T,p}\frac{dS}{dt} + \left(\frac{\partial\theta}{\partial T}\right)_{S,p}\frac{dT}{dt} + \left(\frac{\partial\theta}{\partial p}\right)_{T,S}\frac{dp}{dt}. \tag{8.111}$$

By differentiating (8.110), it can be shown that

$$\left(\frac{\partial\theta}{\partial S}\right)_{T,p} = \frac{\theta}{c_p^0}\left(\left(\frac{\partial s}{\partial S}\right)_{p,T} - \left(\frac{\partial s^0}{\partial S}\right)_{p,T}\right), \quad \left(\frac{\partial\theta}{\partial T}\right)_{S,p} = \frac{c_p}{c_p^0}\frac{\theta}{T}, \quad \left(\frac{\partial\theta}{\partial p}\right)_{T,S} = \frac{\theta}{c_p^0}\left(\frac{\partial s}{\partial p}\right)_{T,S} \tag{8.112}$$

where the specific heat capasity evaluated at the refrence pressure is

$$c_p^0 = \theta\left(\frac{\partial s^0}{\partial\theta}\right)_{p_0,S}. \tag{8.113}$$

The Maxwell relation also gives

$$\left(\frac{\partial s}{\partial p}\right)_{T,S} = -\frac{\beta_T}{\rho}, \quad \left(\frac{\partial S}{\partial T}\right)_{p,T} = -\left(\frac{\partial\Delta\mu}{\partial T}\right)_{p,S}, \tag{8.114}$$

which implise that

$$\frac{\left(\frac{\partial\theta}{\partial p}\right)_{T,S}}{\left(\frac{\partial\theta}{\partial T}\right)_{S,p}} = -\Gamma \tag{8.115}$$

Therefor, equation (8.111) can be written as

$$\frac{d\theta}{dt} = \frac{1}{c_p^0\rho}\frac{\theta}{T}\left[\left(\Delta\mu - \Delta h - T\left(\frac{\partial\Delta\mu^0}{\partial T}\right)_{S,p}\right)\nabla\cdot\mathbf{J}_S + c_p\rho\left(\frac{dT}{dt} - \Gamma\frac{dp}{dt}\right)\right] \tag{8.116}$$

where we have used (8.53) and (8.8). By using (8.95) it follows that the equation for the potential temperature is

$$\rho\frac{d\theta}{dt} = F_\theta \tag{8.117}$$

$$F_\theta = \frac{\theta}{c_p^0 T}\left[\left(\Delta\mu - T\left(\frac{\partial\Delta\mu^0}{\partial T}\right)_{S,p}\right)\nabla\cdot\mathbf{J}_S + \left(\boldsymbol{\sigma}' : \mathsf{D} - \nabla\cdot\mathbf{q} - \nabla\cdot\left(\Delta h\mathbf{J}_S\right)\right)\right] \tag{8.118}$$

Under adiabatic conditions, the potential temperature is conserved. When (p, θ, S) are the independent variables. The equations of motion reads

$$\frac{d\rho}{dt} = -\rho \nabla \cdot \mathbf{u} \tag{8.119}$$

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla p + \nabla \cdot \boldsymbol{\sigma}' + \mathbf{f} \tag{8.120}$$

$$\rho \frac{d\theta}{dt} = F_\theta \tag{8.121}$$

$$\rho \frac{dS}{dt} = -\nabla \cdot \mathbf{J}_S \tag{8.122}$$

$$\rho = \rho\left(p, \theta, S\right) \tag{8.123}$$

Chapter 9

Appendix B

9.1 Spherical coordinates

Spherical coordinates consist of the radial distance r, the polar angle ϑ, and the azimuthal angle ϕ. Choose the z-axis as the polar axis and x-axis as the origin for azimuthal angle, then the transformation from spherical to Cartesian coordinates becomes

$$x = r\sin\vartheta\cos\phi, \quad y = r\sin\vartheta\sin\phi, \quad z = r\cos\vartheta, \tag{9.1}$$

with domains $0 \leq r < \infty$, $0 \leq \vartheta \leq \pi$ and $0 \leq \phi < 2\pi$. Therfor, it follows that the lokal unit vectors in spherical coordinates are given by

$$\begin{aligned} \widehat{\mathbf{r}} &= \sin\vartheta\cos\phi\widehat{\mathbf{x}} + \sin\vartheta\sin\phi\widehat{\mathbf{y}} + \cos\vartheta\widehat{\mathbf{z}}, && (9.2)\\ \widehat{\boldsymbol{\vartheta}} &= \cos\vartheta\cos\phi\widehat{\mathbf{x}} + \cos\vartheta\sin\phi\widehat{\mathbf{y}} - \sin\vartheta\widehat{\mathbf{z}}, && (9.3)\\ \widehat{\boldsymbol{\phi}} &= -\sin\phi\widehat{\mathbf{x}} + \cos\phi\widehat{\mathbf{y}}, && (9.4) \end{aligned}$$

where $\widehat{\mathbf{x}}$, $\widehat{\mathbf{y}}$ and $\widehat{\mathbf{z}}$ are the unit vectors in the Cartesian coordinate system. The unit vectors in spherical coordinates are such that $\widehat{\mathbf{r}}$, $\widehat{\boldsymbol{\vartheta}}$ and $\widehat{\boldsymbol{\phi}}$ form a right handed basis, which is orthogonal. Therefor, an arbitrary vector field can be written out as

$$\mathbf{A} = A_r\,\widehat{\mathbf{r}} + A_\vartheta\,\widehat{\boldsymbol{\vartheta}} + A_\phi\,\widehat{\boldsymbol{\phi}}, \tag{9.5}$$

where A_r, A_ϑ and A_ϕ are projections of $\mathbf{A}$ on the spherical basis. Since the unit vectors are a locally basis, which depends on the position, the derivatives are non-vanishing. This non-vanishing derivatives reads

$$\begin{array}{lll} \frac{\partial\widehat{\mathbf{r}}}{\partial r} = \mathbf{0}, & \frac{\partial\widehat{\mathbf{r}}}{\partial\vartheta} = \widehat{\boldsymbol{\vartheta}}, & \frac{\partial\widehat{\mathbf{r}}}{\partial\phi} = \sin\vartheta\widehat{\boldsymbol{\phi}}, \\ \frac{\partial\widehat{\boldsymbol{\vartheta}}}{\partial r} = \mathbf{0}, & \frac{\partial\widehat{\boldsymbol{\vartheta}}}{\partial\vartheta} = -\widehat{\mathbf{r}}, & \frac{\partial\widehat{\boldsymbol{\vartheta}}}{\partial\phi} = \cos\vartheta\widehat{\boldsymbol{\phi}}, \\ \frac{\partial\widehat{\boldsymbol{\phi}}}{\partial r} = \mathbf{0}, & \frac{\partial\widehat{\boldsymbol{\phi}}}{\partial\vartheta} = \mathbf{0}, & \frac{\partial\widehat{\boldsymbol{\phi}}}{\partial\phi} = -\sin\vartheta\widehat{\mathbf{r}} - \cos\vartheta\widehat{\boldsymbol{\vartheta}}, \end{array} \tag{9.6}$$

The position may also depend on time, which implise that the unit vectors do it also. So, the totale time derivate of the unit vectors are

$$\frac{d\widehat{\mathbf{r}}}{dt} = \frac{d\phi}{dt}\sin\vartheta\,\widehat{\boldsymbol{\phi}} + \frac{d\vartheta}{dt}\,\widehat{\boldsymbol{\vartheta}}, \tag{9.7}$$

$$\frac{d\widehat{\boldsymbol{\vartheta}}}{dt} = \frac{d\phi}{dt}\cos\vartheta\,\widehat{\boldsymbol{\phi}} - \frac{d\vartheta}{dt}\,\widehat{\mathbf{r}}, \tag{9.8}$$

$$\frac{d\widehat{\boldsymbol{\phi}}}{dt} = -\frac{d\phi}{dt}\left(\sin\vartheta\,\widehat{\mathbf{r}} + \cos\vartheta\,\widehat{\boldsymbol{\vartheta}}\right). \tag{9.9}$$

Now we have all relations to transform from a Cartesian coordinate system to a spherical coordinat system. It is straigt forward to show that the del-operator in spherical coordinat system is given by

$$\nabla = \widehat{\mathbf{r}}\frac{\partial}{\partial r} + \widehat{\boldsymbol{\vartheta}}\frac{1}{r}\frac{\partial}{\partial\vartheta} + \widehat{\boldsymbol{\phi}}\frac{1}{r\sin\vartheta}\frac{\partial}{\partial\phi}. \tag{9.10}$$

In geophysical fluid dynamics is more common to use $\widehat{\boldsymbol{\phi}}$, $\widehat{\boldsymbol{\theta}}$ and $\widehat{\mathbf{r}}$ as a right handed orthogonal basis. Where $\theta = \frac{\pi}{2} - \vartheta$ is the latitude angle, which varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ in direction of the unit vector $\widehat{\boldsymbol{\theta}} = -\widehat{\boldsymbol{\vartheta}}$. Therefor, an arbitrary vector field can be written out as

$$\mathbf{A} = A_\phi\,\widehat{\boldsymbol{\phi}} + A_\theta\,\widehat{\boldsymbol{\theta}} + A_r\,\widehat{\mathbf{r}}, \tag{9.11}$$

where A_ϕ, A_θ and A_r are projections of $\mathbf{A}$ on the new spherical basis. In the following, we will use $\widehat{\boldsymbol{\phi}}$, $\widehat{\boldsymbol{\theta}}$ and $\widehat{\mathbf{r}}$ as a right handed basis. It is just to use that $\sin\vartheta = \cos\theta$, $\cos\vartheta = \sin\theta$ and $\frac{\partial}{\partial\vartheta} = -\frac{\partial}{\partial\theta}$. The del-operator is given by

$$\nabla = \widehat{\boldsymbol{\phi}}\frac{1}{r\cos\theta}\frac{\partial}{\partial\phi} + \widehat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta} + \widehat{\mathbf{r}}\frac{\partial}{\partial r}, \tag{9.12}$$

and the Laplacian operator is

$$\nabla^2 = \frac{1}{r^2\cos^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{r^2\cos\theta}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) \tag{9.13}$$

$$= \frac{1}{r^2\cos^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2} - \tan\theta\frac{\partial}{\partial\theta}\right) + \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) \tag{9.14}$$

Below there is listed some well-known results from vector calculus based on these operators:

$$\nabla\cdot\mathbf{A} = \frac{1}{r\cos\theta}\frac{\partial A_\phi}{\partial\phi} + \frac{1}{r\cos\theta}\frac{\partial}{\partial\theta}\left(\cos\theta A_\theta\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) \tag{9.15}$$

$$= \frac{1}{r\cos\theta}\frac{\partial A_\phi}{\partial\phi} + \frac{1}{r}\left(\frac{\partial}{\partial\theta} - \tan\theta\right)A_\theta + \left(\frac{\partial}{\partial r} + \frac{2}{r}\right)A_r \tag{9.16}$$

9.1.1 Material derivativ of the velocity field

The velocity field in spherical coordinates is given by

$$\begin{aligned}
\mathbf{u} &= \frac{d\mathbf{r}}{dt} \\
&= r\frac{d\widehat{\mathbf{r}}}{dt} + \frac{dr}{dt}\widehat{\mathbf{r}} \\
&= r\cos\theta\frac{d\phi}{dt}\,\widehat{\boldsymbol{\phi}} + r\frac{d\theta}{dt}\,\widehat{\boldsymbol{\theta}} + \frac{dr}{dt}\,\widehat{\mathbf{r}}, \qquad (9.17) \\
& \qquad (9.18)
\end{aligned}$$

where $r\cos\theta\frac{d\phi}{dt} = u$ is the zonal velocity, $r\frac{d\theta}{dt} = v$ is the meridional velocity and $\frac{dr}{dt} = w$ is the radial velocity. By using these relations, the total derivative of the unit vectors becomes

$$\frac{d\widehat{\boldsymbol{\phi}}}{dt} = \frac{u\tan\theta}{r}\,\widehat{\boldsymbol{\theta}} - \frac{u}{r}\,\widehat{\mathbf{r}}, \qquad (9.19)$$

$$\frac{d\widehat{\boldsymbol{\theta}}}{dt} = -\frac{u\tan\theta}{r}\,\widehat{\boldsymbol{\phi}} - \frac{v}{r}\,\widehat{\mathbf{r}}, \qquad (9.20)$$

$$\frac{d\widehat{\mathbf{r}}}{dt} = \frac{u}{r}\,\widehat{\boldsymbol{\phi}} + \frac{v}{r}\,\widehat{\boldsymbol{\theta}}. \qquad (9.21)$$

Therefor, the total derivative of the velocity field, which we need in the Navier-Stokes equations, is given by

$$\begin{aligned}
\frac{d\mathbf{u}}{dt} &= \frac{d}{dt}\left(u\,\widehat{\boldsymbol{\phi}} + v\,\widehat{\boldsymbol{\theta}} + w\,\widehat{\mathbf{r}}\right) \\
&= \widehat{\boldsymbol{\phi}}\frac{du}{dt} + \widehat{\boldsymbol{\theta}}\frac{dv}{dt} + \widehat{\mathbf{r}}\frac{dw}{dt} + u\frac{d\widehat{\boldsymbol{\phi}}}{dt} + v\frac{d\widehat{\boldsymbol{\theta}}}{dt} + w\frac{d\widehat{\mathbf{r}}}{dt} \\
&= \widehat{\boldsymbol{\phi}}\left(\frac{du}{dt} + \frac{uw}{r} - \frac{uv\tan\theta}{r}\right) + \widehat{\boldsymbol{\theta}}\left(\frac{dv}{dt} + \frac{vw}{r} + \frac{u^2\tan\theta}{r}\right) \\
&\quad + \widehat{\mathbf{r}}\left(\frac{dw}{dt} - \frac{u^2+v^2}{r}\right), \qquad (9.22)
\end{aligned}$$

where the operator for the total derivative is the material derivative given by

$$\begin{aligned}
\frac{d}{dt} &= \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla \\
&= \frac{\partial}{\partial t} + \frac{u}{r\cos\theta}\frac{\partial}{\partial\phi} + \frac{v}{r}\frac{\partial}{\partial\theta} + w\frac{\partial}{\partial r}. \qquad (9.23)
\end{aligned}$$

In geophysical fluid dynamics, the dynamics in horizontal and vertical directions are very different. Therefore it will be advantageous to split everything into a horizontal component and a vertical component. Hence, equation (9.22) can be splitted as

$$\left(\frac{d\mathbf{u}}{dt}\right)_{\perp} = \left(\left.\frac{d\mathbf{u}_{\perp}}{dt}\right|_{\mathbf{e}_i}\right) + \frac{w}{r}\mathbf{u}_{\perp} + \frac{u}{r}\tan\theta\,\widehat{\mathbf{r}}\times\mathbf{u}_{\perp} \tag{9.24}$$

$$\left(\frac{d\mathbf{u}}{dt}\right)_{\parallel} = \left(\left.\frac{d\mathbf{u}_{\parallel}}{dt}\right|_{\mathbf{e}_i}\right) - \frac{\mathbf{u}_{\perp}}{r}\widehat{\mathbf{r}}, \tag{9.25}$$

where the vertical line symbolizes that the unit vectors remain constant during the differentiation.

9.1.2 Local Cartesian system

If one wants to describe phenomena on small scale in relation to the earth radius, it will be advantageous to introduce a local Cartesian system (X, Y, Z) fixed on the Earth's surface. Let the origin to this local coordinate system be given by the position vector $\phi_0\widehat{\boldsymbol{\phi}}_0 + \theta_0\widehat{\boldsymbol{\theta}}_0 + r_0\widehat{\mathbf{r}}_0$, where the unit vectors which spand out the coordinate system is defined by

$$\widehat{\mathbf{X}} \equiv \widehat{\boldsymbol{\phi}}_0 = -\sin\phi_0\widehat{\mathbf{x}} + \cos\phi_0\widehat{\mathbf{y}}, \tag{9.26}$$

$$\widehat{\mathbf{Y}} \equiv \widehat{\boldsymbol{\theta}}_0 = -\sin\theta_0\cos\phi_0\widehat{\mathbf{x}} - \sin\theta_0\sin\phi_0\widehat{\mathbf{y}} + \cos\theta_0\widehat{\mathbf{z}}, \tag{9.27}$$

$$\widehat{\mathbf{Z}} \equiv \widehat{\mathbf{r}}_0 = \cos\theta_0\cos\phi_0\widehat{\mathbf{x}} + \cos\theta_0\sin\phi_0\widehat{\mathbf{y}} + \sin\theta_0\widehat{\mathbf{z}}. \tag{9.28}$$

For small excursions on the plane, the geometry gives that the coordinate is related to spherical coordinates by

$$\begin{aligned}\mathbf{X} &= X\,\widehat{\mathbf{X}} + Y\,\widehat{\mathbf{Y}} + Z\,\widehat{\mathbf{Z}},\\ &= (\phi-\phi_0)\,r_0\cos\theta_0\,\widehat{\mathbf{X}} + (\theta-\theta_0)\,r_0\,\widehat{\mathbf{Y}} + (r-r_0)\,\widehat{\mathbf{Z}}.\end{aligned} \tag{9.29}$$

By using the chain rule it follows that the relation between the derivatives in the lokal coordinate system and the spherical coordinate system is

$$\frac{\partial}{\partial\phi} = r_0\cos\theta_0\frac{\partial}{\partial X}, \quad \frac{\partial}{\partial\theta} = r_0\frac{\partial}{\partial Y}, \quad \frac{\partial}{\partial r} = \frac{\partial}{\partial Z} \tag{9.30}$$

and

$$\frac{d\phi}{dt} = \frac{1}{r_0\cos\theta_0}\frac{dX}{dt}, \quad \frac{d\theta}{dt} = \frac{1}{r_0}\frac{dY}{dt}, \quad \frac{dr}{dt} = \frac{dZ}{dt}. \tag{9.31}$$

The del-operator in this coordinate system is

$$\nabla = \widehat{\mathbf{X}}\frac{r_0\cos\theta_0}{r\cos\theta}\frac{\partial}{\partial X} + \widehat{\mathbf{Y}}\frac{r_0}{r}\frac{\partial}{\partial Y} + \widehat{\mathbf{Z}}\frac{\partial}{\partial Z}, \tag{9.32}$$

and the velocity is

$$\mathbf{u} = \frac{r\cos\theta}{r_0\cos\theta_0}\frac{dX}{dt}\,\widehat{\mathbf{X}} + \frac{r}{r_0}\frac{dY}{dt}\,\widehat{\mathbf{Y}} + \frac{dZ}{dt}\,\widehat{\mathbf{Z}} \tag{9.33}$$

9.2 From inertial systems to non-intertial systems

Let the Cartesian coordinate system in section 9.1 rotate with an angular velocity $\mathbf{\Omega}$ along the polar axis $\widehat{\mathbf{z}}$. Then, an arbitary vector field $\mathbf{A}$ represented in these Cartesian coordinate system

$$\mathbf{A} = A_x\widehat{\mathbf{x}} + A_y\widehat{\mathbf{y}} + A_z\widehat{\mathbf{z}}, \tag{9.34}$$

will change with time in relation to an observer fixed in the rotating frame with

$$\left(\frac{d\mathbf{A}}{dt}\right)_{\Omega} = \frac{dA_x}{dt}\widehat{\mathbf{x}} + \frac{dA_y}{dt}\widehat{\mathbf{y}} + \frac{dA_z}{dt}\widehat{\mathbf{z}}. \tag{9.35}$$

Note that the unit vectors in this frame are fixed in length and direction. For an observer fixed in an non-rotating frame of reference, both the components of the vector field $\mathbf{A}$ and the unit vectors will change with time according to

$$\left(\frac{d\mathbf{A}}{dt}\right)_{I} = \frac{dA_x}{dt}\widehat{\mathbf{x}} + \frac{dA_y}{dt}\widehat{\mathbf{y}} + \frac{dA_z}{dt}\widehat{\mathbf{z}} + A_x\frac{d\widehat{\mathbf{x}}}{dt} + A_y\frac{d\widehat{\mathbf{y}}}{dt} + A_z\frac{d\widehat{\mathbf{z}}}{dt}. \tag{9.36}$$

Since both $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{y}}$ will move in the azimuthal direction $\widehat{\boldsymbol{\phi}}$ with angular velocity Ω. It follows that $\frac{d\widehat{\mathbf{x}}}{dt} = \Omega\widehat{\mathbf{z}}\times\widehat{\mathbf{x}}$ and $\frac{d\widehat{\mathbf{y}}}{dt} = \Omega\widehat{\mathbf{z}}\times\widehat{\mathbf{y}}$. The unit vector $\widehat{\mathbf{z}}$ along the polar axis will not move, but of course $\frac{d\widehat{\mathbf{z}}}{dt} = \Omega\widehat{\mathbf{z}}\times\widehat{\mathbf{z}} = \mathbf{0}$. Therefor, the change with time in the non-rotating frame is given by

$$\left(\frac{d\mathbf{A}}{dt}\right)_{I} = \left(\frac{d\mathbf{A}}{dt}\right)_{\Omega} + \mathbf{\Omega}\times\mathbf{A}, \tag{9.37}$$

where we have used (9.35) and $\mathbf{\Omega} = \Omega\widehat{\mathbf{z}}$.

Let for instance the vector field $\mathbf{A}$ be the positon vector $\mathbf{r}$ to an arbitary fluid element. According to (9.37) the velocity field observed in the non-rotating frame is given by

$$\mathbf{u}_I = \mathbf{u}_\Omega + \mathbf{\Omega}\times\mathbf{r}, \tag{9.38}$$

where $\mathbf{u}_I = \left(\frac{d\mathbf{r}}{dt}\right)_I$ and $\mathbf{u}_\Omega = \left(\frac{d\mathbf{r}}{dt}\right)_\Omega$. The rate of change of (9.38) observed in the non-rotating frame is

$$\begin{aligned}
\left(\frac{d\mathbf{u}_I}{dt}\right)_I &= \left(\frac{d\mathbf{u}_I}{dt}\right)_\Omega + \boldsymbol{\Omega}\times\mathbf{u}_I \\
&= \left(\frac{d\left(\mathbf{u}_\Omega + \boldsymbol{\Omega}\times\mathbf{r}\right)}{dt}\right)_\Omega + \boldsymbol{\Omega}\times\left(\mathbf{u}_\Omega + \boldsymbol{\Omega}\times\mathbf{r}\right) \\
&= \left(\frac{d\mathbf{u}_\Omega}{dt}\right)_\Omega + \left(\frac{d\boldsymbol{\Omega}}{dt}\right)_\Omega\times\mathbf{r} + \boldsymbol{\Omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_\Omega + \boldsymbol{\Omega}\times\left(\mathbf{u}_\Omega + \boldsymbol{\Omega}\times\mathbf{r}\right) \\
&= \left(\frac{d\mathbf{u}_\Omega}{dt}\right)_\Omega + 2\boldsymbol{\Omega}\times\mathbf{u}_\Omega + \boldsymbol{\Omega}\times\left(\boldsymbol{\Omega}\times\mathbf{r}\right) + \left(\frac{d\boldsymbol{\Omega}}{dt}\right)_\Omega\times\mathbf{r}. \qquad (9.39)
\end{aligned}$$

The second term is known as the Coriolis acceleration, the third term is known as the centrifugal acceleration and the last term is the acceleration due to the change of rate of the angular velocity vector. For our case, the angular velocity is constant along the polar axis. Therefor, the last term will vanish.

9.2.1 Pseudo acceleration in spherical coordinates

The angular velocity vector $\boldsymbol{\Omega} = \Omega\widehat{\mathbf{z}}$ of the earth, point along the polar axis. In spherical coordinat it is given by

$$\boldsymbol{\Omega} = \Omega\left(\cos\theta\,\widehat{\boldsymbol{\theta}} + \sin\theta\,\widehat{\mathbf{r}}\right). \qquad (9.40)$$

An observer who is standing in a rotating frame of reference, will observe moving objects to deflect according to this rotation. In this frame of reference, there exist two pseudo-forces who will act on the object according to (9.39). In spherical coordinates the Coriolis acceleration reads

$$2\boldsymbol{\Omega}\times\mathbf{u} = 2\left(w\Omega\cos\theta - v\Omega\sin\theta\right)\widehat{\boldsymbol{\phi}} + 2u\Omega\sin\theta\,\widehat{\boldsymbol{\theta}} - 2u\Omega\cos\theta\,\widehat{\mathbf{r}}, \qquad (9.41)$$

and the centrifugal acceleration reads

$$\boldsymbol{\Omega}\times\left(\boldsymbol{\Omega}\times\mathbf{r}\right) = \Omega^2 r\cos\theta\left(\sin\theta\,\widehat{\boldsymbol{\theta}} + \cos\theta\,\widehat{\mathbf{r}}\right). \qquad (9.42)$$

The Coriolis acceleration can be written as

$$2\boldsymbol{\Omega}\times\mathbf{u} = \left(2\boldsymbol{\Omega}\times\mathbf{u}\right)_\perp + \left(2\boldsymbol{\Omega}\times\mathbf{u}\right)_\parallel, \qquad (9.43)$$

where horizontal and vertical components reads

$$\begin{aligned}
\left(2\boldsymbol{\Omega}\times\mathbf{u}\right)_\perp &= l\widehat{\boldsymbol{\theta}}\times\mathbf{u}_\parallel + f\widehat{\mathbf{r}}\times\mathbf{u}_\perp \qquad (9.44)\\
\left(2\boldsymbol{\Omega}\times\mathbf{u}\right)_\parallel &= l\widehat{\boldsymbol{\theta}}\times\mathbf{u}_\perp, \qquad (9.45)
\end{aligned}$$

here, $f = 2\left|\boldsymbol{\Omega}\right|\sin\theta$ and $l = 2\left|\boldsymbol{\Omega}\right|\cos\theta$.

9.3 Poisson brackets

In a Cartesian coordinat system the Poisson bracket of two dynamical variabels A and B is defined as

$$\{A, B\} \equiv \left(\frac{\partial A}{\partial x}\frac{\partial B}{\partial y} - \frac{\partial A}{\partial y}\frac{\partial B}{\partial x}\right). \tag{9.46}$$

The Poisson bracket can also be written as

$$\{A, B\} = \widehat{\mathbf{z}} \cdot (\nabla A \times \nabla B), \tag{9.47}$$

or equivalent as

$$\{A, B\} = (\widehat{\mathbf{z}} \times \nabla A) \cdot \nabla B. \tag{9.48}$$

In two-dimensional nearly incompressible flow the advective operator $\mathbf{u} \cdot \nabla$ have the structure $\{\psi, \cdot\}$, where ψ is a stream function. Therefor, introducing Poisson bracket will simplify the notation. From the definition of the Poisson bracket, (9.46), it follows that the Poisson brackets have the following properties:

$$\{A, A\} = 0 \tag{9.49}$$
$$\{A, B\} = -\{B, A\} \tag{9.50}$$
$$\{A + B, C\} = \{A, C\} + \{B, C\} \tag{9.51}$$
$$\{AB, C\} = B\{A, C\} + A\{B, C\} \tag{9.52}$$

It should be pointet out that it is often more convenient to write out the Poisson bracket in flux form as

$$\{A, B\} = \frac{\partial}{\partial x}\left(A\frac{\partial B}{\partial y}\right) - \frac{\partial}{\partial y}\left(A\frac{\partial B}{\partial x}\right), \tag{9.53}$$

or as

$$\{A, B\} = \frac{\partial}{\partial y}\left(B\frac{\partial A}{\partial x}\right) - \frac{\partial}{\partial x}\left(B\frac{\partial A}{\partial y}\right), \tag{9.54}$$

9.4 Reynholds stress tensor in spherical coordinates

The momentum exchange between turbulent small-scale flow and large-scale flow is given by the divergence of the Reynholds stress tensor

$$\mathbf{\Pi} \equiv -\rho_0 \langle \widetilde{\mathbf{u}}\widetilde{\mathbf{u}} \rangle, \tag{9.55}$$

where $\widetilde{\mathbf{u}}$ is the velocity field assosiated with the small-scale motion. Since the equation which describe this small-scale field is very complex, it is natural to parameterize the stress tensor in terms of the large-scale velocity $\mathbf{u}$. In many text books they do it in a Cartisian coordinate system. We will try in this section to do it in the spherical coordinate system. From the definition of the stress tensor, it follows that the tensor is a rank two tensor, which is symmetric. According to [6, p.58] the stress tensor is parameterized as follows

$$\mathbf{\Pi} = \rho \left[A_\perp \left(\nabla_\perp \otimes \mathbf{u} + (\nabla_\perp \otimes \mathbf{u})^T \right) + A_\parallel \left(\nabla_\parallel \otimes \mathbf{u} + \left(\nabla_\parallel \otimes \mathbf{u}\right)^T \right) \right] \tag{9.56}$$

where the coefficients $A_\perp$ and $A_\parallel$ are the horizontal and vertical turbulent viscosity coefficients respectively, and where $\nabla_\perp$ is the horizontal gradient operator, $\nabla_\parallel$ is the vertical gradient operator and where $\otimes$ is the dyadic product. In spherical coordinates, the products in the tensor reads

$$\begin{aligned}
\nabla_\perp \otimes \mathbf{u} &= \left(\widehat{\boldsymbol{\phi}} \frac{1}{r\cos\theta}\frac{\partial}{\partial\phi} + \widehat{\boldsymbol{\theta}} \frac{1}{r}\frac{\partial}{\partial\theta} \right) \otimes \left(u\,\widehat{\boldsymbol{\phi}} + v\,\widehat{\boldsymbol{\theta}} + w\,\widehat{\mathbf{r}} \right) \\
&= \widehat{\boldsymbol{\phi}} \otimes \widehat{\boldsymbol{\phi}} \frac{1}{r\cos\theta}\frac{\partial u}{\partial\phi} + \widehat{\boldsymbol{\phi}} \otimes \left(-\cos\theta\widehat{\mathbf{r}} + \sin\theta\widehat{\boldsymbol{\theta}} \right) \frac{u}{r\cos\theta} \\
&+ \widehat{\boldsymbol{\phi}} \otimes \widehat{\boldsymbol{\theta}} \frac{1}{r\cos\theta}\frac{\partial v}{\partial\phi} - \widehat{\boldsymbol{\phi}} \otimes \widehat{\boldsymbol{\phi}} \frac{\tan\theta}{r} v \\
&+ \widehat{\boldsymbol{\phi}} \otimes \widehat{\mathbf{r}} \frac{1}{r\cos\theta}\frac{\partial w}{\partial\phi} + \widehat{\boldsymbol{\phi}} \otimes \widehat{\boldsymbol{\phi}} \frac{w}{r} \\
&+ \widehat{\boldsymbol{\theta}} \otimes \widehat{\boldsymbol{\phi}} \frac{1}{r}\frac{\partial u}{\partial\theta} \\
&+ \widehat{\boldsymbol{\theta}} \otimes \widehat{\boldsymbol{\theta}} \frac{1}{r}\frac{\partial v}{\partial\theta} - \widehat{\boldsymbol{\theta}} \otimes \widehat{\mathbf{r}} \frac{v}{r} \\
&+ \widehat{\boldsymbol{\theta}} \otimes \widehat{\mathbf{r}} \frac{1}{r}\frac{\partial w}{\partial\theta} + \widehat{\boldsymbol{\theta}} \otimes \widehat{\boldsymbol{\theta}} \frac{w}{r} \\
&= \begin{bmatrix} \frac{1}{r\cos\theta}\frac{\partial u}{\partial\phi} - \frac{v\tan\theta}{r} + \frac{w}{r} & \frac{1}{r\cos\theta}\frac{\partial v}{\partial\phi} + \frac{u\tan\theta}{r} & \frac{1}{r\cos\theta}\frac{\partial w}{\partial\phi} - \frac{u}{r} \\ \frac{1}{r}\frac{\partial u}{\partial\theta} & \frac{1}{r}\frac{\partial v}{\partial\theta} + \frac{w}{r} & \frac{1}{r}\frac{\partial w}{\partial\theta} - \frac{v}{r} \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned} \tag{9.57}$$

$$
\begin{aligned}
\nabla_{\|} \otimes \mathbf{u} &= \left(\widehat{\mathbf{r}}\frac{\partial}{\partial r}\right) \otimes \left(u\,\widehat{\boldsymbol{\phi}} + v\,\widehat{\boldsymbol{\theta}} + w\,\widehat{\mathbf{r}}\right) \\
&= \widehat{\mathbf{r}} \otimes \widehat{\boldsymbol{\phi}}\frac{\partial u}{\partial r} + \widehat{\mathbf{r}} \otimes \widehat{\boldsymbol{\theta}}\frac{\partial v}{\partial r} + \widehat{\mathbf{r}} \otimes \widehat{\mathbf{r}}\frac{\partial w}{\partial r} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial u}{\partial r} & \frac{\partial v}{\partial r} & \frac{\partial w}{\partial r} \end{bmatrix}
\end{aligned} \tag{9.58}
$$

Therefor, it follows that the components of the stress tensor is given by

$$\Pi_{\phi\phi} = 2A_{\perp}\left(\frac{1}{r\cos\theta}\frac{\partial u}{\partial\phi} + \frac{w}{r} - \frac{v\tan\theta}{r}\right) \tag{9.59}$$

$$\Pi_{\phi\theta} = A_{\perp}\left(\frac{1}{r}\frac{\partial u}{\partial\theta} + \frac{1}{r\cos\theta}\frac{\partial v}{\partial\phi} + \frac{u\tan\theta}{r}\right) \tag{9.60}$$

$$\Pi_{\phi r} = A_{\perp}\left(\frac{1}{r\cos\theta}\frac{\partial w}{\partial\phi} - \frac{u}{r}\right) + A_{\|}\frac{\partial u}{\partial r} \tag{9.61}$$

$$\Pi_{\theta\theta} = 2A_{\perp}\left(\frac{1}{r}\frac{\partial v}{\partial\theta} + \frac{w}{r}\right) \tag{9.62}$$

$$\Pi_{\theta r} = A_{\perp}\left(\frac{1}{r}\frac{\partial w}{\partial\theta} - \frac{v}{r}\right) + A_{\|}\frac{\partial v}{\partial r} \tag{9.63}$$

$$\Pi_{rr} = 2A_{\|}\frac{\partial w}{\partial r} \tag{9.64}$$

Since the momentum transfer is given by the divergence to the tensor, we need to calculate these components. According to [4, p.260] is given by

$$
\begin{aligned}
\nabla \cdot \boldsymbol{\Pi} &= \widehat{\boldsymbol{\phi}}\left(\frac{1}{r\cos\theta}\frac{\partial\Pi_{\phi\phi}}{\partial\phi} + \frac{1}{r}\frac{\partial\Pi_{\phi\theta}}{\partial\theta} + \frac{\partial\Pi_{\phi r}}{\partial r} + \frac{2\Pi_{\phi r}}{r} + \frac{\Pi_{r\phi}}{r} - \frac{\tan\theta}{r}\Pi_{\theta\phi} - \frac{\tan\theta}{r}\Pi_{\phi\theta}\right) \\
&+ \widehat{\boldsymbol{\theta}}\left(\frac{1}{r\cos\theta}\frac{\partial\Pi_{\theta\phi}}{\partial\phi} + \frac{1}{r}\frac{\partial\Pi_{\theta\theta}}{\partial\theta} + \frac{\partial\Pi_{\theta r}}{\partial r} + \frac{2\Pi_{\theta r}}{r} + \frac{\Pi_{r\theta}}{r} - \frac{\tan\theta}{r}\Pi_{\theta\theta} + \frac{\tan\theta}{r}\Pi_{\phi\phi}\right) \\
&+ \widehat{\mathbf{r}}\left(\frac{1}{r\cos\theta}\frac{\partial\Pi_{r\phi}}{\partial\phi} + \frac{1}{r}\frac{\partial\Pi_{r\theta}}{\partial\theta} + \frac{\partial\Pi_{rr}}{\partial r} + \frac{2\Pi_{rr}}{r} - \frac{\tan\theta}{r}\Pi_{r\theta} - \frac{\Pi_{\theta\theta}}{r} - \frac{\Pi_{\phi\phi}}{r}\right)
\end{aligned} \tag{9.65}
$$

By substituting the components of the stress tensor, we get

$$
\begin{aligned}
\nabla\cdot\mathbf{\Pi} &= \widehat{\boldsymbol{\phi}}\left(A_\perp\nabla_\perp^2 u + A_\parallel\nabla_\parallel^2 u + \frac{A_\perp}{r\cos\theta}\frac{\partial}{\partial\phi}\nabla\cdot\mathbf{u} + \frac{A_\parallel - A_\perp}{r}\frac{\partial u}{\partial r}\right.\\
&\quad\left. + \frac{2A_\perp}{r^2\cos\theta}\left(-\tan\theta\frac{\partial v}{\partial\phi} + \frac{\partial w}{\partial\phi}\right) - \frac{A_\perp u}{r^2\cos^2\theta}\right)\\
&+ \widehat{\boldsymbol{\theta}}\left(A_\perp\nabla_\perp^2 v + A_\parallel\nabla_\parallel^2 v + \frac{A_\perp}{r}\frac{\partial}{\partial\theta}\nabla\cdot\mathbf{u} + \frac{A_\parallel - A_\perp}{r}\frac{\partial v}{\partial r}\right.\\
&\quad\left. + \frac{2A_\perp}{r^2\cos\theta}\left(\tan\theta\frac{\partial u}{\partial\phi} + \cos\theta\frac{\partial w}{\partial\theta}\right) - \frac{A_\perp v}{r^2\cos^2\theta}\right)\\
&+ \widehat{\mathbf{r}}\left(A_\perp\nabla_\perp^2 w + A_\parallel\nabla_\parallel^2 w + A_\perp\frac{\partial}{\partial r}\nabla\cdot\mathbf{u} + \frac{2\left(A_\parallel - 2A_\perp\right)}{r^2}w\right.\\
&+ \left.\frac{A_\parallel - 3A_\perp}{r^2\cos\theta}\frac{\partial u}{\partial\phi} + \frac{A_\parallel - 3A_\perp}{r^2\cos\theta}\frac{\partial}{\partial\theta}\left(v\cos\theta\right)\right), \qquad (9.66)
\end{aligned}
$$

where the horizontal and vertical Laplacian operators are define as

$$
\nabla_\perp^2 = \frac{1}{r^2\cos^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2} - \tan\theta\frac{\partial}{\partial\theta}\right) \qquad (9.67)
$$

$$
\nabla_\parallel^2 = \nabla^2 - \nabla_\perp^2. \qquad (9.68)
$$

In the limit where the turbulent mixing is isotrophic, i.e. $A_\perp = A_\parallel = A$, and the fluid is incompressible, (9.66) reduced to

$$
\begin{aligned}
\nabla\cdot\mathbf{\Pi} &= \widehat{\boldsymbol{\phi}}A\left(\nabla^2 u + \frac{2}{r^2\cos\theta}\left(-\tan\theta\frac{\partial v}{\partial\phi} + \frac{\partial w}{\partial\phi}\right) - \frac{u}{r^2\cos^2\theta}\right)\\
&+ \widehat{\boldsymbol{\theta}}A\left(\nabla^2 v + \frac{2}{r^2\cos\theta}\left(\tan\theta\frac{\partial u}{\partial\phi} + \cos\theta\frac{\partial w}{\partial\theta}\right) - \frac{v}{r^2\cos^2\theta}\right)\\
&+ \widehat{\mathbf{r}}A\left(\nabla^2 w - \frac{2}{r^2}w - \frac{2}{r^2\cos\theta}\frac{\partial u}{\partial\phi} - \frac{2}{r^2\cos\theta}\frac{\partial}{\partial\theta}\left(v\cos\theta\right)\right)\\
&= A\nabla^2\mathbf{u}, \qquad (9.69)
\end{aligned}
$$

which is the same result given by Batchelor [7, p.617]. Note that Batchelor use the polar angle ϑ instead instead of the latitude angle θ. By introducing a horizontal velocity $\mathbf{u}_\perp = u\widehat{\boldsymbol{\phi}} + v\widehat{\boldsymbol{\theta}}$ and a vertical velocity $\mathbf{u}_\parallel = w\widehat{\mathbf{r}}$, the following identities can be shown

$$
\begin{aligned}
\nabla_\perp^2 \mathbf{u}_\perp &= \widehat{\boldsymbol{\phi}} \left(\nabla_\perp^2 u - \frac{u}{r^2 \cos\theta} - \frac{2\tan\theta}{r^2\cos\theta} \frac{\partial v}{\partial \phi} \right) \\
&+ \widehat{\boldsymbol{\theta}} \left(\nabla_\perp^2 v - \frac{v}{r^2 \cos\theta} + \frac{2\tan\theta}{r^2\cos\theta} \frac{\partial u}{\partial \phi} \right) \\
&- \widehat{\mathbf{r}} \left(\frac{2}{r} \nabla_\perp \cdot \mathbf{u}_\perp \right) && (9.70) \\
\nabla_\parallel^2 \mathbf{u}_\perp &= \widehat{\boldsymbol{\phi}} \nabla_\parallel^2 u + \widehat{\boldsymbol{\theta}} \nabla_\parallel^2 v && (9.71) \\
\nabla_\perp^2 \mathbf{u}_\parallel &= \widehat{\mathbf{r}} \left(\nabla_\perp^2 w - \frac{2w}{r^2} \right) + \widehat{\boldsymbol{\phi}} \frac{2}{r^2\cos\theta} \frac{\partial w}{\partial \phi} + \widehat{\boldsymbol{\theta}} \frac{2}{r^2} \frac{\partial w}{\partial \theta} && (9.72) \\
\nabla_\parallel^2 \mathbf{u}_\parallel &= \widehat{\mathbf{r}} \nabla_\parallel^2 w, && (9.73)
\end{aligned}
$$

which provides the stress tensor,(9.66) in a more compact form the stress tensor:

$$
\begin{aligned}
\nabla \cdot \boldsymbol{\Pi} &= A_\perp \nabla_\perp^2 \mathbf{u}_\perp + A_\parallel \nabla_\parallel^2 \mathbf{u}_\perp + A_\perp \nabla_\perp^2 \mathbf{u}_\parallel + A_\parallel \nabla_\parallel^2 \mathbf{u}_\parallel + A_\perp \nabla_\perp (\nabla \cdot \mathbf{u}) + A_\perp \nabla_\parallel (\nabla \cdot \mathbf{u}) \\
&+ \frac{A_\parallel - A_\perp}{r} \left(\frac{\partial}{\partial r} \left(\mathbf{u}_\perp - \mathbf{u}_\parallel \right) + \nabla \cdot \mathbf{u} \right). && (9.74)
\end{aligned}
$$

9.5 Viscous Stress Tensor

The part of the momentum flux that is not due to the direct transfer of momentum with the mass of moving fluid is given by the minus of the divergence of the stress tensor

$$\boldsymbol{\sigma} = p\mathsf{I} - \boldsymbol{\sigma}', \tag{9.75}$$

where p is the pressure, I is the unit tensor and $\boldsymbol{\sigma}'$ is the viscous stress tensor which gives the irreversible viscous transfer of momentum in the fluid. According to Chapman-Enskog method[1] the viscous stress tensor is given by

$$\boldsymbol{\sigma}' = \eta \left(\nabla \otimes \mathbf{u} + (\nabla \otimes \mathbf{u})^T - \frac{2}{3} (\nabla \cdot \mathbf{u}) \mathsf{I} \right) + \zeta (\nabla \cdot \mathbf{u}) \mathsf{I}, \tag{9.76}$$

where η and ζ are the coefficients of viscosity which both are independent of velocity and positive quantities. In the case where the coefficients of viscosity are constant, the divergence of the stress tensor reads

$$\nabla \cdot \boldsymbol{\sigma} = \nabla p - \eta \nabla^2 \mathbf{u} - \left(\zeta + \frac{1}{3} \eta \right) \nabla (\nabla \cdot \mathbf{u}), \tag{9.77}$$

[1] See [8] for how to derive the viscous stress tensor from the Chapman-Enskog method , or see [5].

where we have used that the diadic product can be written as $\nabla \otimes \mathbf{u} = \nabla \mathbf{u}$ and the identety $\nabla \cdot \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T \right) = \nabla^2 \mathbf{u} + \nabla (\nabla \cdot \mathbf{u})$. In spherical coordinates ϕ, θ, r the components of the stress tensor are equal to the components of the Reynholds stress tensor with $\rho A_\perp = \rho A_{||} = \eta$, when the fluid is incompressible.

9.6 Helmholtz's theorem

Helmholtz's theorem states that a vector field $\mathbf{u}$ can be decomposed into one irrotational vector field $\mathbf{u}_{\mathrm{div}}$ and one divergence free vector field $\mathbf{u}_{\mathrm{rot}}$ as

$$\mathbf{u} = \mathbf{u}_{\mathrm{div}} + \mathbf{u}_{\mathrm{rot}}. \tag{9.78}$$

Where the decomposition must satisfy

$$\nabla \cdot \mathbf{u}_{\mathrm{rot}} = 0, \quad \nabla \times \mathbf{u}_{\mathrm{div}} = \mathbf{0}, \tag{9.79}$$

which implies that

$$\mathbf{u}_{\mathrm{rot}} = \nabla \times \mathbf{A}, \quad \mathbf{u}_{\mathrm{div}} = \nabla \lambda, \tag{9.80}$$

where $\mathbf{A}$ and λ are respectively the vector potential and the scalar potential. As a consequence of, (9.78), it follows for a two-dimensional vector field $\mathbf{u}_\perp$ that

$$\mathbf{u}_\perp = \mathbf{u}_{\perp,\mathrm{div}} + \mathbf{u}_{\perp,\mathrm{rot}} \tag{9.81}$$

where

$$\nabla_\perp \cdot \mathbf{u}_{\perp,\mathrm{rot}} = 0, \quad \nabla_\perp \times \mathbf{u}_{\perp,\mathrm{div}} = \mathbf{0}. \tag{9.82}$$

This implies that

$$\mathbf{u}_{\perp,\mathrm{rot}} = \nabla_\perp \times \mathbf{A}, \quad \mathbf{u}_{\perp,\mathrm{div}} = -\nabla_\perp \lambda, \tag{9.83}$$

however, since $\mathbf{u}_{\perp,\mathrm{rot}}$ should only have horizontal components, it follows that $\mathbf{A}$ must be given by,

$$\mathbf{A} = \psi \widehat{\mathbf{n}}, \tag{9.84}$$

where $\widehat{\mathbf{n}}$ is the unit vector in the vertical direction.

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