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Research Article

Some New Iterated Hardy-Type Inequalities

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We characterize the validity of the Hardy-type inequality $\| \| \int_s^\infty h(z)dz \|_{p,u,(0,t)} \|_{q,w,(0,\infty)} \le c \, \|h\|_{\theta,v(0,\infty)}$, where $0 < p < \infty$, $0 < q \le \infty$, $1 < \theta \le \infty$, u, w, and v are weight functions on $(0,\infty)$. Some fairly new discretizing and antidiscretizing techniques of independent interest are used.

1. Introduction

Everywhere in the paper, u, v, and w are weights, that is, locally integrable nonnegative functions on $(0,\infty)$, and we denote

$$U(s) = \int_0^s u(t)dt,$$
$$V_\theta(t) = \begin{cases} \displaystyle\int_t^\infty v(s)^{1-\theta'} ds & \text{for } 1 < \theta < \infty, \\ \displaystyle\int_t^\infty \frac{ds}{v(s)} & \text{for } \theta = \infty, \end{cases} \tag{1.1}$$

where

$$\theta' := \begin{cases} \infty & \text{if } \theta = 1, \\ \dfrac{\theta}{\theta - 1} & \text{if } 1 < \theta < \infty, \\ 1 & \text{if } \theta = \infty. \end{cases} \tag{1.2}$$

We assume that u is such that $U(t) > 0$ for every $t \in (0, \infty)$.

For $0 < p < \infty$ and w, a weight function on $(a, b) \subseteq (0, \infty)$, let us denote by $L_{p,w}(a, b)$ the weighted Lebesgue space defined as the set of all measurable functions u on (a, b) for which the quantity

$$\|u\|_{p,w,(a,b)} = \begin{cases} \left(\displaystyle\int_a^b |u(x)|^p w(x) dx \right)^{1/p} & \text{for } 1 \le p < \infty, \\ \operatorname*{ess\,sup}_{a<x<b} |u(x)| w(x) & \text{for } p = \infty \end{cases} \tag{1.3}$$

is finite.

In this paper we characterize the validity of the inequality

$$\left\| \left\| \int_s^\infty h(z) dz \right\|_{p,u,(0,t)} \right\|_{q,w,(0,\infty)} \le c \|h\|_{\theta,v,(0,\infty)}, \tag{1.4}$$

where $0 < p < \infty$, $0 < q \le \infty$, $1 < \theta \le \infty$, u, w, and v are weight functions on $(0, \infty)$. Note that inequality (1.4) has been considered in the case $p = 1$ in [1] (see also [2]), where the result is presented without proof, in the case $p = \infty$ in [3] and in the case $\theta = 1$ in [4, 5], where weight functions v of special type were considered. For general weight functions v, the characterization of the inequality (1.4) in the case $\theta = 1$ does not follow directly by this method (there are some technical problems) and we are working on it.

It is worth to mention that, by Fubini's theorem,

$$\int_0^x u(t) \left(\int_t^\infty g(s) ds \right) dt \approx U(x) \int_0^\infty \frac{g(s) U(s) ds}{U(x) + U(s)}. \tag{1.5}$$

Hence, we see that the inequality (1.4) (with $p = 1$) is equivalent with the following inequality:

$$\|Sh\|_{q,U^q w,(0,\infty)} \le c \|h\|_{\theta,U^{-\theta} v,(0,\infty)}, \tag{1.6}$$

where the operator S defined by

$$(Sh)(x) = \int_0^\infty \frac{h(t) dt}{U(x) + U(t)} \tag{1.7}$$

for all nonnegative measurable functions h on $(0, \infty)$. We call this operator the generalized Stieltjes transform; the usual Stieltjes transform is obtained on putting $U(x) \equiv x$.

In the case $U(x) \equiv x^{\lambda}$, $\lambda > 0$, the boundedness of the operator S between weighted L^p and L^q spaces was investigated in [6] (when $1 \leq p \leq q \leq \infty$) and in [7, 8] (when $1 \leq q < p \leq \infty$).

Our approach is based on discretization and antidiscretization methods developed in [4, 9, 10]. Some basic facts concerning these methods and other preliminaries are presented in Section 2. The main results (Theorems 3.1 and 3.2) are stated and proved in Section 3.

Throughout the paper, we always denote by c or C a positive constant which is independent of the main parameters, but it may vary from line to line. However a constant with subscript such as c_1 does not change in different occurrences. By $a \lesssim b$, $(b \gtrsim a)$, we mean that $a \leq \lambda b$, where $\lambda > 0$ depends on inessential parameters. If $a \lesssim b$ and $b \lesssim a$, we write $a \approx b$ and say that a and b are equivalent. We put $1/\infty = 0$, $0 \cdot \infty = 0$, $0/0 = 0$, and $\infty/\infty = 0$.

2. Preliminaries

Let us now recall some definitions and basic facts concerning discretization and antidiscretization which can be found in [4, 9, 10].

Definition 2.1. Let $\{a_k\}$ be a sequence of positive real numbers. One says that $\{a_k\}$ is strongly increasing or strongly decreasing and write $a_k \uparrow\uparrow$ or $a_k \downarrow\downarrow$ when

$$\inf_{k\in\mathbb{Z}} \frac{a_{k+1}}{a_k} > 1 \quad \text{or} \quad \sup_{k\in\mathbb{Z}} \frac{a_{k+1}}{a_k} < 1, \tag{2.1}$$

respectively.

Definition 2.2. Let U be a continuous strictly increasing function on $[0, \infty)$ such that $U(0) = 0$ and $\lim_{t\to\infty} U(t) = \infty$. Then One says that U is admissible.

Let U be an admissible function. We say that a function φ is U-quasiconcave if φ is equivalent to an increasing function on $(0, \infty)$ and φ/U is equivalent to a decreasing function on $(0, \infty)$. We say that a U-quasiconcave function φ is nondegenerate if

$$\lim_{t\to 0+} \varphi(t) = \lim_{t\to\infty} \frac{1}{\varphi(t)} = \lim_{t\to\infty} \frac{\varphi(t)}{U(t)} = \lim_{t\to 0+} \frac{U(t)}{\varphi(t)} = 0. \tag{2.2}$$

The family of nondegenerate U-quasiconcave functions will be denoted by Ω_U. We say that φ is quasiconcave when $\varphi \in \Omega_U$ with $U(t) = t$. A quasiconcave function is equivalent to a concave function. Such functions are very important in various parts of analysis. Let us just mention that, for example, the Hardy operator $Hf(x) = \int_0^x f(t)dt$ of a decreasing function, the Peetre K-functional in interpolation theory, and the fundamental function $\|\chi_E\|_X$, X is a rearrangement invariant space, all are quasiconcave.

Definition 2.3. Assume that U is admissible and $\varphi \in \Omega_U$. One says that $\{x_k\}_{k\in\mathbb{Z}}$ is a discretizing sequence for φ with respect to U if

(i) $x_0 = 1$ and $U(x_k) \uparrow\uparrow$;

(ii) $\varphi(x_k) \uparrow\uparrow$ and $\varphi(x_k)/U(x_k) \downarrow\downarrow$;

(iii) there is a decomposition $\mathbb{Z} = \mathbb{Z}_1 \cup \mathbb{Z}_2$ such that $\mathbb{Z}_1 \cap \mathbb{Z}_2 = \emptyset$ and for every $t \in [x_k, x_{k+1}]$

$$\begin{gathered}\varphi(x_k) \approx \varphi(t) \quad \text{if } k \in \mathbb{Z}_1, \\ \frac{\varphi(x_k)}{U(x_k)} \approx \frac{\varphi(t)}{U(t)} \quad \text{if } k \in \mathbb{Z}_2.\end{gathered} \tag{2.3}$$

Let us recall (see [9, Lemma 2.7]) that if $\varphi \in \Omega_U$, then there always exists a discretizing sequence for φ with respect to U.

Definition 2.4. Let U be an admissible function, and let ν be a nonnegative Borel measure on $[0, \infty)$. We say that the function φ defined by

$$\varphi(t) = U(t) \int_{[0,\infty)} \frac{d\nu(s)}{U(s) + U(t)}, \quad t \in (0, \infty), \tag{2.4}$$

is the fundamental function of the measure ν with respect to U. One will also say that ν is a representation measure of φ with respect to U.

We say that ν is nondegenerate if the following conditions are satisfied for every $t \in (0, \infty)$:

$$\int_{[0,\infty)} \frac{d\nu(s)}{U(s) + U(t)} < \infty, \qquad \int_{[0,1]} \frac{d\nu(s)}{U(s)} = \int_{[1,\infty)} d\nu(s) = \infty. \tag{2.5}$$

We recall from [9, Remark 2.10] that

$$\varphi(t) \approx \int_{[0,t]} d\nu(s) + U(t) \int_{[t,\infty)} U(s)^{-1} d\nu(s), \quad t \in (0, \infty). \tag{2.6}$$

Corollary 2.5 (see [10, Lemma 1.5]). *Let u, w be weights, and let φ be defined by*

$$\varphi(t) = \operatorname*{ess\,sup}_{s \in (0,t)} U(s)^{1/p} \operatorname*{ess\,sup}_{\tau \in (s,\infty)} \frac{w(\tau)}{U(\tau)^{1/p}}, \quad t \in (0, \infty). \tag{2.7}$$

Then φ is the least $U^{1/p}$-quasiconcave majorant of w, and

$$\begin{aligned}&\sup_{t \in (0,\infty)} \varphi(t) \left(\frac{1}{U(t)} \int_0^t \left(\int_s^\infty h(z)dz \right)^p u(s)ds \right)^{1/p} \\ &\quad = \operatorname*{ess\,sup}_{t \in (0,\infty)} w(t) \left(\frac{1}{U(t)} \int_0^t \left(\int_s^\infty h(z)dz \right)^p u(s)ds \right)^{1/p}\end{aligned} \tag{2.8}$$

for any nonnegative measurable h on $(0,\infty)$. Further, for $t \in (0,\infty)$,

$$\varphi(t) = \operatorname*{ess\,sup}_{\tau\in(0,\infty)} w(\tau) \min\left\{1, \left(\frac{U(t)}{U(\tau)}\right)^{1/p}\right\} = U(t)^{1/p} \operatorname*{ess\,sup}_{s\in(t,\infty)} \frac{1}{U(s)^{1/p}} \operatorname*{ess\,sup}_{\tau\in(0,s)} w(\tau),$$
$$\varphi(t) \approx \operatorname*{ess\,sup}_{s\in(0,\infty)} w(s) \left(\frac{U(t)}{U(s)+U(t)}\right)^{1/p}. \tag{2.9}$$

Theorem 2.6 (see [9, Theorem 2.11]). *Let $p,q,r \in (0,\infty)$. Assume that U is an admissible function, ν is a nonnegative nondegenerate Borel measure on $[0,\infty)$, and φ is the fundamental function of ν with respect to U^q and $\sigma \in \Omega_{U^p}$. If $\{x_k\}$ is a discretizing sequence for φ with respect to U^q, then*

$$\int_{[0,\infty)} \frac{\varphi(t)^{(r/q)-1}}{\sigma(t)^{r/p}} d\nu(t) \approx \sum_{k\in\mathbb{Z}} \frac{\varphi(x_k)^{r/q}}{\sigma(x_k)^{r/p}}. \tag{2.10}$$

Lemma 2.7 (see [9, Corollary 2.13]). *Let $q \in (0,\infty)$. Assume that U is an admissible function, $f \in \Omega_U$, ν is a nonnegative nondegenerate Borel measure on $[0,\infty)$, and φ is the fundamental function of ν with respect to U^q. If $\{x_k\}$ is a discretizing sequence for φ with respect to U^q, then*

$$\left(\int_{[0,\infty)} \left(\frac{f(t)}{U(t)}\right)^q d\nu(t)\right)^{1/q} \approx \left(\sum_{k\in\mathbb{Z}} \left(\frac{f(x_k)}{U(x_k)}\right)^q \varphi(x_k)\right)^{1/q}. \tag{2.11}$$

Lemma 2.8 (see [9, Lemma 3.5]). *Let $p,q,r \in (0,\infty)$. Assume that U is an admissible function, $\varphi \in \Omega_{U^q}$, and $g \in \Omega_{U^p}$. If $\{x_k\}$ is a discretizing sequence for φ with respect to U^q and $\{\lambda_k\}$ is a discretizing sequence of g with respect to U^p, then*

$$\sum_{k\in\mathbb{Z}} \frac{\varphi(x_k)^{r/q}}{g(x_k)^{r/p}} \approx \sum_{\ell\in\mathbb{Z}} \frac{\varphi(\lambda_\ell)^{r/q}}{g(\lambda_\ell)^{r/p}},$$
$$\sup_{t\in(0,\infty)} \frac{\varphi(t)^{1/q}}{g(t)^{1/p}} \approx \sup_{k\in\mathbb{Z}} \frac{\varphi(x_k)^{1/q}}{g(x_k)^{1/p}} \approx \sup_{\ell\in\mathbb{Z}} \frac{\varphi(\lambda_\ell)^{1/q}}{g(\lambda_\ell)^{1/p}}. \tag{2.12}$$

Lemma 2.9 (see [4, Lemma 2.5]). *If $\tau_k \downarrow\downarrow$, then*

$$
\begin{gathered}
\sum_{k\in\mathbb{Z}}\left(\int_0^{x_k} h\right)^q \tau_k \approx \sum_{k\in\mathbb{Z}}\left(\int_{x_{k-1}}^{x_k} h\right)^q \tau_k,\\
\sup_{k\in\mathbb{Z}}\left(\int_0^{x_k} h\right)^q \tau_k \approx \sup_{k\in\mathbb{Z}}\left(\int_{x_{k-1}}^{x_k} h\right)^q \tau_k,\\
\sum_{k\in\mathbb{Z}}\left(\int_{x_k}^{\infty} h\right)^q \tau_k^{-1} \approx \sum_{k\in\mathbb{Z}}\left(\int_{x_k}^{x_{k+1}} h\right)^q \tau_k^{-1},\\
\sup_{k\in\mathbb{Z}}\left(\int_{x_k}^{\infty} h\right)^q \tau_k^{-1} \approx \sup_{k\in\mathbb{Z}}\left(\int_{x_k}^{x_{k+1}} h\right)^q \tau_k^{-1}.
\end{gathered}
\tag{2.13}
$$

Lemma 2.10 (see [9, Lemma 3.6]). *Let $q \in (0,\infty)$. Assume that U is an admissible function, ν is a nondegenerate nonnegative Borel measure on $[0,\infty)$, φ is the fundamental function of ν with respect to U^q, and f is a measurable function on $[0,\infty)$. If $\{x_k\}$ is a discretizing sequence for φ with respect to U^q, then*

$$
\begin{aligned}
&\int_0^\infty\left(\int_0^\infty \frac{|f(t)|dt}{U(t)+U(x)}\right)^q d\nu(x)\\
&\quad\approx \sum_{k\in\mathbb{Z}}\left(\int_0^\infty \frac{|f(t)|dt}{U(t)+U(x_k)}\right)^q \varphi(x_k)\\
&\quad\approx \sum_{k\in\mathbb{Z}}\left(U^{-1}(x_k)\int_{x_{k-1}}^{x_k} |f(y)|dy + \int_{x_k}^{x_{k+1}} |f(y)|U^{-1}(y)dy\right)^q \varphi(x_k)\\
&\quad\approx \sum_{k\in\mathbb{Z}}\left(\int_{x_k}^{x_{k+1}} |f(y)|U^{-1}(y)\varphi(y)^{1/q}dy\right)^q.
\end{aligned}
\tag{2.14}
$$

Lemma 2.11 (see [9, Lemma 3.7]). *Let $q \in (0,\infty)$. Assume that U is an admissible function, ν is a nondegenerate nonnegative Borel measure on $[0,\infty)$, φ is the fundamental function of ν with respect to U^q, and f is a measurable function on $[0,\infty)$. If $\{x_k\}$ is a discretizing sequence for φ with respect to U^q, then*

$$
\begin{aligned}
&\int_{[0,\infty)}\left(\operatorname*{ess\,sup}_{y\in(0,\infty)} \frac{|f(y)|}{U(x)+U(y)}\right)^q d\nu(x)\\
&\quad\approx \sum_{k\in\mathbb{Z}}\left(\operatorname*{ess\,sup}_{y\in(0,\infty)} \frac{|f(y)|}{U(x_k)+U(y)}\right)^q \varphi(x_k)
\end{aligned}
$$

$$
\begin{aligned}
&\approx \sum_{k\in\mathbb{Z}}\left(U^{-1}(x_k)\operatorname*{ess\,sup}_{x_{k-1}\le y<x_k}|f(y)| + \operatorname*{ess\,sup}_{x_k\le y<x_{k+1}}|f(y)|U^{-1}(y)\right)^q\varphi(x_k)\\
&\approx \sum_{k\in\mathbb{Z}}\operatorname*{ess\,sup}_{x_k\le y<x_{k+1}}|f(y)|^q U^{-q}(y)\varphi(y).
\end{aligned}
\tag{2.15}
$$

Lemma 2.12 (see [9, Lemma 3.8]). *Let $q \in (0,\infty)$. Assume that U is an admissible function, $\varphi \in \Omega_{U^q}$, $\{x_k\}$ is a discretizing sequence for φ with respect to U^q, and f is a measurable function on $[0,\infty)$. Then*

$$
\begin{aligned}
&\sup_{x\in(0,\infty)}\left(\int_0^\infty \frac{|f(t)|dt}{U(t)+U(x)}\right)^q\varphi(x)\\
&\quad\approx \sup_{k\in\mathbb{Z}}\left(\int_0^\infty \frac{|f(t)|dt}{U(t)+U(x_k)}\right)^q\varphi(x_k)\\
&\quad\approx \sup_{k\in\mathbb{Z}}\left(U^{-1}(x_k)\int_{x_{k-1}}^{x_k}|f(y)|dy + \int_{x_k}^{x_{k+1}}|f(y)|U^{-1}(y)dy\right)^q\varphi(x_k)\\
&\quad\approx \sup_{k\in\mathbb{Z}}\left(\int_{x_k}^{x_{k+1}}|f(y)|U^{-1}(y)\varphi(y)^{1/q}dy\right)^q.
\end{aligned}
\tag{2.16}
$$

Lemma 2.13 (see [9, Lemma 3.9]). *Let U be an admissible function, $\varphi \in \Omega_U$, $\{x_k\}$ be a discretizing sequence for φ with respect to U, and f be a measurable function on $[0,\infty)$. Then*

$$
\begin{aligned}
&\sup_{x\in(0,\infty)}\varphi(x)\operatorname*{ess\,sup}_{y\in(0,\infty)}\frac{|f(y)|}{U(x)+U(y)}\\
&\quad\approx \sup_{k\in\mathbb{Z}}\varphi(x_k)\operatorname*{ess\,sup}_{y\in(0,\infty)}\frac{|f(y)|}{U(x_k)+U(y)}\\
&\quad\approx \sup_{k\in\mathbb{Z}}\varphi(x_k)U(x_k)^{-1}\operatorname*{ess\,sup}_{x_{k-1}\le y<x_k}|f(y)| + \sup_{k\in\mathbb{Z}}\varphi(x_k)\operatorname*{ess\,sup}_{x_k\le y<x_{k+1}}|f(y)|U(y)^{-1}\\
&\quad\approx \sup_{k\in\mathbb{Z}}\operatorname*{ess\,sup}_{x_k\le y<x_{k+1}}|f(y)|U(y)^{-1}\varphi(y).
\end{aligned}
\tag{2.17}
$$

Proposition 2.14 (see [9, Proposition 4.1]). *Let $\{\omega_k\}$ and $\{v_k\}$, $k \in \mathbb{Z}$, be two sequences of positive real numbers. Let $p, q \in (0,\infty)$, and assume that the inequality*

$$
\left(\sum_{k\in\mathbb{Z}}a_k^q v_k\right)^{1/q} \le c\left(\sum_{k\in\mathbb{Z}}a_k^p\omega_k\right)^{1/p}
\tag{2.18}
$$

is satisfied for every sequence $\{a_k\}$ of positive real numbers.

(i) *If* $p \le q$, *then*

$$\sup_{k\in\mathbb{Z}} \omega_k^{-q/p} \upsilon_k < \infty. \tag{2.19}$$

(ii) *If* $p > q$ *and* $r = pq/(p-q)$, *then*

$$\left(\sum_{k\in\mathbb{Z}} \omega_k^{-r/p} \upsilon_k^{r/q}\right)^{1/r} < \infty. \tag{2.20}$$

Lemma 2.15. *One has the following Hardy-type inequalities.*

(a) *Let* $1 < \theta \le p < \infty$. *Then the inequality*

$$\left\|\int_s^{x_k} h(z)dz\right\|_{p,u,(x_{k-1},x_k)} \le c\|h\|_{\theta,v,(x_{k-1},x_k)} \tag{2.21}$$

holds for all nonnegative measurable h *if and only if*

$$A := \sup_{x_{k-1}<t<x_k} \left(\int_{x_{k-1}}^t u(s)ds\right)^{1/p} \left(\int_t^{x_k} v(s)^{1-\theta'} ds\right)^{1/\theta'} < \infty, \tag{2.22}$$

and the best constant in (2.21) *satisfies* $c \approx A$.

(b) *Let* $1 < \theta < \infty$, $p = \infty$. *Then the inequality* (2.21) *holds if and only if*

$$B := \sup_{x_{k-1}<t<x_k} \left(\operatorname*{ess\,sup}_{x_{k-1}<s<t} u(s)\right) \left(\int_t^{x_k} v(s)^{1-\theta'} ds\right)^{1/\theta'} < \infty, \tag{2.23}$$

and the best constant in (2.21) *satisfies* $c \approx B$.

(c) *Let* $\theta = p = \infty$. *Then the inequality* (2.21) *holds if and only if*

$$C := \sup_{x_{k-1}<t<x_k} \left(\operatorname*{ess\,sup}_{x_{k-1}<s<t} u(s)\right) \int_t^{x_k} \frac{ds}{v(s)} < \infty, \tag{2.24}$$

and the best constant in (2.21) *satisfies* $c \approx C$.

(d) *Let* $1 < \theta < \infty$, $0 < p < \theta$, *and* $1/r = 1/p - 1/\theta$. *Then the inequality* (2.21) *holds if and only if*

$$D := \left(\int_{x_{k-1}}^{x_k} \left(\int_{x_{k-1}}^t u(s)ds\right)^{r/p} \left(\int_t^{x_k} v(s)^{1-\theta'} ds\right)^{r/p'} v(t)^{1-\theta'} dt\right)^{1/r} < \infty, \tag{2.25}$$

and the best constant in (2.21) *satisfies* $c \approx D$.

(e) *Let $\theta = \infty$, $0 < p < \infty$. Then the inequality* (2.21) *holds if and only if*

$$E := \left(\int_{x_{k-1}}^{x_k}\left(\int_{x_{k-1}}^{t} u(s)ds\right)\left(\int_{t}^{x_k}\frac{ds}{v(s)}\right)^{p-1}\frac{dt}{v(t)}\right)^{1/p} < \infty, \tag{2.26}$$

and the best constant in (2.21) *satisfies $c \approx E$.*

These results are just classical results of Maz'ja [11] and Sinnamon [12] (cf. also [13, 14]).

3. The Main Results

In this section we characterize the validity of the inequalities

$$\left(\int_0^\infty\left(\frac{1}{U(t)}\int_0^t\left(\int_s^\infty h(z)dz\right)^p u(s)ds\right)^{q/p} w(t)dt\right)^{1/q} \le c\|h\|_{\theta,v,(0,\infty)}, \tag{3.1}$$

$$\operatorname*{ess\,sup}_{t\in(0,\infty)} w(t)\left(\frac{1}{U(t)}\int_0^t\left(\int_s^\infty h(z)dz\right)^p u(s)ds\right)^{1/p} \le c\|h\|_{\theta,v,(0,\infty)}. \tag{3.2}$$

Denote

$$\mathcal{U}(x,t) := \frac{U(x)}{U(t)+U(x)}. \tag{3.3}$$

First we characterize (3.1) as follows.

Theorem 3.1. *Let $0 < q < \infty$, $0 < p < \infty$, $1 < \theta \le \infty$, and let u, v, w be weights. Assume that u is such that $U^{q/p}$ is admissible and the measure $w(t)dt$ is nondenerate with respect to $U^{q/p}$. Then the inequality* (3.1) *holds for every measurable function f on $(0,\infty)$ if and only if*

(i) $1 < \theta \le p < \infty$, $\theta \le q$

$$A_1 := \sup_{x\in(0,\infty)}\left(\int_0^\infty \mathcal{U}(x,t)^{q/p}w(t)dt\right)^{1/q} U(x)^{-1/p}\sup_{t\in(0,x)} U(t)^{1/p}V_\theta(t)^{1/\theta'} < \infty. \tag{3.4}$$

Moreover, the best constant c in (3.1) *satisfies $c \approx A_1$.*

(ii) $1 < \theta \le p < \infty$, $0 < q < \theta < \infty$, $l = \theta q/(\theta - q)$

$$A_2 := \left(\int_0^\infty\left(\int_0^\infty \mathcal{U}(x,t)^{q/p}w(t)dt\right)^{(l-q)/q} w(x)U(x)^{-l/p}\sup_{t\in(0,x)} U(t)^{l/p}V_\theta(t)^{l/\theta'}dx\right)^{1/l} < \infty. \tag{3.5}$$

Moreover, the best constant c in (3.1) *satisfies $c \approx A_2$.*

(iii) $0 < p < \theta < \infty$, $1 < \theta \leq q < \infty$, $r = \theta p/(\theta - p)$

$$A_3 := \sup_{x\in(0,\infty)} \left(\int_0^\infty \mathcal{U}(x,t)^{q/p} w(t)dt\right)^{1/q} \left(\int_0^\infty \mathcal{U}(t,x)^{r/p} V_\theta(t)^{r/p'} v(t)^{1-\theta'} dt\right)^{1/r} < \infty. \tag{3.6}$$

Moreover, the best constant c in (3.1) *satisfies* $c \approx A_3$.

(iv) $0 < p < \theta < \infty$, $1 < \theta < \infty$, $q < \theta$, $r = \theta p/(\theta - p)$, $l = \theta q/(\theta - q)$,

$$A_4 := \left(\int_0^\infty \left(\int_0^\infty \mathcal{U}(x,t)^{q/p} w(t)dt\right)^{(l-q)/q} w(x) \left(\int_0^\infty \mathcal{U}(t,x)^{r/p} V_\theta(t)^{r/p'} v(t)^{1-\theta'} dt\right)^{l/r} dx\right)^{1/l} < \infty. \tag{3.7}$$

Moreover, the best constant c in (3.1) *satisfies* $c \approx A_4$.

(v) *Let* $\theta = \infty$, $0 < p < \infty$, $0 < q < \infty$,

$$A_5 := \left(\int_0^\infty \left(\mathcal{U}(t,x) V_\theta(t)^{p-1} \frac{dt}{v(t)}\right)^{q/p} w(x)dx\right)^{1/q} < \infty. \tag{3.8}$$

Moreover, the best constant c in (3.1) *satisfies* $c \approx A_5$.

Proof. Define

$$\varphi(x) = \int_0^\infty \mathcal{U}(x,s)^{q/p} w(s)ds. \tag{3.9}$$

Then $\varphi \in \Omega_{U^{q/p}}$, and therefore there exists a discretizing sequence for φ with respect to $U^{q/p}$. Let $\{x_k\}$ be one such sequence. Then $\varphi(x_k) \uparrow\uparrow$ and $\varphi(x_k)U^{-q/p} \downarrow\downarrow$. Furthermore, there is a decomposition $\mathbb{Z} = \mathbb{Z}_1 \cup \mathbb{Z}_2$, $\mathbb{Z}_1 \cap \mathbb{Z}_2 = \emptyset$ such that for every $k \in \mathbb{Z}_1$ and $t \in [x_k, x_{k+1}]$, $\varphi(x_k) \approx \varphi(t)$ and for every $k \in \mathbb{Z}_2$ and $t \in [x_k, x_{k+1}]$, $\varphi(x_k)U(x_k)^{-q/p} \approx \varphi(t)U(t)^{-q/p}$.

For the left-hand side of (3.1), by using Lemma 2.7 with

$$d\nu(t) = w(t)dt, \qquad f(t) = \int_0^t \left(\int_s^\infty h(z)dz\right)^p u(s)ds, \tag{3.10}$$

we get that

$$\begin{aligned} J &:= \left(\int_0^\infty \left(\frac{1}{U(t)} \int_0^t \left(\int_s^\infty h(z)dz\right)^p u(s)ds\right)^{q/p} w(t)dt\right)^{1/q} \\ &\approx \left(\sum_{k\in\mathbb{Z}} \left(\int_0^{x_k} \left(\int_s^\infty h(z)dz\right)^p u(s)ds\right)^{q/p} \frac{\varphi(x_k)}{U^{q/p}(x_k)}\right)^{1/q}. \end{aligned} \tag{3.11}$$

Moreover, by using Lemma 2.9, we get that

$$
\begin{aligned}
J &\approx \left(\sum_{k\in\mathbb{Z}}\left(\int_{x_{k-1}}^{x_k}\left(\int_s^{\infty} h(z)dz\right)^p u(s)ds\right)^{q/p}\frac{\varphi(x_k)}{U^{q/p}(x_k)}\right)^{1/q} \\
&= \left(\sum_{k\in\mathbb{Z}}\left(\int_{x_{k-1}}^{x_k}\left(\int_s^{x_k} h(z)dz+\int_{x_k}^{\infty} h(z)dz\right)^p u(s)ds\right)^{q/p}\frac{\varphi(x_k)}{U^{q/p}(x_k)}\right)^{1/q} \\
&\approx \left(\sum_{k\in\mathbb{Z}}\left(\int_{x_{k-1}}^{x_k}\left(\int_s^{x_k} h(z)dz\right)^p u(s)ds\right)^{q/p}\frac{\varphi(x_k)}{U^{q/p}(x_k)}\right)^{1/q} \\
&\quad + \left(\sum_{k\in\mathbb{Z}}\left(\int_{x_{k-1}}^{x_k}\left(\int_{x_k}^{\infty} h(z)dz\right)^p u(s)ds\right)^{q/p}\frac{\varphi(x_k)}{U^{q/p}(x_k)}\right)^{1/q} \\
&\approx \left(\sum_{k\in\mathbb{Z}}\left(\int_{x_{k-1}}^{x_k}\left(\int_s^{x_k} h(z)dz\right)^p u(s)ds\right)^{q/p}\frac{\varphi(x_k)}{U^{q/p}(x_k)}\right)^{1/q} \\
&\quad + \left(\sum_{k\in\mathbb{Z}}\left(\int_{x_k}^{\infty} h(z)dz\right)^q\left(\int_{x_{k-1}}^{x_k} u(s)ds\right)^{q/p}\frac{\varphi(x_k)}{U^{q/p}(x_k)}\right)^{1/q}.
\end{aligned}
\tag{3.12}
$$

By now using the fact that $\int_{x_{k-1}}^{x_k} u(s)ds = U(x_k)-U(x_{k-1}) \approx U(x_k)$, we find that

$$
\begin{aligned}
J \approx{}& \left(\sum_{k\in\mathbb{Z}}\left(\int_{x_{k-1}}^{x_k}\left(\int_s^{x_k} h(z)dz\right)^p u(s)ds\right)^{q/p}\frac{\varphi(x_k)}{U^{q/p}(x_k)}\right)^{1/q} \\
&+ \left(\sum_{k\in\mathbb{Z}}\left(\int_{x_k}^{\infty} h(z)dz\right)^q\varphi(x_k)\right)^{1/q},
\end{aligned}
\tag{3.13}
$$

is, by using Lemma 2.9 on the second term,

$$
\begin{aligned}
J \approx{}& \left(\sum_{k\in\mathbb{Z}}\left(\int_{x_{k-1}}^{x_k}\left(\int_s^{x_k} h(z)dz\right)^p u(s)ds\right)^{q/p}\frac{\varphi(x_k)}{U^{q/p}(x_k)}\right)^{1/q} \\
&+ \left(\sum_{k\in\mathbb{Z}}\left(\int_{x_k}^{x_{k+1}} h(z)dz\right)^q\varphi(x_k)\right)^{1/q} := I + II.
\end{aligned}
\tag{3.14}
$$

Now we will distinguish several cases. We start with the case $1 < \theta \le p < \infty$. Then, by using Lemma 2.15, we get that

$$\begin{aligned} I &= \left(\sum_{k\in\mathbb{Z}}\left(\int_{x_{k-1}}^{x_k}\left(\int_s^{x_k} h(z)dz\right)^p u(s)ds\right)^{q/p}\frac{\varphi(x_k)}{U^{q/p}(x_k)}\right)^{1/q} \\ &\le \left(\sum_{k\in\mathbb{Z}}\frac{\varphi(x_k)}{U^{q/p}(x_k)}\sup_{x_{k-1}<t<x_k} U(t)^{q/p}V_\theta(t)^{q/\theta'}\left(\int_{x_{k-1}}^{x_k} h(z)^\theta v(z)dz\right)^{q/\theta}\right)^{1/q}. \end{aligned} \tag{3.15}$$

Moreover, by applying Hölder's inequality for II, we find that

$$\begin{aligned} II &= \left(\sum_{k\in\mathbb{Z}}\varphi(x_k)\left(\int_{x_k}^{x_{k+1}} h(z)dz\right)^q\right)^{1/q} \\ &\le \left(\sum_{k\in\mathbb{Z}}\varphi(x_k)\left(\int_{x_k}^{x_{k+1}} v(z)^{1-\theta'}dz\right)^{q/\theta'}\left(\int_{x_k}^{x_{k+1}} h(z)^\theta v(z)dz\right)^{q/\theta}\right)^{1/q} \\ &\le \left(\sum_{k\in\mathbb{Z}}\varphi(x_k)V_\theta(x_k)^{q/\theta'}\left(\int_{x_k}^{x_{k+1}} h(z)^\theta v(z)dz\right)^{q/\theta}\right)^{1/q}. \end{aligned} \tag{3.16}$$

(i) In the case $q/\theta \ge 1$, according to (3.15), we have that

$$I \le \sup_{k\in\mathbb{Z}}\frac{\varphi(x_k)^{1/q}}{U^{1/p}(x_k)}\sup_{x_{k-1}<t<x_k} U(t)^{1/p}V_\theta(t)^{1/\theta'}\|h\|_{\theta,v,(0,\infty)}. \tag{3.17}$$

Similarly, if $q/\theta \ge 1$, then, according to (3.16), we obtain that

$$II \le \sup_{k\in\mathbb{Z}}\varphi(x_k)^{1/q}V_\theta(x_k)^{1/\theta'}\|h\|_{\theta,v,(0,\infty)}, \tag{3.18}$$

and, finally, by using (3.9), Lemma 2.13, and (3.14), we get that

$$\begin{aligned} J &\lesssim \sup_{k\in\mathbb{Z}}\varphi(x_k)^{1/q}\left(U^{-1/p}(x_k)\sup_{x_{k-1}<t<x_k} U(t)^{1/p}V_\theta(t)^{1/\theta'} + V_\theta(x_k)^{1/\theta'}\right)\|h\|_{\theta,v,(0,\infty)} \\ &\approx \sup_{x\in(0,\infty)}\varphi(x)^{1/q}\sup_{t\in(0,\infty)}\mathcal{U}(t,x)^{1/p}V_\theta(t)^{1/\theta'}\|h\|_{\theta,v,(0,\infty)} \\ &= \sup_{x\in(0,\infty)}\left(\int_0^\infty \mathcal{U}(x,t)^{q/p}w(t)dt\right)^{1/q}\sup_{t\in(0,\infty)}\mathcal{U}(t,x)^{1/p}V_\theta(t)^{1/\theta'}\|h\|_{\theta,v,(0,\infty)} \\ &\approx \sup_{x\in(0,\infty)}\left(\int_0^\infty \mathcal{U}(x,t)^{q/p}w(t)dt\right)^{1/q}U(x)^{-1/p}\sup_{t\in(0,x)} U(t)^{1/p}V_\theta(t)^{1/\theta'}\|h\|_{\theta,v,(0,\infty)}. \end{aligned} \tag{3.19}$$

(ii) For the case $0 < q < \theta < \infty$, $l = \theta q/(\theta - q)$, by applying Hölder's inequality for sums to the right-hand side of (3.15) and (3.16) with exponents θ/q and l/q, we find that

$$\begin{aligned} I &\le \left(\sum_{k\in\mathbb{Z}} \frac{\varphi(x_k)^{l/q}}{U^{l/p}(x_k)} \sup_{x_{k-1}<t<x_k} U(t)^{l/p} V_\theta(t)^{l/\theta'}\right)^{1/l} \|h\|_{\theta,v,(0,\infty)}, \\ II &\le \left(\sum_{k\in\mathbb{Z}} \varphi(x_k)^{l/q} V_\theta(x_k)^{l/\theta'}\right)^{1/l} \|h\|_{\theta,v,(0,\infty)}. \end{aligned} \tag{3.20}$$

Therefore, we get that

$$I + II \le \left(\sum_{k\in\mathbb{Z}} \varphi(x_k)^{l/q} \left(U(x_k)^{-l/p} \sup_{x_{k-1}<t<x_k} U(t)^{l/p} V_\theta(t)^{l/\theta'} + V_\theta(x_k)^{l/\theta'}\right)\right)^{1/l} \|h\|_{\theta,v,(0,\infty)}, \tag{3.21}$$

so that, in view of Lemma 2.11, Theorem 2.6, and (3.14),

$$\begin{aligned} J &\lesssim \left(\sum_{k\in\mathbb{Z}} \varphi(x_k)^{l/q} \sup_{t\in(0,\infty)} \mathcal{U}(t,x_k)^{l/p} V_\theta(t)^{l/\theta'}\right)^{1/l} \|h\|_{\theta,v,(0,\infty)} \\ &\approx \left(\int_0^\infty \varphi(x)^{(l/q)-1} \sup_{t\in(0,\infty)} \mathcal{U}(t,x)^{l/p} V_\theta(t)^{l/\theta'} w(x)\,dx\right)^{1/l} \|h\|_{\theta,v,(0,\infty)} \\ &\approx \left(\int_0^\infty \left(\int_0^\infty \mathcal{U}(x,t)^{q/p} w(t)\,dt\right)^{(l-q)/q} w(x) U(x)^{-l/p} \sup_{t\in(0,x)} U(t)^{l/p} V_\theta(t)^{l/\theta'}\,dx\right)^{1/l} \|h\|_{\theta,v,(0,\infty)}. \end{aligned} \tag{3.22}$$

Now let us assume that $0 < p < \theta < \infty$, $1 < \theta < \infty$, $1/r = 1/p - 1/\theta$. By Lemma 2.15, we have that

$$\begin{aligned} I \le &\left(\sum_{k\in\mathbb{Z}} \frac{\varphi(x_k)}{U^{q/p}(x_k)} \left(\int_{x_{k-1}}^{x_k} \left(\int_{x_{k-1}}^{t} u(s)\,ds\right)^{r/p} \left(\int_t^{x_k} v(s)^{1-\theta'}\,ds\right)^{r/p'} v(t)^{1-\theta'}\,dt\right)^{q/r}\right. \\ &\left.\times \left(\int_{x_{k-1}}^{x_k} h(z)^\theta v(z)\,dz\right)^{q/\theta}\right)^{1/q}. \end{aligned} \tag{3.23}$$

Moreover, by applying Hölder's inequality for II, we find that

$$II \le \left(\sum_{k\in\mathbb{Z}} \varphi(x_k) V_\theta(x_k)^{q/\theta'} \left(\int_{x_k}^{x_{k+1}} h(z)^\theta v(z)\,dz\right)^{q/\theta}\right)^{1/q}. \tag{3.24}$$

(iii) Now, we assume that $q/\theta \geq 1$. Then, according to (3.23) and (3.24), we obtain that

$$I \lesssim \sup_{k\in\mathbb{Z}} \frac{\varphi(x_k)^{1/q}}{U^{1/p}(x_k)} \left(\int_{x_{k-1}}^{x_k} U(t)^{r/p} V_\theta(t)^{r/p'} v(t)^{1-\theta'} dt \right)^{1/r} \|h\|_{\theta,v,(0,\infty)},$$
$$II \leq \sup_{k\in\mathbb{Z}} \varphi(x_k)^{1/q} V_\theta(x_k)^{1/\theta'} \|h\|_{\theta,v,(0,\infty)}. \tag{3.25}$$

Hence, using Lemmas 2.9 and 2.12, and (3.14), we get that

$$\begin{aligned}
J &\lesssim \sup_{k\in\mathbb{Z}} \varphi(x_k)^{1/q} \left(U^{-r/p}(x_k) \int_{x_{k-1}}^{x_k} U(t)^{r/p} V_\theta(t)^{r/p'} v(t)^{1-\theta'} dt + \int_{x_k}^{\infty} V_\theta(t)^{r/p'} v(t)^{1-\theta'} dt \right)^{1/r} \\
&\quad \times \|h\|_{\theta,v,(0,\infty)} \\
&\approx \sup_{k\in\mathbb{Z}} \varphi(x_k)^{1/q} \left(U^{-r/p}(x_k) \int_{x_{k-1}}^{x_k} U(t)^{r/p} V_\theta(t)^{r/p'} v(t)^{1-\theta'} dt + \int_{x_k}^{x_{k+1}} V_\theta(t)^{r/p'} v(t)^{1-\theta'} dt \right)^{1/r} \\
&\quad \times \|h\|_{\theta,v,(0,\infty)} \\
&\approx \sup_{k\in\mathbb{Z}} \varphi(x_k)^{1/q} \left(\int_0^\infty \mathcal{U}(t,x_k)^{r/p} V_\theta(t)^{r/p'} v(t)^{1-\theta'} dt \right)^{1/r} \|h\|_{\theta,v,(0,\infty)} \\
&\approx \sup_{x\in(0,\infty)} \varphi(x)^{1/q} \left(\int_0^\infty \mathcal{U}(t,x)^{r/p} V_\theta(t)^{r/p'} v(t)^{1-\theta'} dt \right)^{1/r} \|h\|_{\theta,v,(0,\infty)} \\
&= \sup_{x\in(0,\infty)} \left(\int_0^\infty \mathcal{U}(x,t)^{q/p} w(t) dt \right)^{1/q} \left(\int_0^\infty \mathcal{U}(t,x)^{r/p} V_\theta(t)^{r/p'} v(t)^{1-\theta'} dt \right)^{1/r} \\
&\quad \times \|h\|_{\theta,v,(0,\infty)}.
\end{aligned} \tag{3.26}$$

(iv) Next, we consider the case $0 < q < \theta$, $1/l = 1/q - 1/\theta$. By using Hölder's inequality for sums to the right-hand side of (3.23) and (3.24) with exponents θ/q and l/q, we get that

$$I \leq \left(\sum_{k\in\mathbb{Z}} \frac{\varphi(x_k)^{l/q}}{U^{l/p}(x_k)} \left(\int_{x_{k-1}}^{x_k} U(t)^{r/p} V_\theta(t)^{r/p'} v(t)^{1-\theta'} dt \right)^{l/r} \right)^{1/l} \|h\|_{\theta,v,(0,\infty)},$$
$$II \leq \left(\sum_{k\in\mathbb{Z}} \varphi(x_k)^{l/q} V_\theta(x_k)^{l/\theta'} \right)^{1/l} \|h\|_{\theta,v,(0,\infty)}. \tag{3.27}$$

Therefore, using Lemmas 2.9 and 2.10, Theorem 2.6, and (3.14), we find that

$$\begin{aligned}
J &\lesssim \left(\sum_{k\in\mathbb{Z}} \varphi(x_k)^{l/q}\left(U^{-l/p}(x_k)\left(\int_{x_{k-1}}^{x_k} U(t)^{r/p}V_\theta(t)^{r/p'}v(t)^{1-\theta'}dt\right)^{l/r} + V(x_k)^{l/\theta'}\right)\right)^{1/l} \\
&\quad \times \|h\|_{\theta,v,(0,\infty)} \\
&\approx \left(\sum_{k\in\mathbb{Z}} \varphi(x_k)^{l/q}\left(U^{-r/p}(x_k)\int_{x_{k-1}}^{x_k} U(t)^{r/p}V_\theta(t)^{r/p'}v(t)^{1-\theta'}dt + \int_{x_k}^{\infty} V_\theta(t)^{r/p'}v(t)^{1-\theta'}dt\right)^{l/r}\right)^{1/l} \\
&\quad \times \|h\|_{\theta,v,(0,\infty)} \\
&\approx \left(\sum_{k\in\mathbb{Z}} \varphi(x_k)^{l/q}\left(U^{-r/p}(x_k)\int_{x_{k-1}}^{x_k} U(t)^{r/p}V_\theta(t)^{r/p'}v(t)^{1-\theta'}dt + \int_{x_k}^{x_{k+1}} V_\theta(t)^{r/p'}v(t)^{1-\theta'}dt\right)^{l/r}\right)^{1/l} \\
&\quad \times \|h\|_{\theta,v,(0,\infty)} \\
&\approx \left(\sum_{k\in\mathbb{Z}} \varphi(x_k)^{l/q}\left(\int_0^{\infty} \mathcal{U}(t,x_k)^{r/p}V_\theta(t)^{r/p'}v(t)^{1-\theta'}dt\right)^{l/r}\right)^{l/l} \|h\|_{\theta,v,(0,\infty)} \\
&\approx \left(\int_0^{\infty} \varphi(t)^{l/q-1}\left(\int_0^{\infty} \mathcal{U}(t,x)^{r/p}V_\theta(t)^{r/p'}v(t)^{1-\theta'}dt\right)^{l/r} w(x)dx\right)^{l/l} \|h\|_{\theta,v,(0,\infty)} \\
&= \left(\int_0^{\infty}\left(\int_0^{\infty} \mathcal{U}(x,t)^{q/p}w(t)dt\right)^{(l-q)/q} w(x)\left(\int_0^{\infty} \mathcal{U}(t,x)^{r/p}V_\theta(t)^{r/p'}v(t)^{1-\theta'}dt\right)^{l/r} dx\right)^{1/l} \\
&\quad \times \|h\|_{\theta,v,(0,\infty)}.
\end{aligned} \tag{3.28}$$

(v) Let $\theta = \infty$, $0 < p < \infty$, $0 < q < \infty$. According to Lemma 2.15, we have that

$$I \le \left(\sum_{k\in\mathbb{Z}} \frac{\varphi(x_k)}{U^{q/p}(x_k)}\left(\int_{x_{k-1}}^{x_k}\left(\int_{x_{k-1}}^{t} u(s)ds\right)\left(\int_t^{x_k}\frac{dz}{v(z)}\right)^{p-1}\frac{dt}{v(t)}\right)^{q/p}\right)^{1/q} \|h\|_{\theta,v,(0,\infty)}. \tag{3.29}$$

Moreover, it yields that

$$II \le \left(\sum_{k\in\mathbb{Z}} \varphi(x_k)\left(\int_{x_k}^{x_{k+1}}\frac{dz}{v(z)}\right)^{q}\right)^{1/q} \|h\|_{\theta,v,(0,\infty)}. \tag{3.30}$$

Hence, by integrating by parts, using Lemmas 2.9 and 2.10, and (3.14), we get that

$$\begin{aligned}
J &\lesssim \left(\sum_{k\in\mathbb{Z}}\varphi(x_k)\left(U(x_k)^{-1}\int_{x_{k-1}}^{x_k} U(s)\left(\int_s^\infty \frac{dz}{v(z)}\right)^{p-1}\frac{ds}{v(s)}+\int_{x_k}^{\infty}\left(\int_s^\infty \frac{dz}{v(z)}\right)^{p-1}\frac{ds}{v(s)}\right)^{q/p}\right)^{1/q} \\
&\quad\times \|h\|_{\infty,v,(0,\infty)} \\
&\approx \left(\sum_{k\in\mathbb{Z}}\varphi(x_k)\left(U(x_k)^{-1}\int_{x_{k-1}}^{x_k} U(s)\left(\int_s^\infty \frac{dz}{v(z)}\right)^{p-1}\frac{ds}{v(s)}+\int_{x_k}^{x_{k+1}}\left(\int_s^\infty \frac{dz}{v(z)}\right)^{p-1}\frac{ds}{v(s)}\right)^{q/p}\right)^{1/q} \\
&\quad\times \|h\|_{\infty,v,(0,\infty)} \\
&\approx \left(\int_0^\infty\left(\mathcal{U}(t,x)\left(\int_t^\infty \frac{dz}{v(z)}\right)^{p-1}\frac{dt}{v(t)}\right)^{q/p}\omega(x)dx\right)^{1/q}\|h\|_{\infty,v,(0,\infty)}.
\end{aligned} \tag{3.31}$$

Now we prove the lower bounds (necessity). Let $0<q<\infty$ and $\{x_k\}$ be a discretizing sequence for φ from (3.9). Then, by (3.14), we find that

$$\begin{aligned}
&\left(\sum_{k\in\mathbb{Z}}\left(\int_{x_{k-1}}^{x_k}\left(\int_s^{x_k} h(z)dz\right)^p u(s)ds\right)^{q/p}\frac{\varphi(x_k)}{U^{q/p}(x_k)}\right)^{1/q} \\
&\quad+\left(\sum_{k\in\mathbb{Z}}\left(\int_{x_k}^{x_{k+1}} h(z)dz\right)^q\varphi(x_k)\right)^{1/q} \\
&\lesssim \left(\sum_{k\in\mathbb{Z}}\int_{x_{k-1}}^{x_k} h(z)^\theta v(z)dz\right)^{1/\theta}.
\end{aligned} \tag{3.32}$$

Let $1<\theta\le p<\infty$. For $k\in\mathbb{Z}$, let h_k be functions that saturate the Hardy inequality (2.21) and Hölder's inequality, that is, functions h_k satisfying

$$\begin{gathered}
\operatorname{supp} h_k\subset[x_{k-1},x_k], \\
\int_{x_{k-1}}^{x_k} h_k(t)^\theta v(t)dt=1, \\
\left\|\int_s^{x_k} h_k(z)dz\right\|_{p,u,(x_{k-1},x_k)}\gtrsim \sup_{x_{k-1}<t<x_k}\left(\int_{x_{k-1}}^t u(s)ds\right)^{1/p}\left(\int_t^{x_k} v(s)^{1-\theta'}ds\right)^{1/\theta'}, \\
\int_{x_{k-1}}^{x_k} h_k(t)dt\gtrsim\left(\int_{x_{k-1}}^{x_k} v(t)^{1-\theta'}dt\right)^{1/\theta'}.
\end{gathered} \tag{3.33}$$

Now we define the test function

$$h(t) = \sum_{k\in\mathbb{Z}} a_k h_k(t), \tag{3.34}$$

where $\{a_k\}$ is a sequence of positive real numbers. Thus, using test function (3.34) in (3.32), we get that

$$\begin{aligned}
&\left(\sum_{k\in\mathbb{Z}} a_k^q \frac{\varphi(x_k)}{U^{q/p}(x_k)} \sup_{x_{k-1}<t<x_k} \left(\int_{x_{k-1}}^t u(s)ds\right)^{q/p} \left(\int_t^{x_k} v(s)^{1-\theta'} ds\right)^{q/\theta'}\right)^{1/q} \\
&\quad + \left(\sum_{k\in\mathbb{Z}} a_k^q \varphi(x_k) \left(\int_{x_k}^{x_{k+1}} v(t)^{1-\theta'} dt\right)^{q/\theta'}\right)^{1/q} \lesssim \left(\sum_{k\in\mathbb{Z}} a_k^\theta\right).
\end{aligned} \tag{3.35}$$

Now using Proposition 2.14 for the case $\theta \le q$, we obtain that

$$\begin{aligned}
&\sup_{k\in\mathbb{Z}} \varphi(x_k)^{1/q} \left(\int_{x_k}^{x_{k+1}} v(t)^{1-\theta'} dt\right)^{1/\theta'} \\
&\quad + \sup_{k\in\mathbb{Z}} \frac{\varphi(x_k)^{1/q}}{U^{1/p}(x_k)} \sup_{x_{k-1}<t<x_k} \left(\int_{x_{k-1}}^t u(s)ds\right)^{1/p} \left(\int_t^{x_k} v(s)^{1-\theta'} ds\right)^{1/\theta'} < \infty.
\end{aligned} \tag{3.36}$$

On the other hand, using Lemma 2.9, we get that

$$\begin{aligned}
A_1 &\lesssim \sup_{k\in\mathbb{Z}} \frac{\varphi(x_k)^{1/q}}{U^{1/p}(x_k)} \sup_{x_{k-1}<t<x_k} U(t)^{1/p} V_\theta(t)^{1/\theta'} + \sup_{k\in\mathbb{Z}} \varphi(x_k)^{1/q} V_\theta(x_k)^{1/\theta'} \\
&\lesssim \sup_{k\in\mathbb{Z}} \frac{\varphi(x_k)^{1/q}}{U^{1/p}(x_k)} \sup_{x_{k-1}<t<x_k} U(t)^{1/p} \left(\int_t^{x_k} v(s)^{1-\theta'} ds\right)^{1/\theta'} \\
&\quad + \sup_{k\in\mathbb{Z}} \varphi(x_k)^{1/q} V_\theta(x_k)^{1/\theta'} \\
&\lesssim \sup_{k\in\mathbb{Z}} \frac{\varphi(x_k)^{1/q}}{U^{1/p}(x_k)} \sup_{x_{k-1}<t<x_k} \left(\int_{x_{k-1}}^t u(s)ds\right)^{1/p} \left(\int_t^{x_k} v(s)^{1-\theta'} ds\right)^{1/\theta'} \\
&\quad + \sup_{k\in\mathbb{Z}} \varphi(x_k)^{1/q} V_\theta(x_k)^{1/\theta'} \\
&\lesssim \sup_{k\in\mathbb{Z}} \frac{\varphi(x_k)^{1/q}}{U^{1/p}(x_k)} \sup_{x_{k-1}<t<x_k} \left(\int_{x_{k-1}}^t u(s)ds\right)^{1/p} \left(\int_t^{x_k} v(s)^{1-\theta'} ds\right)^{1/\theta'} \\
&\quad + \sup_{k\in\mathbb{Z}} \varphi(x_k)^{1/q} \left(\int_{x_k}^{x_{k+1}} v(s)^{1-\theta'} ds\right)^{1/\theta'} < \infty.
\end{aligned} \tag{3.37}$$

Let $0 < q < \theta < \infty$. From (3.35) and Proposition 2.14, we obtain that

$$\left(\sum_{k\in\mathbb{Z}}\varphi(x_k)^{l/q}\left(\int_{x_k}^{x_{k+1}} v(t)^{1-\theta'}dt\right)^{l/\theta'}\right)^{1/l} + \left(\sum_{k\in\mathbb{Z}}\frac{\varphi(x_k)^{l/q}}{U^{l/p}(x_k)}\sup_{x_{k-1}<t<x_k}\left(\int_{x_{k-1}}^{t} u(s)ds\right)^{l/p}\left(\int_{t}^{x_k} v(s)^{1-\theta'}ds\right)^{l/\theta'}\right)^{1/l} < \infty. \tag{3.38}$$

Since

$$\begin{aligned} A_2 &\approx \left(\sum_{k\in\mathbb{Z}}\varphi(x_k)^{l/q}V_\theta(x_k)^{l/\theta'}\right)^{1/l} + \left(\sum_{k\in\mathbb{Z}}\frac{\varphi(x_k)^{l/q}}{U(x_k)^{l/p}}\sup_{x_{k-1}<t<x_k}U(t)^{l/p}V_\theta(t)^{l/\theta'}\right)^{1/l} \\ &\lesssim \left(\sum_{k\in\mathbb{Z}}\varphi(x_k)^{l/q}V_\theta(x_k)^{l/\theta'}\right)^{1/l} + \left(\sum_{k\in\mathbb{Z}}\frac{\varphi(x_k)^{l/q}}{U(x_k)^{l/p}}\sup_{x_{k-1}<t<x_k}U(t)^{l/p}\left(\int_{t}^{x_k} v(s)^{1-\theta'}ds\right)^{l/\theta'}\right)^{1/l} \\ &\lesssim \left(\sum_{k\in\mathbb{Z}}\varphi(x_k)^{l/q}V_\theta(x_k)^{l/\theta'}\right)^{1/l} + \left(\sum_{k\in\mathbb{Z}}\frac{\varphi(x_k)^{l/q}}{U(x_k)^{l/p}}\sup_{x_{k-1}<t<x_k}\left(\int_{x_{k-1}}^{t} u(s)ds\right)^{l/p}\left(\int_{t}^{x_k} v(s)^{1-\theta'}ds\right)^{l/\theta'}\right)^{1/l}, \end{aligned} \tag{3.39}$$

by Lemma 2.9, we arrive at

$$A_2 \lesssim \left(\sum_{k\in\mathbb{Z}}\varphi(x_k)^{l/q}\left(\int_{t}^{x_k} v(s)^{1-\theta'}ds\right)^{l/\theta'}\right)^{1/l} + \left(\frac{\varphi(x_k)^{l/q}}{U(x_k)^{l/p}}\sup_{x_{k-1}<t<x_k}\left(\int_{x_{k-1}}^{t} u(s)ds\right)^{l/p}\left(\int_{t}^{x_k} v(s)^{1-\theta'}ds\right)^{l/\theta'}\right)^{1/l} < \infty. \tag{3.40}$$

Let $1 < \theta < \infty$, $0 < p < \theta$. For $k \in \mathbb{Z}$, let h_k be functions that saturate the Hardy inequality (2.21) and Hölder's inequality, that is, functions h_k satisfying

$$\begin{gathered}
\operatorname{supp} h_k \subset [x_{k-1}, x_k], \\
\int_{x_{k-1}}^{x_k} h_k(t)^\theta v(t)\,dt = 1, \\
\left\| \int_s^{x_k} h_k(z)\,dz \right\|_{p,u,(x_{k-1},x_k)} \gtrsim \left(\int_{x_{k-1}}^{x_k} \left(\int_{x_{k-1}}^{t} u(s)\,ds \right)^{r/p} \left(\int_t^{x_k} v(s)^{1-\theta'}\,ds \right)^{r/p'} v(t)^{1-\theta'}\,dt \right)^{1/r}, \\
\int_{x_{k-1}}^{x_k} h_k(t)\,dt \gtrsim \left(\int_{x_{k-1}}^{x_k} v(t)^{1-\theta'}\,dt \right)^{1/\theta'}.
\end{gathered} \tag{3.41}$$

Now we define the test function

$$h(t) = \sum_{k\in\mathbb{Z}} a_k h_k(t), \tag{3.42}$$

where $\{a_k\}$ is a sequence of positive real numbers. Thus, using test function (3.42) in (3.32), we get that

$$\begin{aligned}
&\left(\sum_{k\in\mathbb{Z}} a_k^q \frac{\varphi(x_k)}{U^{q/p}(x_k)} \left(\int_{x_{k-1}}^{x_k} \left(\int_{x_{k-1}}^{t} u(s)\,ds \right)^{r/p} \left(\int_t^{x_k} v(s)^{1-\theta'}\,ds \right)^{r/p'} v(t)^{1-\theta'}\,dt \right)^{q/r} \right)^{1/q} \\
&\quad + \left(\sum_{k\in\mathbb{Z}} a_k^q \varphi(x_k) \left(\int_{x_k}^{x_{k+1}} v(t)^{1-\theta'}\,dt \right)^{q/\theta'} \right)^{1/q} \lesssim \left(\sum_{k\in\mathbb{Z}} a_k^\theta \right)^{1/\theta}.
\end{aligned} \tag{3.43}$$

Now using Proposition 2.14 for the case $\theta \le q$, we obtain that

$$\begin{aligned}
&\sup_{k\in\mathbb{Z}} \frac{\varphi(x_k)^{1/q}}{U^{1/p}(x_k)} \left(\int_{x_{k-1}}^{x_k} \left(\int_{x_{k-1}}^{t} u(s)\,ds \right)^{r/p} \left(\int_t^{x_k} v(s)^{1-\theta'}\,ds \right)^{r/p'} v(t)^{1-\theta'}\,dt \right)^{1/r} \\
&\quad + \sup_{k\in\mathbb{Z}} \varphi(x_k)^{1/q} \left(\int_{x_k}^{x_{k+1}} v(t)^{1-\theta'}\,dt \right)^{1/\theta'} < \infty.
\end{aligned} \tag{3.44}$$

Since

$$
\begin{aligned}
A_3 &\approx \sup_{k\in\mathbb{Z}} \varphi(x_k)^{1/q}\left(U^{-r/p}(x_k)\int_{x_{k-1}}^{x_k} U(t)^{r/p}V_\theta(t)^{r/p'}v(t)^{1-\theta'}dt + \int_{x_k}^{\infty} V_\theta(t)^{r/p'}v(t)^{1-\theta'}dt\right)^{1/r} \\
&\lesssim \sup_{k\in\mathbb{Z}} \frac{\varphi(x_k)^{1/q}}{U^{1/p}(x_k)}\left(\int_{x_{k-1}}^{x_k} U(t)^{r/p}V_\theta(t)^{r/p'}v(t)^{1-\theta'}dt\right)^{1/r} \\
&\quad + \sup_{k\in\mathbb{Z}} \varphi(x_k)^{1/q}V_\theta(x_k)^{1/\theta'} \\
&\lesssim \sup_{k\in\mathbb{Z}} \frac{\varphi(x_k)^{1/q}}{U^{1/p}(x_k)}\left(\int_{x_{k-1}}^{x_k}\left(\int_{x_{k-1}}^{t} u(s)ds\right)^{r/p} V_\theta(t)^{r/p'}v(t)^{1-\theta'}dt\right)^{1/r} \\
&\quad + \sup_{k\in\mathbb{Z}} \varphi(x_k)^{1/q}V_\theta(x_k)^{1/\theta'},
\end{aligned}
\tag{3.45}
$$

by integrating by parts, we get that

$$
\begin{aligned}
A_3 &\lesssim \sup_{k\in\mathbb{Z}} \frac{\varphi(x_k)^{1/q}}{U^{1/p}(x_k)}\left(\int_{x_{k-1}}^{x_k}\left(\int_{t}^{\infty} v(s)^{1-\theta'}ds\right)^{(r/p')+1} u(t)\left(\int_{x_{k-1}}^{t} u(s)ds\right)^{(r/p)-1} dt\right)^{1/r} \\
&\quad + \sup_{k\in\mathbb{Z}} \varphi(x_k)^{1/q}V_\theta(x_k)^{1/\theta'} \\
&\lesssim \sup_{k\in\mathbb{Z}} \frac{\varphi(x_k)^{1/q}}{U^{1/p}(x_k)}\left(\int_{x_{k-1}}^{x_k}\left(\int_{t}^{x_k} v(s)^{1-\theta'}ds\right)^{(r/p')+1} u(t)\left(\int_{x_{k-1}}^{t} u(s)ds\right)^{(r/p)-1} dt\right)^{1/r} \\
&\quad + \sup_{k\in\mathbb{Z}} \varphi(x_k)^{1/q}V_\theta(x_k)^{1/\theta'}.
\end{aligned}
\tag{3.46}
$$

Again integrating by parts, we arrive at

$$
\begin{aligned}
A_3 &\lesssim \sup_{k\in\mathbb{Z}} \frac{\varphi(x_k)^{1/q}}{U^{1/p}(x_k)}\left(\int_{x_{k-1}}^{x_k}\left(\int_{x_{k-1}}^{t} u(s)ds\right)^{r/p}\left(\int_{t}^{x_k} v(s)^{1-\theta'}ds\right)^{r/p'} v(t)^{1-\theta'}dt\right)^{1/r} \\
&\quad + \sup_{k\in\mathbb{Z}} \varphi(x_k)^{1/q}V_\theta(x_k)^{1/\theta'} < \infty.
\end{aligned}
\tag{3.47}
$$

Now let $1 < \theta < \infty$, $0 < p < \theta$, $q < \theta$. By using (3.43) and Proposition 2.14, we obtain

$$\left(\sum_{k\in\mathbb{Z}} \frac{\varphi(x_k)^{l/p}}{U^{1/p}(x_k)} \left(\int_{x_{k-1}}^{x_k} \left(\int_{x_{k-1}}^{t} u(s)ds\right)^{r/p} \left(\int_t^{x_k} v(s)^{1-\theta'} ds\right)^{r/p'} v(t)^{1-\theta'} dt\right)^{l/r}\right)^{1/l}$$
$$+ \left(\sum_{k\in\mathbb{Z}} \varphi(x_k)^{l/p} \left(\int_{x_k}^{x_{k+1}} v(t)^{1-\theta'} dt\right)^{l/\theta'}\right)^{1/l} < \infty. \tag{3.48}$$

Since

$$A_4 \approx \left(\sum_{k\in\mathbb{Z}} \varphi(x_k)^{l/q} \left(U^{-l/p}(x_k) \left(\int_{x_{k-1}}^{x_k} U(t)^{r/p} V_\theta(t)^{r/p'} v(t)^{1-\theta'} dt\right)^{l/r} + V_\theta(x_k)^{l/\theta'}\right)\right)^{1/l}$$
$$\lesssim \left(\sum_{k\in\mathbb{Z}} \frac{\varphi(x_k)^{l/q}}{U^{l/p}(x_k)} \left(\int_{x_{k-1}}^{x_k} U(t)^{r/p} V_\theta(t)^{r/p'} v(t)^{1-\theta'} dt\right)^{l/r}\right)^{1/l} + \left(\sum_{k\in\mathbb{Z}} \varphi(x_k)^{l/q} V_\theta(x_k)^{l/\theta'}\right)^{1/l}$$
$$\lesssim \left(\sum_{k\in\mathbb{Z}} \frac{\varphi(x_k)^{l/q}}{U^{l/p}(x_k)} \left(\int_{x_{k-1}}^{x_k} \left(\int_{x_{k-1}}^{t} u(s)ds\right)^{r/p} V_\theta(t)^{r/p'} v(t)^{1-\theta'} dt\right)^{l/r}\right)^{1/l}$$
$$+ \left(\sum_{k\in\mathbb{Z}} \varphi(x_k)^{l/q} V_\theta(x_k)^{l/\theta'}\right)^{1/l}, \tag{3.49}$$

integrating by parts, we find that

$$A_4 \lesssim \left(\sum_{k\in\mathbb{Z}} \frac{\varphi(x_k)^{l/q}}{U^{l/p}(x_k)} \left(\int_{x_{k-1}}^{x_k} \left(\int_t^{\infty} v(s)^{1-\theta'} ds\right)^{(r/p')+1} u(t) \left(\int_{x_{k-1}}^{t} u(s)ds\right)^{(r/p)-1} dt\right)^{l/r}\right)^{1/l}$$
$$+ \left(\sum_{k\in\mathbb{Z}} \varphi(x_k)^{l/q} V_\theta(x_k)^{l/\theta'}\right)^{1/l}$$
$$\lesssim \left(\sum_{k\in\mathbb{Z}} \frac{\varphi(x_k)^{l/q}}{U^{l/p}(x_k)} \left(\int_{x_{k-1}}^{x_k} \left(\int_t^{x_k} v(s)^{1-\theta'} ds\right)^{(r/p')+1} u(t) \left(\int_{x_{k-1}}^{t} u(s)ds\right)^{(r/p)-1} dt\right)^{l/r}\right)^{1/l}$$
$$+ \left(\sum_{k\in\mathbb{Z}} \varphi(x_k)^{l/q} V_\theta(x_k)^{l/\theta'}\right)^{1/l}. \tag{3.50}$$

Again integrating by parts, we arrive at

$$A_4 \lesssim \left(\sum_{k\in\mathbb{Z}} \frac{\varphi(x_k)^{l/q}}{U^{l/p}(x_k)} \left(\int_{x_{k-1}}^{x_k} \left(\int_{x_{k-1}}^{t} u(s)ds \right)^{r/p} \left(\int_{t}^{x_k} v(s)^{1-\theta'} ds \right)^{r/p'} v(t)^{1-\theta'} dt \right)^{l/r} \right)^{1/l}$$
$$+ \left(\sum_{k\in\mathbb{Z}} \varphi(x_k)^{l/q} V_\theta(x_k)^{l/\theta'} \right)^{1/l} < \infty. \tag{3.51}$$

Now let $0 < p < \infty$, $\theta = \infty$, $0 < q < \infty$. For $k \in \mathbb{Z}$, let h_k be functions that saturate the Hardy inequality (2.21) and Hölder's inequality for $\theta = \infty$, that is, functions h_k satisfying

$$\begin{gathered} \operatorname{supp} h_k \subset [x_{k-1}, x_k], \\ \|h_k\|_{\infty,v,(0,\infty)} = 1, \\ \left\| \int_s^{x_k} h_k(z)dz \right\|_{p,u,(x_{k-1},x_k)} \gtrsim \left(\int_{x_{k-1}}^{x_k} \left(\int_s^{x_k} \frac{dz}{v(z)} \right)^p u(s)ds \right)^{1/p}, \\ \int_{x_{k-1}}^{x_k} h_k(t)dt \gtrsim \int_{x_k}^{x_{k+1}} \frac{dz}{v(z)}. \end{gathered} \tag{3.52}$$

Now we define the test function

$$h(t) = \sum_{k\in\mathbb{Z}} a_k h_k(t), \tag{3.53}$$

where $\{a_k\}$ is a sequence of positive real numbers. Thus, using test function (3.53) in (3.32), we get

$$\left(\sum_{k\in\mathbb{Z}} a_k^q \frac{\varphi(x_k)}{U(x_k)^{q/p}} \left(\int_{x_{k-1}}^{x_k} \left(\int_s^{x_k} \frac{dz}{v(z)} \right)^p u(s)ds \right)^{q/p} \right)^{1/q}$$
$$+ \left(\sum_{k\in\mathbb{Z}} a_k^q \varphi(x_k) \left(\int_{x_k}^{x_{k+1}} \frac{dz}{v(z)} \right)^q \right)^{1/q} \lesssim \sup_{k\in\mathbb{Z}} a_k. \tag{3.54}$$

Hence, by Proposition 2.14, we have that

$$\left(\sum_{k\in\mathbb{Z}} \frac{\varphi(x_k)}{U(x_k)^{q/p}} \left(\int_{x_{k-1}}^{x_k} \left(\int_s^{x_k} \frac{dz}{v(z)} \right)^p u(s)ds \right)^{q/p} \right)^{1/q} + \left(\sum_{k\in\mathbb{Z}} \varphi(x_k) \left(\int_{x_k}^{x_{k+1}} \frac{dz}{v(z)} \right)^q \right)^{1/q} < \infty. \tag{3.55}$$

On the other hand,

$$A_5 \approx \left(\sum_{k\in\mathbb{Z}}\varphi(x_k)\left(U(x_k)^{-1}\int_{x_{k-1}}^{x_k}U(s)\left(\int_s^\infty\frac{dz}{v(z)}\right)^{p-1}\frac{ds}{v(s)}+\int_{x_k}^\infty\left(\int_s^\infty\frac{dz}{v(z)}\right)^{p-1}\frac{ds}{v(s)}\right)^{q/p}\right)^{1/q}$$
$$\approx\left(\sum_{k\in\mathbb{Z}}\frac{\varphi(x_k)}{U(x_k)^{q/p}}\left(\int_{x_{k-1}}^{x_k}U(s)\left(\int_s^\infty\frac{dz}{v(z)}\right)^{p-1}\frac{ds}{v(s)}\right)^{q/p}\right)^{1/q}+\left(\sum_{k\in\mathbb{Z}}\varphi(x_k)\left(\int_{x_k}^\infty\frac{dz}{v(z)}\right)^{q}\right)^{1/q}. \tag{3.56}$$

Integrating by part and using Lemma 2.9, we get that

$$A_5 \lesssim \left(\sum_{k\in\mathbb{Z}}\frac{\varphi(x_k)}{U(x_k)^{q/p}}\left(\int_{x_{k-1}}^{x_k}\left(\int_s^\infty\frac{dz}{v(z)}\right)^{p}u(s)ds\right)^{q/p}\right)^{1/q}+\left(\sum_{k\in\mathbb{Z}}\varphi(x_k)\left(\int_{x_k}^\infty\frac{dz}{v(z)}\right)^{q}\right)^{1/q}$$
$$\lesssim \left(\sum_{k\in\mathbb{Z}}\frac{\varphi(x_k)}{U(x_k)^{q/p}}\left(\int_{x_{k-1}}^{x_k}\left(\int_s^{x_k}\frac{dz}{v(z)}\right)^{p}u(s)ds\right)^{q/p}\right)^{1/q}+\left(\sum_{k\in\mathbb{Z}}\varphi(x_k)\left(\int_{x_k}^{x_{k+1}}\frac{dz}{v(z)}\right)^{q}\right)^{1/q}$$
$$<\infty. \tag{3.57}$$

The proof is complete. □

We now state the announced characterization of (3.2).

Theorem 3.2. *Let $0 < p < \infty$, $1 < \theta \le \infty$, and let u, v, w be weights. Assume that u is such that $U^{1/p}$ is admissible and the measure $w(t)dt$ is nondenerate with respect to $U^{1/p}$. Then the inequality* (3.2) *holds for every measurable function f on $(0,\infty)$ if and only if*

(i) $1 < \theta \le p < \infty$ *and*

$$B_1 := \sup_{x\in(0,\infty)}\operatorname*{ess\,sup}_{s\in(0,\infty)} w(s)\mathcal{U}(x,s)^{1/p}\sup_{t\in(0,\infty)}V_\theta(t)^{1/\theta'}\mathcal{U}(t,x)^{1/p}<\infty. \tag{3.58}$$

Moreover, the best constant c in (3.2) *satisfies $c \approx B_1$.*

(ii) $0 < p < \theta < \infty$, $1 < \theta < \infty$, $r = \theta p/(\theta - p)$ *and*

$$B_2 := \sup_{x\in(0,\infty)}\operatorname*{ess\,sup}_{s\in(0,\infty)} w(s)\mathcal{U}(x,s)^{1/p}\left(\int_0^\infty \mathcal{U}(t,x)^{r/p}V_\theta(t)^{r/p'}v(t)^{1-\theta'}dt\right)^{1/r}<\infty. \tag{3.59}$$

Moreover, the best constant c in (3.2) *satisfies $c \approx B_2$.*

(iii) $0 < p < \infty$, $\theta = \infty$ *and*

$$B_3 := \sup_{x\in(0,\infty)} \left(\int_0^\infty \mathcal{U}(s,t)\left(\int_s^\infty \frac{dz}{v(z)}\right)^{p-1} \frac{ds}{v(s)}\right)^{1/p} \operatorname*{ess\,sup}_{s\in(0,\infty)} w(s)\mathcal{U}(x,s)^{1/p} < \infty. \tag{3.60}$$

Moreover, the best constant c in (3.2) *satisfies $c \approx B_3$.*

Proof. Using Corollary 2.5, Lemmas 2.8 and 2.9, we obtain for the left-hand side J_0 of (3.2) that (φ is defined by (2.7))

$$\begin{aligned}
J_0 &\approx \sup_{t\in(0,\infty)} \frac{\varphi(t)}{U(t)^{1/p}}\left(\int_0^t \left(\int_s^\infty h(z)dz\right)^p u(s)ds\right)^{1/p} \\
&\approx \sup_{k\in\mathbb{Z}} \frac{\varphi(x_k)}{U(x_k)^{1/p}}\left(\int_0^{x_k} \left(\int_s^\infty h(z)dz\right)^p u(s)ds\right)^{1/p} \\
&\approx \sup_{k\in\mathbb{Z}} \frac{\varphi(x_k)}{U(x_k)^{1/p}}\left(\int_{x_{k-1}}^{x_k} \left(\int_s^{x_k} h(z)dz\right)^p u(s)ds\right)^{1/p} \\
&\quad + \sup_{k\in\mathbb{Z}} \varphi(x_k)\int_{x_k}^{x_{k+1}} h(z)dz := III + IV.
\end{aligned} \tag{3.61}$$

(i) For the case $1 < \theta \le p < \infty$, by using Lemma 2.15 for III and applying Hölder's inequality for IV, we arrive at

$$\begin{aligned}
III &\lesssim \sup_{k\in\mathbb{Z}} \frac{\varphi(x_k)}{U(x_k)^{1/p}} \sup_{x_{k-1}<t<x_k} U(t)^{1/p} V_\theta(t)^{1/\theta'}\left(\int_{x_{k-1}}^{x_k} h(z)^\theta v(z)dz\right)^{1/\theta}, \\
IV &\lesssim \sup_{k\in\mathbb{Z}} \varphi(x_k) V_\theta(x_k)^{1/\theta'}\left(\int_{x_{k-1}}^{x_k} h(z)^\theta v(z)dz\right)^{1/\theta},
\end{aligned} \tag{3.62}$$

so that, by Lemma 2.13 and (3.61), we obtain that

$$\begin{aligned}
J_0 &\lesssim \left(\sup_{k\in\mathbb{Z}} \frac{\varphi(x_k)}{U(x_k)^{1/p}} \sup_{x_{k-1}<t<x_k} U(t)^{1/p} V_\theta(t)^{1/\theta'} + \sup_{k\in\mathbb{Z}} \varphi(x_k) V_\theta(x_k)^{1/\theta'}\right)\|h\|_{\theta,v,(0,\infty)} \\
&\approx \sup_{x\in(0,\infty)} \varphi(x) \sup_{0<t<\infty} \mathcal{U}(t,x)^{1/p} V_\theta(t)^{1/\theta'} \|h\|_{\theta,v,(0,\infty)} \\
&= \sup_{x\in(0,\infty)} \operatorname*{ess\,sup}_{s\in(0,\infty)} w(s)\mathcal{U}(x,s)^{1/p} \sup_{0<t<\infty} V_\theta(t)^{1/\theta'}\, \mathcal{U}(t,x)^{1/p}\|h\|_{\theta,v,(0,\infty)}.
\end{aligned} \tag{3.63}$$

(ii) Let now $0 < p < \theta < \infty$, $1 < \theta < \infty$, $r = \theta p/(\theta - p)$. By using Lemma 2.15 for III and applying Hölder's inequality for IV, we find that

$$III \lesssim \sup_{k\in\mathbb{Z}} \frac{\varphi(x_k)}{U(x_k)^{1/p}} \left(\int_{x_{k-1}}^{x_k} \left(\int_{x_{k-1}}^{t} u(s)ds \right)^{r/p} \left(\int_t^{x_k} v(s)^{1-\theta'} ds \right)^{r/p'} v(t)^{1-\theta'} dt \right)^{1/r}$$
$$\times \left(\int_{x_{k-1}}^{x_k} h(z)^{\theta} v(z) dz \right)^{1/\theta}, \tag{3.64}$$
$$IV \lesssim \sup_{k\in\mathbb{Z}} \varphi(x_k) V_{\theta}(x_k)^{1/\theta'} \left(\int_{x_{k-1}}^{x_k} h(z)^{\theta} v(z) dz \right)^{1/\theta}$$

and, by Lemmas 2.9 and 2.12, and (3.61), we get that

$$J_0 \lesssim \sup_{k\in\mathbb{Z}} \varphi(x_k) \left(U(x_k)^{-r/p} \int_{x_{k-1}}^{x_k} U(t)^{r/p} V_{\theta}(t)^{r/p'} v(t)^{1-\theta'} dt + \int_{x_k}^{\infty} V_{\theta}(t)^{r/p'} v(t)^{1-\theta'} dt \right)^{1/r}$$
$$\times \|h\|_{\theta,v,(0,\infty)}$$
$$\approx \sup_{k\in\mathbb{Z}} \varphi(x_k) \left(U(x_k)^{-r/p} \int_{x_{k-1}}^{x_k} U(t)^{r/p} V_{\theta}(t)^{r/p'} v(t)^{1-\theta'} dt + \int_{x_k}^{x_{k+1}} V_{\theta}(t)^{r/p'} v(t)^{1-\theta'} dt \right)^{1/r}$$
$$\times \|h\|_{\theta,v,(0,\infty)}$$
$$\approx \sup_{x\in(0,\infty)} \varphi(x) \left(\int_0^{\infty} \mathcal{U}(t,x)^{r/p} V_{\theta}(t)^{r/p'} v(t)^{1-\theta'} dt \right)^{1/r} \|h\|_{\theta,v,(0,\infty)}$$
$$= \sup_{x\in(0,\infty)} \operatorname*{ess\,sup}_{s\in(0,\infty)} w(s) \mathcal{U}(x,s)^{1/p} \left(\int_0^{\infty} \mathcal{U}(t,x)^{r/p} V_{\theta}(t)^{r/p'} v(t)^{1-\theta'} dt \right)^{1/r} \|h\|_{\theta,v,(0,\infty)}. \tag{3.65}$$

(iii) Now let $0 < p < \infty$, $\theta = \infty$. By using Lemma 2.15 for III, we deduce that

$$III \lesssim \sup_{k\in\mathbb{Z}} \frac{\varphi(x_k)}{U(x_k)^{1/p}} \left(\int_{x_{k-1}}^{x_k} \left(\int_s^{x_k} \frac{dz}{v(z)} \right)^p u(s) ds \right)^{1/p} \|h\|_{\infty,v,(0,\infty)}. \tag{3.66}$$

Moreover, for IV, it yields that

$$IV \lesssim \sup_{k\in\mathbb{Z}} \varphi(x_k) \int_{x_k}^{x_{k+1}} \frac{dz}{v(z)} \|h\|_{\infty,v,(0,\infty)}. \tag{3.67}$$

Therefore, by using integration by parts, Lemma 2.12, and (3.61), we get that

$$\begin{aligned} J_0 &\lesssim \sup_{k\in\mathbb{Z}} \varphi(x_k)\left(U(x_k)^{-1}\int_{x_{k-1}}^{x_k} U(s)\left(\int_s^\infty \frac{dz}{v(z)}\right)^{p-1}\frac{ds}{v(s)} + \int_{x_k}^\infty\left(\int_s^\infty \frac{dz}{v(z)}\right)^{p-1}\frac{ds}{v(s)}\right)^{1/p} \\ &\quad \times \|h\|_{\infty,v,(0,\infty)} \\ &\approx \sup_{x\in(0,\infty)}\left(\int_0^\infty \mathcal{U}(s,t)\left(\int_s^\infty \frac{dz}{v(z)}\right)^{p-1}\frac{ds}{v(s)}\right)^{1/p}\varphi(x)\|h\|_{\infty,v,(0,\infty)} \\ &= \sup_{x\in(0,\infty)}\left(\int_0^\infty \mathcal{U}(s,t)\left(\int_s^\infty \frac{dz}{v(z)}\right)^{p-1}\frac{ds}{v(s)}\right)^{1/p}\operatorname*{ess\,sup}_{s\in(0,\infty)} w(s)\mathcal{U}(x,s)^{1/p}\|h\|_{\infty,v,(0,\infty)}. \end{aligned} \tag{3.68}$$

Now we prove the lower bounds (necessity). Let $\{x_k\}$ be a discretizing sequence for φ defined by (2.7). Then, by (3.61), we have

$$\begin{aligned} &\sup_{k\in\mathbb{Z}}\frac{\varphi(x_k)}{U(x_k)^{1/p}}\left(\int_{x_{k-1}}^{x_k}\left(\int_s^{x_k} h(z)dz\right)^p u(s)ds\right)^{1/p} + \sup_{k\in\mathbb{Z}}\varphi(x_k)\int_{x_k}^{x_{k+1}} h(z)dz \\ &\quad\lesssim \left(\sum_{k\in\mathbb{Z}}\int_{x_{k-1}}^{x_k} h(z)^\theta v(z)dz\right)^{1/\theta}. \end{aligned} \tag{3.69}$$

Let $1<\theta\le p<\infty$. If we use in (3.69) the test function defined by (3.34), we obtain that

$$\begin{aligned} &\sup_{k\in\mathbb{Z}} a_k\frac{\varphi(x_k)}{U(x_k)^{1/p}}\sup_{x_{k-1}<t<x_k}\left(\int_{x_{k-1}}^t u(s)ds\right)^{1/p}\left(\int_t^{x_k} v(s)^{1-\theta'}ds\right)^{1/\theta'} \\ &\quad + \sup_{k\in\mathbb{Z}} a_k\varphi(x_k)\left(\int_{x_k}^{x_{k+1}} v(t)^{1-\theta'}dt\right)^{1/\theta'} \lesssim \left(\sum_{k\in\mathbb{Z}} a_k^\theta\right)^{1/\theta}. \end{aligned} \tag{3.70}$$

Therefore, by Proposition 2.14, we have that

$$\begin{aligned} &\sup_{k\in\mathbb{Z}}\frac{\varphi(x_k)}{U(x_k)^{1/p}}\sup_{x_{k-1}<t<x_k}\left(\int_{x_{k-1}}^t u(s)ds\right)^{1/p}\left(\int_t^{x_k} v(s)^{1-\theta'}ds\right)^{1/\theta'} \\ &\quad + \sup_{k\in\mathbb{Z}}\varphi(x_k)\left(\int_{x_k}^{x_{k+1}} v(t)^{1-\theta'}dt\right)^{1/\theta'} < \infty. \end{aligned} \tag{3.71}$$

Since

$$\begin{aligned} B_1 &\approx \sup_{k\in\mathbb{Z}} \frac{\varphi(x_k)}{U(x_k)^{1/p}} \sup_{x_{k-1}<t<x_k} U(t)^{1/p} V_\theta(t)^{1/\theta'} + \sup_{k\in\mathbb{Z}} \varphi(x_k) V_\theta(x_k)^{1/\theta'} \\ &\lesssim \sup_{k\in\mathbb{Z}} \frac{\varphi(x_k)}{U(x_k)^{1/p}} \sup_{x_{k-1}<t<x_k} \left(\int_{x_{k-1}}^{t} u(s)ds\right)^{1/p} \left(\int_t^{x_k} v(s)^{1-\theta'} ds\right)^{1/\theta'} + \sup_{k\in\mathbb{Z}} \varphi(x_k) V_\theta(x_k)^{1/\theta'}, \end{aligned} \tag{3.72}$$

by Lemma 2.9, we get that

$$\begin{aligned} B_1 \lesssim \sup_{k\in\mathbb{Z}} &\frac{\varphi(x_k)}{U(x_k)^{1/p}} \sup_{x_{k-1}<t<x_k} \left(\int_{x_{k-1}}^{t} u(s)ds\right)^{1/p} \left(\int_t^{x_k} v(s)^{1-\theta'} ds\right)^{1/\theta'} \\ &+ \sup_{k\in\mathbb{Z}} \varphi(x_k) \left(\int_{x_k}^{x_{k+1}} v(t)^{1-\theta'} dt\right)^{1/\theta'} < \infty. \end{aligned} \tag{3.73}$$

Now let $0 < p < \theta < \infty$, $1 < \theta < \infty$, $r = \theta p/(\theta - p)$. By using in (3.69) the test function defined by (3.42), we obtain that

$$\begin{aligned} \sup_{k\in\mathbb{Z}} a_k &\frac{\varphi(x_k)}{U(x_k)^{1/p}} \left(\int_{x_{k-1}}^{x_k} \left(\int_{x_{k-1}}^{t} u(s)ds\right)^{r/p} \left(\int_t^{x_k} v(s)^{1-\theta'} ds\right)^{r/p'} v(t)^{1-\theta'} dt\right)^{1/r} \\ &+ \sup_{k\in\mathbb{Z}} a_k \varphi(x_k) \left(\int_{x_k}^{x_{k+1}} v(t)^{1-\theta'} dt\right)^{1/\theta'} \lesssim \left(\sum_{k\in\mathbb{Z}} a_k^\theta\right)^{1/\theta}. \end{aligned} \tag{3.74}$$

Then, by Proposition 2.14, we get that

$$\begin{aligned} \sup_{k\in\mathbb{Z}} &\frac{\varphi(x_k)}{U(x_k)^{1/p}} \left(\int_{x_{k-1}}^{x_k} \left(\int_{x_{k-1}}^{t} u(s)ds\right)^{r/p} \left(\int_t^{x_k} v(s)^{1-\theta'} ds\right)^{r/p'} v(t)^{1-\theta'} dt\right)^{1/r} \\ &+ \sup_{k\in\mathbb{Z}} \varphi(x_k) \left(\int_{x_k}^{x_{k+1}} v(t)^{1-\theta'} dt\right)^{1/\theta'} < \infty. \end{aligned} \tag{3.75}$$

Since

$$
\begin{aligned}
B_2 &\approx \sup_{k\in\mathbb{Z}} \varphi(x_k)\left(U(x_k)^{-r/p}\int_{x_{k-1}}^{x_k} U(t)^{r/p}V_\theta(t)^{r/p'}v(t)^{1-\theta'}dt + \int_{x_k}^{\infty} V_\theta(t)^{r/p'}v(t)^{1-\theta'}dt\right)^{1/\theta'} \\
&\lesssim \sup_{k\in\mathbb{Z}} \frac{\varphi(x_k)}{U(x_k)^{1/p}}\left(\int_{x_{k-1}}^{x_k} U(t)^{r/p}V_\theta(t)^{r/p'}v(t)^{1-\theta'}dt\right)^{1/r} + \sup_{k\in\mathbb{Z}} \varphi(x_k)V_\theta(x_k)^{1/\theta'} \\
&\lesssim \sup_{k\in\mathbb{Z}} \frac{\varphi(x_k)}{U(x_k)^{1/p}}\left(\int_{x_{k-1}}^{x_k} \left(\int_{x_{k-1}}^{t} u(s)ds\right)^{r/p} V_\theta(t)^{r/p'}v(t)^{1-\theta'}dt\right)^{1/r} + \sup_{k\in\mathbb{Z}} \varphi(x_k)V_\theta(x_k)^{1/\theta'},
\end{aligned}
\tag{3.76}
$$

by integrating by parts, we find that

$$
\begin{aligned}
B_2 &\lesssim \sup_{k\in\mathbb{Z}} \frac{\varphi(x_k)}{U(x_k)^{1/p}}\left(\int_{x_{k-1}}^{x_k} \left(\int_{t}^{\infty} v(s)^{1-\theta'}ds\right)^{(r/p')+1} u(t)\left(\int_{x_{k-1}}^{t} u(s)ds\right)^{(r/p)-1} dt\right)^{1/r} \\
&\quad + \sup_{k\in\mathbb{Z}} \varphi(x_k)V_\theta(x_k)^{1/\theta'} \\
&\lesssim \sup_{k\in\mathbb{Z}} \frac{\varphi(x_k)}{U(x_k)^{1/p}}\left(\int_{x_{k-1}}^{x_k} \left(\int_{t}^{x_k} v(s)^{1-\theta'}ds\right)^{(r/p')+1} u(t)\left(\int_{x_{k-1}}^{t} u(s)ds\right)^{(r/p)-1} dt\right)^{1/r} \\
&\quad + \sup_{k\in\mathbb{Z}} \varphi(x_k)V_\theta(x_k)^{1/\theta'}.
\end{aligned}
\tag{3.77}
$$

Moreover, by again integrating by parts, we arrive at

$$
\begin{aligned}
B_2 &\lesssim \sup_{k\in\mathbb{Z}} \frac{\varphi(x_k)}{U(x_k)^{1/p}}\left(\int_{x_{k-1}}^{x_k} \left(\int_{x_{k-1}}^{t} u(s)ds\right)^{r/p} \left(\int_{t}^{x_k} v(s)^{1-\theta'}ds\right)^{r/p'} v(t)^{1-\theta'}dt\right)^{1/r} \\
&\quad + \sup_{k\in\mathbb{Z}} \varphi(x_k)V_\theta(x_k)^{1/\theta'} < \infty.
\end{aligned}
\tag{3.78}
$$

Finally, let $0 < p < \infty$, $\theta = \infty$. By using the test function defined by (3.53) in (3.69), we get that

$$
\sup_{k\in\mathbb{Z}} a_k \frac{\varphi(x_k)}{U(x_k)^{1/p}}\left(\int_{x_{k-1}}^{x_k} \left(\int_{s}^{x_k} \frac{dz}{v(z)}\right)^{p} u(s)ds\right)^{1/p} + \sup_{k\in\mathbb{Z}} a_k\varphi(x_k)\int_{x_k}^{x_{k+1}} \frac{dz}{v(z)} \lesssim \sup_{k\in\mathbb{Z}} a_k.
\tag{3.79}
$$

Hence, by Proposition 2.14, we have that

$$\sup_{k\in\mathbb{Z}} \frac{\varphi(x_k)}{U(x_k)^{1/p}} \left(\int_{x_{k-1}}^{x_k} \left(\int_s^{x_k} \frac{dz}{v(z)} \right)^p u(s)ds \right)^{1/p} + \sup_{k\in\mathbb{Z}} \varphi(x_k) \int_{x_k}^{x_{k+1}} \frac{dz}{v(z)} < \infty. \tag{3.80}$$

Since

$$\begin{aligned} B_3 &\approx \sup_{k\in\mathbb{Z}} \varphi(x_k) \left(U(x_k)^{-1} \int_{x_{k-1}}^{x_k} U(s) \left(\int_s^\infty \frac{dz}{v(z)} \right)^{p-1} \frac{ds}{v(s)} + \int_{x_k}^\infty \left(\int_s^\infty \frac{dz}{v(z)} \right)^{p-1} \frac{ds}{v(s)} \right)^{1/p} \\ &\lesssim \sup_{k\in\mathbb{Z}} \frac{\varphi(x_k)}{U(x_k)^{1/p}} \left(\int_{x_{k-1}}^{x_k} U(s) \left(\int_s^\infty \frac{dz}{v(z)} \right)^{p-1} \frac{ds}{v(s)} \right)^{1/p} + \sup_{k\in\mathbb{Z}} \varphi(x_k) \int_{x_k}^\infty \frac{dz}{v(z)} \\ &\lesssim \sup_{k\in\mathbb{Z}} \frac{\varphi(x_k)}{U(x_k)^{1/p}} \left(\int_{x_{k-1}}^{x_k} U(s) \left(\int_s^{x_k} \frac{dz}{v(z)} \right)^{p-1} \frac{ds}{v(s)} \right)^{1/p} + \sup_{k\in\mathbb{Z}} \varphi(x_k) \int_{x_k}^\infty \frac{dz}{v(z)}, \end{aligned} \tag{3.81}$$

by integrating by parts and using Lemma 2.9, we obtain that

$$B_3 \lesssim \sup_{k\in\mathbb{Z}} \frac{\varphi(x_k)}{U(x_k)^{1/p}} \left(\int_{x_{k-1}}^{x_k} \left(\int_s^{x_k} \frac{dz}{v(z)} \right)^p u(s)ds \right)^{1/p} + \sup_{k\in\mathbb{Z}} \varphi(x_k) \int_{x_k}^{x_{k+1}} \frac{dz}{v(z)} < \infty. \tag{3.82}$$

The proof is complete. □

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