UiT THE ARCTIC UNIVERSITY OF NORWAY

FACULTY OF SCIENCE AND TECHNOLOGY

Department of Mathematics and Statistics

Symmetry transformation groups and differential invariants

—

Eivind Schneider

MAT-3900 Master's Thesis in Mathematics

November 2014

UNIVERSITY OF TROMSØ

Abstract

There exists a local classification of finite-dimensional Lie algebras of vector fields on $\mathbb{C}^2$. We lift the Lie algebras from this classification to the bundle $\mathbb{C}^2 \times \mathbb{C}$ and compute differential invariants of these lifts.

Acknowledgements

I am very grateful to Boris Kruglikov for coming up with an interesting assignment and for guiding me through the work on it. I also extend my sincere gratitude to Valentin Lychagin for his help through many insightful lectures and conversations. A heartfelt thanks to you both, for sharing your knowledge and for the inspiration you give me.

I would like to thank Henrik Winther for many helpful discussions.

Finally, I thank my family, and especially my wife, Susann, for their support and encouragement.

Contents

Chapter 1

Introduction

Consider the problem of classifying all complex analytic scalar partial differential equations of two independent variables with finite-dimensional symmetry groups. The local action of a symmetry group can be described in terms of its Lie algebra of infinitesimal generators. We say that the vector field $X \in \mathfrak{D}(\mathbb{C}^3(x, y, u))$ is an (infinitesimal) symmetry for the equation $F(x, y, u, u_x, u_y, ..., u_{y^k}) = 0$ if

$$X^{(k)}(F) = \lambda F$$

where λ is a smooth function of $x, y, ..., u_{y^k}$. Given a differential equation $F = 0$, we can find its symmetries by solving for X. These symmetries form a Lie algebra. We can also go the other way: Given a Lie algebra of symmetries $\mathfrak{g} \subset \mathfrak{D}(\mathbb{C}^3(x, y, u))$, we can solve for F to find all differential equations with the given Lie algebra as its symmetry algebra.

There exists a local classification of finite-dimensional Lie algebras of vector fields on $\mathbb{C}^2$. By using this, one can get a local description of all scalar ODEs with finite-dimensional Lie algebras of symmetries, up to point transformations.

For $\mathbb{C}^3$ there exists no complete classification of Lie algebras of vector fields, and therefore we cannot classify scalar PDEs of two independent variables in the same way. In this thesis we take the Lie algebras of vector fields on $\mathbb{C}^2$ from the classification, and lift them on the bundle $\mathbb{C}^2 \times \mathbb{C}$. This gives us a subset of all Lie algebras of vector fields on $\mathbb{C}^3$.

A subproblem of finding all differential equations with a given Lie algebra $\mathfrak{g}$ of symmetries, is to find functions F that satisfy

$$X^{(k)}(F) = 0 \qquad \text{for every} \qquad X \in \mathfrak{g}.$$

We call such functions differential invariants (of order k). For each of the lifted vector fields in $\mathbb{C}^2 \times \mathbb{C}$, we will compute differential invariants.

Structure of the thesis

We begin by describing the Jet space $J^k(\pi)$ for the bundle $\pi\colon \mathbb{C}^2 \times \mathbb{C} \to \mathbb{C}^2$ in 2.1. In section 2.2 we recall a classification of Lie algebras of vector fields on $\mathbb{C}$ and $\mathbb{C}^2$. Then, in 2.3 we describe the procedure we use for lifting vector fields on $\mathbb{C}^2$ to $\mathbb{C}^2 \times \mathbb{C}$, and discuss how the lifts correspond to cohomology groups. In section 2.4 we define differential invariants and invariant derivations, and state the Lie-Tresse theorem.

In chapter 3 we take the classification of Lie algebras of vector fields on $\mathbb{C}$, and lift each Lie algebra into a Lie algebra of vector fields on $\mathbb{C} \times \mathbb{C}$. We also find the differential invariants for the lifts. In this chapter the calculations are described in much more detail than in the later part of the thesis, and the chapter can therefore be considered as continuation of the introduction in 2.3 and 2.4.

Chapter 4 contains the main results. Lists of the lifts and their differential invariants are given in 4.1 and 4.3, respectively. In section 4.2 we look at the dimension of a generic orbit of some of the lifts and their jet-prolongations, and we use this to count how many independent differential invariants we expect to find.

Finally, in chapter 5, we look back on our computations and discuss possible applications of our results and some interesting properties of our lifts of Lie algebras. In 5.1 we discuss algebraicity of Lie algebra actions by looking at examples from our computations, and we see how this relates to the form of the differential invariants. In 5.2 we introduce the notion of projectable Lie algebra of vector fields, and discuss the surprising fact that all our lifts has at least one differential invariant of order two. In 5.3 we give an example of how our results can be usefull in the study of differential equations.

The appendix contains a list of differential invariants and invariant derivations that were to long to fit into 4.3 in a reasonable way.

Chapter 2

Preliminaries

2.1 Jet spaces and prolongation of vector fields

We start by fixing some notations regarding Jet spaces. For a more general and detailed description, see for example [KL08], [KV99] or [ALV91].

Jet spaces

Consider the trivial bundle $\pi\colon \mathbb{C}^2 \times \mathbb{C} \to \mathbb{C}^2$. Let x^1, x^2 be coordinates on $\mathbb{C}^2$ and let u be a coordinate on $\mathbb{C}$. We call x^1, x^2 independent coordinates, and u the dependent coordinate.[1] Let $s\colon \mathbb{C}^2 \to \mathbb{C}^2 \times \mathbb{C}$ be a section of the bundle. We can describe this section by a function $f\colon \mathbb{C}^2 \to \mathbb{C}$ in the following way: $s(x^1, x^2) = (x^1, x^2, f(x^1, x^2))$. We say that two sections s_1, s_2 are tangent up to order k at $a \in \mathbb{C}^n$ if $\frac{\partial^{|\sigma|} f_1}{\partial x^\sigma}(a) = \frac{\partial^{|\sigma|} f_2}{\partial x^\sigma}(a)$ for $0 \leq |\sigma| \leq k$, where σ is a multi-index. Denote by $[s]_a^k$ the equivalence class of all sections which are tangent up to order k to s at $a \in \mathbb{C}^2$. We call this the k-jet of s at a. Let $J_a^k(\mathbb{C}^2 \times \mathbb{C}) = J_a^k(\pi)$ be the set of k-jets of sections on π at the point $a \in \mathbb{C}^2$, and $J^k(\pi) = \cup_{a \in \mathbb{C}^2} J_a^k(\pi)$. This set is naturally equipped with the structure of a smooth manifold.

This description of the k-jets of sections, gives a natural set of coordinates x^i, u, u_σ on $J^k(\pi)$:

$$x^i([s]_a^k) = a^i, \quad u([s]_a^k) = f(a), \quad u_\sigma([s]_a^k) = \frac{\partial^{|\sigma|} f}{\partial x^\sigma}, \quad 1 \leq |\sigma| \leq k$$

We will also use the notation $u_0 = u$.

[1] We will for the most part be naming the coordinates x, y, u, but for general discussion it's convenient to use indices.

The map $(a, f(a)) \mapsto [s]_a^0$ gives us the identification $J^0(\mathbb{C}^2 \times \mathbb{C}) = \mathbb{C}^2 \times \mathbb{C}$. The projections $\pi_{k,l}\colon J^k(\pi) \to J^l(\pi)$ defined by $[s]_a^k \mapsto [s]_a^l$ for $k \geq l$ give a tower structure:

$$\mathbb{C}^2 \times \mathbb{C} = J^0(\pi) \xleftarrow{\pi_{1,0}} J^1(\pi) \xleftarrow{\pi_{2,1}} \cdots \xleftarrow{\pi_{k,k-1}} J^k(\pi) \xleftarrow{\pi_{k+1,k}} \cdots$$

We denote the inverse limit of this system of maps by $J^\infty(\pi)$.

Let $\mathcal{F}_k$ be the algebra of analytic functions on $J^k(\pi)$. Through the projections $\pi_{k,k-1}$, we get a filtering of the function algebras:

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_k \subset \cdots \subset \mathcal{F}_\infty$$

Prolongation of vector fields

Consider a diffeomorphism $\phi\colon \mathbb{C}^2 \times \mathbb{C} \to \mathbb{C}^2 \times \mathbb{C}$. The kth prolongation of ϕ is the diffeomorphism $\phi^{(k)}\colon J^k(\pi) \to J^k(\pi)$ defined by $\phi^{(k)}([s]_a^k) = [\phi \circ s]_{\phi(a)}^k$. If $X \in \mathfrak{D}(\mathbb{C}^2 \times \mathbb{C})$ is a vector field, then we can define the kth prolongation $X^{(k)}$ to be the vector field on $J^k(\pi)$ which is generated by the kth prolongation of the flow of X. We will be working with vector fields, so it's useful to have a coordinate description of prolongations of vector fields.

Given a vector field $X \in \mathfrak{D}(\mathbb{C}^2 \times \mathbb{C})$, the prolonged field $X^{(k)}$ can be computed in terms of the generating function φ of X, defined by $\varphi = \omega_0(X)$ where $\omega_0 = du - u_i dx^i$ (we use the Einstein summation convention).

The generating function gives us a nice formula for computing the prolongation of a vector field. If the vector field X has generating function φ, then its kth prolongation is given by

$$X^{(k)} = \sum_{|\sigma| \leq k} \mathcal{D}_\sigma(\varphi)\partial_{u_\sigma} - \sum_{i=1}^{2} \partial_{u_i}(\varphi)\, \mathcal{D}_i^{(k+1)}$$

where

$$\mathcal{D}_i^{(k+1)} = \partial_{x^i} + \sum_{|\sigma|=0}^{k} u_{\sigma i}\partial_{u_\sigma}$$

is the total derivative with respect to x^i restricted to $J^k(\pi)$. Let $\mathfrak{Э}_\varphi = \sum_{|\sigma|=0}^{\infty} \mathcal{D}_\sigma(\varphi)\partial_{u_\sigma}$. This is called the evolutionary derivation with generating function φ. The infinite prolongation $X^{(\infty)} \in \mathfrak{D}(J^\infty(\pi))$ of X is given by

$$X^{(\infty)} = \mathfrak{Э}_\varphi - \sum_{i=1}^{n} \partial_{u_i}(\varphi)\, \mathcal{D}_i$$

where $\partial_{x^i} + \sum_{|\sigma|=0}^{\infty} u_{\sigma i}\partial_{u_\sigma}$ is the total derivative.

Remark 1. *The differential forms $\omega_\sigma = du_\sigma - u_{\sigma i}dx^i$ for $|\sigma| \leq k-1$ determine a distribution on $J^k(\pi)$ called the Cartan distribution. Using this we can define a Lie field as a vector field on $J^k(\pi)$ that preserves the Cartan distribution. If $X \in \mathfrak{D}(\mathbb{C}^2 \times \mathbb{C})$, then $X^{(k)}$ is the unique Lie field that projects to X through $d\pi_{k,0}$. The Lie-Bäcklund theorem tells us that all Lie fields are prolongations of Lie (or, in other words, contact) fields on $J^1(\pi)$. In this sense there are more Lie fields on $J^k(\pi)$ than those prolonged from vector fields on $\mathbb{C}^2 \times \mathbb{C}$.*

2.2 Classification of Lie algebras of vector fields in one and two dimensions

Let G be a Lie group acting on a manifold M, and let $\mathfrak{g}$ be the Lie algebra corresponding to G. The infinitesimal generators of the action of G on M are given by a Lie algebra homomorphism $\rho\colon \mathfrak{g} \to \mathfrak{D}(M)$. The image $\rho(\mathfrak{g}) \in \mathfrak{D}(M)$ is a Lie algebra algebra of vector fields.

The Lie group G acts locally effectively on M if and only if ρ is injective. If G does not act effectively, then the quotient group G/G_M, where G_M is the global isotropy group, does act effectively with the action $(g+G_M)\cdot x = g \cdot x$. So instead of considering G, we can consider G/G_M with Lie algebra $\hat{\mathfrak{g}}$.

Hence every Lie algebra of vector fields on a manifold M can be described by a injective Lie algebra homomorphism $\hat{\rho}\colon \hat{\mathfrak{g}} \to \mathfrak{D}(M)$ of an abstract Lie algebra $\hat{\mathfrak{g}}$. We will usually use $\hat{\mathfrak{g}}$ to denote the image $\hat{\rho}(\hat{\mathfrak{g}}) \in \mathfrak{D}(M)$.

Definition 1. We say that two Lie algebras of vector fields $\mathfrak{g} \in \mathfrak{D}(M), \mathfrak{g}' \in \mathfrak{D}(M')$ are locally equivalent if there exist open sets $U \subset M$ and $U' \subset M'$, and a local biholomorphism $f\colon U \to U'$ such that $df(\mathfrak{g}|_U) = \mathfrak{g}'|_{U'}$.

In [Lie70] Sophus Lie gave local classifications (up to local equivalence) of all nonsingular finite-dimensional Lie algebras of analytic vector fields in one and two complex dimensions (page 6 and 71, respectively). Nonsingular means that there are no fixed points.

Classification of Lie algebras of vector fields on $\mathbb{C}$

Any nonsingular finite-dimensional Lie algebra of analytic vector fields on $\mathbb{C}$ is locally equivalent to one of the following:

$$\mathfrak{g}_1 = \langle \partial_x \rangle, \qquad \mathfrak{g}_2 = \langle \partial_x, x\partial_x \rangle, \qquad \mathfrak{g}_3 = \langle \partial_x, x\partial_x, x^2\partial_x \rangle.$$

Classification of Lie algebras of vector fields on $\mathbb{C}^2$

Any nonsingular finite-dimensional Lie algebra of analytic vector fields on $\mathbb{C}^2$ is locally equivalent to one of the following:

Primitive and locally transitive

$\mathfrak{g}_1 = \langle \partial_x, \partial_y, x\partial_x, x\partial_y, y\partial_x, y\partial_y, x^2\partial_x + xy\partial_y, xy\partial_x + y^2\partial_y \rangle$

$\mathfrak{g}_2 = \langle \partial_x, \partial_y, x\partial_x, x\partial_y, y\partial_x, y\partial_y \rangle$

$\mathfrak{g}_3 = \langle \partial_x, \partial_y, x\partial_y, y\partial_x, x\partial_x - y\partial_y \rangle$

Nonprimitive, locally transitive ($r = \dim \mathfrak{g}_i$)

$\mathfrak{g}_4 = \langle \partial_x, e^{\alpha_1 x}\partial_y, xe^{\alpha_1 x}\partial_y, ..., x^{m_1-1}e^{\alpha_1 x}\partial_y, xe^{\alpha_2 x}\partial_y, ..., x^{m_s-1}e^{\alpha_s x}\partial_y \rangle,$

where $m_i \in \mathbb{N}, \alpha_i \in \mathbb{C}, i = 1, ..., s, \sum_{i=1}^{s} m_i + 1 = r \geq 2$

$\mathfrak{g}_5 = \langle \partial_x, y\partial_y, e^{\alpha_1 x}\partial_y, xe^{\alpha_1 x}\partial_y, ..., x^{m_1-1}e^{\alpha_1 x}\partial_y, xe^{\alpha_2 x}\partial_y, ..., x^{m_s-1}e^{\alpha_s x}\partial_y \rangle,$

where $m_i \in \mathbb{N}, \alpha_i \in \mathbb{C}, i = 1, ..., s, \sum_{i=1}^{s} m_i + 2 = r \geq 2$

$\mathfrak{g}_6 = \langle \partial_x, \partial_y, y\partial_y, y^2\partial_y \rangle$

$\mathfrak{g}_7 = \langle \partial_x, \partial_y, x\partial_x, x^2\partial_x + x\partial_y \rangle$

$\mathfrak{g}_8 = \langle \partial_x, \partial_y, x\partial_y, ..., x^{r-3}\partial_y, x\partial_x + \lambda y\partial_y \rangle$ for $\lambda \in \mathbb{C} \setminus \{r-2\}, r \geq 3$

$\mathfrak{g}_9 = \langle \partial_x, \partial_y, x\partial_y, ..., x^{r-3}\partial_y, x\partial_x + \left((r-2)y + x^{r-2}\right)\partial_y \rangle, r \geq 3$

$\mathfrak{g}_{10} = \langle \partial_x, \partial_y, x\partial_y, ..., x^{r-4}\partial_y, x\partial_x, y\partial_y \rangle, r \geq 4$

$\mathfrak{g}_{11} = \langle \partial_x, x\partial_x, \partial_y, y\partial_y, y^2\partial_y \rangle$

$\mathfrak{g}_{12} = \langle \partial_x, x\partial_x + \partial_y \rangle$

$\mathfrak{g}_{13} = \langle \partial_x, \partial_y, x\partial_y, ..., x^{r-4}\partial_y, x^2\partial_x + (r-4)xy\partial_y, x\partial_x + \frac{r-4}{2}y\partial_y \rangle, r > 4$

$\mathfrak{g}_{14} = \langle \partial_x, \partial_y, ..., x^{r-5}\partial_y, y\partial_y, x\partial_x, x^2\partial_x + (r-5)xy\partial_y \rangle, r > 5$

$\mathfrak{g}_{15} = \langle \partial_x, x\partial_x + \partial_y, x^2\partial_x + 2x\partial_y \rangle$

$\mathfrak{g}_{16} = \langle x^2\partial_x + y^2\partial_y, x\partial_x + y\partial_y, \partial_x + \partial_y \rangle$

$\mathfrak{g}_{17} = \langle \partial_x, x\partial_x, x^2\partial_x, \partial_y, y\partial_y, y^2\partial_y \rangle$

Nonprimitive and locally intransitive

$\mathfrak{g}_{18} = \langle \partial_x \rangle$
$\mathfrak{g}_{19} = \langle \partial_x, x\partial_x \rangle$
$\mathfrak{g}_{20} = \langle \partial_x, x\partial_x, x^2\partial_x \rangle$
$\mathfrak{g}_{21} = \langle \partial_y, \phi_2(x)\partial_y, ..., \phi_r(x)\partial_y \rangle$, where ϕ_i are analytic functions
$\mathfrak{g}_{22} = \langle \partial_y, y\partial_y, \phi_3(x)\partial_y, ..., \phi_r(x)\partial_y \rangle$, where ϕ_i are analytic functions

This list is copied from [GOV93].

2.3 Lifts

Given a vector bundle $\pi\colon E \to M$, a projectable vector field on E is a vector field X that projects to a vector field $d\pi(X)$ on M. A projectable vector field on the bundle $\mathbb{C}^2 \times \mathbb{C} \to \mathbb{C}^2$ with coordinates x, y and u can be expressed on the form $a(x,y)\partial_x + b(x,y)\partial_y + c(x,y,u)\partial_u$. We denote the set of projectable vector fields on $\mathbb{C}^2 \times \mathbb{C}$ by $\mathfrak{D}^{\text{proj}}(\mathbb{C}^2 \times \mathbb{C})$

Definition 2. A lift of $\mathfrak{g} \subset \mathfrak{D}(\mathbb{C}^2)$ on the bundle $\pi\colon \mathbb{C}^2 \times \mathbb{C} \to \mathbb{C}^2$ is a Lie algebra homomoorphism $\rho'\colon \mathfrak{g} \to \mathfrak{D}^{\text{proj}}(\mathbb{C}^2 \times \mathbb{C})$ such that the following diagram commutes:

$$\begin{tikzcd} & \mathfrak{D}^{\text{proj}}(\mathbb{C}^2 \times \mathbb{C}) \arrow[d, "d\pi"] \\ \mathfrak{g} \arrow[ur, "\rho'"] \arrow[r, "\rho"] & \mathfrak{D}(\mathbb{C}^2) \end{tikzcd}$$

We do not consider the most general lift in this paper, but only lifts that are constant on the fibers. Then the lift of $a(x,y)\partial_x + b(x,y)\partial_y$ has the form $a(x,y)\partial_x + b(x,y)\partial_y + c(x,y)\partial_u$. We call this a "constant" lift.

Definition 3. A constant lift of $\mathfrak{g} \subset \mathfrak{D}(\mathbb{C}^2)$ on $\pi\colon \mathbb{C}^2 \times \mathbb{C} \to \mathbb{C}^2$ is a lift that is constant on the fibers.

Let $\mathfrak{g} = \langle Y_1, ..., Y_r \rangle$ be a Lie algebra of vector fields on $\mathbb{C}^2$ with commutation relations $[Y_i, Y_j] = C_{ij}^k Y_k$. The generators are of the form $Y_i = a_i(x,y)\partial_x + b_i(x,y)\partial_y$. We will consider lifts of Y_i to $\mathfrak{D}(\mathbb{C}^2 \times \mathbb{C})$ of the form $Y_i^{(0)} = a_i(x,y)\partial_x + b_i(x,y)\partial_y + c_i(x,y)\partial_u$. We must have $[Y_i^{(0)}, Y_j^{(0)}] = C_{ik}^k Y_k^{(0)}$, with the same structure constants as for $\mathfrak{g}$, for the diagram above to commute. The commutation relations for the lifted generators give some differential equations containing the functions c_i. By solving these differential equations, we find all the possible lifts. Since $\mathbb{C}^2 \times \mathbb{C} = J^0(\mathbb{C}^2 \times \mathbb{C})$, it seems natural to denote the lift of this algebra by $\mathfrak{g}^{(0)}$, and the lift of $X \in \mathfrak{g}$ by $X^{(0)}$. Our first objective is to apply this procedure to all the Lie algebras from the classifications in 2.2.

2.3.1 Cohomology

If we allow for coordinate transformations of the form $(x, y, u) \mapsto (x, y, u - U(x, y))$, and say that two lifts are equivalent if one can be transformed into the other by such a transformation, then the different lifts are described in terms of Lie algebra cohomology.

Let $X, Y \in \mathfrak{g}$. Then the lifts are given by $X^{(0)} = X + \psi_X \partial_u$ and $Y^{(0)} = Y + \psi_Y \partial_u$, where ψ is a $C^\omega(\mathbb{C}^2)$-valued one-form on $\mathfrak{g}$. The commutator is given by

$$[X^{(0)}, Y^{(0)}] = [X + \psi_X \partial_u, Y + \psi_Y \partial_u] = [X, Y] + (X(\psi_Y) - Y(\psi_X))\partial_u.$$

We see that the one-form ψ defines a lift if and only if the equality $X(\psi_Y) - Y(\psi_X) = \psi_{[X,Y]}$ holds. Now, consider the following complex.

$$C^\omega(\mathbb{C}^2) \xrightarrow{d} \mathfrak{g}^* \otimes C^\omega(\mathbb{C}^2) \xrightarrow{d} \wedge^2 \mathfrak{g}^* \otimes C^\omega(\mathbb{C}^2)$$

where d is defined by

$$\begin{aligned} df(X) &= X(f), \quad f \in C^\omega(\mathbb{C}^2) \\ d\psi(X, Y) &= X(\psi_Y) - Y(\psi_X) - \psi_{[X,Y]}, \quad \psi \in \mathfrak{g}^* \otimes C^\omega(\mathbb{C}^2). \end{aligned}$$

The one-form ψ defines a lift if and only if $d\psi = 0$.

Now, two lifts are equivalent if there exists a biholomorphism

$$\phi\colon (x, y, u) \mapsto (x, y, u - U(x, y))$$

on $\mathbb{C}^2 \times \mathbb{C}$ that brings one to the other. The expression

$$d\phi\colon X + \psi_X \partial_u \mapsto X + (\psi_X - dU(X))\partial_u$$

for the differential of ϕ shows us that two lifts $\psi, \tilde{\psi}$ are equivalent if and only if $\psi_X - \tilde{\psi}_X = dU(X)$ for some $U \in C^\omega(\mathbb{C}^2)$. This means that the different lifts are encoded in terms of the cohomology group

$$H^1(\mathfrak{g}, C^\omega(\mathbb{C}^2)) = \{\psi \in \mathfrak{g}^* \otimes C^\omega(\mathbb{C}^2) \mid d\psi = 0\}/\{dU \mid U \in C^\omega(\mathbb{C}^2)\}.$$

Hence we have the following theorem.

Theorem 1. *There is a one-to-one corresponcence between the set of constant lifts of the Lie algebra $\mathfrak{g} \subset \mathfrak{D}(\mathbb{C}^2)$ (up to equivalence) and the cohomology group $H^1(\mathfrak{g}, C^\omega(\mathbb{C}^2))$.*

2.3.2 Coordinate change

To begin with, we will consider the lifts of Lie algebras up to coordinate transformations of the form $(x, y, u) \mapsto (X(x, y), Y(x, y), u - U(x, y))$. Since the Lie algebras of vector fields on $\mathbb{C}^2$ are already in normal form, we can let $X = x$ and $Y = y$. After finding the lifts we will also apply a transformation of the form $u \mapsto Cu$. Note that transformations of the form $(x, y, u) \mapsto (X(x, y), Y(x, y), Cu - U(x, y))$ preserve the set of vector fields that are constant on fibers, and these are actually the only transformations that do that.

The fact that we consider lifts up to suitable coordinate transformations significantly simplifies the expressions we get for the lifts, and it also simplifies the differential equations we have to solve in order to find the lifts.

Example 1. *Consider abelian Lie algebra $\langle X, Y\rangle \subset \mathfrak{D}(\mathbb{C}^2)$, where $X = \partial_x, y = \partial_y$. The lifts of the generators take the form $X^{(0)} = \partial_x + a(x, y)\partial_u$ and $Y^{(0)} = \partial_y + b(x, y)\partial_u$. The holomorphic transformation $u \mapsto u - \int a(x, y)dx$ brings $X^{(0)}$ to the form ∂_x. After this coordinate change, the expression for $Y^{(0)}$ will change, but it will still be of the same form, just with a different function b. The lift is a Lie algebra homomorphism, so we must have $[X^{(0)}, Y^{(0)}] = \partial_x(b)\partial_u = 0$. Hence $b = b(y)$. The holomorphic transformation $u \mapsto u - \int b(y)dy$ maps $Y^{(0)} = \partial_y + b(y)\partial_u$ to ∂_y. Hence all lifts of $\langle X, Y\rangle$ are trivial, up to a triangular transformation.*

This example is very useful since most of the Lie algebras we work with contain $\langle X, Y\rangle$ as a subalgebra. The simple forms of $X^{(0)}, Y^{(0)}$ simplify the differential equations for the lift of the rest of the generators.

2.4 Differential invariants

In this section we will state some definitions and results regarding differential invariants. At the end of this section, we state the Lie-Tresse theorem, which is of great importance for the calculation of differential invariants. Usually these definitions and results are stated using Lie group actions (or pseudogroup actions), while computations are usually done by considering the Lie algebra of infinitesimal generators of the Lie group action. Since our starting point is the classifications of Lie group actions in terms of infinitesimal generators (i.e. Lie algebras of vector fields), we will define everything in terms of these. See for example [Olv96] for an introduction to differential invariants for Lie group actions.

Definition 4. Let $\mathfrak{g} \subset \mathfrak{D}(\mathbb{C}^2 \times \mathbb{C})$. A function $I \in \mathcal{F}_k = C^\omega(J^k(\mathbb{C}^2 \times \mathbb{C}))$ is a differential invariant of order k if

$$X^{(k)}(I) = 0 \text{ for every } X \in \mathfrak{g}.$$

We do not usually require I to be defined at all points. The common approach is to consider an open set in $J^k(\pi)$ on which I is defined. We call this the micro-local approach.

Since prolongation is a Lie algebra homomorphism, we only need to check this equation on the generators of $\mathfrak{g}$. In other words, to find differential invariants of order k of the algebra generated by $X_1, ..., X_r$ we must solve r linear first-order differential equations:

$$X_i^{(k)}(I) = 0, \quad i = 1, ..., r.$$

With pointwise addition and multiplication, the differential invariants of order k make up an algebra, $\mathcal{A}_k$. It's obvious that all differential invariants of order k are also differential invariants of order $k+1$. Hence, we get a filtering

$$\mathcal{A}_0 \subset \mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots .$$

2.4.1 Determining the number of differential invariants of order k

Often when we have a Lie group acting on a manifold, we want to know what the orbits of the group action look like. This question is closely related to the question about invariant functions on the manifold, i.e. functions that are constant on the orbits of the group action. Locally, around generic points, these questions can be answered by Frobenius' theorem.

For us, the Lie group action will always be given in terms of the Lie algebra $\langle X_1, ..., X_r \rangle$ of infinitesimal generators. In the neighborhood of a generic point (a point where the dimension of $\langle X_1, ..., X_r \rangle$ is maximal), the Lie algebra of infinitesimal generators determines a distribution on the manifold which, by Frobenius' theorem, is integrable.

Theorem 2 (Frobenius)**.** *Let P be an s-dimensional distribution on an n-dimensional manifold. There exist local coordinates $w^1, ..., w^n$ such that $P = \langle \partial_{w^1}, ..., \partial_{w^s} \rangle$ if and only if $[X, Y] \in P$ for every $X, Y \in P$.*

In these coordinates the integral manifolds (which are the orbits of the group action) are given by $w^{s+1} = c_{s+1}, ..., w^n = c_n$ where $c_{s+1}, ..., c_n \in \mathbb{C}$, which means that there are $n-s$ functionally independent invariant functions: $w^{s+1}, ..., w^n$.

A differential invariant of order k of $\mathfrak{g}$ is the same as an invariant function of $\mathfrak{g}^{(k)}$. In the neighborhood of a generic point, the Lie algebra $\mathfrak{g}^{(k)}$ determines an s-dimensional distribution. In this neighborhood $J^k(\pi)$ is foliated by s-dimensional submanifolds that are the orbits of $\mathfrak{g}^{(k)}$. Hence there are $\dim J^k(\pi) - s$ functionally independent differential invariants of order k. We say that $I \in \mathcal{A}_k$ is a differential invariant of strict order k if $I \notin \mathcal{A}_{k-1}$.

Definition 5. If $I, J \in \mathcal{A}_k$ are differential invariants of strict order k, we say that they are strictly independent if the functions $I, J, x, y, u, ..., u_{y^{k-1}}$ are functionally independent.

Our goal is to find all differential invariants for the lifts of Lie algebras. This problem may seem too difficult, since there are infinitely many functionally independent differential invariants in $\mathcal{A}_\infty$. However, the Lie-Tresse theorem tells us that every differential invariant is generated by a finite number of differential invariants and invariant derivations.

2.4.2 Invariant derivations

Definition 6. An invariant derivation is a horizontal vector field $\nabla = \alpha\,\mathcal{D}_x + \beta\,\mathcal{D}_y \in \mathfrak{D}(J^\infty(\pi))$, where $\alpha, \beta \in \mathcal{F}_k$ for some k, that commutes with the infinite prolongation of all vector fields in $\mathfrak{g}$, i.e. $[\nabla, X^{(\infty)}] = 0$ for every $X \in \mathfrak{g}$.

We say that ∇ is of order k if $\alpha, \beta \in \mathcal{F}_k$. Given a differential invariant I and an invariant derivation ∇, the product $I \cdot \nabla$ is again an invariant derivation. Hence the invariant derivations form a module over the algebra of differential invariants. We say that ∇_1 and ∇_2 are independent if they are linearly independent in this module. Since the base space of our bundle is two-dimensional, we only need two independent invariant derivations to generate all of them.

One way to find invariant derivations is to solve the commutation equations. Let $\mathfrak{g} = \langle X_1, ..., X_r \rangle$ and $\nabla = \alpha\,\mathcal{D}_x + \beta\,\mathcal{D}_y$ for $\alpha, \beta \in \mathcal{F}_k$ for some k. ∇ is an invariant derivation if the following equations hold:

$$[\nabla, X_i^{(\infty)}] = 0, \quad i = 1, 2, ..., r.$$

We'll rewrite these equations. Let x^1, x^2, u be coordinates on $\mathbb{C}^2 \times \mathbb{C}$. If $X_i = a_i^j(x)\partial_{x^j} + b_i(x)\partial_u$, then $X_i^{(\infty)} = a_i^j(x)\,\mathcal{D}_{x^j} + \ni_\phi$. The evolutionary derivative commutes with total derivatives.

$$\begin{aligned}[\nabla, X_i^{(\infty)}] &= [\alpha^l \mathcal{D}_{x^l}, a_i^j \mathcal{D}_{x^j} + \ni_\phi] \\ &= \alpha^l \partial_{x^l}(a_i^j) \mathcal{D}_{x^j} - (a_i^j \mathcal{D}_{x^j} + \ni_\phi)(\alpha^l) \mathcal{D}_{x^l} \\ &= \alpha^l \partial_{x^l}(a_i^j) \mathcal{D}_{x^j} - X_i^{(\infty)}(\alpha^j) \mathcal{D}_{x^j} \\ &= 0\end{aligned}$$

So for each generator X_i we get a set of 2 linear first-order differential equations of the form

$$X_i^{(\infty)}(\alpha^j) = \alpha^l \partial_{x^l}(a_i^j).$$

Note that since α^j is a function on some finite-order Jet space $J^k(\mathbb{C}^2 \times \mathbb{C})$, we have $X_i^{(\infty)}(\alpha^j) = X_i^{(k)}(\alpha^j)$.

2.4.3 Tresse derivatives

In some cases, the commutation equations are difficult to solve, and we need another method of finding invariant derivations. The following method requires that we have found two functionally independent differential invariants f_1, f_2. In local coordinates, we can define the horizontal differential $\hat{d}\colon C^\omega(J^k(\pi)) \to \Omega^1(J^{k+1}(\pi))$ in the following way:

$$\hat{d}f = \mathcal{D}_{x^i}(f) dx^i$$

If f_1, f_2 are functionally independent, then

$$\hat{d}f_1 \wedge \hat{d}f_2 \neq 0.$$

This means that the total Jacobian matrix

$$DF = \begin{pmatrix} \mathcal{D}_{x^1}(f_1) & \mathcal{D}_{x^1}(f_2) \\ \mathcal{D}_{x^2}(f_1) & \mathcal{D}_{x^2}(f_2) \end{pmatrix}$$

is nondegenerate. For any other differential invariant f, we have

$$\hat{d}f = \frac{\hat{\partial}_i}{\hat{\partial} f_i}(f) \hat{d}f_i.$$

Thus

$$\hat{d} = dx^i \otimes \mathcal{D}_{x^i} = \hat{d}f_i \otimes \frac{\hat{\partial}_i}{\hat{\partial} f_i}.$$

This gives us the expression of Tresse derivatives $\hat{\partial}_i = \hat{\partial}_i/\hat{\partial} f_i$:

$$\begin{pmatrix} \hat{\partial}_1 \\ \hat{\partial}_2 \end{pmatrix} = \begin{pmatrix} \mathcal{D}_{x^1}(f_1) & \mathcal{D}_{x^1}(f_2) \\ \mathcal{D}_{x^2}(f_1) & \mathcal{D}_{x^2}(f_2) \end{pmatrix}^{-1} \begin{pmatrix} \mathcal{D}_{x^1} \\ \mathcal{D}_{x^2} \end{pmatrix}$$

These are two independent invariant derivations that also have the property that they commute with eachother: $[\hat{\partial}_i, \hat{\partial}_j] = 0$.

See [KL06] for more details.

2.4.4 The Lie-Tresse theorem

The Lie-Tresse theorem is a theorem motivated by Lie and Tresse ([Lie93, p. 760] and [Tre94]) that states, loosely speaking, that all differential invariants of a finite-dimensional Lie group of point transformations are generated by a finite number of differential invariants and invariant derivations.

The theorem was rigorously proved in [Kum75a] and [Kum75b] for actions of pseudogroups, micro-locally on generic orbits. In [KL08] it was generalized for pseudogroup actions on differential equations.

For us, the following version will be sufficient.

Theorem 3 (Lie-Tresse)**.** *Let $\mathfrak{g} \subset \mathfrak{D}(\mathbb{C}^2 \times \mathbb{C})$. There exist two invariant derivations ∇_1, ∇_2 and a finite number of differential invariants $I_1, ..., I_q$ such that, micro-locally, any other differential invariant can be written as a function of $I_1, ..., I_q$ and $\nabla_{j_k} \cdots \nabla_{j_1}(I_i)$ for some integer k, where $j_l \in \{1,2\}$ for $l \in \{1, ..., k\}$.*

By adding some conditions for the group action and the manifold it acts on, we can obtain a global version of the Lie Tresse theorem (see [KL13]). We saw earlier that Frobenius' theorem guaranteed enough functionally independent invariants to separate the orbits locally. If a Lie group is acting algebraically on an irreducible algebraic variety, then Rosenlicht's theorem does the same thing, only globally.

Theorem 4 (Rosenlicht)**.** *For an algebraic action of a Lie group on an irreducible variety X, a finite set of rational invariants separates generic orbits.*

Proof. See [Ros56], theorem 2 or [PV94], theorem 2.3. $\square$

We will discuss the topic of algebraic group actions further in 5.1.

Chapter 3

Warm-up: Differential invariants of lifts of Lie algebras in $\mathfrak{D}(\mathbb{C})$

Any nonzero, nonsingular finite-dimensional Lie algebra of analytic vector fields on $\mathbb{C}$ is locally equivalent to one of the following:

$$\mathfrak{g}_1 = \langle \partial_x \rangle, \qquad \mathfrak{g}_2 = \langle \partial_x, x\partial_x \rangle, \qquad \mathfrak{g}_3 = \langle \partial_x, x\partial_x, x^2\partial_x \rangle.$$

In this chapter we will find all constant lifts of these three Lie algebras to $\mathbb{C} \times \mathbb{C}$, and compute the differential invariants of these lifts. Since there are only three cases, we will do a much more detailed description of the calculations here, than we will do for the Lie algebras of vector fields on $\mathbb{C}^2$. Hence this chapter can be viewed as an elementary introduction to the techniques we use for the Lie algebras of vector fields on $\mathbb{C}^2$. The reader not interested in the details can jump to section 3.4 for a summary.

3.1 $\mathfrak{g}_1 = \langle \partial_x \rangle$

Let $X = \partial_x$. The lift of X has the form $X^{(0)} = \partial_x + a(x)\partial_u$. By the coordinate transformation $u \mapsto u - \int a(x)dx$ it can be brought to the form $X^{(0)} = \partial_x$.

The kth prolongation is $X^{(k)} = \partial_x$ for $k = 0, 1, 2, \ldots.$ Every function that does not depend on x is a differential invariant. Thus the differential invariants of order k are generated by $u, u_x, u_{xx}, \ldots, u_{x^k}$.

Since the base space of $\mathbb{C} \times \mathbb{C}$ is one-dimensional, we need only one invariant derivation. The vector field $\nabla = \alpha\, \mathcal{D}_x$ is an invariant derivation if it commutes with $X^{(\infty)} = \partial_x$, i.e. if α is a solution to the equation

$$[\alpha\, \mathcal{D}_x, \partial_x] = -\alpha_x\, \mathcal{D}_x = 0.$$

The function $\alpha = 1$ is obviously a solution. And since $\mathcal{D}_x(u_{x^i}) = u_{x^{i+1}}$ for $i = 0, 1, 2, \ldots$, every differential invariant is generated by u and $\mathcal{D}_x$.

3.2 $\mathfrak{g}_2 = \langle \partial_x, x\partial_x \rangle$

3.2.1 Lift of $\mathfrak{g}_2$ to $\mathfrak{D}(\mathbb{C} \times \mathbb{C})$ and invariants on $\mathbb{C} \times \mathbb{C}$

Let $X_0 = \partial_x, X_1 = x\partial_x$. The lifts of these to $\mathfrak{D}(\mathbb{C} \times \mathbb{C})$ have the form

$$X_0^{(0)} = \partial_x + a_0(x)\partial_u, \quad X_1^{(0)} = x\partial_x + a_1(x)\partial_u.$$

We can straighten out $X_0^{(0)}$ like we did in the last section, so the lifts get the following form:

$$X_0^{(0)} = \partial_x, \quad X_1^{(0)} = x\partial_x + a(x)\partial_u.$$

The commutation relation for $\mathfrak{g}_2$ is $[X_0, X_1] = X_0$. Let's impose the corresponding equation equation on $X_0^{(0)}$ and $X_1^{(0)}$.

$$X_0^{(0)} = [X_0^{(0)}, X_1^{(0)}] = [\partial_x, x\partial_x + a(x)\partial_u] = \partial_x + a'(x)\partial_u$$

It follows that $a' = 0$, so $a = C$ is constant. Hence

$$X_0^{(0)} = \partial_x, \quad X_1^{(0)} = x\partial_x + C\partial_u.$$

If $C = 0$, the invariants are generated by u. If $C \neq 0$, $\mathfrak{g}_2^{(0)} = \langle X_0^{(0)}, X_1^{(0)} \rangle$ is transitive on $\mathbb{C} \times \mathbb{C}$, so there are no invariants on $\mathbb{C} \times \mathbb{C}$. Note also that when $C \neq 0$, the coordinate transformation $u \mapsto u/C$ normalizes the constant. So we can assume that $C = 0$ or $C = 1$.

3.2.2 Differential invariants of first order

Now, let's prolong $\mathfrak{g}_2^{(0)}$ to $\mathfrak{D}(J^1(\mathbb{C} \times \mathbb{C}))$. We get

$$X_0^{(1)} = \partial_x, \qquad X_1^{(1)} = x\partial_x + C\partial_u - u_x\partial_{u_x}$$

where $C = 0$ or $C = 1$. We find the differential invariants of first order (the invariants of $\mathfrak{g}_2^{(1)}$ on $J^1(\mathbb{C} \times \mathbb{C})$) by solving the system

$$\begin{cases} X_0^{(1)}(f) = 0 \\ X_1^{(1)}(f) = 0 \end{cases}$$

where $f = f(x, u, u_x)$. The system is equivalent to the equation

$$C\partial_u f(u, u_x) - u_x\partial_{u_x} f(u, u_x) = 0.$$

If $C = 1$, the general solution to this equation is

$$f(x, u, u_x) = F\left(u_x e^u\right).$$

Hence the algebra of differential invariants of first order is generated by

$$I_1 = u_x e^u.$$

If $C = 0$, the equation reduces to $\partial_{u_x} f(u, u_x) = 0$ which tells us that $f = f(u)$. Hence, in this case, there are no new differential invariants of first order, and the algebra of differential invariants of first order is generated by u.

3.2.3 Differential invariants of higher order

A generic orbit of $\mathfrak{g}_2^{(1)}$ is two-dimensional. This is also true for $\mathfrak{g}_2^{(k)}$ for $k > 1$. Frobenius' theorem tells us that locally, there are $\dim J^k(\mathbb{C} \times \mathbb{C}) - 2 = k$ functionally independent differential invariants of order k for $k \geq 1$. This in turn implies that there is maximally one strictly independent differential invariant of strict order k for $k \geq 2$.

In the next section we find an invariant derivation that, together with the differential invariant we have found, generates all differential invariants for the cases $C = 0$ and $C = 1$, respectively.

3.2.4 Invariant derivations

The vector field $\nabla = \alpha \, \mathcal{D}_x$ is an invariant derivation if it commutes with $X_0^{(\infty)}$ and $X_1^{(\infty)}$:

$$\begin{aligned}
[\nabla, X_0^{(\infty)}] &= [\alpha \, \mathcal{D}_X \, x, \partial_x] = -\alpha_x \, \mathcal{D}_x = 0 \\
[\nabla, X_1^{(\infty)}] &= [\alpha \, \mathcal{D}_x, x \, \mathcal{D}_x + \ni_{C - x u_x}] = \alpha \, \mathcal{D}_x - X_1^{(\infty)}(\alpha) \, \mathcal{D}_x = 0
\end{aligned}$$

The first equation tells us that α does not depend on x. Let first $C = 1$. If we assume that $\alpha = \alpha(u)$, the second equation is equivalent to $\alpha_u = \alpha$. One solution to this equation is $\alpha = e^u$, and hence

$$\nabla = e^u \, \mathcal{D}_x$$

is an invariant derivation.

If $C = 0$, we try with $\alpha = \alpha(u_x, u_{xx})$. Then the second equation is equivalent to $u_x \alpha_{u_x} + 2 u_{xx} \alpha_{u_{xx}} + \alpha = 0$. The function u_{xx}/u_x^3 is a solution to this equation, and thus

$$\hat{\nabla} = \frac{u_{xx}}{u_x^3} \, \mathcal{D}_x$$

is an invariant derivation.

The algebra of differential invariants is generated by I_1 and ∇ when $C = 1$ and by u and $\hat{\nabla}$ when $C = 0$.

3.3 $\mathfrak{g}_3 = \langle \partial_x, x\partial_x, x^2\partial_x \rangle$

3.3.1 Lift of $\mathfrak{g}_3$ to $\mathfrak{D}(\mathbb{C} \times \mathbb{C})$ and invariants on $\mathbb{C} \times \mathbb{C}$

Let $X_0 = \partial_x, X_1 = x\partial_x, X_2 = x^2\partial_x$. By the same argument that we used in the last section, the lift of these vector fields can be brought to the form

$$X_0^{(0)} = \partial_x, \quad X_1^{(0)} = x\partial_x + A\partial_u, \quad X_2^{(0)} = x^2\partial_x + a(x)\partial_u$$

by a change of coordinates. The commutation relations for $\mathfrak{g}_3$ are

$$[X_0, X_1] = X_0, \quad [X_0, X_2] = 2X_1, \quad [X_1, X_2] = X_2.$$

The equation $[X_0^{(0)}, X_1^{(0)}] = X_0^{(0)}$ was used get $X_0^{(0)}$ and $X_1^{(0)}$ to their current forms. The second commutation relation, gives us the equation

$$2x\partial_x + A\partial_u = 2X_1^{(0)} = [X_0^{(0)}, X_2^{(0)}] = 2x\partial_x + a'(x)\partial_u.$$

This implies that $a'(x) = 2A$, and therefore that $a(x) = 2xA + B$. From the last commutation relation we get the following equation:

$$\begin{aligned} x^2\partial_x + (2xA + B)\partial_u &= X_2^{(0)} \\ &= [X_1^{(0)}, X_2^{(0)}] \\ &= x^2\partial_x + x\partial_x(a)\partial_u \\ &= x^2\partial_x + 2Ax\partial_u \end{aligned}$$

Hence $B = 0$, and a constant lift of $\mathfrak{g}_2$ is generated by

$$\begin{aligned} X_0^{(0)} &= \partial_x, \\ X_1^{(0)} &= x\partial_x + A\partial_u, \\ X_2^{(0)} &= x^2\partial_x + 2Ax\partial_u. \end{aligned}$$

Also here, we can normalize the constant so that $A = 0$ or $A = 1$ by a coordinate transformation. If $A = 0$, then u is an invariant. If $A = 1$, there are no invariants on $\mathbb{C} \times \mathbb{C}$.

3.3.2 Differential invariants of first and second order

There are no differential invariants of strict order one. If $A = 0$ there are no differential invariants of strict order two. If $A = 1$, there is one differential invariant of second order:

$$I_2 = \left(u_{xx} + u_x^2/2\right) e^{2u}$$

3.3.3 Differential invariants of higher order

A generic orbit of $\mathfrak{g}_2^{(2)}$ is three-dimensional. This is also true for $\mathfrak{g}_2^{(k)}$ for $k > 2$. Therefore there are $\dim J^k(\mathbb{C} \times \mathbb{C}) - 3 = k - 1$ functionally independent differential invariants of order k for $k \geq 2$. This in turn implies that there is maximally one strictly independent differential invariant of strict order k for $k \geq 3$.

In the next section we find an invariant derivation that, together with the differential invariant we have found, generates all differential invariants for the cases $A = 0$ and $A = 1$, respectively.

3.3.4 Invariant derivations

The vector field $\nabla = \alpha\, \mathcal{D}_x$ is an invariant derivation if it commutes with $X_0^{(\infty)}$, $X_1^{(\infty)}$ and $X_2^{(\infty)}$:

$$\begin{aligned}
[\nabla, X_0^{(\infty)}] &= [\alpha\, \mathcal{D}_x, \partial_x] = -\alpha_x\, \mathcal{D}_x = 0 \\
[\nabla, X_1^{(\infty)}] &= [\alpha\, \mathcal{D}_x, x\, \mathcal{D}_x + \ni_{A - xu_x}] = \alpha\, \mathcal{D}_x - X_1^{(\infty)}(\alpha)\, \mathcal{D}_x = 0 \\
[\nabla, X_2^{(\infty)}] &= [\alpha\, \mathcal{D}_x, x^2\, \mathcal{D}_x + \ni_{2Ax - x^2 u_x}] = 2x\alpha\, \mathcal{D}_x - X_2^{(\infty)}(\alpha)\, \mathcal{D}_x = 0
\end{aligned}$$

The first equation tells us that α does not depend on x. Let's assume that $A = 1$ and try with $\alpha = \alpha(u)$. Then the system is equivalent to the equation $\alpha_u = \alpha$, which has the solution

$$\alpha = e^u.$$

Hence we get an invariant derivation

$$\nabla = e^u\, \mathcal{D}_x\, .$$

If $A = 0$, we let $\alpha = \alpha(u, u_x, u_{xx}, u_{xxx})$ and get the equations

$$\begin{cases} -u_x \alpha_{u_x} - 2u_{xx}\alpha_{u_{xx}} - 3u_{xxx}\alpha_{u_{xxx}} = \alpha \\ -2xu_x\alpha_{u_x} - (2u_x + 4xu_{xx})\alpha_{u_{xx}} - 6(u_{xx} + xu_{xxx})\alpha_{u_{xxx}} = 2x\alpha \end{cases}.$$

It's easily checked that the function

$$\alpha = \frac{2u_{xxx}u_x - 3u_{xx}^2}{u_x^5}$$

is a solution to this system, and thus we get an invariant derivation

$$\hat{\nabla} = \frac{2u_{xxx}u_x - 3u_{xx}^2}{u_x^5}\,\mathcal{D}_x\,.$$

All differential invariants are generated by ∇ and I_2 when $A \neq 0$ and by $\hat{\nabla}$ and u when $A = 0$.

3.4 Summary

The constant lifts of $\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3$ are of the form

$$\begin{aligned}
\mathfrak{g}_1^{(0)} &= \langle \partial_x \rangle \\
\mathfrak{g}_2^{(0)} &= \langle \partial_x, x\partial_x + C\partial_u \rangle \\
\mathfrak{g}_3^{(0)} &= \langle \partial_x, x\partial_x + C\partial_u, x^2\partial_x + 2Cx\partial_u \rangle
\end{aligned}$$

after a suitable change of coordinates of the form $(x, u) \mapsto (x, u - U(x))$. As a corollary we get the following cohomology groups:

$$H^1(\mathfrak{g}_1, C^\omega(\mathbb{C})) = \{0\}, \qquad H^1(\mathfrak{g}_2, C^\omega(\mathbb{C})) = \mathbb{C}, \qquad H^1(\mathfrak{g}_3, C^\omega(\mathbb{C})) = \mathbb{C}$$

By a scaling of u, we can normalize the constant so that $C = 0$ or $C = 1$. The differential invariants of the lifts are generated by the following differential invariants and invariant derivations.

	Differential invariants	Invariant derivation
$\mathfrak{g}_1$	u	$\mathcal{D}_x$
$\mathfrak{g}_2$, $C = 0$	u	$\frac{u_{xx}}{u_x^3}\,\mathcal{D}_x$
$\mathfrak{g}_2$, $C = 1$	$u_x e^u$	$e^u\,\mathcal{D}_x$
$\mathfrak{g}_3$, $C = 0$	u	$\frac{2u_{xxx}u_x - 3u_{xx}^2}{u_x^5}\,\mathcal{D}_x$
$\mathfrak{g}_3$, $C = 1$	$(u_{xx} + u_x^2/2)e^{2u}$	$e^u\,\mathcal{D}_x$

Remark 2. *In the cases where $C = 0$, we could have chosen the simpler invariant derivation $\nabla = \frac{1}{u_x}\mathcal{D}_x$. This is the Tresse derivative with respect to u, which means that $\nabla(u) = 0$. Because of this we would need one more differential invariant (of higher order) to generate all differential invariants.*

Chapter 4

Differential invariants of lifts of Lie algebras in $\mathfrak{D}(\mathbb{C}^2)$

In [Olv96] there is a complete list of differential invariants of the Lie algebras of vector fields from the classification (Olver uses a slightly different classification than we use here) taken as vector fields on $\mathbb{C} \times \mathbb{C}$. In this case x is an independent variable, and y is a dependent variable. In [Nes06] the same is done for a classification of vector fields on $\mathbb{R}^2$.

In this chapter we'll do the same for the classification of Lie algebras of vector fields on $\mathbb{C}^2$ as we did in the previous chapter for the classification of Lie algebras of vector fields on $\mathbb{C}$. We will first find all constant lifts of the Lie algebras to $\mathbb{C}^2 \times \mathbb{C}$, and then find the differential invariants of these lifts.

4.1 Lifts to $\mathfrak{D}(\mathbb{C}^2 \times \mathbb{C})$

The computations of the lifts consists of two parts. First we change coordinates, so that the lifts of one or two of the generators get a simpler form. Then we solve the differential equations given by the commutation relations.

4.1.1 Coordinate change

It was described in 2.3.2 how one can change coordinates $u \mapsto u - U(x, y)$ so that the lift of $\langle \partial_x, \partial_y \rangle$ is the same as the trivial lift. This means that when we consider the lifts of Lie algebras that contain $X = \partial_x$ and $Y = \partial_y$, we can change coordinates so that $X^{(0)} = \partial_x$ and $Y^{(0)} = \partial_y$. There are four other cases where we use other coordinate changes, namely for $\mathfrak{g}_4, \mathfrak{g}_5, \mathfrak{g}_{12}, \mathfrak{g}_{16}$.

The cases $\mathfrak{g}_4$ and $\mathfrak{g}_5$ are handled similarly. We may assume without loss of generality that $m_1 \geq m_2 \geq \cdots \geq m_s$. Let $X = \partial_x, Y = e^{\alpha_1 x}\partial_y$. As before, we can rectify the lift of X by using a suitable coordinate transformation, so that $X^{(0)} = \partial_x$. The general lift of Y is of the form $e^{\alpha_1 x}\partial_y + b(x,y)\partial_u$. The commutation relation $[X^{(0)}, Y^{(0)}] = \alpha_1 Y^{(0)}$ tells us that $b(x,y) = c(y)e^{\alpha_1 x}$. By changing coordinates $u \mapsto u - \int c(y)dy$, we get $Y^{(0)} = e^{\alpha_1 x}\partial_y$.

Consider now $\mathfrak{g}_{12}$. Let $X = \partial_x, Y = x\partial_x + \partial_y$. After a change of coordinates we have $X^{(0)} = \partial_x, Y^{(0)} = x\partial_x + \partial_y + b(x,y)\partial_u$. The commutation relation $[X^{(0)}, Y^{(0)}] = X^{(0)}$ tells us that b does not depend on x. After changing coordinates $u \mapsto u - \int b(y)dy$, we get $Y^{(0)} = x\partial_x + \partial_y$.

Lastly, consider $\mathfrak{g}_{16}$. Let $X = \partial_x, Y = x\partial_x + y\partial_y$. After a change of coordinates we get $X^{(0)} = \partial_x, Y^{(0)} = x\partial_x + y\partial_y + b(x,y)\partial_u$. The commutation relation $[X^{(0)}, Y^{(0)}] = X^{(0)}$ tells us that b does not depend on x. Write $b(y) = B + y\tilde{b}(y)$ where $\tilde{b}$ is an analytic function. The coordinate transformation $u \mapsto u - \int \tilde{b}(y)dy$ transforms the lift of Y to the form $Y^{(0)} = x\partial_x + y\partial_y + B\partial_u$.

4.1.2 Solving the differential equations

It was described in 2.3 how finding the general lift of a Lie algebra of vector fields on $\mathbb{C}^2$ corresponds to solving a set of differential equations. We also saw how this worked in the previous chapter. Here we will only look closely at one case, $\mathfrak{g}_8$.

Let $X_0 = \partial_x$, $X_1 = \partial_y$, $X_2 = x\partial_x + \lambda y\partial_y$ and $Y_i = x^i\partial_y$. We have the following commutation relations:

$$
\begin{gathered}
[X_0, X_1] = 0, \quad [X_0, X_2] = X_0, \quad [X_1, X_2] = \lambda X_1 \\
[X_0, Y_1] = X_1, \quad [X_0, Y_i] = iY_{i-1}, \quad i = 2, 3, ..., r-3, \\
[X_2, Y_i] = (i-\lambda)Y_i, \quad i = 1, 2, ..., r-3.
\end{gathered}
$$

Every other Lie bracket vanishes.

After we straighten out $X_0^{(0)}$ and $X_1^{(0)}$, the lifts of the generators are of the following forms:

$$
\begin{aligned}
X_0^{(0)} &= \partial_x \\
X_1^{(0)} &= \partial_y \\
X_2^{(0)} &= x\partial_x + \lambda y\partial_y + a_2(x,y)\partial_u \\
Y_i^{(0)} &= x^i\partial_y + b_i(x,y)\partial_u.
\end{aligned}
$$

The commutation relations

$$
\begin{aligned}
&[X_0^{(0)}, X_2^{(0)}] = X_0^{(0)}, \\
&[X_1^{(0)}, X_2^{(0)}] = \lambda X_1^{(0)}, \\
&[X_0^{(0)}, Y_1^{(0)}] = X_1^{(0)}, \\
&[X_0^{(0)}, Y_i^{(0)}] = iY_{i-1}, \quad i = 2, 3, ..., r-3, \\
&[X_1^{(0)}, Y_i^{(0)}] = 0, \quad i = 1, 2, ..., r-3, \\
&[X_2^{(0)}, Y_i^{(0)}] = (i-\lambda)Y_i^{(0)}
\end{aligned}
$$

are equivalent to the following set of differential equations:

$$
\begin{aligned}
\partial_x(a_2) &= 0 \\
\partial_y(a_2) &= 0 \\
\partial_x(b_1) &= 0 \\
\partial_x(b_i) &= ib_{i-1} \\
\partial_y(b_i) &= 0 \\
x\partial_x(b_i) + \lambda y \partial_y(b_i) - x^i \partial_y(a_2) &= (i-\lambda)b_i
\end{aligned}
$$

The first two equations tells us that

$$a_2 = A$$

is constant. By combining the equations, the last one simplifies to $xib_{i-1} = (i-\lambda)b_i$. We must consider two cases.

- If $\lambda = k \in \{1, 2, ..., r-3\}$, then $xkb_{k-1} = 0$ if $k > 1$, and by using the last equations, we see that $b_i = 0$ for $i = 1, 2, ..., k-1$. The equation $\partial_x(b_k) = ib_{k-1}$ (or $\partial_x(b_1) = 0$ if $k = 1$) together with $\partial_y(b_k) = 0$ implies that

$$b_k = B$$

  is constant. The rest of the coefficients are given by $b_{k+l} = \binom{k+l}{k} x^l B$.

- If $\lambda \notin \{1, 2, ..., r-3\}$, then we have $b_i = 0$ for $i = 1, 2, ..., r-3$. To see this, we use the equation $x\partial_x(b_i) = (i-\lambda)b_i$. For $i = 1$, this reduces to $(1-\lambda)b_1 = 0$, and since $\lambda \neq 1$, $b_1 = 0$. The equation $xib_{i-1} = (i-\lambda)b_i$ then implies that $b_i = 0$ for every i.

4.1.3 List of lifts and cohomologies

Here we give the complete list of the lifts. From these lifts, we can at once read off the cohomology groups $H^1(\mathfrak{g}_i, C^\omega(\mathbb{C}^2))$. It's not difficult to see that we cannot simplify the expressions more by using coordinate transformations of the form $u \mapsto u - U(x,y)$.

i	Generators for $\mathfrak{g}_i^{(0)}$	$H^1(\mathfrak{g}_i, C^\omega(\mathbb{C}^2))$
1	$\partial_x,\ \partial_y,\ y\partial_x,\ x\partial_y,\ x\partial_x + C\partial_u,\ y\partial_y + C\partial_u,$ $xy\partial_x + y^2\partial_y + 3Cy\partial_u,\ x^2\partial_x + xy\partial_y + 3Cx\partial_u$	$\mathbb{C}$
2	$\partial_x,\ \partial_y,\ y\partial_x,\ x\partial_y,\ x\partial_x + C\partial_u,\ y\partial_y + C\partial_u$	$\mathbb{C}$
3	$\partial_x,\ \partial_y,\ x\partial_y,\ y\partial_x,\ x\partial_x - y\partial_y$	$\{0\}$
6	$\partial_x,\ \partial_y,\ y\partial_y + C\partial_u,\ y^2\partial_y + 2Cy\partial_u$	$\mathbb{C}$
7	$\partial_x,\ \partial_y,\ x\partial_x + C\partial_u,\ x^2\partial_x + x\partial_y + 2Cx\partial_u$	$\mathbb{C}$
11	$\partial_x,\ \partial_y,\ x\partial_x + A\partial_u,\ y\partial_y + B\partial_u,\ y^2\partial_y + 2By\partial_u$	$\mathbb{C}^2$
12	$\partial_x,\ x\partial_x + \partial_y$	$\{0\}$
15	$\partial_x,\ x\partial_x + \partial_y,\ x^2\partial_x + 2x\partial_y + Ce^y\partial_u$	$\mathbb{C}$
16	$2\partial_x,\ x\partial_x + y\partial_y + A\partial_u,$ $\frac{x^2+y^2}{2}\partial_x + xy\partial_y + (Ax + By)\partial_u$	$\mathbb{C}^2$
17	$\partial_x,\ x\partial_x + A\partial_u,\ x^2\partial_x + 2Ax\partial_u,$ $\partial_y,\ y\partial_y + B\partial_u,\ y^2\partial_y + 2By\partial_u$	$\mathbb{C}^2$
18	∂_x	$\{0\}$
19	$\partial_x,\ x\partial_x + a(y)\partial_u$	$C^\omega(\mathbb{C})$
20	$\partial_x,\ x\partial_x + a(y)\partial_u,\ x^2\partial_x + 2xa(y)\partial_u$	$C^\omega(\mathbb{C})$

i	Generators for $\mathfrak{g}_i^{(0)}$	$H^1(\mathfrak{g}_i, C^\omega(\mathbb{C}^2))$
4	∂_x, $x^i e^{\alpha_j x}\partial_y + b_{j,i}(x)\partial_u$ for $i = 0, 1, ..., m_j - 1$, $j = 1, 2, ..., s$ where $b_{1,0} = 0$, $b_{1,i} = e^{\alpha_1 x}\sum_{k=1}^{i}\binom{i}{k}C_{1,k}x^{i-k}$ and $b_{j,i} = e^{\alpha_j x}\sum_{k=0}^{i}\binom{i}{k}C_{j,k}x^{i-k}$. In addition $m_i \geq m_{i+1}$.	$\mathbb{C}^{m_1+m_2+\cdots+m_s-1}$
5	∂_x, $y\partial_y + C\partial_u$, $x^i e^{\alpha_j x}\partial_y$ for $j = 1, 2, ..., s$, $i = 0, 1, ..., m_j - 1$	$\mathbb{C}$
8	∂_x, ∂_y, $x\partial_x + \lambda y\partial_y + A\partial_u$, $x^i\partial_y + b_i(x)\partial_u$, $i = 1, 2, ..., r-3$ where $\lambda \in \mathbb{C}$, $\lambda \neq r-2$. If $\lambda = k \in \{1, 2, ..., r-3\}$, then $b_i = 0$ for $i = 1, 2, ..., k-1$, $b_k = B \in \mathbb{C}$ and $b_{k+l} = \binom{k+l}{k}x^l B$ for $l = 1, 2, ..., r-3-k$. If else $b_i = 0$ for $i = 1, 2, ..., r-3$.	$\mathbb{C}^2$
9	∂_x, ∂_y, $x\partial_x + \left((r-2)y + x^{r-2}\right)\partial_y + C\partial_u$, $x^i\partial_y$, $i = 1, 2, ..., r-3$	$\mathbb{C}$
10	∂_x, ∂_y, $x\partial_x + A\partial_u$, $y\partial_y + B\partial_u$, $x^i\partial_y$, $i = 1, 2, ..., r-4$	$\mathbb{C}^2$
13	∂_x, ∂_y, $x^i\partial_y$, $i = 1, 2, ..., r-4$, $x\partial_x + \frac{r-4}{2}y\partial_y + C\partial_u$, $x^2\partial_x + (r-4)xy\partial_x + 2xC\partial_u$	$\mathbb{C}$
14	∂_x, ∂_y, $x\partial_x + A\partial_u$, $y\partial_y + B\partial_u$, $x^2\partial_x + (r-5)xy\partial_y + (2A + (r-5)B)x\partial_u$, $x^i\partial_y$, $i = 1, 2, ..., r-5$	$\mathbb{C}^2$
21	∂_y, $\phi_i(x)\partial_y + a_i(x)\partial_u$, $i = 2, 3, ..., r$	$C^\omega(\mathbb{C}, \mathbb{C}^{r-2})$
22	∂_y, $y\partial_y + b(x)\partial_u$, $\phi_i(x)\partial_y$, $i = 3, 4, ..., r$	$C^\omega(\mathbb{C})$

Most of the lifts depend on one or two constants. If they depend on one constant, we can always normalize it by a change of coordinates.[1] If $C \neq 0$, we can use the transformation $u \mapsto u/C$ to remove the constant. This means that we can always make C equal to either 0 or 1 by change of coordinates. For the lifts that depend on two constant we can always, for the same reason, make one of them equal to 1.

[1]Note that this transformation has nothing to do with the cohomology groups. For the cohomology we only considered transformations on the form $u \mapsto u - U(x, y)$.

4.2 Counting differential invariants

Given an algebra $\mathfrak{g} \subset \mathfrak{D}(\mathbb{C}^2 \times \mathbb{C})$ we can find the number of strictly independent differential invariants of order k in the neighborhood of a generic point for any k. The dimension of $J^k(\pi)$ is $2 + \binom{k+2}{2}$. Let

$$q_k = \dim J^k(\pi) - \dim J^{k-1}(\pi) = \binom{k+2}{2} - \binom{k+1}{2} = \binom{k+1}{1} = k + 1.$$

This is the number of derivative coordinates of order exactly k.

Let s_k denote the dimension of a generic orbit of $\mathfrak{g}^{(k)}$. Then the number of functionally independent differential invariants of order k is $i_k = 2+\binom{k+2}{2} - s_k$. The number of strictly independent differential invariants of order k is $j_k = i_k - i_{k-1} = q_k - s_k + s_{k-1}$. Hence we only need to calculate s_k and s_{k-1} to count the number of strictly independent differential invariants of order k.

This calculation can be automated in Maple.

	Trivial lift				
Order	0	1	2	3	4
$\mathfrak{g}_3$	1	0	2	4	5
$\mathfrak{g}_{12}$	1	2	3	4	5
$\mathfrak{g}_{18}$	2	2	3	4	5

	$C = 0$					$C = 1$				
Order	0	1	2	3	4	0	1	2	3	4
$\mathfrak{g}_1$	1	0	0	3	5	0	0	1	3	5
$\mathfrak{g}_2$	1	0	1	4	5	0	0	2	4	5
$\mathfrak{g}_6$	1	1	2	4	5	0	1	3	4	5
$\mathfrak{g}_7$	1	1	2	4	5	0	1	3	4	5
$\mathfrak{g}_{15}$	1	1	3	4	5	0	2	3	4	5

	$A=0,B=0$					$A=1,B=0$					$A\in\mathbb{C},B=1$				
Order	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4
$\mathfrak{g}_{11}$	1	0	2	4	5	0	1	2	4	5	0	0	3	4	5
$\mathfrak{g}_{16}$	1	1	3	4	5	1	1	3	4	5	0	2	3	4	5
$\mathfrak{g}_{17}$	1	0	1	4	5	0	0	2	4	5	0	0	2	4	5

	Trivial lifts					$a(y)\neq 0$				
Order	0	1	2	3	4	0	1	2	3	4
$\mathfrak{g}_{19}$	2	1	3	4	5	1	2	3	4	5
$\mathfrak{g}_{20}$	2	1	2	4	5	1	1	3	4	5

For the families of Lie algebras, the number of differential invariants depends on the dimension of the algebra. The bold numbers are numbers that does not change when we increase the algebra dimension.

$\mathfrak{g}_4$	$C_{i,j}=0$
Order	0 1 2 3 4
$r=2$	**1** 2 3 4 5
$r=3$	**1 1** 3 4 5
$r=4$	**1 1 2** 4 5
$r=5$	**1 1 2 3** 5
$r=6$	**1 1 2 3 4**

The lift of $\mathfrak{g}_4$ is much more complicated than the other lifts, so it's difficult to find a pattern like we do for most of the other lifts. But we can count the number of differential invariants for some cases, and we see that this number depends on the constants.

$\mathfrak{g}_4$	$C_{1,1} \neq 0$	$C_{2,0} \neq 0$	$C_{1,1} \neq 0$ and $C_{2,0} \neq 0$
Order	0 1 2 3 4	0 1 2 3 4	0 1 2 3 4
$m_1 = 1, m_2 = 1$		0 2 3 4 5	
$m_1 = 2$	0 2 3 4 5		
$m_1 = 2, m_2 = 1$	0 1 3 4 5	0 1 3 4 5	0 1 3 4 5
$m_1 = 2, m_2 = 2$	0 1 2 4 5	0 1 2 3 4	0 1 2 4 5
$m_1 = 2, m_2 = 2, m_3 = 1$	0 1 2 3 5	0 1 1 4 5	0 1 1 4 5

$\mathfrak{g}_5$	Trivial lift	Nontrivial lift
Order	0 1 2 3 4	0 1 2 3 4
$r = 2$	**1** 2 3 4 5	1 2 3 4 5
$r = 3$	**1** 1 3 4 5	**0** 2 3 4 5
$r = 4$	**1 0** 3 4 5	**0 1** 3 4 5
$r = 5$	**1 0 2** 4 5	**0 1 2** 4 5
$r = 6$	**1 0 2 3** 5	**0 1 2 3** 5
$r = 7$	**1 0 2 3 4**	**0 1 2 3 4**

$\mathfrak{g}_8$	Trivial, $\lambda = 0$	Trivial, $\lambda \neq 0$	$A = 1$, $b_i = 0$
Order	0 1 2 3 4	0 1 2 3 4	0 1 2 3 4
$r = 3$	**1 1** 3 4 5	**1** 1 3 4 5	**0** 2 3 4 5
$r = 4$	**1 1** 2 4 5	**1 0** 3 4 5	**0 1** 3 4 5
$r = 5$	**1 1 1** 4 5	**1 0 2** 4 5	**0 1 2** 4 5
$r = 6$	**1 1 1 3** 5	**1 0 2 3** 5	**0 1 2 3** 5
$r = 7$	**1 1 1 3 4**	**1 0 2 3 4**	**0 1 2 3 4**

$\mathfrak{g}_8$	$\lambda = 1$, $b_1 \neq 0$	$\lambda = 2$, $b_2 \neq 0$	$\lambda = 3$, $b_3 \neq 0$	$\lambda = 4$, $b_4 \neq 0$
Order	0 1 2 3 4	0 1 2 3 4	0 1 2 3 4	0 1 2 3 4
$r = 4$	**0** 1 3 4 5			
$r = 5$	**0 0** 3 4 5	**0 0** 3 4 5		
$r = 6$	**0 0 2** 4 5	**0 0** 2 4 5	**0 0** 2 4 5	
$r = 7$	**0 0 2 3** 5	**0 0 1** 4 5	**0 0 1** 4 5	**0 0** 2 3 5
$r = 8$	**0 0 2 3 4**	**0 0 1 3** 5	**0 0 1** 3 5	**0 0 1** 3 5
$r = 9$		**0 0 1 3 4**	**0 0 1 2** 5	**0 0 1 2** 5
$r = 10$			**0 0 1 2 4**	**0 0 1 2** 4
$r = 11$				**0 0 1 2 3**

$\mathfrak{g}_9$	Trivial lift	Nontrivial lift
Order	0 1 2 3 4	0 1 2 3 4
$r = 3$	**1** 1 3 4 5	**0** 2 3 4 5
$r = 4$	**1 0** 3 4 5	**0 1** 3 4 5
$r = 5$	**1 0 2** 4 5	**0 1 2** 4 5
$r = 6$	**1 0 2 3** 5	**0 1 2 3** 5
$r = 7$	**1 1 2 3 4**	**0 1 2 3 4**

$\mathfrak{g}_{10}$	Trivial	$A = 0$, $B = 1$	$A = 1$, $B \in \mathbb{C}$
Order	0 1 2 3 4	0 1 2 3 4	0 1 2 3 4
$r = 4$	**1 0** 3 4 5	**0 1** 3 4 5	**0** 1 3 4 5
$r = 5$	**1 0** 2 4 5	**0 1** 2 4 5	**0 0** 3 4 5
$r = 6$	**1 0 1** 4 5	**0 1 1** 4 5	**0 0 2** 4 5
$r = 7$	**1 0 1 3** 5	**0 1 1 3** 5	**0 0 2 3** 5
$r = 8$	**1 0 1 3 4**	**0 1 1 3 4**	**0 0 2 3 4**

$\mathfrak{g}_{13}$	Trivial lift	Nontrivial lift
Order	0 1 2 3 4	0 1 2 3 4
$r=4$	**1** 1 2 4 5	**0 1** 3 4 5
$r=5$	**1 0** 2 4 5	**0 1** 2 4 5
$r=6$	**1 0 1** 4 5	**0 1 1** 4 5
$r=7$	**1 0 1 3** 5	**0 1 1 3** 5
$r=8$	**1 1 2 3 4**	**0 1 1 3 4**

$\mathfrak{g}_{14}$	Trivial	$A=0$, $B=1$	$A=1$, $B\in\mathbb{C}$
Order	0 1 2 3 4	0 1 2 3 4	0 1 2 3 4
$r=5$	**1 0** 2 4 5	**0 1** 2 4 5	**0 0** 3 4 5
$r=6$	**1 0 1** 4 5	**0 1 1** 4 5	**0 0** 2 4 5
$r=7$	**1 0 1 3** 5	**0 1 1** 3 5	**0 0 1** 4 5
$r=8$	**1 0 1 3 4**	**0 1 1 2** 5	**0 0 1 3** 5
$r=9$		**0 1 1 2 4**	**0 0 1 3 4**

$\mathfrak{g}_{21}$	Trivial	$a_2\neq 0$	$a_2\neq 0$ and $a_3\neq 0$
Order	0 1 2 3 4	0 1 2 3 4	0 1 2 3 4
$r=2$	**2 1** 3 4 5	**1** 2 3 4 5	
$r=3$	**2 1 2** 4 5	**1 1** 3 4 5	**1 1** 3 4 5
$r=4$	**2 1 2 3** 5	**1 1 2** 4 5	**1 1** 2 4 5
$r=5$	**2 1 2 3 4**	**1 1 2 3** 5	**1 1 1** 4 5
$r=6$		**1 1 2 3 4**	**1 1 1 3** 5
$r=7$			**1 1 1 3 4**

$\mathfrak{g}_{22}$	Trivial lift	Nontrivial lift
Order	0 1 2 3 4	0 1 2 3 4
$r = 3$	**2 0** 3 4 5	**1 1** 3 4 5
$r = 4$	**2 0 2** 4 5	**1 1 2** 4 5
$r = 5$	**2 0 2 3** 5	**1 1 2 3** 5
$r = 6$	**2 0 2 3 4**	**1 1 2 3 4**

We have only listed the number of the invariants of order up to 4. When we compute the differential invariants, it will usually be clear how the pattern continues.

When we prolong a Lie algebra of vector fields, the dimension of its generic orbits will, in general, increase. The dimension of an orbit is bounded by the dimension of the Lie algebra. When the Lie algebra is of finite dimension, the orbit dimension must stabilize at some point. The following theorem, due to Ovsiannikov, tells us that for Lie algebras of vector fields, the orbit dimension stabilizes when it reaches the dimension of the Lie algebra (see [Olv96, p. 143]).

Theorem 5. *A Lie group of point or contact transformations acts locally effectively if and only if its stable orbit dimension equals its dimension.*

Remember that for each Lie algebra of vector fields, there exists an effective Lie group whose local action coincide with the Lie algebra.

If a generic orbit of $\mathfrak{g}^{(k)}$ has dimension equal to the dimension of $\mathfrak{g}$, then the pattern of differential invariants of strict order greater than k, is very simple. We will get the maximal number of differential invariants of strict order l for $l > k$. In other words, there will be $l + 1$ differential invariants of strict order l for $l > k$.

4.3 List of differential invariants

In this section, we give a list of differential invariants and invariant derivations. In many cases these generate all differential invariants for the given Lie algebra of vector fields. For the families of Lie algebras from the classification, these generating differential invariants can be of arbitrarily high order, and even though there seems to be some pattern, we have not been able to express these in general. Therefore, the list is complete only for the Lie algebras of low order.

An interesting property of the Lie algebras we consider is that all of them have differential invariants of order two. And in most cases, even in cases where we have not found all differential invariants, we know the order of the missing generating differential invariants.

It turns out that the following functions appear in many of the differential invariants we find:

$$J_{ki} = \sum_{j=0}^{i-1} (-1)^j \binom{i-1}{j} \frac{u_{x^j y^{k-j}}}{u_x^j u_y^{k-j}}$$

Therefore it will be useful to express many of the differential invariants in terms of them.

Many of the differential invariants and invariant derivations are, due to their lengthy expressions, gathered in the appendix. They appear in the tables in this sections as letters $I, K, L, M, \alpha, \beta$ with indices. All calculations were done using Maple with the packages "DifferentialGeometry" and "JetCalculus".

4.3.1 $\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3$

For $\mathfrak{g}_1^{(0)}$ with $C = 1$ we're only able to find a differential invariant of order two. We know that micro-locally there are three differential invariants of order three and $k+1$ differential invariants of order k for $k > 3$.

	Differential invariants	Invariant derivations
$\mathfrak{g}_1^{(0)}$ $C=0$	$u, I_{31}, I_{32}, I_{33}, I_{41}$	$\alpha_1\left(\mathcal{D}_x-\frac{u_x}{u_y}\,\mathcal{D}_y\right)+\frac{1}{u_y}\,\mathcal{D}_y$, $\alpha_2\left(\mathcal{D}_x-\frac{u_x}{u_y}\,\mathcal{D}_y\right)$
$\mathfrak{g}_1^{(0)}$ $C=1$	$(J_{23}+3J_{21}J_{23}-3J_{22}^2)(u_xu_ye^u)^2$	
$\mathfrak{g}_2^{(0)}$ $C=0$	$u, K_2, K_{31}, K_{32}, K_{33}, K_{34}$	$\alpha_5\left(\mathcal{D}_x-\frac{u_x}{u_y}\,\mathcal{D}_y\right)$, $\alpha_6\left(\mathcal{D}_x-\frac{u_x}{u_y}\,\mathcal{D}_y\right)+\frac{1}{u_y}\,\mathcal{D}_y$
$\mathfrak{g}_2^{(0)}$ $C=1$	$J_{23}\left(u_xu_ye^u\right)^2$, $(J_{23}J_{21}-J_{22}^2)\left(u_xu_ye^u\right)^2$, $K_{31}, K_{32}, K_{33}, K_{34}$	$\alpha_7\,\mathcal{D}_x+\beta_7\,\mathcal{D}_y,\ \alpha_8\,\mathcal{D}_x+\beta_8\,\mathcal{D}_y$
$\mathfrak{g}_3^{(0)}$	$u, J_{23}u_x^2u_y^2, K_2, K_{32}, K_{33}$	$\frac{J_{33}+2J_{21}J_{23}-2J_{22}^2}{J_{34}u_x}\left(\mathcal{D}_x-\frac{u_x}{u_y}\,\mathcal{D}_y\right)+\frac{1}{u_y}\,\mathcal{D}_y,\ \frac{1}{J_{34}u_x^3u_y^2}\left(\mathcal{D}_x-\frac{u_x}{u_y}\,\mathcal{D}_y\right)$

4.3.2 $\mathfrak{g}_6, \mathfrak{g}_7, \mathfrak{g}_{11}, \mathfrak{g}_{12}, \mathfrak{g}_{15}, \mathfrak{g}_{16}, \mathfrak{g}_{17}, \mathfrak{g}_{18}, \mathfrak{g}_{19}, \mathfrak{g}_{20}$

	Differential invariants	Invariant Derivations
$\mathfrak{g}_6^{(0)}$ $C=0$	u, $\frac{u_{yyy}}{u_y^3} - \frac{3}{2}\left(\frac{u_{yy}}{u_y^2}\right)^2$	$\mathcal{D}_x$, $\frac{1}{u_y}\,\mathcal{D}_y$
$\mathfrak{g}_6^{(0)}$ $C=1$	$u_x, (u_{yy}+u_y^2/2)e^{2u}$	$\mathcal{D}_x, e^u\,\mathcal{D}_y$
$\mathfrak{g}_7^{(0)}$ $C=0$	u, $\frac{J_{23}-1/u_y}{J_{22}^2}$	$\frac{u_y}{u_x u_{yy}-u_y u_{xy}}\left(\mathcal{D}_x - \frac{u_x}{u_y}\,\mathcal{D}_y\right)$, $\mathcal{D}_y$
$\mathfrak{g}_7^{(0)}$ $C=1$	u_y, $\left(J_{23}+\frac{4J_{22}-1}{u_y-2}+\frac{4J_{21}}{(u_y-2)^2}\right)(u_x e^u)^2$	$e^u\left(\mathcal{D}_x - \frac{u_x}{u_y-2}\,\mathcal{D}_y\right)$, $\mathcal{D}_y$
$\mathfrak{g}_{11}^{(0)}$ $A=0$ $B=0$	u, $\frac{u_{xx}}{u_x^2}$, $\frac{u_{xy}}{u_x u_y}$, $\frac{u_{yyy}}{u_y^3} - \frac{3}{2}\left(\frac{u_{yy}}{u_y^2}\right)^2$	$\frac{1}{u_x}\,\mathcal{D}_x$, $\frac{1}{u_y}\,\mathcal{D}_y$
$\mathfrak{g}_{11}^{(0)}$ $B=0$ $A=1$	$u_x e^u$, $J_{31} - \frac{3}{2}J_{21}^2$	$\frac{1}{u_x}\,\mathcal{D}_x$, $\frac{1}{u_y}\,\mathcal{D}_y$
$\mathfrak{g}_{11}^{(0)}$ $B=1$ $A\in\mathbb{C}$	$\frac{u_{xx}}{u_x^2}$, $u_{xy}u_x^{A-1}e^u$, $(u_{yy}-u_y^2/2)u_x^{2A}e^{2u}$	$\frac{1}{u_x}\,\mathcal{D}_x, u_x^A e^u\,\mathcal{D}_y$
$\mathfrak{g}_{12}^{(0)}$	u	$e^y\,\mathcal{D}_x$, $\mathcal{D}_y$
$\mathfrak{g}_{15}^{(0)}$ $C=0$	u, $\left(J_{23}+\frac{1}{2u_y}\right)(u_x e^y)^2$	$e^y\left(\mathcal{D}_x - \frac{u_x}{u_y}\,\mathcal{D}_y\right)$, $\mathcal{D}_y$
$\mathfrak{g}_{15}^{(0)}$ $C=1$	$u_y - u$, $2uu_y + u_x e^y - u^2$	$e^y\,\mathcal{D}_x + 2u\,\mathcal{D}_y$, $\mathcal{D}_y$

	Differential invariants	Invariant derivations
$\mathfrak{g}_{16}^{(0)}$ $A=0$ $B=0$	$u, y^2(u_y^2-u_x^2), y^2(u_{yy}-u_{xx})$	$\frac{1}{u_x+u_y}\left(\mathcal{D}_x+\mathcal{D}_y\right)$, $\frac{1}{u_x-u_y}\left(\mathcal{D}_x-\mathcal{D}_y\right)$
$\mathfrak{g}_{16}^{(0)}$ $B=0$ $A=1$	$e^u/y,\ y^2(u_{yy}-u_{xx})$	$(1+y(u_x-u_y))y(\mathcal{D}_x+\mathcal{D}_y)$, $\frac{y}{1+y(u_x-u_y)}(\mathcal{D}_x-\mathcal{D}_y)$
$\mathfrak{g}_{16}^{(0)}$ $B=1$ $A\in\mathbb{C}$	$\left((u_x+u_y)-\frac{1+A}{y}\right)y^{1-A}e^u$, $\left(u_x-u_y-\frac{1-A}{y}\right)y^{A+1}e^{-u}$	$y^{A+1}e^{-u}\left(\mathcal{D}_x-\mathcal{D}_y\right)$, $y^{1-A}e^u\left(\mathcal{D}_x+\mathcal{D}_y\right)$
$\mathfrak{g}_{17}^{(0)}$ $A=0$ $B=0$	$u, \frac{u_{xy}}{u_xu_y}, \frac{u_{xxx}}{u_x^3}-\frac{3}{2}\left(\frac{u_{xx}}{u_x^2}\right)^2$, $\frac{u_{yyy}}{u_y^3}-\frac{3}{2}\left(\frac{u_{yy}}{u_y^2}\right)^2$	$\frac{1}{u_x}\mathcal{D}_x,\ \frac{1}{u_y}\mathcal{D}_y$
$\mathfrak{g}_{17}^{(0)}$ $B=0$ $A=1$	$(u_x^2+2u_{xx})e^{2u}, \frac{u_{xy}}{u_y}e^u$, $J_{31}-\frac{3}{2}J_{21}^2$	$e^u\mathcal{D}_x,\ \frac{1}{u_y}\mathcal{D}_y$
$\mathfrak{g}_{17}^{(0)}$ $B=1$ $A\in\mathbb{C}$	$u_{xy}(u_x^2+2Au_{xx})^{(A-1)/2}e^u$, $(u_y^2+2u_{yy})(u_x^2+2Au_{xx})^Ae^u$	$\frac{1}{\sqrt{u_x^2+2Au_{xx}}}\mathcal{D}_x,\ \frac{\sqrt{u_x^2+2Au_{xx}}}{u_{xy}}\mathcal{D}_y$
$\mathfrak{g}_{18}^{(0)}$	y, u	$\mathcal{D}_x,\ \mathcal{D}_y$
$\mathfrak{g}_{19}^{(0)}$ $a=0$	$y, u, \frac{u_{xx}}{u_x^2}$	$\frac{1}{u_x}\mathcal{D}_x,\ \mathcal{D}_y$
$\mathfrak{g}_{19}^{(0)}$	$y,\ u_xe^{\frac{u}{a(y)}},\ u_y-u\frac{a'(y)}{a(y)}$	$\frac{1}{u_x}\mathcal{D}_x, \mathcal{D}_y$
$\mathfrak{g}_{20}^{(0)}$ $a=0$	$y, u, \frac{u_{xxx}}{u_x^3}-\frac{3}{2}\left(\frac{u_{xx}}{u_x^2}\right)^2$	$\frac{1}{u_x}\mathcal{D}_x,\ \mathcal{D}_y$
$\mathfrak{g}_{20}^{(0)}$	$y,\ u_y-u\frac{a'(y)}{a(y)}$, $\left(\frac{u_x^2}{a(y)}+2u_{xx}\right)e^{\frac{2u}{a(y)}}$	$e^{\frac{u}{a(y)}}\mathcal{D}_x, \mathcal{D}_y$

4.3.3 $\mathfrak{g}_8, \mathfrak{g}_9, \mathfrak{g}_{10}, \mathfrak{g}_{13}, \mathfrak{g}_{14}$

For families of Lie algebras $(\mathfrak{g}_4^{(0)}, \mathfrak{g}_5^{(0)}, \mathfrak{g}_8^{(0)}, \mathfrak{g}_9^{(0)}, \mathfrak{g}_{10}^{(0)}, \mathfrak{g}_{13}^{(0)}, \mathfrak{g}_{14}^{(0)}, \mathfrak{g}_{21}^{(0)}, \mathfrak{g}_{22}^{(0)})$, there are always some differential invariants and invariant derivations, that every Lie algebra in the family have in common. Because of this, we can almost always determine the number of differential invariants of each order. The exceptions are the nontrivial lifts of $\mathfrak{g}_4$, $\mathfrak{g}_8$ and $\mathfrak{g}_{21}$.

In order to generate all differential invariants we need, in addition to the mentioned differential invariants and invariant derivations, one or two differential invariants that depend on the dimension of the Lie algebra. In the tables, r is always the dimension of the Lie algebra.

$\mathfrak{g}_8$	$r = 3, 4, ...$	$\frac{1}{u_x u_{yy} - u_y u_{xy}} \left(\mathcal{D}_x - \frac{u_{xy}}{u_{yy}} \mathcal{D}_y\right), \mathcal{D}_y$
$\lambda = 0$		u, $\frac{J_{33} - J_{21} J_{23}}{J_{22}^2}$
$A = 0$	$r = 3$	J_{22}, $\frac{J_{23}}{J_{22}^2}$
$B = 0$	$r = 4$	$\frac{J_{23}}{J_{22}^2}$
	$r = 5$	$\frac{J_{34} - 3J_{22} J_{23}}{J_{22}^3}$
	$r = 6$	$\frac{J_{45} - 6J_{33} J_{23} - 4J_{22} J_{34} + 12J_{22}^2 J_{23} + 3J_{21} J_{23}^2}{J_{22}^4}$
	$r = k$	one differential invariant of order $k - 2$

$\mathfrak{g}_8$	$r = 3, 4, ...$	$\frac{u_y^2}{u_x u_{yy} - u_y u_{xy}} \left(\mathcal{D}_x - \frac{u_x}{u_y} \mathcal{D}_y\right), \frac{1}{u_y} \mathcal{D}_y$
$\lambda \neq 0$		u, J_{21}, $J_{22} u_x u_y^{-1/\lambda}$
$A = 0$	$r = 3$	$u_x u_y^{-1/\lambda}$
$B = 0$	$r = 4$	$\frac{J_{23}}{J_{22}^2}$
	$r = 5$	$\frac{J_{34} - 3J_{22} J_{23}}{J_{22}^3}$
	$r = 6$	$\frac{J_{45} - 6J_{33} J_{23} - 4J_{22} J_{34} + 12J_{22}^2 J_{23} + 3J_{21} J_{23}^2}{J_{22}^4}$
	$r = k$	one differential invariant of order $k - 2$

$\mathfrak{g}_8$	$r = 3, 4, ...$	$e^u \left(\mathcal{D}_x - \frac{u_x}{u_y} \mathcal{D}_y\right), \frac{1}{u_y} \mathcal{D}_y$
$A = 1$		$u_y e^{\lambda u}$
$B = 0$	$r = 3$	$u_x e^u$
	$r = 4$	$J_{23} (u_x e^u)^2$
	$r = 5$	$(J_{34} - 3J_{22} J_{23}) (u_x e^u)^3$
	$r = 6$	$\left(J_{45} - 6J_{33} J_{23} - 4J_{22} J_{34} + 12J_{22}^2 J_{23} + 3J_{21} J_{23}^2\right) (u_x e^u)^4$
	$r = k$	one differential invariant of order $k - 2$

In each table, the top box on the right-hand side contains invariant derivations and differential invariants that hold for every r. In addition we must specify one, or sometimes two, additional differential invariants that depend on r. Consider for example $\mathfrak{g}_8^{(0)}$ with $\lambda = A = B = 0$. When $r = 5$, all differential invariants are generated by the two differential invariants and invariant derivations in the box labeled "$r = 3, 4, ...$", in addition to the differential invariant $\frac{J_{34} - 3J_{22}J_{23}}{J_{22}^2}$.

$\mathfrak{g}_8$	$r = 4, 5, ...$	$\frac{1}{u_y^{1/k}}\left(\mathcal{D}_x - \frac{u_{xy}}{u_{yy}}\mathcal{D}_y\right), \frac{1}{u_y}\mathcal{D}_y$
$A = 1$	$k = 1, ..., r-3$	J_{21}
	$k = 1$	$\frac{u_{yy}}{u_y^2}(\log(u_y) + u) + B\frac{u_{xy}}{u_y^2}$
	$k = 2$	$B\frac{u_{yyy}u_{xy}^2}{u_y^2 u_{yy}^2} - 2B\frac{u_{xyy}u_{xy}}{u_y^2 u_{yy}} + B\frac{u_{xxy}}{u_y^2} + \frac{u_{yy}}{u_y^2}(2u + \log(u_y))$
$b_k = B$	$k = 1, r = 4$	$\frac{u_x}{u_y}B + u + \log(u_y)$
$B \neq 0$	$r = 5$	$\frac{u_{yy}}{u_y^2}\left(\log(u_y) + u\right)^2 + 2B\frac{u_{xy}}{u_x u_y}\left(\log(u_y) + u\right)\frac{u_x}{u_y} + B^2\frac{u_{xx}}{u_x^2}\frac{u_x^2}{u_y^2} + (\log(u_2) + u) + B\frac{u_x}{u_y}$
$\lambda = k$		
	$r = 6$	$\frac{u_{yyy}}{u_y^3}(\log(u_y) + u)^3 + 3B\frac{u_{xyy}}{u_y^3}(\log(u_y) + u)^2 + 3B^2\frac{u_{xxy}}{u_y^3}(\log(u_y) + u) + B^3\frac{u_{xxx}}{u_y^3} + 4\frac{u_{yy}}{u_y^2}(\log(u_y) + u)^2 + 5B\frac{u_{xy}}{u_y^2}(\log(u_y) + u) + B^2\frac{u_{xx}}{u_y^2} + (\log(u_y) + u) + \frac{u_x}{u_y}\left(3B\frac{u_{yy}}{u_y^2}(\log(u_y) + u) + 3B^2\frac{u_{xy}}{u_y^2} + B\right)$
	$k = 2, r = 5$	$u_x u_y^{-1/2} J_{22}$, $B\frac{u_x^2}{u_y}J_{23} + 2u + \log(u_y)$
	$r = 6$	$\frac{u_{yy}}{u_y^2}(\log(u_y) + 2u) - B\frac{u_{yy}u_{xx} - u_{xy}^2}{u_y^3}$, $\frac{u_x}{u_y^{3/2}J_{21}^3}\big(BJ_{34}J_{21}^3u_x^2 - 3BJ_{33}J_{22}^2J_{21}u_x^2 - 3J_{32}J_{21}(BJ_{21}J_{23}u_x^2 - 2BJ_{22}^2 + 2J_{21}uu_y + J_{21}\log(u_y)u_y) + J_{31}J_{22}(3BJ_{21}J_{23}u_x^2 + 6J_{21}uu_y + 3J_{21}u_y\log(u_y) - 4BJ_{22}^2u_x^2) - 2J_{21}^2J_{22}u_y\big)$
	$k > 2$, $r = k + 3$	$\frac{u_x^k}{u_y}J_{22}^k$, one differential invariant of order k

For the cases when $B \neq 0$, it's more complicated because we have an extra parameter k. In addition to the one differential invariant that holds for every r and k, we usually need two more. In the tables we have added another box that contains one differential invariant which holds for every r when $k = 1$ and when $k = 2$. Then, for these two cases, we need only one more differential invariant that depends on r. If we follow the pattern, it looks like when $k = s$ we have to specify one differential invariant of order $s + 1$ in addition to one differential invariant that depends on r. We've also added a box at the bottom, in order to list all differential invariants of order two.

$\mathfrak{g}_8$	$r = 4, 5, ...$	$\frac{1}{u_y^{1/k}}\left(\mathcal{D}_x - \frac{u_{xy}}{u_{yy}}\mathcal{D}_y\right), \frac{1}{u_y}\mathcal{D}_y$
$A = 0$	$k = 1, ..., r-3$	J_{21}
	$k = 1$	$\frac{uu_{yy}+u_{xy}}{u_y^2}$
	$k = 2$	$\frac{u_{yyy}u_{xy}^2}{u_{yy}^2u_y^2} - 2\frac{u_{xyy}u_{xy}}{u_{yy}u_y^2} + \frac{u_{xxy}}{u_y^2} + 2u\frac{u_{yy}}{u_y^2}$
$b_k = B$	$k = 1, r = 4$	$\frac{u_x}{u_y} + u$
$B = 1$	$r = 5$	$\frac{u^2u_{yy}+2uu_{xy}+u_{xx}+uu_y^2+u_xu_y}{u_y^2}$
$\lambda = k$	$r = 6$	$\frac{u^3u_{yyy}+u^2(3u_{xyy}+4u_yu_{yy})}{u_y^3} + \frac{u(3u_{xxy}+3u_xu_{yy}+5u_yu_{xy}+u_y^3)+u_{xxx}+3u_xu_{xy}+u_yu_{xx}+u_xu_y^2}{u_y^3}$
	$k = 2, r = 5$	$\frac{u_x^2}{u_y}J_{22}^2,\ \frac{u_x^2}{u_y}J_{23} + 2u$
	$r = 6$	$\frac{u_{yy}(u_{xx}+2uu_y)-u_{xy}^2}{u_y^3}, \quad \frac{1}{u_{yy}^6u_y^3}\big(u_{yyy}(3u_{xy}u_{xx}u_{yy} - 4u_{xy}^3 + 6uu_yu_{xy}u_{yy}) + u_{xyy}(6u_{xy}^2u_yy - 3u_{xx}u_{yy}^2 - 6uu_yu_{yy}^2) - 3u_{xxy}u_{xy}u_{yy}^2 + u_{xxx}u_{yy}^3 + 2u_xu_yu_{yy}^3 - 2u_y^2u_{yy}^2u_{xy}\big)^2$
	$k = 3, r = 6$	$\frac{u_x^3}{u_y}J_{22}^3,\ \frac{u_x^3}{u_y}(J_{34} - 3J_{23}J_{22}) - 6u$
	$k = 4, r = 7$	$\frac{u_x^4}{u_y}J_{22}^4,\ \frac{u_x^4}{u_y}\big(J_{45} - 6J_{33}J_{23} - 4J_{22}J_{34} + 12J_{22}^2J_{23} + 3J_{21}J_{23}^2\big) + 24u$
	$k > 4$, $r = k + 3$	$\frac{u_x^k}{u_y}J_{22}^k$, one differential invariant of order k

The structure of the differential invariants for $\mathfrak{g}_9^{(0)}, \mathfrak{g}_{10}^{(0)}, \mathfrak{g}_{13}^{(0)}$ and $\mathfrak{g}_{14}^{(0)}$ are very simple. One property they have in common (except for $\mathfrak{g}_{10}^{(0)}$) is that the differential invariants and invariant derivations in the dop box depens on the dimension r. Note also the similarities of the differential invariants for the different Lie algebras. This is not surprising since all of them contain the vector fields $x^i\partial_y$ when r is sufficiently large.

$\mathfrak{g}_9$	$r = 3, 4, \ldots$	$\frac{1}{u_y^{\frac{1}{r-2}}}\left(\mathcal{D}_x - \frac{u_x}{u_y}\mathcal{D}_y\right), \frac{1}{u_y}\mathcal{D}_y$
$C = 0$		$u, J_{21}, J_{22}u_x u_y^{\frac{-1}{r-2}}$
	$r = 3$	$\frac{u_x}{u_y} - \log(u_y)$
	$r = 4$	$J_{23}\frac{u_x^2}{u_y} - \log(u_y)$
	$r = 5$	$(J_{34} - 3J_{22}J_{23})\frac{u_x^3}{u_y} + 2\log(u_y)$
	$r = 6$	$\frac{(J_{45} - 6J_{33}J_{23} - 4J_{22}J_{34} + 12J_{22}^2J_{23} + 3J_{21}J_{23}^2)u_x^4}{u_y} - 6\log(u_y)$
	$r = k$	one differential invariant of order $k-2$
$\mathfrak{g}_9$	$r = 3, 4, \ldots$	$e^u\left(\mathcal{D}_x - \frac{u_x}{u_y}\mathcal{D}_y\right), \frac{1}{u_y}\mathcal{D}_y$
$C = 1$		$u_y e^{(r-2)u}$
	$r = 3$	$(u_x + u_y u)e^u$
	$r = 4$	$(J_{23}u_x^2 + 2u_y u)e^{2u}$
	$r = 5$	$\left((J_{34} - 3J_{22}J_{23})u_x^3 + 6u_y u\right)e^{3u}$
	$r = 6$	$\left((J_{45} - 6J_{33}J_{23} - 4J_{22}J_{34} + 12J_{22}^2J_{23} + 3J_{21}J_{23}^2)u_x^4 + 24u_y u\right)e^{4u}$
	$r = k$	one differential invariant of order $k-2$

$\mathfrak{g}_{10}$	$r=4,5,...$	$\frac{u_y^2}{u_x u_{yy}-u_y u_{xy}}\left(\mathcal{D}_x-\frac{u_{xy}}{u_{yy}}\,\mathcal{D}_y\right),\ \frac{1}{u_y}\,\mathcal{D}_y$
$A=0$		$u,\ \frac{J_{33}-J_{21}J_{23}}{J_{22}^2}$
$B=0$	$r=4$	$J_{22},\ \frac{J_{23}}{J_{22}^2}$
	$r=5$	$\frac{J_{23}}{J_{22}^2}$
	$r=6$	$\frac{J_{34}-3J_{22}J_{23}}{J_{22}^3}$
	$r=7$	$\frac{J_{45}-6J_{33}J_{23}-4J_{22}J_{34}+12J_{22}^2J_{23}+3J_{21}J_{23}^2}{J_{22}^4}$
	$r=k$	one differential invariant of order $k-3$
$\mathfrak{g}_{10}$	$r=4,5,...$	$\frac{u_y^2}{u_x u_{yy}-u_y u_{xy}}\left(\mathcal{D}_x-\frac{u_{xy}}{u_{yy}}\,\mathcal{D}_y\right),\ \frac{1}{u_y}\,\mathcal{D}_y$
$A=0$		$u_y e^u,\ \frac{J_{33}-J_{21}J_{23}}{J_{22}^2}$
$B=1$	$r=4$	$J_{22},\ \frac{J_{23}}{J_{22}^2}$
	$r=5$	$\frac{J_{23}}{J_{22}^2}$
	$r=6$	$\frac{J_{34}-3J_{22}J_{23}}{J_{22}^3}$
	$r=7$	$\frac{J_{45}-6J_{33}J_{23}-4J_{22}J_{34}+12J_{22}^2J_{23}+3J_{21}J_{23}^2}{J_{22}^4}$
	$r=k$	one differential invariant of order $k-3$
$\mathfrak{g}_{10}$	$r=4,5,...$	$u_y^B e^u\left(\mathcal{D}_x-\frac{u_x}{u_y}\,\mathcal{D}_y\right),\ \frac{1}{u_y}\,\mathcal{D}_y$
$A=1$		$J_{21},\ J_{22}u_x u_y^B e^u$
$B\in\mathbb{C}$	$r=4$	$u_x u_y^B e^u$
	$r=5$	$J_{23}(u_x u_y^B e^u)^2$
	$r=6$	$(J_{34}-3J_{22}J_{23})\,(u_x u_y^B e^u)^3$
	$r=7$	$(J_{45}-6J_{33}J_{23}-4J_{22}J_{34}+12J_{22}^2J_{23}+3J_{21}J_{23}^2)(u_x u_y^B e^u)^4$
	$r=k$	one differential invariant of order $k-3$

$\mathfrak{g}_{13}$ $C=0$	$r=5,6,...$	$\frac{1}{u_y^{\frac{2}{r-4}}}\left(\mathcal{D}_x-\frac{u_x}{u_y}\,\mathcal{D}_y\right),\,\frac{1}{u_y}\,\mathcal{D}_y$ $u,\,J_{21},\,\left(J_{33}-\frac{r-3}{r-4}J_{22}^2-J_{21}J_{23}\right)\left(\frac{u_x}{u_y^{\frac{2}{r-4}}}\right)^2$
	$r=5$ $r=6$ $r=7$ $r=k$	$J_{23}\frac{u_x^2}{u_y^4}$ $(J_{34}-3J_{22}J_{23})\,\frac{u_x^3}{u_y^3}$ $\left(J_{45}-6J_{33}J_{23}-4J_{22}J_{34}+12J_{22}^2J_{23}+3J_{21}J_{23}^2\right)\frac{u_x^4}{u_y^{8/3}}$ one differential invariant of order $k-3$
$\mathfrak{g}_{13}$ $C=1$	$r=5,6,...$	$e^u\left(\mathcal{D}_x-\frac{2u_{xy}+(r-4)u_xu_y}{2u_{yy}+(r-4)u_y^2}\,\mathcal{D}_y\right),\,\frac{1}{u_y}\,\mathcal{D}_y$ $u_y^2e^{(r-4)u},\ \frac{u_x^2}{(r-4+2J_{21})^2u_y^{\frac{4}{r-4}}}\big((-r^2+8r-16)J_{33}+(-4r+16)J_{21}J_{33}+(4r-16)J_{32}J_{22}+8J_{32}J_{21}J_{22}-4J_{31}J_{22}^2+(r^2-7r+12)J_{22}^2+(18-4r)J_{21}J_{22}^2+(r^2-8r+16)J_{21}J_{23}+(4r-16)J_{21}^2J_{23}-4J_{21}^2J_{22}^2+4J_{21}^3J_{23}-4J_{33}J_{21}^2\big)$
	$r=5$ $r=6$ $r=k$	$\frac{u_x^2}{u_y^4}\left(J_{23}-2\frac{J_{22}^2}{1+2J_{21}}\right)$ $\frac{u_x^3}{u_y^3}\left(J_{34}-3\frac{(J_{33}+J_{23})J_{22}}{1+J_{21}}+3\frac{(J_{32}+J_{22})J_{22}^2}{(1+J_{22})^2}+2\frac{J_{22}^3}{(1+J_{21})^3}\right)$ one differential invariant of order $k-3$

$\mathfrak{g}_{14}$ $A=0$ $B=0$	$r=6,7,\dots$	$\frac{1}{u_x(J_{32}-2J_{22}J_{21})}\left(\mathcal{D}_x-\frac{u_x}{u_y}\,\mathcal{D}_y\right),\ \frac{1}{u_y}\,\mathcal{D}_y$ $u,\ J_{21},\ \frac{J_{33}-\frac{r-4}{r-5}J_{22}^2-J_{21}J_{23}}{(J_{32}-2J_{22}J_{21})^2}$
	$r=6$	$\frac{J_{32}-2J_{21}J_{22}}{\sqrt{J_{23}}},\ \frac{J_{34}-6J_{23}J_{22}}{J_{23}^{3/2}}$
	$r=k$	one differential invariant of order $k-4$
$\mathfrak{g}_{14}$ $A=0$ $B=1$	$r=6,7,\dots$	$\frac{1+J_{21}}{u_x(2J_{22}J_{21}+J_{22}J_{31}-J_{32}(1+J_{21}))}\left(\mathcal{D}_x+\frac{u_x}{u_y}\big(\frac{J_{22}}{1+J_{21}}-1\big)\,\mathcal{D}_y\right),$ $\frac{1}{u_y}\,\mathcal{D}_y$ $u_ye^u,\ \frac{J_{33}+J_{22}^2-J_{21}J_{23}+\frac{(J_{31}-2)J_{22}^2}{(1+J_{21})^2}-\frac{\left(2J_{32}+\frac{1}{r-5}J_{22}\right)J_{22}}{1+J_{21}}}{(J_{31}J_{22}-J_{32}-J_{21}J_{32}+2J_{22}J_{21})^2}$
	$r=6$	$\frac{2J_{22}J_{21}+J_{22}J_{31}-J_{32}(1+J_{21})}{(1+J_{21})\sqrt{J_{23}(1+J_{21})-J_{22}^2}},\quad \frac{1}{(J_{23}+J_{21}J_{23}-J_{22}^2)^{3/2}}\Big(J_{34}\ -$ $3J_{33}\left(\frac{J_{22}}{1+J_{21}}\right)\ +\ 3J_{32}\left(\frac{J_{22}}{1+J_{21}}\right)^2\ -\ J_{31}\left(\frac{J_{22}}{1+J_{21}}\right)^3\ -$ $6J_{22}\big(\frac{J_{23}+J_{21}J_{23}-J_{22}^2-(J_{21}+1/3)J_{22}^2}{(1+J_{21})^3}+\frac{J_{21}J_{23}}{(1+J_{21})^2}\big)\Big)$
	$r=k$	one differential invariant of order $k-4$
$\mathfrak{g}_{14}$ $A=1$ $B\in\mathbb{C}$	$r=6,7,\dots$	$u_y^Be^u\left(\mathcal{D}_x-\frac{u_x}{u_y}\left(1-\frac{(2+(r-5)B)J_{22}}{r-5+(2+(r-5)B)J_{21}}\right)\mathcal{D}_y\right),\ \frac{1}{u_y}\,\mathcal{D}_y$ $J_{21},\ \Bigg(J_{33}\ -\ J_{21}J_{23}\ +\ J_{22}^2\ -\ \frac{2(2+B(r-5))J_{32}J_{22}+J_{22}^2}{(2+B(r-5))J_{21}+r-5}\ +$ $\frac{\left(\left(2+B(r-5)\right)^2J_{31}-2(r-5)^2\right)J_{22}^2}{\left((2+B(r-5))J_{21}+r-5\right)^2}\Bigg)\left(u_xu_y^Be^u\right)^2$
	$r=6$	$\left(J_{23}-\frac{(2+B)J_{22}^2}{1+(2+B)J_{21}}\right)\left(u_xu_y^Be^u\right)^2$
	$r=7$	$\Bigg(J_{34}-3\frac{((B+1)J_{33}+J_{23})J_{22}}{(B+1)J_{21}+1}+\frac{\left(3(B+1)^2J_{32}+5(B+1)J_{22}\right)J_{22}^2}{\left((B+1)J_{21}+1\right)^2}\ -$ $\frac{\left((B+1)^3J_{31}+2(B+1)^2J_{21}\right)J_{22}^3}{\left((B+1)J_{21}+1\right)^3}\Bigg)\left(u_xu_y^Be^u\right)^3$
	$r=k$	one differential invariant of order $k-4$

4.3.4 $\mathfrak{g}_4$

$\mathfrak{g}_4$ $C_{i,j} = 0$	$\forall s, m_i$	$\mathcal{D}_x - \frac{u_x}{u_y}\mathcal{D}_y, \mathcal{D}_y$ u
	$s = 1, m_1 = 1$	$\alpha_1 y u_y + u_x$
	$s = 1, m_1 = 2$	$J_{23}u_x^2 - \alpha_1^2 y u_y - 2\alpha_1 u_x$
	$s = 2, m_1 = m_2 = 1$	$J_{23}u_x^2 - \alpha_1\alpha_2 y u_y - (\alpha_1 + \alpha_2)u_x$
	$r = \sum_{i=1}^{s} m_i + 1$	one differential invariant of order $r - 1$

The most difficult case to handle is the family $\mathfrak{g}_4^{(0)}$ because these algebras may contain an arbitrarily high number of constants. Independent of the value of the constants, the following differential invariants and invariant derivations hold for any s and m_i.

$\mathfrak{g}_4$	$\forall s, m_i$	$\mathcal{D}_x - \frac{u_{xy}}{u_{yy}}\mathcal{D}_y, \mathcal{D}_y$ u_y

In the following cases which we consider, all differential invariants can be generated from differential invariants of order two. We consider cases with one, two and three nonzero constants, respectively.

Only one nonzero constant

In the upper right box we have differential invariants that hold in many different cases, while the lower box contains invariants holding only for one particular combination of values of m_i.

$C_{1,1} \neq 0$	$s=1,\, \forall m_1$	$uu_{yy} + C_{1,1}(\alpha_1 y u_{yy} + u_{xy})$
$C_{1,k} \neq 0$	$m_1 = 1+k,\, \forall s, m_i$	$\frac{\alpha_1 u - u_x}{u_y} u_{yy} + u_{xy}$
$C_{2,0} \neq 0$	$m_2 = 1,\, \forall s, m_i$	$\frac{\alpha_2 u - u_x}{u_y} u_{yy} + u_{xy}$
$C_{1,1} \neq 0$	$s=1, m_1=2$	$u_x + \alpha_1(y u_y - u) + \frac{u u_y}{C_{1,1}}$
	$s=2, m_1=2, m_2=1$	L_{22}^{21}
	$s=1, m_1=3$	L_{22}^{3}
	$s=2, m_1=3, m_2=1$	L_{22}^{3}
$C_{1,2} \neq 0$	$s=1, m_1=3$	L_{22}^{32}
	$s=1, m_1=4$	L_{2}^{4}
$C_{2,0} \neq 0$	$s=2, m_1=1, m_2=1$	$(u_x + \alpha_1 y u_y - \alpha_2 u) C_{2,0} + (\alpha_2 - \alpha_1) u u_y$
	$s=3, m_1=1, m_2=1, m_3=1$	L_{22}^{111}
	$s=2, m_1=2, m_2=1$	L_{22}^{211}
	$s=2, m_1=2, m_2=2$	L_{2}^{220}

Two nonzero constants

$C_{1,1}, C_{1,2} \neq 0$	$s=1, m_1=3$	L_{21}^{33}, L_{22}^{33}
	$s=1, m_1=4$	L_{22}^{33}
	$s=2, m_1=3, m_2=1$	L_{2}^{31}
$C_{1,1}, C_{1,3} \neq 0$	$s=1, m_1=4$	L_{2}^{42}
$C_{1,2}, C_{1,3} \neq 0$	$s=1, m_1=4$	L_{2}^{43}
$C_{1,1}, C_{2,0} \neq 0$	$s=2, m_1=2, m_2=1$	$L_{21}^{212}, L_{22}^{212}$
	$s=2, m_1=2, m_2=2$	L_{22}^{212}
	$s=3, m_1=2, m_2=1, m_3=1$	L_{2}^{212}
	$s=2, m_1=3, m_2=1$	L_{2}^{310}
$C_{1,1}, C_{2,1} \neq 0$	$s=2, m_1=2, m_2=2$	L_{2}^{22}
$C_{1,2}, C_{2,0} \neq 0$	$s=2, m_1=3, m_2=1$	L_{2}^{312}
$C_{2,0}, C_{2,1} \neq 0$	$s=2, m_1=2, m_2=2$	L_{2}^{222}
$C_{2,0}, C_{3,0} \neq 0$	$s=3, m_1=1, m_2=1, m_3=1$	$L_{21}^{112}, L_{22}^{112}$
	$s=3, m_1=2, m_2=1, m_3=1$	L_{22}^{112}

Three nonzero constants

$C_{1,1}, C_{1,2}, C_{1,3} \neq 0$	$m_1=4$	L_{2}^{44}
$C_{1,1}, C_{1,2}, C_{2,0} \neq 0$	$m_1=3, m_2=1$	L_{2}^{313}
$C_{1,1}, C_{2,0}, C_{2,1} \neq 0$	$m_1=2, m_2=2$	L_{2}^{223}
$C_{1,1}, C_{2,0}, C_{3,0} \neq 0$	$m_1=2, m_2=1, m_3=1$	L_{2}^{221}
$C_{2,0}, C_{3,0}, C_{4,0} \neq 0$	$m_1=m_2=m_3=m_4=1$	L_{2}^{1111}

4.3.5 $\mathfrak{g}_5$

$\mathfrak{g}_5$ $C=0$	$\forall s, m_i$	$\mathcal{D}_x - \frac{u_x}{u_y}\mathcal{D}_y, \frac{1}{u_y}\mathcal{D}_y$ $u, J_{21}, J_{22}u_x$
	$s=0$	yu_y, u_x
	$s=1, m_1=1$	$\alpha_1 yu_y + u_x$
	$s=1, m_1=2$	$J_{23}u_x^2 - \alpha_1^2 yu_y - 2\alpha_1 u_x$
	$s=2, m_1=m_2=1$	$J_{23}u_x^2 - \alpha_1\alpha_2 yu_y - (\alpha_1+\alpha_2)u_x$
	$r=\sum_{i=1}^s m_i + 2$	one differential invariant of order $r-2$
$\mathfrak{g}_5$ $C=1$	$\forall s, m_i$	$\mathcal{D}_x - \frac{u_x}{u_y}\mathcal{D}_y, \frac{1}{u_y}\mathcal{D}_y$ $u_y e^u$
	$s=0$	e^u/y
	$s=1, m_1=1$	$\alpha_1 yu_y + u_x$
	$s=1, m_1=2$	$J_{23}u_x^2 - \alpha_1^2 yu_y - 2\alpha_1 u_x$
	$s=2, m_1=m_2=1$	$J_{23}u_x^2 - \alpha_1\alpha_2 yu_y - (\alpha_1+\alpha_2)u_x$
	$r=\sum_{i=1}^s m_i + 2$	one differential invariant of order $r-2$

4.3.6 $\mathfrak{g}_{21}, \mathfrak{g}_{22}$

$\mathfrak{g}_{21}$ $a_i=0$	$r=2,3,\ldots$	$\mathcal{D}_x - \frac{u_x}{u_y}\mathcal{D}_y, \mathcal{D}_y$ x, u
	$r=2$	$J_{23}u_x^2 - \frac{\phi_2''(x)}{\phi_2'(x)}u_x$
	$r=k$	One differential invariant of strict order k

For the nontrivial lift of $\mathfrak{g}_{21}$ we again have to find two differential invariants in addition to the two that hold for every r. We can reorder the generators of the algebra so that only the first ones have nonzero function a_i.

$\mathfrak{g}_{21}$	$r = 2, 3, ...$	$\mathcal{D}_x - \frac{u_{xy}}{u_{yy}}\,\mathcal{D}_y, \mathcal{D}_y$
		x, u_y
	$a_i = 0,\ i > 2$	$u_x u_y J_{22} - u u_y J_{21} \frac{a_2'(x)}{a_2(x)}$
	$a_i = 0,\ i > 3$	M_3
	$a_2 \neq 0,\ r = 2$	$\frac{u}{a_2(x)}(a_2'(x) - \phi_2'(x) u_y) - u_x$
	$r = 3$	$\frac{-a_2'(x)\phi_3''(x)u + \phi_2'(x)\phi_3''(x)uu_y + a_2(x)\phi_3''(x)u_x}{a_2(x)\phi_3'(x)} +$ $\frac{-\phi_3'(x)\phi_2''(x)uu_y + \phi_3'(x)a_2''(x)u}{a_2(x)\phi_3'(x)} +$ $\frac{-a_2'(x)^2 J_{21} u^2 + 2a_2'(x)a_2(x)uu_x J_{22} - a_2(x)^2 u_x^2 J_{23}}{a_2(x)^2}$
	$r = k$	one differential invariant of strict order $k-1$
	$a_3 \neq 0,\ r = 3$	M_{21}, M_{22}
	$r = 4$	M_{21}
	$r = k$	one differential invariants of strict order $k-2$
	$a_4 \neq 0,\ r = 4$	M_{23}, one differential invariant of strict order 3

$\mathfrak{g}_{22}$	$r = 3, 4, ...$	$\mathcal{D}_x - \frac{u_x}{u_y}\,\mathcal{D}_y,\ \frac{1}{u_y}\,\mathcal{D}_y$
$b = 0$		$x, u, J_{21}, J_{22}u_x$
	$r = 3$	$J_{23}u_x^2 - \frac{\phi_3''(x)}{\phi_3'(x)}u_x$
	$r = k$	one differential invariant of strict order $k-1$

$\mathfrak{g}_{22}$	$r = 3, 4, ...$	$\mathcal{D}_x + \left(\frac{b'(x)u}{b(x)u_y} - \frac{u_x}{u_y}\right)\mathcal{D}_y,\ \frac{1}{u_y}\,\mathcal{D}_y$
$b \neq 0$		$x, u_y e^{u/b(x)}$
	$r = 3$	$\frac{-b(x)\phi_3''(x)u_x + b'(x)\phi_3''(x)u - b''(x)\phi_3'(x)u}{b(x)\phi_3'(x)} +$ $\frac{b'(x)^2u^2J_{21} - 2b'(x)b(x)uu_xJ_{22} + b(x)^2u_x^2J_{23}}{b(x)^2}$
	$r = k$	one differential invariant of strict order $k-1$

Chapter 5

General remarks and applications

In this chapter we discuss the notion of algebraic actions of Lie algebras of vector fields by using examples from our computations. Then we state some results based on the fact that all our lifts have differential invariants of order two. We end by describing how we can use differential invariants for constructing invariant differential equations.

5.1 Algebraic actions

In section 2.4 we discussed how Frobenius' theorem guarantees the existence of differential invariants on some micro-local neighborhood of a generic point of a Lie algebra of vector fields. Many of the differential invariants we found in section 4.3 are given by rational functions. Some differential invariants contain logarithms or exponential functions, and some contain expressions like u_x/u_y^λ where λ is a complex number.

In this section we will look at a few examples where Rosenlicht's theorem (theorem 4 on page 13) guarantees that there exists a complete set of differential invariants which are rational in derivative coordinates $(u_x, u_y, u_{xx}, ...)$. We will consider $\mathfrak{g} = \mathfrak{g}_8^{(0)}$ with $r = 4$ in detail. It has a basis consisting of ∂_x, ∂_y, $x\partial_x + y\partial_y + A\partial_u$ and $x\partial_y + B\partial_u$.

Since the action of $\mathfrak{g}$ is transitive on $\mathbb{C}^3$, all orbits of $\mathfrak{g}^{(k)}$ project onto $\mathbb{C}^3$. Hence, to study the space of orbits on the level of k-jets we can restrict to a fiber $\pi_{k,0}^{-1}(0)$ and the action of the isotropy algebra of 0: $\mathfrak{g}_0 = \langle B(x\partial_x + y\partial_y) - A(x\partial_y)\rangle$. Consider the group of 1-jets of local diffeomorphisms of $\mathbb{C}^3$ at the point $0 \in \mathbb{C}^3$, denoted by $J_0^1(\mathbb{C}^3, \mathbb{C}^3)$, and let X_i, Y_i, U_i be coordinates, for $i = 1, 2, 3$.

Consider the left action of $\mathfrak{g}_0$ on $J_0^1(\mathbb{C}^3, \mathbb{C}^3)$. It is easily checked that the functions

$$\frac{X_2}{X_1}, \quad \frac{X_3}{X_1}, \quad \frac{Y_2}{X_1} - \frac{Y_1 X_2}{X_1^2}, \quad \frac{Y_3}{X_1} - \frac{Y_1 X_3}{X_1^2},$$
$$U_1, \quad U_2, \quad U_3, \quad X_1^A e^{B \frac{Y_1}{X_1}}$$

are invariants of $\mathfrak{g}_0^{(1)}$. The orbit of $\mathfrak{g}_0^{(1)}$ going through $\mathrm{id}^{(1)}$ is given by the equations

$$\frac{X_2}{X_1} = \frac{X_3}{X_1} = \frac{Y_3}{X_1} - \frac{Y_1 X_1}{X_1^2} = U_1 = U_2 = 0,$$
$$\frac{Y_2}{X_1} - \frac{Y_1 X_2}{X_1^2} = U_3 = X_1^A e^{B \frac{Y_1}{X_1}} = 1.$$

We identify this orbit with the group corresponding to $\mathfrak{g}_0^{(1)}$. We want to determine whether this orbit is algebraic, i.e. given by rational equations. The only equation that may potentially prevent this is

$$X_1^A e^{B \frac{Y_1}{X_1}} = 1. \tag{5.1}$$

Now we will study this orbit for different values of A and B.

Case 1 Let $B = 0$ and $A \neq 0$. Then (5.1) reduces to $X_1^A = 1$, which we rewrite to $X_1 = 1$. Thus the orbit is defined by rational equations, which means that $\mathfrak{g}_0^{(1)}$ acts algebraically. This means that there exists differential invariants which are rational in derivative coordinates. This is in correspondence with the differential invariants we found. For $A = 1$ they were generated by

$$u_y e^u, \quad \left(\frac{u_{yy}}{u_y^2} - 2 \frac{u_{xy}}{u_x u_y} + \frac{u_{xx}}{u_x^2} \right) (u_x e^u)^2, \quad e^u \left(\mathcal{D}_x - \frac{u_x}{u_y} \mathcal{D}_y \right), \quad \frac{1}{u_y} \mathcal{D}_y \,.$$

Case 2 Now, let both A and B be different from 0. If we fix all coordinates except for Y_1, the equation $X_1^A e^{B \frac{Y_1}{X_1}} = 1$ has infinitely many zeroes, and hence the orbit cannot be expressed by rational equations alone. Hence the action of $\mathfrak{g}_0^{(1)}$ is not algebraic, and we may expect to get differential invariants that are not rational in derivative coordinates. When $A = 1$, the differential invariants are generated by

$$B \frac{u_x}{u_y} + u + \log(u_y), \quad \frac{u_{yy}}{u_y^2}, \quad \frac{1}{u_y} \left(\mathcal{D}_x - \frac{u_{xy}}{u_{yy}} \mathcal{D}_y \right), \quad \frac{1}{u_y} \mathcal{D}_y \,.$$

Case 3 Finally, let $A = 0$ and $B \neq 0$. Now equation (5.1) reduces to $e^{B\frac{Y_1}{X_1}} = 1$, which is equivalent to $\frac{Y_1}{X_1} = 0$. Hence the action is algebraic. If $B = 1$, the differential invariants are generated by

$$\frac{u_x}{u_y} + u, \quad \frac{u_{yy}}{u_y^2}, \quad \frac{1}{u_y}\left(\mathcal{D}_x - \frac{u_{xy}}{u_{yy}}\mathcal{D}_y\right), \quad \frac{1}{u_y}\mathcal{D}_y\,.$$

We summarize these results in a theorem.

Theorem 6. *Consider the Lie algebra $\mathfrak{g} = \langle \partial_x, \partial_y, x\partial_x + y\partial_y + A\partial_u, x\partial_y + B\partial_u\rangle$. When $A = 0$ and $B \neq 0$, or when $A \neq 0$ and $B = 0$, then $\mathfrak{g}^{(k)}$ acts algebraically on derivative coordinates. When $AB \neq 0$, then $\mathfrak{g}^{(k)}$ does not act algebraically in derivative coordinates.*

In fact, when $A = 0$, the action of $\mathfrak{g}^{(k)}$ is not only algebraic in derivative coordinates, but also in base coordinates (x, y, u). However, a change of coordinates does not in general preserve such algebraicity in base coordinates. Algebraicity in derivative coordinates, on the other hand, is preserved. To illustrate this, we look at the first prolongation of a point transformation: $(x, y, u) \mapsto (a(x, y, u), b(x, y, u), c(x, y, u))$. The derivative coordinates u_x, u_y transforms according to the following formula:

$$\begin{pmatrix} u_x \\ u_y \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{D}_x^{(1)}(a) & \mathcal{D}_x^{(1)}(b) \\ \mathcal{D}_y^{(1)}(a) & \mathcal{D}_y^{(1)}(b) \end{pmatrix}^{-1} \begin{pmatrix} \mathcal{D}_x^{(1)}(c) \\ \mathcal{D}_y^{(1)}(c) \end{pmatrix}$$

Recall that $\mathcal{D}_x^{(1)} = \partial_x + u_x\partial_u$ and $\mathcal{D}_y^{(1)} = \partial_y + u_y\partial_u$. We see that expression on the right-hand side is rational in u_x and u_y. In [KL13] the notion of algebraic actions is used to formulate a global version of the Lie-Tresse theorem.

5.2 Projectable Lie algebras of vector fields

Consider the trivial bundle $\pi\colon \mathbb{C}^2 \times \mathbb{C} \to \mathbb{C}^2$. Recall that a vector field X on $\mathbb{C}^2 \times \mathbb{C}$ is projectable if there exists a vector field Y on $\mathbb{C}^2$ such that the following diagram commutes.

$$\begin{tikzcd} \mathbb{C}^2 \times \mathbb{C} \arrow[r, "X"] \arrow[d, "\pi"] & T(\mathbb{C}^2 \times \mathbb{C}) \arrow[d, "\pi_*"] \\ \mathbb{C}^2 \arrow[r, "Y"] & T(\mathbb{C}^2) \end{tikzcd}$$

If x, y, u are coordinates on the bundle, then a projectable vector field is of the form

$$a(x,y)\partial_x + b(x,y)\partial_y + c(x,y,u)\partial_u.$$

Now, consider an r-dimensional Lie algebra $\mathfrak{g}$ of vector fields on $\mathbb{C}^2 \times \mathbb{C}$. We call $\mathfrak{g} = \langle X_1, ..., X_r \rangle$ projectable if it consists only of projectable vector fields, and regularly projectable if in addition $d\pi \colon \mathfrak{g} \to \mathfrak{D}(\mathbb{C}^2)$ is injective.

By lifting the algebras from the classifications of Lie, we have found a local description of all regularly projectable Lie algebras of vector fields on $\mathbb{C}^2 \times \mathbb{C}$ that are constant on the fibers of π.

Theorem 7. *Every regularly projectable finite-dimensional Lie algebra of vector fields on the trivial bundle $\mathbb{C}^2 \times \mathbb{C} \to \mathbb{C}^2$ that is constant on the fiber has, on some open set, a differential invariant of order two.*

Proof. Away from singular points, such a Lie algebra of vector fields is locally equivalent to $\mathfrak{g}_i^{(0)}$ for one of the Lie algebras in the classification. The theorem follows from our list of differential invariants. □

This is an interesting property, and for general Lie algebras of vector fields on $\mathbb{C}^2 \times \mathbb{C}$, there is no upper bound on the order of the lowest-order differential invariant.

Consider for example the Lie algebra

$$\mathfrak{h}_s = \langle \partial_x, \partial_y, y\partial_x, x\partial_y, x\partial_x - y\partial_y, x^i y^j \partial_u \mid i + j = 0, 1, ..., s \rangle.$$

Notice that this is constant on fibers, and projects (nonregularly) to $\mathfrak{g}_3$.

Theorem 8. *The Lie algebra $\mathfrak{h}_s$ has no differential invariants of order s.*

Proof. If f is a differential invariant of $\mathfrak{h}_s$, then $f = f(u_x, u_y, ...)$. Assume that $f = f(u_{x^k}, u_{x^{k-1}y}, u_{x^{k-2}y^2}, ...)$. In other words, f is constant on $J^{k-1}(\pi) \subset J^k(\pi)$. The Lie algebra $\mathfrak{h}_k$ contains the vector fields

$$X_i = x^i y^{k-i} \partial_u, \qquad i = 0, 1, ..., k.$$

We know that $X_i^{(\infty)} = \text{Э}_\varphi = \mathcal{D}_\sigma(\varphi)\partial_{u_\sigma}$ where $\varphi = (du - u_x dx - u_y dy)(X) = x^i y^{k-i}$. Hence

$$X_i^{(\infty)} = \sum_{m=0}^{k-i} \sum_{n=0}^{i} \frac{i!(k-i)!}{(i-n)!(k-i-m)!} x^{i-n} y^{k-i-m} \partial_{u_{x^n y^m}}.$$

Since $\partial_{u_{x^n y^m}}(f) = 0$ for $n + m < k$, we get

$$X_i^{(\infty)}(f) = X_i^{(k)}(f) = i!(k-i)!\partial_{u_{x^i y^{k-i}}}(f).$$

For f to be a differential invariant of the $\mathfrak{h}_s$, we must have $X_i^{(k)}(f) = 0$ for $i = 0, 1, ..., k$. This means that f does not depend on $x^k, x^{k-1}y, ..., y^k$. By induction, $\mathfrak{h}_k$ has no differential invariants of order k. □

Hence the lowest-order differential invariant of a nonregularly projectable Lie algebra $\mathfrak{g}$ of vector fields which is constant on fibers can be of arbitrarily high order k. But the dimension of $\ker d\pi|_{\mathfrak{g}}$ gives an upper bound on k.

From the list of differential invariants, we see that all the Lie algebras has at least $\binom{k}{2}$ differential invariants of order k for $k \geq 2$. We get the following corollary.

Corollary 1. *Let $\mathfrak{g} \in \mathfrak{D}(\mathbb{C}^2 \times \mathbb{C})$ be a projectable finite-dimensional Lie algebra of vector fields which is constant on the fibers of $\pi \colon \mathbb{C}^2 \times \mathbb{C} \to \mathbb{C}^2$ and let $k \geq 2$. If $\dim(\ker d\pi|_{\mathfrak{g}}) < \binom{k}{2}$, then $\mathfrak{g}$ has, on some open set, a differential invariant of order k.*

Proof. Locally, $\operatorname{Im} d\pi|_{\mathfrak{g}} = \mathfrak{g}_i$ where $\mathfrak{g}_i \subset \mathfrak{D}(\mathbb{C}^2)$ is an algebra from the classification. Hence $\mathfrak{g} = \mathfrak{g}_i^{(0)} \cup \ker d\pi|_{\mathfrak{g}}$. If $\dim(\ker d\pi|_{\mathfrak{g}}) < \binom{k}{2}$, then $\ker d\pi|_{\mathfrak{g}}$ cannot kill all the differential invariants of $\mathfrak{g}_i^{(0)}$ of order k. □

5.3 Differential equations and their symmetries

One important property of differential invariants is that they generate invariant differential equations.

A scalar differential equation of two independent variables can be considered as a subset $\mathcal{E} \subset J^k(\mathbb{C}^2 \times \mathbb{C})$ given by an equation:

$$\mathcal{E} = \{F(x, y, u, u_x, u_y, ..., u_{y^k}) = 0\}.$$

We will always assume that $\mathcal{E}$ is regular, i.e. that the differential dF is nonzero in a neighborhood of $\mathcal{E}$. In this case, $\mathcal{E} \subset J^k(\mathbb{C}^2 \times \mathbb{C})$ is a regular submanifold of dimension $\dim J^k(\mathbb{C}^2 \times \mathbb{C}) - 1$.

The Lie algebra $\mathfrak{g} \subset \mathfrak{D}(\mathbb{C}^2 \times \mathbb{C})$ is a symmetry algebra of $\mathcal{E}$ if

$$X^{(k)}(F) = \lambda F \qquad \text{for every} \qquad X \subset \mathfrak{g} \tag{5.2}$$

where $\lambda \in C^\omega(J^k(\mathbb{C}^2 \times \mathbb{C}))$. If we set $\lambda = 0$, the above equation tells us that F is a differential invariant of $\mathfrak{g}$. In this way, the differential invariants of $\mathfrak{g}$ gives us invariant differential equations.

However, we do not get all invariant differential equations. In order to get all of them, we should also consider cases where $\lambda \neq 0$. Still, there is a sense in which the differential invariants locally determine all invariant differential equations.

Theorem 9. *Let $\theta \in J^k(\mathbb{C}^2 \times \mathbb{C})$ be a generic point of $\mathfrak{g}^{(k)}$. There exists a neighborhood U of θ such that every $\mathfrak{g}$-invariant differential equation of order k in U is given by $F(I_1, ..., I_r)$, where I_i are differential invariants of $\mathfrak{g}$ of order k.*

Proof. Let $\mathfrak{g}$ be a Lie algebra of vector fields on $\mathbb{C}^2 \times \mathbb{C}$. In a neighborhood of a generic point $\theta \in J^k(\mathbb{C}^2 \times \mathbb{C})$, the prolonged Lie algebra $\mathfrak{g}^{(k)}$ determines an integrable distribution P whose maximal integral manifolds are the orbits of $\mathfrak{g}^{(k)}$. By Frobenius' theorem there exists an open set $W \subset J^k(\mathbb{C}^2 \times \mathbb{C})$ containing θ and local coordinates $w^1, ..., w^m$ on W such that $P = \langle \partial_{w^1}, ..., \partial_{w^s} \rangle$. In these coordinates maximal integral manifolds are given by equations $w^{s+1} = c_1, ..., w^m = c_{m-s}$ where $c_i \in \mathbb{C}$. Now, consider a $\mathfrak{g}$-invariant differential equation $\mathcal{E} = \{F(w^1, ..., w^m) = 0\} \subset W$. Since $\mathcal{E}$ is $\mathfrak{g}$-invariant, F does not depend on $w^1, ..., w^m$, and hence $\mathcal{E} = \{F(w^{s+1}, ..., w^m) = 0\}$. The coordinates $w^{s+1}, ..., w^m$ correspond to differential invariants of order k of the Lie algebra $\mathfrak{g}$. □

Because of this we can use the differential invariants to locally describe all $\mathfrak{g}$-invariant differential equations of order k in the neighborhood of a generic point of $\mathfrak{g}^{(k)}$.

Example 2. *Consider the Lie algebra $\mathfrak{g}_{15}^{(0)} = \langle \partial_x, x\partial_x + \partial_y, x^2\partial_x + 2x\partial_y + e^y\partial_u \rangle$ ($C = 1$). The generators span a three-dimensional vector space at every point, and so does the generators of all prolongations of $\mathfrak{g}_{15}^{(0)}$. Every differential invariant of this Lie algebra is generated by the differential invariants $u_y - u, 2uu_y + u_x e^y - u^2$ and the invariant derivations $e^y \mathcal{D}_x + 2u \mathcal{D}_y, \mathcal{D}_y$. In particular there are five functionally independent differential invariants of order two:*

$$\begin{aligned} &I_1 = u_y - u, \qquad I_2 = 2uu_y + u_x e^y - u^2, \\ &I_3 = u_{yy} - u_y, \qquad I_4 = u_{xy}e^y + 2uu_{yy} - u^2, \\ &I_5 = u_{xx}e^y - 2u(-e^y(2u_{xy} + u_x) + u(u - u_y - 2u_{yy})) \end{aligned}$$

Let $\mathcal{E} \subset J^2(\mathbb{C}^2 \times \mathbb{C})$ be an invariant differential equation. In some neighborhood U of any point in $J^2(\mathbb{C}^2 \times \mathbb{C})$, we have $\mathcal{E} \cap U = \{F(I_1, I_2, I_3, I_4, I_5) = 0\}$.

Chapter 6

Appendix

Some of the expressions for the differential invariants are collected here.

$$
\begin{aligned}
I_{31} &= \frac{J_{34}^2}{J_{23}^3} + 12\left(\frac{J_{33} + J_{22}^2}{J_{23}} - J_{21} - \frac{J_{22}J_{34}}{J_{23}^2}\right) \\
I_{32} &= \tfrac{(-54J_{23}^4J_{32}+18(J_{21}J_{34}+6J_{22}J_{33})J_{23}^3-18J_{34}(J_{33}+4J_{22}^2)J_{23}^2+18J_{22}J_{23}J_{34}^2-J_{34}^3)^2}{J_{23}^9} \\
I_{33} &= \big((-72J_{21}^2 + 108J_{31})J_{23}^6 + (-324J_{22}J_{32} + 144J_{22}^2J_{21} - 180J_{21}J_{33})J_{23}^5 \\
&\quad + (54J_{34}J_{32} + 180J_{21}J_{22}J_{34} + 504J_{33}J_{22}^2 - 72J_{22}^4 + 90J_{33}^2)J_{23}^4 \\
&\quad - 24J_{34}(J_{21}J_{34} + 12J_{22}J_{33} + 12J_{22}^3)J_{23}^3 + J_{34}^2(168J_{22}^2 + 24J_{33})J_{23}^2 \\
&\quad - 24J_{22}J_{23}J_{34}^3 + J_{34}^4\big)/J_{23}^6 \\
I_{41} &= \frac{J_{45} - 12(J_{22}J_{34} + J_{23}^2J_{21} - J_{22}^2J_{23} - J_{33}J_{23})}{J_{23}^2} \\
K_2 &= J_{21} - \frac{J_{22}^2}{J_{23}} \\
K_{34} &= \frac{J_{34}^2}{J_{23}^3} \\
K_{33} &= \frac{J_{33}}{J_{23}} - \frac{J_{34}J_{22}}{J_{23}^2} \\
K_{32} &= \frac{1}{J_{23}}\left(J_{32} - \frac{2J_{33}J_{22} + J_{34}J_{21}}{J_{23}} + 2\frac{J_{34}J_{22}^2}{J_{32}^2}\right)^2 \\
K_{31} &= J_{31} - 3\frac{J_{33}J_{21} + J_{32}J_{22}}{J_{23}} + 3\frac{2J_{33}J_{22}^2 + J_{34}J_{21}J_{22}}{J_{23}^2} - 4\frac{J_{34}J_{22}^3}{J_{23}^3}
\end{aligned}
$$

$$
\begin{aligned}
\alpha_1 = \big(&-12J_{31}J_{23}^4 + 12J_{43}J_{23}^3 - 12J_{33}^2J_{23}^2 + 24J_{21}^2J_{23}^4 - 3J_{34}^2J_{33} \\
&+ 48J_{33}J_{23}^2J_{22}^2 - 48J_{34}J_{23}J_{22}^3 - 12J_{44}J_{23}^2J_{22} - 48J_{21}J_{23}^3J_{22}^2 \\
&+ 6J_{34}^2J_{22}^2 + 24J_{23}^2J_{22}^4 - 12J_{34}J_{32}J_{23}^2 + 2J_{44}J_{34}J_{23} - 12J_{21}J_{33}J_{23}^3 \\
&+ 24J_{21}J_{34}J_{23}^2J_{22} + 18J_{34}J_{33}J_{23}J_{22}\big)/\big(u_x(12J_{44}J_{23}^3 - 36J_{32}J_{23}^4 \\
&- 3J_{34}^3 - 12J_{45}J_{23}^2J_{22} - 30J_{34}J_{33}J_{23}^2 + 72J_{33}J_{23}^3J_{22} + 12J_{21}J_{34}J_{23}^3 \\
&- 48J_{34}J_{23}^2J_{22}^2 + 30J_{34}^2J_{23}J_{22} + 2J_{45}J_{34}J_{23})\big) \\
\alpha_2 = \frac{J_{23}^4}{u_x}\big(&12J_{44}J_{23}^3 - 36J_{32}J_{23}^4 - 3J_{34}^3 - 12J_{45}J_{23}^2J_{22} - 30J_{34}J_{33}J_{23}^2 \\
&+ 72J_{33}J_{23}^3J_{22} + 12J_{21}J_{34}J_{23}^3 - 48J_{34}J_{23}^2J_{22}^2 + 30J_{34}^2J_{23}J_{22} + 2J_{45}J_{34}J_{23}\big)^{-1} \\
\alpha_5 = &\frac{J_{23}^2}{u_x(J_{34}J_{22}^2 + J_{32}J_{23}^2 - 2J_{33}J_{23}J_{22})} \\
\alpha_6 = &\frac{2J_{21}^2J_{23}^2 - J_{31}J_{23}^2 - 4J_{21}J_{23}J_{22}^2 + 2J_{32}J_{23}J_{22} + 2J_{22}^4 - J_{33}J_{22}^2}{u_x(J_{34}J_{22}^2 + J_{32}J_{23}^2 - 2J_{33}J_{23}J_{22})} \\
\alpha_7 = &\Big(2J_{21}J_{23} - 2J_{22}^2 + J_{21}J_{33} - 2J_{32}J_{22} + J_{31}J_{23}\Big)e^{-2u}\Big/\Big(u_y^2u_x^3(4J_{33}J_{22}^3 \\
&+ 2J_{34}J_{22}^2 - 2J_{33}^2J_{22} + 2J_{34}J_{32}J_{22} - 2J_{21}J_{34}J_{22}^2 - 2J_{32}J_{23}J_{22}^2 \\
&- 4J_{33}J_{23}J_{22} + 2J_{32}J_{23}^2 + 2J_{21}J_{32}J_{23}^2 + J_{33}J_{32}J_{23} \\
&- J_{31}J_{34}J_{23} + 2J_{21}^2J_{34}J_{23} - 4J_{21}J_{33}J_{23}J_{22})\Big) \\
\beta_7 = -&\Big(2J_{21}J_{23} - 2J_{22}^2 - J_{21}J_{34} + J_{21}J_{33} + 2J_{33}J_{22} - 2J_{32}J_{22} \\
&- J_{32}J_{23} + J_{31}J_{23}\Big)e^{-2u}\Big/\Big(u_y^3u_x^2(4J_{33}J_{22}^3 + 2J_{34}J_{22}^2 - 2J_{33}^2J_{22} + 2J_{34}J_{32}J_{22} \\
&- 2J_{21}J_{34}J_{22}^2 - 2J_{32}J_{23}J_{22}^2 - 4J_{33}J_{23}J_{22} + 2J_{32}J_{23}^2 + 2J_{21}J_{32}J_{23}^2 + J_{33}J_{32}J_{23} \\
&- J_{31}J_{34}J_{23} + 2J_{21}^2J_{34}J_{23} - 4J_{21}J_{33}J_{23}J_{22})\Big) \\
\alpha_8 = &\Big(2J_{23} - 2J_{22}^2 + 2J_{21}J_{23} + J_{33}\Big)e^{-2u}\Big/\Big(u_y^2u_x^3(4J_{33}J_{22}^3 + 2J_{34}J_{22}^2 - 2J_{33}^2J_{22} \\
&+ 2J_{34}J_{32}J_{22} - 2J_{21}J_{34}J_{22}^2 - 2J_{32}J_{23}J_{22}^2 - 4J_{33}J_{23}J_{22} + 2J_{32}J_{23}^2 + 2J_{21}J_{32}J_{23}^2 \\
&+ J_{33}J_{32}J_{23} - J_{31}J_{34}J_{23} + 2J_{21}^2J_{34}J_{23} - 4J_{21}J_{33}J_{23}J_{22})\Big) \\
\beta_8 = -&\Big(2J_{23} + 2J_{21}J_{23} - 2J_{22}^2 - J_{34} + J_{33}\Big)e^{-2u}\Big/\Big(u_y^3u_x^2(4J_{33}J_{22}^3 + 2J_{34}J_{22}^2 \\
&- 2J_{33}^2J_{22} + 2J_{34}J_{32}J_{22} - 2J_{21}J_{34}J_{22}^2 - 2J_{32}J_{23}J_{22}^2 - 4J_{33}J_{23}J_{22} + 2J_{32}J_{23}^2 \\
&+ 2J_{21}J_{32}J_{23}^2 + J_{33}J_{32}J_{23} - J_{31}J_{34}J_{23} + 2J_{21}^2J_{34}J_{23} - 4J_{21}J_{33}J_{23}J_{22})\Big)
\end{aligned}
$$

$$L_{22}^{3} = \frac{1}{u_{2,2}}\Big(\big(\left(-{\alpha_1}^2 u + 2\,u_1\alpha_1 + u_2\alpha_2\alpha_1 y - u_{1,1}\right)u_{2,2}$$
$$- u_{1,2}\left(u_2\left(\alpha_1-\alpha_2\right) - u_{1,2}\right)\big)C_{1,1} + u_2\left(\left(u\alpha_2 - u_1\right)u_{2,2} + u_2u_{1,2}\right)\Big)$$
$$L_{22}^{21} = \frac{1}{{u_2}^2}\Big(\big(-{u_2}^3\alpha_2\alpha_1 y + \left(\left(u\alpha_2-u_1\right)\alpha_1 - \alpha_2u_1 + u_{1,1}\right){u_2}^2$$
$$+ 2\,u_{1,2}\left(-u_1+\alpha_1 u\right)u_2 + u_{2,2}\left(-u_1+\alpha_1u\right)^2\big)C_{1,1} + {u_2}^3u\left(\alpha_1-\alpha_2\right)\Big)$$
$$L_{22}^{111} = \frac{1}{{u_2}^2C_{2,0}}\Big(\left(-C_{2,0}\alpha_3\alpha_1 y - u\left(-\alpha_3+\alpha_2\right)\left(\alpha_1-\alpha_2\right)\right){u_2}^3$$
$$+ C_{2,0}\left(\alpha_2\left(-\alpha_2+\alpha_3+\alpha_1\right)u + u_{1,1} - u_1\alpha_1 - \alpha_3u_1\right){u_2}^2$$
$$+ 2\,u_{1,2}C_{2,0}\left(u\alpha_2-u_1\right)u_2 + u_{2,2}C_{2,0}\left(u\alpha_2-u_1\right)^2\Big)$$
$$L_{22}^{211} = \frac{1}{u_2^2}\Bigg(\Big(-{u_2}^3y{\alpha_1}^2 + \Big(\left(-{\alpha_2}^2+2\,\alpha_2\alpha_1\right)u - 2\,u_1\alpha_1 + u_{1,1}\Big){u_2}^2$$
$$+ 2\,u_{1,2}\left(u\alpha_2-u_1\right)u_2 + u_{2,2}\left(u\alpha_2-u_1\right)^2\Big)C_{2,0} + {u_2}^3u\left(\alpha_1-\alpha_2\right)^2\Bigg)$$
$$L_{2}^{4} = \Big(\left(\left(-u+yu_2\right){\alpha_1}^2 + 2\,u_1\alpha_1 - u_{1,1}\right)u_{2,2} + {u_{1,2}}^2\Big)\,C_{1,2} - 2\,u_{2,2}uu_2$$
$$L_{21}^{33} = \frac{1}{2\,{C_{1,1}}^2 + u_2C_{1,2}}\Big(\left(2\,u_{1,2} + 2\,\alpha_1u_{2,2}y\right){C_{1,1}}^2 + 2\,u_{2,2}uC_{1,1}$$
$$+ \left(\left(-u_1+\alpha_1u\right)u_{2,2} + u_2u_{1,2}\right)\Big)C_{1,2}$$
$$L_{22}^{33} = \frac{1}{u_{2,2}}\Bigg(\Big(\left(\left(-u+yu_2\right){\alpha_1}^2 + 2\,u_1\alpha_1 - u_{1,1}\right)u_{2,2} + {u_{1,2}}^2\Big)\,{C_{1,2}}^2$$
$$+ \Bigg(\Big(\left(\left(-2\,u - 2\,yu_2\right)\alpha_1 + 2\,u_1\right)C_{1,1} - 2\,u_2u\Big)\,u_{2,2}$$
$$- 4\,C_{1,1}u_2u_{1,2}\Bigg)C_{1,2} - 4\,\Big(\left(C_{1,1}\alpha_1y + u\right)u_{2,2} + u_{1,2}C_{1,1}\Big)\,{C_{1,1}}^2\Bigg)$$
$$L_{2}^{31} = \frac{1}{u_{2,2}}\Big(2\,\left(C_{1,1} - 1/2\,C_{1,2}\left(\alpha_1-\alpha_2\right)\right)u_{1,2}{u_2}^2$$
$$+ \Big(\left(2\,\alpha_2\alpha_1yu_{2,2} - 2\,u_{1,2}\left(\alpha_1-\alpha_2\right)\right){C_{1,1}}^2 + 2\,\left(u\alpha_2-u_1\right)u_{2,2}C_{1,1}$$
$$- C_{1,2}u_{2,2}\left(\alpha_1-\alpha_2\right)\left(-u_1+\alpha_1u\right)\Big)u_2$$
$$- 2\,\Big(\left({\alpha_1}^2u - 2\,u_1\alpha_1 + u_{1,1}\right)u_{2,2} - {u_{1,2}}^2\Big)\,{C_{1,1}}^2\Big)$$

$$
\begin{aligned}
L_2^{42} = \frac{1}{u_{2,2}}\Big(& \left(6\,\alpha_1 u_{2,2} y + 6\,u_{1,2}\right) {C_{1,1}}^3 + 6\,{C_{1,1}}^2 u_{2,2} u \\
& + \Big(\left(\left(-u + y u_2\right) {\alpha_1}^2 + 2\,u_1 \alpha_1 - u_{1,1}\right) u_{2,2} + {u_{1,2}}^2\Big) C_{1,3} C_{1,1} \\
& + u_2 \left(\left(-u_1 + \alpha_1 u\right) u_{2,2} + u_2 u_{1,2}\right) C_{1,3}\Big) \\
L_2^{43} = \frac{1}{u_{2,2}}\Big(\Big(& \left(-3\,{\alpha_1}^2 u + 6\,u_1 \alpha_1 + 3\,{\alpha_1}^2 y u_2 - 3\,u_{1,1}\right) {C_{1,2}}^2 - 6\,u_2 u C_{1,2} \\
& - 2\,u_2 C_{1,3} \left(-u_1 + \alpha_1 u\right)\Big) u_{2,2} - 2\,{u_2}^2 C_{1,3} u_{1,2} + 3\,{C_{1,2}}^2 {u_{1,2}}^2\Big) \\
L_{21}^{212} = \frac{1}{\left(\alpha_1 - \alpha_2\right)\left(u_2 - C_{2,0}\right) C_{1,1} + C_{2,0} u_2}\Big(& \left(\alpha_2 - \alpha_1\right)\Big(\left(\alpha_1 u_{2,2} y + u_{1,2}\right) C_{2,0} \\
& + u_1 u_{2,2} - \alpha_1 u u_{2,2} - u_2 u_{1,2}\Big) C_{1,1} + C_{2,0}\left(u_{2,2} u \alpha_2 - u_1 u_{2,2} + u_2 u_{1,2}\right)\Big) \\
L_{22}^{212} = \frac{1}{u_{2,2}}\Big(\Big(& -{\alpha_1}^3 y C_{1,1} u_{2,2} + \Big(\left(2\,y u_{2,2} \alpha_2 - u_{1,2}\right) C_{1,1} + u_{2,2} u_2 y\Big) {\alpha_1}^2 \\
& + \Big(-\alpha_2 \left(y u_{2,2} \alpha_2 - 2\,u_{1,2}\right) C_{1,1} + 2\,u_2 u_{1,2}\Big) \alpha_1 - u_{1,2} C_{1,1} {\alpha_2}^2 \\
& + \left(-u_{1,1} - {\alpha_2}^2 u + 2\,\alpha_2 u_1\right) u_{2,2} - 2\,\alpha_2 u_2 u_{1,2} + {u_{1,2}}^2\Big) {C_{2,0}}^2 \\
& + \left(\alpha_1 - \alpha_2\right)\Big(C_{1,1} u_{2,2}\left(u + y u_2\right) {\alpha_1}^2 + \Big(\Big(\left(\left(-u - y u_2\right)\alpha_2 - u_1\right) u_{2,2} \\
& + 2\,u_2 u_{1,2}\Big) C_{1,1} - u_{2,2} u u_2\Big) \alpha_1 + \alpha_2 \left(u_1 u_{2,2} - 2\,u_2 u_{1,2}\right) C_{1,1} \\
& - u_2 \left(\left(-2\,u_1 + u \alpha_2\right) u_{2,2} + 2\,u_2 u_{1,2}\right)\Big) C_{2,0} \\
& - C_{1,1} u_2 \left(\alpha_1 - \alpha_2\right)^2 \left(\alpha_1 u u_{2,2} - u_1 u_{2,2} + u_2 u_{1,2}\right)\Big) \\
L_2^{212} = \frac{1}{\left(\alpha_1 - \alpha_2\right) u_{2,2}}\Big(& \left(\alpha_1 - \alpha_2\right)\Big(u_{1,2} {u_2}^2 \left(-\alpha_3 + \alpha_2\right) \\
& + \Big(u_{2,2}\Big(u \alpha_2 + \alpha_3\left(-u + C_{2,0} y\right)\Big) \alpha_1 + \left(\alpha_3 - \alpha_2\right)\left(u_{1,2} C_{2,0} + u_1 u_{2,2}\right)\Big) u_2 \\
& - \left(\left(u \alpha_2 - u_1\right) u_{2,2} \alpha_1 - u_1 u_{2,2} \alpha_2 - {u_{1,2}}^2 + u_{1,1} u_{2,2}\right) C_{2,0}\Big) C_{1,1} \\
& + u_2 \left(\alpha_1 - \alpha_3\right)\left(\left(u \alpha_2 - u_1\right) u_{2,2} + u_2 u_{1,2}\right) C_{2,0}\Big) \\
L_2^{22} = \frac{1}{u_{2,2}}\Big(\Big(\Big(& \left(u_2 \alpha_2 \alpha_1 y + \left(u_1 - u \alpha_2\right)\alpha_1 + \alpha_2 u_1 - u_{1,1}\right) u_{2,2} + {u_{1,2}}^2\Big) C_{2,1} \\
& + u_2\big(\left(\alpha_1 u - u_1\right) u_{2,2} + u_2 u_{1,2}\big)\Big) C_{1,1} + u_2 C_{2,1}\big(\left(u \alpha_2 - u_1\right) u_{2,2} + u_2 u_{1,2}\big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
L_2^{312} = {} & \frac{1}{(\alpha_1-\alpha_2)\,u_{2,2}}\Big(\,(\alpha_1-\alpha_2)\,\Big(u_{2,2}u_2\,(-u+C_{2,0}y)\,{\alpha_1}^2 \\
& +\Big(\big((u_1-u\alpha_2)\,C_{2,0}+u_2\,(u\alpha_2+u_1)\big)\,u_{2,2}-u_2u_{1,2}\,(u_2-C_{2,0})\Big)\,\alpha_1 \\
& +\Big(\big(-u_{1,1}+\alpha_2u_1\big)\,C_{2,0}-u_1u_2\alpha_2\Big)\,u_{2,2}+u_{1,2}\big(\,(-\alpha_2u_2+u_{1,2})\,C_{2,0} \\
& +\alpha_2{u_2}^2)\Big)C_{1,2}+2\,u_2\,\big((u\alpha_2-u_1)\,u_{2,2}+u_2u_{1,2}\big)\,C_{2,0}\Big) \\
L_2^{222} = {} & \frac{1}{u_{2,2}}\Big(-2\,u_{1,2}\,\big(C_{2,0}+1/2\,C_{2,1}\,(\alpha_1-\alpha_2)\big)\,(\alpha_1-\alpha_2)\,{u_2}^2 \\
& +\Big(\big({C_{2,0}}^2{\alpha_1}^2y-(\alpha_1-\alpha_2)\,(\alpha_1u-2\,u_1+u\alpha_2)\,C_{2,0} \\
& -C_{2,1}\,(\alpha_1-\alpha_2)^2\,(u\alpha_2-u_1)\,\big)u_{2,2}+2\,{C_{2,0}}^2u_{1,2}\,(\alpha_1-\alpha_2)\,\Big)u_2 \\
& -{C_{2,0}}^2\,\Big(\big({\alpha_2}^2u+u_{1,1}-2\,\alpha_2u_1\big)\,u_{2,2}-{u_{1,2}}^2\Big)\,\Big) \\
L_{21}^{112} = {} & \frac{1}{\big((-\alpha_2+\alpha_3)\,C_{3,0}+u_2\,(\alpha_1-\alpha_3)\big)\,C_{2,0}-C_{3,0}u_2\,(\alpha_1-\alpha_2)}\Big(\\
& \Big(-(-\alpha_3+\alpha_2)\,\big(\alpha_1u_{2,2}y+u_{1,2}\big)\,C_{3,0}+(\alpha_1-\alpha_3)\,\big(\,(u\alpha_2-u_1)\,u_{2,2} \\
& +u_2u_{1,2})\Big)C_{2,0}-\big((-u_1+\alpha_3u)\,u_{2,2}+u_2u_{1,2}\big)\,(\alpha_1-\alpha_2)\,C_{3,0}\Big) \\
L_{22}^{112} = {} & \frac{1}{(-\alpha_3+\alpha_2)\,u_{2,2}}\Big(\Big(\,(-\alpha_3+\alpha_2)\,\Big(\big(y{\alpha_1}^2u_{2,2}+2\,u_{1,2}(-1/2\,\alpha_2+\alpha_1 \\
& -1/2\,\alpha_3)\big)u_2+\big((-\alpha_3u+u_1)\,\alpha_2-u_{1,1}+\alpha_3u_1\big)\,u_{2,2}+{u_{1,2}}^2\Big)C_{3,0} \\
& -u_2\,\big((u\alpha_2-u_1)\,u_{2,2}+u_2u_{1,2}\big)\,(\alpha_1-\alpha_3)^2\,\Big)C_{2,0}+u_2\big(\,(-u_1+\alpha_3u)\,u_{2,2} \\
& +u_2u_{1,2}\big)\,(\alpha_1-\alpha_2)^2\,C_{3,0}\Big) \\
L_2^{44} = {} & \frac{1}{\big(2\,C_{1,3}C_{1,1}-3\,{C_{1,2}}^2\big)\,u_{2,2}}\Big(\Big(12\,\alpha_1y{C_{1,1}}^3+12\,u{C_{1,1}}^2 \\
& +\Big(\big((6\,u+6\,yu_2)\alpha_1-6\,u_1\big)C_{1,2}+2\,C_{1,3}((-u+yu_2)\,{\alpha_1}^2+2\,u_1\alpha_1 \\
& -u_{1,1})\Big)C_{1,1}+\big((3\,u-3\,yu_2)\,{\alpha_1}^2-6\,u_1\alpha_1+3\,u_{1,1}\big)\,{C_{1,2}}^2+6\,u_2uC_{1,2} \\
& +2\,u_2C_{1,3}\,(-u_1+\alpha_1u)\,\Big)u_{2,2}+2\,u_{1,2}\Big(6\,{C_{1,1}}^3+\big(C_{1,3}u_{1,2}+6\,u_2C_{1,2}\big)\,C_{1,1} \\
& -3/2\,u_{1,2}{C_{1,2}}^2+{u_2}^2C_{1,3}\Big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
L_2^{313} = {} & \frac{1}{\left(2\,C_{2,0}C_{1,1} - C_{1,2}C_{2,0}\alpha_2 + C_{1,2}C_{2,0}\alpha_1 + 2\,{C_{1,1}}^2\alpha_1 - 2\,{C_{1,1}}^2\alpha_2\right)u_{2,2}}\Big(\\
& u_{2,2}\left(2\,{C_{1,1}}^2 + u_2C_{1,2}\right)\left(-u + C_{2,0}y\right){\alpha_1}^3 + \Big(\big((2\,yu_2 + 2\,u - 4C_{2,0}y){C_{1,1}}^2 \\
& + C_{1,2}\left(\left(u + yu_2\right)C_{2,0} - 2\,u_2u\right)\big)u_{2,2}\alpha_2 + u_1\big(4\,{C_{1,1}}^2 \\
& + C_{1,2}(u_2 + C_{2,0})\big)u_{2,2} - u_{1,2}\left(2\,{C_{1,1}}^2 + u_2C_{1,2}\right)\left(u_2 - C_{2,0}\right)\Big){\alpha_1}^2 \\
& + \Big(-2\,\left(1/2\,C_{1,2}u + y{C_{1,1}}^2\right)u_{2,2}\left(u_2 - C_{2,0}\right){\alpha_2}^2 + \Big(\big(-4\,{C_{1,1}}^2u_1 \\
& + \left((-4\,u + 2\,yu_2)\,C_{2,0} + 2\,u_2u\right)C_{1,1} - 2\,u_2C_{1,2}u_1\big)u_{2,2} + 2\,u_{1,2}(2\,{C_{1,1}}^2 \\
& + u_2C_{1,2})\left(u_2 - C_{2,0}\right)\Big)\alpha_2 + u_{2,2}\Big(-2\,u_{1,1}{C_{1,1}}^2 - 2\,u_1\left(u_2 - 2\,C_{2,0}\right)C_{1,1} \\
& - u_{1,1}C_{1,2}C_{2,0}\Big) + 2u_{1,2}\left(u_{1,2}C_{1,1}^2 + u_2\left(u_2 - C_{2,0}\right)C_{1,1} + \tfrac{u_{1,2}C_{1,2}C_{2,0}}{2}\right)\Big)\alpha_1 \\
& - 2\,\left(u_2 - C_{2,0}\right)\Big(\left(-C_{1,2}u_1/2 + uC_{1,1}\right)u_{2,2} + \tfrac{u_{1,2}}{2}\left(2\,{C_{1,1}}^2 + u_2C_{1,2}\right)\Big){\alpha_2}^2 \\
& + \Big(\Big(2\,u_{1,1}{C_{1,1}}^2 + 2\,C_{1,1}u_2u_1 + C_{2,0}\left(u_{1,1}C_{1,2} + 2\,u_2u\right)\Big)u_{2,2} \\
& - 2\left(u_{1,2}{C_{1,1}}^2 + u_2\left(u_2 - C_{2,0}\right)C_{1,1} + 1/2\,u_{1,2}C_{1,2}C_{2,0}\right)u_{1,2}\Big)\alpha_2 \\
& - 2\,C_{2,0}\big(\left(u_1u_2 + u_{1,1}C_{1,1}\right)u_{2,2} - u_{1,2}{u_2}^2 - {u_{1,2}}^2C_{1,1}\big)\Big) \\
L_2^{221} = {} & \frac{1}{\left(\alpha_1 - \alpha_2\right)u_{2,2}}\Big(\left(\alpha_1 - \alpha_2\right)\Big(u_{1,2}{u_2}^2\left(-\alpha_3 + \alpha_2\right) + \big(u_{2,2}\big(u\alpha_2 \\
& + \alpha_3\left(-u + C_{2,0}y\right)\big)\alpha_1 - \left(-\alpha_3 + \alpha_2\right)\left(u_{1,2}C_{2,0} + u_1u_{2,2}\right)\big)u_2 \\
& - \left(\left(u\alpha_2 - u_1\right)u_{2,2}\alpha_1 - u_1u_{2,2}\alpha_2 - {u_{1,2}}^2 + u_{1,1}u_{2,2}\right)C_{2,0}\Big)C_{1,1} \\
& + u_2\left(\alpha_1 - \alpha_3\right)\left(\left(u\alpha_2 - u_1\right)u_{2,2} + u_2u_{1,2}\right)C_{2,0}\Big) \\
L_2^{310} = {} & \frac{1}{u_{2,2}\left(\left(\alpha_1 - \alpha_2\right)C_{1,1} + C_{2,0}\right)C_{1,1}}\Big(\left(\alpha_1 - \alpha_2\right)\Big(u_{2,2}\left(-u + yC_{2,0}\right){\alpha_1}^2 \\
& + \Big(\Big(y\left(u_2 - C_{2,0}\right)\alpha_2 + 2\,u_1\Big)u_{2,2} - u_{1,2}\left(u_2 - C_{2,0}\right)\Big)\alpha_1 - u_{2,2}u_{1,1} \\
& + \Big(\left(u_2 - C_{2,0}\right)\alpha_2 + u_{1,2}\Big)u_{1,2}\Big){C_{1,1}}^2 + \Big(\Big(\Big(\big(\big((-2\,u + yu_2)C_{2,0} \\
& + uu_2\big)\alpha_2 - u_1\left(-2\,C_{2,0} + u_2\right)\big)u_{2,2} + u_2u_{1,2}\left(u_2 - C_{2,0}\right)\Big)\alpha_1 \\
& + \Big(-u\left(u_2 - C_{2,0}\right){\alpha_2}^2 + u_2\alpha_2u_1 - u_{1,1}C_{2,0}\Big)u_{2,2} \\
& - \Big(u_2\left(u_2 - C_{2,0}\right)\alpha_2 - u_{1,2}C_{2,0}\Big)u_{1,2}\Big)C_{1,1} + C_{2,0}\left(\left(u\alpha_2 - u_1\right)u_{2,2} + u_2u_{1,2}\right)u_2\Big)
\end{aligned}
$$

$$
\begin{aligned}
L_2^{220} &= \frac{1}{u_{2,2}}\Bigg(\bigg(\Big({\alpha_1}^2 y C_{2,0} - (\alpha_1-\alpha_2)(\alpha_1 u - 2\,u_1 + u\alpha_2)\Big)u_2 \\
&- C_{2,0}\left(-2\,\alpha_2 u_1 + u_{1,1} + u{\alpha_2}^2\right)\bigg)u_{2,2} - 2\,u_{1,2}\left((\alpha_1-\alpha_2)\,{u_2}^2 - C_{2,0}\,(\alpha_1-\alpha_2)\,u_2 - 1/2\,u_{1,2}C_{2,0}\right)\Bigg) \\
L_{22}^{32} &= \frac{1}{u_2^2}\bigg(\Big(-{u_2}^3 y{\alpha_1}^2 + \left({\alpha_1}^2 u - 2\,u_1\alpha_1 + u_{1,1}\right){u_2}^2 + 2\,u_{1,2}\,(-u_1+\alpha_1 u)\,u_2 \\
&+ u_{2,2}\,(-u_1+\alpha_1 u)^2\Big)C_{1,2} + 2\,{u_2}^3 u\bigg) \\
L_2^{223} &= \frac{1}{\left(-C_{1,1}{\alpha_2}^2 C_{2,1} + {C_{2,0}}^2 - C_{1,1}{\alpha_1}^2 C_{2,1} + 2\,\alpha_2 C_{2,1} C_{1,1}\alpha_1\right)u_{2,2}}\bigg(u_{2,2}\bigg(\\
&- C_{2,1}\,(yu_2-u)\,\alpha_2 + \left(C_{2,0}y - u\right)u_2 - C_{2,1}u_1 - y{C_{2,0}}^2 + uC_{2,0}\bigg)C_{1,1}{\alpha_1}^3 \\
&+ \bigg(2\,C_{2,1}C_{1,1}u_{2,2}(yu_2-u)\alpha_2^2 - 2\,u_{2,2}\bigg(\Big(((C_{2,0}y-u)u_2 + uC_{2,0} - y{C_{2,0}}^2 \\
&- C_{2,1}u_1/2\Big)C_{1,1} + 1/2\,uC_{2,1}u_2\bigg)\alpha_2 + \Big(\left(-C_{2,0}u_1 + u_1u_2 + C_{2,1}u_{1,1}\right)u_{2,2} \\
&- u_{1,2}\left(C_{2,1}u_{1,2} - 2\,C_{2,0}u_2 + {C_{2,0}}^2 + {u_2}^2\right)\Big)C_{1,1} + u_2\big((C_{2,1}u_1 + yC_{2,0}^2 \\
&- uC_{2,0})u_{2,2} - C_{2,1}u_{1,2}u_2)\big)\bigg){\alpha_1}^2 + \bigg(-C_{2,1}C_{1,1}u_{2,2}\,(-u+yu_2)\,{\alpha_2}^3 \\
&+ u_{2,2}\Big(((-u+C_{2,0}y)u_2 + uC_{2,0} - y{C_{2,0}}^2 + C_{2,1}u_1)C_{1,1} + 2\,uC_{2,1}u_2\Big){\alpha_2}^2 \\
&+ \bigg(\bigg((2\,C_{2,0}u_1 - 2\,u_1u_2 - 2\,C_{2,1}u_{1,1})u_{2,2} + 2\,u_{1,2}(C_{2,1}u_{1,2} - 2\,C_{2,0}u_2 \\
&+ {C_{2,0}}^2 + {u_2}^2)\bigg)C_{1,1} - 2\,u_2C_{2,1}\left(-u_2u_{1,2} + u_1u_{2,2}\right)\bigg)\alpha_2 + 2\,C_{2,0}u_2\big(u_1u_{2,2} \\
&- u_{1,2}(u_2 - C_{2,0}))\big)\bigg)\alpha_1 - u_{2,2}C_{2,1}\left(u_2u + u_1C_{1,1}\right){\alpha_2}^3 + \Big(\big((-C_{2,0}u_1 \\
&+ u_1u_2 + C_{2,1}u_{1,1})u_{2,2} - u_{1,2}(C_{2,1}u_{1,2} - 2\,C_{2,0}u_2 + {C_{2,0}}^2 + {u_2}^2)\big)C_{1,1} \\
&+ \Big(\left(uC_{2,0} + C_{2,1}u_1\right)u_2 - {C_{2,0}}^2 u\Big)u_{2,2} - C_{2,1}{u_2}^2 u_{1,2}\Big){\alpha_2}^2 \\
&- 2\,C_{2,0}\left(u_2 - C_{2,0}\right)\left(-u_2u_{1,2} + u_1u_{2,2}\right)\alpha_2 - {C_{2,0}}^2\left(u_{1,1}u_{2,2} - {u_{1,2}}^2\right)\bigg)
\end{aligned}
$$

$$
\begin{aligned}
L_2^{1111} = -u_2\Big(&\big(\left(-\alpha_3+\alpha_2\right)\left(u_{1,2}+\alpha_4 y u_{2,2}\right)C_{3,0}-\left(\alpha_2-\alpha_4\right)(u_{1,2}\\
&+u_{2,2}y\alpha_3)C_{4,0}+\left(-\alpha_4+\alpha_3\right)\left(u_{2,2}u\alpha_2-u_{2,2}u_1+u_2u_{1,2}\right)\big)C_{2,0}\\
&+\big(\left(-\alpha_4+\alpha_3\right)\left(u_{1,2}+\alpha_2u_{2,2}y\right)C_{4,0}-\left(\alpha_2-\alpha_4\right)(\alpha_3u_{2,2}u-u_{2,2}u_1\\
&+u_2u_{1,2})\big)C_{3,0}+C_{4,0}\left(-\alpha_3+\alpha_2\right)\left(u_2u_{1,2}-u_{2,2}u_1+\alpha_4uu_{2,2}\right)\Big){\alpha_1}^2+\\
&\Big(\Big(\left(-\alpha_3+\alpha_2\right)\big(-u_{2,2}y\left(-\alpha_4+\alpha_3\right)\left(\alpha_2-\alpha_4\right)C_{4,0}+(\alpha_3u_{2,2}u-u_{2,2}u_1\\
&+u_2u_{1,2})\alpha_2+\left(u_2u_{1,2}-u_{2,2}u_1\right)\alpha_3-{u_{1,2}}^2+{\alpha_4}^2u_2yu_{2,2}+u_{2,2}u_{1,1}\big)C_{3,0}\\
&-\Big(\left(u_2u_{1,2}-u_{2,2}u_1+\alpha_4uu_{2,2}\right)\alpha_2+{\alpha_3}^2u_2yu_{2,2}+\left(u_2u_{1,2}-u_{2,2}u_1\right)\alpha_4\\
&-{u_{1,2}}^2+u_{2,2}u_{1,1}\Big)\left(\alpha_2-\alpha_4\right)C_{4,0}+u_2\left(-\alpha_4+\alpha_3\right)\left(\alpha_3+\alpha_4\right)\big(u_{2,2}u\alpha_2\\
&-u_{2,2}u_1+u_2u_{1,2}\big)\Big)C_{2,0}+\Big(\big(u_2{\alpha_2}^2yu_{2,2}+\left(u_2u_{1,2}-u_{2,2}u_1+\alpha_4uu_{2,2}\right)\alpha_3\\
&+\left(u_2u_{1,2}-u_{2,2}u_1\right)\alpha_4-{u_{1,2}}^2+u_{2,2}u_{1,1}\big)\left(-\alpha_4+\alpha_3\right)C_{4,0}\\
&-u_2\left(\alpha_2-\alpha_4\right)\left(\alpha_2+\alpha_4\right)\left(\alpha_3u_{2,2}u-u_{2,2}u_1+u_2u_{1,2}\right)\Big)C_{3,0}\\
&+u_2C_{4,0}\left(-\alpha_3+\alpha_2\right)\left(\alpha_2+\alpha_3\right)\left(u_2u_{1,2}-u_{2,2}u_1+\alpha_4uu_{2,2}\right)\Big)\alpha_1\\
&+\Big(-\Big(u_{1,2}\left(-\alpha_4+\alpha_3\right)\left(\alpha_2-\alpha_4\right)C_{4,0}+\alpha_4\big((\alpha_3u_{2,2}u-u_{2,2}u_1\\
&+u_2u_{1,2})\alpha_2+\left(u_2u_{1,2}-u_{2,2}u_1\right)\alpha_3-u_2\alpha_4u_{1,2}+u_{2,2}u_{1,1}\\
&-{u_{1,2}}^2\big)\Big)\left(-\alpha_3+\alpha_2\right)C_{3,0}-\Big(-\left(\alpha_2-\alpha_4\right)\big((u_2u_{1,2}-u_{2,2}u_1\\
&+\alpha_4uu_{2,2})\alpha_2-u_2\alpha_3u_{1,2}+\left(u_2u_{1,2}-u_{2,2}u_1\right)\alpha_4-{u_{1,2}}^2+u_{2,2}u_{1,1}\big)C_{4,0}\\
&+u_2\alpha_4\left(-\alpha_4+\alpha_3\right)\left(u_{2,2}u\alpha_2-u_{2,2}u_1+u_2u_{1,2}\right)\Big)\alpha_3\Big)C_{2,0}\\
&+\Big(\Big(-\big(-u_2u_{1,2}\alpha_2+(u_2u_{1,2}-u_{2,2}u_1+\alpha_4uu_{2,2})\alpha_3+(u_2u_{1,2}\\
&-u_{2,2}u_1)\alpha_4-{u_{1,2}}^2+u_{2,2}u_{1,1}\big)\left(-\alpha_4+\alpha_3\right)C_{4,0}\\
&+u_2\alpha_4\left(\alpha_2-\alpha_4\right)\left(\alpha_3u_{2,2}u-u_{2,2}u_1+u_2u_{1,2}\right)\Big)C_{3,0}\\
&-u_2C_{4,0}\alpha_3\left(-\alpha_3+\alpha_2\right)\left(u_2u_{1,2}-u_{2,2}u_1+\alpha_4uu_{2,2}\right)\Big)\alpha_2
\end{aligned}
$$

$$
\begin{aligned}
M_{21} = &\frac{1}{\left(\frac{d}{dx}\phi_4(x)\right)\left(-a_3(x)\frac{d}{dx}a_2(x)+a_2(x)\frac{d}{dx}a_3(x)\right)u_{2,2}}\Bigg(\\
&-u_2\Bigg(\Bigg(u_{2,2}u\frac{d}{dx}\phi_3(x)+a_3(x)u_{1,2}\Bigg)\frac{d}{dx}a_2(x)+\Bigg(-u_{1,2}a_2(x)\\
&-u\left(\frac{d}{dx}\phi_2(x)\right)u_{2,2}\Bigg)\frac{d}{dx}a_3(x)-\Bigg(-a_3(x)\frac{d}{dx}\phi_2(x)\\
&+a_2(x)\frac{d}{dx}\phi_3(x)\Bigg)\left(-u_2u_{1,2}+u_1u_{2,2}\right)\Bigg)\frac{d^2}{dx^2}\phi_4(x)+\left(\frac{d}{dx}\phi_4(x)\right)\Bigg(\\
&-u_2\left(u_{2,2}u\frac{d}{dx}a_3(x)-a_3(x)\left(-u_2u_{1,2}+u_1u_{2,2}\right)\right)\frac{d^2}{dx^2}\phi_2(x)\\
&+u_2\left(u_{2,2}u\frac{d}{dx}a_2(x)-a_2(x)\left(-u_2u_{1,2}+u_1u_{2,2}\right)\right)\frac{d^2}{dx^2}\phi_3(x)\\
&+\left(u_{2,2}u\frac{d}{dx}a_3(x)-a_3(x)\left(-u_2u_{1,2}+u_1u_{2,2}\right)\right)\frac{d^2}{dx^2}a_2(x)\\
&+\left(-u_{2,2}u\frac{d}{dx}a_2(x)+a_2(x)\left(-u_2u_{1,2}+u_1u_{2,2}\right)\right)\frac{d^2}{dx^2}a_3(x)\\
&-\left(-a_3(x)\frac{d}{dx}a_2(x)+a_2(x)\frac{d}{dx}a_3(x)\right)\left(u_{1,1}u_{2,2}-{u_{1,2}}^2\right)\Bigg)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
M_{22} = &\frac{1}{-a_2(x)\frac{d}{dx}a_3(x)+a_2(x)\left(\frac{d}{dx}\phi_3(x)\right)u_2+a_3(x)\frac{d}{dx}a_2(x)-a_3(x)\left(\frac{d}{dx}\phi_2(x)\right)u_2}\Bigg(\\
&\Bigg(-u_{2,2}u\frac{d}{dx}a_3(x)+a_3(x)\left(-u_2u_{1,2}+u_{2,2}u_1\right)\Bigg)\frac{d}{dx}\phi_2(x)\\
&+\left(u_{2,2}u\frac{d}{dx}a_2(x)-a_2(x)\left(-u_2u_{1,2}+u_{2,2}u_1\right)\right)\frac{d}{dx}\phi_3(x)\\
&-u_{1,2}\left(-a_3(x)\frac{d}{dx}a_2(x)+a_2(x)\frac{d}{dx}a_3(x)\right)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
M_{23} = \Big(&u_{2,2}(((\tfrac{d}{dx}a_4(x))a_3(x) - a_4(x)\tfrac{d}{dx}a_3(x))\tfrac{d}{dx}\phi_2(x) + ((\tfrac{d}{dx}a_2(x))a_4(x) \\
&- (\tfrac{d}{dx}a_4(x))a_2(x))\tfrac{d}{dx}\phi_3(x) + (\tfrac{d}{dx}\phi_4(x))(-a_3(x)\tfrac{d}{dx}a_2(x) \\
&+ a_2(x)\tfrac{d}{dx}a_3(x)))\Big)^{-1}\Big(u_2((-u(\tfrac{d}{dx}a_4(x))u_{2,2} + a_4(x)(-u_2u_{1,2} \\
&+ u_1u_{2,2}))\tfrac{d}{dx}\phi_3(x) + (u_{2,2}u\tfrac{d}{dx}a_3(x) - a_3(x)(-u_2u_{1,2} + u_1u_{2,2}))\tfrac{d}{dx}\phi_4(x) \\
&- u_{1,2}((\tfrac{d}{dx}a_4(x))a_3(x) - a_4(x)\tfrac{d}{dx}a_3(x)))\tfrac{d^2}{dx^2}\phi_2(x) - u_2((-u(\tfrac{d}{dx}a_4(x))u_{2,2} \\
&+ a_4(x)(-u_2u_{1,2} + u_1u_{2,2}))\tfrac{d}{dx}\phi_2(x) + (u_{2,2}u\tfrac{d}{dx}a_2(x) - a_2(x)(-u_2u_{1,2} \\
&+ u_1u_{2,2}))\tfrac{d}{dx}\phi_4(x) - u_{1,2}((\tfrac{d}{dx}a_4(x))a_2(x) - (\tfrac{d}{dx}a_2(x))a_4(x)))\tfrac{d^2}{dx^2}\phi_3(x) \\
&+ u_2((-u_{2,2}u\tfrac{d}{dx}a_3(x) + a_3(x)(-u_2u_{1,2} + u_1u_{2,2}))\tfrac{d}{dx}\phi_2(x) + (u_{2,2}u\tfrac{d}{dx}a_2(x) \\
&- a_2(x)(-u_2u_{1,2} + u_1u_{2,2}))\tfrac{d}{dx}\phi_3(x) - u_{1,2}(-a_3(x)\tfrac{d}{dx}a_2(x) \\
&+ a_2(x)\tfrac{d}{dx}a_3(x)))\tfrac{d^2}{dx^2}\phi_4(x) + ((u(\tfrac{d}{dx}a_4(x))u_{2,2} - a_4(x)(-u_2u_{1,2} \\
&+ u_1u_{2,2}))\tfrac{d}{dx}\phi_3(x) + (-u_{2,2}u\tfrac{d}{dx}a_3(x) + a_3(x)(-u_2u_{1,2} + u_1u_{2,2}))\tfrac{d}{dx}\phi_4(x) \\
&+ u_{1,2}((\tfrac{d}{dx}a_4(x))a_3(x) - a_4(x)\tfrac{d}{dx}a_3(x)))\tfrac{d^2}{dx^2}a_2(x) + ((-u(\tfrac{d}{dx}a_4(x))u_{2,2} \\
&+ a_4(x)(-u_2u_{1,2} + u_1u_{2,2}))\tfrac{d}{dx}\phi_2(x) + (u_{2,2}u\tfrac{d}{dx}a_2(x) - a_2(x)(-u_2u_{1,2} \\
&+ u_1u_{2,2}))\tfrac{d}{dx}\phi_4(x) - u_{1,2}((\tfrac{d}{dx}a_4(x))a_2(x) - (\tfrac{d}{dx}a_2(x))a_4(x)))\tfrac{d^2}{dx^2}a_3(x) \\
&+ ((u_{2,2}u\tfrac{d}{dx}a_3(x) - a_3(x)(-u_2u_{1,2} + u_1u_{2,2}))\tfrac{d}{dx}\phi_2(x) + (-u_{2,2}u\tfrac{d}{dx}a_2(x) \\
&+ a_2(x)(-u_2u_{1,2} + u_1u_{2,2}))\tfrac{d}{dx}\phi_3(x) + u_{1,2}(-a_3(x)\tfrac{d}{dx}a_2(x) \\
&+ a_2(x)\tfrac{d}{dx}a_3(x)))\tfrac{d^2}{dx^2}a_4(x) + (((\tfrac{d}{dx}a_4(x))a_3(x) - a_4(x)\tfrac{d}{dx}a_3(x))\tfrac{d}{dx}\phi_2(x) \\
&+ ((\tfrac{d}{dx}a_2(x))a_4(x) - (\tfrac{d}{dx}a_4(x))a_2(x))\tfrac{d}{dx}\phi_3(x) + (\tfrac{d}{dx}\phi_4(x))(-a_3(x)\tfrac{d}{dx}a_2(x) \\
&+ a_2(x)\tfrac{d}{dx}a_3(x)))(u_{1,1}u_{2,2} - {u_{1,2}}^2)\Big)
\end{aligned}
$$

$$
\begin{aligned}
M_3 = &\frac{1}{u_2{u_{2,2}}^2\left(-a_3(x)\frac{d}{dx}a_2(x) + a_2(x)\frac{d}{dx}a_3(x)\right)}\Big({u_{2,2}}^2\Big(u_{2,2}u\tfrac{d}{dx}a_3(x) \\
&- a_3(x)\left(-u_2u_{1,2} + u_1u_{2,2}\right)\Big)\tfrac{d^2}{dx^2}a_2(x) + \Big(-{u_{2,2}}^3u\tfrac{d}{dx}a_2(x) \\
&+ a_2(x){u_{2,2}}^2\left(-u_2u_{1,2} + u_1u_{2,2}\right)\Big)\tfrac{d^2}{dx^2}a_3(x) + \Big(-{u_{2,2}}^3u_{1,1} \\
&+ \left({u_{1,2}}^2 + u_2u_{1,1,2}\right){u_{2,2}}^2 - 2\,u_2u_{1,2}u_{1,2,2}u_{2,2} + u_2u_{2,2,2}{u_{1,2}}^2\Big)\Big(\\
&- a_3(x)\tfrac{d}{dx}a_2(x) + a_2(x)\tfrac{d}{dx}a_3(x)\Big)\Big)
\end{aligned}
$$

Bibliography

[ALV91] D. V. Alekseevskij, V. V. Lychagin, and A. M. Vinogradov. Basic ideas and concepts of differential geometry. In R. V. Gamkrelidze, editor, *Geometry I*, volume 28 of *Encyclopaedia of Mathematical Sciences*. Springer, 1991.

[GOV93] V.V. Gorbatsevich, A.L. Onishchik, and E.B. Vinberg. *Lie Groups and Lie Algebras I: Foundations of Lie Theory Lie Transformation Groups*, volume 20 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, 1993.

[KL06] Boris Kruglikov and Valentin Lychagin. Invariants of pseudogroup actions: homological methods and finiteness theorem. *International Journal of Geometric Methods in Modern Physics*, 3:1131–1165, 2006.

[KL08] Boris Kruglikov and Valentin Lychagin. *Handbook of Global Analysis*, chapter Geometry of differential equations, pages 725–771. Elsevier, 2008.

[KL13] Boris Kruglikov and Valentin Lychagin. Global lie-tresse theorem. arXiv:1111.5480v2 [math.DG], November 2013.

[Kum75a] A. Kumpera. Invariants différentiels d'un pseudogroup de Lie. I. *Journal of Differential Geometry*, 10(2):289–345, 1975.

[Kum75b] A. Kumpera. Invariants différentiels d'un pseudogroup de Lie. II. *Journal of Differential Geometry*, 10(3):347–416, 1975.

[KV99] I. S. Krasilshchik and A. M. Vinogradov. *Symmetries and Conservation Laws for Differential Equations of Mathematical Physics*. American Mathematical Society, 1999.

[Lie93] Sophus Lie. *Vorlesungen über Continuierliche Gruppen mit Geometrischen und Anderen Anwendungen*. B.G. Teubner, 1893.

[Lie70] Sophus Lie. *Transformationsgruppen.* Chelsea Publishing Company, 1893 (1970).

[Nes06] Maryna Nesterenko. Transformation groups on real plane and their differential invariants. *International Journal of Mathematics and Mathematical Sciences*, vol. 2006, Article ID 17410, 17 pages, 2006.

[Olv96] Peter J. Olver. *Equivalence, Invariants, and Symmetry.* Cambridge University Press, 1996.

[PV94] V. L. Popov and E. B. Vinberg. Invariant theory. In A. N. Parshin and I. R. Shafarevich, editors, *Algebraic Geometry IV*, volume 55 of *Encyclopaedia of Mathematical Sciences*, pages 123–278, 1994.

[Ros56] Maxwell Rosenlicht. Some basic theorems on algebraic groups. *American Journal of Mathematics*, 78(2):401–443, April 1956.

[Tre94] A. Tresse. Sur les invariants différentiels des groupes continus de transformations. *Acta Mathematica*, 18(1):1–88, 1894.