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# Towers of Betti Numbers of Matroids and Weight Distribution of Linear Codes and their Duals

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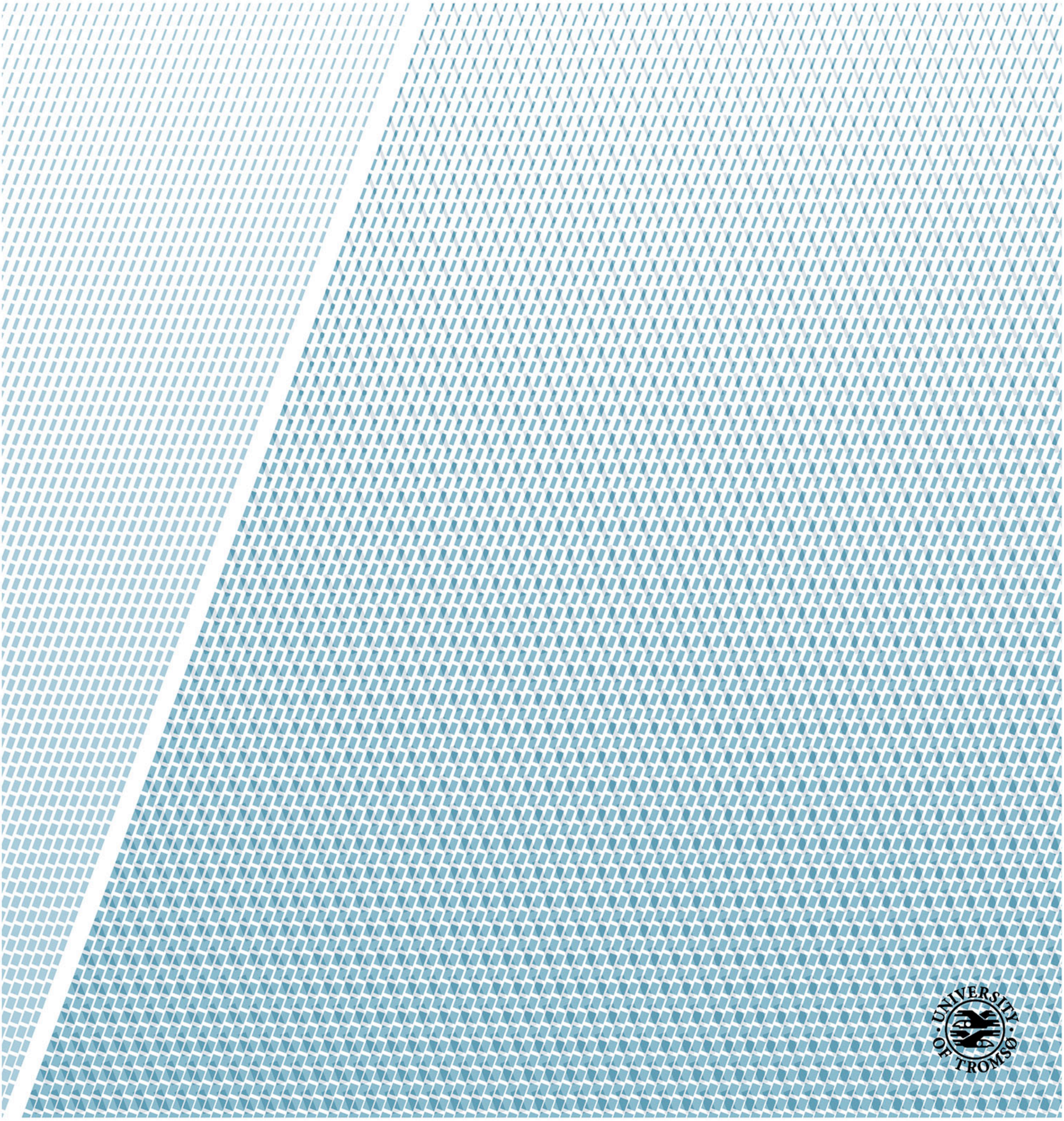

## Abstract

The main notion behind the study of matroids is linear dependence. In this thesis, we give a survey of the concepts and properties of linear error-correcting codes over finite fields being dependent only on the matroids derived from these codes. In particular, the weight distributions of linear codes, and their extensions, over bigger fields are only dependent on the $\mathbb{N}$-graded Betti numbers of these matroids and their so-called elongations. We will use this fact to find the weight distributions for some important codes as constant weight codes and Hamming codes. In addition, the connection between the Betti tower of a matroid and its dual tower will be studied for general matroids.

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Finally, to my +1, who would come with me to the other side of the world.

To Uma,
who does not care
about matroids at all.

# Contents

# Introduction

In this thesis we will look into some deep connections that exist between coding theory, combinatorics, algebraic topology and homological algebra.

Matroid theory was introduced in 1935 by Hassler Whitney, an American mathematician who dedicated himself to graph theory, differential geometry, cohomological theory and algebraic topology. It was also independently studied by Takeo Nakasama, but his work was not recognized until years after he died.

We will begin with a compilation of elementary definitions and properties that codes satisfy. The main concepts for codes introduced in Chapter 1 are the Hamming distance, the weight of a codeword, what a linear code is and the equivalence of linear codes. When dualizing a linear code, we obtain an important result about weight hierarchies: Wei's duality. After coding theory, we will give a introductory overview of matroid theory. There are many equivalent ways to define a matroid, but the principal one is about independence. Afterwards, we will show the relation between codes and matroids, a matroid associated to a linear code will be defined by the parity check matrix of the code.

In Chapter 2 we will establish the main base so that we can work with matroids as topological objects (simplicial complexes). A very basic background in monomial ideals will be given so that it can be applied to matroids. A matroid, seen as a simplicial complex, gives rise to a monomial ideal of a polynomial ring, the Stanley-Reisner ideal, which is the ideal generated by squarefree monomials corresponding to non-faces of the simplicial complex. As a result, the Stanley-Reisner ideal has minimal free resolutions, and this defines its Betti numbers.

In Chapter 3 we will extend the definition of weight enumerator of a code $\mathcal{C}$, given in Chapter 1, to the extended code $\mathcal{C} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^r}$. It has recently been shown that the extended weight enumerator of a code can be entirely expressed with the help of Betti numbers of the matroid and its elongations. The formula given in [18] for computing the coefficients for the GWP has special importance in this thesis since it will be used frequently in chapters 4 and 5. MacWilliams identity will be mentioned here as well, since it will be relevant for computing the dual Betti numbers.

Throughout Chapter 4 the focus will be on constant weight codes. They have pure resolutions, so the Betti numbers for its Stanley-Reisner ring satisfy certain properties. By using some results given in [5] and the Herzog-Kühl equations, we will get a general formula for computing all the Betti tables for a matroid $\mathcal{M}$ associated to a constant weight code and its elongations just from its parameters. Furthermore, we will get a general formula from the GWP for computing all the Betti tables of $\mathcal{M}$ and its elongations and we will give the last columns and some more entries for the duals of the Betti tables.

In Chapter 5, we will leave aside the condition for matroids coming from constant weight codes and take general matroids. We will try to obtain as much information as possible from the Betti tower of $\mathcal{M}$ to achieve the Betti numbers of the dual tower. This will result in a formula for the last Betti number of each dual table. If the GWP are given, we will always be able to compute the Betti numbers for the last complete column of each dual Betti table, in addition to the first and the next-to-last entries as well as the position of certain zeroes. Finally, an example will be given to illustrate how this process works.

# Chapter 1

# Basic definitions

In this first chapter we will introduce some basic concepts so that we can later work with matroids as representants for linear codes. We will define here what is a code, its length, its weight, its minimun distance, etc. Afterwards, we will define what a parity check matrix is, by dualizing, and we will formulate the Wei's duality theorem. Next, we will go through the basic definitions for matroids (via independent sets, bases, circuits and rank function), providing some examples. Finally, we will end up showing the relation between codes and matroids.

## 1.1 Coding theory

### 1.1.1 Elementary definitions

In this section we will provide a really basic overview of what is a code, its parameters and its basic properties.

**Definition 1.1.1.** An *alphabet* is a finite set of symbols. We will denote it by $\Sigma$.

**Definition 1.1.2.** A *code* $\mathcal{C}$ is a subset of all posible words.

$$C \subset \bigcup_{n=0}^{\infty} \Sigma^n = \{\{\varnothing\} \cup \Sigma \cup \Sigma^2 \cup \Sigma^3 \cup \ldots\}$$

**Definition 1.1.3.** Let q be an integer. A *q-ary block code* $\mathcal{C}$ is a set of $r$-tuples $(a_1,...,a_r)$, where $a_i \in \Sigma$, alphabet of cardinality $q$. An element in this set is called a codeword.

**Definition 1.1.4.** The *length* of a block code is the length of any codeword from the code.

**Example 1.1.1.** (Difference between code and block code) The set of all English words is a 26-ary code, but not a block code, since all words do not have the same length. The set of all Norwegian ID numbers is a 10-ary block code of length 11.

From now on we will just work with block codes.

**Definition 1.1.5.** The *Hamming distance* between two codewords $x = (x_1,...,x_n)$ and $y = (y_1,...y_n)$ is:

$$d(x,y) = \#\{i, x_i \neq y_i\}$$

If the alphabet is $\mathbb{F}_q$, then we can define the *weight* of a codeword $x$ as :

$$\mathrm{wt}(x) = \#\{i \ ; \ x_i \neq 0\}$$

**Example 1.1.2.** If $x = (10111), y = (01101)$ the Hamming distance between them is $d(x,y) = 3$.

**Definition 1.1.6.** The *minimun distance* of a code $\mathcal{C}$ is

$$d = \min\{d(x,y); \ x,y \in \mathcal{C}, x \neq y\}$$

**Definition 1.1.7.** Two different block codes of length $n$ over the alphabet $\Sigma$ are *equivalent* if we can obtain one from the other by using the following operations:

1. Permutation of the positions of the code.
2. Permutation of the symbols appearing in a fixed position.

***Notation.*** A $(n, M, d)$ code is a code of length $n$, size (number of codewords) $M$ and minimun Hamming distance $d$.

**Theorem 1.1.1.** *A $q$-ary $(n, M, d)$-code satisfies*

$$M\left(\sum_{i=0}^{t}\binom{n}{i}(q-1)^i\right) \leq q^n$$

*where $t = \lfloor \frac{d-1}{2} \rfloor$.*

*Proof.* [7, Theorem 2.16] □

**Definition 1.1.8.** A $q$-ary code for which there is equality in Theorem 1.1.1 is called *perfect*.

**Definition 1.1.9.** A *linear code* of length $n$ and rank $k$ is a linear subspace with dimension $k$ of the vector space $\mathbb{F}_q^n$, where $\mathbb{F}_q$ is the finite field with $q$ elements.

***Remark.*** In a linear code, any linear combination of codewords is again a codeword.

***Remark.*** The zero vector is necessarily a codeword for any linear code.

From now on we will just work with linear codes.

**Definition 1.1.10.** The *support* of a codeword $x$ is

$$Supp(x) = \{i, x_i \neq 0\}$$

The support of a set of codewords, $S$, is the union of the supports of all codewords in $S$.

$$Supp(S) = \bigcup_{x \in S} Supp(x) = \{i, \exists x \in S, x_i \neq 0\}$$

***Property.***

$$d = \min\{\#Supp(\mathcal{D}) \mid \mathcal{D} \text{ subcode of } \mathcal{C} \text{ of dimension } 1\}$$

*Proof.* [20, Proposition 2 Section 3.1] □

**Definition 1.1.11.** The *generalized Hamming weights* of a $[n, k]$-code $\mathcal{C}$ are:

$$d_i = \min\{\#Supp(\mathcal{D}) \mid \mathcal{D} \text{ subcode of } \mathcal{C} \text{ of dimension } i\}$$

where $1 \leq i \leq k$. The sequence $(d_1, ..., d_k)$ is called the *weight hierarchy* of the code.

***Notation.*** $[n, k]_q$ describes the code: length $n$, dimension $k$ and alphabet size $q$.

**Definition 1.1.12.** A *generator matrix* of a $[n, k]_q$-code $\mathcal{C}$ is a $k \times n$ matrix over $\mathbb{F}_q$ whose rows form a basis of $\mathcal{C}$.

**Definition 1.1.13.** Two different linear block codes of length $n$ over a field $\mathbb{F}_q$ are *equivalent* if we can obtain the one from the other by using the following operations:

1. Permutation of the positions of the code.
2. Multiplication of the symbols appearing in a fixed position by a non-zero scalar.

**Example 1.1.3.**

Let

$$B = \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 & 2 \end{bmatrix}$$

be a generator matrix of a linear code over $\mathbb{F}_3$.

If we multiply the fourth column by 2, we will obtain an equivalent code.

$$\left[\begin{array}{ccc|c|c} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 & 2 \end{array}\right] \xrightarrow{\times 2} \left[\begin{array}{ccc|c|c} 1 & 2 & 0 & 2 & 2 \\ 0 & 0 & 1 & 4 & 2 \end{array}\right] \overset{\mathbb{F}_3}{=} \begin{bmatrix} 1 & 2 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$$

***Remark.*** If two different linear codes are equivalent in the sense of Definition 1.1.13, they are, obviously, also equivalent in the sense of Definition 1.1.7.

In the sequel, equivalence of linear codes will always be in the sense of Definition 1.1.13.

**Proposition 1.1.2.** *Two equivalent linear codes have the same parameters: length, cardinality and minimal distance.*

*Proof.* [20, Proposition 7] □

**Theorem 1.1.3.** *Let $G$ be a generator matrix of an $[n, k]_q$-code. Then we can find an equivalent linear code with generator matrix of the form*

$$[\ I_k \mid A\ ]$$

*where $I_k$ is the $k \times k$ identity matrix and $A$ is a $k \times (n - k)$ matrix.*

*Proof.* [7, Theorem 5.5] □

**Definition 1.1.14.** A generator matrix of the form

$$[\, I_k \,\big|\, A \,]$$

where $I_k$ is the $k \times k$ identity matrix and A is a $k \times (n-k)$ matrix, is called generator matrix under *standard form*.

**Example 1.1.4.** Let us take a code $\mathcal{C}$ over $\mathbb{F}_2$ generated by the codewords $x = \{0,1,1,0,1\}$ and $y = \{1,0,1,1,1\}$. A generator matrix of $\mathcal{C}$ is:

$$G = \begin{bmatrix} 0\,1\,1\,0\,1 \\ 1\,0\,1\,1\,1 \end{bmatrix}$$

We can swap the rows and obtain a generator matrix in standard form:

$$G' = \begin{bmatrix} 1\,0\,1\,1\,1 \\ 0\,1\,1\,0\,1 \end{bmatrix}$$

***Remark***. Generator matrices under standard form are not unique for equivalent codes.

**Example 1.1.5.** In $\mathbb{F}_3$, $G_1$ is a generator matrix for a code and $G_2$ a generator matrix for an equivalent code. Then, their generator matrices under standard form do not need to coincide:

$$\begin{tikzcd}
G_1 = \begin{bmatrix} 0 & 1 & 2 & 2 & 1 \\ 2 & 2 & 2 & 2 & 1 \end{bmatrix} \arrow[d] \arrow[r, phantom, "\sim"] & G_2 = \begin{bmatrix} 1 & 2 & 2 & 2 & 1 \\ 0 & 1 & 2 & 2 & 1 \end{bmatrix} \arrow[d] \\
G_1' = \begin{bmatrix} 1 & 0 & 2 & 2 & 1 \\ 0 & 1 & 2 & 2 & 1 \end{bmatrix} \arrow[r, phantom, "\neq"] & G_2' = \begin{bmatrix} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 2 & 2 & 1 \end{bmatrix}
\end{tikzcd}$$

## 1.1.2 Duality

Let us first introduce some algebraic concepts that will provide the base for the definition of dual objects related with coding theory.

**Definition 1.1.15.** Let $u = (u_1, ..., u_n), v = (v_1, ..., v_n) \in \mathbb{F}_q^n$ be two vectors. Then the *inner product* is

$$u \cdot v = \sum_{i=1}^{n} u_i v_i$$

The inner product is a bilinear form, that is, it is linear on each component of the cartesian product (bilinear), and its target is the set of scalars of the vector space (form).

**Definition 1.1.16.** A bilinear form $f : V \times V \longrightarrow \mathbb{K}$ is said to be:

- *Symmetric* if $f(x, y) = f(y, x) \ \ \forall x, y \in V$,
- *Nondegenerate* if $f(x, y) = 0 \, \forall y \in V \Rightarrow x = 0$ and $f(x, y) = 0 \, \forall x \in V \Rightarrow y = 0$.

**Definition 1.1.17.** Let $V$ be a $\mathbb{K}$ vector space, and $\phi : V \times V \longrightarrow \mathbb{K}$ be a symmetric bilinear form. Let $W \subset V$ be a subspace. We define the *orthogonal* of $W$ as:

$$W^{\perp} = \{v \in V \ ; \ \phi(v, w) = 0 \, \forall w \in W\}$$

**Theorem 1.1.4.** *Let $V$ be a $\mathbb{K}$ vector space, and $\phi : V \times V \longrightarrow \mathbb{K}$ be a symmetric bilinear form. Let $W \subset V$ be a subspace. Then $W^{\perp}$ is a vector subspace of $V$ . Moreover, if $V$ is finite dimensional and $\phi$ is nondegenerate, then $W^{\perp}$ is finite dimensional and*

$$dim_{\mathbb{K}}(W^{\perp}) + dim_{\mathbb{K}}(W) = dim_{\mathbb{K}}(V)$$

*Proof.* [13, Theorem 2.3] □

***Remark.*** Even if $\dim_{\mathbb{K}}(W^\perp)+\dim_{\mathbb{K}}(W) = \dim_{\mathbb{K}}(V)$, we do not usually have $W \oplus W^\perp = V$.

Let $\mathcal{C}$ be a $[n,k]_q$ code with generator matrix $G$. Let $\mathcal{C}^\perp$ be the orthogonal of the code for the usual inner product. Since the inner product is a nondegenerate symmetric bilinear form, we know that $\mathcal{C}^\perp$ is a $[n, n-k]_q$ code. A generator matrix $H$ of $\mathcal{C}^\perp$ is therefore a $(n-k)\times n$ matrix with entries in $\mathbb{F}_q$, and whose rows are a basis of $\mathcal{C}^\perp$.

**Definition 1.1.18.** Let $\mathcal{C}$ be a $[n,k]_q$ linear code. Then, the $[n,n-k]_q$ linear code $\mathcal{C}^\perp$ is called the *dual code*.

**Definition 1.1.19.** A *parity check matrix* of a linear code $\mathcal{C}$ is a generator matrix of the dual code $\mathcal{C}^\perp$.

***Remark.*** It describes the linear relations that the components of a codeword from $\mathcal{C}$ must satisfy, since the rows of a parity check matrix are the coefficients of the parity check equations, defining linear combinations of codewords. It can be used in decoding algorithms and also to decide if a particular vector is a codeword: $x$ is a codeword in $\mathcal{C}$ iff $Hx^t = 0$.

**Definition 1.1.20.** A parity check matrix of the form

$$\left[\begin{array}{c|c} B & I_{n-k} \end{array}\right]$$

is said to be under *standard form*.

**Theorem 1.1.5.** *Let $\mathcal{C}$ be a linear $[n,k]_q$ code given by a generator matrix $G$ under standard form, say*

$$G = \left[\ I_k \ \middle|\ A\ \right]$$

*Then, a parity check matrix for $\mathcal{C}$ is given by*

$$H = \left[\ -A^t \ \middle|\ I_{n-k}\ \right]$$

*Proof.* [7, Theorem 7.6] □

**Proposition 1.1.6.** *If $G$,$H$ are a generator matrix and a parity check matrix for $\mathcal{C}$ respectively, then they are a parity check matrix and a generator matrix for $\mathcal{C}^{\perp}$ respectively.*

*Proof.* It is clear from the definition of parity check matrix. Since $H$ generates the dual code, being a parity check matrix for $\mathcal{C}$, then, a parity check matrix for $\mathcal{C}^{\perp}$ will generate $(\mathcal{C}^{\perp})^{\perp} = \mathcal{C}$. □

**Example 1.1.6.** Let us take $G$ a generator matrix in standard form over $\mathbb{F}_2$ as in Example 1.1.4

$$G = \begin{bmatrix} 1\,0\,1\,1\,1 \\ 0\,1\,1\,0\,1 \end{bmatrix}$$

From the standard generator matrix we can obtain a parity check matrix:

$$H = \begin{bmatrix} 1\,1\,1\,0\,0 \\ 1\,0\,0\,1\,0 \\ 1\,1\,0\,0\,1 \end{bmatrix}$$

That is a generator matrix for the dual code of $\mathcal{C}$.

**Theorem 1.1.7** (Wei's duality). *Let $\mathcal{C}$ be a $[n,k]_q$ linear code, and $\mathcal{C}^\perp$ its dual code. Let $d_1 < \ldots < d_k$ and $e_1 < \ldots < e_{n-k}$ the weight hierarchies of $\mathcal{C}$ and $\mathcal{C}^\perp$ respectively. Then,*

$$\{d_1, ..., d_k, n+1-e_1, ..., n+1-e_{n-k}\} = \{1, ..., n\}$$

*Proof.* [21, Theorem 3] □

### 1.1.3 Weight enumerator for a linear code

**Definition 1.1.21.** Let $\mathcal{C}$ be a linear $[n,k]$-code over $\mathbb{F}_q$. The *weight enumerator* of $\mathcal{C}$ is defined as the following polynomial:

$$W_{\mathcal{C}}(Z) = \sum_{j=0}^{n} A_{\mathcal{C},j} Z^j$$

where $A_{\mathcal{C},j}$ denotes the number of codewords in $\mathcal{C}$ of weight $j$.

***Remark.*** Another way of express $W_{\mathcal{C}}(Z)$ is

$$W_{\mathcal{C}}(Z) = \sum_{x \in \mathcal{C}} Z^{wt(x)}$$

**Example 1.1.7.** Let $\mathcal{C}$ be the binary even-weight linear code of length 3, i.e $\mathcal{C} = \{000, 011, 101, 110\}$.
Then, $W_{\mathcal{C}}(Z) = 1 + 3Z^2$

**Definition 1.1.22.** The *homogeneous weight enumerator* of $\mathcal{C}$ is defined as:

$$W_{\mathcal{C}}(X,Y) = \sum_{j=0}^{n} A_{\mathcal{C},j} X^{n-j} Y^j$$

***Remark***. Another way of express $W_{\mathcal{C}}(X,Y)$ is

$$W_{\mathcal{C}}(X,Y)=\sum_{x\in\mathcal{C}} X^{n-wt(x)}Y^{wt(x)}$$

***Remark***. Note that $W_{\mathcal{C}}(Z)$ and $W_{\mathcal{C}}(X,Y)$ are equivalent in representing weight information. They determine each other uniquely by the following equations:

$$W_{\mathcal{C}}(Z)=W_{\mathcal{C}}(1,Z)$$
$$W_{\mathcal{C}}(X,Y)=X^nW_{\mathcal{C}}(X^{-1}Y)$$

Also note that $W_{\mathcal{C}}(X,Y)$ is not the ordinary homogeneization of $W_{\mathcal{C}}(Z)$ as usually described. In the case of the code from Example 1.1.7, we obtain $W_{\mathcal{C}}(X,Y)=X^3+3XY^2$, but if we just homogenize as usual, we obtain $W^h_{\mathcal{C}}(Z,T)=T^2+3Z^2$, which is different from $W_{\mathcal{C}}(Z)$ .

**Example 1.1.8.** Let us compute the weight enumerator for Example 1.1.4

$$G=\begin{bmatrix}1\,0\,1\,1\,1\\0\,1\,1\,0\,1\end{bmatrix}$$

The generators of the code $\mathcal{C}$ over $\mathbb{F}_2$ are the words $x=\{1,0,1,1,1\}$ and $y=\{0,1,1,0,1\}$, therefore all the words $w\in\mathcal{C}$ can be written as $w=\alpha x+\beta y$ where $\alpha,\beta\in\mathbb{F}_2$.

| $\alpha$ | $\beta$ | $w=\alpha x+\beta y$ | $wt(w)$ |
|---|---|---|---|
| 0 | 0 | $\{0,0,0,0,0\}$ | 0 |
| 0 | 1 | $\{0,1,1,0,1\}$ | 3 |
| 1 | 0 | $\{1,0,1,1,1\}$ | 4 |
| 1 | 1 | $\{1,1,0,1,0\}$ | 3 |

Then,

$$\begin{aligned} W_{\mathcal{C}}(Z) &= \sum_{x\in\mathcal{C}} Z^{wt(x)} = 1 + 2Z^3 + Z^4 \\ W_{\mathcal{C}}(X,Y) &= \sum_{j=0}^{n} A_{\mathcal{C},j} X^{n-j} Y^j = X^5 + 2X^2Y^3 + XY^4 \\ W_{\mathcal{C}}(Z) &= W_{\mathcal{C}}(1,Z) = 1 + 2Z^3 + Z^4 \end{aligned}$$

It has important applications in the theory of error-correcting codes. Knowledge of the weight enumerator of a code enables us to calculate the probabilty of having undetected errors, as shown in [7, Theorem 6.14].

## 1.2 Matroids

### 1.2.1 Equivalent definitions

There are many equivalent ways to define a matroid. In this section some alternative ways to define them will be given among some properties.

#### 1.2.1.1 Via independent sets

**Definition 1.2.1.** A *finite matroid* $\mathcal{M}$ is a pair $(E, \mathcal{I})$, where $E$ is a finite set (called the ground set) and $\mathcal{I}$ is a family of subsets of $E$ (called the independent sets) satisfying the following axioms:

$(I_1)$ $\emptyset \in \mathcal{I}$.

$(I_2)$ If $I_1 \in \mathcal{I}$ and $I_2 \subset I_1$, then $I_2 \in \mathcal{I}$.

$(I_3)$ If $I_1$, $I_2 \in \mathcal{I}$ with $|I_1| < |I_2|$ then there exists $x \in I_2 \backslash I_1$ such that $I_1 \cup \{x\} \in \mathcal{I}$.

**Example 1.2.1.** Let $\mathcal{I}(\mathcal{M})$ be

$$\begin{aligned}\mathcal{I}(\mathcal{M}) &= \{\{\emptyset\},\{1\},\{2\},\{3\},\{4\},\{5\},\{1,2\},\{1,3\},\{1,4\},\{1,5\},\\ &\quad \{2,3\},\{2,4\},\{2,5\},\{3,4\},\{3,5\},\{4,5\},\{1,2,3\},\{1,2,5\},\\ &\quad \{1,3,4\},\{1,3,5\},\{1,4,5\},\{2,3,4\},\{2,4,5\},\{3,4,5\}\}\end{aligned}$$

It is straightforward to check that this is a matroid. We will see in the sequel that this is the matroid associated to the code defined in Example 1.1.4.

**Definition 1.2.2.** A subset of the ground set $E$ that is not independent is called *dependent*.

#### 1.2.1.2 Via bases

**Definition 1.2.3.** A *basis* for a matroid is a maximal independent set for inclusion, i.e., an independent set which becomes dependent on adding any element of $E$. The set of bases will be denoted as $\mathcal{B}$.

**Example 1.2.2.** The set of bases for the matroid given in Example 1.2.1 is

$\mathcal{B} = \{\{1,2,3\},\{1,2,5\},\{1,3,4\},\{1,3,5\},\{1,4,5\},\{2,3,4\},\{2,4,5\},\{3,4,5\}\}$

**Proposition 1.2.1.** *All the bases of a matroid have the same cardinality.*

*Proof.* [17, Lemma 1.2.4] □

**Proposition 1.2.2.** *Let $\mathcal{B} \subset 2^E$ be a set of bases. $\mathcal{B}$ satisfies the following properties:*

*($B_1$)* $\mathcal{B} \neq \emptyset$.

*($B_2$)* *(Base change)* $\forall B_1, B_2 \in \mathcal{B}$, $\forall x \in B_2 \backslash B_1$, $\exists y \in B_1 \backslash B_2$ *such that* $(B_2 \cup \{y\}) \backslash \{x\} \in \mathcal{B}$ .

*Proof.* [17, Lemma 1.2.2] □

**Theorem 1.2.3.** {*1,2,5*} *Let $\mathcal{B}$ be a set of subsets of $E$ satisfying* $(B_1)$ *and* $(B_2)$*. Let*

$$\mathcal{I} = \{\sigma \subset B \mid B \in \mathcal{B}\}$$

*Then, $\mathcal{M}(\mathcal{B}) = (E, \mathcal{I})$ is a matroid, whose set of bases is $\mathcal{B}$.*

*Proof.* [17, Theorem 1.2.3] □

### 1.2.1.3 Via circuits

**Definition 1.2.4.** A *circuit* of a matroid $\mathcal{M}$ is a minimal dependent subset of $E$ (for inclusion), i.e, a dependent set whose proper subsets are all independent. The set of circuits will be denoted as $\mathcal{C}$.

**Proposition 1.2.4.** *The set of circuits of a matroid satisfy the following properties:*

*($C_1$)* $\emptyset \notin \mathcal{C}$.

*($C_2$)* *If $C_1, C_2 \in \mathcal{C}$ with $C_1 \subset C_2$, then $C_1 = C_2$.*

*($C_3$)* (*Global elimination axiom*) *If $C_1, C_2 \in \mathcal{C}$ are distinct and $C_1 \cap C_2 \neq \emptyset$, then $\forall e \in C_1 \cap C_2$ $\exists C_3 \in \mathcal{C}$ such that $C_3 \subset (C_1 \cup C_2) \backslash \{e\}$ .*

*Proof.* [17, Lemma 1.1.3] □

**Definition 1.2.5.** An element that does not belong to any independent set is called a *loop*, i.e, if $\{e\} \in \mathcal{C}$ then $e$ is a loop.

**Theorem 1.2.5.** *A matroid over the ground set $E$ is entirely defined by its set of bases, or by its set of circuits. Namely we have:*

$$\mathcal{I} = \{\sigma \subset E \ ; \ \exists B \in \mathcal{B}, \ \sigma \subset B\}$$

*and*

$$\mathcal{I} = \{\sigma \subset E, \ \forall \tau \in \mathcal{C}, \ \tau \not\subset \sigma\}$$

*Proof.* [17, Theorem 1.1.4] □

***Remark.*** While the bases of a matroid have all the same cardinality, circuits might not.

**Example 1.2.3.** The set of circuits for the matroid given in Example 1.2.1 is

$$\mathcal{C}(\mathcal{M}) = \{\{1,2,4\},\{1,3,4,5\},\{2,3,5\}\}$$

**Proposition 1.2.6.** *Let $E$ be a finite set and $\mathcal{C}$ a set of subsets of $E$. Let $(C_3')$ be the following property:*

*($C_3'$)* *(Strong circuit elimination axiom) If $C_1, C_2 \in \mathcal{C}$ are distinct and $C_1 \cap C_2 \neq \emptyset$, then $\forall e \in C_1 \cap C_2, \ \forall f \in C_1 \backslash C_2, \ \exists C_3 \in \mathcal{C}$ such that $f \in C_3 \subset (C_1 \cup C_2) \backslash \{e\}$ .*

*Then, the properties $(C_1), (C_2), (C_3)$ are equivalent to the properties $(C_1), (C_2), (C_3')$.*

*Proof.* [17, Proposition 1.4.11 + Corollary 1.4.12] □

**Theorem 1.2.7.** *Let $E$ be a finite set, and $\mathcal{C} \subset 2^E$ satisfy the axioms $(C_1), (C_2), (C_3)$. Let*

$$\mathcal{I} = \{\sigma \subset E, \ \nexists \tau \in \mathcal{C}, \ \tau \subset \sigma\}$$

*Then $(E, \mathcal{I})$ is a matroid whose set of circuits is $\mathcal{C}$.*

*Proof.* [17, Theorem 1.1.4] □

#### 1.2.1.4 Via rank function

**Definition 1.2.6.** Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid.

The *rank function* of the matroid $\mathcal{M}$ is :

$$\begin{array}{cccc} r: & 2^E & \longrightarrow & \mathbb{N} \\ & \sigma & \longmapsto & r(\sigma) = \max\{\ |I|\ \text{ s.t }\ I \subset \sigma,\ I \in \mathcal{I}\} \end{array}$$

The *nullity function* of the matroid $\mathcal{M}$ is :

$$\begin{array}{cccc} n: & 2^E & \longrightarrow & \mathbb{N} \\ & \sigma & \longmapsto & n(\sigma) = |\sigma| - r(\sigma) \end{array}$$

**Proposition 1.2.8.** *Let $\sigma \subset E$, then*

$$r(\sigma) = \max\{\ |\sigma \cap B|\ ;\ B \in \mathcal{B}\}$$

*Proof.* [20, Proposition16] □

**Definition 1.2.7.** The rank of a matroid $\mathcal{M}$ over $E$ is defined as

$$r(\mathcal{M}) = r(E)$$

**Example 1.2.4.** Let us take Example 1.2.1 and calculate the rank and nullity for some sets:

| $\sigma$ | $\mathbf{r}(\sigma)$ | $\mathbf{n}(\sigma)$ |
|---|---|---|
| $\emptyset$ | 0 | 0 |
| {1} | 1 | 0 |
| {1,2} | 2 | 0 |
| {1,2,4} | 2 | 1 |
| {1,2,3} | 3 | 0 |
| {1,2,3,4} | 3 | 1 |
| {1,2,3,4,5} | 3 | 2 |

**Proposition 1.2.9.** *The rank function of a matroid $\mathcal{M} = (E,\mathcal{I})$ satisfies the following properties:*

*($R_1$)* $r(\emptyset) = 0$.

*($R_2$) If* $\sigma \subset E,\ x \in E$, *then* $r(\sigma) \leq r(\sigma \cup \{x\}) \leq r(\sigma) + 1$.

*($R_3$) If* $\sigma \subset E,\ x, y \in E$ *are such that* $r(\sigma \cup \{x\}) = r(\sigma \cup \{y\}) = r(\sigma)$ *then* $r(\sigma \cup \{x, y\}) = r(\sigma)$.

*Proof.* [17, Theorem 1.4.14] □

**Proposition 1.2.10.** *Let* $r :\ 2^E \longrightarrow \mathbb{N}$ *be a function. Then, the three following properties are equivalent to the ones that the rank function satisfies* ( $(R_1), (R_2), (R_3)$ ).

*($R'_1$)* $0 \leq r(\sigma) \leq |\sigma|$.

*($R'_2$) If* $\sigma \subset \tau \subset E,\ r(\sigma) \leq r(\tau)$.

*($R'_3$) If* $\sigma, \tau \subset E,\ r(\sigma \cap \tau) + r(\sigma \cup \tau) \leq r(\sigma) + r(\tau)$.

*Proof.* [17, Lemma 1.3.1] □

**Theorem 1.2.11** (Matroid Via rank function.)**.** *Let $E$ be a finite set and* $r :\ 2^E \longrightarrow \mathbb{N}$ *a function satisfying* $((R1), (R2), (R3))$ *(or alternatively* $((R'_1), (R'_2), (R'_3))$*, and*

$$\mathcal{I} = \{I \in 2^E,\ r(I) = |I|\}$$

*Then, $(E,\mathcal{I})$ is a matroid with set of bases*

$$\mathcal{B} = \{I \in 2^E,\ r(E) = r(I) = |I|\}$$

*and rank $r$.*

*Proof.* [17, Theorem 1.3.2] □

**Corollary.** *If $\mathcal{M} = (E, \mathcal{I})$ is a matroid with rank function $r$, $\sigma \subset E$ is dependent if and only if*

$$r(\sigma) \leq |\sigma| - 1$$

*In particular, if $\sigma$ is a circuit, then*

$$r(\sigma) = |\sigma| - 1$$

*Proof.* [17, Proposition 1.3.5] □

### 1.2.2 Duality

**Theorem 1.2.12.** *Let $\mathcal{M}$ be a matroid on the ground set $E$ with set of bases $\mathcal{B}$. Let $\bar{\mathcal{B}}$ be*

$$\bar{\mathcal{B}} = \{E \backslash B,\ B \in \mathcal{B}\}$$

*Then $\bar{\mathcal{B}}$ is the set of bases of a matroid over $E$.*

*Proof.* [17, Theorem 2.1.1] □

**Definition 1.2.8.** Let $\mathcal{M}$ be a matroid on the ground set $E$ and set of bases $\mathcal{B}$. Then the matroid on $E$ and set of bases $\bar{\mathcal{B}}$ is called the *dual* of $\mathcal{M}$, and denoted by $\mathcal{M}^*$.

***Remark.*** $(\mathcal{M}^*)^* = \mathcal{M}$

**Example 1.2.5.** The set of bases for the dual matroid from Example 1.2.1 is

$$\bar{\mathcal{B}} = \{\{1,2\},\{1,3\},\{1,5\},\{2,3\},\{2,4\},\{2,5\},\{3,4\},\{4,5\}\} = \mathcal{B}(\mathcal{M}^*)$$

**Definition 1.2.9.** Let $\mathcal{M}$ be a matroid. Then

- The elements of $\mathcal{I}(\mathcal{M}^*)$ are the *coindependent sets* of $\mathcal{M}$.
- The elements of $\mathcal{B}(\mathcal{M}^*)$ are the *cobases* of $\mathcal{M}$.
- The elements of $\mathcal{C}(\mathcal{M}^*)$ are the *cocircuits* of $\mathcal{M}$.
- The rank function of $\mathcal{M}^*$ is the *corank* function of $\mathcal{M}$.
- A coloop of $\mathcal{M}$ is a loop of $\mathcal{M}^*$.

***Remark.*** The coindependent sets are not the complements in $E$ of the independent sets. There is not a nice description of the cocircuits of a matroid. We cannot even predict the number of cocircuits from the number of bases or the number of circuits.

**Proposition 1.2.13.** *Let $\mathcal{M}$ be a matroid of rank $r$ on the ground set $E$. Then the rank of $\mathcal{M}^*$ (or the corank of $\mathcal{M}$) is $|E| - r$.*

*Proof.* The rank of $\mathcal{M}$ is equal to the cardinality of any base. Then, the cardinality of any base of $\mathcal{M}^*$ is equal to $|E| - r$. □

**Theorem 1.2.14.** *Let $\mathcal{M}$ be a matroid of rank function $r$. Then the rank function $r^*$ of $\mathcal{M}^*$ is given by*

$$r^*(\sigma) = |\sigma| + r(E \backslash \sigma) - r(E).$$

*Proof.* [17, Proposition 2.1.9] □

**Definition 1.2.10.** Let $\mathcal{M}$ be a matroid over the ground set $E$ with rank function $r$. Let $1 \leq i \leq \ |E| - r(E)$. Then the *i-th generalized Hamming weight* of $\mathcal{M}$ is

$$d_i = \min\{|\sigma|;\ n(\sigma) = i\}$$

**Proposition 1.2.15.** *Let $\mathcal{M}$ be a matroid of rank $r$ on the ground set $E$. Then, we have*

$$d_1 < \ldots < d_{\#E-r}$$

*Proof.* [21, Theorem 1] □

**Theorem 1.2.16.** *Let $\mathcal{M}$ be a matroid on the ground set $E$ and rank $r$. Let $d_1 < \ldots < d_{\#E-r}$ be its weight hierarchy. Let $e_1 < \ldots < e_r$ be the weight hierarchy of $\mathcal{M}^*$. Then, we have*

$$\{d_1, \ldots, d_{\#E-r}\} \cup \{n+1-e_1, \ldots, n+1-e_r\} = \{1, \ldots, n\}$$

*and the union is disjoint.*

*Proof.* [14, Proposition 5.18] □

## 1.3 Relation between codes and matroids

### 1.3.1 Representable matroids

**Definition 1.3.1.** Two matroids $(E, \mathcal{I}_1)$, $(E, \mathcal{I}_2)$ are *isomorphic* if exists a bijection $\phi : E \longrightarrow F$ such that $\sigma \in \mathcal{I}_1 \Longleftrightarrow \phi(\sigma) \in \mathcal{I}_2$.

**Definition 1.3.2.** A *vectorial matroid* over a field $\mathbb{F}$ is a matroid obtained from a finite set $\vec{v_1}, \ldots, \vec{v_n}$ in some finite dimensional vector space $W$ over $\mathbb{F}$ such that $E = [1, \ldots, n]$, and $\sigma = \{i_1, ..., i_m\} \in \mathcal{I}$ if and only if $\vec{v_{i_1}}, ..., \vec{v_{i_m}}$ are linearly independent vectors over $\mathbb{F}$.

**Definition 1.3.3.** A matroid is *representable* if it is a vectorial matroid over some field $\mathbb{F}$.

**Example 1.3.1.** $V = \mathbb{K}^t$, $v_i = \begin{bmatrix} v_{1i} \\ \vdots \\ v_{ti} \end{bmatrix}$.

From these vectors we obtain a matrix $A = \begin{bmatrix} v_{11} & \dots & v_{1n} \\ \vdots & & \vdots \\ v_{t1} & \dots & v_{tn} \end{bmatrix} = \begin{bmatrix} v_1, \dots, v_n \end{bmatrix}$.

The independent sets of the vector matroid $\mathcal{M}[A]$ are

$$\mathcal{I}_{\mathcal{M}[A]} = \{\{i_1, ..., i_m\} \ ; \ v_{i_1}, ..., v_{i_m} \text{ lin.indep.}\}$$

***Remark.*** A set of columns in $A$ is linearly independent (as vectors) if and only if the corresponding set is independent in $\mathcal{M}[A]$.

**Proposition 1.3.1.** *Let $A$ be a $k \times n$ matrix with $k \leq n$, $X \subset E = [1, \dots, n]$.*
*Then, the rank function of the matroid $\mathcal{M}[A]$ is given by*

$$r(X) = rank(A[X])$$

*where $A[X]$ is the vector matrix formed by the columns of $A$ indexed by $X$.*

*Proof.* It comes directly from the definition of rank since, for $X \subset E$, the rank of the matrix $A[X]$ is the rank of the vector space spanned by $X$. □

**Theorem 1.3.2.** *If $\mathcal{M}$ is the vector matroid of $[\ I \mid A\ ]$, then $\mathcal{M}^*$ is the vector matroid of $[\ -A^t \mid I\ ]$.*

*Proof.* [17, Theorem 2.2.8] □

**Corollary.** *If $\mathcal{M}$ is representable over the field $\mathbb{F}$, then $\mathcal{M}^*$ is also representable over $\mathbb{F}$.*

**Proposition 1.3.3.** *Let $\mathcal{M}$ be a matroid having fewer than 8 elements. Then, $\mathcal{M}$ is representable.*

*Proof.* [17, Proposition 6.4.10] □

### 1.3.2 Non-representable matroids

All matroids that come from a code are vector (representable) matroids. However, there are some matroids that are non-representable, so they do not came from any code. Let us see an example:

**Example 1.3.2** ($V_8$ - *Vámos* matroid)**.**

The *Vámos* matroid $\mathcal{V}$ is defined on the set $V = \{v_1, ..., v_8\}$. Its independent sets are all the subsets of cardinality $\leqslant 4$, except for five: $\{v_1, v_2, v_3, v_4\}$, $\{v_1, v_2, v_5, v_6\}$, $\{v_3, v_4, v_5, v_6\}$, $\{v_3, v_4, v_7, v_8\}$, $\{v_5, v_6, v_7, v_8\}$.

It is the smallest known matroid that is non-representable over any field, as shown in [17, Proposition 6.1.10].

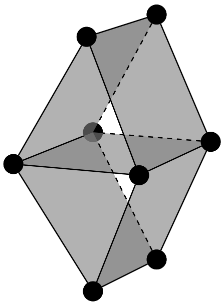


Figure 1.1: Vámos matroid

### 1.3.3 How to obtain a matroid from a code

**Proposition 1.3.4.** *Let $H_1$ and $H_2$ be two different parity check matrices from the same linear code. Then,*

$$\mathcal{M}[H_1] = \mathcal{M}[H_2]$$

*Proof.* $H_1$ and $H_2$ are parity check matrices from the same code. It means that we can obtain one from the other by row operations. Since the matroid $\mathcal{M}[H_1]$ describes the independence between columns in $H_1$ and the matroid $\mathcal{M}[H_2]$ describes the independence between columns in $H_2$, both of them will define the same exact independent sets. □

**Definition 1.3.4.** A matroid from a $[n, k]_q$ linear code $\mathcal{C}$ is defined as

$$\mathcal{M}_{\mathcal{C}} = \mathcal{M}[H]$$

where $H$ is a parity check matrix.

***Remark***. Although a matroid can also be defined using the generator matrix, we will use the parity check matrix. The reason comes from the definition of weight hierarchy of a code.

$$d_i(\mathcal{C}) \;=\; \min\{|\sigma|;\; n(\sigma) = i\} \;=\; d_i(\mathcal{M}_H)$$

**Example 1.3.3.**

$$H_1 = \begin{bmatrix} 0\,1\,1\,1\,0 \\ 1\,0\,1\,0\,0 \\ 0\,0\,1\,0\,1 \end{bmatrix} \qquad H_2 = \begin{bmatrix} 1\,0\,0\,1\,0 \\ 0\,1\,0\,1\,1 \\ 0\,0\,1\,0\,1 \end{bmatrix}$$

Both matrices give us the same matroid.

***Remark.*** Two different isomorphic matroids may come from two codes that are not equivalent.

**Example 1.3.4.** By using MAGMA ( [2]), we can find two different isomorphic matroids such that come from two $[5, 3, 2]$-codes over $\mathbb{F}_7$ that are not equivalent:

$$G_{\mathcal{M}_1} = \begin{bmatrix} 1 & 0 & 0 & 4 & 5 \\ 0 & 1 & 0 & 4 & 3 \\ 0 & 0 & 1 & 4 & 0 \end{bmatrix} \qquad G_{\mathcal{M}_2} = \begin{bmatrix} 1 & 0 & 0 & 2 & 5 \\ 0 & 1 & 0 & 6 & 5 \\ 0 & 0 & 1 & 5 & 0 \end{bmatrix}$$

**Theorem 1.3.5.** *Let $\mathcal{C}$ be a linear $[n, k]_q$-code. Then, $\mathcal{M}_{\mathcal{C}}$ is a matroid on $E = [1, \ldots, n]$ of rank $n - k$ and the dual of this matroid is $\mathcal{M}^*_{\mathcal{C}} = \mathcal{M}_{\mathcal{C}^\perp}$*

*Proof.* [20, Theorem 7.12] □

**Theorem 1.3.6.** *Let $\mathcal{C}$ be a linear code, and $\mathcal{M}_{\mathcal{C}}$ its associated matroid.*

*Then,*

$$d_i(\mathcal{C}) = d_i(\mathcal{M}_{\mathcal{C}})$$

*where $d_i(\mathcal{C})$ and $d_i(\mathcal{M}_{\mathcal{C}})$ are the generalized Hamming weights of $\mathcal{C}$ and $\mathcal{M}_{\mathcal{C}}$ respectively.*

*Proof.* [20, Theorem 7.14] □

**Definition 1.3.5.** (*Shortening*) Let $\mathcal{C}$ be a $[n, k]_q$ linear code with generator matrix $G$ and parity check matrix $H$. Let $J \subset [1, \ldots, n]$. Consider the set $\mathcal{C}_J$, obtained from $\mathcal{C}$ by taking all the words from $\mathcal{C}$ that are equal to 0 on $J$, and then delete the zeroes at $J$.

**Proposition 1.3.7.** *The code $\mathcal{C}_J$ is a linear code of length $n-|J|$, with parity check matrix $H_J$ obtained from $H$ by deleting the columns corresponding to $J$. $C_J$ is called the shortened code.*

*Proof.* [20, Proposition 10] □

***Remark.*** Shortening involves the throwing out of codewords and deleting coordinate positions.

Generalizing the definition of shortening for codes we get to the concept of *restriction* of a matroid.

**Proposition 1.3.8.** *Let $M$ be a matroid on the ground set $E$, with set of independent sets $I$, and let $F \subset E$. Let*

$$J = \{X \subset F \mid X \in I\}$$

*$\mathcal{N} = (F, J)$ is a matroid.*

*Proof.* [20, Proposition 24] □

**Definition 1.3.6.** The matroid $\mathcal{N}$ on the ground set $F$ with set of independent sets $J$ is called the *restriction* of $\mathcal{M}$ to $F$ or the deletion of $E \setminus F$ from $\mathcal{M}$, and denoted by either $\mathcal{M}|_F$ or $\mathcal{M} \setminus (E \setminus F)$

### 1.3.4 Elongation

**Definition 1.3.7.** For $0 \leq i \leq n - r(E)$, we define the *i-th elongation* of $\mathcal{M}$ as the matroid $\mathcal{M}_{(i)}$ whose set of independent sets are

$$\mathcal{I}(\mathcal{M}_{(i)}) = \{\sigma \subseteq E \ ; \ n(\sigma) \leq i\}$$

***Remark.*** It is not difficult to see that $\mathcal{M}_{(i)}$ is, indeed, a matroid:

$(I_1)$ $\emptyset \in \mathcal{I}(\mathcal{M}_{(i)})$, since $n(\emptyset) = 0$.

$(I_2)$ If $I_1 \in \mathcal{I}(\mathcal{M}_{(i)}), I_2 \subset I_1$, then

$$|\mathcal{I}_2| < |I_1| \text{ and } r(I_2) < r(I_1)$$

By the property $(R_2)$,

$$r(I_2) + |I_1 \setminus I_2| \geq r(I_1)$$

Then,

$$n(I_2) = |I_2| - r(I_2) \leq |I_2| - r(I_1) + |I_1| - |I_2| = n(I_1) \leq i$$

Therefore, $I_2 \in \mathcal{I}(\mathcal{M}_{(i)})$.

$(I_3)$ Let $I_1, I_2 \in \mathcal{I}(\mathcal{M}_{(i)})$ such that $|I_1| < |I_2|$ and assume that $\forall x \in I_2 \setminus I_1,\ \ n(I_1 \cup \{x\}) > i$. Then,

$$i < n(I_1 \cup \{x\}) = |I_1| + 1 - r(I_1 \cup \{x\})$$

$$r(I_1 \cup \{x\}) < |I_1| + 1 - i$$

We have that $r(I_1) \geq |I_1| - i$ and $r(I_2) \geq |I_2| - i$. Then,

$$|I_1| - i \leq r(I_1) \leq r(I_1 \cup \{x\}) \leq |I_1| - i$$

Thereore, $\quad r(I_1 \cup \{x\}) = r(I_1) = |I_1| - i$.

Since $\mathcal{M}$ is a matroid, the properties for the rank function $r$ are satisfied, and we asumed that $n(I_1 \cup \{x\}) > i \ \ \forall x \in I_2 \setminus I_1$. Then, by repeated aplications of $(R_2)$,

$$r(I_1) = r(I_1 \cup \{x\}) = r(I_1 \cup \{x, y\}) = \ldots = r(I_2) = |I_1| - i$$

We get

$$n(I_2) = |I_2| - r(I_2) = |I_2| - r(I_1) = |I_2| - |I_1| + i \Rightarrow n(I_2) > i$$

that is absurd.

**Proposition 1.3.9.** *Let $\mathcal{M}$ be a matroid and let $r$ and $n$ be its rank and nullity functions. Then, for $\sigma \in E$, the rank and nullity functions for its elongations are:*

$$r_{(i)}(\sigma) = \begin{cases} r(\sigma) + i & \textit{if} \;\; n(\sigma) > i \\ |\sigma| & \textit{if} \;\; n(\sigma) \leq i \end{cases}$$

*and*

$$n_{(i)}(\sigma) = \begin{cases} n(\sigma) - i & \textit{if} \;\; n(\sigma) > i \\ 0 & \textit{if} \;\; n(\sigma) \leq i \end{cases}$$

*Proof.* We know that the rank of an independent set is equal to its cardinality. Therefore, $r_{(i)}(\sigma) = |\sigma|$, when $n(\sigma) \leq i$.

For dependent sets we need to find one $I \in \mathcal{I}(\mathcal{M}_{(i)})$ such that $I \subset \sigma$ and $|I| = r(\sigma) + i$. Let $\sigma \subset E$ such that $n(\sigma) = |\sigma| - r(\sigma) > i$.

Let now $I \in \mathcal{I}(\mathcal{M})$ such that $I \subset \sigma$ is maximal. Then, $r(\sigma) = r(I) = |I|$.

Since $\sigma$ is dependent,

$$\begin{aligned} r(\sigma) < |\sigma| - i \;\; &\Rightarrow \;\; |\sigma| - i > r(\sigma) = r(I) = |I| \\ &\Rightarrow \;\; |\sigma| - |I| > i \end{aligned}$$

Then, $\exists \tau \subset \sigma \setminus I$ such that $|\tau| = i$.

Let $J = I \cup \tau$. Then, $n(J) = |J| - r(J) = |I| + i - |I| = i$. Therefore, $J \in \mathcal{I}(\mathcal{M}_{(i)})$.

We have $I \subset J \subset \sigma$ and $J \in \mathcal{I}(\mathcal{M}_{(i)})$. Therefore,

$$r_{(i)}(\sigma) = \max\{|I| \;; I \subset \sigma, I \in \mathcal{I}(\mathcal{M}_{(i)})\} \geq |J| = |I| + i = r(\sigma) + i$$

Assume now that $\exists J \in \mathcal{I}(\mathcal{M}_{(i)})$ such that $J \subset \sigma$ and $|J| = r(\sigma) + i + 1$. Then,

$$n(J) = |J| - r(J) = r(\sigma) + i + 1 - r(J) \geq r(J) + i + 1 - r(J) > i$$

that is absurd.

In the case of nullity function for elongations, the definition reads:

$$n_{(i)}(\sigma) = |\sigma| - r_{(i)}(\sigma)$$

Therefore,
If $n(\sigma) > i$, $n_{(i)}(\sigma) = |\sigma| - (r(\sigma) + i) = |\sigma| - r(\sigma) - i = n(\sigma) - i$.
If $n(\sigma) \leq i$, $n_{(i)}(\sigma) = |\sigma| - |\sigma| = 0$.

□

**Corollary.** $r_{(i)}(\mathcal{M}_{(i)}) = r(E) + i = r(E) + i$.

***Remark***. The matoid $\mathcal{M}_{(i)}$ is commonly referred to as the elongation of $\mathcal{M}$ to rank $r(\mathcal{M}) + i$.

**Lemma 1.3.10.** *Let $r$ be the rank funcion of a matroid $\mathcal{M}$. Then, the rank function for its elongations satisfies:*

$$r_{(i)}(\sigma) = \min\{|\sigma|, r(\sigma) + i\}$$

*Proof.*

$$r_{(i)}(\sigma) = \begin{cases} r(\sigma) + i & \text{if} \quad n(\sigma) > i \Leftrightarrow r(\sigma) + i < |\sigma| \\ |\sigma| & \text{if} \quad n(\sigma) \leq i \Leftrightarrow |\sigma| \leq r(\sigma) + i \end{cases}$$

Therefore, $r_{(i)}(\sigma) = \min\{|\sigma|, r(\sigma) + i\}$.

□

**Corollary.** *Let $r$ be the nullity funcion of a matroid $\mathcal{M}$. Then, the nullity function for its elongations satisfies:*

$$n_{(i)}(\sigma) = \max\{0, n(\sigma) - i\}$$

*Proof.* By the definition of nullity function,

$$\begin{aligned} n_{(i)}(\sigma) &= |\sigma| - r_{(i)}(\sigma) = |\sigma| - \min\{|\sigma|, r(\sigma) + i\} \\ &= \max\{0, |\sigma| - r(\sigma) - i\} = \max\{0, n(\sigma) - i\} \end{aligned}$$

□

### 1.3.5 Truncation

**Definition 1.3.8.** The *i-th truncation* of $\mathcal{M}$ is the matroid $\mathcal{M}_{(i)}$ whose set of independent sets are

$$\mathcal{I}(\mathcal{M}^{(i)}) = \{\sigma \in \mathcal{I} \ ; \ |\sigma| \leq r(E) - i\}$$

***Remark.*** It is not difficult to see that $\mathcal{M}(i)$ is a matroid:

($I_1$) $\emptyset \in \mathcal{I}(\mathcal{M}^{(i)})$.

($I_2$) If $I_1 \in \mathcal{I}(\mathcal{M}^{(i)}), I_2 \subset I_1$, then $|I_2| \leq |I_1 \leq r(E) - i$. Therefore, $I_2 \in \mathcal{I}(\mathcal{M}^{(i)})$.

($I_3$) Let $I_1, I_2 \in \mathcal{I}(\mathcal{M}^{(i)})$ such that $|I_1| < |I_2|$. Take $x \in I_2 \setminus I_1$, then $|I_1 \cup \{x\}| = |I_1| + 1 \leq |I_2| \leq r(E) - i$. Therefore, $I_1 \cup \{x\} \in \mathcal{I}(\mathcal{M}^{(i)})$.

***Remark.*** Equivalents ways to define truncations are:

$$\mathcal{I}(\mathcal{M}^{(i)}) = \{\sigma \in \mathcal{I} \ ; \ r(\sigma) \leq r(E) - i\}$$

$$\mathcal{B}(\mathcal{M}^{(i)}) = \{\sigma \in \mathcal{I} \ ; \ r(\sigma) = r(E) - i\}$$

**Proposition 1.3.11.** *For $\sigma \in E$, the rank function for truncations is:*

$$r^{(i)}(\sigma) = \begin{cases} r(E) - i & \text{if } \ r(\sigma) > r(E) - i \\ r(\sigma) & \text{if } \ r(\sigma) \leq r(E) - i \end{cases}$$

*Proof.* It is sufficient to prove it for $i = 1$.

For independent sets in $\mathcal{M}^{(1)}$, $r(\sigma) \leq r(E) - 1$. Let $J$ be a maximal independent subset in $M$ such that $J \subset \sigma$. Then, $|J| = r(\sigma) \leq r(E) - 1$.

Therefore, $J \in \mathcal{I}(\mathcal{M}^{(1)})$, and so, $r^{(1)}(\sigma) \geq |J| = r(\sigma)$.

On the other hand, by definition, $r^{(1)}(\sigma) = \max\{|J| \ ; \ J \subset \sigma, J \in \mathcal{I}(\mathcal{M}^{(i)})\} \leq \max\{|J| \ ; \ J \subset \sigma, J \in \mathcal{I}(\mathcal{M})\} = r(\sigma)$.

Thus, $r^{(1)}(\sigma) = r(\sigma)$ for $r(\sigma) \leq r(E) - 1$.

If $\sigma$ is dependent in $\mathcal{M}^{(1)}$, $r(\sigma) > r(E) - 1$. Therefore, $r(\sigma) = r(E)$.

We have that $r^{(1)}(\sigma) \leq r^{(1)}(E) = r(E) - 1$.

Now, If $r(\sigma) = r(E)$ then $\exists B \subset \mathcal{B}$ such that $B \in \sigma$, so $B \setminus x \subset \sigma$ $\forall x \in B$. Since $B \in \mathcal{I}(\mathcal{M})$ we get that $B \setminus x \in \mathcal{I}(\mathcal{M})$.

$$\begin{array}{rcccl} r(B) & = & r(E) & = & |B| \\ r(B \setminus x) & = & r(B) - 1 & = & r(E) - 1 \\ r(B \setminus x) & = & |B \setminus x| & = & r(B) - 1 \end{array}$$

Then, $B \setminus x \in \mathcal{I}(\mathcal{M}^{(1)})$. So $r^{(1)}(\sigma) \geq |B \setminus x| = r(E) - 1$.

Thus, $r^{(1)}(\sigma) = r(E) - 1$ for $r(\sigma) > r(E) - 1$. □

**Corollary.** *Let $r$ be the rank function of a matroid $\mathcal{M}$. Then, for $\sigma \in E$, the rank function for its truncations satisfies:*

$$r^{(i)}(\sigma) = \min\{r(E) - i, r(\sigma)\}$$

*Proof.* It comes directly from the previous proposition. □

**Proposition 1.3.12.**

$$(\mathcal{M}^{(i)})^* = (\mathcal{M}^*)_{(i)}$$

*Proof.*

$$\begin{array}{rcl} (r^{(i)})^*(\sigma) & = & |\sigma| + r^{(i)}(E\backslash\sigma) - r^{(i)}(E) \\ & = & |\sigma| + \min\{r(E\backslash\sigma), r(E) - i\} - \min\{r(E), r(E) - i\} \\ & = & |\sigma| + \min\{r(E\backslash\sigma), r(E) - i\} - (r(E) - i) \end{array}$$

If $r(E\backslash\sigma) < r(E) - i$, $(r^{(i)})^*(\sigma) = |\sigma| + r(E\backslash\sigma) - (r(E) - i) = r^*(\sigma) + i$.

If $r(E\backslash\sigma) \geq r(E) - i$, $(r^{(i)})^*(\sigma) = |\sigma|$

On the other hand,

$$\begin{array}{rcl} (r^*)_{(i)}(\sigma) & = & \min\{\ r^*(\sigma) + i,\ |\sigma|\ \} \\ & = & \min\{\ |\sigma| + r(E\backslash\sigma) - r(E) + i\ ,\ |\sigma|\ \} \end{array}$$

If $r(E\backslash\sigma) < r(E) - i$, $(r^*)_{(i)})(\sigma) = |\sigma| + r(E\backslash\sigma) - (r(E) - i) = r^*(\sigma) + i$.

If $r(E\backslash\sigma) \geq r(E) - i$, $(r^*)_{(i)})(\sigma) = |\sigma|$

which are the results we were looking for. □

# Chapter 2

# Betti numbers associated to matroids

In this chapter we will define what a simplicial complex is and what its homological properties are. This will establish the base for its later use in chapters 4 and 5.

## 2.1 Simplicial complexes

**Definition 2.1.1.** A simplicial complex $\Delta$ on the vertex set $E = \{1, \ldots, n\}$ is a family of subsets of $E$, called faces, that satisfy the following condition:

- If $\sigma_1 \in \Delta$ and $\sigma_2 \subset \sigma_1 \Rightarrow \sigma_2 \in \Delta$

The set of maximal faces (for inclusion), called *facets*, is denoted as $\mathcal{F}(\Delta)$. The set of minimal non-faces (for inclusion) is denoted as $\mathcal{N}(\Delta)$.

**Definition 2.1.2.** The dimension of a face $F$ is $|F| - 1$, and the dimension of the simplicial complex $\Delta$ is the maximun dimension of its faces, i.e.

$$\dim(\Delta) = \max\{|\sigma| - 1 \ , \ \sigma \in \Delta\}$$

**Definition 2.1.3.** A subcomplex of $\Delta$ is a simplicial complex such that every of its faces belongs to $\Delta$.

**Definition 2.1.4.** The $k$-skeleton of $\Delta$ is the subcomplex of $\Delta$ consisting of all of the faces of $\Delta$ that have dimension at most $k$.

**Definition 2.1.5.** $\Delta$ is said to be *pure* if every facet has the same dimension.

**Definition 2.1.6.** Let $\Delta$ be a pure simplicial complex. A *shelling* on $\Delta$ is a total order on the facets $F_1 < \ldots < F_t$ such that $\forall\ 1 \leq i < j \leq t\ \ \exists\ k < j$ such that $F_i \cap F_j \subset F_k \cap F_j = F_j \backslash \{x\}$ for $x \in F_j$.

## 2.2 Monomial ideals

Let $\mathbb{K}$ be a field, and let $S = \mathbb{K}[x_1, \ldots, x_n]$ the polynomial ring in $n$ variables over $\mathbb{K}$.

**Definition 2.2.1.** A *monomial* is a polynomial of the form

$$\underline{x}^{\underline{a}} = \prod_{i=1}^{n} x_i^{a_i}$$

where $a = (a_1, \ldots, a_n)$ and $a_i \in \mathbb{N}$ .

***Property.***

$$\underline{x}^{\underline{a}} \cdot \underline{x}^{\underline{b}} = \underline{x}^{\underline{a}+\underline{b}}$$

***Remark.*** The set $\mathrm{Mon}(S)$ of monomials of $S$ is a $\mathbb{K}$-basis of $S$, i.e any polynomial $f \in S$ is a unique finite $\mathbb{K}$-linear combination of monomials.

$$f = \sum_{u \in Mon(S)} a_u u \qquad \text{where } a_u \in \mathbb{K}$$

**Definition 2.2.2.** The set $\mathrm{supp}(f) = \{u \in \mathrm{Mon}(S)\ ;\ a_u \neq 0\}$ is called the *support* of $f$

**Definition 2.2.3.** An ideal $I$ is called a *monomial ideal* if it is generated by monomials.

**Theorem 2.2.1.** *The set of monomials belonging to a monomial ideal $I$ is a $\mathbb{K}$-basis of $I$.*

*Proof.* [6, Theorem 1.1.2] □

***Remark.*** This is not true for all ideals.
If we take $I =< x_1 + x_2x_3 >$, the set of monomials belonging to $I$ are not a $\mathbb{K}$-basis for it since $Mon(I) = \emptyset$.

**Definition 2.2.4.** For a monomial ideal $I$,

$$\mathrm{Mon}(I) = \mathrm{Mon}(S) \cap I$$

**Corollary.** *Let $I$ be a monomial ideal. The residue classes of the monomials not belonging to $I$ form a $\mathbb{K}$-basis of the residue class ring $S/I$.*

$$\mathcal{B}_{S/I} = \{\bar{u}\ ;\ u \in Mon(S),\ u \notin Mon(I)\}$$

*Proof.* [6, Corollary 1.1.4] □

**Definition 2.2.5.** A monomial $\underline{x}^{\underline{a}}$ is called *squarefree* if the components of $\underline{a}$ are 0 or 1.

**Definition 2.2.6.** A monomial ideal is *squarefree* if it is generated by squarefree monomials.

## 2.3 Gradings

Let $\mathbb{K}$ be a field, and let $S = \mathbb{K}[x_1, \dots, x_n]$ the polynomial ring in $n$ variables over $\mathbb{K}$.

Any nonzero polynomial $f \in \mathbb{K}$ can be decomposed, in an unique way, as sum of homogeneous polynomials of different degrees.

**Definition 2.3.1.** The polynomials from the descomposition of $f$ in the sum of homogenous polynomials are called the *homogeneous components* of $f$.

**Definition 2.3.2.** An ideal $I \subset S$ is graded if, whenever $f \in I$, all homogeneous components of $f$ belong to $I$.

Let $G$ be an abelian group.

**Definition 2.3.3.** A *G-graded ring* $R$ is a ring such that $R = \bigoplus_{g \in G} R_g$ where:

- $(R_g, +)$ is a subgroup of $(R, +) \quad \forall g \in G$
- $R_g \cdot R_h \subset R_{g+h} \quad \forall g, h \in G$

***Remark.*** $1 \in R_0$

**Definition 2.3.4.** A *G-graded module* $M$ on a $G$-graded ring is such that $M = \bigoplus_{g \in G} M_g$ where:

- $(M_g, +)$ is a subgroup of $(M, +) \quad \forall g \in G$
- $R_g \cdot M_h \subset M_{g+h} \quad \forall g, h \in G$

**Definition 2.3.5.** Elements in $M_g$ are called *homogeneous elements* of degree $g$.

**Definition 2.3.6.** The standard structure of the polynomial ring $S$ as a $\mathbb{Z}$-graded ring is the following:

$$S_i = \begin{cases} \emptyset & \text{if} \quad i{<}0 \\ \mathbb{K} & \text{if} \quad i = 0 \\ \{\text{homogeneous polynomials of degree } i\} & \text{if} \quad i \geq 1 \end{cases}$$

**Definition 2.3.7.** The standard structure of the polynomial ring $S$ as a $\mathbb{Z}^n$-graded ring is the following:
Let $\underline{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$. Then,

$$S_{\underline{a}} = \begin{cases} \emptyset & \text{if} \quad \exists a_i{<}0 \\ \mathbb{K} & if \quad \underline{a} = \underline{0} \\ \mathbb{K} \cdot x^{\underline{a}} & if \quad a_i \geq 0 \end{cases}$$

**Definition 2.3.8.** A *G-graded (or homogeneous) morphism* of graded $R$-modules

$$\varphi : M = \bigoplus_{g \in G} M_g \longrightarrow N = \bigoplus_{h \in G} N_h$$

of degree $d$ is a morphism of $R$-modules such that $\varphi(M_g) \subset N_{g+d} \ \ \forall g$.

**Definition 2.3.9.** A *graded (or homogeneous) morphism* of graded rings

$$\varphi : R = \bigoplus_{g \in G} R_g \longrightarrow S = \bigoplus_{h \in G} S_h$$

is a morphism of rings such that $\varphi(R_n) \subset S_n \ \ \forall n$.

**Definition 2.3.10.** If $M$ is a graded module then, the *shift* of $M$ by $d \in G$ is $M(d)_g = M_{g+d}$.

***Remark.*** If $\varphi : M \longrightarrow N$ is a graded morphism of degree $d$, then $\varphi : M(-d) \longrightarrow N$ is a degree 0 homomorphism.

From now on, we will consider all the morphisms as morphisms of degree 0.

**Definition 2.3.11.** A *chain complex* $(F_\bullet, \delta_\bullet)$ is a sequence of modules connected by homomorphisms $\delta_i : F_i \longrightarrow F_{i-1}$ (called *boundary operators*) such that $\delta_i \circ \delta_{i+1} = 0$. We will denote it as $\mathbb{F}$.

$$\mathbb{F} : \ldots \xrightarrow{\delta_{i+1}} F_i \xrightarrow{\delta_i} F_{i-1} \xrightarrow{\delta_{i-1}} \ldots \xrightarrow{\delta_2} F_1 \xrightarrow{\delta_1} F_0 \xrightarrow{\delta_0} F_{-1} \xrightarrow{\delta_{-1}} F_{-2} \xrightarrow{\delta_{-3}} \ldots$$

It is exact when $Im(F_{i+1}) = \ker(F_i)$.

Due to Hilbert's Basis theorem, $S$ is a Noetherian $\mathbb{Z}$-graded ring. Let $M$ be a finitely generated $\mathbb{Z}$-graded $S$-module.

**Theorem 2.3.1** (Hilbert Syzygy Theorem). *Let $\mathbb{K}$ be a field and $M$ a finitely generated module over the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$. Then, exists a free resolution of $M$ of length at most $n$.*

*Proof.* [4, Theorem 1.13 and Chapter 19] □

**Definition 2.3.12.** A *free resolution of a $G$-graded module $M$* over $S$ is an exact complex of $S$-modules

$$\mathbb{F} : \ldots \xrightarrow{\delta_{i+1}} F_i \xrightarrow{\delta_i} F_{i-1} \xrightarrow{\delta_{i-1}} \ldots \xrightarrow{\delta_2} F_1 \xrightarrow{\delta_1} F_0 \xrightarrow{\delta_0} M \longrightarrow 0$$

such that each map $\delta_i$ is a $G$-graded morphism, and each $F_i$ is a free $S$-module.

It is called *minimal* if $Im(F_{i+1}) \subset \mathfrak{M} F_i$, where $\mathfrak{M} =< x_1, \ldots, x_n >$.

**Proposition 2.3.2.** *Minimal graded free resolutions of a graded module are unique up to isomorphisms.*

*Proof.* [6, Prop.A.2.3] □

**Proposition 2.3.3.** *Let $M$ be a finitely generated $\mathbb{Z}^n$-graded $S$-module and*

$$\mathbb{F} : \ldots \xrightarrow{\pi_{i+1}} F_i \xrightarrow{\pi_i} F_{i-1} \xrightarrow{\pi_{i-1}} \ldots \xrightarrow{\pi_2} F_1 \xrightarrow{\pi_1} F_0 \xrightarrow{\pi_0} M \longrightarrow 0$$

*a minimal graded free resolution of $M$ with*

$$F_i = \bigoplus_{\underline{a} \in \mathbb{Z}^n} S(-\underline{a})^{\beta_{i\underline{a}}} \quad \forall i$$

*Then,*

$$\beta_{i\underline{a}} = \dim_{\mathbb{K}} Tor_i^S(\mathbb{K}, M)_{\underline{a}} \qquad \forall i, \underline{a}$$

*Proof.* [16, Lemma 1.32] □

**Proposition 2.3.4.** *Let $M$ be a finitely generated $\mathbb{Z}$-graded $S$-module and*

$$\mathbb{F} : \ldots \xrightarrow{\pi_{i+1}} F_i \xrightarrow{\pi_i} F_{i-1} \xrightarrow{\pi_{i-1}} \ldots \xrightarrow{\pi_2} F_1 \xrightarrow{\pi_1} F_0 \xrightarrow{\pi_0} M \longrightarrow 0$$

*a minimal graded free resolution of $M$ with*

$$F_i = \bigoplus_{j} S(-j)^{\beta_{ij}} \quad \forall i$$

*Then,*

$$\beta_{ij} = \dim_{\mathbb{K}} Tor_i(\mathbb{K}, M)_j \qquad \forall i, j$$

*Proof.* [6, Proposotion A.2.2] □

***Remark.*** $\mathrm{Tor}_i^S(\mathbb{K}, -)$ are the $i$-th derived functors of the functor $S/\mathfrak{M} \otimes_S -$. The definition and properties can be found in [8, Chapter 3, Section 8].

**Definition 2.3.13.** The numbers

$$\beta_{i\underline{a}} = \dim_{\mathbb{K}} \operatorname{Tor}_i^S(\mathbb{K}, M)_{\underline{a}}$$

are called the $\mathbb{Z}^n$*-graded* or *multigraded Betti numbers* of $M$, and

$$\beta_i = \sum_j \beta_{ij}$$

where $\beta_{i,j} = \sum_{|\underline{a}|=j} \beta_{i,\underline{a}}$ is called the *i-th* or *ungraded Betti number* of $M$.

***Remark.*** We also must observe that the $\beta_{ij}$ are independent of the minimal free resolutions, and that we hence can find them by studying one explicit such resolution.

**Definition 2.3.14.** Let $\Delta$ be a simplicial complex. The chain complex associated to $\Delta$ is

$$\mathbb{F} : 0 \longrightarrow \ldots \xrightarrow{\delta_{i+1}} \mathbb{K}^{F_i(\Delta)} \xrightarrow{\delta_i} \mathbb{K}^{F_{i-1}(\Delta)} \xrightarrow{\delta_{i-1}} \ldots \xrightarrow{\delta_1} \mathbb{K}^{F_0(\Delta)} \xrightarrow{\delta_0} \mathbb{K}^{F_{-1}(\Delta)} \longrightarrow 0$$

where

$$F_i(\Delta) = \{\sigma \in \Delta \ ; \ \dim(\sigma) = i\}.$$

and $\delta_i$ are defined as follows:

$$\begin{array}{rcccl} \delta_i & : & \mathbb{K}^{F_i(\Delta)} & \longrightarrow & \mathbb{K}^{F_{i-1}(\Delta)} \\ & & \delta_i(x^\sigma) & \longmapsto & \sum_{1 \le j \le i} (-1)^{j+1} x^{a_1,\ldots,a_{j-1},a_{j+1},\ldots,a_n} \end{array}$$

where $x^\sigma = x^{a_1,\ldots,a_n}$, $a_1 < \ldots < a_n$.

Then, the *i-th reduced homology* is defined as

$$\widetilde{H}_i(\delta, \mathbb{K}) = \ker(\delta_i)/Im(\delta_{i+1})$$

**Definition 2.3.15.**

$$\widetilde{h}_i(M, \mathbb{K}) = \dim(\widetilde{H}_i(M, \mathbb{K}))$$

**Definition 2.3.16.** Let $A = \bigoplus\limits_{n \geq 0} A_n$ be a finitely generated graded $\mathbb{K}$-algebra.

The *Hilbert function* of $A$ is defined as

$$\mathcal{H}(A, n) = \dim_{\mathbb{K}}(A_n)$$

where $\dim_{\mathbb{K}}(A_n)$ is the dimension of the vector space $A_n$ over $\mathbb{K}$.

If $I = \underset{n \geq 0}{\oplus} I_n$ is a homogeneous ideal of $A$, we can also define

$$\mathcal{H}(I, n) = \dim(I_n)$$

**Definition 2.3.17.** Let $A = \bigoplus\limits_{n \geq 0} A_n$ be a finitely generated $\mathbb{K}$-algebra.

The *Hilbert series* of $A$ is defined to be the generating function

$$\mathcal{F}(A, t) = \sum_{n=0}^{\infty} \mathcal{H}(A, n) \cdot t^n$$

Similarly, if $I$ is a homogeneous ideal of $A$, then the Hilbert series of $I$ is

$$\mathcal{F}(I, t) = \sum_{n=0}^{\infty} \mathcal{H}(I, n) \cdot t^n$$

## 2.4 The Stanley-Reisner ideal for a Simplicial complex

Let $\Delta$ be a simplicial complex on $E = \{1, \ldots, n\}$ and $S = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring in $n$ variables over a field $\mathbb{K}$.

***Notation.*** We will use the following notation:

$$x^F = \prod_{i \in F} x_i \qquad \text{for } F \subset E$$

***Remark.*** This is a squarefree monomial ideal.

**Definition 2.4.1.** Let $\Delta$ be a simplicial complex on the ground set $E$. The *Stanley-Reisner ideal* of $\Delta$ is the monomial ideal generated by its minimal non-faces, i.e,

$$I_\Delta = <x^F; F \in \mathcal{N}(\Delta)> = <x^F; F \notin \Delta>$$

***Remark.*** $I_\Delta$ is a squarefree monomial ideal.

**Definition 2.4.2.** The *Stanley-Reisner ring* is

$$\mathbb{K}[\Delta] = S/I_\Delta$$

**Definition 2.4.3.** The *facet ideal* of $\Delta$ is

$$I(\Delta) = <x^F \ ; \ F \in \mathcal{F}(\Delta)>$$

where $\mathcal{F}(\Delta)$ is the set of facets of $\Delta$

***Remark.*** $I(\Delta)$ is also a squarefree monomial ideal.

**Definition 2.4.4.** The *Facet Ring* of $\Delta$ is

$$S[\Delta] = S/I(\Delta)$$

**Definition 2.4.5.** Given a simplicial complex $\Delta$ on $E$, we define its *Alexander Dual* as

$$\Delta^\vee = \{E \backslash F \ ; \ F \notin \Delta\}$$

**Proposition 2.4.1.** $\Delta^\vee$ *is also a simplicial complex.*

*Proof.* [6, Lemma 1.5.3] □

**Theorem 2.4.2.** *(Hochster's formula) The nonzero Betti numbers of $I_\Delta$ and $S/\mathcal{I}_\Delta$ lie only in squarefree degrees $\sigma$, and we have*

$$\beta_{i,\sigma}(S/\mathcal{I}_\Delta) = \beta_{i-1}(I_\Delta) = \overset{\sim}{h}_{|\sigma|-i-1}(\Delta_{|\sigma})$$

*Proof.* [16, Corollary 5.12] □

It is quite easy to verify that the definition of a matroid $\mathcal{M}$ ensures that the set $\mathcal{I}(\mathcal{M})$ of independent sets satisfies the conditions of a simplicial complex. Consequently, the *Stanley-Reisner ring* and the *Stanley-Reisner ideal* of a matroid $\mathcal{M}$ can, and will be defined as those ones from the simplicial complex $\Delta = \mathcal{I}(\mathcal{M})$.

Since all matroids are shellable simplicial complexes we can apply the results from algebraic topology to them.

## 2.5 Matroids as simplicial complexes

### 2.5.1 Free resolutions for matroids

Let $\mathbb{K}$ be a field and $S$ the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$. Let also $\mathcal{M}$ be a matroid on the ground set $E = \{1, \ldots, n\}$.

As we saw before, matroids are simplicial complexes, so the definition of the Stanley-Reisner ideal is the same as for matroids.

**Definition 2.5.1.** The *Stanley-Reisner* ideal of $\mathcal{M}$ is the ideal in $S$ generated by monomials corresponding to its circuits.

$$I_{\mathcal{M}} = <x^\sigma \in S \ ; \ \sigma \text{ circuit of } \mathcal{M}> = <\prod_{i\in\sigma} x_i \ ; \ \sigma \in \mathcal{C}>$$

**Definition 2.5.2.** A free resolution of $I_{\mathcal{M}}$ is

$$0 \longrightarrow \bigoplus_{d} S(-d)^{\beta_{\rho,d}(I_{\mathcal{M}})} \longrightarrow \ldots \longrightarrow \bigoplus_{d} S(-d)^{\beta_{0,d}(I_{\mathcal{M}})} \longrightarrow I_{\mathcal{M}} \longrightarrow 0$$

It said to be *pure* if it has the form:

$$0 \longrightarrow S(-d_l)^{\beta_{l,d_l}} \longrightarrow \ldots \longrightarrow S(-d_0)^{\beta_{0,d_0}} \longrightarrow I_{\mathcal{M}} \longrightarrow 0$$

where $d = (d_0, ..., d_l)$ is a strictly increasing sequence of integers.

It is said to be *linear* if exists $d \in \mathbb{Z}$ such that

$$0 \longrightarrow S(-d-\rho)^{\delta_\rho} \longrightarrow \ldots \longrightarrow S(-d-1)^{\delta_1} \longrightarrow S(-d)^{\delta_0} \longrightarrow I_{\mathcal{M}} \longrightarrow 0$$

**Example 2.5.1.**

For the matroid $\mathcal{M}$ in Example 1.2.1, its circuits were

$$\mathcal{C}(\mathcal{M}) = \{\{1,2,4\},\{1,3,4,5\},\{2,3,5\}\}$$

The Stanley-Reisner ideal of $\mathcal{M}$ is therefore

$$I_{\mathcal{M}} =< x_1x_2x_4, x_1x_3x_4x_5, x_2x_3x_5 >$$

A free resolution for $I_{\mathcal{M}}$ is therefore

$$0 \longrightarrow S(-5)^2 \longrightarrow S(-4) \oplus S(-3)^2 \longrightarrow I_{\mathcal{M}} \longrightarrow 0$$

### 2.5.2 Weight hierarchy

Let $\mathcal{M}$ be a matroid on the ground set $E = \{1, \dots, n\}$. Recall from Chapter 1 that the Hamming weights of a matroid are defined as

$$d_i = \min\{\ |\sigma|\ ;\ n_{\mathcal{M}}(\sigma) = i\} \qquad \text{for} \quad 1 \leq i \leq n - r(E)$$

Let us define them in terms of Betti numbers.

**Theorem 2.5.1.** *Let $\mathcal{M}$ be a matroid on E. Then, the generalized Hamming weights are given by*

$$d_i = \min\{d\ ;\ \beta_{i,d} \neq 0\} \quad \textit{for } 1 \leq i \leq n - r(E)$$

*Proof.* [9, Theorem 2] □

**Proposition 2.5.2.** *Let $\mathcal{M}$ be a matroid associated to a code and $\beta_{i,\sigma}$ the Betti numbers associated to a free resolution for its Stanley Reisner ideal. Then,*

$$\beta_{i,\sigma}(I_{\mathcal{M}}) \neq 0 \iff \beta_{i-1,\sigma}(I_{\mathcal{M}_{(1)}}) \neq 0$$

*for $1 \leq i \leq |E| - r(\mathcal{M})$.*

*Proof.* In [9, Th.1] is proved that $\beta_{i,\sigma} \neq 0 \Leftrightarrow \sigma$ is minimal for inclusion in $N_i$, where $N_i = \{\sigma\ ;\ \mathrm{n}(\sigma) = i\}$. Let $0 \leq i \leq |E| - r(\mathcal{M})$.

Let us take $\sigma \in N_i(\mathcal{M})$. The nullity function for its first elongation is

$$\mathrm{n}_{(1)}(\sigma) = \max\{0,\ \mathrm{n}(\sigma) - 1\} = \max\{0,\ i - 1\}$$

Therefore, $\sigma$ belongs to $N_{i-1}(\mathcal{M}_{(1)})$.

Let us now take $\sigma \in N_{i-1}(\mathcal{M}_{(1)})$. The nullity function now is

$$\mathrm{n}_{(1)} = i - 1 = \max\{0,\ \mathrm{n}(\sigma) - 1\}$$

so $\mathrm{n}(\sigma) = i$. Therefore, $\sigma$ belongs to $N_i(\mathcal{M})$.

Since $N_i(\mathcal{M})$ and $N_{i-1}(\mathcal{M}_{(1)})$ have exactly the same elements, if we take $\sigma$ minimal from $N_i(\mathcal{M})$ it will be minimal in $N_{i-1}(\mathcal{M}_{(1)})$ and vice versa. Therefore,

$$\beta_{i,\sigma}(I_{\mathcal{M}}) \neq 0 \iff \beta_{i-1,\sigma}(I_{\mathcal{M}_{(1)}}) \neq 0$$

□

**Corollary.** *All $\sigma \in N_i = \{\sigma \mid n(\sigma) = i\}$ have the same cardinality.*

*Proof.* Let $\sigma \in N_i$. By [10, Lemma 3.1], for $\sigma \in N_i$, there exists a subcode $C'$ of dimension $i$ such that $\sigma = \text{supp}(C')$. Then, due to [15, Theorem 1], $|\sigma| = |\text{supp}(C')| = Wt(C') = d_i = d \cdot \frac{q^i-1}{q^{i-1}(q-1)}$.

□

**Corollary.** *If the Stanley-Reisner ring of a matroid $\mathcal{M}$ has a pure resolution, then its elongations $\mathcal{M}_{(k)}$ also have pure resolutions $\forall\ 1 \leq k \leq |E| - r(\mathcal{M})$.*

**Proposition 2.5.3.** *For $0 \leq i \leq |E| - r(\mathcal{M})$,*

$$d_i(\mathcal{M}_{(j+1)}) = d_{i+1}(\mathcal{M}_{(j)})$$

*Proof.*

$$\begin{aligned} d_i(\mathcal{M}_{(k+1)}) &= \min\{|\sigma| \ ; \ n_{(j+1)}(\sigma) = i\} \\ &= \min\{|\sigma| \ ; \ n(\sigma) - (j+1) = i\} \\ &= \min\{|\sigma| \ ; \ n(\sigma) - j = i + 1\} \\ &= \min\{|\sigma| \ ; \ n_{(j)}(\sigma) = i\} \qquad = \ d_{i+1}(\mathcal{M}_{(j)}) \end{aligned}$$

□

# Chapter 3

# Weight enumerator

In this chapter we will define the weight enumerator for extended codes and for matroids. In addition, we will write about the relation between a code and its dual code as shown in the MacWilliams identity.

## 3.1 Weight enumerator for codes and extended codes

Let $\mathcal{C}$ be an $[n, k]$-code over $\mathbb{F}_q$. Just as a reminder from the first chapter, its weight enumerator is defined as:

**Definition 3.1.1.**

$$W_{\mathcal{C}}(X, Y) = \sum_{j=0}^{n} A_{\mathcal{C},j} X^{n-j} Y^{j}$$

where $A_{\mathcal{C},j}$ denotes the number of codewords in $\mathcal{C}$ of weight $j$.

Let $\mathcal{C}$ be an $[n, k]$-code over $\mathbb{F}_q$ with generator matrix $G$. The set of all $\mathbb{F}_{q^r}$ linear combinations of words of $\mathcal{C}$ is itself a linear code, named *extension of* $\mathcal{C}$ *to* $\mathbb{F}_{q^r}$, denoted by $\mathcal{C} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^r}$. A codeword $w$ belonging to the extended code $\mathcal{C} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^r}$ can be expressed as $w = a \cdot G$, where $a = (a_1, ..., a_k) \in \mathbb{F}_{q^r}^k$ and $G$ is a generator matrix of $\mathcal{C}$ . We will see that the number of words of weight $j$ in $\mathcal{C} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^r}$ can be expressed in terms of the initial code $\mathcal{C}$ as a polynomial in $q^r$, whose coefficients depend only on $\mathcal{C}$.

**Lemma 3.1.1.** *Let $G = [c_1|...|c_n]$ be a generator matrix for the code $\mathcal{C}$. Let $a \in \mathbb{F}_Q^k$, where $Q = q^r$, and $A_i(Q) = \{a \in \mathbb{F}_Q^k; c_i^t a = 0\}$. Then, for $i_1 < \ldots < i_j$,*

$$\#(A_{i_1}(Q) \cap \ldots \cap A_{i_m}(Q)) = Q^{k-rank([c_{i_1}|...|c_{i_j}])}$$

*Proof.* Let $\varphi_{i,j}$ be the linear function generated by the columns $[c_{i_1}|\ldots|c_{i_j}]$ from $G$. The cardinality of the intersection is

$$\begin{aligned}
\#(A_{i_1}(Q) \cap \ldots \cap A_{i_m}(Q)) &= \# \ker \varphi_{i,j} \\
&= Q^{\dim \ker(\varphi_{i,j})} \\
&= Q^{k-\dim(Im\varphi_{i,j})} \\
&= Q^{k-\text{rank}([c_{i_1}|...|c_{i_j}])}
\end{aligned}$$

□

**Lemma 3.1.2.** *Let $\mathcal{C}$ be a $[n, k]$ linear code over $\mathbb{F}_q$. Then, exists $P \in \mathbb{Z}[T]$ of degree at most $k$ such that $\forall r$,*

$$P(q^r) = A_{\mathcal{C},n}(q^r) = \#\{\textit{codewords of weight } n \textit{ in } \mathcal{C} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^r}\}$$

*Proof.* Let $G = [c_1|c_2|\ldots|c_n]$ be a generator matrix of $\mathcal{C}$, where $c_j$ denotes the corresponding column $j$ in $G$. Let $i \in \{1, ..., n\}$ and $q^r = Q$.

For $0 \leq m \leq n$, let $A_{C,m}(Q)$ denotes the number of words of weight $m$ in $\mathcal{C} \otimes_{\mathbb{F}_q} \mathbb{F}_Q$.

Let $a = (a_1, ..., a_k) \in \mathbb{F}_Q^k$, then $w = a \cdot G$ is a codeword in $\mathcal{C} \otimes_{\mathbb{F}_q} \mathbb{F}_Q$.

The weight of a codeword in $\mathcal{C} \otimes_{\mathbb{F}_q} \mathbb{F}_Q$ is $n$ if and only if $w_i \neq 0 \ \forall i$, i.e. $c_i^t \cdot a \neq 0 \ \forall i$.

The number of words of weight $n$ is therefore

$$A_{C,n}(Q) = \#\{a \in \mathbb{F}_Q^k \ ; \ c_i^t \cdot a \neq 0 \ \forall i\}$$

Those are all words, except the ones that have weight different from $n$.

The words that have weight different from $n$ satisfy $c_i^t \cdot a = 0$ for some $i$, and the cardinality of such set of words is

$$\#\{a \in \mathbb{F}_Q^k \ ; \ \exists\, i \ \text{ s.t } c_i^t \cdot a = 0\} = \# \bigcup_{i=1}^{n} \{a \in \mathbb{F}_Q^k \ ; \ c_i^t \cdot a = 0\}$$

By using the inclusion/exclusion principle and the previous lemma,

$$\#(\bigcup_{i=1}^{n} A_i(Q)) = \sum_{i=1}^{n} \#(A_i(Q)) - \sum_{i<j} \#(A_i(Q) \cap A_j(Q)) + \ldots + (-1)^n \sum \#(\bigcap_{i=1}^{n} A_i(Q))$$

$$\begin{aligned} A_{C,n}(Q) &= Q^k - \#(\bigcup_{i=0}^{n} A_i(Q)) = \\ &= Q^k - \sum_{i\in\{1,\ldots,n\}} Q^{k-\text{rank}([c_i])} + \sum_{i<j} Q^{k-\text{rank}([c_i,c_j])} - \ldots + \\ &+ (-1)^l \sum_{i_1<\ldots<i_l} Q^{k-\text{rank}([c_{i_1},\ldots,c_{i_l}])} - \ldots + (-1)^n Q^{k-\text{rank}(G)} \end{aligned}$$

□

**Corollary.** *$P$ depends only on the matroid associated to $\mathcal{C}$.*

*Proof.*

$$\begin{aligned} \sum_{i_1<\ldots<i_l} Q^{k-\text{rank}([c_{i_1},\ldots,c_{i_l}])} &= \sum_{i_1<\ldots<i_l} Q^{k-r_{\mathcal{M}[G]}(\{i_1,\ldots,i_l\})} \\ &= \sum_{|\sigma|=l} Q^{k-r_{\mathcal{M}[G]}(\sigma)} \\ &= \sum_{|\sigma|=l} Q^{n_{\mathcal{M}[H]}(E\setminus\sigma)} \\ &= \sum_{|\sigma|=n-l} Q^{n_{\mathcal{M}[H]}(\sigma)} \end{aligned}$$

Therefore,

$$A_{C,n}(Q) = (-1)^n \sum_{\sigma\subseteq E} (-1)^{|\sigma|} Q^{n_{\mathcal{M}[H]}(\sigma)}$$

□

**Lemma 3.1.3.** *For $\sigma \subset E$, let $a_{\mathcal{C},\sigma}(Q) = \#\{w \in \mathcal{C} \otimes_{\mathbb{F}_q} \mathbb{F}_Q \ ; \ supp(w) = \sigma\}$. Then,*

$$a_{\mathcal{C},\sigma}(Q) = (-1)^{|\sigma|} \sum_{\gamma \subseteq \sigma} (-1)^{|\gamma|} Q^{n_{\mathcal{M}[H]}(\gamma)}$$

*Proof.* Let $C_\sigma(Q)$ denotes the shortening of $\mathcal{C} \otimes_{\mathbb{F}_q} \mathbb{F}_Q$ in $E \backslash \sigma$, and let $H|_\sigma$ denotes the restriction of $H$ to the columns indexed by $\sigma$.

Since the shortening of $\mathcal{C}$ by $J = \{j_1, \ldots, j_n\}$ is define as the code with parity check matrix obtained by deleting the $c_{j_i}$ columns from $H$, we have that $H|_\sigma$ is a parity check matrix for $C_\sigma(Q)$.

It is clear that $a_{\mathcal{C},\sigma}(Q) = a_{\mathcal{C}|_\sigma,\sigma}(Q)$.

Since $\mathcal{M}[H]|_\sigma \simeq \mathcal{M}[H|_\sigma]$, and because of $n_{\mathcal{M}[H]}|_\sigma(\gamma) = n_{\mathcal{M}[H]}(\gamma)$, $\forall \gamma \subseteq \sigma$, aplying the latest corollary:

$$a_{\mathcal{C},\sigma}(Q) = (-1)^{|\sigma|} \sum_{\gamma \subseteq \sigma} (-1)^{|\gamma|} Q^{n_{\mathcal{M}[H]|_\sigma}(\gamma)}$$

□

**Proposition 3.1.4.** *Let $\mathcal{C}$ be a $[n,k]$ linear code over $\mathbb{F}_q$ and $j \in [0, \ldots, n]$. Then, $\exists P_j \in \mathbb{Z}[T]$ of degree at most $k$ such that $\forall r$,*

$$P_j(q^r) = A_{\mathcal{C},j}(q^r) = \#\{\textit{codewords of weight } j \textit{ in } \mathcal{C} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^r}\}$$

*Proof.* For $1 \leq m \leq n$ we have,

$$\begin{aligned} A_{\mathcal{C},m}(Q) &= \sum_{|\sigma|=m} a_{\mathcal{C},\sigma}(Q) \\ &= (-1)^m \sum_{|\sigma|=m} \sum_{\gamma \subseteq \sigma} (-1)^{|\gamma|} Q^{n_{\mathcal{M}[H]}(\gamma)} \end{aligned}$$

□

***Remark.*** As before, $P_j$ depends only on the matroid $\mathcal{M}$.

**Definition 3.1.2.** The *extended weight enumerator* for the code $\mathcal{C}$ is the polynomial

$$W_{\mathcal{C}}(X,Y,T) = \sum_{j=0}^{n} A_j(T) X^{n-j} Y^j$$

where $A_j(T)$ are integral polynomials in $T$ and $A_j(q^r)$ are the number of codewords of weight $j$ in $\mathcal{C} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^r}$.

## 3.2 Weight enumerator for matroids

Let $\mathcal{M}$ be a matroid over the ground set $E = \{1, ..., n\}$.

**Definition 3.2.1.** The *Tutte polynomial* of $\mathcal{M}$ is defined by

$$T_{\mathcal{M}}(X,Y) = \sum_{\sigma \subseteq E} (X-1)^{r(E)-r(\sigma)} (Y-1)^{n(\sigma)}$$

***Remark.***

$T_{\mathcal{M}}(1,1)$ gives us the number of bases of $\mathcal{M}$.

$T_{\mathcal{M}}(2,1)$ gives us the number of independent sets of $\mathcal{M}$.

**Definition 3.2.2.** The *Generalized weight Polynomials (GWP)* are defined as follows:

$$\begin{aligned} P_{\mathcal{M},0}(T) &= 1 \\ P_{\mathcal{M},j}(T) &= (-1)^j \sum_{|\sigma|=j} \sum_{\gamma \subseteq \sigma} (-1)^{|\gamma|} T^{n_{\mathcal{M}}(\gamma)} \qquad \text{for } 1 \leq j \leq n \end{aligned}$$

and we call them the *j-th generalized weight polynomials* or just *GWP* of $\mathcal{M}$.

**Example 3.2.1.** Let $\mathcal{C}$ be the code

$$\mathcal{C} = \{\{0,0,0\},\{0,1,1\},\{1,0,1\},\{1,1,0\}\}$$

A generator matrix for $\mathcal{C}$ is

$$G = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

The parity check matrix for $\mathcal{C}$ from $G$ is

$$H = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

Therefore, the set of bases of its associated matroid $\mathcal{M}$ is

$$\mathcal{B}(\mathcal{M}) = \{\{1\},\{2\},\{3\}\}$$

Let us write the following table for easier computations:

| $\sigma$ | $r(\sigma)$ | $n(\sigma)$ |
|---|---|---|
| $\emptyset$ | 0 | 0 |
| $\{1\}$ | 1 | 0 |
| $\{2\}$ | 1 | 0 |
| $\{3\}$ | 1 | 0 |
| $\{1,2\}$ | 1 | 1 |
| $\{1,3\}$ | 1 | 1 |
| $\{2,3\}$ | 1 | 1 |
| $\{1,2,3\}$ | 1 | 2 |

Let us now apply the formula of the GWP to this concrete matroid:

$$\begin{aligned}
P_{\mathcal{M},0}(T) &= 1\\
P_{\mathcal{M},1}(T) &= (-1)^1 \sum_{|\sigma|=1} \sum_{\gamma\subseteq\sigma} (-1)^{|\gamma|} T^{n_{\mathcal{M}}(\gamma)} =\\
&= (-1)\cdot\Big([(-1)^{|\emptyset|}T^{n(\emptyset)} + (-1)^{|\{1\}|}T^{n(\{1\})}] +\\
&+ [(-1)^{|\emptyset|}T^{n(\emptyset)} + (-1)^{|\{2\}|}T^{n(\{2\})}] +\\
&+ [(-1)^{|\emptyset|}T^{n(\emptyset)} + (-1)^{|\{3\}|}T^{n(\{3\})}]\Big) =\\
&= (-1)\cdot 0 = 0
\end{aligned}$$

$$\begin{aligned}
P_{\mathcal{M},2}(T) &= (-1)^2 \sum_{|\sigma|=2} \sum_{\gamma\subseteq\sigma} (-1)^{|\gamma|} T^{n_{\mathcal{M}}(\gamma)} =\\
&= (-1)^2\cdot\Big([(-1)^{|\emptyset|}T^{n(\emptyset)} + (-1)^{|\{1\}|}T^{n(\{1\})} +\\
&+ (-1)^{|\{2\}|}T^{n(\{2\})} + (-1)^{|\{1,2\}|}T^{n(\{1,2\})}] +\\
&+ [(-1)^{|\emptyset|}T^{n(\emptyset)} + (-1)^{|\{1\}|}T^{n(\{1\})} +\\
&+ (-1)^{|\{3\}|}T^{n(\{3\})} + (-1)^{|\{1,3\}|}T^{n(\{1,3\})}] +\\
&+ [(-1)^{|\emptyset|}T^{n(\emptyset)} + (-1)^{|\{2\}|}T^{n(\{2\})} + (-1)^{|\{3\}|}T^{n(\{3\})} +\\
&+ (-1)^{|\{2,3\}|}T^{n(\{2,3\})}]\Big) = 3T - 3
\end{aligned}$$

$$\begin{aligned}
P_{\mathcal{M},3}(T) &= (-1)^3 \sum_{|\sigma|=3} \sum_{\gamma\subseteq\sigma} (-1)^{|\gamma|} T^{n_{\mathcal{M}}(\gamma)} =\\
&= (-1)^3\cdot\Big((-1)^{|\emptyset|}T^{n(\emptyset)} + (-1)^{|\{1\}|}T^{n(\{1\})} +\\
&+ (-1)^{|\{2\}|}T^{n(\{2\})} + (-1)^{|\{3\}|}T^{n(\{3\})} +\\
&+ (-1)^{|\{1,2\}|}T^{n(\{1,2\})} + (-1)^{|\{1,3\}|}T^{n(\{1,3\})} +\\
&+ (-1)^{|\{2,3\}|}T^{n(\{2,3\})} + (-1)^{|\{1,2,3\}|}T^{n(\{1,2,3\})}\Big) =\\
&= T^2 - 3T + 2
\end{aligned}$$

Just in order to check the computations, we can replace $T$ for $q^r$ when $q = 2$ and $r = 1$, and look at the results for our code $\mathcal{C}$.

$$\begin{array}{rcll}
P_{\mathcal{M},0}(2) & = & 1 & \rightsquigarrow \text{ 1 word of weight } 0: \{0,0,0\}. \\
P_{\mathcal{M},1}(2) & = & 0 & \rightsquigarrow \text{ No words of weight 1.} \\
P_{\mathcal{M},2}(2) & = & 3\cdot 2-3=3 & \rightsquigarrow \text{ 3 words of weight } 2: \{0,1,1\},\{1,0,1\},\{1,1,0\} \\
P_{\mathcal{M},3}(2) & = & 2^2-3\cdot 2+2=0 & \rightsquigarrow \text{ No words of weight 3 .}
\end{array}$$

The GWP can be expressed in terms of Betti numbers as follows:

**Theorem 3.2.1.** *For each $1 \leq j \leq n$, and $l \in [0, n-r(E)]$, the coefficient of $T^l$ in $P_{\mathcal{M},j}$ is equal to*

$$\sum_{i=0}^{n}(-1)^i(\beta_{i,j}(I_{\mathcal{M}_{(l-1)}}) - \beta_{i,j}(I_{\mathcal{M}_{(l)}}))$$

*Proof.* [18, Theorem 5.1] □

***Remark.*** In our case, the formula will be used as

$$\sum_{i=1}^{n}(-1)^{i+1}(\beta_{i,j}(I_{\mathcal{M}_{(l-1)}}) - \beta_{i,j}(I_{\mathcal{M}_{(l)}}))$$

since we will use this theorem for computing the Betti numbers for the Stanley-Reisner ring instead of the ones from the Stanley-Reisner ideal, and they satisfy

$$\beta_{i,j}(I_{\mathcal{M}}) = \beta_{i+1,j}(S/I_{\mathcal{M}})$$

**Example 3.2.2.** For $\mathcal{M}$ from Example 3.2.1,

Free resolutions for $I_{\mathcal{M}}$ and $I_{\mathcal{M}_{(1)}}$, respectively, are:

$$0 \to S(-3)^2 \to S(-2)^3 \to I_{\mathcal{M}} \to 0$$

$$0 \to S(-3) \to I_{\mathcal{M}_{(1)}} \to 0$$

So, the Betti numbers $\beta_{i,d_i}(\mathcal{I}_{\mathcal{M}_{(l)}})$ that are different from 0 are

$$\beta_{1,2}(I_{\mathcal{M}}) = 3 \quad \beta_{2,3}(I_{\mathcal{M}}) = 2 \quad \beta_{1,3}(I_{\mathcal{M}_{(1)}}) = 1$$

Also assume that $\beta_{i,j}(I_{\mathcal{M}_{(l)}}) = 0$ whenever $l \notin [0,2]$ .

Then, the coefficients for $T^l$ in $P_{\mathcal{M},j}(T)$ are:

$$\begin{array}{lcll}
\mathrm{Coeff}_{j=1}(T^0) & = & \sum_{i=0}^{3}(-1)^i(\beta_{i,1}(I_{\mathcal{M}_{(-1)}}) - \beta_{i,1}(I_{\mathcal{M}})) & = 0 \\
\mathrm{Coeff}_{j=1}(T^1) & = & \sum_{i=0}^{3}(-1)^i(\beta_{i,1}(I_{\mathcal{M}}) - \beta_{i,1}(I_{\mathcal{M}_{(1)}})) & = 0 \\
\mathrm{Coeff}_{j=1}(T^2) & = & \sum_{i=0}^{3}(-1)^i(\beta_{i,1}(I_{\mathcal{M}_{(1)}}) - \beta_{i,1}(I_{\mathcal{M}_{(2)}})) & = 0 \\
\mathrm{Coeff}_{j=2}(T^0) & = & \sum_{i=0}^{3}(-1)^i(\beta_{i,2}(I_{\mathcal{M}_{(-1)}}) - \beta_{i,2}(I_{\mathcal{M}})) & = -3 \\
\mathrm{Coeff}_{j=2}(T^1) & = & \sum_{i=0}^{3}(-1)^i(\beta_{i,2}(I_{\mathcal{M}}) - \beta_{i,2}(I_{\mathcal{M}_{(1)}})) & = 3 \\
\mathrm{Coeff}_{j=2}(T^2) & = & \sum_{i=0}^{3}(-1)^i(\beta_{i,2}(I_{\mathcal{M}_{(1)}}) - \beta_{i,2}(I_{\mathcal{M}_{(2)}})) & = 0 \\
\mathrm{Coeff}_{j=3}(T^0) & = & \sum_{i=0}^{3}(-1)^i(\beta_{i,3}(I_{\mathcal{M}_{(-1)}}) - \beta_{i,3}(I_{\mathcal{M}})) & = 2 \\
\mathrm{Coeff}_{j=3}(T^1) & = & \sum_{i=0}^{3}(-1)^i(\beta_{i,3}(I_{\mathcal{M}}) - \beta_{i,3}(I_{\mathcal{M}_{(1)}})) & = -3 \\
\mathrm{Coeff}_{j=3}(T^2) & = & \sum_{i=0}^{3}(-1)^i(\beta_{i,3}(I_{\mathcal{M}_{(1)}}) - \beta_{i,3}(I_{\mathcal{M}_{(2)}})) & = 1
\end{array}$$

**Definition 3.2.3.** The *matroid enumerator* is the polynomial:

$$W_{\mathcal{M}}(X,Y,T) = \sum_{j=0}^{n} P_{\mathcal{M},j}(T) X^{n-j} Y^j$$

***Property***. If $\mathcal{C}$ is a linear code with parity check matrix $H$ and extended weight enumerator $W_{\mathcal{C}}(X,Y,T)$ then,

$$W_{\mathcal{C}}(X,Y,T) = W_{\mathcal{M}[H]}(X,Y,T)$$

## 3.3 Duality

The weight enumerator of a code and the weight enumerator of its dual code are related by the MacWilliams identity as follows:

**Theorem 3.3.1.** *Let $\mathcal{C}$ be a linear code and let $\mathcal{C}^\perp$ be its dual. Then, the weight enumerator of $\mathcal{C}$ completely determines the weight enumerator of $\mathcal{C}^\perp$ and vice versa, via the following formula:*

$$W_{\mathcal{C}^\perp}(X,Y) = q^{-k} \cdot W_{\mathcal{C}}(X+(q-1)Y, X-Y)$$

*Proof.* [12, Theorem 5.13] □

**Example 3.3.1.**

Let $\mathcal{C}$ be the following constant weight code:

$$\mathcal{C} = \{\{0,0,0\},\{0,1,1\},\{1,0,1\},\{1,1,0\}\}$$

and $\mathcal{C}^\perp$ its dual.

$$\mathcal{C}^\perp = \{\{0,0,0\},\{1,1,1\}\}$$

Then, their weight enumerators are, respectively:

$$\begin{aligned} W_{\mathcal{C}}(X,Y) &= X^3 + 3XY^2 \\ W_{\mathcal{C}}^\perp(X,Y) &= X^3 + Y^3 \end{aligned}$$

When we use the MacWilliams identity we observe that:

$$\begin{aligned} W_{\mathcal{C}}^\perp(X,Y) &= \frac{1}{4} W_{\mathcal{C}}(X+Y, X-Y) \\ &= \frac{1}{4}[(X+Y)^3 + 3(X+Y)(X-Y)^2] \\ &= X^3 + Y^3 \end{aligned}$$

and

$$\begin{aligned} W_{\mathcal{C}}(X,Y) &= \frac{1}{2} W_{\mathcal{C}}^\perp(X+Y, X-Y) \\ &= \frac{1}{2}[(X+Y)^3 + (X-Y)^3] \\ &= X^3 + 3XY^2 \end{aligned}$$

MacWilliams identity is also true for the extended weight enumerator.

**Theorem 3.3.2.** *Let $\mathcal{C}$ be a linear code and let $\mathcal{C}^{\perp}$ be its dual. Then, the extended weight enumerator of $\mathcal{C}$ completely determines the extended weight enumerator of $\mathcal{C}^{\perp}$ and vice versa, via the following formula:*

$$W_{\mathcal{C}^{\perp}}(X,Y,T) = T^{-k} \cdot W_{\mathcal{C}}(X+(T-1)Y, X-Y, T)$$

*Proof.* [12, Theorem 5.13] □

# Chapter 4

# Constant weight codes

In this chapter we will focus on constant weight codes. We will find a formula for the Betti numbers of elongations from the formula given in [5]. The $\mathbb{N}$-graded resolutions associated to constant weight codes are pure, but not linear ( [10]) and we will also see that it is enough to know the first Betti number to know the rest of them. From the Betti numbers we can also obtain the weight hierarchy and vice versa.

## 4.1 Definition and properties

**Definition 4.1.1.** A *constant weight code* is a code where all non zero codewords have the same Hamming weight.

**Example 4.1.1.** The following matrix generates a constant weight code over $\mathbb{F}_2$:

$$G = \begin{bmatrix} 0\,0\,0\,1\,1\,1\,1 \\ 0\,1\,1\,0\,0\,1\,1 \\ 1\,0\,1\,0\,1\,0\,1 \end{bmatrix}$$

An associated matroid to its parity check matrix is:
$\mathcal{M} = \{\{2,4,6,7\},\{3,4,6,7\},\{3,5,6,7\},\{2,3,5,6\},\{1,2,4,5\},\{1,4,6,7\},$
$\{1,4,5,7\},\{1,5,6,7\},\{2,4,5,7\},\{1,2,6,7\},\{2,3,4,6\},\{3,4,5,6\},\{2,3,5,7\},$
$\{2,5,6,7\},\{2,4,5,6\},\{1,2,3,7\},\{1,2,3,5\},\{1,2,3,6\},\{1,3,4,7\},\{1,2,5,7\},$
$\{1,3,5,6\},\{1,2,3,4\},\{2,3,4,7\},\{1,3,6,7\},\{1,4,5,6\},\{3,4,5,7\},\{1,3,4,5\},$
$\{1,2,4,6\}\}$

A minimal free resolution for its Stanley-Reisner ideal

$I_{\mathcal{M}} =< x_1x_3x_5x_7, x_2x_3x_4x_5, x_4x_5x_6x_7, x_1x_2x_5x_6, x_1x_3x_4x_6, x_2x_3x_6x_7, x_1x_2x_4x_7 >$

is:

$$0 \to S(-7)^8 \to S(-6)^{14} \to S(-4)^7 \to I_{\mathcal{M}} \to 0$$

which is pure, but not linear.

A minimal free resolution for its Alexander dual is:

$$0 \to S(-7)^8 \to S(-6)^{42} \to S(-5)^{84} \to S(-4)^{77} \to S(-3)^{28} \to I_{\mathcal{M}^\vee} \to 0$$

which is pure and linear, as we expected. ( [3, Theorem 3]

**Theorem 4.1.1.** *Let $\mathcal{C}$ be a $[n,k,d]$-code over $\mathbb{F}_q$. Let $1 \leq s \leq k-1$. Suppose that all the $s$-dimensional linear subcodes of $\mathcal{C}$ have the same weight $d_s$. Then, for every $0 \leq t \leq k$, and every subcode $D_t$ of dimension $t$ of $\mathcal{C}$, we have:*

$$wt(D_t) = d_t = d_s \frac{q^k - q^{k-t}}{q^k - q^{k-s}}$$

*Proof.* [15, Theorem 1] □

***Remark.*** This theorem shows that being of constant weight is the same as being of constant weight in any dimension, except in dimension 0 and dimension $k$.

The hierarchy $(d_1, ..., d_k)$ of a constant weight code is given in ( [10]) by

$$d_i = d\frac{q^i - 1}{q^{i-1}(q-1)} \tag{4.1}$$

where $d$ is the minimun distance.

Conversely, a $[n, k, d]_q$-code where $d_k = d_i \frac{q^k-1}{q^{k-i}(q^i-1)}$, for some $1 \leq i < k$ is a constant weight code of weight $d_k \frac{q^{k-1}(q-1)}{q^k-1}$.

**Example 4.1.2.** The weight hierarchy of the code from Example 4.1.1, is:

$$\begin{array}{rcl} d_1 &=& 4 \\ d_2 &=& 4 \cdot \frac{2^2-1}{2} &=& 6 \\ d_3 &=& 4 \cdot \frac{2^3-1}{2^2} &=& 7 \end{array}$$

## 4.2 Betti Numbers for the Stanley-Reisner ring

Let $\mathbb{K}$ be a field, $S = \mathbb{K}[x_1, \ldots, x_n]$ and $\Delta$ a simplicial complex such that $\dim(\Delta) = d - 1$. Let $I_\Delta$ be its Stanley-Reisner ideal. We know that there exists a minimal free resolution for $S/I_\Delta$, and, due to Hilbert Syzygy's theorem, we know that the resolution is finite and its length is less or equal to $n$. Let us take $I_\Delta$ such that $\text{proj.dim}(S/I_\Delta) = k$, which means that the minimal length among all finite projective resolutions (or, equivalently, by Quillen-Suslin theorem, free resolutions) is $k$.

### 4.2.1 Simplicial complexes with pure resolution

Let us consider a particular case, when $S/I_\Delta$ has a pure resolution. Then, there exists a strictly increasing sequence of integers $d_0 < \ldots < d_k$ such that the resolution of $S/I_\Delta$ has the form

$$0 \to S(-d_k)^{\beta_k} \to \ldots \to S(-d_1)^{\beta_1} \to S(-d_0)^{\beta_0} \to S/I_\Delta \to 0$$

We will show that the Betti numbers $\beta_i$ for $1 \leq i \leq k$ satisfy a set of equations.

The Hilbert series and the Betti numbers are related by

$$H_{S/I_\Delta}(t) = \frac{\sum_{i=0}^{k} \beta_i t^{d_i}}{(1-t)^n}$$

as shown in ( [6, Section 6.1.3])

and so,

$$\sum_{i=0}^{k} (-1)^i \beta_i \; t^{d_i} = H_{S/I_\Delta}(t)(1-t)^n$$

We also know, due to Hilbert's theorem, [6, Th.6.1.3], that there exists a Laurent polynomial $Q_\Delta(t) \in \mathbb{Z}[t, t^{-1}]$ with $Q_\Delta(1) > 0$ such that

$$H_{S/I_\Delta}(t) = \frac{Q_\Delta(t)}{(1-t)^d}$$

where $d-1$ is the dimension of the simplicial complex.

In consequence,

$$\sum_{i=0}^{k} (-1)^i \beta_i \; t^{d_i} = Q_\Delta(t)(1-t)^{n-d}$$

If we differenciate $m$ times, for $0 \leq m < n-d$, we obtain

$$\sum_{i=0}^{k} (-1)^i \beta_i \binom{d_i}{m} t^{d_i - m} = \frac{\partial^m}{\partial t^m}[(1-t)^{n-d} Q_\Delta(t)]$$

where $\binom{d_i}{m} = 0 \;\; \forall d_i < m$.

Taking $t = 1$ in every derivation, we will obtain the *Herzog-Kühl* equations:

$$\sum_{i=0}^{k}(-1)^i\beta_i d_i^m = 0$$

***Remark.***

The Herzog-Kühl equations are not obtained directly from the substitution of $t$ for 1, but the computations we need are simple:

If we take $t = 1$ in

$$\sum_{i=0}^{k}(-1)^i\beta_i \; t^{d_i} = Q_\Delta(t)(1-t)^{n-d}$$

we obtain

$$\sum_{i=0}^{k}(-1)^i\beta_i = 0$$

If we take $t = 1$ in the first derivation, we obtain

$$\sum_{i=0}^{k}(-1)^i\beta_i d_i = 0$$

If we take $t = 1$ in the second derivation, we obtain

$$\sum_{i=0}^{k}(-1)^i\beta_i d_i(d_i - 1) = 0$$

Therefore,

$$\sum_{i=0}^{k}(-1)^i\beta_i d_i^2 - \sum_{i=0}^{k}(-1)^i\beta_i d_i = \sum_{i=0}^{k}(-1)^i\beta_i d_i^2 - 0 = 0$$

which gives us the third Herzog-Kühl equation

$$\sum_{i=0}^{k}(-1)^i\beta_i d_i^2 = 0$$

Proceedig as before, we will get all of them.

From the Herzog-Kühl equations, we obtain the following system:

$$\begin{bmatrix} d_0^0 & -d_1^0 & d_2^0 & \dots & (-1)^k d_k^0 \\ d_0^1 & -d_1^1 & d_2^1 & \dots & (-1)^k d_k^1 \\ d_0^2 & -d_1^2 & d_2^2 & \dots & (-1)^k d_k^2 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ d_0^{n-d-1} & -d_1^{n-d-1} & d_2^{n-d-1} & \dots & (-1)^k d_k^{n-d-1} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Taking $\beta_i' = (-1)^k \beta_i$ we obtain the following *Vandermonde* system of equations:

$$\begin{bmatrix} d_0^0 & \dots & d_k^0 \\ \vdots & & \vdots \\ d_0^{n-d-1} & \dots & d_k^{n-d-1} \end{bmatrix} \begin{bmatrix} \beta_0' \\ \vdots \\ \beta_k' \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

### 4.2.2 Cohen Macaulay simplexes with pure resolution

In adittion of having pure resolution, we consider now $\Delta$ as a Cohen-Macaulay simplicial complex.

**Definition 4.2.1.** A simplicial complex $\Delta$ is said to be *Cohen-Macaulay* if its Stanley-Reisner ring $S/I_\Delta$ is a Cohen-Macaulay ring, i.e, if

$$\text{krull.dim}(S/I_\Delta) = \text{depth}(S/I_\Delta)$$

where the krull dimension is the supremum of the lengths of all chains of prime ideals in $S/I_\Delta$ and the depth is the longest regular sequence for $S/I_\Delta$.

Then, we have

$$\text{depth}(S/I_\Delta) = \text{krull.dim}(S/I_\Delta) = \dim(\Delta) + 1 = d$$

Due to *Auslander-Buchsbaum* formula ( [6, Corollary A.4.3])

$$\operatorname{depth}(S/I_\Delta) + \operatorname{proj.dim}(S/I_\Delta) = \operatorname{depth}(S)$$

so we get

$$\operatorname{depth}(S/I_\Delta) = n - k$$

**Proposition 4.2.1.** *Let $\mathcal{M}$ be a matroid over $E = \{1, \dots, n\}$ of rank $r$. Then, $\mathcal{M}$ is Cohen Macaulay.*

*Proof.* Since the rank of $\mathcal{M}$ is $r$, it makes $\mathcal{I}(\mathcal{M})$ to be a simplicial complex of dimension $r - 1$. Due to [6, Corolary 6.6.2], $\operatorname{krull.dim}(S/I_{\mathcal{M}}) = r$, so $\operatorname{proj.dim}(S/I_{\mathcal{M}}) = n - r$.

Auslander-Buchsbaum formula gives us the desired result:

$$\operatorname{depth}(S/I_{\mathcal{M}}) = n - (n - r) = r = \operatorname{krull.dim}(S/I_{\mathcal{M}})$$

□

Now, if we allow the previous Cohen-Macaulay simplicial complex to be a matroid $\mathcal{M}$ of rank $r$, we have that $\dim(\mathcal{M}) = r - 1$, so $\operatorname{depth}(S/I_{\mathcal{M}}) = r$ and, as we considered before, $\operatorname{proj.dim}(S/I_{\mathcal{M}}) = k$.

At this point, we get the following set of equations:

$$\begin{bmatrix} d_0^0 & \dots & d_{n-d}^0 \\ \vdots & & \vdots \\ d_0^{n-d-1} & \dots & d_{n-d}^{n-d-1} \end{bmatrix} \begin{bmatrix} \beta_0' \\ \vdots \\ \beta_{n-d}' \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

which has $n - d + 1$ columns and $n - d$ rows. Since it is a non degenerate Vandermonde matrix, it is of maximal rank, $n - d$, and so, the dimension of the kernel is 1.

Supposing that $[\gamma_0, ..., \gamma_k]$ is a non-zero solution for the non-degenerate Vandermonde system, any solution will be of the form $[q\gamma_0, ..., q\gamma_k]$ for $q \in \mathbb{K}$. As a result, all $\beta_i$ are given $\forall i \in \{1, ..., k\}$, since we know $\beta_0 = 1$ and $d_0 = 0$.

The formula given in [5, Section 1.4] finds the Betti numbers by computing the solutions of the previous Vandermonde determinants:

$$\begin{aligned} \beta_{i,d_i} &= (-1)^i \cdot t \cdot \prod_{s\neq i} \frac{1}{d_s - d_i} \qquad\qquad \text{for some } t \\ \beta_{0,0} &= t \cdot \prod_{s\neq 0} \frac{1}{d_s - d_0} = 1 \end{aligned}$$

This is in particular true if $\Delta$ is a matroid having a pure resolution, since all matroids have a Cohen-Macaulay Stanley-Reisner ring. So, if $\mathcal{M}$ is a matroid of rank $r$, it has dimension $r-1$. We get the following system:

$$\begin{bmatrix} d_0^0 & \dots & d_{n-r}^0 \\ \vdots & & \vdots \\ d_0^{n-r-1} & \dots & d_{n-r}^{n-r-1} \end{bmatrix} \begin{bmatrix} \beta_0' \\ \vdots \\ \beta_{n-r}' \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

and

$$\beta_{i,d_i} = (-1)^i \cdot t \cdot \prod_{s\neq i} \frac{1}{d_s - d_i} \qquad\qquad \text{for some } t \tag{4.2}$$

$$t = \prod_{s\neq 0} d_s$$

### 4.2.3 Matroids associated to constant weight codes

Let $\mathcal{M}$ be a matroid associated to a constant weight code. Then, we know that it is Cohen-Macaulay and has pure resolution.

Replacing $t$ and $d_i$ in ( 4.2), (using ( 4.1))

$$\begin{aligned}
\beta_{i,d_i} &= (-1)^i \cdot \prod_{s=1}^{k} d_s \cdot \prod_{s\neq i} \frac{1}{d_s - d_i} \\
&= (-1)^i \cdot \prod_{s=1}^{k} \frac{d(q^s-1)}{q^{s-1}(q-1)} \cdot \prod_{s=0}^{i-1} \frac{q^{s+i-1}(q-1)}{d(q^s-q^i)} \cdot \prod_{s=i+1}^{k} \frac{q^{s+i-1}(q-1)}{d(q^s-q^i)} \\
&= (-1)^i \cdot \prod_{s=1}^{k} \frac{q^s-1}{q^{s-1}} \cdot \prod_{s=0}^{i-1} \frac{q^{i-1}}{1-q^{i-s}} \cdot \prod_{s=i+1}^{k} \frac{q^{s-1}}{q^{s-i}-1} \\
&= (-1)^i \cdot \prod_{s=1}^{k} \frac{q^s-1}{q^{s-1}} \cdot \prod_{m=1}^{i} \frac{q^{i-1}}{1-q^m} \cdot \prod_{m=1}^{k-i} \frac{q^{m+i-1}}{q^m-1} \\
&= \frac{\prod_{s=1}^{k} q^s - 1}{\prod_{m=1}^{i} q^m - 1 \cdot \prod_{m=1}^{k-i} q^m - 1} \cdot \frac{\prod_{m=1}^{i} q^{i-1} \cdot \prod_{m=1}^{k-i} q^{m+i-1}}{\prod_{s=1}^{k} q^s - 1}
\end{aligned}$$

The $q$ *binomial* is defined as

$$\begin{bmatrix} k \\ i \end{bmatrix}_q = \frac{f(k,q)}{f(r,q)f(k-r,q)}$$

where $f(n,q) = \prod_{i=1}^{n}(q^i-1)$..

Then,

$$\beta_{i,d_i} = \begin{bmatrix} k \\ i \end{bmatrix}_q q^{\frac{i(i-1)}{2}}$$

Since all the elongations will have pure resolutions, we can apply to them the same strategy that we have applied to $\mathcal{M}$.

Let us extend the definition of weight hierarchy to $d_0$.

$$d_i = \min\{|\sigma| \ ; \ n(\sigma) = i\}, \quad \text{for } 0 \leq i \leq n - r(\mathcal{M})$$

Let $d_{i(j)}$ denotes the weight hierarchy for the $j$-th elongation of the matroid $\mathcal{M}$ associated to the constant weight code and $\beta_{i,d_{i(j)}}$ its Betti numbers. We have:

Let $0 \leq j \leq k$.

$$\begin{aligned} t_{(j)} &= \prod_{s\neq 0} d_{s(j)} &&= \prod_{s=1}^{k-j} d_s \\ d_{s(j)} &= d_{s+j} &&= d\tfrac{q^{s+j}-1}{q^{s+j-1}(q-1)} \qquad \text{for } 1 \leq s \leq k-j \end{aligned}$$

$$\begin{aligned} \beta_{i,d_{i(j)}} &= (-1)^i \cdot \prod_{s=1}^{k-j} d_s \cdot \prod_{s\neq i} \frac{1}{d_{s(j)} - d_{i(j)}} \\ &= (-1)^{i+1} \cdot \prod_{s=1}^{k-j} \frac{q^{s+j}-1}{q^{s+j-1}} \cdot \frac{q^{i+j-1}}{q^{i+j}-1} \cdot \prod_{s=1}^{i-1} \frac{q^{i+j-1}}{1-q^{i-s}} \cdot \prod_{s=i+1}^{k-j} \frac{q^{s+j-1}}{q^{s-i}-1} \\ &= \prod_{s=1}^{k-j} \frac{q^{s+j}-1}{q^{s+j-1}} \cdot \frac{q^{i+j-1}}{q^{i+j}-1} \cdot \prod_{m=1}^{i-1} \frac{q^{i+j-1}}{q^m-1} \cdot \prod_{m=1}^{k-i-j} \frac{q^{m+i+j-1}}{q^m-1} \\ &= \frac{q^{i+j-1} \cdot \prod_{m=1}^{i-1} q^{i+j-1} \cdot \prod_{m=1}^{k-i-j} q^{m+i+j-1}}{\prod_{s=1}^{k-j} q^{s+j-1}} \cdot \frac{\prod_{s=1}^{k-j} (q^{s+j}-1)}{(q^{i+j}-1) \cdot \prod_{m=1}^{i-1} (q^m-1) \cdot \prod_{m=1}^{k-i-j} (q^m-1)} \\ &= q^{\frac{i(i-1)}{2}} \cdot \frac{\prod_{m=1}^{k} (q^m-1) \cdot \prod_{m=i}^{i+j-1} (q^m-1)}{\prod_{m=1}^{j} (q^m-1) \cdot \prod_{m=1}^{i+j} (q^m-1) \cdot \prod_{m=1}^{k-i-j} (q^m-1)} \\ &= q^{\frac{i(i-1)}{2}} \cdot \frac{\prod_{m=i}^{i+j-1} (q^m-1)}{\prod_{m=1}^{j} (q^m-1)} \cdot \begin{bmatrix} k \\ i+j \end{bmatrix}_q \\ &= q^{\frac{i(i-1)}{2}} \cdot \begin{bmatrix} i+j-1 \\ j \end{bmatrix}_q \cdot \begin{bmatrix} k \\ i+j \end{bmatrix}_q \end{aligned}$$

**Example 4.2.1.** Using this formulas is easy to compute the Betti numbers for the resolutions for the elongations of the matroid from Example 4.1.1. The resolutions for the Stanley Reisner ideals for the elongations of $\mathcal{M}$ are:

$$0 \to S(-7)^8 \to S(-6)^{14} \to S(-4)^7 \to I_{\mathcal{M}} \to 0$$

$$0 \to S(-7)^6 \to S(-6)^7 \to I_{\mathcal{M}_{(1)}} \to 0$$

$$0 \to S(-7) \to I_{\mathcal{M}_{(2)}} \to 0$$

## 4.3 Weight enumerator

In this section we will analyze the weight enumerator of a constant weight code, from which we will try to obtain as much information as possible. In addition, we will find out that the dual of a Hamming code is, in fact, a constant weight code. Hence, given a Hamming code, we will be able to obtain its weight enumerator just by using the information from its parameters.

Let us write, as a reminder, the definition of weight enumerator.

$$W_{\mathcal{M}}(X,Y,T) = \sum_{j=0}^{n} P_{\mathcal{M},j}(T)X^{n-j}Y^j$$

### 4.3.1 Duality - Hamming codes

**Definition 4.3.1.** Let $\mathbb{K}$ be a finite field, and let $r > 1$ be an integer. Consider $\mathbb{PK}_q^{r-1} = (\mathbb{K}_q^r \backslash 0)/\mathbb{F}_q^*$. It has $\frac{q^r-1}{q-1}$ elements. Choose a column vector in $\mathbb{K}_q^r$ for every class. Let $H$ be the $\frac{q^r-1}{q-1} \times r$ matrix whose columns are precisely these vectors. Then $H$ is a parity check matrix of a code $\mathcal{C}$ called a $(r,q)$ Hamming code.

**Proposition 4.3.1.** *Let $r > 2$. Then, the $q$-ary Hamming code $Ham(r, q)$ has parameters $[\frac{q^r-1}{q-1}, \frac{q^r-1}{q-1} - r, 3]_q$.*

*Proof.* [11, Proposition 5.7] □

**Theorem 4.3.2.** *Let $\mathcal{C}$ be a $[n, k]_q$ code given by a parity check matrix $H$. Then, the minimum distance of $\mathcal{C}$ is $d$ if and only if*

*(1) Any $d - 1$ columns of $H$ are linearly independent.*

*(2) There exist $d$ columns that are linearly dependent.*

*Proof.* [11, Proposition 5.6] □

**Corollary.** *Let $\mathcal{C}$ be a code given by a parity check matrix $H$ . Then, $\mathcal{C}$ has minimum distance 1 if and only if $H$ has a 0 column.*

*Proof.* [20, Corollary 5.3] □

**Theorem 4.3.3.** *The dual code of a Hamming code $Ham(r, q)$ is a $[\frac{q^r-1}{q-1}, r]_q$ code such that all the non-zero codewords have the same weight: $q^{r-1}$. The weight hierarchy of such a code is $d_1 = q^{r-1}$ and $d_{i+1} = d_i \frac{q^{i+1}-1}{q(q^i-1)}$*

*Proof.* [1, Theorem 2.15] □

Thanks to this last theorem, we can get the weight enumerator for a Hamming code using the MacWilliams formula for dual codes given in Theorem 3.3.1.

**Example 4.3.1.** Given the Hamming code $\mathcal{C}_{Ham}$
$\mathcal{C}_{Ham} = \{\{1,0,0,0,0,1,1\},\{0,1,0,0,1,0,1\},\{0,0,1,0,1,1,0\},\{0,0,0,1,1,1,1\}\}$
Generator and parity check matrices are:

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \qquad H = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

It is a Ham(3, 2)-code, so $r = 3, q = 2, n = 7, k = 4$. Consequently, its dual code will be a $[7,3]_2$ constant weight code of weight 4.

***Notation***. We will use the following notation for Betti numbers:

$$\beta[\mathcal{M}] = \begin{bmatrix} \beta_{1,d_1} & \beta_{2,d_1+1} & \beta_{3,d_1+2} & \cdots & & \\ \beta_{1,d_1+1} & \beta_{2,d_1+2} & \cdots & & & \\ \beta_{1,d_1+2} & \cdots & & & & \vdots \\ \vdots & & & & \cdots & \beta_{k,d_k-1} \\ & & & \cdots & \beta_{k-1,d_k-1} & \beta_{k,d_k} \end{bmatrix}_{d_1}$$

which gives us $\beta_{i,j}(S/I_\Delta)$ for $1 \leq i \leq k$. It is known as *Betti table*.

The Betti tables for a matroid $\mathcal{M}$ from $\mathcal{C}_{cwc}$ and its elongations, computed by using the formulas from the last section, are:

$$\beta[\mathcal{M}] = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 14 & 8 \end{bmatrix}_4 \quad \beta[\mathcal{M}_{(1)}] = \begin{bmatrix} 7 & 6 \end{bmatrix}_6 \quad \beta[\mathcal{M}_{(2)}] = \begin{bmatrix} 1 \end{bmatrix}_7$$

Then, a table for the coefficients of $Z^l$ in GWP is:

| $\sum_{i=0}^{7}(-1)^{i+1}\big(\beta_{i,j}(\mathcal{M}_{(l-1)})-\beta_{i,j}(\mathcal{M}_{(l)})\big)$ | | | | |
|---|---|---|---|---|
| | $l=0$ | $l=1$ | $l=2$ | $l=3$ |
| $j=1$ | $0$ | $0$ | $0$ | $0$ |
| $j=2$ | $0$ | $0$ | $0$ | $0$ |
| $j=3$ | $0$ | $0$ | $0$ | $0$ |
| $j=4$ | $-7$ | $7$ | $0$ | $0$ |
| $j=5$ | $0$ | $0$ | $0$ | $0$ |
| $j=6$ | $14$ | $-21$ | $7$ | $0$ |
| $j=7$ | $-8$ | $14$ | $-7$ | $1$ |

The GWP is therefore

$$P_{M_{\mathcal{C}},j}(Z)=\begin{cases} 1 & \text{if } j=0 \\ 0 & \text{if } 1\leq j\leq 3 \\ 7Z-7 & \text{if } j=4 \\ 7Z^2-21Z+14 & \text{if } j=6 \\ Z^3-7Z^2+14Z-8 & \text{if } j=7 \end{cases}$$

Its weight enumerator is

$$\begin{aligned} W_{\mathcal{C}_{cwc}}(X,Y) &= \sum_{j=0}^{n} A_{\mathcal{C}_{cwc},j}X^{n-j}Y^j \\ &= X^7+7X^3Y^4 \end{aligned}$$

So the weight enumerator for its dual code will be

$$\begin{aligned} W_{\mathcal{C}_{Ham}}(X,Y) &= q^{-k}\cdot W_{cwc}(X+(q-1)Y,X-Y) \\ &= 2^{-3}\cdot W_{cwc}(X+Y,X-Y) \\ &= 2^{-3}\big((X+Y)^7+7(X+Y)^3(X-Y)^4\big) \end{aligned}$$

Its extended weight enumerator is

$$\begin{aligned}
W_{\mathcal{C}_{cwc}}(X,Y,T) &= \sum_{j=0}^{n} P_{\mathcal{M},j}(T)X^{n-j}Y^{j} \\
&= X^7 + (7T-7)X^3Y^4 + (7T^2-21T+14)XY^6 + \\
&+ (T^3-7T^2+14T-8)Y^7
\end{aligned}$$

So the extended weight enumerator for its dual code will be

$$\begin{aligned}
W_{\mathcal{C}_{Ham}}(X,Y,T) &= T^{-k}\cdot W_{cwc}(X+(T-1)Y, X-Y, T) \\
&= T^{-3}\Big(\big(X+(T-1)Y\big)^7 + (7T-7)\big(X+(T-1)Y\big)^3(X-Y)^4 + \\
&+ (7T^2-21T+14)\big(X+(T-1)Y\big)(X-Y)^6 + (T^3-7T^2+ \\
&+ 14T-8)(X-Y)^7\Big) \\
&= X^7 + 7(T-1)X^4Y^3 + 7(T-1)X^3Y^4 + 21(T^2-3T+2)X^2Y^5 + \\
&+ 7(T^3-6T^2+11T-6)XY^6 + (T^4-7T^3+21T^2-28T+13)Y^7
\end{aligned}$$

***Remark***. When we replace $X, Y$ in $W(X,Y,T)$ for 0 and 1 respectively, we obtain

$$W_{\mathcal{M}}(0,1,T) = \sum_{j=0}^{n} 0^{n-j}\cdot 1^{j}\cdot P_{\mathcal{M},j}(T) = 0^1\cdot 1^n\cdot P_{\mathcal{M},n}(T) = P_{\mathcal{M},n}(T)$$

That will coincide with $P_{\mathcal{M},d_k}(T)$ when the code is non-degenerate.

### 4.3.2 Betti tables for constant weight codes

***Notation***. We are going to make a slightly change of notation for the Betti numbers of elongations so that they become easier to read.

$$\beta_{i,j}(I_{\mathcal{M}_{(m)}}) = \beta_{i,j}^{(m)}$$

Let us consider a matroid $\mathcal{M}$ from a constant weight code with weight enumerator $W(X,Y,T)$.

**Lemma 4.3.4.** *Let $P_{\mathcal{M},j}(T)$ be the GWP of $\mathcal{M}$. Then,*

$$d_i = \min\{j \ ; \ \deg(P_{\mathcal{M},j}) = i\}$$

*Proof.* We know that $d_i = \min\{|\sigma| \ ; \ n(\sigma) = i\}$.

If $j < d_i$, $\forall |\sigma| = j < d_i$ we have $n(\sigma) < i$. It is also clear that $\forall \gamma \subset \sigma$, $n(\gamma) < i$. Therefore, $P_{\mathcal{M},j}(T) = (-1)^j \sum\limits_{|\sigma|=j} \sum\limits_{\gamma\subset\sigma} (-1)^{|\gamma|} T^{n(\gamma)}$ has degree less than $i$.

If $j = d_i$

$$\begin{aligned} P_{\mathcal{M},j}(T) &= (-1)^j \sum_{\substack{|\sigma|=j \\ n(\sigma)=i}} \left( \sum_{\gamma\subsetneq\sigma} (-1)^{|\gamma|} T^{n(\gamma)} + (-1)^{|\sigma|} T^{n(\sigma)} \right) \\ &+ (-1)^j \sum_{\substack{|\sigma|=j \\ n(\sigma)<i}} \sum_{\gamma\subset\sigma} (-1)^{|\gamma|} T^{n(\gamma)} \end{aligned}$$

which has degree $i$, since there is at least one $\sigma$ such that $|\sigma| = j, n(\sigma) = i$ and the two last summands have degree at most $i-1$, i.e.,

$$P_{\mathcal{M},j}(T) = \sum_{\substack{|\sigma|=j \\ n(\sigma)=i}} T^i + (-1)^j \sum_{\substack{|\sigma|=j \\ n(\sigma)=i}} \sum_{\gamma\subsetneq\sigma} (-1)^{|\gamma|} T^{n(\gamma)} + (-1)^j \sum_{\substack{|\sigma|=j \\ n(\sigma)<i}} \sum_{\gamma\subset\sigma} (-1)^{|\gamma|} T^{n(\gamma)}$$

□

Taking $W(X,Y,T)$ from the begining of the section, we have now that

$$W(X,Y,T) = P_{\mathcal{M},d_0} X^n + P_{\mathcal{M},d_1}(T) X^{n-d_1} Y^{d_1} + \ldots + P_{\mathcal{M},d_k}(T) X^{n-d_k} Y^{d_k}$$

So, when we replace $X$ and $Y$ by 0 and 1 respectively, we get

$$W(0,1,T) = P_{\mathcal{M},n}(T) = P_{\mathcal{M},d_k}(T)$$

since $d_k = n$ for non-degenerate codes.

From here, by using the formulas given in Theorem 3.2.1 and Proposition 2.5.3 we can obtain all the Betti numbers for the last position in the last column of the Betti table for each elongation of $\mathcal{M}$, as we will now explain.

$$P_{\mathcal{M},n}(T) = P_{\mathcal{M},d_k}(T) = a_k T^k + \ldots + a_1 T + a_0$$

$$\begin{array}{lrcl}
\beta[\mathcal{M}] \rightsquigarrow & \displaystyle\sum_{i=0}^{n}(-1)^{i+1}\big(\beta_{i,n}^{(-1)} - \beta_{i,n}^{(0)}\big) & = & a_0 \\
 & \beta_{k,d_k}^{(0)} & = & -a_0 \\
 & & & \\
\beta[\mathcal{M}_{(1)}] \rightsquigarrow & \displaystyle\sum_{i=0}^{n}(-1)^{i+1}\big(\beta_{i,n}^{(0)} - \beta_{i,n}^{(1)}\big) & = & a_1 \\
 & (-1)^{k+1}\big(\beta_{k,d_k}^{(0)} + \beta_{k-1,d_k}^{(1)}\big) & = & a_1 \\
 & \beta_{k-1,d_k}^{(1)} & = & (-1)^{k-1}(a_1 + a_0) \\
 & & & \\
\beta[\mathcal{M}_{(m)}] \rightsquigarrow & \displaystyle\sum_{i=0}^{n}(-1)^{i+1}\big(\beta_{i,n}^{(m-1)} - \beta_{i,n}^{(m)}\big) & = & a_m \\
 & (-1)^{k-m-1}\big(\beta_{k-m+1,d_k}^{(m-1)} + \beta_{k-m,d_k}^{(m)}\big) & = & a_m \\
 & \beta_{k-m,d_k}^{(m)} & = & (-1)^{k-m}(a_m + \ldots a_0)
\end{array}$$

Since all the resolutions considered are pure, we could get all the other Betti numbers by taking the other $P_{\mathcal{M},d_s}$ for $1 \leq s \leq k$.

The general formula for getting the $\beta_{i,j}$ in $\beta[\mathcal{M}_{(m)}]$ from the weight enumerator for $0 \leq m \leq k$ is therefore

$$\beta_{s-m,d_s}^{(m)} = (-1)^{s-m}(a_m^{(s)} + \ldots + a_0^{(s)}) \tag{4.3}$$

where $a_i^{(s)}$ are the coefficients of $T^s$ in $P_{\mathcal{M},d_s}(T)$.

**Example 4.3.2.** Let

$W(X,Y,T) = X^7+(7T-7)X^3Y^4+(7T^2-21T+14)XY^6+(T^3-7T^2+14T-8)Y^7$

be the weight enumerator for a matroid $\mathcal{M}$ from a constant weight code.

$$W(0,1,T) = P_{\mathcal{M},n} = T^3 - 7T^2 + 14T - 8$$

where $T^{\max} = T^k$. So $k = 3$ and therefore, $\mathcal{C}$ is a constant weight $[7,3]$-code.

The $j's$ that are different from 0 in $W(X,Y,T)$ give us the weight hierarchy, so $(d_1, d_2, d_3) = (4, 6, 7)$.

The Betti numbers that are different from 0 are therefore,

$$\begin{array}{lll}
\mathcal{M} & \mathcal{M}_{(1)} & \mathcal{M}_{(2)} \\
\beta_{1,d_1} = \beta_{1,4} & \beta_{1,d_{1(1)}} = \beta_{1,d_2} = \beta_{1,6} & \beta_{1,d_{1(2)}} = \beta_{1,d_3} = \beta_{1,7} \\
\beta_{2,d_2} = \beta_{2,6} & \beta_{2,d_{2(1)}} = \beta_{2,d_3} = \beta_{2,7} & \\
\beta_{3,d_3} = \beta_{3,7} & &
\end{array}$$

By using the formula (4.3),

$$\begin{array}{lll}
\beta_{1,4}(I_{\mathcal{M}}) = 7 & \beta_{1,6}(I_{\mathcal{M}_{(1)}}) = 7 & \beta_{1,7}(I_{\mathcal{M}_{(2)}}) = 1 \\
\beta_{2,6}(I_{\mathcal{M}}) = 14 & \beta_{2,7}(I_{\mathcal{M}_{(1)}}) = 6 & \\
\beta_{3,7}(I_{\mathcal{M}}) = 8 & &
\end{array}$$

So the Betti tables look like this:

$$\beta[\mathcal{M}] = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 14 & 8 \end{bmatrix}_4 \quad \beta[M_{(1)}] = \begin{bmatrix} 7 & 6 \end{bmatrix}_6 \quad \beta[M_{(2)}] = \begin{bmatrix} 1 \end{bmatrix}_7$$

which can be compared with the Betti tables we have from before, in Example 4.3.1.

# Chapter 5

# Duals of towers of Betti tables

As seen in last chapter, for a constant weight code, we can get all the Betti numbers of the matroid associated to the code and its elongations from its GWP. This works fine for all matroids having pure resolutions, but not in general.

## 5.1 Betti table for a general matroid

Let $\mathcal{M}$ be a matroid of rank $r$ over the ground set $E = \{1, \dots, n\}$. Let $k = n - r$. Let $\sigma \subset E$. The ungraded $\beta_{i,\sigma}$ was defined in Chapter 2 as $\dim_{\mathbb{K}} \operatorname{Tor}_i^S(\mathbb{K}, S/I_{\mathcal{M}})_\sigma$

As a reminder, the notation we use for the Betti table is the following

$$\beta[\mathcal{M}] = \begin{bmatrix} \beta_{1,d_1} & \beta_{2,d_1+1} & \beta_{3,d_1+2} & \cdots & & \\ \beta_{1,d_1+1} & \beta_{2,d_1+2} & \cdots & & & \\ \beta_{1,d_1+2} & \cdots & & & & \vdots \\ \vdots & & & & \cdots & \beta_{k,n-1} \\ & & & \cdots & \beta_{k-1,n-1} & \beta_{k,n} \end{bmatrix}_{d_1}$$

where $\beta_{i,j}(\mathcal{M}) = \beta_{i,j}(S/I_{\mathcal{M}}) \qquad \text{for } 1 \le i \le k.$

The Betti numbers listed in the table refer to the sum of $\beta_{i,\sigma}$ for a certain cardinality $j$, i.e, $\beta_{i,j} = \sum_{|\sigma|=j} \beta_{i,\sigma}$, and each $\beta_{i,i+j-1}$ is displayed in the $i$-th column and $(j-d+1)$-th row, where $d$ is the minimal Hamming distance.

**Lemma 5.1.1.** *For $i \notin [1,k]$, $\beta_{i,j}(\mathcal{M}) = 0$.*

*Proof.* It comes directly from the definition of Betti number for the resolution of $I_{\mathcal{M}}$. It is obvious for $i \leq 0$. We also have that proj.dim$(S/I_{\mathcal{M}}) = k$, which gives us $\beta_{i,j} = 0$ for $i \geq k+1$.

□

**Lemma 5.1.2.** *If $j \geq d_k - k + i + 1$, $\beta_{i,j}(\mathcal{M}) = 0$.*

*Proof.* As said in [9, Corollary 5], $d_k = |\bigcup_{\tau\in\mathcal{C}} \tau| = |E| \setminus \{\text{coloops of } \mathcal{M}\}$, where $\mathcal{C}$ denote the set of circuits of $\mathcal{M}$.

Let $F$ be the union of all circuits of $\mathcal{M}$. Then, $E \setminus F$ is the set of coloops of $\mathcal{M}$, i.e, the set of elements of $\mathcal{M}$ that are in all the bases.

Let $\sigma \subseteq F$. Then, $\mathcal{M}|_{\sigma} = \mathcal{M}|_{F_{|\sigma}}$, so

$$\beta_{i,\sigma}(\mathcal{M}) = \beta_{i,\sigma}(\mathcal{M}_F) = \widetilde{h}_{|\sigma|-i-1}(\mathcal{M}_{|\sigma}, \mathbb{K})$$

Let $\sigma \nsubseteq F$. Then, $\exists x \in E \setminus F$ such that $x \in \sigma$. Then, $\mathcal{M}_{|\sigma}$ has, at least one coloop.

$$\begin{array}{rcl} r^*_{\mathcal{M}_{|\sigma}}(x) & = & |x| - r_{\mathcal{M}_{|\sigma}}(\sigma \setminus x) - r_{\mathcal{M}_{|\sigma}}(\sigma) \\ & = & 1 - r_{\mathcal{M}}(\sigma \setminus x) - r_{\mathcal{M}}(\sigma) \\ x \text{ coloop} & \rightsquigarrow & r_{\mathcal{M}}(\sigma \setminus x) = r_{\mathcal{M}}(\sigma) - 1 \\ r^*_{\mathcal{M}_{|\sigma}}(x) & = & 0 \end{array}$$

Therefore, $x$ is still a coloop in $M_{|\sigma}$.

It is proven that a simplicial complex with an isthmus (an element that is in all the facets) has no homology. Then, due to Hochster's formula,

$$\beta_{i,\sigma}(I_{\mathcal{M}}) = \widetilde{h}_{|\sigma|-i-1}(\mathcal{M}_{|\sigma}, \mathbb{K}) = 0$$

Therefore,

$$\beta[\mathcal{M}_{|F}|\sigma] = \beta[\mathcal{M}_\sigma]$$

We can therefore assume that $n = d_k$. The lemma is then given in [19, Lemma 3.2]. □

**Proposition 5.1.3.** *Let $(d_1, \dots, d_k)$ be the weight hierarchy of $\mathcal{M}$. Then, if $j < d_i$, $\beta_{i,j}(I_\mathcal{M}) = 0$*

*Proof.* By definition of $d_i = \min\{|\sigma|; n(\sigma) = i\}$, for $j < d_i$ it is not possible to find any $\sigma$ such that $n(\sigma) = i$ and $|\sigma| < \min\{|\sigma|; n(\sigma) = i\}$. Therefore, $\beta_{i,j} = 0$ for those $j$.

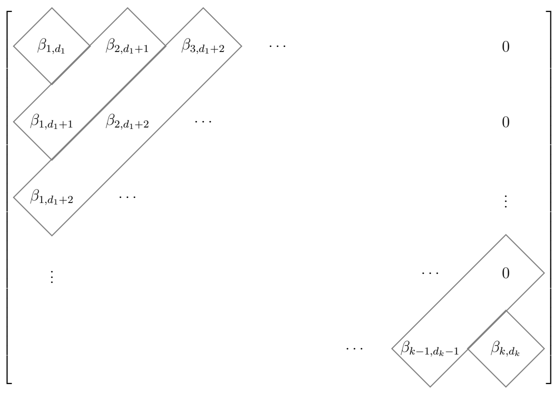
□

This way, the Betti table for any Stanley Reisner ideal of a matroid has zeroes at the left ($i \leq 0$) and right ($i > k$) sides, as well as at the top ($j < d_i$) and bottom ($j \geq d_k - k + i + 1$) of the table, so that we can just focus in what is inside these bounds.

## 5.2 Weight enumerator

The Betti table is distributed in such a way that the Betti numbers associated to a given cardinality of $\sigma \subset E$, $|\sigma| = d_1 + i$, are in the same diagonal. In addition, from the previous lemmas and proposition we know that, in certain positions of the table, $\beta_{i,j} = 0$, so, for now, we have the following

$\beta[\mathcal{M}] =$



where each "diagonal" corresponds to a given cardinality of $\sigma$, and therefore, to a given polynomial $P_{\mathcal{M},j}$. As we see, for the diagonals corresponding to $|\sigma| = d_1$, $|\sigma| = d_k - 1$ and $|\sigma| = d_k$, at most one of the $\beta_{i,j}$ is different from 0, so these can be retrieved from the GWP as we did for constant weight codes.

For computing $\beta_{k,d_k}$ and $\beta_{k-1,d_k-1}$ we can use the formula ( 4.3), given in Chapter 4.

Let $0 \leq m \leq k$. Then,

$$\beta^{(m)}_{k_{(m)},d_k} = \beta^{(m)}_{k-m,d_k} = (-1)^{k-m}(a^{(k)}_m + \ldots + a^{(k)}_0) \tag{5.1}$$

$$\beta^{(m)}_{k_{(m)}-1,d_k} = \beta^{(m)}_{k-m-1,d_k} = (-1)^{k-m-1}(a^{(k-1)}_m + \ldots + a^{(k-1)}_0) \tag{5.2}$$

where $a^{(s)}_i$ are the coefficients of $T^i$ in $P_{\mathcal{M},s}(T)$, for $s = d_k$ and $s = d_k - 1$ respectively.

***Remark***. Note that every time a matroid is elongated, its rank increases by one. Then, its corresponding $k = n - r$ decreases by one.

$$\beta^{(m)}_{k_{(m)},d_k} = \beta^{(m)}_{k-m,d_k}$$

For the computations for the first Betti number of each elongation $\beta^{(m)}_{1,d_{1(m)}}$ we must consider the polynomials $P_{\mathcal{M},d_{m+1}}$ for each $m^{\text{th}}$-elongation, since $d_{1(m)} = d_{m+1}$.

From Theorem 3.2.1, we have

$$\sum_{i=1}^{n}(-1)^{i+1}\big(\beta^{(m)}_{i,d_{m+1}} - \beta^{(m+1)}_{i,d_{m+1}}\big) = a^{(m+1)}_{m+1}$$

Since the weight hierarchy increases, $d_1 < d_2 < ... < d_k$, we get that

$$d_{1(m)} = d_{m+1} < d_{m+2} = d_{1(m+1)}$$

Then,

$$\beta^{(m+1)}_{i,d_{m+1}} = 0 \;\; \forall i, m$$

We also know that $\beta^{(m)}_{1,d_{1(m)}} \neq 0$ . It is the first coordinate in the Betti table for the $m^{\text{th}}$-elongation. By Proposition 5.1.3,

$$\beta^{(m)}_{i,d_{1(m)}} = 0 \;\; \forall m, i \neq 1$$

Thus,

$$\beta^{(m)}_{1,d_{1(m)}} = \beta^{(m)}_{1,d_{m+1}} = a^{(m+1)}_{m+1} \tag{5.3}$$

where $a^{(m+1)}_{m+1}$ is the coefficient of $T^{m+1}$ in $P_{\mathcal{M},d_{m+1}}(T)$.

Summing up, from the GWP of a matroid $\mathcal{M}$ of rank $r$ over $E$, we can figure out

$$\beta[\mathcal{M}_{(m)}] = \begin{bmatrix} * & & & 0 \\ & & & \vdots \\ & & & \vdots \\ & & & 0 \\ & & * & * \end{bmatrix}_{d_1}$$

## 5.3 Dual Betti tables

Given all the Betti numbers for an unknown matroid $\mathcal{M}$ and its elongations, we will try to get as much information as we can about the Betti tables for its dual matroid and elongations.

Let us begin with an example about the difficulties when computing the duals of Betti tables. There exists matroids with equal Betti tables, but different dual Betti tables (see example below). Because of this reason, we need at least the Betti tables for the elongations so that we can get more information about the duals. It is not known if knowing the Betti tables for all the elongations is enough to know the Betti tables of the dual.

**Example 5.3.1.** Let

$$\begin{aligned}\mathcal{M}_1 &= \{\{1,3,4,6,7\},\{1,2,3,6,8\},\{1,2,3,4,8\},\{1,2,3,5,8\},\{1,2,5,6,8\},\\ &\quad \{1,2,3,4,7\},\{1,2,3,5,7\},\{1,2,5,6,7\},\{1,3,4,5,7\},\{1,3,4,6,8\},\\ &\quad \{1,2,4,6,8\},\{1,2,4,6,7\},\{1,3,4,5,8\},\{1,2,4,5,7\},\{1,4,5,6,7\},\\ &\quad \{1,2,3,6,7\},\{1,3,5,6,7\},\{1,4,5,6,8\},\{1,3,5,6,8\},\{1,2,4,5,8\}\}\\ \mathcal{M}_2 &= \{\{1,3,4,6,7\},\{1,2,3,4,8\},\{1,2,3,5,8\},\{1,2,5,6,8\},\{1,2,3,4,7\},\\ &\quad \{1,2,3,5,7\},\{1,2,5,6,7\},\{1,3,4,5,7\},\{1,3,4,6,8\},\{1,2,4,6,8\},\\ &\quad \{1,2,4,6,7\},\{1,3,4,5,8\},\{1,2,4,5,7\},\{1,3,4,5,6\},\{1,2,4,5,6\},\\ &\quad \{1,3,5,6,7\},\{1,2,3,5,6\},\{1,2,3,4,6\},\{1,3,5,6,8\},\{1,2,4,5,8\}\}\end{aligned}$$

Then,

$$\beta[\mathcal{M}_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 5 & 4 & 0 \\ 0 & 5 & 4 \end{bmatrix}_2 = \beta[\mathcal{M}_2]$$

but

$$\beta[\mathcal{M}_1^*] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 10 & 25 & 21 & 6 & 0 \\ 0 & 10 & 25 & 21 & 6 \end{bmatrix}_1 \quad \beta[\mathcal{M}_2^*] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 \\ 4 & 14 & 13 & 4 & 0 \\ 4 & 22 & 38 & 27 & 7 \end{bmatrix}_1$$

Consider $\mathcal{M}$ a matroid of rank $r$ over $E = \{1, \dots, n\}$ and let $k = n - r$. Consider its tower of Betti tables.

$$\beta[\mathcal{M}_{(m)}] = \begin{bmatrix} \beta^{(m)}_{1,d_{1+m}} & \beta^{(m)}_{2,d_{2+m}} & \dots & & 0 \\ \beta^{(m)}_{1,d_{2+m}} & & & & \vdots \\ \vdots & & & & 0 \\ & & \dots & \beta^{(m)}_{k-m-1,d_k-1} & \beta^{(m)}_{k-m,d_k} \end{bmatrix}_{d_{1+m}}$$

If we know all the $\beta^{(m)}_{i,j}$, we can get the GWP thanks to the formula in Theorem 3.2.1 given in Chapter 3.

$$\begin{aligned} P_{\mathcal{M},0} &= 1 \\ P_{\mathcal{M},j} &= \sum_{m=0}^{n} \Big( \sum_{i=1}^{n} (-1)^{i+1} \big( \beta_{i,j}(\mathcal{I}_{\mathcal{M}_{(m-1)}}) - \beta_{i,j}(\mathcal{I}_{\mathcal{M}_{(m)}}) \big) \Big) T^m \end{aligned}$$

Then, the coefficient for $T^m$ in $P_{\mathcal{M},j}$ is

$$c_m = \sum_{i=1}^{k+1} (-1)^{i+1} \big( \beta^{(m-1)}_{i,j} - \beta^{(m)}_{i,j} \big)$$

The weight enumerator was defined as

$$W_{\mathcal{M}}(X, Y, T) = \sum_{j=0}^{n} P_{\mathcal{M},j}(T) X^{n-j} Y^j$$

Replacing,

$$W_{\mathcal{M}}(X, Y, T) = \sum_{j=0}^{n} \sum_{m=0}^{n} c_m T^m X^{n-j} Y^j$$

If $X = 0$ and $Y = 1$ we have

$$W_{\mathcal{M}^*}(0, 1, T) = P_{\mathcal{M}^*,n}(T)$$

MacWilliams identity reads

$$W_{\mathcal{M}^*}(X,Y,T) = T^{-k} \cdot W_{\mathcal{M}}(X+(T-1)Y, X-Y, T)$$

so

$$P_{\mathcal{M}^*,n}(T) = W_{\mathcal{M}^*}(0,1,T) = T^{-k} \cdot W_{\mathcal{M}}(T-1,-1,T)$$

By using the formula for the coefficients in $P_{\mathcal{M},j}$,

$$W_{\mathcal{M}}(T-1,-1,T) \quad = \quad (T-1)^n + \sum_{j=1}^{n} \left( \sum_{m=0}^{k+1} c_m T^m \right) (T-1)^{n-j}(-1)^j \tag{5.4}$$

Note that

$$(T-1)^{n-j} = \sum_{\alpha=0}^{n-j} \binom{n-j}{\alpha} T^{\alpha}(-1)^{n-j-\alpha}$$

The coefficient of $T^t$ in ( 5.4) is determined by the values $m$ and $\alpha = t-m$ when $m$ varies from 0 to $t$, along with the convention $\binom{r}{s} = 0$ when $s < 0$ or $s > r$. Then,

$$e_t \quad = \quad (-1)^{n-t}\left( \binom{n}{t} + \sum_{m=0}^{t}(-1)^m \sum_{j=1}^{n} c_m \binom{n-j}{t-m} \right)$$

where $\binom{n-j}{t-m}$ is a polynomial in $j$ of degree $t-m$.

**Lemma 5.3.1.** *For all polyomial $P \in \mathbb{Q}[j]$ of degree at most r-m-1, then*

$$\sum_{i=0}^{n}\sum_{j=0}^{n}(-1)^i P(j)\beta_{i,j}(S/I_{\mathcal{M}_{(m)}}) = 0$$

*and*

$$\sum_{i\neq 0}^{n}\sum_{j=0}^{n}(-1)^i P(j)\beta_{i,j}(\mathcal{M}_{(m)}) = -P(0)$$

*Proof.* The first part is true because of the Herzog-Kühl equations, that affirm

$$\sum_{i,j}(-1)^i\beta_{i,j}(S/I_{\mathcal{M}_{(m)}})j^l = 0$$

for $0 \leq l \leq r - m - 1 = n - \text{krull.dim}(S/I_{\mathcal{M}_{(m)}}) - 1$.

For the second part, we have the following

$$\begin{aligned}\sum_{i=0}^{n}\sum_{j=0}^{n}(-1)^iP(j)\beta_{i,j}(S/I_{\mathcal{M}_{(m)}}) &= \sum_{i\neq 0}^{n}\sum_{i=0}^{n}(-1)^iP(j)\beta_{i,j}(S/I_{\mathcal{M}_{(m)}}) + P(0)\beta_{0,0}(S/I_{\mathcal{M}_{(m)}}) \\ &= \sum_{i\neq 0}^{n}\sum_{i=0}^{n}(-1)^iP(j)\beta_{i,j}(S/I_{\mathcal{M}_{(m)}}) + P(0)\cdot 1\end{aligned}$$

Then,

$$\sum_{i\neq 0}^{n}\sum_{j=0}^{n}(-1)^iP(j)\beta_{i,j}(\mathcal{M}_{(m)}) = -P(0)$$

□

**Lemma 5.3.2.** *Let $k = n - r$. Then, for $t < k$, $e_t = 0$.*

*Proof.*

$$\begin{aligned}e_t &= (-1)^{n-t}\tbinom{n}{t} + \sum_{m=0}^{t}\sum_{j=1}^{n}\sum_{i=1}^{n}(-1)^{i+1}\big(\beta_{i,j}^{(m-1)} - \beta_{i,j}^{(m)}\big)\binom{n-j}{t-m}(-1)^{n-t+m} \\ &= (-1)^{n-t}\tbinom{n}{t} + \sum_{m=1}^{t}\sum_{j=1}^{n}\sum_{i=1}^{n}(-1)^i\beta_{i,j}^{(m-1)}\binom{n-j}{t-m}(-1)^{n-t+m+1} - \\ &- \sum_{m=0}^{t}\sum_{j=1}^{n}\sum_{i=1}^{n}(-1)^i\beta_{i,j}^{(m)}\binom{n-j}{t-m}(-1)^{n-t+m+1}\end{aligned}$$

where $\binom{n-j}{t-m}$ is a polynomial in $j$ of degree $t - m \leq k - m - 1$.

Applying Lemma 5.3.1 to $e_t$,

$$\begin{aligned}e_t &= (-1)^{n-t}\tbinom{n}{t} - \sum_{j=1}^{n}\sum_{i=1}^{n}(-1)^i\beta_{i,j}^{(0)}\binom{n-j}{t}(-1)^{n-t+1} \\ &= (-1)^{n-t}\tbinom{n}{t} - \tbinom{n}{t}(-1)^{n-t} = 0\end{aligned}$$

□

***Remark.*** This was expected since $W_{\mathcal{M}^*}$ is known to be a polynomial.

For $t \geq k$ we have therefore

$$P_{\mathcal{M}^*,n}(T) = T^{-k} \cdot \sum_{t=k}^{n} e_t T^t = \sum_{t=k}^{n} e_t T^{t-k}$$

From here, we can apply the formula obtained before for computing the Betti numbers from the GWP.

***Remark.*** Note that, for the dual matroid, $\mathcal{M}^*$, the parameters are

$$\begin{array}{rclcl} k^* & = & n-k & = & r \\ r^* & = & n-r & = & k \\ n^* & = & n & & \end{array}$$

The last Betti number for the duals are therefore,

$$\begin{array}{rcl} {\beta^*}^{(m)}_{r-m,d^*_r} & = & (-1)^{r-m}\big(e_{k+m} + \ldots + e_k\big) \\ & = & (-1)^{2r+m} \displaystyle\sum_{z=0}^{m} (-1)^z \left( \binom{n}{k+z} + \sum_{l=0}^{k+z} (-1)^l \sum_{j=1}^{n} c_l \binom{n-j}{k+z-l} \right) \end{array}$$

***Remark.*** Even if the matroid is degenerated, this formula holds for ${\beta^*}^{(m)}_{r-m,n}$ since we know that the only Betti numbers that might be different from 0 are those with $i = r-m$ and $j = n$.

***Remark.*** It is not possible to give such formulas for ${\beta^*}^{(m)}_{r-m-1,n-1}$ and${\beta^*}_{1,d^*_{1+m}}$ since $P_{\mathcal{M}^*,n-1}$ depends only on ${\beta^*}^{(m)}_{r-m-1,n-1}$ and ${\beta^*}^{(m)}_{r-m,n-1}$, and we generally do not know which one is 0. We know that one of them is 0, but we cannot decide generally which one.

$$\begin{array}{lcl} \mathcal{M}^* \text{ degenerate } \ (d_1^* = 1) & \Rightarrow & {\beta^*}^{(m)}_{r-m-1,n-1} = 0 \\ \mathcal{M}^* \text{ non-degenerate } (d_1^* \neq 1) & \Rightarrow & {\beta^*}^{(m)}_{r-m,n-1} = 0 \end{array}$$

Summarizing, for given $\beta^{(m)}_{i,j}$ of $\mathcal{M}$, a non degenerated matroid of rank $r$ over $E$, we have a formula for computing the Betti numbers in the last, next-to-last and first position in every Betti table for the dual of a matroid $\mathcal{M}^*$. Along with the results in the previous lemmas and proposition at the begining of the chapter, we know additionaly, all the elements in the last columns, i.e.,

$$\beta[\mathcal{M}_{(m)}] = \begin{bmatrix} & & \\ & \beta^{(m)}_{i,j} & \\ & & \end{bmatrix}_{d_{1+m}} \rightsquigarrow \beta[\mathcal{M}^*_{(m)}] = \begin{bmatrix} * & & 0 \\ & & \vdots \\ & * & * \end{bmatrix}_{d^*_{1+m}}$$

Besides, we can obtain some certain positions for zeroes in the dual Betti tables. Since the $\beta^{(m)}_{i,j}$ are given, we know the weight hirarchy $(d_1, \ldots, d_k)$, and, therefore, the dual weight hierarchy, $({d_1}^*, \ldots, {d_r}^*)$, thanks to the Wei's duality theorem, given in Chapter 1, that affirms

$$\{d_1, \ldots, d_k, n+1-d_1^*, \ldots, n+1-d_r^*\} = \{1, \ldots, n\}$$

Proposition 5.1.3 will gives us the zeroes in the dual Betti tables.

## 5.4 Examples

Let's give some examples in order to clarify what has been said so far.

**Example 5.4.1.** Let $\mathcal{M}$ over $E = \{1, \ldots, 6\}$ with the following Betti tower:

$$\beta[\mathcal{M}] = \begin{bmatrix} 1 & 0 & 0 \\ 7 & 12 & 5 \end{bmatrix}_2 \quad \beta[\mathcal{M}_{(1)}] = \begin{bmatrix} 5 & 4 \end{bmatrix}_4 \quad \beta[\mathcal{M}_{(2)}] = \begin{bmatrix} 1 \end{bmatrix}_5$$

The rank of the matroid is $r = 3$.

Since $d_1(\mathcal{M}) = 2, d_1(\mathcal{M}_{(1)}) = 4$ and $d_1(\mathcal{M}_{(2)}) = 5$, we get the weight hierarchy for $\mathcal{M}$:

$$(d_1, d_2, d_3) = (2, 4, 5)$$

The rank of $\mathcal{M}^*$ is $n - r = 3$.

In consequence of Wei's duality theorem, the weight hierarchy of $\mathcal{M}^*$ is

$$(d_1^*, d_2^*, d_3^*) = (1, 4, 6)$$

The weight enumerator of $\mathcal{M}$ is

$$\begin{aligned} W_{\mathcal{M}}(X, Y, T) &= X^6 + (T-1)X^4Y^2 + (7T-7)X^3Y^3 + \\ &+ (5T^2 - 17T + 12)X^2Y^4 + (T^3 - 5T^2 + 9T - 5)XY^5 \end{aligned}$$

The weight enumerator of $\mathcal{M}^*$ is

$$\begin{aligned} W_{\mathcal{M}^*}(X, Y, T) &= T^{-3} \cdot W_{\mathcal{M}}(X + (T-1)Y, X - Y, T) \\ &= X^6 + (T-1)X^5Y + (T-1)X^3Y^3 + (T^2 + T - 2)X^2Y^4 + \\ &+ (4T^2 - 10T + 6)XY^5 + (T^3 - 5T^2 + 7T - 3)Y^6 \end{aligned}$$

Then, by using the formulas ( 5.3) ,( 5.2) and ( 5.1):

$$\begin{array}{lll} {\beta^*}^{(0)}_{1,1} = 1 & {\beta^*}^{(1)}_{1,4} = 1 & {\beta^*}^{(2)}_{1,6} = 1 \\ {\beta^*}^{(0)}_{2,6} = 6 & {\beta^*}^{(1)}_{1,5} = 4 & \\ {\beta^*}^{(0)}_{3,6} = 3 & {\beta^*}^{(1)}_{2,6} = 4 & \end{array}$$

Since we know that, for $j < d_i$, $\beta_{i,j} = 0$, we get the following tables:

$$\beta[\mathcal{M}^*] = \begin{bmatrix} 1 & 0 & 0 \\ ? & 0 & 0 \\ ? & ? & 0 \\ ? & 6 & 3 \end{bmatrix}_1 \quad \beta[\mathcal{M}^*_{(1)}] = \begin{bmatrix} 1 & 0 \\ 4 & 4 \end{bmatrix}_4 \quad \beta[\mathcal{M}^*_{(2)}] = \begin{bmatrix} 1 \end{bmatrix}_6$$

**Example 5.4.2.** Let us take an example where the matroid is non-representable. Let $\mathcal{M}$ be the *non-Pappus* matroid. Its set of independent sets are all the subsets of $E = \{1, \ldots, 9\}$ with cardinality $|\sigma| \leq 3$ except for $\{1,2,3\}, \{1,5,7\}, \{1,6,8\}, \{2,4,7\}$, $\{2,6,9\}, \{3,4,8\}, \{3,5,9\}$ and $\{4,5,6\}$. Its betti tower is:

$$\begin{aligned} \beta[\mathcal{M}] &= \begin{bmatrix} 8 & 0 & 0 & 0 & 0 & 0 \\ 78 & 384 & 680 & 600 & 267 & 48 \end{bmatrix}_3 \\ \beta[\mathcal{M}_{(1)}] &= \begin{bmatrix} 126 & 420 & 540 & 315 & 70 \end{bmatrix}_5 \\ \beta[\mathcal{M}_{(2)}] &= \begin{bmatrix} 84 & 216 & 189 & 56 \end{bmatrix}_6 \\ \beta[\mathcal{M}_{(3)}] &= \begin{bmatrix} 36 & 63 & 28 \end{bmatrix}_7 \\ \beta[\mathcal{M}_{(4)}] &= \begin{bmatrix} 9 & 8 \end{bmatrix}_8 \\ \beta[\mathcal{M}_{(5)}] &= \begin{bmatrix} 1 \end{bmatrix}_9 \end{aligned}$$

The rank of the matroid is $r = 6$, and its weight hierarchy $(3, 5, 6, 7, 8, 9)$. Therefore, the dual weight hierarchy is $(6, 8, 9)$.

The weight enumerators for $\mathcal{M}$ and $\mathcal{M}^*$ are

$$\begin{aligned}
W_{\mathcal{M}}(X,Y,T) &= X^9 + (8T-8)X^6Y^3 + (78T-78)X^5Y^4 + \\
&+ (126T^2 - 510T + 384)X^4Y^5 + \\
&+ (84T^3 - 504T^2 + 1100T - 680)X^3Y^6 + \\
&+ (36T^4 - 252T^3 + 756T^2 - 1140T + 600)X^2Y^7 + \\
&+ (9T^5 - 72T^4 + 252T^3 - 504T^2 + 582T - 267)XY^8 + \\
&+ (T^6 - 9T^5 + 36T^4 - 84T^3 + 126T^2 - 118T + 48)Y^9
\end{aligned}$$

$$\begin{aligned}
W_{\mathcal{M}^*}(X,Y,T) &= X^9 + (8T-8)X^3Y^6 + (12T-12)X^2Y^7 + \\
&+ (9T^2 - 48T + 39)XY^8 + (T^3 - 9T^2 + 28T - 20)Y^9
\end{aligned}$$

Then, by using the formulas ( 5.3) ,( 5.2) and ( 5.1):

$$\begin{array}{lll}
{\beta^*}^{(0)}_{1,6} = 8 & {\beta^*}^{(1)}_{1,8} = 9 & {\beta^*}^{(2)}_{1,6} = 1 \\
{\beta^*}^{(0)}_{2,8} = 39 & {\beta^*}^{(1)}_{2,9} = 8 & \\
{\beta^*}^{(0)}_{3,9} = 20 & &
\end{array}$$

Then,

$$\beta[\mathcal{M}^*] = \begin{bmatrix} 8 & 0 & 0 \\ ? & 39 & 20 \end{bmatrix}_6 \quad \beta[\mathcal{M}^*_{(1)}] = \begin{bmatrix} 9 & 8 \end{bmatrix}_8 \quad \beta[\mathcal{M}^*_{(2)}] = \begin{bmatrix} 1 \end{bmatrix}_9$$

Actually, we can know $\beta^{(0)}_{1,7}$ too by using Herzog-Kühl equations, since $\beta^{(0)}_{2,7} = 0$.

$$\beta^{(0)}_{1,7} = -\beta^{(0)}_{1,6} + \beta^{(0)}_{2,8} - \beta^{(0)}_{3,9} - \beta^{(0)}_{0,0} = 12 \tag{5.5}$$

Then,

$$\beta[\mathcal{M}^*] = \begin{bmatrix} 8 & 0 & 0 \\ 12 & 39 & 20 \end{bmatrix}_6 \quad \beta[\mathcal{M}^*_{(1)}] = \begin{bmatrix} 9 & 8 \end{bmatrix}_8 \quad \beta[\mathcal{M}^*_{(2)}] = \begin{bmatrix} 1 \end{bmatrix}_9$$

This example shows that the method works for general matroids. There is no necessity that the matroid is associated to a linear code.

# Bibliography

[1] Bierbrauer, J.,*Introduction to coding theory.* Discrete Mathematics and its Applications (Boca Raton). Chapman & Hall/CRC, Boca Raton, FL, 2005

[2] Bosma. W., Cannon,J., Playoust, C., *The Magma algebra system. I. The user language, J. Symbolic Comput.*, 24 (1997), 235–265.

[3] Eagon, J.A., Reiner, V., *Resolutions of Stanley-Reisner Rings and Alexander Duality.* J. Pure Appl. Algebra 130 (1998), no. 3, 265–275.

[4] Eisenbud, D., *Commutative Algebra. With a view toward algebraic geometry.* Graduate Texts in Mathematics, 150. Springer-Verlag New York, 1995.

[5] G. Fløystad., *Boij-Söderberg theory: introduction and survey.* Progress in commutative algebra 1, 1–54, de Gruyter, Berlin, 2012.

[6] Herzog, J., Hibi, T., *Monomial Ideals*, Graduate Texts in Mathematics, 260. Springer-Verlag London Limited, Springer, 2011.

[7] Hill, R., *A First Course in Coding Theory*, Oxford Applied Mathematics and Computing Science Series. The Clarendon Press, Oxford University Press, New York, 1986.

[8] Hilton P.J., Stammbach, U., *A course in Homological Algebra.* Graduate Text in Mathematics, 4. Second edition. Springer-Verlag, New York, 1997.

[9] Johnsen, T., Verdure, H., *Hamming weights and Betti numbers of Stanley-Reisner rings associated to matroids.* Appl. Algebra Engrg. Comm. Comput. 24 (2013), no. 1, 73–93.

[10] Johnsen, T., Verdure, H., *Stanley–Reisner resolution of constant weight linear codes*, Des. Codes Cryptogr. 72 (2014), no. 2, 471–481.

[11] Jurrius, R. *Weight enumeration of codes from finite spaces.* Des. Codes Cryptogr. 63 (2012), no. 3, 321–330.

05B35

[12] Jurrius, R., Pelikaan, R., *Codes, arrangements and matroids.* Algebraic geometry modeling in information theory, 219–325, Ser. Coding Theory Cryptol., 8, World Sci. Publ., Hackensack, NJ, 2013.

[13] Lang, S. *Linear Algebra* Undergraduate Texts in Mathematics. Springer-Verlag, New York, 1989.

[14] Larsen, A.H., *Matroider og lineære koder*, Master in algebra, University of Bergen, 2005.

[15] Liu, Z., Chen, C., *Notes on the value function* Des. Codes Cryptogr. 54, 11–19 (2010).

[16] Miller, E., Sturmfels, B. *Combinatorial Commutative Algebra.* Graduate Texts in Mathematics 227. Springer-Verlag, New York, 2005.

[17] Oxley, J., *Matroid theory.* Second edition. Oxford Graduate Texts in Mathematics, 21. Oxford University Press, Oxford, 2011.

[18] Roksvold, J., *A generalization of weight polynomials to matroids.* . arXiv:1311.6291v1

[19] Roksvold, J., Verdure, H., *Betti numbers of skeletons* . arXiv:1502.05670v2

[20] Verdure, H., *Code and Matroid theory.* Lecture Notes, University of Tromsø, 2013.

[21] Wei, V., *Generalized Hamming weights for linear codes.* IEEE Trans. Inform. Theory 37 (1991), no. 5, 1412–1418.

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