UiT The Arctic University of Norway

Faculty of Engineering Science and Technology
Department of Computer Science and Computational Engineering

Approximation properties of Cesàro means of Vilenkin-Fourier series

Tsitsino Tepnadze

A dissertation for the degree of Philosophiae Doctor January 2022

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Contents

0 Introduction **11**

1 Approximation Properties of Cesàro Means of the one-dimensional Vilenkin-Fourier Series **33**

1.1 Formulation of the main results 33
1.2 Auxiliary results 35
1.3 Proofs of the main results 44

2 Approximation Properties of Cesàro Means of the two-dimensional Vilenkin-Fourier Series **47**

2.1 Formulation of the main results 47
2.2 Auxiliary results 49
2.3 Proofs of the main results 60

3 Approximation Properties of Cesàro Means of the quadratic partial sums of double Vilenkin-Fourier series **65**

3.1 Formulation of the main results 65
3.2 Auxiliary results 68
3.3 Proofs of the main results 79

Bibliography **93**

Key words: Fourier analysis, Vilenkin system, Vilenkin-Fourier series, Cesàro means, Modulus of continuity, Convergence in norm, Inequalities, Approximation.

To my family

Abstract

This PhD thesis focuses on the investigation of approximation properties of Cesàro means of the Vilenkin-Fourier series. In particular, we obtain some new inequalities related to the rate of L^p approximation by Cesàro means of the Vilenkin-Fourier series of functions from L^p. These inequalities imply sufficient conditions for the convergence of Cesàro means of the Vilenkin-Fourier series in the L^p-metric in terms of the modulus of continuity. Furthermore, we also proved the sharpness of these conditions. In particular, we find a continuous function under some condition of the modulus of continuity, for which Cesàro means of the Vilenkin-Fourier series diverge in the L^p- metric.

This PhD thesis consists of three main Chapters, based on five papers. At first, we have an Introduction, where we give a general overview of fundamental definitions and notations, followed by historical and new results, on which our study is based and inspired. We also give a formulation of our main results in this general frame and review some auxiliary results, that are significant for the proofs of our new theorems in the next main chapters.

In Chapter 1, we investigate the approximation properties of Cesàro means of negative order of the one-dimensional Vilenkin-Fourier Series. In particular, we derive sufficient conditions for the convergence of the means $\sigma_n^{-\alpha}(f,x)$ to $f(x)$ in the L^p- metric in terms of the modulus of continuity. Moreover, we prove the sharpness of these conditions.

Chapter 2 is focused on a new approach to investigate the rate of L^p approximation by Cesàro means of negative order of the two-dimensional Vilenkin-Fourier Series of functions from L^p. In particular, we derived a necessary and sufficient condition for the convergence of Cesàro $(C,-\alpha,-\beta)$ means with α,β ϵ $(0,1)$ in terms of the modulus of continuity. Some corresponding sharpness results are proved also in this case.

Chapter 3 is devoted to deriving some new results concerning the behavior of Cesàro $(C,-\alpha)$ means of the quadratic partial sums of double Vilenkin-Fourier series. The new results are sharp also in this case.

Remark: All main results in this PhD thesis are also published in international journals.

Preface

This PhD thesis is written as a monograph and is based on the following papers (papers **A**-**E**):

A T. Tepnadze, On the approximation properties of Cesàro means of negative order of Vilenkin-Fourier series. Studia Sci. Math. Hung. 53 (2016), no 4, 532-544.

B T. Tepnadze, On the approximation properties of Cesàro means of negative order of double Vilenkin-Fourier series. Ukrainian Math. J. 72 (2020), no. 3, 446–463.

C T. Tepnadze and L. E. Persson, Some inequalities for Cesàro means of double Vilenkin-Fourier series. J. Inequal. Appl. 2018, Paper no. 352, 17 pp.

D T. Tepnadze, Cesàro means of negative order of the quadratic partial sums of double Vilenkin-Fourier series. *To appear in Nonlinear Studies in February 2022, 13 pp.*

E T. Tepnadze, Cesàro means of negative order of the one-dimensional Vilenkin-Fourier series. Bulletin of the L.N. Gumilyov Eurasian National University. Mathematics. Computer science. Mechanics Series 135 (2021), no. 2, 6-11.

The main results in these publications are put into a more general frame in an Introduction. In particular, the Introduction contains a brief overview of the necessary definitions and a formulation of the main results from all five publications and related results. Furthermore, there are some applied problems in the Introduction included, which are related to the research in this PhD thesis.

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Chapter 0

Introduction

Definitions and Notations

Classical Fourier analysis is very important for various applications in engineering science. In this PhD thesis, we derive some new sharp results in a modern form of Fourier analysis, where the classical orthonormal systems are replaced by some orthonormal systems from the point of view of the structure of a topological group.

This PhD thesis in Engineering Science and Mathematics is mainly focused on the investigation of the approximation properties of Cesàro means of the Vilenkin-Fourier series. For this we need to give a brief introduction including the most important definitions e.g. those of the Vilenkin system and the Vilenkin group. The Vilenkin system was introduced by Vilenkin in 1947 (see [135]).

If we denote N_+ as the set of positive integers, then N can be defined as follows: $N := N_+ \cup \{0\}$.

Let $m := (m_0, m_1, ...)$ denote a sequence of positive integers not less then 2. By the set $Z_{m_k} := \{0, 1, ..., m_k - 1\}$ the additive group of integers $\mod m_k$ can be denoted.

Define the group G_m as the complete direct product of the groups Z_{m_j} with the product of the discrete topologies of Z_{m_j}'s.

The direct product μ of the measures

$$\mu_k(\{j\}) := \frac{1}{m_k} \qquad (j \in Z_{m_k})$$

is called the Haar measure on G_m and $\mu(G_m) = 1$.

The elements of G_m can be represented by sequences $x := (x_0, x_1, ..., x_j, ...)$, $(x_j \in Z_{m_j})$.

The group operation $+$ in G_m is given by

$$x + y = ((x_0 + y_0) \, mod \, m_0, ..., (x_k + y_k) \, mod \, m_k, ...),$$

for $x := (x_0, ..., x_k, ...)$ and $y := (y_0, ..., y_k, ...) \in G_m$.

The inverse of $+$ will be denoted by $-$.

If the sequence m is bounded, then G_m is called a bounded Vilenkin group. In this PhD thesis we will consider only bounded Vilenkin groups.

It is easy to give a base for the neighborhoods of G_m :

$$I_0(x) := G_m$$

$$I_n(x) := \{y \in G_m | y_0 = x_0, ..., y_{n-1} = x_{n-1}\},$$

where $x \in G_m$ and $n \in N$. We can define $I_n := I_n(0)$ for $n \in N$.

Set $e_n := (0, ..., 0, 1, 0, ...) \in G_m$, where the $n-$th coordinate is equal to 1 and the rest are zeros $(n \in N)$.

If we define the so-called generalized number system based on m in the following way: $M_0 := 1$, $M_{k+1} := m_k M_k$ $(k \in N)$. Then every $n \in N$ can be uniquely expressed as

$$n = \sum_{j=0}^{\infty} n_j M_j, \text{ with } n_j \in Z_{m_j} \ (j \in N_+),$$

where only a finite number of n_j's differ from zero.

We use the following notation: Let $|n|$:=max$\{k \in N : n_k \neq 0\}$ (that is , $M_{|n|} \leq n < M_{|n|+1}$).

In order to introduce an orthonormal system on G_m, at first we need to define the complex-valued function $r_k(x) : G_m \to C$ (see Paley [93]), which is called the generalized Rademacher function, in the following way:

$$r_k(x) := \exp\left(\frac{2\pi i x_k}{m_k}\right), \qquad \left(i^2 = -1,\ x \in G_m,\ k \in N\right).$$

Now based on the the generalized Rademacher function we define the Vilenkin system $\psi := (\psi_n : n \in N)$ on G_m as follows:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x), \qquad (n \in N).$$

In particular, we call the system the Walsh-Paley if $m = 2$. Each ψ_n is a character of G_m and all characters of G_m are of this norm. Moreover, $\psi_n(-x) = \bar{\psi}_n(x)$. The Vilenkin system is complete in $L^1(G_m)$ (see Vilenkin [135]).

If $f \in L^1(G_m)$ we can establish the following definitions with respect to the Vilenkin system:
Fourier coefficients:

$$\widehat{f}(k) := \int_{G_m} f \overline{\psi_k} d\mu, \qquad (k \in N),$$

Partial sums:

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}(k) \psi_k, \quad (n \in N_+,\ S_0 f := 0),$$

Fejér means:

$$\sigma_n f := \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) \widehat{f}(k) \psi_k, \quad (n \in N_+),$$

Dirichlet kernels:

$$D_n := \sum_{k=0}^{n-1} \psi_k, \quad (n \in N_+),$$

Fejér kernels:

$$K_n(x) := \frac{1}{n} \sum_{k=1}^{n} D_k(x).$$

Recall that (see Golubov, Efimov, and Skvortsov [70] or Schipp, Wade, Simon and Pál[106])

$$D_{M_n}(x) = \begin{cases} M_n, & \text{if } x \in I_n, \\ 0, & \text{if } x \notin I_n. \end{cases} \tag{0.1}$$

It is well known that $\sigma_n f$ can be rewritten as follows:

$$\sigma_n f(x) = \int_{G_m} f(t) K_n(x-t) \, d\mu(t).$$

The $(C, -\alpha)$ means of the Vilenkin-Fourier series are defined by

$$\sigma_n^{-\alpha}(f,x) = \frac{1}{A_n^{-\alpha}} \sum_{k=0}^{n-1} A_{n-k-1}^{-\alpha} \widehat{f}(k) \psi_k(x),$$

where

$$A_0^{\alpha} = 1, \qquad A_n^{\alpha} = \frac{(\alpha+1)\dots(\alpha+n)}{n!}.$$

It is well known that (see Zygmund [159])

$$A_n^{\alpha} = \sum_{k=0}^{n} A_k^{\alpha-1}. \tag{0.2}$$

$$A_n^{\alpha} - A_{n-1}^{\alpha} = A_n^{\alpha-1}. \tag{0.3}$$

$$A_n^{\alpha} \sim n^{\alpha}. \tag{0.4}$$

The $(C, -\alpha)$ means of the Vilenkin-Fourier series can be rewritten as follows:

$$\sigma_n^{-\alpha}(f,x) = \frac{1}{A_n^{-\alpha}} \sum_{k=0}^{n-1} A_{n-k-1}^{-\alpha} (S_{k+1} f(x) - S_k f(x))$$

$$= \frac{1}{A_n^{-\alpha}} \left(\sum_{k=1}^{n} A_{n-k}^{-\alpha} S_k f(x) - \sum_{k=1}^{n-1} A_{n-k-1}^{-\alpha} S_k f(x) \right)$$

$$= \frac{1}{A_n^{-\alpha}} \left(\sum_{k=1}^{n-1} (A_{n-k}^{-\alpha} - A_{n-k-1}^{-\alpha}) S_k f(x) + S_n f(x) \right)$$
$$= \frac{1}{A_n^{-\alpha}} \left(\sum_{k=1}^{n-1} A_{n-k}^{-\alpha-1} S_k f(x) + S_n f(x) \right)$$
$$= \frac{1}{A_n^{-\alpha}} \sum_{k=1}^{n} A_{n-k}^{-\alpha-1} S_k f(x)$$

The norm of the space $L^p(G_m)$ is defined by

$$\|f\|_p := \left(\int_{G_m} |f(x)|^p \, d\mu(x) \right)^{1/p}, \quad (1 \leq p < \infty).$$

Denote by $C(G_m)$ the class of continuous functions on the group G_m, endoved with the supremum norm.

For the sake of brevity in notation, we agree to write $L^\infty(G_m)$ instead of $C(G_m)$.

Let $f \in L^p(G_m), 1 \leq p \leq \infty$. The expression

$$\omega\left(\frac{1}{M_n}, f\right)_p = \sup_{h \in I_n} \|f(\cdot - h) - f(\cdot)\|_p$$

is called the modulus of continuity.

Next, we give an overview of some historical results in this research area.

A number of significant results have been obtained regarding the approximate properties of Cesàro means. In the following, however, only the results are presented that are the most essential for this PhD thesis.

In 1885, Weierstrass [143] proved that if $f \in C(T)$, then

$$E_n(f)_C \to 0, \quad \text{as } n \to \infty,$$

where $E_n(f)_C$ is the best uniform approximation of f by trigonometric polynomial of degree $\leq n$.

Lebesgue [79] was the first, who applied this statement in 1906 to the theory of trigonometric series. He proved that if $f \in C(T)$, then

$$\|S_n(f) - f\|_C \leq A E_n(f)_C \log(n+2), \quad n \in \mathbb{N}.$$

In 1910, he [80] also showed that if $f \in$ Lip α with $\alpha \in]0,1]$, then

$$\|S_n(f) - f\|_C \leq A(f) n^{-\alpha} \log(n+2), \quad \text{for } n \in \mathbb{N}.$$

We can easily verify that if $f \in \mathrm{Lip}\,\alpha$ with $\alpha \in]0,1]$, then, for any $n \in \mathbb{N}$

$$\|\sigma_n^1(f) - f\|_C \leq \begin{cases} A(f) n^{-\alpha} \log(n+2) & \text{if } \alpha \in]0,1[\\ A(f) n^{-\alpha} \log^2(n+2) & \text{if } \alpha = 1 \end{cases}.$$

Later, Bernstein [6] improved these estimations. In particular, in 1912 he showed that if $f \in \mathrm{Lip}\,\alpha$ with $\alpha \in]0,1]$, then, for any $n \in \mathbb{N}$,

$$\left\|\sigma_n^1(f) - f\right\|_C \leq \begin{cases} A(f)n^{-\alpha} & \text{if } \alpha \in]0,1[\\ A(f)n^{-\alpha}\log(n+2) & \text{if } \alpha = 1 \end{cases},$$

where the order can not be improved.

In 1940, Nikolskii [88] proved that if $f \in \mathrm{Lip}\,\alpha$ with $\alpha \in [0,1]$, then, for any $n \geq n_0 > 1$,

$$\sup_{f \in Lip\alpha} \left\|\sigma_n^1(f) - f\right\|_C = \begin{cases} \frac{2}{\pi n}\log n + \gamma_n & \text{if } \alpha = 1 \\ \frac{2\Gamma(\alpha)}{\pi(1-\alpha)n^\alpha}\sin\frac{\alpha\pi}{2} + \gamma_n(\alpha) & \text{if } \alpha \in]0,1[\end{cases},$$

where $n\left|\gamma_n\right| \leq A$ and $n^\alpha\left|\gamma_n(\alpha)\right| \leq A(\alpha)$.

Natanson [87] in 1950 established that if $f \in C(T)$, then

$$\left\|\sigma_n^1(f) - f\right\|_C \leq 3\omega\left[\frac{\pi(1+2\log n)}{4n}, f\right], \quad \text{for } n \in \mathbb{N}.$$

In 1954, Izumi [76] proved that if $f \in \mathrm{Lip}\,\alpha$ with $\alpha \in]0,1]$, then

$$\|\sigma_n^\alpha(f) - f\}_C \leq A(f,\alpha)n^{-\alpha}\log(n+2), \quad n \in \mathbb{N}.$$

In 1956, Flett [17] investigated approximating properties of (C,α)-means at single points and in the norm of $C(T)$, with $\alpha \in [0,+\infty[$. In particular, he generalized some results of Bernstein and others.

In 1960, Taberski [119] proved the following theorems:

a) if $f \in C(T)$, then, for any $n \in \mathbb{N}$,

$$\|\sigma_n^\alpha(f) - f\|_C \leq \begin{cases} A(f,\alpha)\omega\left(n^{-\alpha}, f\right) & \text{if } \alpha \in]0,1[\\ A(f,\alpha)\omega\left[n^{-1}\log(n+2), f\right] & \text{if } \alpha \in [1,\infty[\end{cases}$$

b) if $f \in$ Lip α with $\alpha \in]0,1]$, then for any $n \in \mathbb{N}$,

$$\left\|\sigma_n^\beta(f) - f\right\|_C \leq \begin{cases} A(f,\beta)n^{-\alpha} & \text{if } \alpha \in]0,\beta[\\ A(f,\beta)n^{-\alpha}\log(n+2) & \text{if } \alpha = \beta \\ A(f,\beta)n^{-\beta} & \text{if } \alpha \in]\beta,1[\end{cases}$$

and

$$\left\|\sigma_n^\beta(f) - f\right\|_C \leq \begin{cases} A(f,\beta)n^{-\alpha} & \text{if } \alpha \in]0,1[, \beta \in [1,+\infty[\\ A(f,\beta)n^{-1}\log(n+2) & \text{if } \alpha = 1, \beta \in [1,+\infty[\end{cases}$$

In 1961 , Stechkin [118] showed that if $f \in C(T)$, then

$$\left\|\sigma_{n-1}^1(f) - f\right\|_C \leq An^{-1}\sum_{k=1}^{\infty} E_k(f)_C, \quad \text{for } n \in \mathbb{N}.$$

This inequality also holds for $f \in L^p(T)$ with $p \in [1, +\infty[$.

We should add that for $p \in]1, +\infty[$ we also have that

$$\left\|\sigma_{n-1}^1(f) - f\right\|_p \leq A(p)n^{-1} \sum_{k=1}^{n} E_k(f)_p, \quad \text{for } n \in \mathbb{N}.$$

Thus, the approximating properties of Cesàro means of positive orders in detail were investigated.

In 1925, Zygmund [160] proved that if $f \in \mathrm{Lip}\,\alpha$ with $\alpha \in]0, 1]$, then for any $\beta \in]0, \alpha[$

$$\left\|\sigma_n^{-\beta}(f) - f\right\|_C \leq A(f, \beta)n^{\beta-\alpha}, \quad \text{for} \quad n \in \mathbb{N}.$$

In 1928, Hardy and Littlewood [72] showed that if $f \in C(T) \cap \mathrm{Lip}(\alpha, p)$ with $\alpha \in]0, 1]$ and $p \in]1, +\infty\,[$, then $f \in \mathrm{Lip}\left(\alpha - \frac{1}{p}\right)$. Moreover, in this case we have the following estimation:

$$\|S_n(f) - f\|_C \leq A(p, f)n^{1/p-\alpha} \log(n + 2), \quad \text{for} \quad n \in \mathbb{N}.$$

Later, in 1955, Izumi [77] showed that a more precise estimate holds:

$$\|S_n(f) - f\|_C \leq A(p, f)n^{1/p-\alpha}, \quad \text{for} \quad n \in \mathbb{N}.$$

In 1955, Satô [102]-[103] proved that if $f \in C(T)$ and $\alpha \in]0, 1[$, then

$$\begin{aligned} &\left\|\sigma_n^{-\alpha}(f) - f\right\|_C \\ &\leq A(f, \alpha)\left\{\omega\left(\frac{1}{n}, f\right)_C \left[n\omega\left(\frac{1}{n}, f\right)\right]_1 \left[\omega\left(\frac{1}{n}, f\right)_C^{-1}\right]^\alpha \right.\\ &\left. + n^{-1}\int_{1/n}^{\pi} t^{-2}\omega(t, f)_C dt\right\}. \end{aligned}$$

In his monography [156] Zhizhiashvili investigated the behavior of Cesàro means of negative order for trigonometric Fourier series in detail. In particular, he proved the following estimate:

Theorem Zh1. *[156] Let $f \in L^p(T)$ for some $p \in [1, \infty]$ and $\alpha \in (0, 1)$. Then*

$$\begin{aligned} &\left\|\sigma_n^{-\alpha}(f) - f\right\|_p \\ &\leq A(p, \alpha)\left\{(n^\alpha + 1)\omega^{(2)}\left(\frac{1}{n}, f\right)_p + n^{-1}\int_{1/n}^{\pi} t^{-2}\omega^{(2)}(t, f)_p dt\right\}, \end{aligned}$$

for $n \in N$.

Moreover, he also proved that

Corollary Zh2. *[156] Let $f \in L^p(T)$ for some $p \in [1, \infty]$ and $\alpha \in (0,1)$. Then*

$$\left\|\sigma_n^{-\alpha}(f) - f\right\|_p \leq A(p,\alpha) n^{\alpha} \omega \left(\frac{1}{n}, f\right)_p ,$$

for $n \in N$.

Fine [16] proved that if $f \in L^1$, then the Walsh-Fourier series are almost everywhere (C, α) summable to the function f for all $\alpha > 0$.

In [104] Schipp agrees with the Carleson-Hunt point of view and shows that the maximal function associated with a large class of summability methods is of type (∞, ∞) and of weak type $(1, 1)$.

Pàl and Simon [94] considered the same approach for Vilenkin groups of bounded type and showed the following:
If $\sigma_n f$ represents the n-th partial Cesàro sum of $S_n f$ and if $\sigma^* f := \sup_{n>0} |\sigma_n f|$, then

$$\mu \{x : |\sigma^* f(x)| > y\} \leq C \|f\|_1 / y, \; y > 0.$$

Therefore from $f \in L^1(G_2)$ it follows that $\sigma^* f$ is weakly integrable.

Using atomic $H^1(G)$, Nobuhiko [89] proved that

$$\int\limits_{G_m} |\sigma^* f| \, d\mu \leq C \|f\|_{H^1} .$$

Consequently, if $f \in H^1(G)$, then we have that $\sigma^* f \in L^1(G)$.

Baiarstanova [5] has investigated $(C, 1)$ summability of subsequences of Walsh-Fourier series. She showed that if

$$\omega(\delta, f) = O(1/\sqrt{log(1/\delta)}), \text{ as } \delta \to \infty,$$

then the subsequence of Walsh-Fourier series is uniformly $(C, 1)$ summable.

Schipp [105] proved that if $f \in L^1$, then

$$\frac{1}{m} \sum_{k=0}^{m-1} \{|S_k f - f|^p\}^{1/p}$$

converges to zero almost everywhere, as $m \to \infty$, for any $0 < p < \infty$.

Yano [154] studied the growth of (C, β) sums of Walsh-Fourier series in the L^p space. He proved that if $1 \leq p \leq \infty, 0 < \alpha < 1$ and $f \in Lip(\alpha, L^p)$, then

$$\left\|\sigma_n^{\beta}(f) - f\right\|_p = O(n^{-\alpha}), \text{ as } n \to \infty,$$

where $\beta > \alpha$.

Skvortsov [115] proved that this estimation also holds for $0 < \beta \leq \alpha$. Furthermore, he obtained an order estimation for the limiting case when $\alpha = 1$: if $f \in Lip(1, L^p)$, then

$$\left\|\sigma_n^{\beta}(f) - f\right\|_p = O(\log n / n), \text{ as } n \to \infty.$$

In addition, he proved that there is an absolute constant C such that if $n \geq 0, \beta > 0$ and $f \in L^p$, then

$$\left\|\tilde{\sigma}_n^{\beta}(f) - f\right\|_p \leq C2^{-m} \sum_{k=0}^{m} 2^k \omega_p(2^{-k}, f),$$

where $\tilde{\sigma}_n(f)$ represents the partial (C, β) sum of Walsh-Fourier series in any piecewise linear rearrangement, while m is defined by $2^m < n < 2^{m+1}$.
More precisely, if $f \in L^p, 1 \leq p \leq \infty$, then the (C, β) sum of any piecewise linear rearrangement of the Walsh-Fourier series converges to f in L^p norm.

These results for Vilenkin groups of bounded type were generalized by Skvortsov [116].

The first results concerning Cesàro means of negative order of the Walsh-Fourier series have been studied by Tevzadze [133]. He proved convergence in the norm. In his papers, approximate properties have not been investigated.

Goginava [51] studied the rate of convergence of Cesàro means of negative order of Walsh-Fourier series. These results of Goginava are analogical of the above-mentioned Zhizhiashvili's theorems in the case of the Walsh system. In particular, the following theorem was proved:

Theorem G1. *(See [51]) Let f belong to $L^p\left(G_2\right)$ for some $p \in [1, \infty]$ and $\alpha \in (0, 1)$. Then, for any $2^k \leq n < 2^{k+1}$ $(k, n \in N)$ the inequality*

$$\left\|\sigma_n^{-\alpha}\left(f\right) - f\right\|_p \leq c\left(p, \alpha\right)\left\{2^{k\alpha}\omega\left(1/2^{k-1}, f\right)_p + \sum_{r=0}^{k-2} 2^{r-k}\omega\left(1/2^r, f\right)_p\right\}$$

holds.

Corollary G1. *(See [51]) Let $f(x)$ belong to $L^p[(0, 1)]$ for some $p \in [1, \infty]$ and let*

$$2^{\alpha k}\omega\left(1/2^{k-1}, f\right) \to 0, \text{ as } k \to \infty, \ \alpha \in (0, 1).$$

Then

$$\left\|\sigma_n^{-\alpha}\left(f\right) - f\right\|_p \to 0 \quad \text{as } n \to \infty.$$

For $p = \infty$ the sharpness of this Corollary was proved by Tevzadze [133]. Moreover, Goginava in the following theorem showed that this Corollary can not be improved in the case $p = 1$.

Theorem G2. *(See [51]) For every $\alpha \in (0, 1)$, there exists a function $f_0 \in L^1\left([0, 1]\right)$ for which*

$$\omega\left(\delta, f_0\right) = O\left(\delta^{\alpha}\right),$$

and

$$\limsup_{n\to\infty} \left\|\sigma_{2^n}^{-\alpha}\left(f_0\right) - f_0\right\|_1 > 0.$$

The Summability of Cesàro means of negative order of the Vilenkin-Fourier series has not been investigated yet. One main goal of this PhD thesis is to study approximation properties of Cesàro means of negative order of the Vilenkin-Fourier series. The rate of convergence will be estimated in terms of the modulus of continuity. In particular, we investigate and two-dimensional cases. More preconeisely, in Chapter 1 we are going to establish approximation properties of Cesàro $(C,-\alpha)$ means with $\alpha \epsilon (0,1)$ for the one-dimensional Vilenkin-Fourier series.

Theorem 1. ***(See Paper A)*** *Let f belong to $L^p(G_m)$ for some $p \in [1,\infty]$ and $\alpha \in (0,1)$. Then, for any $M_k \leq n < M_{k+1}$ $(k,n \in N)$, the inequality*

$$\left\|\sigma_n^{-\alpha}(f)-f\right\|_p \leq c(p,\alpha)\left\{M_k^{\alpha}\omega\left(1/M_{k-1},f\right)_p+\sum_{r=0}^{k-2}\frac{M_r}{M_k}\omega\left(1/M_r,f\right)_p\right\}$$

holds.

This result allows us to obtain a condition which is sufficient for the convergence of the means $\sigma_n^{-\alpha}(f,x)$ to $f(x)$ in the L^p-metric.

Corollary 1. ***(See Paper A)*** *Let f belong to $L^p(G_m)$ for some $p \in [1,\infty]$ and let $\alpha \in (0,1)$. If*

$$\omega\left(f,\frac{1}{M_{k-1}}\right)_p = o\left(\frac{1}{M_k^{\alpha}}\right),$$

then

$$\left\|\sigma_n^{-\alpha}(f)-f\right\|_p \to 0 \quad \text{as } n\to\infty.$$

Next, we investigate the sharpness of Corollary 1. In particular, the following Theorem holds:

Theorem 2. ***(See Paper E)*** *For every $\alpha \in (0,1)$, there exists a function $f \in C(G_m)$ for which*

$$\omega\left(f,\frac{1}{M_{k-1}}\right)_C = O\left(\frac{1}{M_k^{\alpha}}\right),$$

and

$$\limsup_{k\to\infty}\left\|\sigma_{M_k}^{-\alpha}(f)-f\right\|_1 > 0.$$

Since for a continuous function we have proved divergence in the space L_1, we can conclude the following corollary:

Corollary 2. ***(See Paper E)*** *For every $\alpha \in (0,1)$, there exists a function $f \in C(G_m)$, for which*

$$\omega\left(f,\frac{1}{M_{k-1}}\right)_p = O\left(\frac{1}{M_k^{\alpha}}\right),$$

and

$$\limsup_{k\to\infty} \left\|\sigma_{M_k}^{-\alpha}(f) - f\right\|_p > 0, \quad \textit{for some } p \in [1,\infty].$$

Chapter 2 studies the rate of L^p approximation by Cesàro means of negative order of rectangular partial sums of the two-dimensional Vilenkin-Fourier Series of functions from L^p. Before we present these results, we need to introduce some notation concerning the theory of two-dimensional Vilenkin system.

Let $\tilde{m}$ be a sequence like m. The relation between the sequences $(\tilde{m}_n)$ and $\left(\tilde{M}_n\right)$ is the same as between sequences (m_n) and (M_n).

The group $G_m \times G_{\tilde{m}}$ is called a two-dimensional Vilenkin group. We also suppose that $m = \tilde{m}$ and $G_m \times G_{\tilde{m}} = G_m^2$.

he normalized Haar measure is denoted by μ as in the one-dimensional case.

The norm of the space $L^p\left(G_m^2\right)$ is defined by

$$\|f\|_p := \left(\int_{G_m^2} |f(x,y)|^p \, d\mu(x,y)\right)^{1/p}, \quad (1 \le p < \infty).$$

Denote by $C\left(G_m^2\right)$ the class of continuous functions on the group G_m^2, endowed with the supremum norm.

For the sake of brevity in notation, we agree to write $L^\infty\left(G_m^2\right)$ instead of $C\left(G_m^2\right)$.

The two-dimensional Fourier coefficients, the rectangular partial sums of the Fourier series, the Dirichlet kernels with respect to the two-dimensional Vilenkin system are respectively defined as follows:

$$\widehat{f}(n_1,n_2) := \int_{G_m^2} f(x,y)\, \bar{\psi}_{n_1}(x)\, \bar{\psi}_{n_2}(y)\, d\mu(x,y),$$

$$S_{n_1,n_2}(x,y,f) := \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} \widehat{f}(k_1,k_2)\, \psi_{k_1}(x)\, \psi_{k_2}(y),$$

$$D_{n_1,n_2}(x,y) := D_{n_1}(x)\, D_{n_2}(y),$$

Moreover, we define

$$S_n^{(1)}(x,y,f) := \sum_{l=0}^{n-1} \widehat{f}(l,y)\, \psi_l(x),$$

$$S_m^{(2)}(x,y,f) := \sum_{r=0}^{m-1} \widehat{f}(x,r)\, \psi_r(y),$$

where

$$\widehat{f}(l,y)=\int\limits_{G_m} f(x,y)\,\psi_l(x)\,d\mu(x)$$

and

$$\widehat{f}(x,r)=\int\limits_{G_m} f(x,y)\,\psi_r(y)\,d\mu(y)\,.$$

The $(C,-\alpha,-\beta)$ means of the two-dimensional Vilenkin-Fourier series are defined as

$$\sigma_{n,m}^{-\alpha,-\beta}(x,y,f)=\frac{1}{A_n^{-\alpha}A_m^{-\beta}}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}A_{n-i-1}^{-\alpha}A_{m-j-1}^{-\beta}\hat{f}(i,j)\ \ \psi_i(u)\,\psi_j(v)\,,$$

The $(C,-\alpha,-\beta)$ means of the two-dimensional Vilenkin-Fourier series can be rewritten as follows:

$$\sigma_{n,m}^{-\alpha,-\beta}(x,y,f):=\frac{1}{A_n^{-\alpha}A_m^{-\beta}}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}A_{n-i-1}^{-\alpha}A_{m-j-1}^{-\beta}$$

$$\times(S_{i+1,j+1}(x,y,f)-S_{i+1,j}(x,y,f)-S_{i,j+1}(x,y,f)-S_{i,j}(x,y,f))$$

$$=\frac{1}{A_n^{-\alpha}A_m^{-\beta}}\left(\sum_{i=1}^{n}\sum_{j=1}^{m}A_{n-i}^{-\alpha}A_{m-j}^{-\beta}S_{i,j}(x,y,f)\right.$$

$$-\sum_{i=1}^{n}\sum_{j=1}^{m-1}A_{n-i}^{-\alpha}A_{m-j-1}^{-\beta}S_{i,j}(x,y,f)$$

$$-\sum_{i=1}^{n-1}\sum_{j=1}^{m}A_{n-i-1}^{-\alpha}A_{m-j}^{-\beta}S_{i,j}(x,y,f)$$

$$\left.-\sum_{i=1}^{n-1}\sum_{j=1}^{m-1}A_{n-i-1}^{-\alpha}A_{m-j-1}^{-\beta}S_{i,j}(x,y,f)\right)$$

$$=\frac{1}{A_n^{-\alpha}A_m^{-\beta}}\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{m-1}A_{n-i}^{-\alpha}(A_{m-j}^{-\beta}-A_{m-j-1}^{-\beta})S_{i,j}(x,y,f)\right.\right.$$

$$\left.+\,A_{n-i}^{-\alpha}S_{i,m}(x,y,f)\right)$$

$$-\sum_{i=1}^{n-1}\left(\sum_{j=1}^{m-1}A_{n-i-1}^{-\alpha}(A_{m-j}^{-\beta}-A_{m-j-1}^{-\beta})S_{i,j}(x,y,f)\right.$$

$$
\begin{aligned}
&\quad + A_{n-i-1}^{-\alpha} S_{i,m}(x,y,f)\big)\\
&= \frac{1}{A_n^{-\alpha} A_m^{-\beta}} \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{m-1} A_{n-i}^{-\alpha} A_{m-j}^{-\beta-1} S_{i,j}(x,y,f) \right.\right.\\
&\quad + A_{n-i}^{-\alpha} S_{i,m}(x,y,f)\big)\\
&\quad - \sum_{i=1}^{n-1} \left(\sum_{j=1}^{m-1} A_{n-i-1}^{-\alpha} A_{m-j}^{-\beta-1} S_{i,j}(x,y,f) \right.\\
&\quad + A_{n-i-1}^{-\alpha} S_{i,m}(x,y,f)\big)\\
&= \frac{1}{A_n^{-\alpha} A_m^{-\beta}} \left(\sum_{i=1}^{n} \sum_{j=1}^{m} A_{n-i}^{-\alpha} A_{m-j}^{-\beta-1} S_{i,j}(x,y,f) \right.\\
&\quad \left. - \sum_{i=1}^{n-1} \sum_{j=1}^{m} A_{n-i-1}^{-\alpha} A_{m-j}^{-\beta-1} S_{i,j}(x,y,f) \right)\\
&= \frac{1}{A_n^{-\alpha} A_m^{-\beta}} \left(\sum_{i=1}^{n-1} \sum_{j=1}^{m} (A_{n-i}^{-\alpha} - A_{n-i-1}^{-\alpha}) A_{m-j}^{-\beta-1} S_{i,j}(x,y,f) \right.\\
&\quad \left. - A_{m-j}^{-\beta-1} S_{n,j}(x,y,f) \right)\\
&= \frac{1}{A_n^{-\alpha} A_m^{-\beta}} \left(\sum_{i=1}^{n-1} \sum_{j=1}^{m} A_{n-i}^{-\alpha-1} A_{m-j}^{-\beta-1} S_{i,j}(x,y,f) \right.\\
&\quad \left. - A_{m-j}^{-\beta-1} S_{n,j}(x,y,f) \right)\\
&= \frac{1}{A_n^{-\alpha} A_m^{-\beta}} \sum_{i=1}^{n} \sum_{j=1}^{m} A_{n-i}^{-\alpha-1} A_{m-j}^{-\beta-1} S_{i,j}(x,y,f).
\end{aligned}
$$

The dyadic partial modulus of continuity ω_1 and ω_2 of a function $f \in L^p\left(G_m^2\right)$ in the L^p-norm are respectively defined by

$$
\omega_1\left(f, \frac{1}{M_n}\right)_p = \sup_{u \in I_n} \| f(\cdot + u, \cdot) - f(\cdot, \cdot) \|_p ,
$$

and

$$
\omega_2\left(f, \frac{1}{M_n}\right)_p = \sup_{v \in I_n} \| f(\cdot, \cdot + v) - f(\cdot, \cdot) \|_p ,
$$

while the dyadic mixed modulus of continuity is defined as follows:

$$
\omega_{1,2}\left(f, \frac{1}{M_n}, \frac{1}{M_m}\right)_p
$$

$$= \sup_{(u,v)\in I_n\times I_m} \left\| f(\cdot+u,\cdot+v) - f(\cdot+u,\cdot) - f(\cdot,\cdot+v) + f(\cdot,\cdot)\right\|_p .$$

The dyadic total modulus of continuity is defined by

$$\omega\left(f,\frac{1}{M_n}\right)_p := \sup_{(u,v)\in I_n\times I_n} \left\| f(\cdot+u,\cdot+v) - f(\cdot,\cdot)\right\|_p$$

It is evident that

$$\omega_i\left(f,\frac{1}{M_n}\right) \leq \omega\left(f,\frac{1}{M_n}\right), \quad i=1,2.$$

Since

$$\omega_{1,2}\left(f,\frac{1}{M_n},\frac{1}{M_m}\right) \leq 2\omega_1\left(f,\frac{1}{M_n}\right),$$

and

$$\omega_{1,2}\left(f,\frac{1}{M_n},\frac{1}{M_m}\right) \leq 2\omega_2\left(f,\frac{1}{M_m}\right),$$

it is clear that

$$\omega_{1,2}\left(f,\frac{1}{M_n},\frac{1}{M_m}\right)_p \leq \omega_1\left(f,\frac{1}{M_n}\right)_p + \omega_2\left(f,\frac{1}{M_m}\right)_p ,$$

and we also have that

$$\omega_{1,2}\left(f,\frac{1}{M_n},\frac{1}{M_m}\right) \leq 2\omega\left(f,\frac{1}{M_n}\right),$$

and

$$\omega_{1,2}\left(f,\frac{1}{M_n},\frac{1}{M_m}\right) \leq 2\omega\left(f,\frac{1}{M_m}\right).$$

It is easy to show that the dyadic total modulus of continuity can be estimated by the dyadic partial modulus of continuity ω_1 and ω_2:

$$\omega\left(f,\frac{1}{M_n}\right)_p = \sup_{(u,v)\in I_n\times I_n} \| f(\cdot+u,\cdot+v) \qquad (0.5)$$

$$-f(\cdot,\cdot+v) + f(\cdot,\cdot+v) - f(\cdot,\cdot)_p$$

$$\leq \sup_{(u,v)\in I_n\times I_n} \left\| f(\cdot+u,\cdot+v) - f(\cdot,\cdot+v)\right\|_p$$

$$+ \sup_{(u,v)\in I_n\times I_n} \left\| f(\cdot,\cdot+v) - f(\cdot,\cdot)\right\|_p$$

$$\leq \omega_1\left(f,\frac{1}{M_n}\right)_p + \omega_2\left(f,\frac{1}{M_n}\right)_p .$$

Suppose that ω is a modulus of continuity. We define

$$H_\infty^\omega := \{f \in C(I^2) : \omega_i(f,\delta)_C = O(\omega(\delta)),\ i = 1,2\}.$$

and

$$H_p^\omega := \{f \in L^p(I^2) : \omega_i(f,\delta)_p = O(\omega(\delta)),\ i = 1,2\}.$$

The problems of summability of partial sums and Cesàro means of negative order for trigonometric Fourier series were investigated in detail by Zhizhiashvili [156].

Moreover, Goginava in [55] proved some new approximation properties of Cesàro $(C,-\alpha,-\beta)$ means with $\alpha,\beta \in (0,1)$ in the case of double Walsh-Fourier series. In particular, the following theorem was proved:

Theorem G3. *(See [55]) Let f belong to $L^p\left(G_2 \times G_2\right)$ for some $p \in [1,\infty]$ and $\alpha,\beta \in (0,1)$. Then, for any $2^k \leq n < 2^{k+1}, 2^l \leq m < 2^{l+1}$ $(k,n \in N)$, the inequality*

$$\left\|\sigma_{n,m}^{-\alpha,-\beta}(f) - f\right\|_p \leq c(\alpha,\beta)\Big(2^{k\alpha}\omega_1\left(f,1/2^{k-1}\right)_p$$

$$+2^{l\beta}\omega_2\left(f,1/2^{l-1}\right)_p + 2^{k\alpha}2^{l\beta}\omega_{1,2}\left(f,1/2^{k-1},1/2^{l-1}\right)_p$$

$$+\sum_{r=0}^{k-2}2^{r-k}\omega_1\left(f,1/2^r\right)_p + \sum_{s=0}^{l-2}2^{s-l}\omega_2\left(f,1/2^s\right)_p\Big)$$

holds.

This theorem implies the following convergence results:

Corollary G2. *(See [55]) Let f belong to $L^p(I^2)$ for some $p \in [1,\infty]$. If*

$$2^{k\alpha}\omega_1\left(f,1/2^k\right)_p \to 0 \ \ as\ k \to \infty\,(0<\alpha<1),$$

$$2^{l\beta}\omega_2\left(f,1/2^l\right)_p \to 0 \ \text{ as }\ l \to \infty\ \ (0<\beta<1),$$

$$2^{k\alpha}2^{l\beta}\omega_{1,2}\left(f,1/2^k,1/2^l\right) \to 0 \quad \text{as} \quad k,l \to \infty,$$

then

$$\left\|\sigma_{n,m}^{-\alpha,-\beta}(f) - f\right\|_p \to 0 \quad \text{as}\ \ n,m \to \infty.$$

Corollary G3. *(See [55]) Let f belong to $L^p\left(I^2\right)$ for some $p \in [1,\infty]$ and let $\alpha,\beta \in (0,1)$, $\alpha+\beta<1$. If*

$$\omega(f,\delta)_p = o\left(\delta^{\alpha+\beta}\right),$$

then

$$\left\|\sigma_{n,m}^{-\alpha,-\beta}(f) - f\right\|_p \to 0 \quad \text{as}\ \ n,m \to \infty.$$

The following theorem shows that this Corollary can not be improved.

Theorem G4. *(See [55]) For every* $\alpha, \beta \in (0,1), \alpha + \beta < 1,$ *there exists a function* $f \in C(I^2)$ *for which*

$$\omega(f, \delta)_p = O\left(\delta^{\alpha+\beta}\right),$$

and

$$\limsup_{n\to\infty} \left\| \sigma_{2^n,2^n}^{-\alpha,-\beta}(f) - f \right\|_1 > 0.$$

In Chapter 2 the rate of convergence for Cesàro means of negative order of rectangular partial sums of the two-dimensional Vilenkin-Fourier Series will be proved in terms of the partial and mixed modulus of continuity. The following theorems are analogous to the above-mentioned theorems of Goginava. We have the following result:

Theorem 3. ***(See Paper B)*** *Let* f *belong to* $L^p\left(G_m^2\right)$ *for some* $p \in [1,\infty]$ *and* $\alpha \in (0,1)$*. Then, for any* $M_k \le n < M_{k+1}$ $M_l \le m < M_{l+1} (k,n,m,l \in N)$ *the inequality*

$$\left\| \sigma_{n,m}^{-\alpha,-\beta}(f) - f \right\|_p \le c(\alpha,\beta) \Big(\omega_1(f, 1/M_{k-1})_p M_k^\alpha$$

$$+\omega_2(f, 1/M_{l-1})_p M_l^\beta + \omega_{1,2}(f, 1/M_{k-1}, 1/M_{l-1})_p M_k^\alpha M_l^\beta$$

$$+\sum_{r=0}^{k-2} \frac{M_r}{M_k} \omega_1(f, 1/M_r)_p + \sum_{s=0}^{l-2} \frac{M_s}{M_l} \omega_2(f, 1/M_s)_p \Big)$$

holds.

From this main result we can obtain the following convergence results:

Corollary 3. ***(See Paper B)*** *Let* f *belong to* L^p *for some* $p \in [1,\infty]$*. If*

$$M_k^\alpha \omega_1\left(f, \frac{1}{M_k}\right)_p \to 0 \;\; as\; k \to \infty \, (0 < \alpha < 1),$$

$$M_l^\beta \omega_2\left(f, \frac{1}{M_l}\right)_p \to 0 \;\; \text{as} \;\; l \to \infty \;\; (0 < \beta < 1),$$

$$M_k^\alpha M_l^\beta \omega_{1,2}\left(f, \frac{1}{M_k}, \frac{1}{M_l}\right)_p \to 0 \quad \text{as} \quad k,l \to \infty,$$

then

$$\left\| \sigma_{n,m}^{-\alpha,-\beta}(f) - f \right\|_p \to 0 \quad \text{as} \;\; n,m \to \infty.$$

Corollary 4. ***(See Paper B)*** *Let f belong to L^p for some $p \in [1,\infty]$ and let $\alpha,\beta \in (0,1)$, $\alpha+\beta<1$. If*

$$\omega\left(f,\frac{1}{M_n}\right)_p = o\left(\left(\frac{1}{M_n}\right)^{\alpha+\beta}\right),$$

then

$$\left\|\sigma_{n,m}^{-\alpha,-\beta}(f)-f\right\|_p \to 0 \quad \text{as } n,m\to\infty.$$

The next main result shows in particular that Corollary 4 cannot be improved.

Theorem 4. ***(See Paper B)*** *For every $\alpha,\beta\in(0,1)$, $\alpha+\beta<1$, there exists a function $f_0 \in C(G_m\times G_m)$ for which*

$$\omega\left(f,\frac{1}{M_n}\right)_C = O\left(\left(\frac{1}{M_n}\right)^{\alpha+\beta}\right),$$

and

$$\limsup_{n\to\infty}\left\|\sigma_{M_n,M_n}^{-\alpha,-\beta}(f)-f\right\|_1 > 0.$$

In Chapter 3 we investigate the behavior of the Cesàro $(C,-\alpha)$ means of the quadratic partial sums of double Vilenkin-Fourier series.

The $(C,-\alpha)$ means of the double Vilenkin-Fourier series are defined as follows

$$\sigma_n^{-\alpha}(f,x,y) := \frac{1}{A_{n-1}^{-\alpha}}\sum_{j=1}^{n} A_{n-j}^{-\alpha-1}S_{j,j}(f,x,y).$$

Next, we note that Zhizhiashvili [156] also investigated the behavior of Cesàro means of negative order for trigonometric Fourier series.

Theorem Zh3. *(See [156]) Let $\alpha\in(-1,0)\cup(0,+\infty)$. Let f belong to $L^p([-\pi,\pi]^2)$ for some $p\in[1,\infty]$. Then*

a) if $\alpha\in(0,+\infty)$, then

$$\left\|\sigma_m^{\alpha}(f)-f\right\|_p \tag{0.6}$$

$$\leq A(p,\alpha)\left\{\frac{1}{m}\left[\int_{\frac{1}{m}}^{\pi} s_1^{-2}\omega_1(s_1,f)_p ds_1 + \int_{\frac{1}{m}}^{\pi} s_2^{-2}\omega_2(s_2,f)_p ds_2\right]\right\},$$

b) if $\alpha\in(0,1)$, then

$$\left\|\sigma_m^{-\alpha}(f)-f\right\|_p \tag{0.7}$$

$$\leq A(p,\alpha)\left\{m^{\alpha}\log(m+2)\left[\omega_1(\frac{1}{m},f)_p+\omega_2(\frac{1}{m},f)_p\right]\right\}.$$

In fact, Zhizhiashvili also proved that inequality (0.6) holds for $p=1$ or $p=\infty$ for the whole space L_p.

Moreover, Goginava [53] proved that under the condition $\omega(\delta)/\delta \to \infty,$ as $\delta \to 0+$, inequality (0.7) cannot be improved on the whole class H_p^ω for $p=1$ or $p=\infty$. In particular, he proved that the following theorem is valid:

Theorem G5. *(See [53]) Suppose that $\alpha \in (0,1)$ and $\omega(\delta)/\delta \to \infty$ as $\delta \to 0+$. Then, for $p=\infty$ or $p=1$ in the class H_p^ω, there exists a function f depending on p such that*

$$\limsup_{n\to\infty} \frac{\|\sigma_n^{-\alpha}(f)-f\|_1}{n^\alpha \omega\left(\frac{1}{n}\right)\log n} > 0.$$

Inequality (0.7) implies (see [156]) an analog of Zygmund's theorem (see [160]) for the means $\sigma_n^{-\alpha}$. In particular, we have the following result:

Theorem Zh4. *(See [156]) Suppose that $f \in H_p^{\delta^\alpha}$ for some $\alpha \in (0,1)$. Then, for any $\beta \in (0,\alpha)$, the following convergence result holds:*

$$\left\|\sigma_m^{-\beta}(f)-f\right\|_p \to 0, \text{ as } m \to \infty.$$

It follows from Theorem $G5$ that for $\beta=\alpha$ Theorem $Zh4$ fails for $p=+\infty$ and $p=1$. Namely, Goginava proved the following result.

Theorem G6. *(See [53]) Let $\alpha \in (0,1)$. Then, for $p=\infty$ or $p=1$ there exists a function f in the class $H_p^{\delta^\alpha}$ depending on p such that*

$$\limsup_{n\to\infty} \left\|\sigma_n^{-\beta}(f)-f\right\|_p = \infty.$$

Moreover, he [56] investigated the behavior of Cesàro $(C,-\alpha)-$means with $\alpha \in (0,1)$ in the case of the quadratic partial sums of Walsh-Fourier series. In particular, the following results are due to him:

Theorem G7. *(See [56]) Let f belong to $L^p(G_2)$ for some $p \in [1,\infty]$ and $\alpha \in (0,1)$. Then, for any $2^k \le n < 2^{k+1}, (k,n \in N)$, the inequality*

$$\begin{aligned}&\left\|\sigma_{2^k}^{-\alpha}(f)-f\right\|_p \\ &\le c(\alpha)\Big\{2^{k\alpha}\omega_1\left(f,1/2^{k-1}\right)_p + 2^{k\alpha}\omega_2\left(f,1/2^{k-1}\right)_p \\ &+\sum_{r=0}^{k-2} 2^{r-k}\omega_1\left(f,1/2^r\right)_p + \sum_{s=0}^{k-2} 2^{s-k}\omega_2\left(f,1/2^s\right)_p\Big\}\end{aligned}$$

holds.

Theorem G8. *(See [56]) Let f belong to $L^p(G_2)$ for some $p \in [1,\infty]$ and $\alpha \in (0,1)$. Then, for any $2^k \le n < 2^{k+1}, (k,n \in N)$, the inequality*

$$\left\|\sigma_n^{-\alpha}(f)-f\right\|_p$$

$$\leq c(\alpha)\Big\{2^{k\alpha}k\omega_1\left(f,1/2^{k-1}\right)_p+2^{k\alpha}k\omega_2\left(f,1/2^{k-1}\right)_p$$

$$+\sum_{r=0}^{k-2}2^{r-k}\omega_1\left(f,1/2^r\right)_p+\sum_{s=0}^{k-2}2^{s-k}\omega_2\left(f,1/2^s\right)_p\Big\}$$

holds.

Theorem G9. *(See [56]) a) Let f belong to H_p^{ω} for some $p\in[1,\infty]$ and $\alpha\in(0,1)$. Then*

$$\left\|\sigma_{2^k}^{-\alpha}(f)-f\right\|_p=O\left(2^{k\alpha}\omega\left(\frac{1}{2^{k-1}}\right)\right).$$

b) Let $\omega(\delta)/\delta\to\infty$ as $\delta\to 0+$. Then there exists function $f\in H_{\infty}^{\omega}$, for which

$$\limsup_{k\to\infty}\frac{\left\|\sigma_{2^k}^{-\alpha}(f)-f\right\|_1}{2^{k\alpha}\omega\left(\frac{1}{2^k}\right)}>0.$$

Theorem G10. *(See [56]) a) Let $f\in H_{\infty}^{\omega}$ and $\alpha\in(0,1)$. Then*

$$\left\|\sigma_n^{-\alpha}(f)-f\right\|_C=O\left(n^{\alpha}\omega\left(\frac{1}{n}\right)\log n\right).$$

b) Let $\omega(\delta)/\delta\to\infty$ as $\delta\to 0+$. Then there exists function $g\in H_{\infty}^{\omega}$, for which

$$\limsup_{n\to\infty}\frac{\|\sigma_n^{-\alpha}(g)-g\|_1}{n^{\alpha}\omega\left(\frac{1}{n}\right)\log n}>0.$$

Theorem G11. *(See [56]) a) Let $f\in H_1^{\omega}$ and $\alpha\in(0,1)$. Then*

$$\left\|\sigma_n^{-\alpha}(f)-f\right\|_1=O\left(n^{\alpha}\omega\left(\frac{1}{n}\right)\log n\right).$$

b) Let $\omega(\delta)/\delta\to\infty$ as $\delta\to 0+$. Then there exists function h belong to $f\in H_1^{\omega}$, for which

$$\limsup_{n\to\infty}\frac{\|\sigma_n^{-\alpha}(h)-h\|_1}{n^{\alpha}\omega\left(\frac{1}{n}\right)\log n}>0.$$

In Chapter 3 we state and prove the analogous results in the case of the double Vilenkin-Fourier series. Our main results read:

Theorem 5. ***(See Paper C)*** *Let f belong to $L^p\left(G_m^2\right)$ for some $p\in[1,\infty]$ and $\alpha\in(0,1)$. Then, for any $M_k\leq n<M_{k+1}\ (k,n\in N)$, the inequality*

$$\left\|\sigma_{M_k}^{-\alpha}(f)-f\right\|_p$$

$$\leq c(\alpha)\Big(\omega_1\left(f,1/M_{k-1}\right)_pM_k^{\alpha}+\omega_2\left(f,1/M_{k-1}\right)_pM_k^{\alpha}$$

$$+\sum_{r=0}^{k-2}\frac{M_r}{M_k}\omega_1\left(f,1/M_r\right)_p+\sum_{s=0}^{k-2}\frac{M_s}{M_k}\omega_2\left(f,1/M_s\right)_p\Bigg)$$

holds.

Theorem 6. ***(See Paper C)*** *Let f belong to $L^p\left(G_m^2\right)$ for some $p\in[1,\infty]$ and $\alpha\in(0,1)$. Then, for any $M_k\leq n<M_{k+1}$ $(k,n\in N)$, the inequality*

$$\left\|\sigma_n^{-\alpha}(f)-f\right\|_p$$

$$\leq c(\alpha)\Big(\omega_1\left(f,1/M_{k-1}\right)_p M_k^{\alpha}\log n+\omega_2\left(f,1/M_{k-1}\right)_p M_k^{\alpha}\log n$$

$$+\sum_{r=0}^{k-2}\frac{M_r}{M_k}\omega_1\left(f,1/M_r\right)_p+\sum_{s=0}^{k-2}\frac{M_s}{M_k}\omega_2\left(f,1/M_s\right)_p\Bigg)$$

holds.

Theorem 5 and Theorem 6 imply the following sufficient conditions for the convergence of Cesàro means of the quadratic partial sums of the double Vilenkin-Fourier series in the norm in terms of the modulus of continuity.

Corollary 5. ***(See Paper D)*** *Let f belong to L^p for some $p\in[1,\infty]$ and let $\alpha\in(0,1)$. If*

$$\omega\left(f,\frac{1}{M_{k-1}}\right)_p=o\left(\frac{1}{M_k^{\alpha}}\right),$$

then

$$\left\|\sigma_{M_k}^{-\alpha}(f)-f\right\|_p\to 0 \quad \textit{as } k\to\infty.$$

Corollary 6. ***(See Paper D)*** *Let f belong to L^p for some $p\in[1,\infty]$ and let $\alpha\in(0,1)$. If*

$$\omega\left(f,\frac{1}{M_{k-1}}\right)_p=o\left(\frac{1}{M_k^{\alpha}\log M_k}\right),$$

then

$$\left\|\sigma_n^{-\alpha}(f)-f\right\|_p\to 0 \quad \textit{as } n\to\infty.$$

Moreover, we have also proved the sharpness of Corollary 5 and Corollary 6. In particular, the following Theorems hold:

Theorem 7. ***(See Paper D)*** *For every $\alpha\in(0,1)$, there exists a function $f_0\in C\left(G_m^2\right)$ for which*

$$\omega\left(f,\frac{1}{M_{k-1}}\right)_C=O\left(\frac{1}{M_k^{\alpha}}\right),$$

and

$$\limsup_{k\to\infty}\left\|\sigma_{M_k}^{-\alpha}(f)-f\right\|_1>0.$$

Since for a continuous function we have proved divergence in the space L_1, we can conclude the following corollary:

Corollary 7. ***(See Paper D)*** *For every* $\alpha \in (0,1)$, *there exists a function* $f_0 \in C\left(G_m^2\right) \subset L_p\left(G_m^2\right)$ *for some* $p \in [1,\infty]$, *for which*

$$\omega\left(f, \frac{1}{M_{k-1}}\right)_p = O\left(\frac{1}{M_k^\alpha}\right),$$

and

$$\limsup_{k\to\infty} \left\|\sigma_{M_k}^{-\alpha}(f) - f\right\|_p > 0.$$

Theorem 8. ***(See Paper D)*** *For every* $\alpha \in (0,1)$, *there exists a function* $g \in C\left(G_m^2\right)$ *for which*

$$\omega\left(g, \frac{1}{M_{k-1}}\right)_C = O\left(\frac{1}{M_k^\alpha \log M_k}\right),$$

and

$$\limsup_{n\to\infty} \left\|\sigma_n^{-\alpha}(g) - g\right\|_C > 0.$$

Theorem 9. ***(See Paper D)*** *For every* $\alpha \in (0,1)$, $p = 1$, *there exists a function* $h \in L_1\left(G_m^2\right)$ *for which*

$$\omega\left(h, \frac{1}{M_{k-1}}\right)_{L_1} = O\left(\frac{1}{M_k^\alpha \log M_k}\right),$$

and

$$\limsup_{n\to\infty} \left\|\sigma_n^{-\alpha}(h) - h\right\|_1 > 0.$$

The summability of the Walsh- and Vilenkin-Fourier series for one and two-dimensional cases have been investigated by a lot of researchers. For instance, we list researches conducted by the following authors: Alotaibi and Mursaleen [3], Avdispahić and Memić [4], Blahota, Gát and Goginava [7] - [8], Blahota and Goginava [9], Blahota and Nagy [10], Blahota, Nagy, Persson and Tephnadze [11], Blahota, Persson and Tephnadze [12], Blahota and Tephnadze [13] - [14], Chandra [15], Fridli [18] - [19], Galoyan and Grigoryan [20], Gát [21] - [28], Gát and Blahota [29], Gát and Goginava [30] - [38], Gát and Nagy [39], Getsadze [40], Ghodadra [41] - [43], Goginava [45] - [65], Goginava, and Gogoladze [66] - [67], Goginava and Nagy [68], Goginava and Sahakian [69], Grigoryan and Sargsyan [71], Iofina and Volosivets [74] - [75], Memić, Simon and Tephnadze [81], Móricz, Schipp and Wade [82], Móricz and Schipp [83], Nadiradze [84], Nagy [85], Nagy and Salim [86], Oniani [90], Oninani and Goginava [91] - [92], Persson [95] - [96], Persson, Tephnadze and Weisz [97], Persson, Schipp, Tephnadze and Weisz [98], Polyakov [99] - [100], Sahoo [101], Simon [108] - [111], Simon and Weisz [112] - [113], Schipp [105], Scheckter and Sukochev [107], Suetin [117], Tephnadze [120] -

[130], Tephnadze and Tutberidze [131], Tevzadze [132], Tukhliev [134], Volosivets [136] - [137], Volosivets and Kuznetsova [138], Voronov [139] - [140], Wade [141] - [142], Weisz [144] - [153], Zhang [155], Zhu [157], Zhu and Zheng [158].

Finally, I would like to mention that my work in this area has been inspired by these researches in classical Fourier analysis.

Chapter 1

Approximation Properties of Cesàro Means of the one-dimensional Vilenkin-Fourier Series

1.1 Formulation of the main results

In this Chapter we establish approximation properties of Cesàro $(C,-\alpha)$ means with $\alpha \epsilon (0,1)$ in the case of the one-dimensional Vilenkin-Fourier Series. Our first main result reads:

Theorem 1.1. ***(See Paper A)*** *Let f belong to $L^p(G_m)$ for some $p \in [1,\infty]$ and $\alpha \in (0,1)$. Then, for any $M_k \leq n < M_{k+1}$ $(k,n \in N)$, the inequality*

$$\left\|\sigma_n^{-\alpha}(f)-f\right\|_p \leq c(p,\alpha)\left\{M_k^{\alpha}\omega\left(1/M_{k-1},f\right)_p+\sum_{r=0}^{k-2}\frac{M_r}{M_k}\omega\left(1/M_r,f\right)_p\right\}$$

holds.

This result allows us to obtain a condition which is sufficient for the convergence of the means $\sigma_n^{-\alpha}(f,x)$ to $f(x)$ in the L^p-metric.

Corollary 1.2. ***(See Paper A)*** *Let f belong to $L^p(G_m)$ for some $p \in [1,\infty]$ and let $\alpha \in (0,1)$. If*

$$\omega\left(f,\frac{1}{M_{k-1}}\right)_p=o\left(\frac{1}{M_k^{\alpha}}\right),$$

then

$$\left\|\sigma_n^{-\alpha}(f)-f\right\|_p \to 0 \quad \text{as } n \to \infty.$$

Next, we state the sharpness of Corollary 1.2. In particular, the following Theorem holds:

Theorem 1.3. ***(See Paper E)*** *For every $\alpha \in (0,1)$, there exists a function $f \in C(G_m)$ for which*

$$\omega\left(f,\frac{1}{M_{k-1}}\right)_C=O\left(\frac{1}{M_k^{\alpha}}\right),$$

and

$$\limsup_{k\to\infty}\left\|\sigma_{M_k}^{-\alpha}(f)-f\right\|_1>0.$$

Since for a continuous function we have proved the divergence in the space L_1, we can conclude the following consequence of this result:

Corollary 1.4. ***(See Paper E)*** *For every* $\alpha \in (0,1)$, *there exists a function* $f \in C(G_m)$, *for which*

$$\omega\left(f, \frac{1}{M_{k-1}}\right)_p = O\left(\frac{1}{M_k^\alpha}\right),$$

and

$$\limsup_{k\to\infty} \left\|\sigma_{M_k}^{-\alpha}(f) - f\right\|_p > 0, \quad \textit{for some } p \in [1,\infty].$$

1.2 Auxiliary results

In order to prove Theorem 1.1 we need the following results:

Lemma 1.1. *(See [2])Let $\alpha_1, ..., \alpha_n$ be real numbers.Then*

$$\frac{1}{n}\int_G \left|\sum_{k=1}^n \alpha_k D_k(x)\right| d\mu(x) \leq \frac{c}{\sqrt{n}}\left(\sum_{k=1}^n \alpha_k^2\right)^{1/2},$$

where c is an absolute constant.

Abel's Transformation. *(See [1]) If $a_1, \ldots, a_N, b_1, \ldots, b_N$, are given complex numbers and we set*

$$B_n = \sum_{i \leq n} b_i,$$

then the summation by parts is the identity

$$\sum_{k=1}^N a_k b_k = a_N B_N - \sum_{k=1}^{N-1} B_k(a_{k+1} - a_k).$$

Generalized Minkowski's Inequality. *(See [73]) Let $f \in L^p(G_m)$ and $1 \leq p \leq \infty$. Then the following inequality holds:*

$$\left(\int_{G_m}\left(\int_{G_m} |f|\, d\mu(x)\right)^p d\mu(y)\right)^{1/p}$$

$$\leq \int_{G_m}\left(\int_{G_m} |f|^p\, d\mu(y)\right)^{1/p} d\mu(x).$$

Lemma 1.2. *Let $f \in L^p(G_m)$ for some $p \in [1, \infty]$. Then, for every $\alpha \in (0, 1)$, the following estimations holds:*

$$\frac{1}{A_n^{-\alpha}}\left\|\int_{G_m} \sum_{v=0}^{M_{k-1}-1} A_{n-v}^{-\alpha}\psi_v(u)\left[f(\cdot + u) - f(\cdot)\right] d\mu(u)\right\|_p$$

$$\leq c(p, \alpha)\sum_{r=0}^{k-1}\frac{M_r}{M_k}\omega(1/M_k, f)_p,$$

where $M_k \leq n \leq M_{k+1}$.

Proof of Lemma 1.2. By applying Abel's transformation, from (0.3) we get

$$\frac{1}{A_n^{-\alpha}}\left\|\int\limits_{G_m}\sum_{v=0}^{M_{k-1}-1}A_{n-v}^{-\alpha}\psi_v(u)\left[f(\cdot+u)-f(\cdot)\right]d\mu(u)\right\|_p \tag{1.1}$$

$$=\frac{1}{A_n^{-\alpha}}\left\|\int\limits_{G_m}\sum_{v=1}^{M_{k-1}}A_{n-v-1}^{-\alpha}\psi_{v-1}(u)\left[f(\cdot+u)-f(\cdot)\right]d\mu(u)\right\|_p$$

$$=\frac{1}{A_n^{-\alpha}}\left\|\int\limits_{G_m}\left(A_{n-M_{k-1}-1}^{-\alpha}D_{M_{k-1}}+\sum_{v=1}^{M_{k-1}-1}A_{n-v-1}^{-\alpha-1}D_v(u)\right)\right.$$
$$\left.\times\left[f(\cdot+u)-f(\cdot)\right]d\mu(u)\right\|_p$$

$$\leq\frac{1}{A_n^{-\alpha}}\left\|\int\limits_{G_m}\sum_{v=1}^{M_{k-1}-1}A_{n-v-1}^{-\alpha-1}D_v(u)\left[f(\cdot+u)-f(\cdot)\right]d\mu(u)\right\|_p$$

$$+\frac{1}{A_n^{-\alpha}}\left\|\int\limits_{G_m}A_{n-M_{k-1}-1}^{-\alpha}D_{M_{k-1}}(u)\left[f(\cdot+u)-f(\cdot)\right]d\mu(u)\right\|_p$$

$$:=I_1+I_2.$$

From the generalized Minkowski's inequality, and by (0.1) and (0.4) we obtain that

$$I_2\leq n^{-\alpha}(n-M_{k-1}-1)^{-\alpha}\int\limits_{I_{k-1}}M_{k-1}\left\|f(\cdot+u)-f(\cdot)\right\|d\mu(u)_p \tag{1.2}$$

$$\leq c(\alpha)M_{k-1}\int\limits_{I_{k-1}}\left\|f(\cdot+u)-f(\cdot)\right\|_p d\mu(u)$$

$$=O\left(\omega\left(1/M_{k-1},f\right)\right)_p.$$

Moreover, we use the addition properly of equality and add and subtract $S_{M_r}(\cdot+u,f)$ and $S_{M_r}(\cdot,f)$ to I_1, so we get that

$$I_1\leq\frac{1}{A_n^{-\alpha}}\sum_{r=0}^{k-2}\left\|\int\limits_{G_m}\sum_{v=M_r}^{M_{r+1}-1}A_{n-v-1}^{-\alpha-1}D_v(u)\right. \tag{1.3}$$

$$\left.\times\left[f(\cdot+u)-f(\cdot)\right]d\mu(u)\right\|_p$$

$$\leq\frac{1}{A_n^{-\alpha}}\sum_{r=0}^{k-2}\left\|\int\limits_{G_m}\sum_{v=M_r}^{M_{r+1}-1}A_{n-v-1}^{-\alpha-1}D_v(u)\right.$$

$$\times \left[f(\cdot+u) - S_{M_r}(\cdot+u,f)\right] d\mu(u)\|_p$$

$$+\frac{1}{A_n^{-\alpha}}\sum_{r=0}^{k-2}\left\|\int\limits_{G_m}\sum_{v=M_r}^{M_{r+1}-1}A_{n-v-1}^{-\alpha-1}D_v(u)\right.$$

$$\times \left[S_{M_r}(\cdot+u,f) - S_{M_r}(\cdot,f)\right] d\mu(u)\|_p$$

$$+\frac{1}{A_n^{-\alpha}}\sum_{r=0}^{k-2}\left\|\int\limits_{G_m}\sum_{v=M_r}^{M_{r+1}-1}A_{n-v-1}^{-\alpha-1}D_v(u)\right.$$

$$\times \left[S_{M_r}(\cdot,f) - f(\cdot)\right] d\mu(u)\|_p$$

$$:= I_{11} + I_{12} + I_{13}.$$

Since

$$\left\|f - S_{M_r}(f)\right\|_p \le \omega(1/M_r, f)_p,$$

by using the generalized Minkowski's inequality, Lemma 1.1 and (0.4) for I_{11}, we can estimate the following:

$$I_{11} \le \frac{1}{A_n^{-\alpha}}\sum_{r=0}^{k-2}\int\limits_{G_m}\left|\sum_{v=M_r}^{M_{r+1}-1}A_{n-v-1}^{-\alpha-1}D_v(u)\right| \tag{1.4}$$

$$\times \left\|f(\cdot+u) - S_{M_r}(\cdot+u,f)\right\|_p d\mu(u)$$

$$\le \frac{1}{A_n^{-\alpha}}\sum_{r=0}^{k-2}\omega(1/M_r,f)_p$$

$$\times \int\limits_{G_m}\left|\sum_{v=M_r}^{M_{r+1}-1}A_{n-v-1}^{-\alpha-1}D_v(u)\right| d\mu(u)$$

$$\le c(\alpha)\, n^{\alpha}\sum_{r=0}^{k-2}\omega(1/M_r,f)_p\sqrt{M_{r+1}}$$

$$\times \left(\sum_{v=M_r}^{M_{r+1}-1}(n-v-1)^{-2\alpha-2}\right)^{1/2}$$

$$\le c(\alpha)\, n^{\alpha}\sum_{r=0}^{k-2}\omega(1/M_r,f)_p$$

$$\times\sqrt{M_{r+1}}\left(n-M_{r+1}\right)^{-\alpha-1}\sqrt{M_{r+1}}$$

$$\leq c\left(\alpha\right)\sum_{r=0}^{k-2}\frac{M_r}{M_k}\omega\left(1/M_r,f\right)_p.$$

Analogously, we can prove that

$$I_{13}\leq c\left(\alpha\right)\sum_{r=0}^{k-2}\frac{M_r}{M_k}\omega\left(1/M_r,f\right)_p. \tag{1.5}$$

Since

$$S_n(\cdot,f)=\int_{G_m}f\left(t\right)D_n\left(\cdot-t\right)d\mu\left(t\right),$$

and by using (0.1), it is evident that

$$\int_{G_m}\sum_{v=M_r}^{M_{r+1}-1}A_{n-v-1}^{-\alpha-1}D_v\left(u\right) \tag{1.6}$$

$$\times\left[S_{M_r}\left(\cdot+u,f\right)-S_{M_r}\left(\cdot,f\right)\right]d\mu\left(u\right)$$

$$=\sum_{v=M_r}^{M_{r+1}-1}A_{n-v-1}^{-\alpha-1}\int_{G_m}S_{M_r}\left(\cdot+u,f\right)D_v\left(u\right)d\mu\left(u\right)$$

$$-\sum_{v=M_r}^{M_{r+1}-1}A_{n-v-1}^{-\alpha-1}S_{M_r}\left(\cdot,f\right)$$

$$=\sum_{v=M_r}^{M_{r+1}-1}A_{n-v-1}^{-\alpha-1}S_v\left(\cdot,S_{M_r}\left(f\right)\right)$$

$$-\sum_{v=M_r}^{M_{r+1}-1}A_{n-v-1}^{-\alpha-1}S_{M_r}\left(\cdot,f\right)$$

$$=\sum_{v=M_r}^{M_{r+1}-1}A_{n-v-1}^{-\alpha-1}S_{M_r}\left(\cdot,f\right)$$

$$-\sum_{v=M_r}^{M_{r+1}-1}A_{n-v-1}^{-\alpha-1}S_{M_r}\left(\cdot,f\right)=0.$$

Hence,

$$I_{12}=0. \tag{1.7}$$

By combining (1.1)-(1.7) we receive the proof of Lemma 1.2. ■

Lemma 1.3. *Let* $f \in L^p(G_m)$ *for some* $p \in [1, \infty]$ *. Then, for every* $\alpha \in (0,1)$*, the following estimations hold*

$$\frac{1}{A_n^{-\alpha}} \left\| \int_{G_m} \sum_{v=M_{k-1}}^{M_k - 1} A_{n-v}^{-\alpha} \psi_v (u) \left[f(\cdot + u) - f(\cdot) \right] d\mu (u) \right\|_p$$

$$\leq c(p, \alpha) \, \omega (1/M_{k-1}, f)_p \, M_k^{\alpha}$$

and

$$\frac{1}{A_n^{-\alpha}} \left\| \int_{G_m} \sum_{v=M_k}^{n} A_{n-v}^{-\alpha} \psi_v (u) \left[f(\cdot + u) - f(\cdot) \right] d\mu (u) \right\|_p$$

$$\leq c(p, \alpha) \, \omega (1/M_k, f)_p \, M_k^{\alpha},$$

where $M_k \leq n < M_{k+1}$.

Proof of Lemma 1.3. We have that

$$II := \frac{1}{A_n^{-\alpha}} \left\| \int_{G_m} \sum_{v=M_{k-1}}^{M_k - 1} A_{n-v}^{-\alpha} \psi_v (u) \left[f(\cdot + u) - f(\cdot) \right] d\mu (u) \right\|_p \tag{1.8}$$

$$= \frac{1}{A_n^{-\alpha}} \left\| \int_{G_m} \sum_{v=M_{k-1}}^{M_k - 1} A_{n-v}^{-\alpha} \psi_v (u) \, f(\cdot + u) \, d\mu (u) \right\|_p$$

$$\leq \frac{1}{A_n^{-\alpha}} \left\| \int_{G_m} \sum_{v=M_{k-1}}^{M_k - 1} A_{n-v}^{-\alpha} \psi_v (u) \right.$$

$$\left. \times \left[f(\cdot + u) - S_{M_{k-1}} (\cdot + u, f) \right] d\mu (u) \right\|_p$$

$$+ \frac{1}{A_n^{-\alpha}} \left\| \int_{G_m} \sum_{v=M_{k-1}}^{M_k - 1} A_{n-v}^{-\alpha} \psi_v (u) \, S_{M_{k-1}} (\cdot + u, f) \, d\mu (u) \right\|_p$$

$$:= II_1 + II_2.$$

Since

$$\int_{G_m} \sum_{v=M_{k-1}}^{M_k - 1} A_{n-v}^{-\alpha} \psi_v (u) \, S_{M_{k-1}} (\cdot + u, f) \, d\mu (u) \tag{1.9}$$

$$= \sum_{j=0}^{M_{k-1} - 1} \widehat{f}(j) \, \psi_j (x) \sum_{v=M_{k-1}}^{M_k - 1} A_{n-v}^{-\alpha}$$

$$\times \int_{G_m} \psi_v(u)\,\psi_j(u)\,d\mu(u) = 0,$$

for II_2 we obtain that

$$II_2 = 0. \tag{1.10}$$

Moreover, by using the generalized Minkowski's inequality and the fact that

$$\left\|f - S_{M_{k-1}}(f)\right\|_p \leq \omega\left(1/M_{k-1}, f\right)_p,$$

we have that

$$II_1 \leq \frac{1}{A_n^{-\alpha}} \int_{G_m} \left| \sum_{v=M_{k-1}}^{M_k-1} A_{n-v}^{-\alpha} \psi_v(u) \right| \tag{1.11}$$

$$\times \left\|f(\cdot + u) - S_{M_{k-1}}(\cdot + u, f)\right\|_p d\mu(u)$$

$$\leq c(\alpha)\, n^{\alpha} \omega\left(1/M_{k-1}, f\right)_p$$

$$\times \int_{G_m} \left| \sum_{v=M_{k-1}}^{M_k-1} A_{n-v}^{-\alpha} \psi_v(u) \right| d\mu(u).$$

Let $t \in I_{A-1} \backslash I_A,\ \ A = 1, 2, ..., k-1$ and $M_k = pM_A + q$, where $0 \leq q < M_A$. Then we find that

$$\sum_{v=M_{k-1}}^{M_k-1} A_{n-v}^{-\alpha} \psi_v(t) \tag{1.12}$$

$$= \sum_{v=M_{k-1}}^{pM_A-1} A_{n-v}^{-\alpha} \psi_v(t) + \sum_{v=pM_A}^{M_k-1} A_{n-v}^{-\alpha} \psi_v(t)$$

$$= \sum_{r=M_{k-1}/M_A}^{p-1} \sum_{v=rM_A}^{(r+1)M_A-1} A_{n-v}^{-\alpha} \psi_v(t)$$

$$+ \sum_{v=0}^{q-1} A_{n-v-pM_A}^{-\alpha} \psi_{v+pM_A}(t)$$

$$= \sum_{r=M_{k-1}/M_A}^{p-1} \sum_{v=0}^{M_A-1} A_{n-v-rM_A}^{-\alpha} \psi_{v+rM_A}(t)$$

$$+ \sum_{v=0}^{q-1} A_{n-v-pM_A}^{-\alpha} \psi_{v+pM_A}(t)$$

$$
\begin{aligned}
&= \sum_{r=M_{k-1}/M_A}^{p-2} \psi_{rM_A}(t) \sum_{v=0}^{M_A-1} A_{n-v-rM_A}^{-\alpha} \psi_v(t) \\
&+\psi_{(p-1)M_A}(t) \sum_{v=0}^{M_A-1} A_{n-v-(p-1)M_A}^{-\alpha} \psi_v(t) \\
&+\psi_{rM_A}(t) \sum_{v=0}^{q-1} A_{q-v}^{-\alpha} \psi_v(t) \\
&:= A_1 + A_2 + A_3.
\end{aligned}
$$

Since $D_{M_A}(t) = 0$, where $t \in I_{A-1} \backslash I_A$ and $|D_k(t)| \leq k$, from Abel's transformation and by using (0.4), it follows that

$$
\begin{aligned}
|A_1| &= \left| \sum_{r=M_{k-1}/M_A}^{p-2} \psi_{rM_A}(t) \sum_{v=1}^{M_A} A_{n-v+1-rM_A}^{-\alpha} \psi_{v-1}(t) \right| \qquad (1.13) \\
&= \left| \sum_{r=M_{k-1}/M_A}^{p-2} \psi_{rM_A}(t) \left(A_{n-M_A+1-rM_A}^{-\alpha} D_{M_A}(t) \right. \right. \\
&\quad \left. \left. + \sum_{v=1}^{M_A-1} A_{n-v+1-rM_A}^{-\alpha-1} D_v(t) \right) \right| \\
&= \left| \sum_{r=M_{k-1}/M_A}^{p-2} \psi_{rM_A}(t) \sum_{v=0}^{M_A-2} A_{n-v-rM_A}^{-\alpha-1} D_{v+1}(t) \right| \\
&\leq c(\alpha) M_A \sum_{r=M_{k-1}/M_A}^{p-2} \sum_{v=0}^{M_A} (n - rM_A - v)^{-\alpha-1} \\
&\leq c(\alpha) M_A (n - (p-1) M_A)^{-\alpha} \\
&\leq c(\alpha) M_A^{1-\alpha}
\end{aligned}
$$

Similarly, for A_2 we have that

$$
\begin{aligned}
|A_2| &\leq c(\alpha) \sum_{v=0}^{M_A-1} (n - (p-1) M_A - v)^{-\alpha} \qquad (1.14) \\
&\leq c(\alpha) \sum_{v=0}^{M_A-1} (M_A + q - v)^{-\alpha} \\
&\leq c(\alpha) M_A^{1-\alpha}.
\end{aligned}
$$

Moreover, by using (0.4) and the fact that $q < M_A$, we get that

$$|A_3| \leq c(\alpha) \sum_{v=0}^{q-1} (q-v)^{-\alpha} \tag{1.15}$$

$$\leq c(\alpha) q^{1-\alpha} \leq c(\alpha) M_A^{1-\alpha}.$$

We can combine (1.12)-(1.15) to obtain that

$$\left| \sum_{v=M_{k-1}}^{M_k-1} A_{n-v}^{-\alpha} \psi_v(u) \right| \leq c(\alpha) M_A^{1-\alpha}, \tag{1.16}$$

where $t \in I_{A-1} \backslash I_A, \ A = 1, ..., k-1.$
From (1.16) it follows that

$$\int_{G_m} \left| \sum_{v=M_{k-1}}^{M_k-1} A_{n-v}^{-\alpha} \psi_v(u) \right| d\mu(u) \tag{1.17}$$

$$= \sum_{A=1}^{k-1} \int_{I_{A-1} \backslash I_A} \left| \sum_{v=M_{k-1}}^{M_k-1} A_{n-v}^{-\alpha} \psi_v(u) \right| d\mu(u)$$

$$+ \int_{I_{k-1}} \left| \sum_{v=M_{k-1}}^{M_k-1} A_{n-v}^{-\alpha} \psi_v(u) \right| d\mu(u)$$

$$\leq c(\alpha) \sum_{A=1}^{k} \frac{1}{M_A} M_A^{1-\alpha} + \frac{c(\alpha)}{M_k} M_k^{1-\alpha}$$

$$\leq c(\alpha).$$

By combining (1.11) with (1.17) we have that

$$II_1 \leq c(\alpha) \omega(1/M_{k-1}, f)_p M_k^{\alpha}. \tag{1.18}$$

Combining (1.8), (1.10) and (1.18) we can conclude that

$$\frac{1}{A_n^{-\alpha}} \left\| \int_{G_m} \sum_{v=M_{k-1}}^{M_k-1} A_{n-v}^{-\alpha} \psi_v(u) [f(\cdot+u) - f(\cdot)] d\mu(u) \right\|_p$$

$$\leq c(\alpha) \omega(1/M_{k-1}, f)_p M_k^{\alpha}.$$

Analogously, we can prove that

$$\frac{1}{A_n^{-\alpha}} \left\| \int\limits_{G_m} \sum_{v=M_k}^{n} A_{n-v}^{-\alpha} \psi_v (u) \left[f (\cdot + u) - f (\cdot) \right] d\mu (u) \right\|_p$$

$$\leq c (\alpha) \, \omega (1/M_k, f)_p \, M_k^{\alpha}.$$

The proof is completed. ■

1.3 Proofs of the main results

Proof of Theorem 1.1. In order to prove Theorem 1.1, we need to estimate the difference $\sigma_n^{-\alpha}(f,x) - f(x)$, which can be presented with three summands in the following way:

$$\sigma_n^{-\alpha}(f,x) - f(x) = \frac{1}{A_n^{-\alpha}} \int\limits_{G_m} \sum_{v=0}^{n} A_{n-v}^{-\alpha} \psi_v(x) \left[f(\cdot + u) - f(\cdot) \right] d\mu(u)$$

$$= \frac{1}{A_n^{-\alpha}} \int\limits_{G_m} \sum_{v=0}^{M_{k-1}-1} A_{n-v}^{-\alpha} \psi_v(x) \left[f(\cdot + u) - f(\cdot) \right] d\mu(u)$$

$$+ \frac{1}{A_n^{-\alpha}} \int\limits_{G_m} \sum_{v=M_{k-1}}^{M_k-1} A_{n-v}^{-\alpha} \psi_v(x) \left[f(\cdot + u) - f(\cdot) \right] d\mu(u)$$

$$+ \frac{1}{A_n^{-\alpha}} \int\limits_{G_m} \sum_{v=M_k}^{n} A_{n-v}^{-\alpha} \psi_v(x) \left[f(\cdot + u) - f(\cdot) \right] d\mu(u)$$

$$:= I + II + III.$$

Since

$$\left\| \sigma_n^{-\alpha}(f,\cdot) - f(\cdot) \right\|_p \leq \|I\|_p + \|II\|_p + \|III\|_p ,$$

and I, II, and III were already estimated in Lemmas 1.2 and 1.3, so the proof is complete. ■

Proof of Theorem 1.3. We define the function

$$f(x) := \sum_{j=1}^{\infty} \frac{1}{M_j^{\alpha}} f_j(x),$$

where

$$f_j(x) = \rho_j(x) = \exp \frac{2\pi i x_j}{m_j}.$$

First, we prove that

$$\omega\left(f, \frac{1}{M_n}\right)_C = O\left(\frac{1}{M_n^{\alpha}}\right). \tag{1.19}$$

Since

$$|f_j(x-t) - f_j(x)| = 0, \quad j = 0, 1, ..., n-1, \quad t \in I_n,$$

we get that

$$
\begin{aligned}
&\left|f(x-t)-f(x)\right| \\
&\leq \sum_{j=1}^{n-1} \frac{1}{M_j^{\alpha}} \left|f_j(x-t)-f_j(x)\right| \\
&+\sum_{j=n}^{\infty} \frac{2}{M_j^{\alpha}} \leq \frac{c}{M_n^{\alpha}},
\end{aligned}
$$

which means that (1.19) holds.

Next, we shall prove that $\sigma_{M_k}^{-\alpha}(f)$ diverges in the L^1 metric. It is clear that

$$
\begin{aligned}
&\left\|\sigma_{M_k}^{-\alpha}(f)-f\right\|_1 \qquad (1.20) \\
&\geq \left| \int_{G_m} \left[\sigma_{M_k}^{-\alpha}(f,x)-f(x)\right] \psi_{M_k}(x)\, d\mu(x)\right| \\
&\geq \left| \int_{G_m} \sigma_{M_k}^{-\alpha}(f,x)\, \psi_{M_k}(x)\, d\mu(x)\right| - \left|\widehat{f}(M_k)\right| \\
&= \left| \frac{1}{A_{M_k}^{-\alpha}} \sum_{i=0}^{M_k} A_{M_k-i}^{-\alpha} \widehat{f}(i) \int_{G_m} \psi_i(x)\, \psi_{M_k}(x)\, d\mu(x)\right| \\
&-\left|\widehat{f}(M_k)\right| = \frac{1}{A_{M_k}^{-\alpha}} \left|\widehat{f}(M_k)\right| - \left|\widehat{f}(M_k)\right|.
\end{aligned}
$$

By the definition of $\widehat{f}(M_k)$, we have

$$
\begin{aligned}
&\widehat{f}(M_k) = \int_{G_m} f(x)\, \bar{\psi}_{M_k}(x)\, d\mu(x) \qquad (1.21) \\
&= \sum_{j=1}^{\infty} \frac{1}{M_j^{\alpha}} \int_{G_m} \rho_j(x)\, \bar{\psi}_{M_k}(x)\, d\mu(x) = \frac{1}{M_k^{\alpha}}.
\end{aligned}
$$

By combining (1.20) with (1.21) we find that

$$
\left\|\sigma_{M_k}^{-\alpha}(f)-f\right\|_1 \geq c(\alpha) > 0, \qquad (1.22)
$$

so also the second statement in Theorem 1.3 is proved. The proof is complete. ■

Chapter 2

Approximation Properties of Cesàro Means of the two-dimensional Vilenkin-Fourier Series

2.1 Formulation of the main results

In this Chapter we investigate approximation properties of Cesàro $(C,-\alpha,-\beta)$ means with $\alpha,\beta\in(0,1)$ in the case of the double Vilenkin-Fourier series. Our first main result reads:

Theorem 2.1. ***(See Paper B)*** *Let f belong to $L^p\left(G_m^2\right)$ for some $p\in[1,\infty]$ and $\alpha\in(0,1)$. Then, for any $M_k\leq n<M_{k+1}$ $M_l\leq m<M_{l+1}(k,n,m,l\in N)$, the following inequality holds:*

$$\left\|\sigma_{n,m}^{-\alpha,-\beta}(f)-f\right\|_p$$

$$\leq c(\alpha,\beta)\Big(\omega_1\left(f,1/M_{k-1}\right)_p M_k^{\alpha}+\omega_2\left(f,1/M_{l-1}\right)_p M_l^{\beta}$$

$$+\omega_{1,2}\left(f,1/M_{k-1},1/M_{l-1}\right)_p M_k^{\alpha}M_l^{\beta}$$

$$+\sum_{r=0}^{k-2}\frac{M_r}{M_k}\omega_1\left(f,1/M_r\right)_p+\sum_{s=0}^{l-2}\frac{M_s}{M_l}\omega_2\left(f,1/M_s\right)_p\Big).$$

As a consequence of Theorem 2.1 we obtain the following convergence results:

Corollary 2.2. ***(See Paper B)*** *Let f belong to L^p for some $p\in[1,\infty]$. If*

$$M_k^{\alpha}\omega_1\left(f,\frac{1}{M_k}\right)_p\to 0 \ as\ k\to\infty\,(0<\alpha<1),$$

$$M_l^{\beta}\omega_2\left(f,\frac{1}{M_l}\right)_p\to 0 \text{ as } l\to\infty\ (0<\beta<1),$$

$$M_k^{\alpha}M_l^{\beta}\omega_{1,2}\left(f,\frac{1}{M_k},\frac{1}{M_l}\right)_p\to 0 \quad\text{as}\quad k,l\to\infty,$$

then

$$\left\|\sigma_{n,m}^{-\alpha,-\beta}(f)-f\right\|_p\to 0 \quad\text{as}\quad n,m\to\infty.$$

Corollary 2.3. ***(See Paper B)*** *Let f belong to L^p for some $p \in [1,\infty]$ and let $\alpha,\beta \in (0,1)$, $\alpha+\beta<1$. If*

$$\omega\left(f,\frac{1}{M_n}\right)_p = o\left(\left(\frac{1}{M_n}\right)^{\alpha+\beta}\right),$$

then

$$\left\|\sigma_{n,m}^{-\alpha,-\beta}(f)-f\right\|_p \to 0 \quad \text{as } n,m\to\infty.$$

The following Theorem shows that Corollary 2.3 cannot be improved.

Theorem 2.4. ***(See Paper B)*** *For every $\alpha,\beta \in (0,1)$, $\alpha+\beta<1$, there exists a function $f_0 \in C(G_m\times G_m)$ for which*

$$\omega\left(f,\frac{1}{M_n}\right)_C = O\left(\left(\frac{1}{M_n}\right)^{\alpha+\beta}\right),$$

and

$$\limsup_{n\to\infty}\left\|\sigma_{M_n,M_n}^{-\alpha,-\beta}(f)-f\right\|_1 > 0.$$

2.2 Auxiliary results

In order to prove Theorem 2.1 we need the following Lemmas of independent interest:

Lemma 2.1. *Let* $f \in L^p(G_m^2)$ *for some* $p \in [1, \infty]$ *. Then, for every* $\alpha, \beta \in (0,1)$ *, the following estimations holds*

$$I := \frac{1}{A_n^{-\alpha} A_m^{-\beta}} \left\| \int_{G_m^2} \sum_{i=0}^{M_{k-1}-1} \sum_{j=0}^{M_{l-1}-1} A_{n-i}^{-\alpha} A_{m-j}^{-\beta} \psi_i(u) \psi_j(v) \right.$$

$$\left. \times [f(\cdot + u, \cdot + v) - f(\cdot, \cdot)] \, d\mu(u,v) \right\|_p$$

$$\leq c(\alpha, \beta) \left(\sum_{r=0}^{k-1} \frac{M_r}{M_k} \omega_1(f, 1/M_r)_p + \sum_{s=0}^{l-1} \frac{M_s}{M_l} \omega_2(f, 1/M_s)_p \right),$$

where $M_k \leq n < M_{k+1}, M_l \leq m < M_{l+1}$.

Proof of Lemma 2.1. By applying Abel's transformation on $\sum_{i=0}^{M_{k-1}-1} A_{n-i}^{-\alpha} \psi_i(u)$, and by using (0.3) we have that

$$\sum_{i=0}^{M_{k-1}-1} A_{n-i}^{-\alpha} \psi_i(u)$$

$$= \sum_{i=1}^{M_{k-1}} A_{n-i+1}^{-\alpha} \psi_{i-1}(u)$$

$$= A_{n-M_{k-1}+1}^{-\alpha} D_{M_{k-1}}(u) + \sum_{i=1}^{M_{k-1}-1} A_{n-i+1}^{-\alpha-1} D_i(u).$$

By using the same transformation on $\sum_{j=0}^{M_{l-1}-1} A_{m-j}^{-\beta} \psi_j(v)$, for I we obtain accordingly the following estimate:

$$I \leq \frac{1}{A_n^{-\alpha} A_m^{-\beta}} \left\| \int_{G_m^2} \sum_{i=1}^{M_{k-1}-1} \sum_{j=1}^{M_{l-1}-1} A_{n-i+1}^{-\alpha-1} A_{m-j+1}^{-\beta-1} D_i(u) D_j(v) \right. \tag{2.1}$$

$$\left. \times [f(\cdot + u, \cdot + v) - f(\cdot, \cdot)] \, d\mu(u,v) \right\|_p$$

$$+\frac{1}{A_n^{-\alpha}A_m^{-\beta}}\left\| \int\limits_{G_m^2} A_{m-M_{l-1}+1}^{-\beta}D_{M_{l-1}}(v)\sum_{i=1}^{M_{k-1}-1}A_{n-i+1}^{-\alpha-1}D_i(u)\right.$$

$$\times\left[f(\cdot+u,\cdot+v)-f(\cdot,\cdot)\right]d\mu(u,v)\|_p$$

$$+\frac{1}{A_n^{-\alpha}A_m^{-\beta}}\left\| \int\limits_{G_m^2} A_{n-M_{k-1}+1}^{-\alpha}D_{M_{k-1}}(u)\sum_{j=1}^{M_{l-1}-1}A_{m-j+1}^{-\beta-1}D_j(v)\right.$$

$$\times\left[f(\cdot+u,\cdot+v)-f(\cdot,\cdot)\right]d\mu(u,v)\|_p$$

$$+\frac{1}{A_n^{-\alpha}A_m^{-\beta}}\left\| \int\limits_{G_m^2} A_{n-M_{k-1}+1}^{-\alpha}A_{m-M_{l-1}+1}^{-\beta}D_{M_{k-1}}(u)D_{M_{l-1}}(v)\right.$$

$$\times\left[f(\cdot+u,\cdot+v)-f(\cdot,\cdot)\right]d\mu(u,v)\|_p$$

$$:=I_1+I_2+I_3+I_4.$$

From the generalized Minkowski's inequality, and by using (0.1) and (0.4) we obtain that

$$I_4\leq\frac{1}{A_n^{-\alpha}A_m^{-\beta}}\int\limits_{G_m^2}\left|A_{n-M_{k-1}+1}^{-\alpha}A_{m-M_{l-1}+1}^{-\beta}D_{M_{k-1}}(u)D_{M_{l-1}}(v)\right| \tag{2.2}$$

$$\times\|f(\cdot+u,\cdot+v)-f(\cdot,\cdot)\|_p\,d\mu(u,v)$$

$$\leq c(\alpha,\beta)M_{k-1}M_{l-1}\int\limits_{I_{k-1}\times I_{l-1}}\|f(\cdot+u,\cdot+v)-f(\cdot,\cdot)\|_p\,d\mu(u,v)$$

$$=O\left(\omega_1(f,1/M_{k-1})_p+\omega_2(f,1/M_{l-1})_p\right).$$

Hence, it is evident that by adding and subtracting $S_{M_r,M_s}(\cdot+u,\cdot+v,f)$ and $S_{M_r,M_s}(\cdot,\cdot,f)$ to I_1 in (2.2), we get that

$$I_1\leq\frac{1}{A_n^{-\alpha}A_m^{-\beta}}\sum_{r=0}^{k-2}\sum_{s=0}^{l-2}\left\|\int\limits_{G_m^2}\sum_{i=M_r}^{M_{r+1}-1}\sum_{j=M_s}^{M_{s+1}-1}A_{n-i+1}^{-\alpha-1}A_{m-j+1}^{-\beta-1}D_i(u)D_j(v)\right. \tag{2.3}$$

$$\times \left[f\left(\cdot + u, \cdot + v \right) - f\left(\cdot, \cdot \right) \right] d\mu\left(u, v \right) \Big\|_p$$

$$\leq \frac{1}{A_n^{-\alpha} A_m^{-\beta}} \sum_{r=0}^{k-2} \sum_{s=0}^{l-2} \left\| \int_{G_m^2} \sum_{i=M_r}^{M_{r+1}-1} \sum_{j=M_s}^{M_{s+1}-1} A_{n-i+1}^{-\alpha-1} A_{m-j+1}^{-\beta-1} D_i\left(u \right) D_j\left(v \right) \right.$$

$$\times \left[f\left(\cdot + u, \cdot + v \right) - S_{M_r, M_s}\left(\cdot + u, \cdot + v, f \right) \right] d\mu\left(u, v \right) \Big\|_p$$

$$+ \frac{1}{A_n^{-\alpha} A_m^{-\beta}} \sum_{r=0}^{k-2} \sum_{s=0}^{l-2} \left\| \int_{G_m^2} \sum_{i=M_r}^{M_{r+1}-1} \sum_{j=M_s}^{M_{s+1}-1} A_{n-i+1}^{-\alpha-1} A_{m-j+1}^{-\beta-1} D_i\left(u \right) D_j\left(v \right) \right.$$

$$\times \left[S_{M_r, M_s}\left(\cdot + u, \cdot + v, f \right) - S_{M_r, M_s}\left(\cdot, \cdot, f \right) \right] d\mu\left(u, v \right) \Big\|_p$$

$$+ \frac{1}{A_n^{-\alpha} A_m^{-\beta}} \sum_{r=0}^{k-2} \sum_{s=0}^{l-2} \left\| \int_{G_m^2} \sum_{i=M_r}^{M_{r+1}-1} \sum_{j=M_s}^{M_{s+1}-1} A_{n-i+1}^{-\alpha-1} A_{m-j+1}^{-\beta-1} D_i\left(u \right) D_j\left(v \right) \right.$$

$$\times \left[S_{M_r, M_s}\left(\cdot, \cdot, f \right) - f\left(\cdot, \cdot \right) \right] d\mu\left(u, v \right) \Big\|_p$$

$$:= I_{11} + I_{12} + I_{13}.$$

Since

$$S_{i,j}\left(\cdot, \cdot, f \right) = \int_{G_m^2} f\left(u, v \right) D_i\left(\cdot - u \right) D_j\left(\cdot - v \right) d\mu\left(u, v \right),$$

it is easy to prove that

$$\int_{G_m^2} \sum_{i=M_r}^{M_{r+1}-1} \sum_{j=M_s}^{M_{s+1}-1} A_{n-i+1}^{-\alpha-1} A_{m-j+1}^{-\beta-1} D_i\left(u \right) D_j\left(v \right)$$

$$\times \left[S_{M_r, M_s}\left(\cdot + u, \cdot + v, f \right) - S_{M_r, M_s}\left(\cdot, \cdot, f \right) \right] d\mu\left(u, v \right)$$

$$= \sum_{i=M_r}^{M_{r+1}-1} \sum_{j=M_s}^{M_{s+1}-1} A_{n-i+1}^{-\alpha-1} A_{m-j+1}^{-\beta-1}$$

$$\times \int_{G_m^2} S_{M_r, M_s}\left(\cdot + u, \cdot + v, f \right) D_i\left(u \right) D_j\left(v \right) d\mu\left(u, v \right)$$

$$-\sum_{i=M_r}^{M_{r+1}-1}\sum_{j=M_s}^{M_{s+1}-1} A_{n-i+1}^{-\alpha-1}A_{m-j+1}^{-\beta-1}S_{M_r,M_s}\left(\cdot,\cdot,f\right)$$

$$=\sum_{i=M_r}^{M_{r+1}-1}\sum_{j=M_s}^{M_{s+1}-1} A_{n-i+1}^{-\alpha-1}A_{m-j+1}^{-\beta-1}S_{i,j}\left(\cdot,\cdot,S_{M_r,M_s}\left(f\right)\right)$$

$$-\sum_{i=M_r}^{M_{r+1}-1}\sum_{j=M_s}^{M_{s+1}-1} A_{n-i+1}^{-\alpha-1}A_{m-j+1}^{-\beta-1}S_{M_r,M_s}\left(\cdot,\cdot,f\right)$$

$$=\sum_{i=M_r}^{M_{r+1}-1}\sum_{j=M_s}^{M_{s+1}-1} A_{n-i+1}^{-\alpha-1}A_{m-j+1}^{-\beta-1}S_{M_r,M_s}\left(\cdot,\cdot,f\right)$$

$$-\sum_{i=M_r}^{M_{r+1}-1}\sum_{j=M_s}^{M_{s+1}-1} A_{n-i+1}^{-\alpha-1}A_{m-j+1}^{-\beta-1}S_{M_r,M_s}\left(\cdot,\cdot,f\right)=0,$$

and accordingly it follows that

$$I_{12}=0. \tag{2.4}$$

Since

$$\left\|f\left(\cdot+u,\cdot+v\right)-S_{M_r,M_s}\left(\cdot+u,\cdot+v,f\right)\right\|_p$$

$$\leq \omega_1(f,1/M_r)_p+\omega_2(f,1/M_s)_p,$$

from the generalized Minkowski's inequality and by using Lemma 1.1 and (0.4) for I_{11} we can write

$$I_{11} \tag{2.5}$$

$$\leq \frac{1}{A_n^{-\alpha}A_m^{-\beta}}\sum_{r=0}^{k-2}\sum_{s=0}^{l-2}\int_{G_m^2}\left|\sum_{i=M_r}^{M_{r+1}-1}\sum_{j=M_s}^{M_{s+1}-1} A_{n-i+1}^{-\alpha-1}A_{m-j+1}^{-\beta-1}D_i\left(u\right)D_j\left(v\right)\right|$$

$$\times\left\|f\left(\cdot+u,\cdot+v\right)-S_{M_r,M_s}\left(\cdot+u,\cdot+v,f\right)\right\|_p d\mu\left(u,v\right)$$

$$\leq c\left(\alpha,\beta\right)n^{\alpha}m^{\beta}\sum_{r=0}^{k-2}\sum_{s=0}^{l-2}\left(\omega_1(f,1/M_r)_p+\omega_2(f,1/M_s)_p\right)$$

$$\times\int_{G_m}\left|\sum_{i=M_r}^{M_{r+1}-1} A_{n-i+1}^{-\alpha-1}D_i\left(u\right)\right|d\mu\left(u\right)$$

$$\times\int_{G_m}\left|\sum_{j=M_s}^{M_{s+1}-1} A_{m-j+1}^{-\beta-1}D_j\left(v\right)\right|d\mu\left(v\right)$$

$$\leq c(\alpha,\beta)\, n^{\alpha} m^{\beta} \sum_{r=0}^{k-2}\sum_{s=0}^{l-2} \left(\omega_1(f,1/M_r)_p + \omega_2(f,1/M_s)_p\right)$$

$$\times \sqrt{M_{r+1}} \left(\sum_{i=M_r}^{M_{r+1}-1} (n-i+1)^{-2\alpha-2}\right)^{1/2}$$

$$\times \sqrt{M_{s+1}} \left(\sum_{j=M_s}^{M_{s+1}-1} (m-j+1)^{-2\beta-2}\right)^{1/2}$$

$$\leq c(\alpha,\beta)\, n^{\alpha} m^{\beta} \sum_{r=0}^{k-2}\sum_{s=0}^{l-2} \left(\omega_1(f,1/M_r)_p + \omega_2(f,1/M_s)_p\right)$$

$$\times \sqrt{M_{r+1}}\, (n - M_{r+1})^{-\alpha-1} \sqrt{M_{r+1}}$$

$$\times \sqrt{M_{s+1}}\, (n - M_{s+1})^{-\beta-1} \sqrt{M_{s+1}}$$

$$\leq c(\alpha,\beta)\, n^{\alpha} m^{\beta} \sum_{r=0}^{k-2}\sum_{s=0}^{l-2} \frac{M_{r+1}}{M_k^{\alpha+1}} \frac{M_{s+1}}{M_l^{\beta+1}}$$

$$\times \left(\omega_1(f,1/M_r)_p + \omega_2(f,1/M_s)_p\right)$$

$$\leq c(\alpha,\beta) \left(\sum_{r=0}^{k-2} \frac{M_{r+1}}{M_k} \omega_1(f,1/M_r)_p \sum_{s=0}^{l-2} \frac{M_{s+1}}{M_l} \right.$$

$$\left. + \sum_{s=0}^{l-2} \frac{M_{s+1}}{M_l} \omega_2(f,1/M_s)_p \sum_{r=0}^{k-2} \frac{M_{r+1}}{M_k} \right)$$

$$\leq c(\alpha,\beta) \left(\sum_{r=0}^{k-2} \frac{M_r}{M_k} \omega_1(f,1/M_r)_p + \sum_{s=0}^{l-2} \frac{M_s}{M_l} \omega_2(f,1/M_s)_p \right).$$

Analogously, we can prove that

$$I_{13} \leq c(\alpha,\beta) \left(\sum_{r=0}^{k-2} \frac{M_r}{M_k} \omega_1(f,1/M_r)_p + \sum_{s=0}^{l-2} \frac{M_s}{M_l} \omega_2(f,1/M_s)_p \right). \tag{2.6}$$

By combining (2.3)-(2.6) for I_1 we obtain the estimate

$$I_1 \leq c(\alpha,\beta) \left(\sum_{r=0}^{k-2} \frac{M_r}{M_k} \omega_1(f,1/M_r)_p + \sum_{s=0}^{l-2} \frac{M_s}{M_l} \omega_2(f,1/M_s)_p \right). \tag{2.7}$$

Next, we shell estimate I_2 and first we note that

$$I_2 \leq \frac{1}{A_n^{-\alpha} A_m^{-\beta}} \left\| \int_{G_m^2} A_{m-M_{l-1}+1}^{-\beta} D_{M_{l-1}}(v) \sum_{i=1}^{M_{k-1}-1} A_{n-i+1}^{-\alpha-1} D_i(u) \right. \tag{2.8}$$

$$\left. \times \left[f(\cdot + u, \cdot + v) - f(\cdot + u, \cdot) \right] d\mu(u,v) \right\|_p$$

$$+ \frac{1}{A_n^{-\alpha} A_m^{-\beta}} \left\| \int_{G_m^2} A_{m-M_{l-1}+1}^{-\beta} D_{M_{l-1}}(v) \sum_{i=1}^{M_{k-1}-1} A_{n-i+1}^{-\alpha-1} D_i(u) \right.$$

$$\left. \times \left[f(\cdot + u, \cdot) - f(\cdot, \cdot) \right] d\mu(u,v) \right\|_p$$

$$:= I_{21} + I_{22}.$$

From the generalized Minkowski's inequality, and by using (0.1) and (0.4) we obtain that

$$I_{21} \leq \frac{(m - M_{l-1} + 1)^{-\beta}}{A_n^{-\alpha} A_m^{-\beta}} \int_{G_m} D_{M_{l-1}}(v) \left(\int_{G_m} \left| \sum_{i=1}^{M_{k-1}-1} A_{n-i+1}^{-\alpha-1} D_i(u) \right| \right. \tag{2.9}$$

$$\left. \times \left\| f(\cdot + u, \cdot + v) - f(\cdot + u, \cdot) \right\|_p d\mu(u) \right) d\mu(v)$$

$$\leq c(\alpha, \beta) \frac{M_{l-1}}{A_n^{-\alpha}} \int_{I_{l-1}} \left(\int_{G_m} \left| \sum_{i=1}^{M_{k-1}-1} A_{n-i+1}^{-\alpha-1} D_i(u) \right| \right.$$

$$\left. \times \left\| f(\cdot + u, \cdot + v) - f(\cdot + u, \cdot) \right\|_p d\mu(u) \right) d\mu(v)$$

$$\leq c(\alpha, \beta) n^\alpha \omega_2(f, 1/M_{l-1})$$

$$\times \int_{G_m} \left| \sum_{i=1}^{M_{k-1}-1} A_{n-i+1}^{-\alpha-1} D_i(u) \right| d\mu(u)$$

$$\leq c(\alpha, \beta) n^\alpha \omega_2(f, 1/M_{l-1})$$

$$\times \sqrt{M_{k-1}} \left(\sum_{i=1}^{M_{k-1}-1} (n - i + 1)^{-2\alpha-2} \right)^{1/2}$$

$$\leq c(\alpha,\beta)\, n^{\alpha}\omega_2\left(f,1/M_{l-1}\right)$$

$$\times\sqrt{M_{k-1}}\left(n-M_{k-1}\right)^{-\alpha-1}\sqrt{M_{k-1}}$$

$$\leq c(\alpha,\beta)\,\omega_2\left(f,1/M_{l-1}\right).$$

The estimation of I_{22} is analogous to the estimation of I_1 and we have that

$$I_{22}\leq c(\alpha,\beta)\sum_{r=0}^{k-2}\frac{M_r}{M_k}\omega_1\left(f,1/M_r\right)_p. \tag{2.10}$$

So, combining (2.9)-(2.10) for I_2 we find that

$$I_2\leq c(\alpha,\beta)\left(\sum_{r=0}^{k-2}\frac{M_r}{M_k}\omega_1\left(f,1/M_r\right)_p+\omega_2\left(f,1/M_{l-1}\right)\right). \tag{2.11}$$

The estimation I_3 is analogous to the estimation of I_2 and we obtain that

$$I_3\leq c(\alpha,\beta)\left(\sum_{s=0}^{l-2}\frac{M_s}{M_l}\omega_2\left(f,1/M_s\right)_p+\omega_1\left(f,1/M_{k-1}\right)\right). \tag{2.12}$$

By combining (2.2), (2.7), (2.11) and (2.12) we obtain for (2.1) that

$$I\leq c(\alpha,\beta)\left(\sum_{r=0}^{k-1}\frac{M_r}{M_k}\omega_1\left(f,1/M_r\right)_p+\sum_{s=0}^{l-1}\frac{M_s}{M_l}\omega_2\left(f,1/M_s\right)_p\right),$$

so the proof is complete. ■

Lemma 2.2. *Let $f\in L^p(G_m)$ for some $p\in[1,\infty]$. Then, for every $\alpha,\beta\in(0,1)$, the following estimations holds*

$$II:=\frac{1}{A_n^{-\alpha}A_m^{-\beta}}\left\|\int_{G_m^2}\sum_{i=M_{k-1}}^{n}\sum_{j=0}^{M_{l-1}-1}A_{n-i}^{-\alpha}A_{m-j}^{-\beta}\psi_i(u)\,\psi_j(v)\right.$$

$$\left.\times\left[f\left(\cdot+u,\cdot+v\right)-f\left(\cdot,\cdot\right)\right]d\mu\left(u,v\right)\right\|_p$$

$$\leq c(\alpha,\beta)\,\omega_2\left(f,1/M_{l-1}\right)_p M_l^{\beta},$$

and

$$III:=\frac{1}{A_n^{-\alpha}A_m^{-\beta}}\left\|\int_{G_m^2}\sum_{i=0}^{M_{k-1}-1}\sum_{j=M_{l-1}}^{m}A_{n-i}^{-\alpha}A_{m-j}^{-\beta}\psi_i(u)\,\psi_j(v)\right.$$

$$\times \left[f\left(\cdot+u,\cdot+v\right)-f\left(\cdot,\cdot\right)\right]d\mu\left(u,v\right)\|_{p}$$

$$\leq c\left(\alpha,\beta\right)\omega_{1}\left(f,1/M_{k-1}\right)_{p}M_{k}^{\alpha},$$

where $M_k \leq n < M_{k+1}, M_l \leq m < M_{l+1}$.

Proof of Lemma 2.2. From the generalized Minkowski's inequality we obtain that

$$II=\frac{1}{A_{n}^{-\alpha}A_{m}^{-\beta}}\left\|\int_{G_{m}^{2}}\sum_{i=M_{k-1}}^{n}\sum_{j=0}^{M_{l-1}-1}A_{n-i}^{-\alpha}A_{m-j}^{-\beta}\psi_{i}\left(u\right)\psi_{j}\left(v\right)\right. \tag{2.13}$$

$$\times f\left(\cdot+u,\cdot+v\right)d\mu\left(u,v\right)\|_{p}$$

$$=\frac{1}{A_{n}^{-\alpha}A_{m}^{-\beta}}\left\|\int_{G_{m}^{2}}\sum_{i=M_{k-1}}^{n}\sum_{j=0}^{M_{l-1}-1}A_{n-i}^{-\alpha}A_{m-j}^{-\beta}\psi_{i}\left(u\right)\psi_{j}\left(v\right)\right.$$

$$\times\left[f\left(\cdot+u,\cdot+v\right)-S_{M_{l-1}}^{(2)}\left(\cdot+u,\cdot+v,f\right)\right]d\mu\left(u,v\right)\Big\|_{p}$$

$$\leq\frac{1}{A_{n}^{-\alpha}A_{m}^{-\beta}}\int_{G_{m}^{2}}\left|\sum_{i=M_{k-1}}^{M_{k}-1}\sum_{j=0}^{M_{l-1}-1}A_{n-i}^{-\alpha}A_{m-j}^{-\beta}\psi_{i}\left(u\right)\psi_{j}\left(v\right)\right|$$

$$\times\left\|f\left(\cdot+u,\cdot+v\right)-S_{M_{l-1}}^{(2)}\left(\cdot+u,\cdot+v,f\right)\right\|_{p}d\mu\left(u,v\right)$$

$$+\frac{1}{A_{n}^{-\alpha}A_{m}^{-\beta}}\int_{G_{m}^{2}}\left|\sum_{i=M_{k}}^{n}\sum_{j=0}^{M_{l-1}-1}A_{n-i}^{-\alpha}A_{m-j}^{-\beta}\psi_{i}\left(u\right)\psi_{j}\left(v\right)\right|$$

$$\times\left\|f\left(\cdot+u,\cdot+v\right)-S_{M_{l-1}}^{(2)}\left(\cdot+u,\cdot+v,f\right)\right\|_{p}d\mu\left(u,v\right)$$

$$:=II_{1}+II_{2}.$$

In the previous Chapter we showed that the inequality

$$\int_{G_{m}}\left|\sum_{v=M_{k-1}}^{M_{k}-1}A_{n-v}^{-\alpha}\psi_{v}(u)\right|d\mu(u)\leq c\left(\alpha\right),\quad\left(k=1,2...\right) \tag{2.14}$$

holds (see equation 1.17).

Since

$$\left\| f(\cdot+u,\cdot+v) - S_{M_{l-1}}^{(2)}(\cdot+u,\cdot+v,f) \right\|_p \le \omega_2(f,1/M_{l-1})_p ,$$

by using Lemma 1.1, in view of (0.4) and (2.14) we estimate II_1 as follows:

$$II_1 \le c(\alpha,\beta)\, n^{\alpha} m^{\beta} \omega_2(f,1/M_{l-1})_p \tag{2.15}$$

$$\times \int_{G_m} \left| \sum_{i=M_{k-1}}^{M_k-1} A_{n-i}^{-\alpha} \psi_i(u) \right| d\mu(u)$$

$$\times \int_{G_m} \left| \sum_{j=1}^{M_{l-1}} A_{m-j+1}^{-\beta} \psi_{j-1}(v) \right| d\mu(v)$$

$$\le c(\alpha,\beta)\, n^{\alpha} m^{\beta} \omega_2(f,1/M_{l-1})_p$$

$$\times \sqrt{M_{l-1}} \left(\sum_{i=1}^{M_{l-1}} (m-j+1)^{-2\beta} \right)^{1/2}$$

$$\le c(\alpha,\beta)\, n^{\alpha} m^{\beta} \omega_2(f,1/M_{l-1})_p$$

$$\times \sqrt{M_{l-1}} (n-M_{l-1})^{-\beta} \sqrt{M_{l-1}}$$

$$\le c(\alpha,\beta)\, \omega_2(f,1/M_{l-1})_p M_l^{\beta}.$$

The estimation of II_2 is analogous to the estimation of II_1 and we have that

$$II_2 \le c(\alpha,\beta)\, \omega_2(f,1/M_{l-1})_p M_l^{\beta}. \tag{2.16}$$

By combining (2.13), (2.15) and (2.16) we conclude that

$$II \le c(\alpha,\beta)\, \omega_2(f,1/M_{l-1})_p M_l^{\beta}. \tag{2.17}$$

Analogously, we can prove that

$$III := \frac{1}{A_n^{-\alpha} A_m^{-\beta}} \left\| \int_{G_m^2} \sum_{i=0}^{M_{k-1}-1} \sum_{j=M_{l-1}}^{m} A_{n-i}^{-\alpha} A_{m-j}^{-\beta} \psi_i(u)\, \psi_j(v) \right. \tag{2.18}$$

$$\left. \times \left[f(\cdot+u,\cdot+v) - f(\cdot,\cdot) \right] d\mu(u,v) \right\|_p$$

$$\le c(\alpha,\beta)\, \omega_1(f,1/M_{k-1})_p M_k^{\alpha}.$$

Finally, we just combine (2.17) with (2.18) and the proof of Lemma 2.2 is complete. ■

Lemma 2.3. *Let $f \in L^p(G_m)$ for some $p \in [1,\infty]$. Then, for every $\alpha, \beta \in (0,1)$, the following estimation holds:*

$$IV := \frac{1}{A_n^{-\alpha} A_m^{-\beta}} \left\| \int_{G_m^2} \sum_{i=M_{k-1}}^{n} \sum_{j=M_{l-1}}^{m} A_{n-i}^{-\alpha} A_{m-j}^{-\beta} \psi_i(u) \psi_j(v) \right.$$

$$\left. \times [f(\cdot + u, \cdot + v) - f(\cdot, \cdot)] \, d\mu(u) \, d\mu(v) \right\|_p$$

$$\leq c(\alpha,\beta) \, \omega_{1,2}(f, 1/M_k, 1/M_l)_p \, M_k^{\alpha} M_l^{\beta},$$

where $M_k \leq n < M_{k+1}, M_l \leq m < M_{l+1}$.

Proof of Lemma 2.3. From the generalized Minkowski's inequality, and by using (0.4) and (2.14) we obtain that

$$IV = \frac{1}{A_n^{-\alpha} A_m^{-\beta}} \left\| \int_{G_m^2} \sum_{i=M_{k-1}}^{n} \sum_{j=M_{l-1}}^{m} A_{n-i}^{-\alpha} A_{m-j}^{-\beta} \psi_i(u) \psi_j(v) \right. \tag{2.19}$$

$$\left. \times f(\cdot + u, \cdot + v) \, d\mu(u,v) \right\|_p$$

$$\leq \frac{1}{A_n^{-\alpha} A_m^{-\beta}} \left\| \int_{G_m^2} \sum_{i=M_{k-1}}^{n} \sum_{j=M_{l-1}}^{m} A_{n-i}^{-\alpha} A_{m-j}^{-\beta} \psi_i(u) \psi_j(v) \right.$$

$$\times \Big[S_{M_{k-1},M_{l-1}}(\cdot + u, \cdot + v, f) - S_{M_{k-1}}^{(1)}(\cdot + u, \cdot + v, f)$$

$$\left. - S_{M_{l-1}}^{(2)}(\cdot + u, \cdot + v, f) + f(\cdot + u, \cdot + v) \Big] \, d\mu(u,v) \right\|_p$$

$$\leq \frac{1}{A_n^{-\alpha} A_m^{-\beta}} \int_{G_m^2} \left| \sum_{i=M_{k-1}}^{n} \sum_{j=M_{l-1}}^{m} A_{n-i}^{-\alpha} A_{m-j}^{-\beta} \psi_i(u) \psi_j(v) \right|$$

$$\times \Big\| S_{M_{k-1},M_{l-1}}(\cdot + u, \cdot + v, f) - S_{M_{k-1}}^{(1)}(\cdot + u, \cdot + v, f)$$

$$- S_{M_{l-1}}^{(2)}(\cdot + u, \cdot + v, f) + f(\cdot + u, \cdot + v) \Big\|_p \, d\mu(u,v)$$

$$\leq c(\alpha,\beta)\, n^{\alpha} m^{\beta} \omega_{1,2}\left(f, 1/M_{k-1}, 1/M_{l-1}\right)_p$$

$$\times \int\limits_{G_m^2} \left| \sum_{i=M_{k-1}}^{n} \sum_{j=M_{l-1}}^{m} A_{n-i}^{-\alpha} A_{m-j}^{-\beta} \psi_i(u)\, \psi_j(v) \right| d\mu(u,v)$$

$$\leq c(\alpha,\beta)\, n^{\alpha} m^{\beta} \omega_{1,2}\left(f, 1/M_{k-1}, 1/M_{l-1}\right)_p$$

$$\times \int\limits_{G_m} \left| \sum_{i=M_{k-1}}^{n} A_{n-i}^{-\alpha} \psi_i(u) \right| d\mu(u)$$

$$\times \int\limits_{G_m} \left| \sum_{j=M_{l-1}}^{m} A_{m-j}^{-\beta} \psi_j(v) \right| d\mu(v)$$

$$\leq c(\alpha,\beta)\, M_k^{\alpha} M_l^{\beta} \omega_{1,2}\left(f, 1/M_{k-1}, 1/M_{l-1}\right)_p .$$

The proof is complete. ■

2.3 Proofs of the main results

Proof of Theorem 2.1. It is evident that the difference $\sigma_{n,m}^{-\alpha,-\beta}(f,x,y)-f(x,y)$ can be written as follows

$$\sigma_{n,m}^{-\alpha,-\beta}(f,x,y)-f(x,y)$$

$$=\frac{1}{A_n^{-\alpha}A_m^{-\beta}}\int\limits_{G_m^2}\sum_{i=0}^{n}\sum_{j=0}^{n}A_{n-i}^{-\alpha}A_{m-j}^{-\beta}\psi_i(u)\psi_j(v)$$

$$\times\left[f(\cdot+u,\cdot+v)-f(\cdot,\cdot)\right]d\mu(u,v)$$

$$=\frac{1}{A_n^{-\alpha}A_m^{-\beta}}\int\limits_{G_m^2}\sum_{i=0}^{M_{k-1}-1}\sum_{j=0}^{M_{l-1}-1}A_{n-i}^{-\alpha}A_{m-j}^{-\beta}\psi_i(u)\psi_j(v)$$

$$\times\left[f(\cdot+u,\cdot+v)-f(\cdot,\cdot)\right]d\mu(u,v)$$

$$+\frac{1}{A_n^{-\alpha}A_m^{-\beta}}\int\limits_{G_m^2}\sum_{i=M_{k-1}}^{n}\sum_{j=0}^{M_{l-1}-1}A_{n-i}^{-\alpha}A_{m-j}^{-\beta}\psi_i(u)\psi_j(v)$$

$$\times\left[f(\cdot+u,\cdot+v)-f(\cdot,\cdot)\right]d\mu(u,v)$$

$$+\frac{1}{A_n^{-\alpha}A_m^{-\beta}}\int\limits_{G_m^2}\sum_{i=0}^{M_{k-1}-1}\sum_{j=M_{l-1}}^{m}A_{n-i}^{-\alpha}A_{m-j}^{-\beta}\psi_i(u)\psi_j(v)$$

$$\times\left[f(\cdot+u,\cdot+v)-f(\cdot,\cdot)\right]d\mu(u,v)$$

$$+\frac{1}{A_n^{-\alpha}A_m^{-\beta}}\int\limits_{G_m^2}\sum_{i=M_{k-1}}^{n}\sum_{j=M_{l-1}}^{m}A_{n-i}^{-\alpha}A_{m-j}^{-\beta}\psi_i(u)\psi_j(v)$$

$$\times\left[f(\cdot+u,\cdot+v)-f(\cdot,\cdot)\right]d\mu(u)\,d\mu(v)$$

$$:=I+II+III+IV.$$

Since

$$\left\|\sigma_{n,m}^{-\alpha,-\beta}(f,x)-f(x)\right\|_p$$

$$\leq\|I\|_p+\|II\|_p+\|III\|_p+\|IV\|_p$$

and I, II, III and IV we have already estimated in Lemmas 2.1-2.3, the proof will be complete by just combining these estimates. ■

Proof of Corollary 2.3. In order to prove Corollary 2.3 we need to use some properties of the modulus of continuity. Namely, since

$$\omega_i\left(f,\frac{1}{M_n}\right)\leq\omega\left(f,\frac{1}{M_n}\right),\quad i=1,2, \tag{2.20}$$

$$\omega_{1,2}\left(f,\frac{1}{M_n},\frac{1}{M_m}\right)\leq 2\omega_1\left(f,\frac{1}{M_n}\right)$$

and

$$\omega_{1,2}\left(f,\frac{1}{M_n},\frac{1}{M_m}\right)\leq 2\omega_2\left(f,\frac{1}{M_m}\right),$$

by (2.20) we obtain that

$$\begin{aligned}&\omega_{1,2}\left(f,\frac{1}{M_n},\frac{1}{M_m}\right) \qquad (2.21)\\
&=\left(\omega_{1,2}\left(f,\frac{1}{M_n},\frac{1}{M_m}\right)\right)^{\frac{\alpha}{\alpha+\beta}}\left(\omega_{1,2}\left(f,\frac{1}{M_n},\frac{1}{M_m}\right)\right)^{\frac{\beta}{\alpha+\beta}}\\
&\leq 2\left(\omega_1\left(f,\frac{1}{M_n}\right)\right)^{\frac{\alpha}{\alpha+\beta}}\left(\omega_2\left(f,\frac{1}{M_m}\right)\right)^{\frac{\beta}{\alpha+\beta}}\\
&\leq 2\left(\omega\left(f,\frac{1}{M_n}\right)\right)^{\frac{\alpha}{\alpha+\beta}}\left(\omega\left(f,\frac{1}{M_m}\right)\right)^{\frac{\beta}{\alpha+\beta}}.\end{aligned}$$

According to Corollary 2.2 we have that if

$$\omega_1\left(f,\frac{1}{M_k}\right)_p=o\left(\left(\frac{1}{M_k}\right)^{\alpha}\right),$$

$$\omega_2\left(f,\frac{1}{M_l}\right)_p=o\left(\left(\frac{1}{M_l}\right)^{\beta}\right),$$

$$\omega_{1,2}\left(f,\frac{1}{M_k},\frac{1}{M_l}\right)_p=o\left(\left(\frac{1}{M_k}\right)^{\alpha}\left(\frac{1}{M_l}\right)^{\beta}\right),$$

then

$$\left\|\sigma_{n,m}^{-\alpha,-\beta}(f)-f\right\|_p\to 0\quad\text{as } n,m\to\infty.$$

Since in (2.21) we estimated the dyadic mixed modulus of continuity with the dyadic total modulus of continuity, the validity of Corollary 2.3 follows immediately from Corollary 2.2, (2.20) and (2.21). ■

Proof of Theorem 2.4. We set

$$f_j\left(x\right)=\rho_j\left(x\right)=\exp\frac{2\pi i x_j}{m_j},$$

and define the function $f\left(x,y\right)$ as follows:

$$f\left(x,y\right):=\sum_{j=1}^{\infty}\frac{1}{M_j^{(\alpha+\beta)}}f_j\left(x\right)f_j\left(y\right).$$

First, we prove that

$$\omega\left(f,\frac{1}{M_n}\right)_C=O\left(\left(\frac{1}{M_n}\right)^{\alpha+\beta}\right). \tag{2.22}$$

Since

$$\left|f_j\left(x+t\right)-f_j\left(x\right)\right|=0,\quad t\in I_n,$$

we find that

$$\begin{aligned}&\left|f\left(x+t,y\right)-f\left(x,y\right)\right|\\ &\leq\sum_{j=1}^{n-1}\frac{1}{M_j^{(\alpha+\beta)}}\left|f_j\left(x+t\right)-f_j\left(x\right)\right|+\sum_{j=n}^{\infty}\frac{2}{M_j^{(\alpha+\beta)}}\\ &\leq\frac{c}{M_n^{(\alpha+\beta)}},\end{aligned}$$

and it follows that

$$\omega_1\left(f,\frac{1}{M_n}\right)=O\left(\left(\frac{1}{M_n}\right)^{\alpha+\beta}\right). \tag{2.23}$$

Analogously, we have that

$$\omega_2\left(f,\frac{1}{M_m}\right)=O\left(\left(\frac{1}{M_m}\right)^{\alpha+\beta}\right). \tag{2.24}$$

Now, by using (2.23) and (2.24), we obtain (2.22) and thus the first part of the theorem is proved.

Next, we shall prove that $\sigma_{M_n,M_n}^{-\alpha,-\beta}\left(f\right)$ diverge in the metric of L^1. It is clear that

$$\left\|\sigma_{M_n,M_n}^{-\alpha,-\beta}\left(f\right)-f\right\|_1\geq\left|\int_{G_m^2}\left[\sigma_{M_n,M_n}^{-\alpha,-\beta}\left(f;x,y\right)-f\left(x,y\right)\right]\right. \tag{2.25}$$

$$\left.\times\psi_{M_k}\left(x\right)\psi_{M_k}\left(y\right)d\mu\left(x,y\right)\right|$$

$$\geq \left| \int_{G_m^2} \sigma_{M_n,M_n}^{-\alpha,-\beta}(f;x,y)\,\psi_{M_k}(x)\,\psi_{M_k}(y)\,d\mu(x)\,d\mu(y) \right|$$

$$-\left|\widehat{f}(M_k,M_k)\right|$$

$$=\left|\frac{1}{A_{M_k}^{-\alpha}A_{M_k}^{-\beta}}\sum_{i=0}^{M_{l_k}}\sum_{j=0}^{M_{l_k}}A_{M_k-i}^{-\alpha}A_{M_k-j}^{-\beta}\hat{f}(i,j)\right.$$

$$\left.\times\int_{G_m^2}\psi_i(x)\,\psi_j(y)\,\psi_{M_k}(x)\,\psi_{M_k}(y)\,d\mu(x,y)\right|$$

$$-\left|\widehat{f}(M_k,M_k)\right|$$

$$=\frac{1}{A_{M_{l_k}}^{-\alpha}A_{M_{l_k}}^{-\beta}}\left|\widehat{f}(M_k,M_k)\right|-\left|\widehat{f}(M_k,M_k)\right|.$$

Moreover,

$$\widehat{f}(M_k,M_k)=\int_{G_m^2}f(x,y)\,\psi_{M_k}(x)\,\psi_{M_k}(y)\,d\mu(x,y)$$

$$=\sum_{j=1}^{\infty}\frac{1}{M_j^{(\alpha+\beta)}}\int_{G_m^2}\rho_j(x)\,\rho_j(y)\,\psi_{M_k}(x)\,\psi_{M_k}(y)\,d\mu(x,y)$$

$$=\sum_{j=1}^{\infty}\frac{1}{M_j^{(\alpha+\beta)}}\int_{G_m}\rho_j(x)\,\psi_{M_k}(x)\,d\mu(x)$$

$$\times\int_{G_m}\rho_j(y)\,\psi_{M_k}(y)\,d\mu(y)$$

$$=\frac{1}{M_k^{(\alpha+\beta)}}.$$

Hence, according to (2.25) we have that

$$\left\|\sigma_{M_n,M_n}^{-\alpha,-\beta}(f)-f\right\|_1\geq c(\alpha,\beta). \tag{2.26}$$

We can conclude that

$$\limsup_{n\to\infty}\left\|\sigma_{M_n,M_n}^{-\alpha,-\beta}(f)-f\right\|_1>0.$$

So also the second part of the Theorem is proved. The proof is complete. ∎

Chapter 3

Approximation Properties of Cesàro Means of the quadratic partial sums of double Vilenkin-Fourier series

3.1 Formulation of the main results

In this Chapter, we discuss the rate of convergence for Cesàro $(C,-\alpha)$ means of the quadratic partial sums of double Vilenkin-Fourier series. Our main results read:

Theorem 3.1. ***(See Paper C)*** *Let f belong to $L^p\left(G_m^2\right)$ for some $p\in[1,\infty]$ and $\alpha\in(0,1)$. Then, for any $M_k\leq n<M_{k+1}\ (k,n\in N)$, the inequality*

$$\left\|\sigma_{M_k}^{-\alpha}(f)-f\right\|_p$$
$$\leq c(\alpha)\Big(\omega_1\left(f,1/M_{k-1}\right)_p M_k^{\alpha}+\omega_2\left(f,1/M_{l-1}\right)_p M_k^{\alpha}+$$
$$+\sum_{r=0}^{k-2}\frac{M_r}{M_k}\omega_1\left(f,1/M_r\right)_p+\sum_{s=0}^{k-2}\frac{M_s}{M_k}\omega_2\left(f,1/M_s\right)_p\Big)$$

holds.

Theorem 3.2. ***(See Paper C)*** *Let f belong to $L^p\left(G_m^2\right)$ for some $p\in[1,\infty]$ and $\alpha\in(0,1)$. Then, for any $M_k\leq n<M_{k+1}\ (k,n\in N)$, the inequality*

$$\left\|\sigma_n^{-\alpha}(f)-f\right\|_p\leq$$
$$c(\alpha)\Big(\omega_1\left(f,1/M_{k-1}\right)_p M_k^{\alpha}\log n+\omega_2\left(f,1/M_{k-1}\right)_p M_k^{\alpha}\log n$$
$$+\sum_{r=0}^{k-2}\frac{M_r}{M_k}\omega_1\left(f,1/M_r\right)_p+\sum_{s=0}^{k-2}\frac{M_s}{M_k}\omega_2\left(f,1/M_s\right)_p\Big)$$

holds.

Theorem 3.1 and Theorem 3.2 imply the following sufficient conditions for the convergence in the norm of Cesàro means of the quadratic partial sums of double Vilenkin-Fourier series in terms of the modulus of continuity.

Corollary 3.3. ***(See Paper D)*** *Let f belong to L^p for some $p \in [1,\infty]$ and let $\alpha \in (0,1)$. If*

$$\omega\left(f, \frac{1}{M_{k-1}}\right)_p = o\left(\frac{1}{M_k^\alpha}\right),$$

then

$$\left\|\sigma_{M_k}^{-\alpha}(f) - f\right\|_p \to 0 \quad \textit{as } k \to \infty.$$

Corollary 3.4. ***(See Paper D)*** *Let f belong to L^p for some $p \in [1,\infty]$ and let $\alpha \in (0,1)$. If*

$$\omega\left(f, \frac{1}{M_{k-1}}\right)_p = o\left(\frac{1}{M_k^\alpha \log M_k}\right),$$

then

$$\left\|\sigma_n^{-\alpha}(f) - f\right\|_p \to 0 \quad \textit{as } n \to \infty.$$

Next, we state some results which show the sharpness of Corollary 3.3 and Corollary 3.4. In particular, the following Theorems hold:

Theorem 3.5. ***(See Paper D)*** *For every $\alpha \in (0,1)$, there exists a function $f \in C\left(G_m^2\right)$ for which*

$$\omega\left(f, \frac{1}{M_{k-1}}\right)_C = O\left(\frac{1}{M_k^\alpha}\right),$$

and

$$\limsup_{k\to\infty} \left\|\sigma_{M_k}^{-\alpha}(f) - f\right\|_1 > 0.$$

Since for a continuous function we have proved divergence in the space L_1, we can conclude the following:

Corollary 3.6. ***(See Paper D)*** *For every $\alpha \in (0,1)$, there exists a function $f \in C\left(G_m^2\right)$, for which*

$$\omega\left(f, \frac{1}{M_{k-1}}\right)_p = O\left(\frac{1}{M_k^\alpha}\right),$$

and

$$\limsup_{k\to\infty} \left\|\sigma_{M_k}^{-\alpha}(f) - f\right\|_p > 0, \quad \textit{for some } p \in [1,\infty].$$

We also have the following sharpness result:

Theorem 3.7. ***(See Paper D)*** *For every $\alpha \in (0,1)$, there exists a function $g \in C\left(G_m^2\right)$ for which*

$$\omega\left(g, \frac{1}{M_{k-1}}\right)_C = O\left(\frac{1}{M_k^\alpha \log M_k}\right),$$

and

$$\limsup_{n\to\infty} \left\| \sigma_n^{-\alpha}(g) - g \right\|_C > 0.$$

Theorem 3.8. ***(See Paper D)*** *For every* $\alpha \in (0,1)$, *there exists a function* $h \in L_1\left(G_m^2\right)$ *for which*

$$\omega\left(h, \frac{1}{M_{k-1}}\right)_C = O\left(\frac{1}{M_k^{\alpha} \log M_k}\right),$$

and

$$\limsup_{n\to\infty} \left\| \sigma_n^{-\alpha}(h) - h \right\|_1 > 0.$$

3.2 Auxiliary results

In order to make the proofs of these Theorems more clear, we formulate some auxiliary Lemmas (see [44] and [30], respectively)

Lemma 3.1. *Let* $\alpha_1, \alpha_2, ..., \alpha_n$ *be real numbers. Then*

$$\frac{1}{n}\int\limits_{G_m^2}\left|\sum_{k=1}^{n}\alpha_k D_k(x) D_k(y)\right| d\mu(x,y) \leq \frac{c}{\sqrt{n}}\left(\sum_{k=1}^{n}\alpha_k^2\right)^{1/2}.$$

Lemma 3.2. *Let* $0 \leq j < n_s M_s$ *and* $0 \leq n_s < m_s$. *Then*

$$D_{n_s M_s - j} = D_{n_s M_s} - \psi_{n_s M_s - 1}\bar{D}_j.$$

We also need the following new Lemmas of independent interest:

Lemma 3.3. *Let* f *belong to* $L^p\left(G_m^2\right)$ *for some* $p \in [1,\infty]$. *Then, for every* $\alpha \in (0,1)$*, the following inequality holds:*

$$I := \frac{1}{A_n^{-\alpha}}\left\|\int\limits_{G_m^2}\sum_{i=1}^{M_{k-1}} A_{n-i}^{-\alpha-1} D_i(u) D_i(v)\left[f(\cdot+u,\cdot+v) - f(\cdot,\cdot)\right] d\mu(u,v)\right\|_p$$

$$\leq \sum_{r=0}^{k-2}\frac{M_r}{M_k}\omega_1(f,1/M_r)_p + \sum_{s=0}^{k-2}\frac{M_s}{M_k}\omega_2(f,1/M_s)_p\,,$$

where $M_k \leq n < M_{k+1}$.

Proof of Lemma 3.3. By applying Abel's transformation, from (0.3) we get that

$$I \leq \frac{1}{A_n^{-\alpha}}\left\|\int\limits_{G_m^2}\sum_{i=1}^{M_{k-1}-1} A_{n-i}^{-\alpha-2}\sum_{l=1}^{i} D_l(u) D_l(v)\right. \tag{3.1}$$

$$\times\left.\left[f(\cdot+u,\cdot+v) - f(\cdot,\cdot)\right] d\mu(u,v)\right\|_p$$

$$+\frac{1}{A_n^{-\alpha}}\left\|\int\limits_{G_m^2} A_{n-M_{k-1}}^{-\alpha-1}\sum_{i=1}^{M_{k-1}} D_i(u) D_i(v)\right.$$

$$\times\left.\left[f(\cdot+u,\cdot+v) - f(\cdot,\cdot)\right] d\mu(u,v)\right\|_p$$

$$:= I_1 + I_2.$$

For I_2 we can estimate as follows:

$$I_2 \leq \frac{1}{A_n^{-\alpha}} \left\| \int\limits_{G_m^2} A_{n-M_{k-1}}^{-\alpha-1} \sum_{r=1}^{k-2} \sum_{i=M_r}^{M_{r+1}-1} D_i(u) D_i(v) \right. \tag{3.2}$$

$$\times \left[f(\cdot + u, \cdot + v) - f(\cdot, \cdot) \right] \Bigg\|_p d\mu(u,v)$$

$$\leq \frac{1}{A_n^{-\alpha}} \left\| \int\limits_{G_m^2} A_{n-M_{k-1}}^{-\alpha-1} \sum_{r=1}^{k-2} \sum_{i=M_r}^{M_{r+1}-1} D_i(u) D_i(v) \right.$$

$$\times \left[f(\cdot + u, \cdot + v) - S_{M_r,M_r}(\cdot + u, \cdot + v, f) \right] d\mu(u,v) \Bigg\|_p$$

$$+ \frac{1}{A_n^{-\alpha}} \left\| \int\limits_{G_m^2} A_{n-M_{k-1}}^{-\alpha-1} \sum_{r=1}^{k-2} \sum_{i=M_r}^{M_{r+1}-1} D_i(u) D_i(v) \right.$$

$$\times \left[S_{M_r,M_r}(\cdot + u, \cdot + v, f) - S_{M_r,M_r}(\cdot, \cdot, f) \right] d\mu(u,v) \Bigg\|_p$$

$$+ \frac{1}{A_n^{-\alpha}} \left\| \int\limits_{G_m^2} A_{n-M_{k-1}}^{-\alpha-1} \sum_{r=1}^{k-2} \sum_{i=M_r}^{M_{r+1}-1} D_i(u) D_i(v) \right.$$

$$\times \left[S_{M_r,M_r}(\cdot, \cdot, f) - f(\cdot, \cdot) \right] d\mu(u,v) \Bigg\|_p$$

$$:= I_{21} + I_{22} + I_{23}.$$

Since

$$S_{i,j}(\cdot, \cdot, f) = \int\limits_{G_m^2} f(u,v) D_i(\cdot + u) D_j(\cdot + v) \, d\mu(u,v),$$

it is evident that

$$\int\limits_{G_m^2} \sum_{i=M_r}^{M_{r+1}-1} D_i(u) D_i(v)$$

$$\times \left[S_{M_r,M_r}(\cdot + u, \cdot + v, f) - S_{M_r,M_r}(\cdot, \cdot, f) \right] d\mu(u,v)$$

$$= \sum_{i=M_r}^{M_{r+1}-1} \left(\int\limits_{G_m^2} D_i(u) D_i(v) S_{M_r,M_r}(\cdot + u, \cdot + v, f) \, d\mu(u,v) \right.$$

$$-S_{M_r,M_r}(\cdot,\cdot,f))$$

$$=\sum_{i=M_r}^{M_{r+1}-1}\left(S_i\left(\cdot,\cdot,S_{M_r,M_r}(f)\right)-S_{M_r,M_r}(\cdot,\cdot,f)\right)$$

$$=\sum_{i=M_r}^{M_{r+1}-1}\left(S_{M_r,M_r}(\cdot,\cdot,f)-S_{M_r,M_r}(\cdot,\cdot,f)\right)=0.$$

Hence,

$$I_{22}=0. \tag{3.3}$$

Moreover, according to the generalized Minkowski's inequality, Lemma 3.1, by (0.1) and (0.4) we obtain that

$$I_{21}\leq\frac{1}{A_n^{-\alpha}}\left|A_{n-M_{k-1}}^{-\alpha-1}\right|\sum_{r=1}^{k-2}\int_{G_m^2}\left|\sum_{i=M_r}^{M_{r+1}-1}D_i(u)\,D_i(v)\right| \tag{3.4}$$

$$\times\left\|f(\cdot+u,\cdot+v)-S_{M_r,M_r}(\cdot+u,\cdot+v,f)\right\|_p d\mu(u,v)$$

$$\leq\frac{c(\alpha)}{M_k}\sum_{r=1}^{k-2}\left(\omega_1(f,1/M_r)_p+\omega_2(f,1/M_r)_p\right)$$

$$\times\int_{G_m^2}\left|\sum_{i=M_r}^{M_{r+1}-1}D_i(x)\,D_i(y)\right|d\mu(u,v)$$

$$\leq\frac{c(\alpha)}{M_k}\sum_{r=1}^{k-2}\left(\omega_1(f,1/M_r)_p+\omega_2(f,1/M_r)_p\right)$$

$$\times\sqrt{M_r}\left(\sum_{i=M_r}^{M_{r+1}-1}1\right)^{1/2}$$

$$\leq c(\alpha)\sum_{r=1}^{k-2}\frac{M_r}{M_k}\left(\omega_1(f,1/M_r)_p+\omega_2(f,1/M_r)_p\right).$$

The estimation of I_{23} is analogous to the estimation of I_{21} and we get that

$$I_{23}\leq c(\alpha)\sum_{r=1}^{k-2}\frac{M_r}{M_k}\left(\omega_1(f,1/M_r)_p+\omega_2(f,1/M_r)_p\right). \tag{3.5}$$

Moreover, in a similar way, we can estimate I_1:

$$I_1 \leq \frac{1}{A_n^{-\alpha}} \sum_{r=1}^{k-2} \left\| \int_{G_m^2} \sum_{i=M_r}^{M_{r+1}-1} A_{n-i}^{-\alpha-2} \sum_{l=1}^{i} D_l(u) D_l(v) \right. \tag{3.6}$$

$$\left. \times \left[f(\cdot+u, \cdot+v) - S_{M_r,M_r}(\cdot+u, \cdot+v, f) \right] d\mu(u,v) \right\|_p$$

$$+ \frac{1}{A_n^{-\alpha}} \sum_{r=1}^{k-2} \left\| \int_{G_m^2} \sum_{i=M_r}^{M_{r+1}-1} A_{n-i}^{-\alpha-2} \sum_{l=1}^{i} D_l(u) D_l(v) \right.$$

$$\left. \times \left[S_{M_r,M_r}(\cdot+u, \cdot+v, f) - S_{M_r,M_r}(\cdot, \cdot, f) \right] \right\|_p d\mu(u,v)$$

$$+ \frac{1}{A_n^{-\alpha}} \sum_{r=1}^{k-2} \left\| \int_{G_m^2} \sum_{i=M_r}^{M_{r+1}-1} A_{n-i}^{-\alpha-2} \sum_{l=1}^{i} D_l(u) D_l(v) \right.$$

$$\left. \times \left[S_{M_r,M_r}(\cdot, \cdot, f) - f(\cdot, \cdot) \right] d\mu(u,v) \right\|_p$$

$$\leq \frac{1}{A_n^{-\alpha}} \sum_{r=1}^{k-2} \int_{G_m^2} \left| \sum_{i=M_r}^{M_{r+1}-1} A_{n-i}^{-\alpha-2} \sum_{l=1}^{i} D_l(u) D_l(v) \right|$$

$$\times \left\| f(\cdot+u, \cdot+v) - S_{M_r,M_r}(\cdot+u, \cdot+v, f) \right\|_p d\mu(u,v)$$

$$+ \frac{1}{A_n^{-\alpha}} \sum_{r=1}^{k-2} \int_{G_m^2} \left| \sum_{i=M_r}^{M_{r+1}-1} A_{n-i}^{-\alpha-2} \sum_{l=1}^{i} D_l(u) D_l(v) \right|$$

$$\times \left\| S_{M_r,M_r}(\cdot, \cdot, f) - f(\cdot, \cdot) \right\|_p d\mu(u,v)$$

$$\leq \frac{1}{A_n^{-\alpha}} \sum_{r=1}^{k-2} \Big(\omega_1(f, 1/M_r)_p + \omega_2(f, 1/M_r)_p \Big)$$

$$\times \sum_{i=M_r}^{M_{r+1}-1} (n-i)^{-\alpha-2} \int_{G_m^2} \left| \sum_{l=1}^{i} D_l(u) D_l(v) \right| d\mu(u,v)$$

$$\leq c(\alpha)\, n^{\alpha} \sum_{r=1}^{k-2} \Big(\omega_1(f, 1/M_r)_p + \omega_2(f, 1/M_r)_p \Big)$$

$$\times \sum_{i=M_r}^{M_{r+1}-1} (n-i)^{-\alpha-2} i$$

$$\leq c(\alpha) n^{\alpha} \sum_{r=1}^{k-2} \left(\omega_1 (f, 1/M_r)_p + \omega_2 (f, 1/M_r)_p\right)$$

$$\times \sum_{i=M_r}^{M_{r+1}+1} (n - M_{r+1} - 1)^{-\alpha-2} i$$

$$\leq c(\alpha) \sum_{r=0}^{k-2} \frac{M_r}{M_k} \left(\omega_1 (f, 1/M_r)_p + \omega_2 (f, 1/M_r)_p\right).$$

By combining (3.1)-(3.6) for I we find that

$$I \leq c(\alpha) \sum_{r=0}^{k-2} \frac{M_r}{M_k} \left(\omega_1 (f, 1/M_r)_p + \omega_2 (f, 1/M_r)_p\right). \tag{3.7}$$

The proof of Lemma 3.3 is complete. ■

Lemma 3.4. *Let* $\alpha \in (0,1)$ *and* $p = M_k, M_k + 1, \ldots$. *Then*

$$II := \int_{G_m^2} \left| \sum_{i=1}^{M_k} A_{p-i}^{-\alpha-1} D_i(u) D_i(v) \right| d\mu(u,v) \leq c(\alpha) < \infty, \ k = 1, 2 \ldots$$

Proof of Lemma 3.4. It is evident that

$$II \leq \int_{G_m^2} \left| \sum_{i=1}^{M_k-1} A_{p-M_k+i}^{-\alpha-1} D_{M_k-i}(u) D_{M_k-i}(v) \right| d\mu(u,v) \tag{3.8}$$

$$+ \left| A_{p-M_k}^{-\alpha-1} \right| \int_{G_m^2} D_{M_k}(u) D_{M_k}(v) \, d\mu(u,v)$$

$$:= II_1 + II_2.$$

From (0.1) and by the fact that $\left| A_{p-M_k}^{-\alpha-1} \right| \leq 1$ it follows that

$$II_2 \leq 1. \tag{3.9}$$

Moreover, by Lemma 3.2 we have that

$$II_1 \leq \int_{G_m^2} \left| \sum_{i=1}^{M_k-1} A_{p-M_k+i}^{-\alpha-1} \bar{D}_i(u) \bar{D}_i(v) \right| d\mu(u,v) \tag{3.10}$$

$$+\int_{G_m^2} D_{M_k}(u)\left|\sum_{i=1}^{M_k-1} A_{p-M_k+i}^{-\alpha-1}\bar{D}_i(v)\right| d\mu(u,v)$$

$$+\int_{G_m^2} D_{M_k}(v)\left|\sum_{i=1}^{M_k-1} A_{p-M_k+i}^{-\alpha-1}\bar{D}_i(u)\right| d\mu(u,v)$$

$$+\left|\sum_{i=1}^{M_k-1} A_{p-M_k+i}^{-\alpha-1}\right|\int_{G_m^2} D_{M_k}(u)\, D_{M_k}(v)\, d\mu(u,v)$$

$$:= II_{11}+II_{12}+II_{13}+II_{14}.$$

From (0.1) and (0.4) it follows that

$$II_{14} \le c(\alpha)\sum_{i=1}^{M_k-1}(p-M_k+i)^{-\alpha-1} \tag{3.11}$$

$$\le c(\alpha)\, M_k (p-1)^{-\alpha-1} < \infty.$$

Moreover, by Applying Abel's transformation, in view of Lemma 3.1 and (0.4) we have that

$$II_{11} \le \int_{G_m^2}\left|\sum_{i=1}^{M_k-2} A_{p-M_k+i}^{-\alpha-2}\sum_{l=1}^{i}\bar{D}_l(u)\,\bar{D}_l(v)\right| d\mu(u,v) \tag{3.12}$$

$$+\int_{G_m^2}\left|A_{p-1}^{-\alpha-1}\sum_{i=1}^{M_k-1}\bar{D}_i(u)\,\bar{D}_i(v)\right| d\mu(u,v)$$

$$\le \sum_{i=1}^{M_k-2}(p-M_k+i)^{-\alpha-2}\int_{G_m^2}\left|\sum_{l=1}^{i}\bar{D}_l(u)\,\bar{D}_l(v)\right| d\mu(u,v)$$

$$+(p-1)^{-\alpha-1}\int_{G_m^2}\left|\sum_{i=1}^{M_k-1}\bar{D}_i(u)\,\bar{D}_i(v)\right| d\mu(u,v)$$

$$\le c(\alpha)\left\{\sum_{i=1}^{M_k-2}(p-M_k+i)^{-\alpha-2}\, i+(p-1)^{-\alpha-1} M_k\right\}$$

$$\le c(\alpha)\left\{\sum_{i=1}^{\infty} i^{-\alpha-1}+M_k^{-\alpha}\right\} < \infty.$$

The estimations of II_{12} and II_{13} are analogous to the estimation of II_{11}. By Applying Abel's transformation, in view of Lemma 1.1 and (0.4) we find that

$$II_{12} \leq \int_{G_m^2} D_{M_k}(u) \left| \sum_{i=1}^{M_k-2} A_{p-M_k+i}^{-\alpha-2} \sum_{l=1}^{i} \bar{D}_l(v) \right| d\mu(u,v) \tag{3.13}$$

$$+ \int_{G_m^2} D_{M_k}(u) \left| A_{p-1}^{-\alpha-1} \sum_{i=1}^{M_k-1} \bar{D}_i(v) \right| d\mu(u,v)$$

$$\leq \int_{G_m} D_{M_k}(u)\, d\mu(u) \times \sum_{i=1}^{M_k-2} (p-M_k+i)^{-\alpha-2} \int_{G_m} \left| \sum_{l=1}^{i} \bar{D}_l(v) \right| d\mu(v)$$

$$+ \int_{G_m} D_{M_k}(u)\, d\mu(u) \times (p-1)^{-\alpha-1} \int_{G_m} \left| \sum_{i=1}^{M_k-1} \bar{D}_i(v) \right| d\mu(v)$$

$$\leq c(\alpha) \sum_{i=1}^{M_k-2} (p-M_k+i)^{-\alpha-2} \sqrt{i} \left(\sum_{l=1}^{i} 1 \right)^{1/2}$$

$$+ c(\alpha)(p-1)^{-\alpha-1} \sqrt{M_k} \left(\sum_{i=1}^{M_k-1} 1 \right)^{1/2}$$

$$\leq c(\alpha) \left\{ \sum_{v=1}^{M_k-2} (p-M_k+i)^{-\alpha-2} i + (p-1)^{-\alpha-1} M_k \right\}$$

$$\leq c(\alpha) \left\{ \sum_{i=1}^{\infty} i^{-\alpha-1} + M_k^{-\alpha} \right\} < \infty,$$

and

$$II_{13} \leq \int_{G_m^2} D_{M_k}(v) \left| \sum_{i=1}^{M_k-2} A_{p-M_k+i}^{-\alpha-2} \sum_{l=1}^{i} \bar{D}_l(u) \right| d\mu(u,v) \tag{3.14}$$

$$+ \int_{G_m^2} D_{M_k}(v) \left| A_{p-1}^{-\alpha-1} \sum_{i=1}^{M_k-1} \bar{D}_i(u) \right| d\mu(u,v)$$

$$\leq c(\alpha) \left\{ \sum_{v=1}^{M_k-2} (p-M_k+i)^{-\alpha-2} i + (p-1)^{-\alpha-1} M_k \right\}$$

$$\leq c(\alpha) \left\{ \sum_{i=1}^{\infty} i^{-\alpha-1} + M_k^{-\alpha} \right\} < \infty.$$

The proof is complete by combining (3.8)-(3.14). ■

Lemma 3.5. *The inequality*

$$III := \int\limits_{G_m^2} \left| \sum_{i=1}^{n} A_{n-i}^{-\alpha-1} D_i\left(u\right) D_i\left(v\right) \right| d\mu(u,v) \leq c\left(\alpha\right) \log n$$

holds.

Proof of Lemma 3.5. Let

$$n = n_{k_1} M_{k_1} + ... + n_{k_s} M_{k_s},\ k_1 > ... > k_s \geq 0.$$

Denote

$$n^{(i)} = n_{k_i} M_{k_i} + ... + n_{k_s} M_{k_s}, \quad i = 1, 2, ...s.$$

Since (see [70])

$$D_{j+n_A M_A} = D_{n_A M_A} + \psi_{n_A M_A} D_j, \tag{3.15}$$

we find that

$$III \leq \int\limits_{G_m^2} \left| \sum_{i=1}^{n_{k_1} M_{k_1}} A_{n-i}^{-\alpha-1} D_i\left(u\right) D_i\left(v\right) \right| d\mu(u,v) \tag{3.16}$$

$$+ \int\limits_{G_m^2} \left| \sum_{i=1}^{n^{(2)}} A_{n^{(2)}-i}^{-\alpha-1} D_{i+n_{k_1} M_{k_1}}\left(u\right) D_{i+n_{k_1} M_{k_1}}\left(v\right) \right| d\mu(u,v)$$

$$\leq \int\limits_{G_m^2} \left| \sum_{i=1}^{n_{k_1} M_{k_1}} A_{n-i}^{-\alpha-1} D_i\left(u\right) D_i\left(v\right) \right| d\mu(u,v)$$

$$+ \int\limits_{G_m^2} \left| \sum_{i=1}^{n^{(2)}} A_{n^{(2)}-i}^{-\alpha-1} D_i\left(u\right) D_i\left(v\right) \right| d\mu(u,v)$$

$$+ \int\limits_{G_m^2} D_{n_{k_1} M_{k_1}}\left(u\right) D_{n_{k_1} M_{k_1}}\left(v\right) \left| \sum_{i=1}^{n^{(2)}} A_{n^{(2)}-i}^{-\alpha-1} \right| d\mu(u,v)$$

$$+ \int\limits_{G_m^2} D_{n_{k_1} M_{k_1}}\left(u\right) \left| \sum_{i=1}^{n^{(2)}} A_{n^{(2)}-i}^{-\alpha-1} D_i\left(v\right) \right| d\mu(u,v)$$

$$+ \int\limits_{G_m^2} D_{n_{k_1} M_{k_1}}\left(v\right) \left| \sum_{i=1}^{n^{(2)}} A_{n^{(2)}-i}^{-\alpha-1} D_i\left(u\right) \right| d\mu(u,v)$$

$$:= III_1 + III_2 + III_3 + III_4 + III_5.$$

By using (0.1) and (0.4) we have that

$$III_3 \leq c(\alpha) < \infty. \tag{3.17}$$

Moreover, by Shavardenidze (see [114]) it was shown that

$$\left| \sum_{i=1}^{n} A_{n-i}^{-\alpha-1} D_i(u) \right| = O\left(|u|^{\alpha-1} \right). \tag{3.18}$$

From (3.18) and by using (0.1) for III_4 we get that

$$III_4 \leq \int_{G_m^2} D_{n_{k_1} M_{k_1}}(u) |v|^{\alpha-1} d\mu(u,v) \tag{3.19}$$

$$\leq \int_{G_m} |v|^{\alpha-1} d\mu(v) = \frac{1}{\alpha} < \infty.$$

Analogously, we find that

$$III_5 \leq \int_{G_m^2} D_{n_{k_1} M_{k_1}}(v) |u|^{\alpha-1} d\mu(u,v) \tag{3.20}$$

$$\leq \int_{G_m} |u|^{\alpha-1} d\mu(v) = \frac{1}{\alpha} < \infty.$$

For $r \in \{0, ...m_A - 1\}, 0 \leq j < M_A$, (see [70]), it holds that

$$D_{j+rM_A} = \left(\sum_{q=0}^{r-1} \psi_{M_A}^q \right) D_{M_A} + \psi_{M_A}^r D_j.$$

Thus, we have that

$$\int_{G_m^2} \sum_{i=1}^{n_{k_1} M_{k_1} - 1} A_{n-i}^{-\alpha-1} D_i(u) D_i(v) d\mu(u,v)$$

$$\leq \int_{G_m^2} \sum_{r=0}^{n_{k_1}-1} \sum_{i=0}^{M_{k_1}-1} A_{n-i-rM_{k_1}}^{-\alpha-1} D_{i+rM_{k_1}}(u) D_{i+rM_{k_1}}(v) d\mu(u,v)$$

$$\leq \int_{G_m^2} \sum_{r=0}^{n_{k_1}-1} \sum_{i=0}^{M_{k_1}-1} A_{n-i-rM_{k_1}}^{-\alpha-1} \left(\sum_{q=0}^{r-1} \psi_{M_{k_1}}^q \right) D_{M_{k_1}}(u)$$

$$\times\left(\sum_{q=0}^{r-1}\psi_{M_{k_1}}^{q}\right)D_{M_{k_1}}(v)\,d\mu(u,v)$$

$$+\int_{G_m^2}\sum_{r=0}^{n_{k_1}-1}\sum_{i=0}^{M_{k_1}-1}A_{n-i-rM_{k_1}}^{-\alpha-1}\left(\sum_{q=0}^{r-1}\psi_{M_{k_1}}^{q}\right)$$

$$\times D_{M_{k_1}}(u)\,\psi_{M_A}^{r}D_i(v)\,d\mu(u,v)$$

$$+\int_{G_m^2}\sum_{r=0}^{n_{k_1}-1}\sum_{i=0}^{M_{k_1}-1}A_{n-i-rM_{k_1}}^{-\alpha-1}\psi_{M_A}^{r}D_i(u)$$

$$\times\left(\sum_{q=0}^{r-1}\psi_{M_{k_1}}^{q}\right)D_{M_{k_1}}(v)\,d\mu(u,v)$$

$$+\int_{G_m^2}\sum_{r=0}^{n_{k_1}-1}\sum_{i=0}^{M_{k_1}-1}A_{n-i-rM_{k_1}}^{-\alpha-1}\psi_{M_A}^{r}$$

$$\times D_i(u)\,\psi_{M_A}^{r}D_i(v)\,d\mu(u,v).$$

Therefore, by using (0.1) and (0.4) we obtain that

$$\int_{G_m^2}A_{n-n_{k_1}M_{k_1}}^{-\alpha-1}D_{n_{k_1}M_{k_1}}(u)\,D_{n_{k_1}M_{k_1}}(v)\,d\mu(u,v)\le c(\alpha)<\infty.$$

Consequently, for III_1 we have the estimate

$$III_1\le\int_{G_m^2}D_{M_{k_1}}(u)\,D_{M_{k_1}}(v)\left|\sum_{r=0}^{n_{k_1}-1}\sum_{i=1}^{M_{k_1}}A_{n-i-rM_{k_1}}^{-\alpha-1}\right|d\mu(u,v) \tag{3.21}$$

$$+\int_{G_m^2}D_{M_{k_1}}(u)\left|\sum_{r=0}^{n_{k_1}-1}\sum_{i=1}^{M_{k_1}}A_{n-i-rM_{k_1}}^{-\alpha-1}D_i(v)\right|d\mu(u,v)$$

$$+\int_{G_m^2}D_{M_{k_1}}(v)\left|\sum_{r=0}^{n_{k_1}-1}\sum_{i=1}^{M_{k_1}}A_{n-i-rM_{k_1}}^{-\alpha-1}D_i(u)\right|d\mu(u,v)$$

$$+\int\limits_{G_m^2}\left|\sum_{r=0}^{n_{k_1}-1}\sum_{i=1}^{M_{k_1}}A_{n-i-rM_{k_1}}^{-\alpha-1}D_i\left(u\right)D_i\left(v\right)\right|d\mu(u,v)+c\left(\alpha\right)$$

$$:=III_{11}+III_{12}+III_{13}+III_{14}+c\left(\alpha\right).$$

From Lemma 3.3 we have that

$$III_{14}\leq c\left(\alpha\right)<\infty. \tag{3.22}$$

The estimation of III_{11} is analogous to the estimation of III_3 and we find that

$$III_{11}\leq c\left(\alpha\right)<\infty. \tag{3.23}$$

The estimations of III_{12} and III_{13} are analogous to the estimation of III_4 and we obtain that

$$III_{12}\leq c\left(\alpha\right)<\infty, \tag{3.24}$$

and

$$III_{13}\leq c\left(\alpha\right)<\infty. \tag{3.25}$$

After substituting (3.17) and (3.19)- (3.25) into (3.16) we conclude that

$$III\leq III_2+c\left(\alpha\right)$$

$$=\int\limits_{G_m^2}\left|\sum_{i=1}^{n^{(2)}}A_{n^{(2)}-i}^{-\alpha-1}D_i\left(u\right)D_i\left(v\right)\right|d\mu(u,v)+c\left(\alpha\right)$$

$$\leq\ldots\leq\int\limits_{G_m^2}\left|\sum_{i=1}^{n^{(s)}}A_{n^{(s)}-i}^{-\alpha-1}D_i\left(u\right)D_i\left(v\right)\right|d\mu(u,v)+c\left(\alpha\right)\left(s-1\right)$$

$$\leq c\left(\alpha\right)+c\left(\alpha\right)\left(s-1\right)\leq c\left(\alpha\right)s.$$

It is easy to see that

$$\log n\approx\log M_k\geq k_1,$$

and since $k_1\geq s$, for III we get the following estimation:

$$III\leq c\left(\alpha\right)s\leq c\left(\alpha\right)\log n.$$

The proof is complete. ■

3.3 Proofs of the main results

Now we are ready to prove the main results.

Proof of Theorem 3.1. It is evident that

$$\left\|\sigma_{M_k}^{-\alpha}(f)-f\right\|_p \tag{3.26}$$

$$=\frac{1}{A_{M_k}^{-\alpha}}\left\|\int_{G_m^2}\sum_{i=1}^{M_k}A_{M_k-i}^{-\alpha-1}D_i(u)D_i(v)\right.$$

$$\times\left[f(\cdot+u,\cdot+v)-f(\cdot,\cdot)\right]d\mu(u,v)\|_p$$

$$\leq\frac{1}{A_{M_k-1}^{-\alpha}}\left\|\int_{G_m^2}\sum_{i=1}^{M_{k-1}}A_{M_k-i}^{-\alpha-1}D_i(u)D_i(v)\right.$$

$$\times\left[f(\cdot+u,\cdot+v)-f(\cdot,\cdot)\right]d\mu(u,v)\|_p$$

$$+\frac{1}{A_{M_k-1}^{-\alpha}}\left\|\int_{G_m^2}\sum_{i=M_{k-1}+1}^{M_k}A_{M_k-i}^{-\alpha-1}D_i(u)D_i(v)\right.$$

$$\times\left[f(\cdot+u,\cdot+v)-f(\cdot,\cdot)\right]d\mu(u,v)\|_p$$

$$:=I+II.$$

From Lemma 3.3 it follows that

$$I\leq c(\alpha)\sum_{r=0}^{k-2}\frac{M_r}{M_k}\left(\omega_1(f,1/M_r)_p+\omega_2(f,1/M_r)_p\right). \tag{3.27}$$

Moreover, for II we have the estimate

$$II\leq\frac{1}{A_{M_k-1}^{-\alpha}}\left\|\int_{G_m^2}\sum_{i=M_{k-1}+1}^{M_k}A_{M_k-i}^{-\alpha-1}D_i(u)D_i(v)\right. \tag{3.28}$$

$$\times\left[f(\cdot+u,\cdot+v)-S_{M_{k-1}}^{(1)}(\cdot+u,\cdot+v,f)\right]d\mu(u,v)\Bigg\|_p$$

$$+\frac{1}{A_{M_k-1}^{-\alpha}}\left\|\int_{G_m^2}\sum_{i=M_{k-1}+1}^{M_k}A_{M_k-i}^{-\alpha-1}D_i(u)D_i(v)\right.$$

$$\times\left[S^{(1)}_{M_{k-1}}(\cdot+u,\cdot+v,f)-f(\cdot,\cdot)\right]d\mu(u,v)\Bigg\|_p$$

$$:=II_1+II_2.$$

In view of generalized Minkowski's inequality, by (0.4) and by using Lemma 3.4 we get that

$$II_1\le\frac{1}{A^{-\alpha}_{M_k-1}}\int_{G^2_m}\left|\sum_{i=M_{k-1}+1}^{M_k}A^{-\alpha-1}_{M_k-i}D_i(u)D_i(v)\right| \tag{3.29}$$

$$\times\left\|f(\cdot+u,\cdot+v)-S^{(1)}_{M_{k-1}}(\cdot+u,\cdot+v,f)\right\|_p d\mu(u,v)$$

$$\le\frac{\omega_1(f,1/M_{k-1})_p}{A^{-\alpha}_{M_k-1}}\int_{G^2_m}\left|\sum_{i=M_{k-1}+1}^{M_k}A^{-\alpha-1}_{M_k-i}D_i(u)D_i(v)\right|d\mu(u,v)$$

$$\le c(\alpha)M_k^{\alpha}\omega_1(f,1/M_{k-1})_p.$$

The estimation of II_2 is analogous to the estimation of II_1 and we find that

$$II_2\le c(\alpha)M_k^{\alpha}\omega_2(f,1/M_{k-1})_p. \tag{3.30}$$

After substituting (3.29)-(3.31) into (3.28), we have

$$II\le c(\alpha)M_k^{\alpha}(\omega_1(f,1/M_{k-1})_p+\omega_2(f,1/M_{k-1})_p). \tag{3.31}$$

The proof is complete by just combining (3.26), (3.27) and (3.31). ■

Proof of Theorem 3.2. It is evident that

$$\left\|\sigma^{-\alpha}_n(f)-f\right\|_p\le\frac{1}{A^{-\alpha}_{n-1}}\Bigg\|\int_{G^2_m}\sum_{i=1}^{M_{k-1}}A^{-\alpha-1}_{n-i}D_i(u)D_i(v) \tag{3.32}$$

$$\times[f(\cdot+u,\cdot+v)-f(\cdot,\cdot)]d\mu(u,v)\|_p$$

$$+\frac{1}{A^{-\alpha}_{n-1}}\Bigg\|\int_{G^2_m}\sum_{i=M_{k-1}+1}^{M_k}A^{-\alpha-1}_{n-i}D_i(u)D_i(v)$$

$$\times[f(\cdot+u,\cdot+v)-f(\cdot,\cdot)]d\mu(u,v)\|_p$$

$$+\frac{1}{A_{n-1}^{-\alpha}}\left\|\int\limits_{G_m^2}\sum_{i=M_k+1}^{n}A_{n-i}^{-\alpha-1}D_i(u)\,D_i(v)\right.$$

$$\times\left[f(\cdot+u,\cdot+v)-f(\cdot,\cdot)\right]d\mu(u,v)\|_p$$

$$:=I+II+III.$$

From Lemma 3.3 it follows that

$$I\leq\ c(\alpha)\sum_{r=0}^{k-2}\frac{M_r}{M_k}\left(\omega_1(f,1/M_r)_p+\omega_2(f,1/M_r)_p\right). \tag{3.33}$$

Next, we repeat the arguments just in the same way as in the proof of Theorem 3.1 and find that

$$II\leq\ c(\alpha)\,M_k^{\alpha}\left(\omega_1(f,1/M_{k-1})_p+\omega_2(f,1/M_{k-1})_p\right). \tag{3.34}$$

Moreover, for III we have that

$$III\leq\frac{1}{A_{n-1}^{-\alpha}}\left\|\int\limits_{G_m^2}\sum_{i=M_k+1}^{n}A_{n-i}^{-\alpha-1}D_i(u)\,D_i(v)\right. \tag{3.35}$$

$$\times\left[f(\cdot+u,\cdot+v)-f(\cdot,\cdot)\right]\Big\|_p\,d\mu(u,v)$$

$$\leq\frac{1}{A_n^{-\alpha}}\left\|\int\limits_{G_m^2}\sum_{i=M_k+1}^{n}A_{n-i}^{-\alpha-1}D_i(u)\,D_i(v)\right.$$

$$\times\left[f(\cdot+u,\cdot+v)-S_{M_k,M_k}(\cdot+u,\cdot+v,f)\right]d\mu(u,v)\Big\|_p$$

$$\leq\frac{1}{A_n^{-\alpha}}\left\|\int\limits_{G_m^2}\sum_{i=M_k+1}^{n}A_{n-i}^{-\alpha-1}D_i(u)\,D_i(v)\right.$$

$$\times\left[S_{M_k,M_k}(\cdot+u,\cdot+v,f)-S_{M_k,M_k}(\cdot,\cdot,f)\right]\,d\mu(u,v)\Big\|_p$$

$$\leq\frac{1}{A_n^{-\alpha}}\left\|\int\limits_{G_m^2}\sum_{i=M_k+1}^{n}A_{n-i}^{-\alpha-1}D_i(u)\,D_i(v)\right.$$

$$\times\left[S_{M_k,M_k}(\cdot,\cdot,f)-f(\cdot,\cdot)\right]\,d\mu(u,v)\Big\|_p$$

$$:= III_1 + III_2 + III_3.$$

Since

$$S_{i,j}(\cdot,\cdot,f) = \int_{G_m^2} f(u,v) D_i(\cdot - u) D_j(\cdot - v) \, d\mu(u,v),$$

we get that

$$\int_{G_m^2} \sum_{i=M_k+1}^{n} A_{n-i}^{-\alpha-1} D_i(u) D_i(v)$$

$$\times \left[S_{M_k,M_k}(\cdot + u, \cdot + v, f) - S_{M_k,M_k}(\cdot,\cdot,f) \right] d\mu(u,v)$$

$$= \sum_{i=M_k+1}^{n} A_{n-i}^{-\alpha-1} \int_{G_m^2} S_{M_k,M_k}(\cdot + u, \cdot + v, f) D_i(u) D_i(v) \, d\mu(u,v)$$

$$- \sum_{i=M_k+1}^{n} A_{n-i}^{-\alpha-1} S_{M_k,M_k}(\cdot,\cdot,f)$$

$$= \sum_{i=M_k+1}^{n} A_{n-i}^{-\alpha-1} S_{i,i}(\cdot,\cdot,S_{M_k,M_k}(f))$$

$$- \sum_{i=M_k+1}^{n} A_{n-i}^{-\alpha-1} S_{M_k,M_k}(\cdot,\cdot,f)$$

$$= \sum_{i=M_k+1}^{n} A_{n-i}^{-\alpha-1} S_{M_k,M_k}(\cdot,\cdot,f)$$

$$- \sum_{i=M_k+1}^{n} A_{n-i}^{-\alpha-1} S_{M_k,M_k}(\cdot,\cdot,f) = 0.$$

Consequently, we have that

$$III_2 = 0. \tag{3.36}$$

According to the generalized Minkowski's inequality and by using Lemma 3.5 for III_1 we obtain that

$$III_1 \le \frac{1}{A_n^{-\alpha}} \int_{G_m^2} \left| \sum_{i=M_k+1}^{n} A_{n-i}^{-\alpha-1} D_i(u) D_i(v) \right| \tag{3.37}$$

$$\times \left\| f(\cdot + u, \cdot + v) - S_{M_r,M_r}(\cdot + u, \cdot + v, f) \right\|_p d\mu(u,v)$$

$$\leq c(\alpha) M_k^{\alpha} \left(\omega_1 (f, 1/M_{k-1})_p + \omega_2 (f, 1/M_{k-1})_p\right)$$

$$\times \int\limits_{G_m^2} \left| \sum_{i=M_k+1}^{n} A_{n-i}^{-\alpha-1} D_i(u) D_i(v) \right| d\mu(u,v)$$

$$\leq c(\alpha) M_k^{\alpha} \log n \left(\omega_1 (f, 1/M_{k-1})_p + \omega_2 (f, 1/M_{k-1})_p\right).$$

The estimation of III_3 is analogous to the estimation of III_1 and we find that

$$III_3 \leq c(\alpha) M_k^{\alpha} \log n \left(\omega_1 (f, 1/M_{k-1})_p + \omega_2 (f, 1/M_{k-1})_p\right). \tag{3.38}$$

By combining (3.35)-(3.38), we get for III the following estimation:

$$III \leq c(\alpha) M_k^{\alpha} \log n \left(\omega_1 (f, 1/M_{k-1})_p + \omega_2 (f, 1/M_{k-1})_p\right). \tag{3.39}$$

After substituting (3.33), (3.34) and (3.39) into (3.32), we receive the inequality stated in Theorem 3.2, so the proof is complete. ■

Proof of Theorem 3.5. We define the functions

$$f_j(x) := \psi_{M_j-1}(x), \text{ with } j = 1, 2, \ldots,$$

and based on these functions we can define the following function:

$$f(x,y) := \sum_{j=1}^{\infty} \frac{1}{M_j^{\alpha}} f_j(x) f_j(y).$$

First, we prove that

$$\omega \left(f, \frac{1}{M_n}\right)_C = O\left(\frac{1}{M_n^{\alpha}}\right). \tag{3.40}$$

Since

$$|f_j(x-t) - f_j(x)| = 0, \quad j = 0, 1, \ldots, n-1, \quad t \in I_n,$$

we find that

$$|f(x-t,y) - f(x,y)| \leq 2\sum_{j=n}^{\infty} \frac{1}{M_j^{\alpha}} \leq \frac{c}{M_n^{\alpha}}.$$

Consequently,

$$\omega_1 \left(f, \frac{1}{M_n}\right)_C = O\left(\frac{1}{M_n^{\alpha}}\right). \tag{3.41}$$

Analogously, we can prove

$$\omega_2\left(f,\frac{1}{M_m}\right)_C=O\left(\frac{1}{M_m^{\alpha}}\right). \tag{3.42}$$

By (0.5), (3.41) and (3.42), we obtain (3.40), which implies that the first part of the theorem is proved.

Next, we shall prove that $\sigma_{M_k}^{-\alpha}(f)$ diverge in the metric of L_1. Indeed,

$$\left\|\sigma_{M_k}^{-\alpha}(f)-f\right\|_1 \tag{3.43}$$

$$\geq\left|\int_{G_m^2}\left[\sigma_{M_k}^{-\alpha}(f;x,y)-f(x,y)\right]\right.$$

$$\left.\times\,\psi_{M_k-1}(x)\,\psi_{M_k-1}(y)\,d\mu(x,y)\right|$$

$$\geq\left|\int_{G_m^2}\sigma_{M_k}^{-\alpha}(f;x,y)\,\psi_{M_k-1}(x)\,\psi_{M_k-1}(y)\,d\mu(x,y)\right|$$

$$-\left|\widehat{f}(M_k-1,M_k-1)\right|$$

$$=\left|\int_{G_m^2}\left(\frac{1}{A_{M_k-1}^{-\alpha}}\sum_{j=1}^{M_k}A_{M_k-j}^{-\alpha-1}S_{j,j}(f;x,y)\right)\right.$$

$$\left.\times\,\psi_{M_k-1}(x)\,\psi_{M_k-1}(y)\,d\mu(x,y)\right|$$

$$-\left|\widehat{f}(M_k-1,M_k-1)\right|$$

$$=\left|\int_{G_m^2}\frac{1}{A_{M_k-1}^{-\alpha}}S_{M_k,M_k}(f;x,y)\right.$$

$$\left.\times\,\psi_{M_k-1}(x)\,\psi_{M_k-1}(y)\,d\mu(x,y)\right|$$

$$-\left|\widehat{f}(M_k-1,M_k-1)\right|$$

$$=\left|\frac{1}{A_{M_k-1}^{-\alpha}}\sum_{k_1=1}^{M_k-1}\sum_{k_2=1}^{M_k-1}\widehat{f}(k_1,k_2)\right.$$

$$\times \int_{G_m^2} \psi_{k_1}(x)\,\psi_{k_2}(y)\,\psi_{M_k-1}(x)\,\psi_{M_k-1}(y)\,d\mu(x,y)\Bigg|$$

$$-\left|\widehat{f}(M_k-1, M_k-1)\right|$$

$$= \frac{1}{A_{M_k-1}^{-\alpha}}\left|\widehat{f}(M_k-1, M_k-1)\right|$$

$$-\left|\widehat{f}(M_k-1, M_k-1)\right|.$$

Moreover, by the definition of the two-dimensional Fourier coefficients we have that

$$\widehat{f}(M_k-1, M_k-1) \tag{3.44}$$

$$= \int_{G_m^2} f(x,y)\,\bar{\psi}_{M_k-1}(x)\,\bar{\psi}_{M_k-1}(y)\,d\mu(x,y)$$

$$= \sum_{j=1}^{\infty} \frac{1}{M_j^{\alpha}} \int_{G_m^2} \psi_{M_j-1}(x)\,\psi_{M_j-1}(y)$$

$$\times\, \bar{\psi}_{M_k-1}(x)\,\bar{\psi}_{M_k-1}(y)\,d\mu(x)\,d\mu(y)$$

$$= \sum_{j=1}^{\infty} \frac{1}{M_j^{\alpha}} \int_{G_m} \psi_{M_j-1}(x)\,\bar{\psi}_{M_k-1}(x)\,d\mu(x)$$

$$\times \int_{G_m} \psi_{M_j-1}(y)\,\bar{\psi}_{M_k-1}(y)\,d\mu(y) = \frac{1}{M_k^{\alpha}}.$$

Combining (3.43) and (3.44) we get

$$\left\|\sigma_{M_k}^{-\alpha}(f) - f\right\|_1 > c(\alpha),$$

where $c(\alpha)$ is a strictly positive constant independent of k.

Thus also the second statement of Theorem 3.5 is proved. So the proof is complete. ■

Proof of Theorem 3.7. Let $\{n_k : k \geq 1\}$ be a sequence of positive integers such that $M_k \leq n_k < M_{k+1}$ $(k, n \in N)$ and $\|D_{n_k}\|_1 \geq c \log n_k$.

We define a function $g(x,y)$ as follows

$$g(x,y) := \sum_{j=1}^{\infty} g_j(x,y),$$

where

$$g_j(x,y) = \frac{1}{M_j^{\alpha} \log M_j} \psi_{n_j - 1}(x) \exp(-i \arg(\bar{D}_{n_j}(y))).$$

First, we prove that

$$\omega\left(g, \frac{1}{M_n}\right)_C = O\left(\frac{1}{M_n^{\alpha} \log M_n}\right). \tag{3.45}$$

Since

$$|g_j(x-t,y) - g_j(x,y)| = 0,$$

and

$$|g_j(x,y-t) - g_j(x,y)| = 0,$$

for $j = 0, 1, ..., n-1,$ and $t \in I_n$, we find that

$$|g(x-t,y) - g(x,y)$$

$$= \left| \sum_{j=n}^{\infty} \frac{1}{M_j^{\alpha} \log M_j} (\psi_{n_j-1}(x-t) - \psi_{n_j-1}(x)) \exp(-i \arg(\bar{D}_{n_j}(y))) \right|$$

$$\leq 2 \sum_{j=n}^{\infty} \frac{1}{M_j^{\alpha} \log M_j} \leq \frac{c}{M_n^{\alpha} \log M_n}.$$

Consequently,

$$\omega_1\left(g, \frac{1}{M_n}\right)_C = O\left(\frac{1}{M_n^{\alpha} \log M_n}\right). \tag{3.46}$$

Analogously, we can prove that

$$\omega_2\left(g, \frac{1}{M_m}\right)_C = O\left(\frac{1}{M_m^{\alpha} \log M_m}\right). \tag{3.47}$$

By (0.5), (3.46) and (3.47), we obtain (3.45) and thus we have proved the first part of the theorem.

Next, we shall prove the divergence statement. It is evident that

$$\left|\sigma_{n_k}^{-\alpha}(g; 0, 0)\right| \geq \left|\sigma_{n_k}^{-\alpha}(g_k; 0, 0)\right| \tag{3.48}$$

$$- \left| \sigma_{n_k}^{-\alpha} \left(\sum_{j=k+1}^{\infty} g_j ; 0, 0 \right) \right| - \left| \sigma_{n_k}^{-\alpha} \left(\sum_{j=1}^{k-1} g_j ; 0, 0 \right) \right|$$

$$:= I - II - III.$$

We have that

$$I = \frac{1}{A_{n_k-1}^{-\alpha}} \left| \sum_{j=1}^{n_k} A_{n_k-j}^{-\alpha-1} S_{j,j} \left(g_k ; 0, 0 \right) \right| \tag{3.49}$$

$$= \frac{1}{A_{n_k-1}^{-\alpha}} \left| \int_{G_m^2} g_k \left(x, y \right) \sum_{j=1}^{n_k} A_{n_k-j}^{-\alpha-1} D_j \left(x \right) D_j \left(y \right) d\mu \left(x, y \right) \right|$$

$$= \frac{1}{A_{n_k-1}^{-\alpha}} \left| \int_{G_m} \sum_{j=1}^{n_k} A_{n_k-j}^{-\alpha-1} D_j \left(x \right) \right.$$

$$\left. \times \left(\int_{G_m} g_k \left(x, y \right) D_j \left(y \right) d\mu \left(y \right) \right) d\mu \left(x \right) \right|$$

$$= \frac{1}{M_k^{\alpha} \log M_k} \frac{1}{A_{n_k-1}^{-\alpha}} \left| \sum_{j=1}^{n_k} A_{n_k-j}^{-\alpha-1} \int_{G_m} \psi_{n_k-1} \left(x \right) D_j \left(x \right) d\mu \left(x \right) \right.$$

$$\left. \times \int_{G_m} \exp(-i \arg \left(\bar{D}_{n_k} \left(y \right) \right)) D_j \left(y \right) d\mu \left(y \right) \right|$$

$$= \frac{1}{M_k^{\alpha} \log M_k} \frac{1}{A_{n_k-1}^{-\alpha}} \int_{G_m} \left| D_{n_k} \left(y \right) \right| d\mu \left(y \right)$$

$$\geq c \left(\alpha \right) \frac{1}{\log n_k} \int_{G_m} \left| D_{n_k} \left(y \right) \right| d\mu \left(y \right) \geq c \left(\alpha \right).$$

It is easy to prove that

$$II = 0. \tag{3.50}$$

From Theorem 3.2 we obtain that

$$III \leq \sum_{j=1}^{k-1} \left\| \sigma_{n_k}^{-\alpha} \left(g_j \right) - g_j \right\|_C$$

$$\leq c \left(\alpha \right) \sum_{j=1}^{k-1} \left(\omega_1 \left(g_j , \frac{1}{M_{k-1}} \right)_p M_k^{\alpha} \log n_k \right.$$

$$+\omega_2\left(g_j,\frac{1}{M_{k-1}}\right)_p M_k^{\alpha}\log n_k$$

$$+\sum_{r=0}^{k-2}\frac{M_r}{M_k}\omega_1\left(g_j,\frac{1}{M_r}\right)_p$$

$$+\sum_{s=0}^{k-2}\frac{M_s}{M_k}\omega_2\left(g_j,\frac{1}{M_s}\right)_p\Bigg).$$

Hence, it is evident that

$$\sum_{j=1}^{k-1}\sum_{r=0}^{k-2}\frac{M_r}{M_k}\omega_1\left(g_j,\frac{1}{M_r}\right)_p \tag{3.51}$$

$$\leq \sum_{j=1}^{k-1}\sum_{r=0}^{j}\frac{M_r}{M_k}\omega_1\left(g_j,\frac{1}{M_r}\right)_p$$

$$\leq \sum_{r=0}^{k-1}\sum_{j=r}^{k-1}\frac{M_r}{M_k}\omega_1\left(g_j,\frac{1}{M_r}\right)_p$$

$$\leq \sum_{r=0}^{k-1}\sum_{j=r}^{k-1}\frac{M_r}{M_k}\frac{1}{M_j^{\alpha}\log M_j}$$

$$\leq c\sum_{r=0}^{k-1}\frac{M_r}{M_k}\frac{1}{M_r^{\alpha}\log M_r}$$

$$\leq \frac{c}{M_k}\sum_{r=0}^{k-1}\frac{M_r^{(1-\alpha)}}{\log M_r}\leq \frac{c}{M_k^{\alpha}}.$$

Analogously we can prove that

$$\sum_{j=1}^{k-1}\sum_{s=0}^{k-2}\frac{M_s}{M_k}\omega_2\left(g_j,\frac{1}{M_s}\right)_p\leq \frac{c}{M_k^{\alpha}}. \tag{3.52}$$

Because of the construction of the function g_j, we have

$$\omega_1\left(g_j,\frac{1}{M_{k-1}}\right)_p=0, \tag{3.53}$$

and

$$\omega_2\left(g_j,\frac{1}{M_{k-1}}\right)_p=0, \tag{3.54}$$

for $j=0,1,...,k-1.$

From $(3.51)-(3.54)$ we get the following estimation of III :

$$III\leq c\left(\alpha\right)\frac{c}{M_{k}^{\alpha}}. \tag{3.55}$$

Combining (3.49) , (3.50) and (3.55) we get that

$$\limsup_{k\rightarrow\infty}\left|\sigma_{n_{k}}^{-\alpha}\left(g;0,0\right)-g\left(0,0\right)\right|>0.$$

Hence, also the second statement of Theorem 3.7 is proved. The proof is complete. ■

Proof of Theorem 3.8. Let $\{n_{k}:k\geq1\}$ be a sequence of positive integers as above $(M_{k}\leq n_{k}<M_{k+1}\ (k,n\in N)$ and $\|D_{n_{k}}\|_{1}\geq c\log n_{k})$.

We define a function $h\left(x,y\right)$ as follows:

$$h\left(x,y\right):=\sum_{l=1}^{\infty}\frac{1}{M_{l}^{\alpha}\log M_{l}}\left[D_{n_{l}}\left(x\right)-D_{n_{l-1}}\left(x\right)\right]\left[D_{n_{l}}\left(y\right)-D_{n_{l-1}}\left(y\right)\right].$$

First we prove that

$$\omega\left(h,\frac{1}{M_{k}}\right)_{C}\leq\frac{c}{M_{k}^{\alpha}\log M_{k}}.$$

Since

$$\int_{I_{n}}\left|D_{n_{l}}\left(x-t\right)-D_{n_{l}}\left(x\right)\right|d\mu\left(x\right)=0,$$

for $l=0,1,...,k-1,\ \ t\in I_{k}$, we have that

$$\int_{G_{m}^{2}}\left|h\left(x-t,y\right)-h\left(x,y\right)\right|d\mu\left(x,y\right) \tag{3.56}$$

$$\leq2\sum_{l=k}^{\infty}\frac{1}{M_{l}^{\alpha}\log M_{l}}\int_{G_{m}}\left|D_{n_{l}}\left(x-t\right)-D_{n_{l}}\left(x\right)\right|d\mu\left(x\right)$$

$$+2\sum_{l=k}^{\infty}\frac{1}{M_{l}^{\alpha}\log M_{l}}\int_{G_{m}}\left|D_{n_{l-1}}\left(x-t\right)-D_{n_{l-1}}\left(x\right)\right|d\mu\left(x\right)$$

$$:=I_{1}+I_{2}.$$

By using (0.1) we get that

$$I_{1}\leq4\sum_{l=k}^{\infty}\frac{1}{M_{l}^{\alpha}\log M_{l}} \tag{3.57}$$

$$\leq c\frac{1}{\log M_k}\sum_{l=k}^{\infty}\frac{1}{M_l^{\alpha}}\leq\frac{c}{M_k^{\alpha}\log M_k}.$$

and

$$I_2\leq\frac{c}{M_k^{\alpha}\log M_k}. \tag{3.58}$$

By combining $(3.56)-(3.58)$ we have that

$$\omega_1\left(h,\frac{1}{M_k}\right)_C\leq\frac{c}{M_k^{\alpha}\log M_k}. \tag{3.59}$$

Analogously, we can prove that

$$\omega_2\left(h,\frac{1}{M_k}\right)_C\leq\frac{c}{M_k^{\alpha}\log M_k}. \tag{3.60}$$

According to (0.5), (3.59) and (3.60) we get the desired first statement:

$$\omega\left(h,\frac{1}{M_k}\right)_C\leq\frac{c}{M_k^{\alpha}\log M_k}.$$

Now, we shall prove the divergence statement. It is evident that

$$\left\|\sigma_{n_k}^{-\alpha}(h)-h\right\|_1 \tag{3.61}$$

$$\geq\int\limits_{G_m}\left|\int\limits_{G_m}\left[\sigma_{n_k}^{-\alpha}(h;x,y)-h(x,y)\right]\psi_{n_k-1}(x)\,d\mu(x)\right|d\mu(y)$$

$$\geq\int\limits_{G_m}\left|\int\limits_{G_m}\sigma_{n_k}^{-\alpha}(h;x,y)\,\psi_{n_k-1}(x)\,d\mu(x)\right|d\mu(y)$$

$$-\int\limits_{G_m}\left|\int\limits_{G_m}h(x,y)\,\psi_{n_k-1}(x)\,d\mu(x)\right|d\mu(y)$$

$$:=II_1-II_2.$$

From the construction of the function h and by using (0.1) we get that

$$II_2\leq c\frac{1}{M_k^{\alpha}\log M_k}. \tag{3.62}$$

Moreover, since

$$\hat{h}(n_k-1,q)=\int\limits_{G_m^2}h(x,y)\,\psi_{n_k-1}(x)\,\psi_q(y)\,d\mu(x,y)$$

$$= \sum_{l=1}^{\infty} \frac{1}{M_l^{\alpha} \log M_l} \int\limits_{G_m} \left[D_{n_l}(x) - D_{n_{l-1}}(x) \right] \psi_{n_k - 1}(x)\, d\mu(x)$$

$$\times \int\limits_{G_m} \left[D_{n_l}(y) - D_{n_{l-1}}(y) \right] \psi_q(y)\, d\mu(y)$$

$$= \frac{1}{M_k^{\alpha} \log M_k} \int\limits_{G_m} \left[D_{n_k}(y) - D_{n_{k-1}}(y) \right] \psi_q(y)\, d\mu(y),$$

for II_1, we find that

$$II_1 = \frac{1}{A_{n_k-1}^{-\alpha}} \int\limits_{G_m} \left| \sum_{j=1}^{n_k} A_{n_k-j}^{-\alpha-1} \sum_{p=0}^{j-1} \sum_{q=0}^{j-1} \widehat{h}(p,q)\, \psi_q(y) \right. \tag{3.63}$$

$$\left. \times \int\limits_{G_m} \psi_p(x)\, \psi_{n_k-1}(x)\, d\mu(x) \right| d\mu(y)$$

$$= \frac{1}{A_{n_k-1}^{-\alpha}} \int\limits_{G_m} \left| \sum_{q=0}^{n_k-1} \widehat{h}(n_k-1,q)\, \psi_q(y) \right| d\mu(y)$$

$$= \frac{1}{A_{n_k-1}^{-\alpha}} \frac{1}{M_k^{\alpha} \log M_k} \int\limits_{G_m} \left| D_{n_k}(y) - D_{n_{k-1}}(y) \right| d\mu(y)$$

$$\geq \frac{1}{A_{n_k-1}^{-\alpha}} \frac{1}{M_k^{\alpha} \log M_k} \left(\int\limits_{G_m} \left| D_{n_k}(y) \right| d\mu(y) - 1 \right).$$

Since $\|D_{n_k}\|_1 \geq c \log n_k$, by combining $(3.61) - (3.63)$ we have that

$$\lim\sup_{k\to\infty} \left\| \sigma_{n_k}^{-\alpha}(h) - h \right\|_1 > 0,$$

and also the divergence statement is proved. The proof is complete. ■

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